Recap

$$p_{X|A}(x) := \frac{p_X(x)}{P(A)} \text{ if } x \in A.$$

$$E[X/A] = \sum_x x p_{X|A}(x).$$

$$p_X(x) = \sum_{i=1}^n \mathbb{P}(A_i) p_{X|A_i}(x)$$

$$E[X] = \sum_{i=1}^n \mathbb{P}(A_i) E[X|A_i]$$

$$p_{X,Y}(x,y) = p_{X|Y}(x|y) p_Y(y)$$

$$p_X(x) = \sum_y p_{X|Y}(x|y) p_Y(y)$$

Conditional expectation $E[X|Y=y]$

It is easy to guess that

$$E[X|Y=y] := \sum_x x p_{X|Y}(x|y)$$
$$E[Y|X=x] := \sum_y y p_{Y|X}(y|x)$$

Can you write $E[X]$ in terms of $E[X|Y=y]$?

$$E[X] = \sum_y p_Y(y) E[X|Y=y]$$

$$\text{Proof: } \sum_y p_Y(y)E[X|Y=y] = \sum_y p_Y(y) \sum_x x p_{X|Y}(x|y)$$
$$= \sum_x \sum_y x p_{X,Y}(x,y)$$
$$= \sum_x x p_X(x)$$
$$= E[X]$$

Summary

$p_{X|A}(x) := \frac{p_X(x)}{P(A)}$ if $x \in A$.

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$E[X] = \sum_{i=1}^n \mathbb{P}(A_i) E[X|A_i]$

$p_{X,Y}(x,y) = p_{X|Y}(x|y) p_Y(y)$

$p_X(x) = \sum_y p_{X|Y}(x|y) p_Y(y)$

$E[X|Y=y] := \sum_x x p_{X|Y}(x|y)$

$E[X] = \sum_y p_Y(y) E[X|Y=y]$

How about all this for continuous X & Y?

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How about all this for continuous X & Y?

$$f_{X|A}(x) = \frac{f_X(x)}{P(A)} \text{ if } x \in A.$$

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$$p_X(x) = \sum_{i=1}^{n} \mathbb{P}(A_i) p_{X|A_i}(x)$$

$$E[X] = \sum_{i=1}^{n} \mathbb{P}(A_i) E[X|A_i]$$

$$p_{X,Y}(x,y) = p_{X|Y}(x|y) p_Y(y)$$

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$$f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$$

$$f_X(x) = \int_y f_{X|Y}(x|y) f_Y(y) dy$$

$$E[X|Y=y] = \int_x x f_{X|Y}(x|y) dx$$

$$E[X] = \int_y E[X|Y=y] f_Y(y) dy$$

Conditional expectation $E[X|Y]$

Recall that

$$E[X|Y=y] := \sum_x x p_{X|Y}(x|y)$$

- $E[X|Y=y]$ is a constant given y.
- $g(y) := E[X|Y=y]$ is a function of y.
- Now consider the random variable $E[X|Y]$.
- When Y takes the value y, (this happens with probability $p_Y(y)$) $E[X|Y]$ takes the value $E[X|Y=y]$.
- $E[X|Y]$ is a function of Y, say $g(Y)$.
- What is the expectation of $E[X|Y]$?

Conditional expectation $E[X|Y]$

- $g(Y) = E[X|Y]$.
- What is $E[g(Y)] = E[E[X|Y]]$?
- $E[g(Y)] = \sum_y g(y)p_Y(y) = \sum_y E[X|Y=y]p_Y(y)$.
- This implies $E[g(Y)] = E[E[X|Y]] = E[X]$. This is the law of iterated expectation.

$$\boxed{E[E[X|Y]] = E[X]}$$

Conditional expectation $E[X|Y]$ – Example

- Consider $Y = \begin{cases} \lambda_1 \text{ with prob } p \\ \lambda_2 \text{ with prob } 1-p \ . \end{cases}$
- Now consider an exponential random variable X with a random parameter Y.
- What is $E[X]$?
- $E[X] = E[E[X|Y]] = \sum_y E[X|Y=y]p_Y(y)$
- We have $X \sim Exp(\lambda_1)$ with probability p when $Y = \lambda_1$.
- Similarly $X \sim Exp(\lambda_2)$ with probability $1-p$ when $Y = \lambda_2$.
- $E[X|Y=\lambda_i] = \frac{1}{\lambda_i}$
- $E[X] = \frac{p}{\lambda_1} + \frac{1-p}{\lambda_2}$.

Conditional expectation $E[X|Y]$ – Example 2

- Consider $Y = X_1 + X_2 + \ldots X_N$ where N is a positive integer valued r.v. with PMF $p_N(\cdot)$ and $X_i's$ are independent and identically distributed (i.i.d) with mean $E[X]$.
- What is $E[Y]$? Use $E[Y] = E[E[Y|N]]$.
- What is $E[Y|N = n]$?
- $E[Y|N = n] = E[X_1 + X_2 + \ldots X_n] = nE[X]$.
- This implies $E[Y|N] = NE[X]$.
- Now $E[Y] = E[E[Y|N]] = E[NE[X]] = E[X]E[N]$.
- What is $Var(Y)$? (section 4.5)

Sums of independent random variable

- Consider $Z = X + Y$. What is the pdf of Z when X and Y?
- What is $p_Z(z)$ or $f_Z(z)$?
- $p_Z(z) = \sum_{\{(x,y):x+y=z\}} p_{X,Y}(x,y)$
- $f_Z(z) = \int_{\{(x,y):x+y=z\}} f_{X,Y}(x,y)dxdy$.
- Integral of a surface over line.
- `https://en.wikipedia.org/wiki/Line_integral`
- Since X and Y are independent $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ and $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. This gives us

> Convolution formula
>
> $$p_Z(z) = \sum_x p_X(x)p_Y(z-x)$$
> $$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

HW: What if X and Y are not independent?

Examples

- EX1: Suppose X and Y are independent and $U[0, 1]$. Find the pdf and CDF of $Z = X + Y$.
- `https://en.m.wikipedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif`
- Ex2: Suppose X and Y are outcomes of independent roll of dice. Find the pmf of $Z = X + Y$.