Warm-up: Basic deterministic under-sampling

In this first notebook, we start with simple deterministic under-sampling using low-freq and high-freq only masks. Hence we want to generate a sampling mask $\Omega \in \{0,1\}^N$ such that $y = F_\Omega x$ where $x \in R^N$ is the original image. The data y is then corrupted by additive noise b:

$$y = F_\Omega x + b$$

where b is a complex-valued circular Gaussian noise of variance σ^2 and $F_\Omega = \Omega F$ in this **Cartesian** setting (i.e. sampling on the grid). We consider:

1. a low-frequency Cartesian mask Ω is defined by the central lines of k-space.
2. a low-frequency Cartesian mask Ω is defined by a square box in k-space centered at $(k_x, k_y) = (0, 0)$.
3. a high-frequency Cartesian mask Ω is defined as the complementary set of the lines defined in to 1)
4. a high-frequency Cartesian mask Ω is defined as the complementary set of the box defined in 2)

Based on these masks we generate the corresponding measured data from a reference MR image and finally perform Cartesian image reconstruction $\hat{x}$ from the under-sampled data y as follows:

$$\hat{x} = \tilde{F}^* y$$

where F^* is the zero-filled inverse Fourier transform (ifft).

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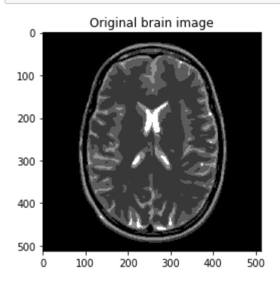
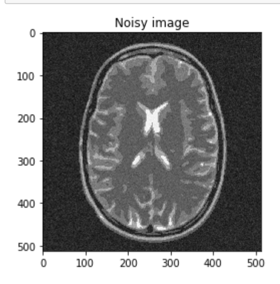
In [70]:
```
#DISPLAY BRAIN PHANTOM
%matplotlib inline
import sys
import os.path as op
import os
import math
import cmath

import numpy as np

import matplotlib.pyplot as plt

from skimage import data, io

# get current working dir
cwd = os.getcwd()
#cwd= "/"
dirimg_2d = op.join(cwd, "../data")
img_size = 512   #256
FOV = 0.2 #field of view in m
pixelSize = FOV/img_size

# load data file corresponding to the target resolution
filename = "BrainPhantom%s.png" % img_size
mri_filename = op.join(dirimg_2d, filename)
mri_img = io.imread(mri_filename)

plt.figure()
plt.title("Brain Phantom, size = %s " % img_size)
if mri_img.ndim == 2:
    plt.imshow(mri_img, cmap=plt.cm.gray)
else:
    plt.imshow(mri_img)
plt.title("Original brain image")
plt.show()
#plt.close()
```

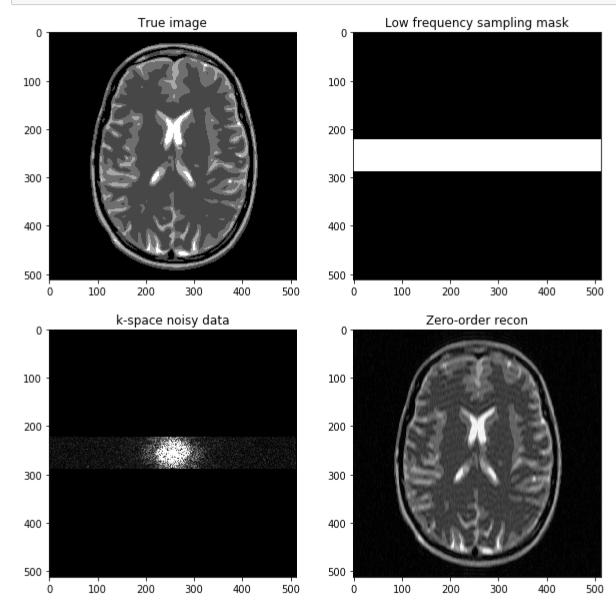


In [71]:
```
# Plot Noisy image
new_img = np.copy(mri_img.astype('float64'))
new_img += np.random.randn(*mri_img.shape) * 10.
#print(mri_img.shape)

if 1:
    plt.figure()
    plt.imshow(new_img, cmap='Greys_r')
    plt.title("Noisy image")
    plt.show()
```



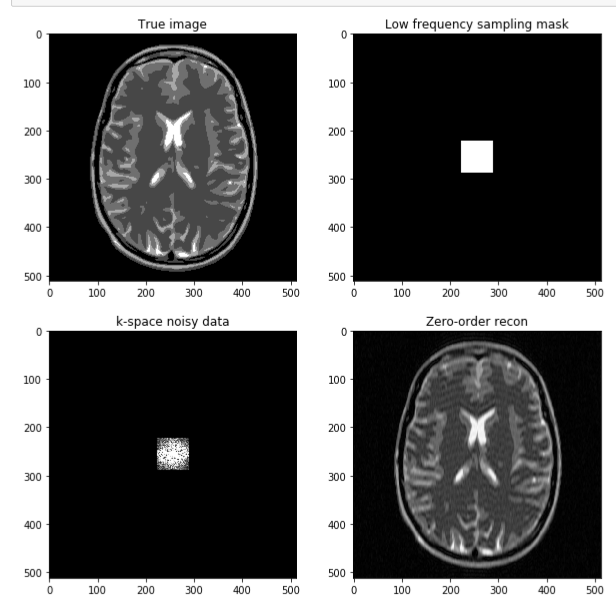
- To start up, the objective is to construct a low-frequency sampling mask that consists of the central k-space lines
- Fourier transform the reference MR image to compute k-space data, add zero-mean Gaussian complex-valued noise
- Mask the data using the above defined mask
- Next, perform zero-filled MR image reconstruction from masked k-space data
- Study the impact of the number of lines in the mask on the final image quality

In [72]:
```
# Generate trivial Cartesian sampling masks
img_size = 512

mask="low_res"
factor_ctr = 8

#Objective: construct a k-space mask that consists of the central lines
# 0 entries: points not sampled, 1 entries: points to be sampled
# Initialize k-space with 0 everywhere and then place the "1" appropriately
kspace_masklines = np.zeros((img_size,img_size), dtype="float64")

low_res_size = img_size // factor_ctr + 1
idx_vec = np.linspace(img_size // 2 - low_res_size // 2, img_size // 2 + low_res_size // 2, low_res_size)
idx_vec_ =  idx_vec.astype("int")
#print(idx_vec_)

#Use fancy indexing to select low frequency lines only
kspace_masklines[idx_vec_, ] = 1

#if 0: # to debug
#    vec =  np.array([idx_vec.astype("int")])
#    zero_vfill = np.zeros((1, (img_size-low_res_size) // 2), dtype="int")
#    line_vec = np.concatenate((zero_vfill, vec, zero_vfill), axis=1)
#    center_lines = np.tile(line_vec, (low_res_size, 1))
#    kspace_mask[center_lines] =1.

if 0:
    plt.figure()
    plt.imshow(kspace_masklines, cmap='Greys_r')
    plt.title("Sampling mask")
    plt.show()

norm = "ortho"
#norm = None

def fft(x):
    return np.fft.fft2(x, norm=norm)

def ifft(x):
    return np.fft.ifft2(x, norm=norm)

# Generate the kspace data: first Fourier transform the image
kspace_data = np.fft.fftshift(fft(mri_img))
#add Gaussian complex-valued random noise
signoise = 10
kspace_data += np.random.randn(*mri_img.shape) * signoise * (1+1j)
# Mask data to perform subsampling
kspace_data *= kspace_masklines

# Zero order solution
image_rec0 = ifft(np.fft.ifftshift(kspace_data))            # get back to the convention

fig, axs = plt.subplots(2, 2, figsize=(10, 10))
axs[0, 0].imshow(mri_img, cmap='Greys_r')
axs[0, 0].set_title("True image")
axs[0, 1].imshow(kspace_masklines, cmap='Greys_r')
axs[0, 1].set_title("Low frequency sampling mask")
axs[1, 0].imshow(np.abs(kspace_data),  cmap='gray', vmax=0.01*np.abs(kspace_data).max())
#axs[1].imshow(np.abs(np.fft.ifftshift(kspace_data)), cmap='Greys_r')
axs[1, 0].set_title("k-space noisy data")
axs[1, 1].imshow(np.abs(image_rec0), cmap='Greys_r')
axs[1, 1].set_title("Zero-order recon")
plt.show()
```



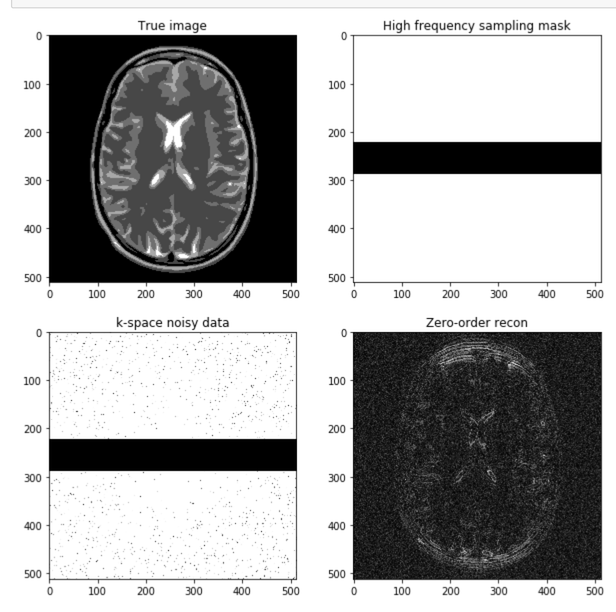
- Question: based on the previous example, try to construct a low-freqency sampling mask defined by a squared box centered around $(k_x, k_y) = (0, 0)$.
- Then, replicate the same steps:
  1. Generate noisy masked data
  2. P zero-folled MR image recon
  3. Visualize results and study the impact of both the noise level and the mask size

In [73]:
```
#Objective: construct a k-space mask consisting of a central box

# Initialize k-space with 0 everywhere and then place the "1" appropriately
kspace_maskbox = np.zeros((img_size,img_size), dtype="float64")
# use fancy indexing along rows
kspace_maskbox[idx_vec_, ] = 1

list_img_size = np.arange(0., img_size).tolist()
filtered_center = [x for x in list_img_size if x not in idx_vec_]
array_idx_center = np.array(filtered_center)
array_idx_center_ = array_idx_center.astype("int")
# use fancy indexing along cols
kspace_maskbox[:, array_idx_center_] = 0

# Note that the combination of these two fancy indexing replaces the poor code below
#for i in idx_vec_:
#    for j in idx_vec_:
#        kspace_mask[i, j] = 1.

if 0:
    plt.figure()
    plt.imshow(kspace_maskbox, cmap='Greys_r')
    plt.title("Sampling mask")
    plt.show()

# Generate the kspace data: first Fourier transform the image
kspace_data = np.fft.fftshift(fft(mri_img))
#add Gaussian complex-valued random noise
signoise = 10
kspace_data += np.random.randn(*mri_img.shape) * signoise * (1+1j)
# Mask data to perform subsampling
kspace_data *= kspace_maskbox

fig, axs = plt.subplots(2, 2, figsize=(10, 10))
axs[0, 0].imshow(mri_img, cmap='Greys_r')
axs[0, 0].set_title("True image")
axs[0, 1].imshow(kspace_maskbox, cmap='Greys_r')
axs[0, 1].set_title("Low frequency sampling mask")
axs[1, 0].imshow(np.abs(kspace_data),  cmap='gray', vmax=0.01*np.abs(kspace_data).max())
#axs[1].imshow(np.abs(np.fft.ifftshift(kspace_data)), cmap='Greys_r')
axs[1, 0].set_title("k-space noisy data")
axs[1, 1].imshow(np.abs(image_rec0), cmap='Greys_r')
axs[1, 1].set_title("Zero-order recon")
plt.show()
```



- Next question: Construct a k-space sampling mask that consists of the high-frequency lines, just discard the central lines
- Study the impact of the number of lines removed

In [74]:
```
# 0 entries: points not sampled, 1 entries: points to be sampled
# Initialize k-space with "1" everywhere and then place the "0" appropriately
kspace_masklines = np.ones((img_size,img_size), dtype="float64")
#low_res_size = img_size // (factor_ctr // 2) + 1
low_res_size = img_size // factor_ctr + 1
idx_vec = np.linspace(img_size // 2 - low_res_size // 2, img_size // 2 + low_res_size // 2, low_res_size)
idx_vec_ =  idx_vec.astype("int")

# set central k-space lines to 0: use fancy indexing along rows
kspace_masklines[idx_vec_, ] = 0

# Generate the kspace data: first Fourier transform the image
kspace_data = np.fft.fftshift(fft(mri_img))
#add Gaussian complex-valued random noise
signoise = 10
kspace_data += np.random.randn(*mri_img.shape) * signoise * (1+1j)
# Mask data to perform subsampling
kspace_data *= kspace_masklines

# Zero order solution
image_rec0 = ifft(np.fft.ifftshift(kspace_data))            # get back to the convention

fig, axs = plt.subplots(2, 2, figsize=(10, 10))
axs[0, 0].imshow(mri_img, cmap='Greys_r')
axs[0, 0].set_title("True image")
axs[0, 1].imshow(kspace_masklines, cmap='Greys_r')
axs[0, 1].set_title("High frequency sampling mask")
axs[1, 0].imshow(np.abs(kspace_data),  cmap='gray', vmax=0.01*np.abs(kspace_data).max())
#axs[1].imshow(np.abs(np.fft.ifftshift(kspace_data)), cmap='Greys_r')
axs[1, 0].set_title("k-space noisy data")
axs[1, 1].imshow(np.abs(image_rec0), cmap='Greys_r')
axs[1, 1].set_title("Zero-order recon")
plt.show()
```



Question: Based on the previous example, try to construct a k-space mask that consists of the removing the central box centered in $(k_x, k_y) = (0, 0)$.

- Then, replicate the same steps:
  1. Generate noisy masked data
  2. Perform zero-filled MR image recon
  3. Visualize results and study the impact of both the noise level and the mask size

In [75]:
```
# 0 entries: points not sampled, 1 entries: points to be sampled
# Initialize k-space with "1" everywhere and then place the "0" appropriately
kspace_maskbox = np.ones((img_size,img_size), dtype="float64")
# use fancy indexing along rows
kspace_maskbox[idx_vec_, ] = 0

list_img_size = np.arange(0., img_size).tolist()
filtered_center = [x for x in list_img_size if x not in idx_vec_]
array_idx_center = np.array(filtered_center)
array_idx_center_ = array_idx_center.astype("int")
# use fancy indexing along cols
kspace_maskbox[:, array_idx_center_] = 1

# Note that the combination of these two fancy indexing replaces the poor code below
#for i in idx_vec_:
#    for j in idx_vec_:
#        kspace_mask[i,j]=0.

# Generate the kspace data: first Fourier transform the image
kspace_data = np.fft.fftshift(fft(mri_img))
#add Gaussian complex-valued random noise
signoise = 10
kspace_data += np.random.randn(*mri_img.shape) * signoise * (1+1j)
# Mask data to perform subsampling
kspace_data *= kspace_maskbox

# Zero order solution
image_rec0 = ifft(np.fft.ifftshift(kspace_data))

fig, axs = plt.subplots(2, 2, figsize=(10, 10) )
axs[0,0].imshow(mri_img, cmap='Greys_r')
axs[0,0].set_title("True image")
axs[0,1].imshow(kspace_maskbox, cmap='Greys_r')
axs[0,1].set_title("High frequency box sampling mask")
axs[1,0].imshow(np.abs(kspace_data),  cmap='gray', vmax=1*np.abs(kspace_data).max())
#axs[1].imshow(np.abs(np.fft.ifftshift(kspace_data)), cmap='Greys_r')
axs[1,0].set_title("k-space noisy data")
axs[1,1].imshow(np.abs(image_rec0), cmap='Greys_r')
axs[1,1].set_title("Zero-order recon")
plt.show()
```