MA 6.101 Probability and Statistics

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Towards E[g(X, Y)]

▶ What about E[aX + bY + c]?

$$E[aX + bY + c] = \sum_{x,y} (ax + by + c) p_{XY}(x,y)$$

$$= a \sum_{xy} x p_{XY}(x,y) + b \sum_{xy} y p_{XY}(x,y)$$

$$+ c \sum_{xy} p_{XY}(x,y)$$

$$= aE[X] + bE[Y] + c.$$

Along similar lines, one would expect:

$$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{XY}(x,y)$$

Finding $p_Z(\cdot)$ where Z = g(X, Y).

- ▶ Suppose Z = g(X). Then what is $p_Z(z)$?
- $\triangleright p_Z(z) = \sum_{\{x:g(x)=z\}} p_X(x).$
- Now suppose Z = g(X, Y) then we have

$$p_Z(z) = \sum_{\{x,y:g(x,y)=z\}} p_{XY}(x,y)$$

E[g(X, Y)]

- ▶ How do we define E[g(X, Y)]?
- ▶ One way is to define Z = g(X, Y) and find $E[Z] = \sum_{z} zp_{z}(z)$
- $\blacktriangleright \text{ Recall } p_Z(z) = \sum_{\{x,y:g(x,y)=z\}} p_{XY}(x,y)$
- This gives us $E[Z] = \sum_{z} \sum_{\{x,y:g(x,y)=z\}} zp_{XY}(x,y)$.
- This is same as $E[g(X,Y)] = \sum_{\{x,y\}} g(x,y) p_{XY}(x,y)$.

$$E[g(X,Y)] = \sum_{xy} g(x,y) p_{XY}(xy)$$

$$\frac{E[g(X,Y)]}{\text{ous r.v}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) dx dy \quad \text{(for continuous r.v)}$$

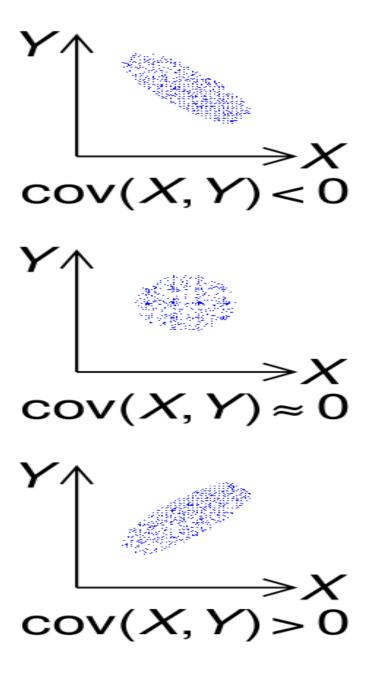
Example of function of Random variables

- Suppose X and Y are continuous independent random variables. Let W = max(X, Y) and Z = min(X, Y) Find the CDF and pdf of Z.
- ► HW: When X and Y are exponential with parameters λ_1 and λ_2 then Z is also exponential with parameter $\lambda_1 + \lambda_2$.

Covariance of X and Y

- ightharpoonup Cov(X, Y) = E[(X E[X])(Y E[Y])] = E[XY] E[X]E[Y].
- ► When Covariance is zero, they are said to be uncorrelated.
- (Section 4.2 Bertsekas)
- https://en.wikipedia.org/wiki/Covariance

Covariance of X and Y



Conditioning with random variables

Today's class

- ▶ Conditioning X on an event $A \in \mathcal{F}$.
- ightharpoonup Conditional Expectation E[X|A].
- ▶ Conditioning X with disjoint partitions $\{A_i\}$ of Ω .
- ▶ Conditioning X on an event $\{X \in A\} \in \mathcal{F}'$
- ightharpoonup Conditioning X on another random variable Y.
- ▶ Conditional expectation E[X|Y=y].

A new running example

- \triangleright Pick 2 integers from $\{1,2,3\}$ without replacement.
- $ightharpoonup \mathbb{P}\{\omega\} = \frac{1}{6} \text{ for all } \omega \in \Omega.$
- \triangleright Denote them by random variables X and Y.
- For $\omega = (1,3) \ X(\omega) = 1$ and $Y(\omega) = 3$.
- ▶ Write down their joint PMF $p_{X,Y}(x,y)$.
- ightharpoonup Write down their marginal PMFs p_X and p_Y ?
- ▶ What is E[X], E[Y] and E[XY]?

Conditioning on an event A

- Consider a discrete r.v. X with pmf $p_X(x)$. Suppose an event A has happened where $A \in \mathcal{F}$.
- Consider event $\{\omega \in \Omega : X(\omega) = x\}$. We will use shorthand $\{X = x\}$.
- ► What is $\mathbb{P}(X = x|A)$? $\mathbb{P}(X = x|A) = \frac{\mathbb{P}(\{X = x\} \cap A)}{\mathbb{P}(A)}$.

$$p_{X|A}(x) := \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}$$

- $\triangleright p_{X|A}(x)$ denotes the conditional PMF of X under event A.
- In the running example say A is the event that the first number is odd and second is even. $A = \{(1,2), (3,2)\}$. Compute $p_{X|A}(\cdot)$.
- ▶ How do we know that it is consistent, i.e., $\sum_{x} p_{X|A}(x) = 1$?