EE 511 Simulation Methods for Stochastic Systems
Project #4 (Chaitra Suresh -7434709345)

[DTFS MC Stationary Distributions]

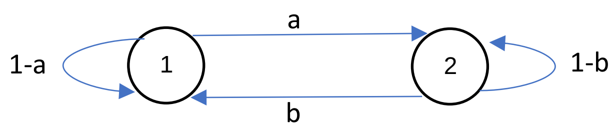
Goal:
To find stationary distributions for a discrete time finite state (DTFS) Markov chain {Xn} having transition matrix:

$$P = \begin{bmatrix} 1\text{-}a & a \\ b & 1\text{-}b \end{bmatrix}$$

when (i) a=1/10 and b=1/15
(ii) a=0.5 and b=0.5
(iii) a=1 and b=1
(iv) a=0 and b=0

Algorithm / Routine:

Step 1: Given the transition matrix $P = \begin{bmatrix} 1\text{-}a & a \\ b & 1\text{-}b \end{bmatrix}$ a,b ϵ (0,1)



a : probability of transition from state 1 to state 2
b : probability of transition from state 2 to state 1
1-a : probability of transition from state 1 to state 1
1-b : probability of transition from state 2 to state 2

The rows of the transition matrix sum up to 1.

Stationary probability distribution function of Markov chain with transition matrix P is a vector with components which determines the probability of occurrence of each state and it satisfies the following property:
$\pi^* = \pi^* \mathrm{P}$
where π^* is the stationary probability distribution function of the transition matrix P.
π^* is a row eigen vector of transition matrix P with eigen value 1

Step 2: Eigen values are calculated by solving for roots of the characteristic equation det(λI – P)

$$\text{Det}(\lambda \text{I} - \text{P}) = \begin{vmatrix} \lambda - (1-a) & \text{-a} \\ \text{-b} & \lambda - (1-b) \end{vmatrix} = (\lambda - 1)(\lambda - 1 + a + b) = 0$$

Eigen vector are found by determining the left-null space of λI – P or null space of P^T – λI.
Row eigen vector with eigen value $\lambda = 1$ and $\sum_{i=1}^{n} \pi^*_j = 1$ determines the stationary pdf π^*

By solving for eigen vector with constraint satisfied, the stationary distribution can be generalized as:
$\pi^* = \left(\frac{b}{a+b}, \frac{a}{a+b} \right)$

Results:

(i) a=1/10 and b=1/15

π^* = ($\frac{b}{a+b}$, $\frac{a}{a+b}$) = ($\frac{(1/15)}{0.1+(1/15)}$, $\frac{0.1}{0.1+(1/15)}$) = (0.4 , 0.6)

A unique stationary distribution exists since it is aperiodic and irreducible (ergodic)

(ii) a=0.5 and b=0.5

π^* = ($\frac{b}{a+b}$, $\frac{a}{a+b}$) = ($\frac{0.5}{0.5+0.5}$, $\frac{0.5}{0.5+0.5}$) = (0.5 , 0.5)

A unique stationary distribution exists since it is aperiodic and irreducible (ergodic)

(iii) a=1 and b=1

π^* = ($\frac{b}{a+b}$, $\frac{a}{a+b}$) = ($\frac{1}{1+1}$, $\frac{1}{1+1}$) = (0.5 , 0.5)

A unique stationary distribution exists since it is aperiodic and irreducible (ergodic)

(iv) a=0 and b=0

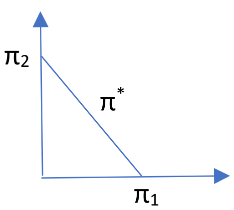
$$P = \begin{bmatrix} 1\text{-}a & a \\ b & 1\text{-}b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigen vector are found by determining the left-null space of $\lambda I - P$ or null space of $P^T - \lambda I$.
Row eigen vector with eigen value $\lambda = 1$ and $\sum_{i=1}^{n} \pi^*_j = 1$ determines the stationary pdf π^*

By solving for eigen vector with constraint satisfied, there exists many (infinitely) stationary pdf with eigen values equal to 1 and $\sum_{i=1}^{n} \pi^*_j = 1$ since the left null space for eigen vector is R^2 or (null space for column eigen vector of P^T is R^2)

A unique stationary distribution does not exist since it is aperiodic and reducible (not ergodic)

Therefore, the stationary probability distribution function can be generalized as:
π^* = (π_1 , π_2) =(a , 1-a) where a is any real number such that: 0 < = a < =1
The stationary distribution π^* lies on the line in the figure below:



[DTFS MC Simulation]

Goal:

To Simulate 10 sample paths of length 500 for the DTFS Markov chains below using the update equation: $\pi(t + 1) = \pi(t)P$

All MCs start with the initial probability vector $\pi(t = 0) = (1, 0)$

P1 = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

P2 = $\begin{bmatrix} 0.75 & 0.25 \\ 0.1 & 0.9 \end{bmatrix}$

P3 = $\begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.22 & 0.7 & 0.08 \\ 0 & 0 & 1 \end{bmatrix}$

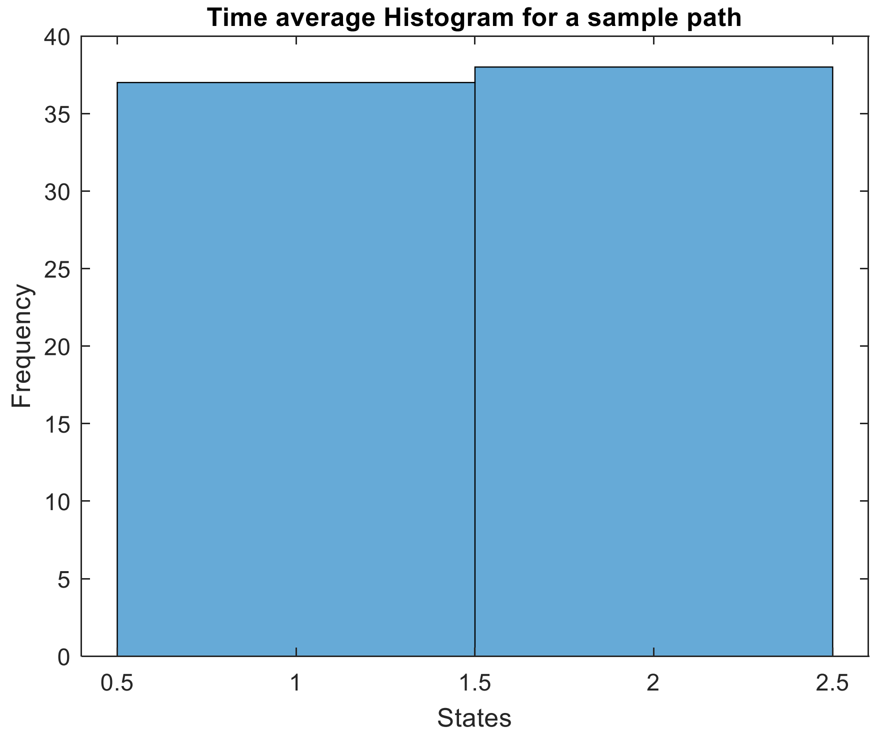
To check for MC convergence by applying a goodness-of-fit test to the last 75 samples of each sample path. Compare the ensemble and time averages for the 3 Markov chains. Based on these averages, which of the chains are ergodic?

Algorithm / Routine:

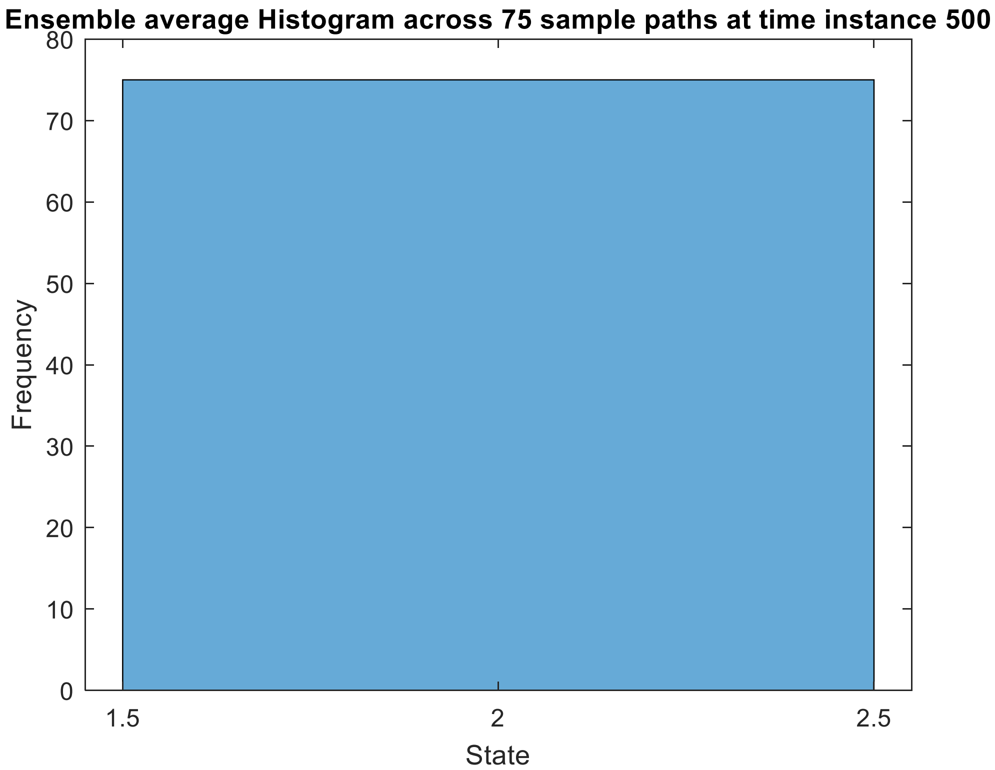
Step 1: Let initial state for P1, P2 be [1 0] and for P3 be [1 0 0]
Step 2: Assign initial state to temporary state distribution
Step 3: Generate next state probability distribution by multiplying the temporary state with transition matrix (P)
Step 4: Generate a uniform random number in the interval [0,1]
Step 5: For case P1 and P2, let the obtained next state probability distribution be [p1, p2]. The interval [0,1] is sub-divided into 2 parts as [0, p1) and [p1, p1+p2(=1)] which belongs 1 and 2 state for next state respectively.
For case P3, let the obtained next state probability distribution be (p1, p2, p3). The interval [0,1] is sub-divided into 3 parts as [0, p1), [p1, p1+p2) and [p1+p2,1(or p1+p2+p3)] which belongs 1, 2 and 3 states for next state respectively.
Step 6: Depending on the sub-interval the generated uniform random number belongs, the next state is 1 / 2 (for P1, P2) or 1 / 2 / 3 (for P3) which is stored as a path for time instance t
Step 7: The obtained next state probability distribution is assigned to temporary state distribution
Step 8: Repeat Steps 2~7 for 500- time instances(t) to generate a sample path
Step 9: Repeat Step 8 for 75 times to generate 75 sample paths.
Step 10: Histogram the states of a sample path of last 75 samples (time average)
Step 11: Histogram the states of 75 sample path at time instance = 500 (ensemble average)

Results:

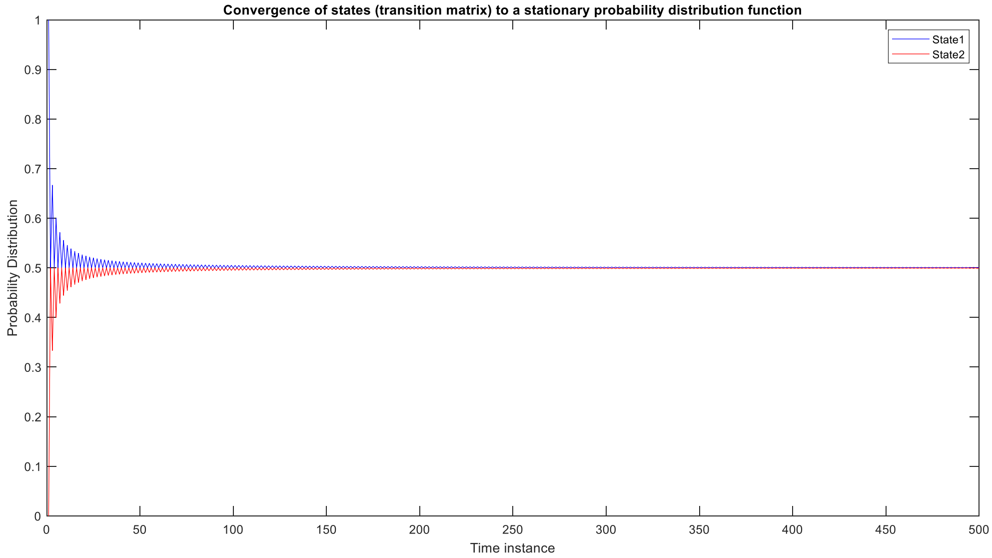
1. Results for P1



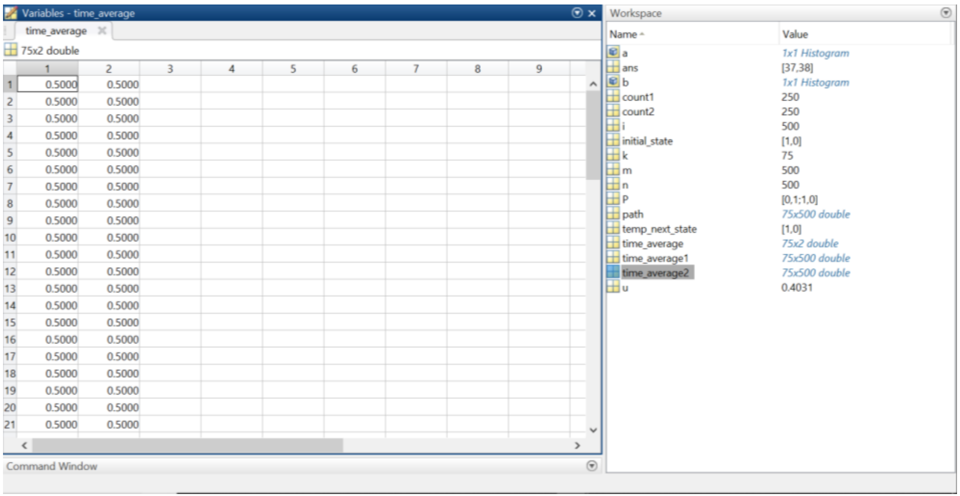
Frequency of State 1 and 2 are 37 and 38 respectively



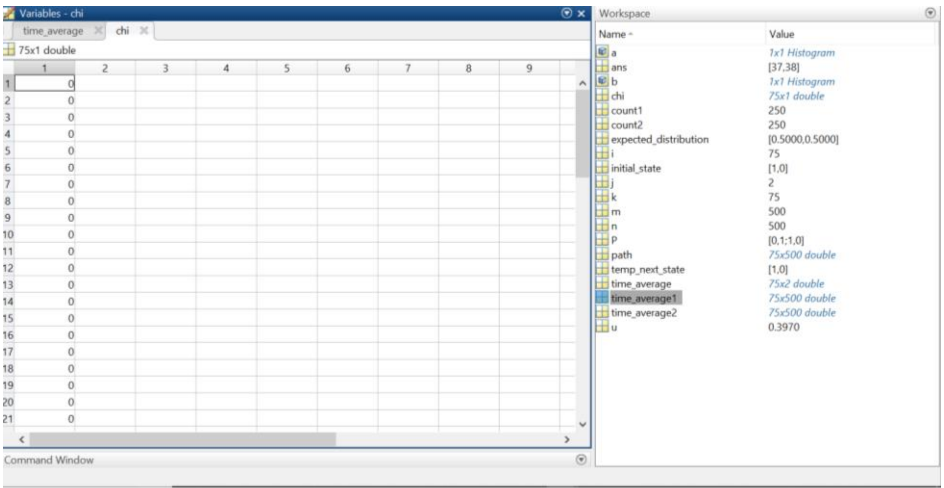
Frequency of State 1 and 2 are 0 and 75 respectively



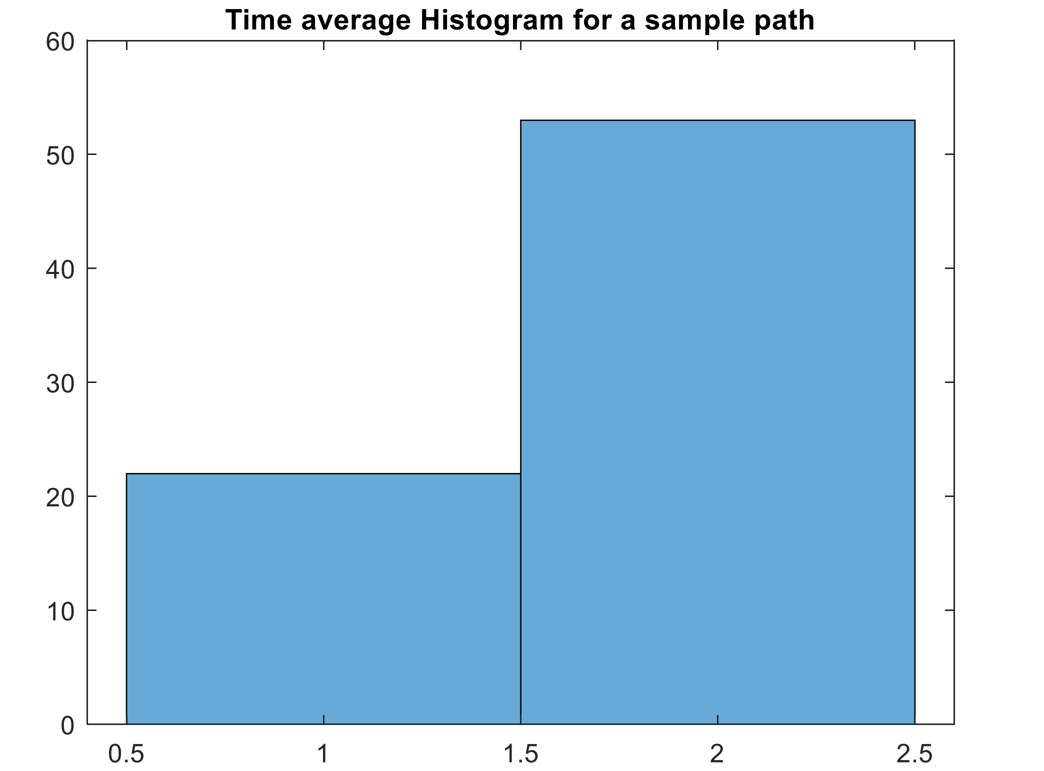
Convergence of time average to theoretical stationary distribution [0.5, 0.5]



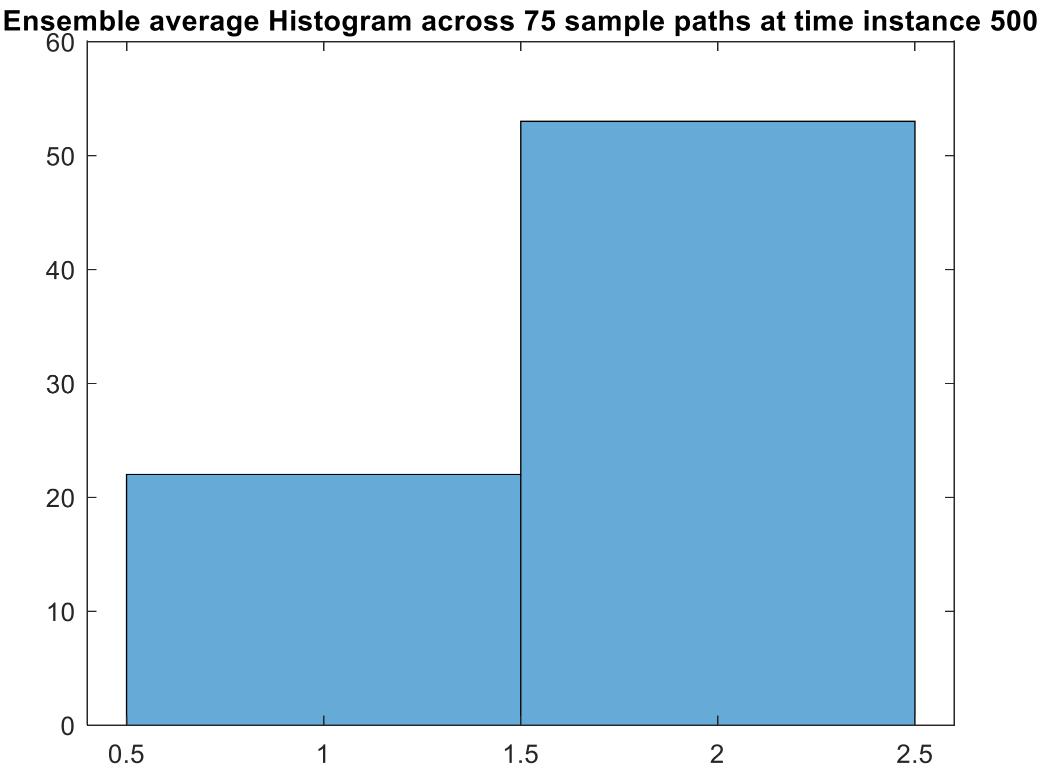
Chi squared Goodness of fit test for each sample path



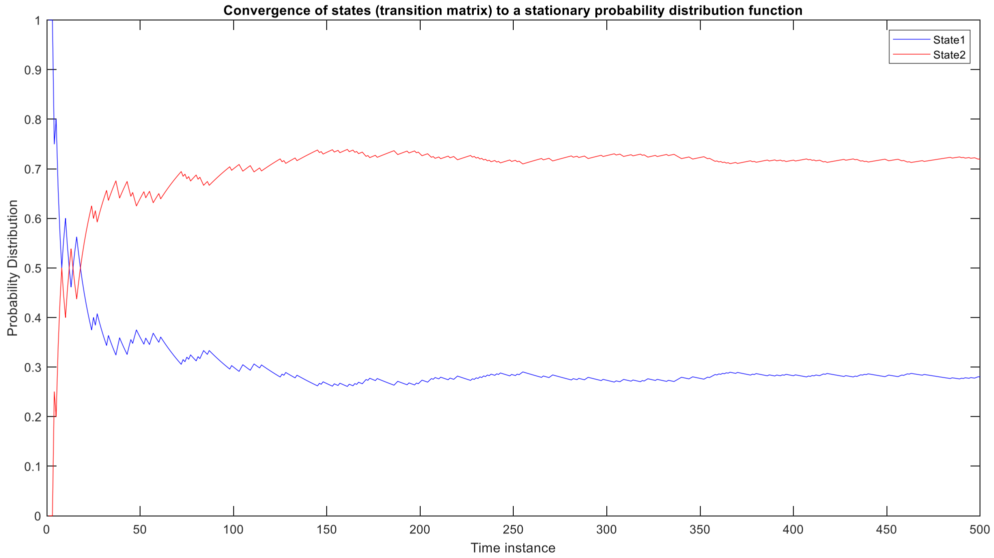
2. Results for P2



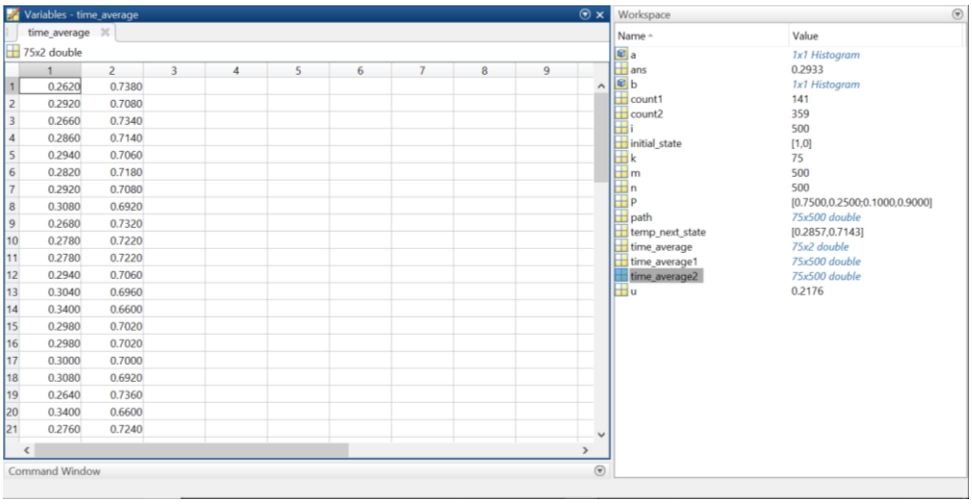
Frequency of State 1 and 2 are 22 and 53 respectively



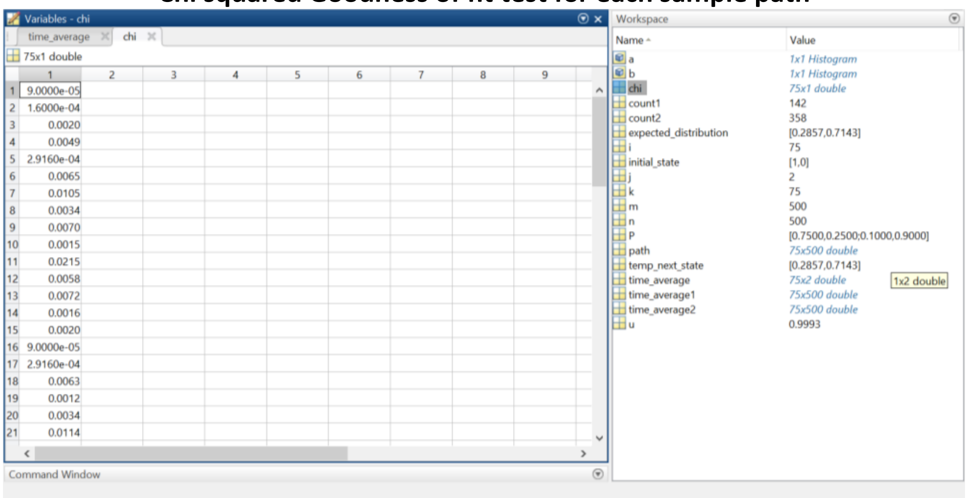
Frequency of State 1 and 2 are 24 and 51 respectively



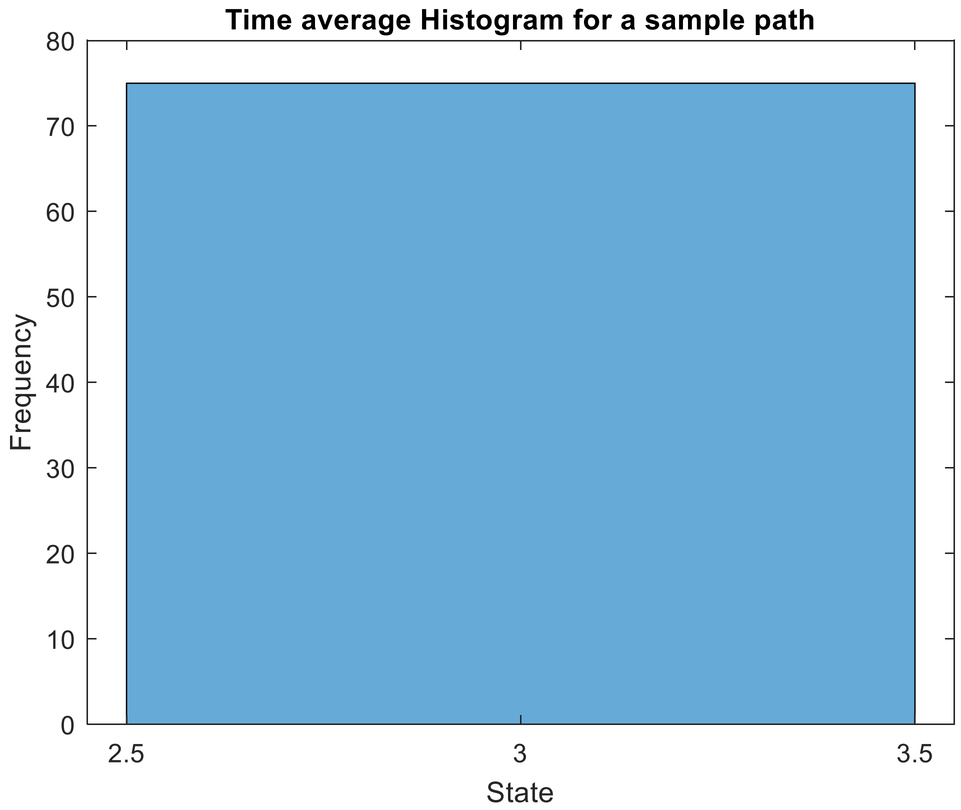
Convergence of time average to theoretical stationary distribution [0.2857, 0.7143]



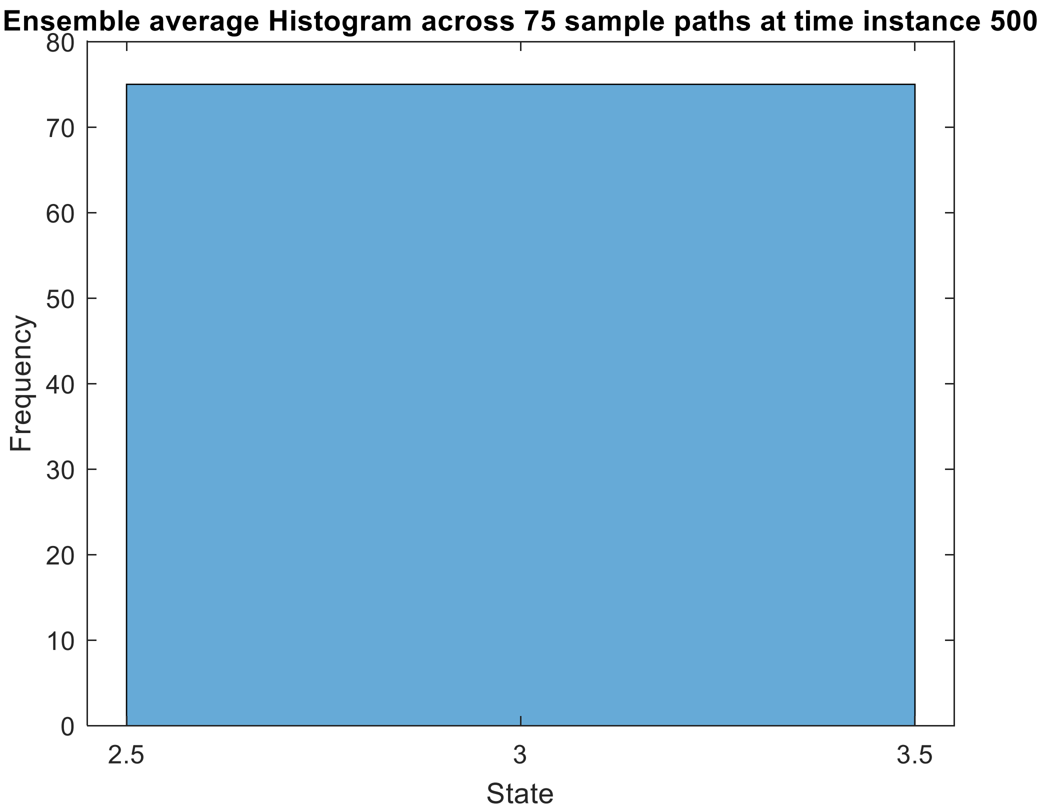
Chi squared Goodness of fit test for each sample path



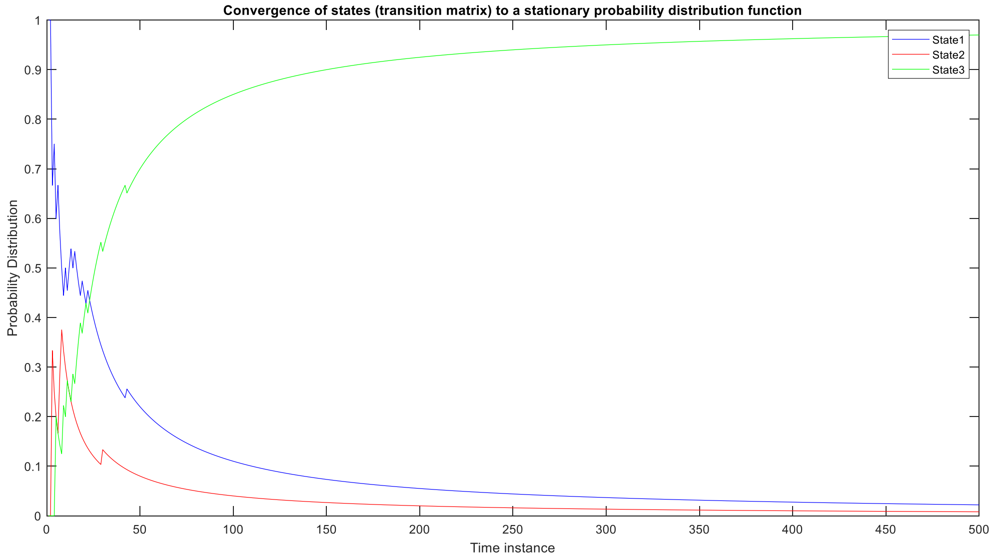
3. Results for P3



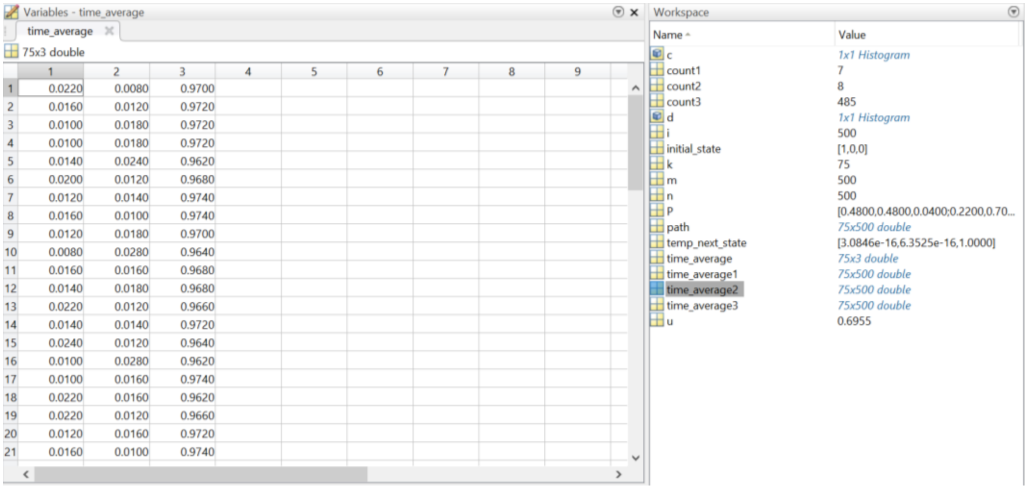
Frequency of State 1,2 and 3 are 0, 0 and 75 respectively



Frequency of State 1,2 and 3 are 0, 0 and 75 respectively



Convergence of time average to theoretical stationary distribution [0,0,1]


Variables - time_average

time_average

75x3 double

	1	2	3	4	5	6	7	8	9
1	0.0220	0.0080	0.9700						
2	0.0160	0.0120	0.9720						
3	0.0100	0.0180	0.9720						
4	0.0100	0.0180	0.9720						
5	0.0140	0.0240	0.9620						
6	0.0200	0.0120	0.9680						
7	0.0120	0.0140	0.9740						
8	0.0160	0.0100	0.9740						
9	0.0120	0.0180	0.9700						
10	0.0080	0.0280	0.9640						
11	0.0160	0.0160	0.9680						
12	0.0140	0.0180	0.9680						
13	0.0220	0.0120	0.9660						
14	0.0140	0.0140	0.9720						
15	0.0240	0.0120	0.9640						
16	0.0100	0.0280	0.9620						
17	0.0100	0.0160	0.9740						
18	0.0220	0.0160	0.9620						
19	0.0220	0.0120	0.9660						
20	0.0120	0.0160	0.9720						
21	0.0160	0.0100	0.9740						

Workspace

Name	Value
c	1x1 Histogram
count1	7
count2	8
count3	485
d	1x1 Histogram
i	500
initial_state	[1,0,0]
k	75
m	500
n	500
P	[0.4800,0.4800,0.0400;0.2200,0.70...
path	75x500 double
temp_next_state	[3.0846e-16,6.3525e-16,1.0000]
time_average	75x3 double
time_average1	75x500 double
time_average2	75x500 double
time_average3	75x500 double
u	0.6955

Discussion:

1. P1 Transition Matrix

P1 = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

State 1 is connected to state 2 with probability 1 and state 2 is connected to state 1 with probability 1.

P1 transition matrix is a **periodic and irreducible** Markov chain since the periodicity of all states is Equal to 2 and all states are connected to each other.

The theoretical stationary probability distribution function for transition matrix P1 with a=1 and b=1 is

$\pi^* = \left(\frac{b}{a+b}, \frac{a}{a+b} \right) = \left(\frac{1}{1+1}, \frac{1}{1+1} \right) = (0.5, 0.5)$

From the results obtained, the mean of time convergence at time instance= 500 over 75 sample path is (0.5, 0.5) is in good fit with theoretical stationary probability distribution (0.5, 0.5). For each sample path, the chi square values are calculated and shown in results.

From the results for P1, it is evident that time average is not equal to the ensemble average and since the transition matrix is periodic and irreducible, the homogeneous Markov chain with transition matrix P1 is not ergodic.

2. P2 Transition Matrix

P2 = $\begin{bmatrix} 0.75 & 0.25 \\ 0.1 & 0.9 \end{bmatrix}$

State 1 is connected to state 2 with probability 0.25.
State 2 is connected to state 1 with probability 0.1.
State 1 is connected to state 1 with probability 0.75.
State 2 is connected to state 2 with probability 0.9.

P2 transition matrix is an **aperiodic and irreducible** Markov chain since the periodicity of all states is 1 and all states are connected to each other.

The theoretical stationary probability distribution function for transition matrix P2 with a=0.25 and b=0.1 is

$\pi^* = \left(\frac{b}{a+b}, \frac{a}{a+b} \right) = \left(\frac{0.1}{0.25+0.1}, \frac{0.25}{0.25+0.1} \right) = (0.2857, 0.7143)$

From the results obtained, the mean of time convergence at time instance= 500 over 75 sample path is (0.2890, 0.7110) is in good fit with theoretical stationary probability distribution (0.2857, 0.7143). For each sample path, the chi square values are calculated and shown in results.

From the results for P2, it is evident that time average is equal to the ensemble average (approximately) and since the transition matrix is an aperiodic and irreducible, the homogeneous Markov chain with transition matrix P2 is ergodic.

3. P3 Transition Matrix

P3 = $\begin{bmatrix} 0.48 & 0.48 & 0.04 \\ 0.22 & 0.7 & 0.08 \\ 0 & 0 & 1 \end{bmatrix}$

State 1 is connected to state 2 with probability 0.48.
State 2 is connected to state 1 with probability 0.22.
State 1 is connected to state 1 with probability 0.48.
State 2 is connected to state 2 with probability 0.7.
State 1 is connected to state 3 with probability 0.04.
State 2 is connected to state 3 with probability 0.08.
State 3 is connected to state 3 with probability 1.

P3 transition matrix is an **aperiodic and reducible** Markov chain since the periodicity of each state is equal to 1 and not all states are reachable from each sate (example: state 3 cannot reach state 1 and state 2).

The theoretical stationary probability distribution function for transition matrix P3 is
π^* = (0,0,1)

From the results obtained, the mean of time convergence at time instance= 500 over 75 sample path is [0.0129, 0.0186, 0.9685] which is in good fit with theoretical stationary probability distribution function [0, 0, 1]

From the results for P3, it is evident that time average is equal to the ensemble average and since the transition matrix is aperiodic and reducible, the homogeneous Markov chain with transition matrix P3 is not ergodic.

We can infer that if Markov chain are ergodic, then time average is equal to the ensemble average. But, it is does not mean/infer that the Markov chain is ergodic. Since, aperiodic and irreducible properties of Markov chain are necessary to be ergodic. But, in this case: state 3 is absorbing and does not communicate with other states. Hence P3 is aperiodic and reducible and not ergodic.

Code:
2.

```
P=[0 1;1 0];
% P=[0.75 0.25;0.1 0.9];
initial_state=[1 0];
for k=1:75
    temp_next_state=initial_state;
for i=1:500
    u=rand;
    if u<temp_next_state(1,1)
     path(k,i)=1;
    else
     path(k,i)=2;
    end
    temp_next_state=temp_next_state*P;
    count1=0;
    count2=0;
    n=i;
    for m=1:n
        if(path(k,m)==1)
            count1=count1+1;
        else
            count2=count2+1;
        end
    end
    time_average1(k,n)=(1/n)*count1;
    time_average2(k,n)=(1/n)*count2;
end
end


figure(1);
plot(time_average1(1,:),'b');
hold on;
plot(time_average2(1,:),'r');
legend('State1','State2');
title('Convergence of states (transition matrix) to a stationary
probability density function');

figure(2);
a=histogram(path(:,500));
title('Ensemble average Histogram across 75 sample paths at time
instance 500');

figure(3);
b=histogram(path(1,426:500));
title('Time average Histogram for a sample path');

expected_distribution=[P(2,1)/(P(1,2)+P(2,1))
P(1,2)/(P(1,2)+P(2,1))]
time_average=[time_average1(:,500) time_average2(:,500)];
for i=1:75
```

```
chi(i,1)=0;
for j=1:2
chi(i,1)=chi(i,1)+((time_average(i,j)-
expected_distribution(1,j)).^2)/expected_distribution(1,j);
end
end


P=[ 0.48 0.48 0.04 ; 0.22 0.7 0.08 ;0 0 1];
initial_state=[1 0 0];
for k=1:75
    temp_next_state=initial_state;
for i=1:500
    u=rand;
    if u<temp_next_state(1,1)
     path(k,i)=1;
    elseif (u>=temp_next_state(1,1) &&
u<(temp_next_state(1,1)+temp_next_state(1,2)))
     path(k,i)=2;
    else
     path(k,i)=3;
    end
    temp_next_state=temp_next_state*P;
    count1=0;
    count2=0;
    count3=0;
    n=i;
    for m=1:n
        if(path(k,m)==1)
            count1=count1+1;
        elseif(path(k,m)==2)
            count2=count2+1;
        else
            count3=count3+1;
        end
    end
    time_average1(k,n)=(1/n)*count1;
    time_average2(k,n)=(1/n)*count2;
    time_average3(k,n)=(1/n)*count3;
end
end
figure(4);
plot(time_average1(1,:),'b');
hold on;
plot(time_average2(1,:),'r');
hold on;
plot(time_average3(1,:),'g');
legend('State1','State2','State3');
title('Convergence of states (transition matrix) to a stationary
probability density function');

figure(5);
c=histogram(path(:,500));
title('Ensemble average Histogram across 75 sample paths at time
instance 500');
```

```
figure(6);
d=histogram(path(1,426:500));
title('Time average Histogram for a sample path');
time_average=[time_average1(:,500) time_average2(:,500)
time_average3(:,500)];
```