Counter-example

▶ Random exp: Pick a number uniformly from the real line.

▶ $\Omega = \mathbb{R}$ and hence $\mathbb{P}(\mathbb{R}) = 1$.

▶ Domain $\mathcal{P}(\mathbb{R})$ which is unimaginably complex!

▶ We have $\mathbb{P} : \mathcal{P}(\mathbb{R}) \to [0, 1]$.

▶ $\mathbb{P}$ has the property that sets of equal 'length' have equal probability.

▶ We know that $\mathbb{R} = \overset{\infty}{\underset{n=-\infty}{\bigcup}} [n, n+1)$ where $[n, n+1) \in \mathcal{P}(\mathbb{R})$.

▶ What is $\mathbb{P}[n, n+1)$?

▶ If we define $\mathbb{P}[n, n+1) = x$ for all $n \in \mathbb{Z}$ then $\mathbb{P}(\mathbb{R}) = \infty$!

▶ If we define $\mathbb{P}[n, n+1) = 0$ for all $n \in \mathbb{Z}$ then $\mathbb{P}(\mathbb{R}) = 0$!

Counter-example

▶ What is the takeaway from the counterexample?

▶ Not all set-functions (or measures) can be calibrated to measure every possible subset of your sample space.

▶ This is like you weighing scale at home, that is not able to weigh a piece of paper!

▶ What is the way out?

▶ Restrict your domain to only measurable sets.

▶ Possible domain for the counter example?

▶ $\mathcal{F} = \{\Phi, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$. There can be many other domains one can define!

Towards sigma-algebra

- $\mathcal{F} = \{\emptyset, \Omega, \mathbb{R}_-, \mathbb{R}_+\}$.

- The domain $\mathcal{F}$ should have some nice and obvious properties.

- For example, $\emptyset$ and Ω in $\mathcal{F}$. . Also if $B \in \mathcal{F}$, then $B^c \in \mathcal{F}$.

- If A_1 and A_2 belong in $\mathcal{F}$, then $A_1 \cup A_2 \in \mathcal{F}$ and $A_1 \cap A_2 \in \mathcal{F}$.

- A domain with such nice properties is called as a *sigma-algebra*.

sigma-algebra as domain for $\mathbb{P}$

▶ Event space or *sigma-algebra* $\mathcal{F}$ associated with a set Ω is a collection of subsets of Ω that satisfy
- $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
- $A_1, A_2, \ldots A_n, \ldots \in \mathcal{F} \implies \cup_{n=1}^{\infty} A_n \in \mathcal{F}$

▶ The $\sigma-$algebra is said to be closed under formation of complements and countable unions.

▶ Is it also closed under the formation of countable intersections?

▶ When Ω is countable and finite, is $\mathcal{P}(\Omega)$ a sigma-algebra? Yes.

When Ω is countable and finite, we will consider power-set $\mathcal{P}(\Omega)$ as the domain.

Formal definition of Probability measure $\mathbb{P}$

Definition

A probability measure $\mathbb{P}$ on the *measurable space* $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ s.t.

1. $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

2. For a disjoint collection of event sets $A_1, A_2, \ldots$ from $\mathcal{F}$ we have
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

(countable additivity)

▶ The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.

▶ Recall that when $|\Omega| < \infty$, we consider $\mathcal{F} = 2^{\Omega}$.

Formal definition of Probability measure $\mathbb{P}$

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(countable additivity)

▶ The trio $(\Omega, \mathcal{F}, \mathbb{P})$ is called as a probability space.

▶ Identify the probability space in the coin and dice experiment.

Probability space for $U[0, 1]$

▶ $\Omega = [0, 1]$.

▶ Suppose $\mathcal{F} = \{\Phi, [0, 1], [0, .5), [.5, 1]\}$. Is there a problem in using this as a sigma-algebra?

▶ We cannot measure probability of sets like $[.25, .75]$ although we know $P([.25, .75]) = .5$.

▶ So lets include $[.25, .75]$ in $\mathcal{F}$.

▶ Now we have $\mathcal{F}^+ = \{\Phi, [0, 1], [0, .5), [.5, 1], [.25, .75]\}$. Is $\mathcal{F}^+$ a sigma-algebra? No.

▶ Can you make it a sigma-algebra by adding missing pieces ?

Probability space for $U[0,1]$

▶ $\mathcal{F}^+ = \{\Phi, [0,1], [0,.5), [.5,1], [.25,.75]\}$

▶ Can you make it a sigma-algebra by adding missing pieces ?

▶ Recall that sigma-algebras are closed under complements, union and intersection.

▶ Intersection and union of $[.25,.75]$ with sets in $\mathcal{F}^+$ gives the collection $\{[.25,.5), [.5,.75], [.25,1], [0,0.75]\}$.

▶ Adding complements, the collection enlarges by $\{[.25,.5), [.5,.75], [.25,1], [0,0.75], [0,.25) \cup [.5,1], [0,.5) \cup (.75,1], [0,.25), (0.75,1]\}$.

▶ Lets call it $\mathcal{F}^{++} =$ $\{\Phi, [0,1], [0,.5), [.5,1], [.25,.75], [.25,.5), [.5,.75], [.25,1], [0,0.75], [0,.25) \cup [.5,1], [0,.5) \cup (.75,1], [0,.25), (0.75,1]\}$

Probability space for $U[0,1]$

- $\mathcal{F}^{++} =$
  $\{\Phi, [0,1], [0,.5), [.5,1], [.25,.75], [.25,.5), [.5,.75], [.25,1],$
  $[0,0.75], [0,.25) \cup [.5,1], [0,.5) \cup (.75,1], [0,.25), (0.75,1]\}$

- Notice different type of sets with different brackets $[), (], ()$ that appear.

- But $\mathcal{F}^{++}$ is still not a sigma-algebra as each red set will demand a furthermore sets to be added.

- This operation we attempted is called generating a sigma-algebra!.

- Continuing on these lines, the resulting sigma algebra is called a borel-sigma algebra $\mathcal{B}[0,1]$.

Borel sigma-algebra $\mathcal{B}[0,1]$

► Borel $\sigma-algebra$ $\mathcal{B}[0,1]$: When $\Omega = [0,1]$ the $\mathcal{B}[0,1]$ is the $\sigma-algebra$ generated by closed sets of the form $[a,b]$ where $a \leq b$ and $a,b \in [0,1]$.

► Does this set contain sets of the form (a,b) or $[a,b)$ or $(a,b]$?

► $$\left| (a,b) = \cup_{n=1}^{\infty}[a + \tfrac{1}{n}, b - \tfrac{1}{n}]. \right| (a,b] = \cap_{n=1}^{\infty}(a, b + \tfrac{1}{n})$$

Borel $\sigma-algebra$ $\mathcal{B}[0,1]$: $\mathcal{B}[0,1]$ is the $\sigma-algebra$ generated by sets of the form $[a,b]$ or (a,b) or $(a,b]$ or even $[a,b)$ where $a \leq b$ and $a,b \in [0,1]$.

Borel sigma-algebra $\mathcal{B}(\mathbb{R})$

▶ Borel sigma-algebra $\mathcal{B}(\mathbb{R})$:
  If $\Omega = \mathbb{R}$, then $\mathcal{B}(\mathbb{R})$ is the sigma-algebra generated by open sets of the form (a, b) where $a \leq b$ and $a, b \in \mathbb{R}$.

▶ $\mathcal{B}(\mathbb{R})$ contains intervals of the form

$$[a, b]$$
$$[a, b)$$
$$(a, \infty)$$
$$[a, \infty)$$
$$(-\infty, b]$$
$$(-\infty, b)$$
$$\{a\}$$

▶ How would you define $\mathcal{B}(\mathbb{R}^2)$?

Consequences of the Probability Axioms

▶ $P(A^c) = 1 - P(A)$

▶ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

▶ If $A \subseteq B$, prove that $P(A) \leq P(B)$. ($A \subseteq B$ has the interpretation that Event A implies event B)

▶ $P(\cup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} P(B_i)$ (Boole's/Bonferroni's inequality). HW

▶ What is $P(A \cup B \cup C)$?

▶ State and prove the inclusion-exclusion principle for $P(\cup_{i=1}^{n} A_i)$

Impossible event v/s Zero prob. event

▶ In $U[0, 1]$ what is $P(\omega = 0.5)$? $= 0$.

▶ Intuitive reasoning for this is that a point has zero length!

▶ If $P(\omega \in [a, b]) = b - a$ then $P([.5, .5]) = P(\{.5\}) = 0$.

▶ This is a zero probability event. In fact, every outcome of this experiment is a zero probability event.

▶ This implies that events of zero probability can happen and they are not impossible events.

▶ $P(\emptyset) = 0$, then is $\emptyset$ also possible ?No!

▶ What is $P(\omega \in [0, .25] \cap [.75, 1])$?

▶ $P([0, .25] \cap [.75, 1]) = P(\emptyset) = 0$ This event will never happen.

Impossible event v/s Zero prob. event

▶ Note that in the $U[0, 1]$ experiment, $\Omega = \bigcup_{\omega \in \Omega} \{\omega\}$

▶ $P(\Omega) = P(\bigcup_{\omega \in \Omega} \{\omega\}) = \sum_{\omega \in \Omega} P(\{\omega\}) = 0.$

▶ What is the problem above ?

▶ Ω is an uncountable set and the probability set-function only has a countable additive property.

▶ $\bigcup_{\omega \in \Omega} \{\omega\}$ is an uncountable disjoint union!

Limits and Continuity

▶ How do we define limit of a sequence $\{a_1, a_2, \ldots, \}$?

▶ Notation: $\lim_{n \to \infty} a_n = L$.

▶ How do you define limit of a function at a point c?

▶ Notation: $\lim_{x \to c} f(x) = L$

▶ How do you define continuity of a function $f(x)$ at c ?

▶ When do you say a function is continuous ?

▶ (ϵ, δ)-definition of limits and continuity?

Limits and Continuity

Definition in terms of limits of sequences.

> For a continuous function $f(\cdot)$, as $x \to c$, we have $f(x) \to f(c)$

> For a continuous set-function S, as $A_n \to A$, we have $S(A_n) \to S(A)$

▶ Recall that $\mathbb{P}$ is a set-function. Is it continuous?

▶ We will see the proof shortly.