- Problem:
Suppose the equations AB, BC and CA are respectively given by

$$\mathbf{n_i}^\top \mathbf{x} = c_i \tag{1}$$

$$i = 1, 2, 3$$

The equations of the respective angle bisectors are then given by

$$\frac{\mathbf{n_i}^\top \mathbf{x} - c_i}{\|\mathbf{n_i}\|} = \pm \frac{\mathbf{n_j}^\top \mathbf{x} - c_j}{\|\mathbf{n_j}\|} \tag{2}$$

$$i \neq j$$

Substitute numerical values and find the equations of the angle bisectors of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

- Solution:
Given $\mathbf{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$
The normal form of a line is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{3}$$

where $\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{m}$ and $\mathbf{m} = (\mathbf{B} - \mathbf{A})$, where $\mathbf{A}$, $\mathbf{B}$ are two points on the line and $\mathbf{m}$ is slope vector of the line.
From problem 1.1.5,

$$CA : \begin{pmatrix} 4 & -4 \end{pmatrix} \mathbf{x} = 8 \tag{4}$$

Similarly;

$$AB : \begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} = 2 \tag{5}$$

$$BC : \begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} = -10 \tag{6}$$

From eq.1, the equations AB, BC and CA are respectively,

$$n_i^\top \mathbf{x} = c_i \tag{7}$$

$$i = 1, 2, 3$$

$$\implies \mathbf{n_1} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$\mathbf{n_2} = \begin{pmatrix} -11 \\ -1 \end{pmatrix}$$

$$\mathbf{n_3} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

and

$$c_1 = 2$$

$$c_2 = -10$$

$$c_3 = 8$$

From eq 2, the equations of the respective angle bisectors are then given by,

$$\frac{\mathbf{n_i}^\top \mathbf{x} - c_i}{\|\mathbf{n_i}\|} = \pm \frac{\mathbf{n_j}^\top \mathbf{x} - c_j}{\|\mathbf{n_j}\|}$$

$$i \neq j$$

For internal angle bisector,

$$\frac{\mathbf{n_i}^\top \mathbf{x} - c_i}{\|\mathbf{n_i}\|} = \frac{\mathbf{n_j}^\top \mathbf{x} - c_j}{\|\mathbf{n_j}\|}$$

$\implies$ Internal angle bisector of $\mathbf{B}$ (i=1 , j=2) :

$$\frac{\begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 2}{\left\| \begin{pmatrix} 7 \\ 5 \end{pmatrix} \right\|} = \frac{\begin{pmatrix} -11 & -1 \end{pmatrix} \mathbf{x} - 10}{\left\| \begin{pmatrix} -11 \\ -1 \end{pmatrix} \right\|}$$

$\because \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = \sqrt{x_1^2 + x_2^2}$:

$$\implies \frac{\begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 2}{\sqrt{74}} = \frac{\begin{pmatrix} 11 & -1 \end{pmatrix} \mathbf{x} - 10}{\sqrt{122}}$$

$$\implies \frac{\begin{pmatrix} 7 & 5 \end{pmatrix} \mathbf{x} - 2}{\sqrt{37}} = \frac{\begin{pmatrix} 11 & -1 \end{pmatrix} \mathbf{x} - 10}{\sqrt{61}}$$

$$\implies \begin{pmatrix} 7\sqrt{61} - 11\sqrt{37} & 5\sqrt{61} + \sqrt{37} \end{pmatrix} \mathbf{x} = 2(\sqrt{61} - 5\sqrt{37})$$

$\therefore$ Internal angle bisector of $\mathbf{B}$:

$$\begin{pmatrix} 7\sqrt{61} - 11\sqrt{37} & 5\sqrt{61} + \sqrt{37} \end{pmatrix} \mathbf{x} = 2(\sqrt{61} - 5\sqrt{37})$$

$$or$$

$$\begin{pmatrix} -12.2386 & 45.1304 \end{pmatrix} \mathbf{x} = -45.2071$$