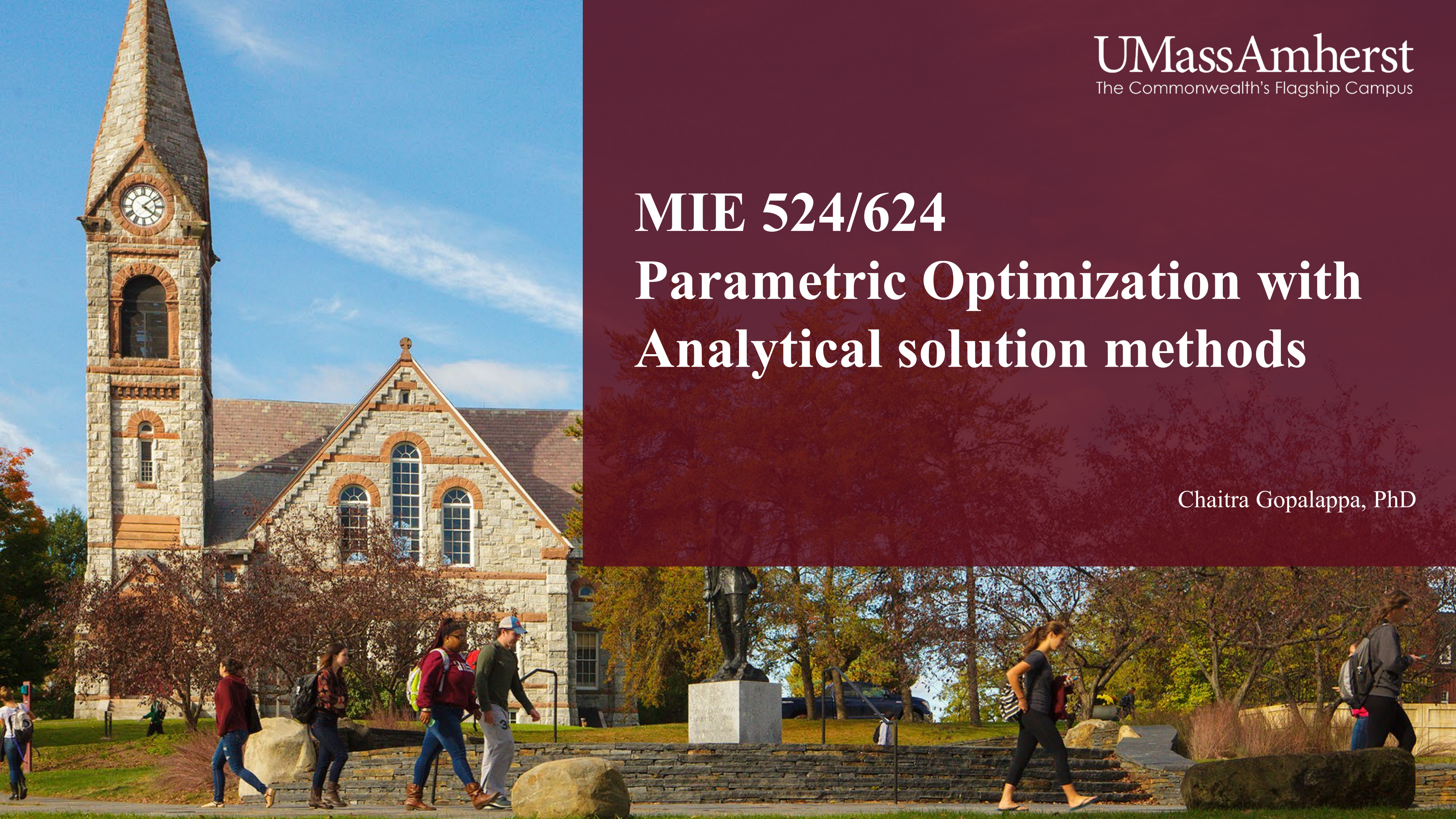

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MIE 524/624
Parametric Optimization with Analytical solution methods

Chaitra Gopalappa, PhD

Reference

- Chapter 8 – Murphy, 2023(Book 1)

Optimization - What?

- What is optimization modeling?

Optimization modeling – How?

- **Formulate problem- determine**
  - Objective function
  - Exogenous (decision) variables
  - Endogenous (input) variables
  - Constraints (if any)
- **Apply suitable solution algorithms**
  - dependent on form of objective function and constraints - linear, integer, nonlinear
    - simplex, interior point method (Linear Programming)
    - branch and bound (Integer programming)
    - Newton's (analytical algorithms; non-linear programming)
    - gradient descent (numerical optimization; non-linear programming)

Problem: Find optimal routing of drones between n cities for aerial assessment of damage after a natural disaster

- Problem formulation
  - **Objective function**
    - $\min \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij}$
  - **Endogenous variables**
    - n = number of cities to visit
    - c_{ij}= cost of travel from location i to location j
  - **Exogenous (decision) variables**
    - x_{ij} = (binary with value 1 if drone should travel directly from location i to location j
  - **Constraints**:
    - s.t, $\sum_{i=0}^{n} x_{ij} = 1, \forall j; \sum_{j=0}^{n} x_{ij} = 1, \forall i;$
- Solution algorithm?

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Linear

  - **Endogenous variables**
    - n = number of cities to visit
    - c_{ij}= cost of travel from location i to location j
  - **Exogenous (decision) variables**

Binary/Integer

    - x_{ij} = (binary with value 1 if drone should travel directly from location i to location j
  - **Constraints**: s.t, $\sum_{i=0}^{n} x_{ij} = 1, \forall j; \sum_{j=0}^{n} x_{ij} = 1, \forall i;$

Linear/combinatorial

- Solution algorithm?

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- Solution algorithm (combinatorial)
  - Branch and bound

Non-linear programming/ continuous

- $Min\ f(\boldsymbol{x})$,
- $\boldsymbol{x} \in R^n$ Continuous variable
- $f(\boldsymbol{x}) = -x_1(70 - 4x_1) - x_2(150 - 15x_2) + 100 + 15\ x_1 + 15\ x_2$

Non-linear objective function

Analytic gradient-based solution

- First and second order necessary and sufficient conditions
- Suppose $f: R^n \rightarrow R$ is twice differentiable at $\vec{x}*$. If $\vec{x}*$ is a **local** minimum then
  - $\nabla f(\vec{x}*) = 0$ (**first-order necessary condition**)
  - $H(\vec{x}*)$ is positive semi-definite (**second-order necessary**)
  - $H(\vec{x}*)$ is positive definite (**second-order sufficient**)
- If f is **convex** and $\vec{x}*$ is a local optima then, $\vec{x}*$ is also a **global** optima
- (note: the arrow in $\vec{x}$ implies x is a vector; typical notation is to use a bold **x** to indicate a vector; when writing by hand on the board I may use $\vec{x}$)

Calculus- basics

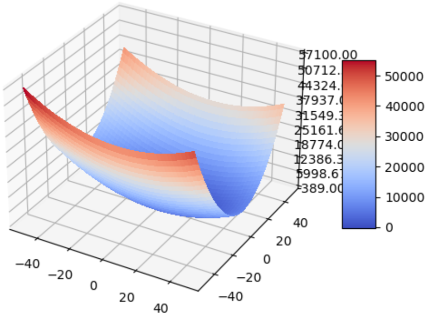
- Hessian (H): Suppose $f: R^n \rightarrow R$ is twice differentiable at some point $\boldsymbol{x}$**,** then **(note:** R^n implies **vector** $\boldsymbol{x}$ is n-dimensional)
  - $H(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}_{\boldsymbol{x}}$
- Matrix H is positive semidefinite, iif (if and only if)
  - $\boldsymbol{z}H\boldsymbol{z}^{\boldsymbol{T}} \geq 0$; $\boldsymbol{z}$ is any non-zero vector; $\boldsymbol{z}^T$ is transpose of $\boldsymbol{z}$;
  - OR, all leading principal minors are ≥ 0; A k^{th} principal minor of a $n \times n$ matrix is the determinant of any $k \times k$ matrix obtained by deleting $\mathrm{n} - k$ rows and corresponding $\mathrm{n} - k$ columns
- Matrix H is positive definite, iff
  - $\boldsymbol{z}H\boldsymbol{z}^{\boldsymbol{T}} > 0$; $\boldsymbol{z}$ is any non-zero vector; $\boldsymbol{z}^{\boldsymbol{T}}$ is transpose of $\boldsymbol{z}$;
  - OR, all leading principal minors are> 0
- $f(\boldsymbol{x})$ is convex iff $H(\boldsymbol{x})$ is positive semidefinite for 'every' point $\boldsymbol{x}$ in the domain of f
- $f(\boldsymbol{x})$ is strictly convex iff $H(\boldsymbol{x})$ is positive definite for 'every' point $\boldsymbol{x}$ in the domain of f

Non-linear programming/ continuous

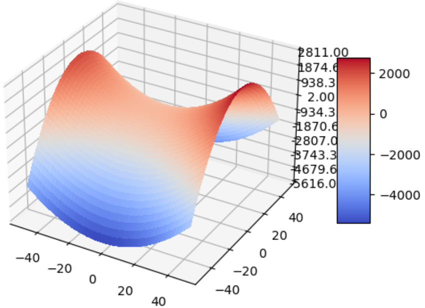
- $Min\ f(\boldsymbol{x}),\ \boldsymbol{x} \in R^n$
- $f(\boldsymbol{x}) = -x_1(70 - 4x_1) - x_2(150 - 15x_2) + 100 + 15\ x_1 + 15\ x_2$
- $f(\boldsymbol{x}) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$

Example of analytic method

- $f(\boldsymbol{x}) = -x_1(70 - 4x_1) - x_2(150 - 15x_2) + 100 + 15\,x_1 + 15\,x_2$
- Solve for $\boldsymbol{x}$ using first order necessary condition, by setting the partial derivatives equal to zero
- $\frac{\partial f}{\partial x_1} = -70 + 8x_1 + 15 = 0 => x_1 = \frac{55}{8}$
- $\frac{\partial f}{\partial x_2} = -150 + 30x_2 + 15 = 0 => x_2 = \frac{9}{2}$
- Test for the second order necessary and sufficient conditions, by checking if the Hessian at $\left(\frac{55}{8}, \frac{9}{2}\right)$ is positive semidefinite or positive definite
- $H = \begin{bmatrix} 8 & 0 \\ 0 & 30 \end{bmatrix}$ (notice it is independent of x)
- 1st leading principal minor: 8>0
- 2nd leading principal minor=determinant $\begin{vmatrix} 8 & 0 \\ 0 & 30 \end{vmatrix} > 0$ => H is positive definite
- Solution: $\left(\frac{55}{8}, \frac{9}{8}\right)$ is a strict local minimum of $f(\boldsymbol{x})$
- Is it global optima?
  - $f(\boldsymbol{x})$ is convex because $H(\boldsymbol{x})$ is positive definite for 'every' point $\boldsymbol{x}$ in the domain of f => above solution is a global optima



Example 2



- $f(\boldsymbol{x}) = \quad 8x_1 + 12x_2 + x_1^2 - 2x_2^2$
- Solve for $\boldsymbol{x}$ using first order necessary condition, by setting the partial derivatives equal to zero
- $\frac{\partial f}{\partial x_1} = 8 + 2x_1 = 0 => x_1 = -4$
- $\frac{\partial f}{\partial x_2} = 12 - 4x_2 = 0 => x_2 = 3$
- Test for the second order necessary and sufficient conditions, by checking if the Hessian at $(-4{,}3)$ is positive semidefinite or positive definite
- $H = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$
- 1^{st} leading principal minor: 2>0
- 2^{nd} leading principal minor=determinant $\begin{vmatrix} 2 & 0 \\ 0 & -4 \end{vmatrix}$ <0 => H is not positive definite or semi definite
- **Solution: (-4,3) is a stationary point but not a minima for $\boldsymbol{f(x)}$.**

This analytic method is insufficient for this problem

If objective function is a maxima

- **Is it a maxima?** Convert max to min (Max $f(x)$~ Min $-f(x)$)

Questions?

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