

RECAP

- ▶ CDF $F_X(a) := P_X(\{X \in (-\infty, a]\})$
- ▶ CDF is non-decreasing and right continuous.
- ▶ A continuous r. v.'s have continuous $F_X(\cdot)$.
- ▶ They also have a pdf $f_X(\cdot)$.
- ▶ $P_X(a \leq X \leq b) = \int_a^b f_X(u)du$. (Area under the curve)
- ▶ $F_X(x) = \int_{u=-\infty}^x f_X(u)du$.

Standard Examples

Uniform random variable ($U[a, b]$)

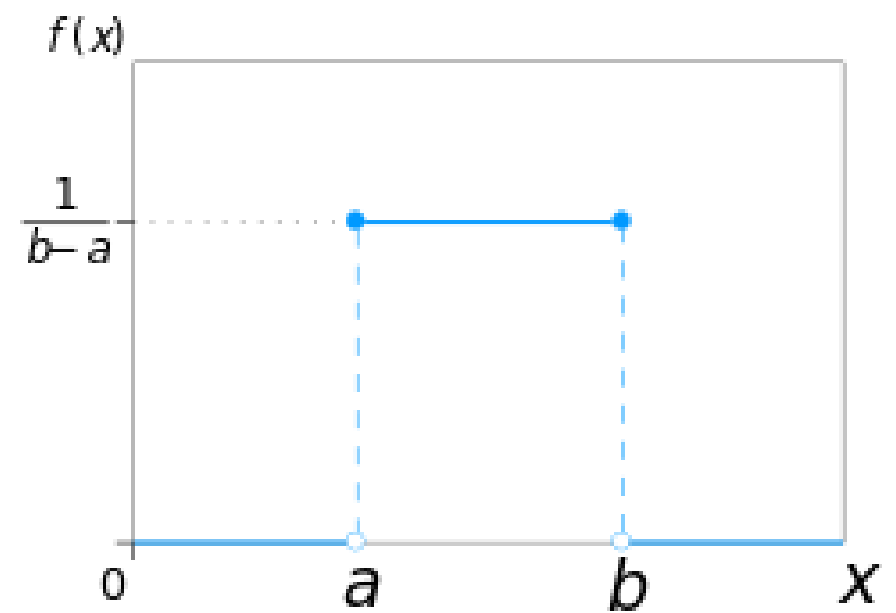
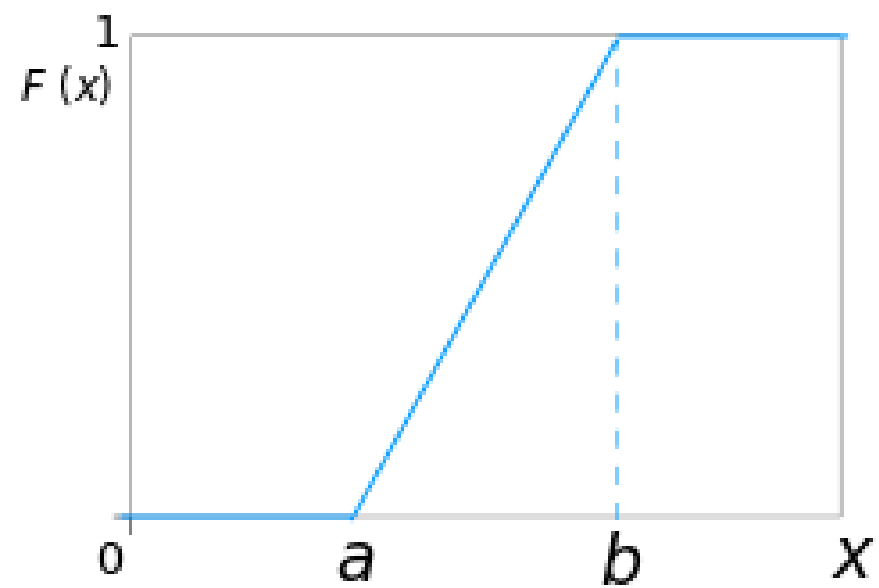
► This is a real valued r.v.

► Its pdf $f_X(x) = \frac{1}{b-a}$ for all $x \in [a, b]$.

► Its CDF is given by $F_X(x) = \begin{cases} 0 & \text{for } x < a. \\ \frac{x-a}{b-a} & \text{for } x \in [a, b] \\ 1 & \text{otherwise.} \end{cases}$

► HW: Verify $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$

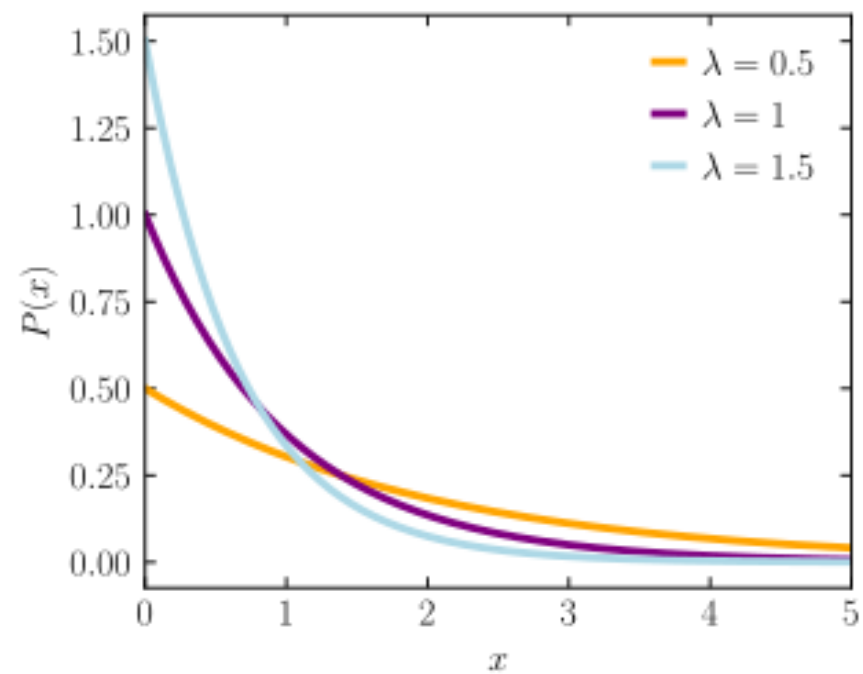
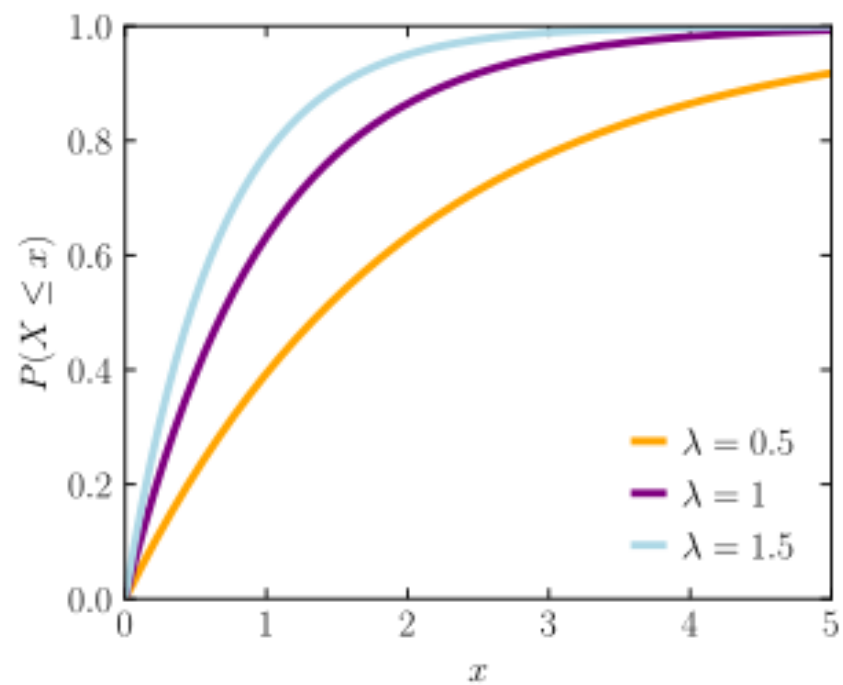
$$U[a, b]$$



Exponential random variable ($Exp(\lambda)$)

- ▶ This is a non-negative r.v. with parameter λ .
- ▶ Its pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$.
- ▶ Its CDF is given by $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.
- ▶ $E[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$
- ▶ $E[X^n] = \frac{n!}{\lambda^n}$

$Exp(\lambda)$



Significance of Exponential r.v.

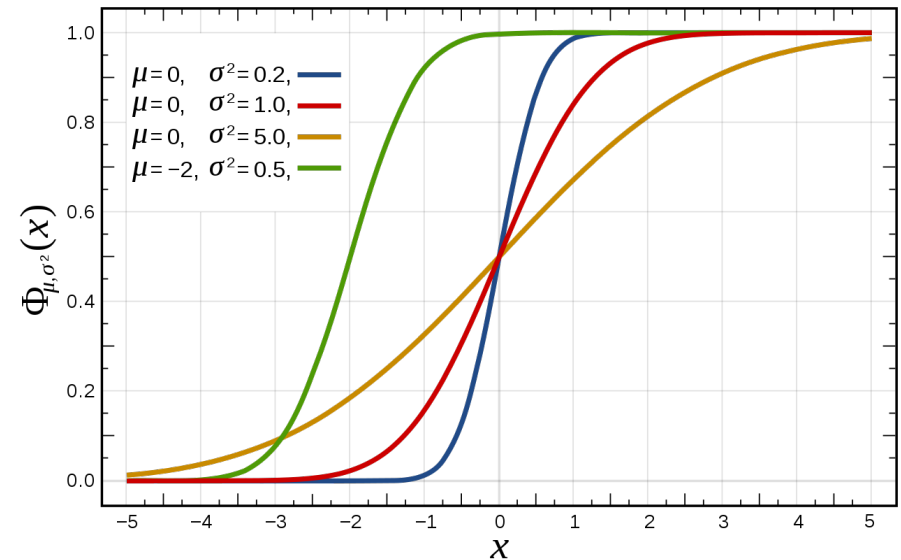
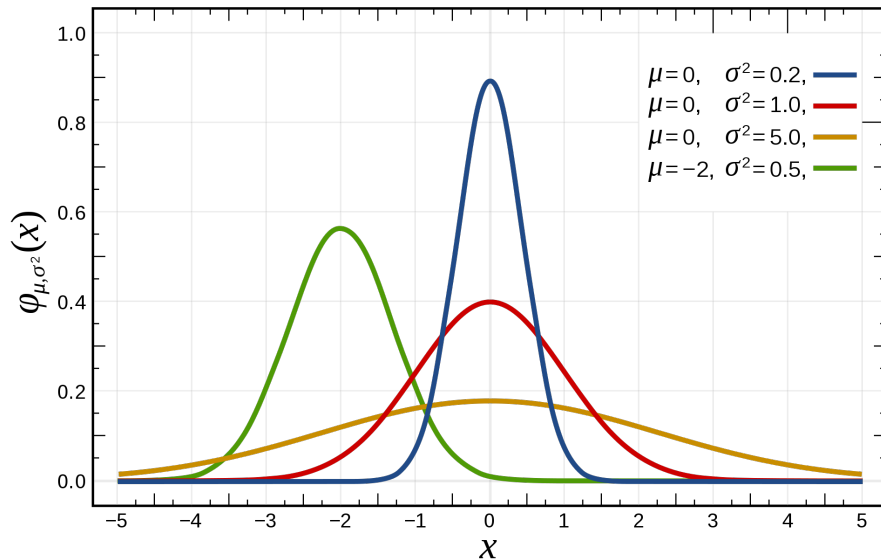
- ▶ Building blocks for Continuous time Markov Chains.
- ▶ Demonstrate memory-less property (to be seen formally soon).
- ▶
$$P(X > a + h | X > a) = \frac{e^{-\lambda(a+h)}}{e^{-\lambda(a)}} = e^{-\lambda(h)} = P(X > h).$$
- ▶ Used extensively in Queueing theory to model inter-arrival time and service time of jobs.

Gaussian random variable ($\mathcal{N}(\mu, \sigma^2)$)

► This is a real valued r.v. with two parameters, μ and σ .

► Its pdf $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for all $x \in \mathbb{R}$.

► Verify: $\int_{-\infty}^{\infty} f_X(x) dx = 1$, $E[X] = \mu$ and $\text{Var}(X) = \sigma^2$.



Standard Normal random variable ($\mathcal{N}(0, 1)$)

- ▶ When $\mu = 0$ and $\sigma = 1$, it is called as a standard normal.

- ▶ In this case $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

- ▶ What is $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$? How do you even solve this? ($= \sqrt{2\pi}$)

- ▶ The CDF of standard normal, denoted by $\Phi(x)$ is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$$

- ▶ $Q(x) := 1 - \Phi(x)$ is the Complimentary CDF ($P(X > x)$).

A closely related cousin in the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

- ▶ Φ = These values are recorded in a table. (Fig. 3.10 in Bertsekas)

- ▶ https://en.wikipedia.org/wiki/Gaussian_function

Normality preserved under Linear Transformations

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b$ is also a normal variable with $E[Y] = a\mu + b$ and variance $a^2\sigma^2$. (To be proved later)

- ▶ Suppose X is standard normal, then find a and b such that $Y \sim \mathcal{N}(\mu, \sigma^2)$
- ▶ In this case, the CDF of Y in terms of X is given by $\Phi\left(\frac{x-\mu}{\sigma}\right)$.

Significance of Gaussian r.v.

- ▶ Key role in Central limit theorem.
- ▶ $\frac{1}{n} \sum_{i=1}^n X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$ where X_i is any random variable with mean μ and variance σ^2 .
- ▶ Building block for multinomial Gaussian vector and Gaussian processes (GP).
- ▶ Gaussian process are used in Bayesian Optimization (black-box optimization).
- ▶ Brownian motion is a type of GP and is used in Finance.
- ▶ GP Regression, Gaussian mixture models, used widely in ML.

List of Probability distributions ...

`https://en.wikipedia.org/wiki/List_of_probability_distributions`

Important ones are Beta, Gamma, Erlang, Logistic, Weibull