MA 6.101
Probability and Statistics

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Logistics

- Feel free to contact me anytime at tejas.bodas@iiit.ac.in.
- Office @ A5304.
- TA list: Around 12 TAs, you will meet them during tutorials
- Lectures: Wednesday and Saturday 10:00 to 11:25
- Tutorial on Friday 8:30 to 10:00.
- Phones in pocket, laptops in bag!

Resources

- Wont be following any one particular book.
- Lecture slides will have material from variety of sources.
- Some popular books
  1. Introduction to probability by Bertsekas and Tsisiklis (Athena Scientific)
  2. Intro. to Probability and Statistics for Engineers and Scientists by Sheldon Ross (Elsevier)
  3. A first course in probability by Sheldon Ross (Prentice Hall)
- Some urls
  1. https://www.probabilitycourse.com/
  2. https://www.statlect.com/
  3. https://www.randomservices.org/

Evaluation scheme

- Quiz 1 : 15%
- Midsem exam: 30%.
- Quiz 2: 10%
- Endsem 35 %.
- Surprise Quiz 10 %

Course Outline

- Module 1 (3 Lectures)
  Motivation & Probability basics
- Module 2 (11 Lectures)
  All about random variables!
- Module 3 (5 Lectures)
  Conditioning, Modes of convergence, Stochastic Simulation
- Module 4 (3 Lectures)
  Random vectors, Random Processes
- Module 5 (4 lectures)
  Probability inequalities and Statistics

Prerequisites

- Set theory
- Limits & Continuity
- Differentiation & Integration
- Matrices and Determinants
- These are clickable links to relevant NCERT resources !

Where is probability & statistics useful?

- Machine Learning
- Reinforcement Learning
- Computer Systems (performance analysis)
- Finance (option pricing)
- Operations Research (Inventory management, dynamic pricing)

Random experiments and Sample space

- Random experiment : Experiment involving randomness
  - Coin toss
  - Roll a dice
  - Pick a number at random from $[0, 1]$.
- Sample space Ω: set of all possible outcomes of the random experiment. It could be a finite or infinite set.
  - $\Omega_c = \{H, T\}$
  - $\Omega_d = \{1, 2, \ldots, 6\}$
  - $\Omega_u = [0, 1]$
  - $\Omega_{2c} = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$

Outcomes and Events

- Element $\omega \in \Omega$ is called a **sample point** or possible outcome.
- A subset $A \subseteq \Omega$ is called an **event**.
- Examples of events
  - Events in the coin experiment: $C_1 = \{T\}$.
  - Events in the dice experiment: $D_1 = \{6\}, D_2 = \{1, 3, 5\}$
  - Events in $U[0, 1]$ experiment: $U_1 = \{0.5\}$, $U_2 = [.25, .75]$.
- In this course, we are interested in probability of events.
- Probability of event A is denoted by $\mathbb{P}(A)$.
- It may not be possible to measure/assign probability for every subset A (more later).
- Any guesses for $\mathbb{P}(C_1), \mathbb{P}(D_1), \mathbb{P}(D_2), \mathbb{P}(U_1)$ and $\mathbb{P}(U_2)$?

Probability theory

$\{$Random experiment, Sample space, Events$\}$ are the key ingredients in probability theory.

In probability theory, we are interested in **measuring** the probability of subsets of Ω (events).

Probability measure $\mathbb{P}$ is a **set function**, i.e. it acts on sets and measures the probability of such sets.

Set theory 101

Visualizing operations on events using Venn diagram!

- Complements: A^c
- $\emptyset$ denotes empty set.$\emptyset \subseteq A$ for all A.
- Union: $A \cup B$
- Intersections: $A \cap B$
- Difference: $A \backslash B$
- Symmetric difference:
- Mutually exclusive or disjoint events A and B:
- Identity laws,Complement laws,Associative, Commutative & Distributive laws, De'Morgans law.

Set theory 101–Cardinality & Countability

- Cardinality of A is denoted by $|A|$.
- Inclusion-exclusion principle $|A \cup B| = |A| + |B| - |A \cap B|$.
- Inclusion-exclusion principle for n sets ?
- Countable sets: Set A is said to be countable if it is either finite or has 1-1 correspondence with natural numbers $\mathbb{N}$.
- Uncountable sets: These are sets which are not countable.

Set theory 101 – Monotone sequence of sets

- Increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \ldots$

- Decreasing sequence $A_1 \supseteq A_2 \supseteq A_3 \ldots$

- Examples from $U[0, 1]$:
  - $I_n = [0, 1 - \frac{1}{n}]$
  - $D_n = [0, \frac{1}{n}]$

Set theory 101 – Cartesian product of sets

- Cartesian product of sets A and B is denoted by $A \times B$.

- $A \times B$ is itself a set whose members are sets of the form (a, b) where $a \in A$ and $b \in B$.

- Suppose $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$ what is $A \times B$?

- What is $[0, 1] \times [0, 1]$? unit square!

Set theory 101 – Powersets

Powerset of A is denoted by $\mathcal{P}(A)$ is a set whose members are all possible subsets of A. ($\mathcal{P}$ and $\mathbb{P}$ are different!)

- What is $\mathcal{P}(\Omega_c)$?
- What is $\mathcal{P}(\Omega_d)$?
- What is $\mathcal{P}(\Omega_u)$?
- What is the cardinality of $\mathcal{P}(\Omega_c), \mathcal{P}(\Omega_d), \mathcal{P}(\Omega_u)$?
- For discrete sets Ω, often the power set is denoted by 2^{Ω}.

functions and set functions

- What are functions? Functions are rules or maps that map elements from a domain $\mathcal{D}$ to elements in the range $\mathcal{R}$.
- $f : \mathcal{D} \to \mathcal{R}$.
- Example: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$.
- Read more on injection, surjection, bijection!
- What are set functions? these are functions that act on sets and hence domain $\mathcal{D}$ is a collection of sets.
- Example: length of closed segments on the real line.
- $l : \mathcal{D} \to \mathbb{R}_+$ where $\mathcal{D} = \{[a, b] : a \leq b, a, b \in \mathbb{R}\}$ and where $l([c, d]) = d - c$.

Back to $\mathbb{P}$

- Why this detour to set theory?
- Recall that Probability measure $\mathbb{P}$ acts on sets and measures the probability of such sets.
- In set theory 101 we looked at operations on sets A and B that gave new sets like $A \cup B$, $A \backslash B$, $A \times B, \mathcal{P}(A)$.
- So given $\mathbb{P}(A)$ and $\mathbb{P}(B)$, can we deduce $\mathbb{P}(A \cup B)$ or $\mathbb{P}(A/B)$?
- We want to understand how the probability measure $\mathbb{P}$ acts on sets such as $A \cup B$, $A \backslash B$, $A \times B$.

$\mathbb{P}$ axioms

Probability measure $\mathbb{P}$ is a **set function**.

Axiom 1: $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$
Axiom 2: For a set $A \subseteq \Omega$ we have $0 \leq \mathbb{P}(A) \leq 1$.
Axiom 3: For a disjoint collection of events $A_1, A_2, \ldots$ (where $A_i \subseteq \Omega$)

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

- What is in general the domain of $\mathbb{P}$? Ω?
- $\mathcal{P}(\Omega)$? Recall $\mathcal{P}(\Omega) = \{A : A \subseteq \Omega\}$. Seems like a great choice!

Towards a formal definition of $\mathbb{P}$

Probability measure $\mathbb{P}$ can be defined as a set-function $\mathbb{P} : \mathcal{P}(\Omega) \to [0, 1]$ that satisfies the following 3 axioms.
Axiom 1: $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$
Axiom 2: For a set $A \subseteq \Omega$ we have $0 \leq \mathbb{P}(A) \leq 1$.
Axiom 3: For a disjoint collection of events $A_1, A_2, \ldots$ (where $A_i \subseteq \Omega$)

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

- Is there a perceivable problem with this definition?
- The following counter-example will construct a set-function $\mathbb{P}$ for which you cannot assign valid probabilities to every subsets in Ω without violating these axioms.