Sequence of sets

- Given $(\Omega, \mathcal{F})$, If $A_1 \subset A_2 \ldots$ is an increasing sequence of events defined on $\mathcal{F}$ and $\cup_{n=1}^{\infty} A_n = A \in \mathcal{F}$, then we say thats the sequence of sets A_n are increasing to A $(A_n \uparrow A)$.

- Similarly when $A_1 \supset A_2 \ldots$ is a decreasing sequence of events and $\cap_{n=1}^{\infty} A_n = A$, then we have $A_n \downarrow A$.

- Alternative notation: For an increasing sequence of sets A_n we often write $\lim\limits_{n\to\infty} A_n$ for $\cup_{n=1}^{\infty} A_n$ and for a decreasing sequence of sets A_n that $\lim\limits_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n$.

Continuity of set-function $\mathbb{P}$

Lemma

For sequence of events of the type $A_n \uparrow A$ or $A_n \downarrow A$, we have $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.

Proof

- Consider increasing sequence first. Similar arguments follow for decreasing seq.
- Define $F_n = A_n - A_{n-1}$
- $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} F_n$.
- $\mathbb{P}(A) = \mathbb{P}(\cup_{n=1}^{\infty} A_n) = \mathbb{P}(\cup_{n=1}^{\infty} F_n)$
- But $\mathbb{P}(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(F_i) = \lim_{n\to\infty} \mathbb{P}(A_n)$. $\square$

Equivalently if $An \to \emptyset$, then $\mathbb{P}(A_n) \to 0$.

Conditional probability

- Given/If dice rolls odd, what is the probability that the outcome is 1?

- Given/If $\bar{\omega} \in [0, 0.5]$ what is the probability that $\bar{\omega} \in [0, 0.25]$?

The conditional probability of event B given event A is defined as $\mathbb{P}(B/A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$ when $\mathbb{P}(A) > 0$.

Conditional probability

- Show that $P(A/B)P(B) = P(B/A)P(A)$.
- Bayes rule: $P(B/A) = \frac{P(A/B)P(B)}{P(A)}$.
- What is $P(A/(B \cap C))$?. This is also denoted as $P(A/BC)$
- Prove the chain rule $P(A \cap B \cap C) = P(A)P(B/A)P(C/(AB))$.

HW: Prove the chain rule for conditional probability given by

$P(A_1 \cap A_2 \dots A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1A_2)\dots P(A_n/A_{n-1}\dots A_1)$.

Conditional probability – Examples

- Suppose you draw 4 cards from a deck at random without replacement. What is the probability that (in order) these cards are 9 of club, 8 of diamond, king of spade and king of club?
- What if you do the above with replacement?
- Consider a finite sample space Ω where each outcome is equally likely. Then what is $P(B/A)$?
- $P(B/A) = \frac{|A \cap B|}{|A|}$.

Law of total probability

- $A = (A \cap B) \cup (A \cap B^c)$. What is $P(A)$?
- $P(A) = P(A \cap B) + P(A \cap B^c)$.
- This is same as $P(A) = P(A/B)P(B) + P(A/B^c)P(B^c)$.
- This formula is useful when $P(A)$ is not given or is difficult to find but $P(B)$ or $P(A/B)$ is readily available.

Let $B_1, B_2, \ldots B_n$ be the partition of the sample space Ω. Then for any event A we have

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i)P(B_i).$$

Example 1

- I have 3 bags that contain M marbles. Bag i has R_i red and B_i blue marbles respectively (for $i = 1, 2, 3$).
- I choose a bag at random and then draw a marble. What is the probability that the chosen marble is red ?
- Solution: $P(Red) = \sum_i P(Red/B_i)P(B_i)$

Example 2

1. If an item is defective, a robot can spot it with 98% accuracy.
2. If an item is not defective, a robot will declare it so with 99% accuracy.
3. A total of 0.1% items are defective.
4. If the robot says that the item you drew at random is defective, what is the probability that the robot is correct?

- $P(\text{defective}/\text{robot says defective}) = \frac{P(\text{robot says defective}/\text{defective})P(\text{defective})}{P(\text{robot says defective})}$

- What is $P(\text{robot says defective})$?

Bayes rule revisited

Let $B_1, B_2, \ldots B_n$ be the partition of the sample space Ω. Then for any event A with $P(A) > 0$ we have

$$P(B_j/A) = \frac{P(A/B_j)P(B_j)}{\sum_{i=1}^{n} P(A/B_i)P(B_i)}.$$

In the marble example, given that the marble drawn is red, what is the probability that bag 1 was chosen ?

Independence

- Consider the experiment of tossing a coin and a dice simultaneously.
- Identify its underlying probability space.
- What is $\mathbb{P}(\{H, 6\})$?
- What is $\mathbb{P}(\{T, \text{ odd }\}) = \mathbb{P}(\{\cup_{i=1,3,5}\{T, i\}\})$?
- In both cases above we have $\mathbb{P}(A \cap B) = P(A)P(B)$.
- This implies that $\mathbb{P}(A/B) = \mathbb{P}(A)$.

> - Two events A, B are independent iff $P(A/B) = P(A)$ and $P(B/A) = P(B)$.
> - Two events A, B are independent iff $P(A \cap B) = P(A)P(B)$.

Independence

- Two events A, B are independent if and only if $P(A \cap B) = P(A)P(B)$.

- If A and B are independent, then are A^c and B^c independent?
- What about A and B^c? Are they independent?
- If $A_1, A_2, \ldots, A_n$ are independent, then prove that

$$P(\cup_{i=1}^{n} A_i) = 1 - \prod_{i=1}^{n} [1 - P(A_i)]$$

Mutual and Pairwise Independence

- A collection of events $\{A_i, i \in I\}$ are said to be **mutually independent** if the $P(\cap_{j \in J} A_j) = \prod_{j \in J} P(A_j)$ for any subset J of I.

- A collection of events $\{A_i, i \in I\}$ are said to be **pairwise independent** if any pair of events from the collection are independent.

- Mutual independence implies pairwise independence but not the other way around.

- HW: Find an example where pairwise independence does not imply mutual independence.

Independence - Example

- Pick a number randomly from the set $\{1, \ldots, 10\}$.
- Event A says that the number is less than 7.
- Event B says that the number is less than 8.
- Event C says that the number is even.
- Are the events mutually independent?
- Which pair of event is independent?

Correlation between events

- Two events A, B are independent iff $P(A \cap B) = P(A)P(B)$.
- Two events A and B are positively correlated iff $P(A/B) > P(A)$.
- Two events A and B are negatively correlated iff $P(A/B) < P(A)$.
- A and B have the same correlation as A^c and B^c.
- A and B have the opposite correlation as A and B^c.