

Mutually exclusive and Independence

- ▶ Two events A and B are mutually exclusive if occurrence of one implies that the other event cannot occur. Are they independent?
- ▶ If A and B are mutually exclusive, then they are not independent (and vice versa). This can be seen as follows.

A and B are Mutually Exclusive

- ▶ $P(A \cap B) = 0$
- ▶ $P(A/B) = 0$
- ▶ $P(A/B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A)}{P(B^c)}$

A and B are Independent

- ▶ $P(A \cap B) = P(A)P(B)$
- ▶ $P(A/B) = P(A)$
- ▶ $P(A/B^c) = P(A)$

- ▶ If $A \subseteq B$, then two events are neither mutually exclusive nor independent.

Zero probability events and Independence

Zero probability events are always independent!

- ▶ Let E be a zero probability event, i.e. $P(E) = 0$.
- ▶ Then for any set F , we want to show that $P(E \cap F) = 0$.
- ▶ Note that $E \cap F \subseteq E$.
- ▶ This implies that $P(E \cap F) \leq P(E)$.

Conditional independence

- ▶ Recall : $P(A/B) = \frac{P(AB)}{P(B)}$.
- ▶ Also recall : $P(A/BC) = \frac{P(AB/C)}{P(B/C)}$
- ▶ This implies $P(AB/C) = P(A/BC)P(B/C)$.

Two events A and B are said to be conditionally independent of event C ($P(C) > 0$) if $P((AB)/C) = P(A/C).P(B/C)$

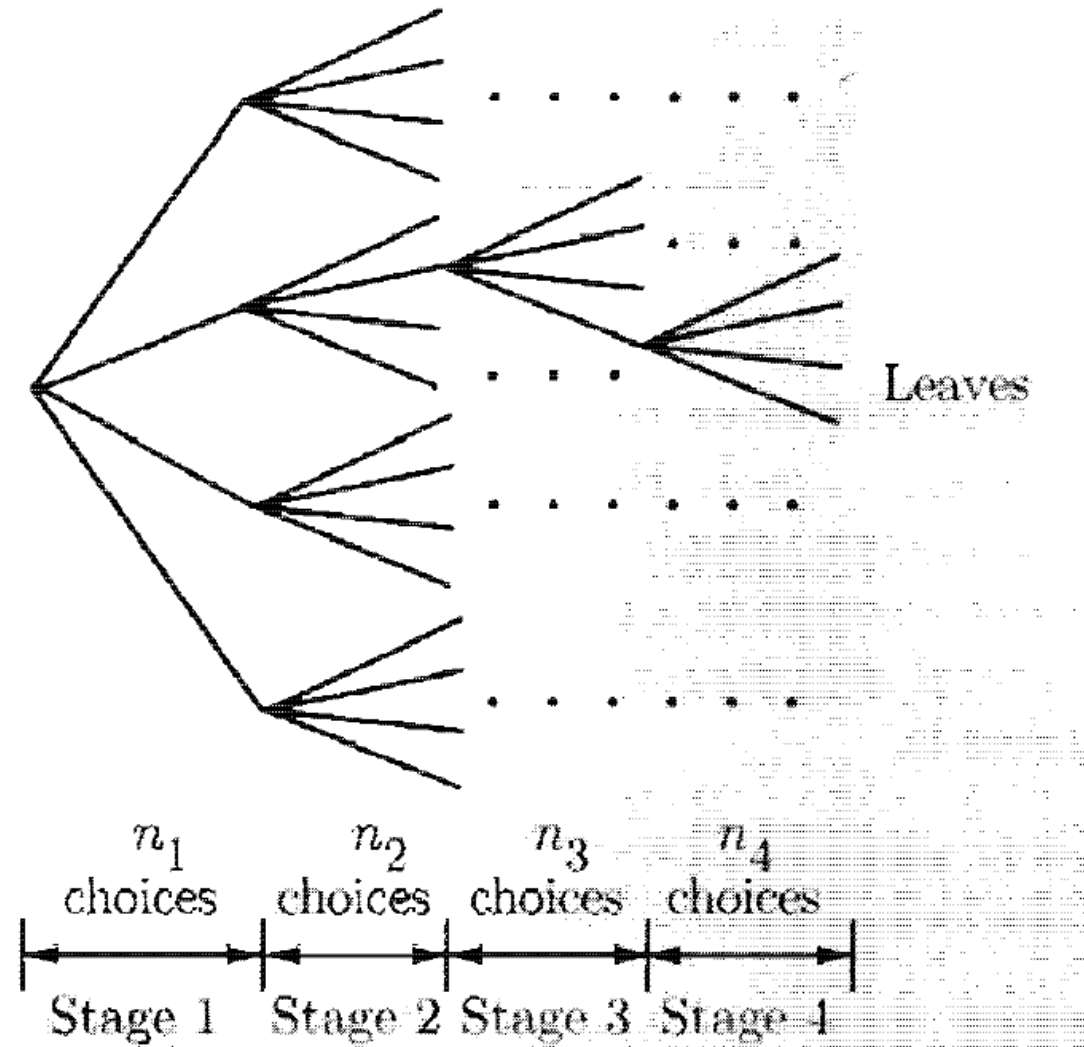
- ▶ As a consequence $P(A/BC) = P(A/C)$

HW: Verify if events A and B are conditionally independent of event C (in the experiment of picking number randomly in $\{1, \dots, 10\}$)

Conditional independence – example

- ▶ There are two coins, one fair and other fake (both heads). The experiment is to choose a coin uniformly and toss twice.
- ▶ Event A: First coin toss results in H. What is its probability? $P(A) = 3/4$.
- ▶ Event B: Second coin toss results in H. What is its probability? $P(B) = 3/4$.
- ▶ Event C: Coin 1 is chosen.
- ▶ What is $P(A/C)$ and $P(B/C)$? $1/2$
- ▶ What is $P((A \cap B)/C)$? $1/4$ Hence A and B are conditionally independent given C.
- ▶ Are A and B independent? HW

First principle of counting



Principle of counting

- ▶ Given n objects, in how many ways can you arrange them? $n!$
- ▶ Given n objects, how many distinct pairs can you form?
 ${}^nC_2 = \binom{n}{2} = \frac{n!}{n-2!2!}.$
- ▶ In general, given n objects, we can make ${}^nC_k = \binom{n}{k} = \frac{n!}{n-k!k!}$ distinct combination of k objects.
- ▶ Note that in each combination or group of k objects, the ordering within each group is immaterial. What if we also want to count this?
- ▶ ${}^nP_k = {}^nC_k \times k!$

Experiments with Sampling

- ▶ Sampling: Sampling from a set means choosing an element from the set.
- ▶ Sampling uniformly at random: All items in the set have equal probability of being chosen.
- ▶ Sampling can be with replacement or without replacement.
- ▶ Sampling can be ordered or unordered.
- ▶ In ordered sampling, $(a, b, c) \neq (c, b, a)$.
- ▶ This leaves us with 4 combinations.
 1. Ordered sampling with replacement
 2. Ordered sampling without replacement
 3. Unordered sampling with replacement
 4. Unordered sampling without replacement

Ordered sampling with replacement

- ▶ Suppose you want to sample k out of n objects with replacement and where the ordering of the k objects matters.
- ▶ Because we sample with replacement, repetition is allowed.
- ▶ How many ways can you choose k objects out of n this way?
- ▶ a) nk ? b) $\binom{n}{k}$ c) k^n d) n^k ?
- ▶ There are k positions and n choices for every position.
- ▶ Total n^k .

Ordered sampling without replacement

- ▶ Suppose you want to sample k out of n objects now without replacement and where the ordering of the k objects matters.
- ▶ Because we sample without replacement, repetition is not allowed.
- ▶ How many ways can you choose k objects out of n this way?
- ▶ a) nk ? b) $\binom{n}{k}$ c) k^n d) none ?
- ▶ There are k positions and $n - i + 1$ choices for every i^{th} position.
- ▶ Total $n \times (n - 1) \times \dots (n - k + 1) = \frac{n!}{(n-k)!} = {}^n P_k$.

Unordered sampling without replacement

- ▶ Here you want to sample k out of n objects without replacement and the ordering of the k objects does not matter.
- ▶ Because we sample without replacement, repetition is not allowed.
- ▶ How many ways can you choose k objects out of n this way?
- ▶ a) nk ? b) $\binom{n}{k}$ c) k^n d) none ?
- ▶ Essentially we want to count distinct k sized subsets from n objects without caring for ordering.
- ▶ nC_k .

Unordered sampling with replacement

- ▶ Here you want to sample k out of n objects with replacement and the ordering of the k objects does not matter.
- ▶ Because we sample with replacement, repetition is allowed.
- ▶ How many ways can you choose k objects out of n this way?
- ▶ In any such sampling, any object i can appear at most k times.
- ▶ Let x_i denote the number of times object i is chosen in k samples.
- ▶ Then any sampling satisfies $\sum_{i=1}^n x_i = k$
- ▶ How many solutions to the above equation tells you how many ways you can do the above sampling.
- ▶ $\binom{n+k-1}{k}$ Think(HW).

Example 1

- ▶ How many different 7-plate licenses are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?
- ▶ ANS: $26^3 10^4$.
- ▶ What if the alphabets and numbers are not to repeat?

Example 2

- ▶ How many functions defined on n points are possible if each functional value is either 0 or 1?
- ▶ ANS: 2^n

Example 3

- ▶ How many different letter arrangements can be formed using the letters PEPPER?
- ▶ If the P's and E's are distinguished as P_1, P_2, P_3 and E_1, E_2, R then $6!$.
- ▶ But we don't want to distinguish the P's and E's.
- ▶ For every indistinguishable arrangement, say PPPREE, there are $3! \times 2!$ different distinguished arrangements.
- ▶ Using principles of counting, the number of indistinguishable arrangements are $\frac{6!}{3!2!1!}$

Example 4

- ▶ How many different permutations of n objects can be formed when n_1 are alike, n_2 are alike, ..., n_r are alike?
- ▶ ANS: $\frac{n!}{n_1!n_2!\dots n_r!}$ where $\sum_{i=1}^r n_i = n$.
- ▶ When $r = 2$, we have $\frac{n!}{n_1!n-n_1!} = {}^nC_{n_1} = {}^nC_{n-n_1}$.
- ▶ Now suppose there are n distinct items and you want to divide them in r groups where group i has size n_i and where $\sum_{i=1}^r n_i = n$. How many ways can you do this in?
- ▶ ANS: $\binom{n}{n_1} \binom{n-n_1}{n_2} \dots \binom{n-n_1-n_2-\dots-n_{r-1}}{n_r}$
- ▶ This is same as $\frac{n!}{n_1!n_2!\dots n_r!}$

Example 5

- ▶ There are n red balls and r bins. How many ways can you put these balls in bins such that no bin is empty ?
- ▶ ANS: This is same as finding the number of solutions to
$$\sum_{i=1}^r x_i = n \text{ where } x_i > 0.$$
- ▶ Arrange all n balls in a line.
- ▶ There are $n - 1$ spaces between these n balls where you want to place $r - 1$ partitions (or sticks).
- ▶ No two partitions can be in the same space else that would mean a bin is empty.
- ▶ Select $r - 1$ out of $n - 1$ (unordered without replacement)
- ▶ $\binom{n-1}{r-1}$.

Example 6

- ▶ There are n red balls and r bins. How many ways can you put these balls in bins such that bins can be empty ?
- ▶ ANS: This is same as finding the number of solutions to
$$\sum_{i=1}^r x_i = n \text{ where } 0 \leq x_i \leq n.$$
- ▶ This is same as finding the number of solutions to
$$\sum_{i=1}^r y_i = n + r \text{ where } 0 < y_i < n. \text{ (substitute } y_i = x_i + 1 \text{ above!)}$$
- ▶ $\binom{n+r-1}{r-1}.$

Example 6 – Alternative solution

- ▶ There are n red balls and r bins. How many ways can you put these balls in bins such that bins can be empty ?
- ▶ ANS: This is same as finding the number of solutions to
$$\sum_{i=1}^r x_i = n \text{ where } 0 \leq x_i \leq n.$$
- ▶ Represent x_i by that many vertical lines.
- ▶ Total n vertical lines and $r - 1$ + signs.
- ▶ $n + r - 1$ objects where n are alike and $r - 1$ are alike.
- ▶ $\binom{n+r-1}{r-1}$.

Example 7

- ▶ Toss a biased coin n times with p as the probability of head. What is the probability that you have k heads ?
- ▶ ANS: $\binom{n}{k} p^k (1 - p)^{n-k}$.
- ▶ When $p = 1$ and $k = n$, we will have the convention that $0^0 = 1!$. Check the following link
- ▶ https://en.wikipedia.org/wiki/Zero_to_the_power_of_zero
- ▶ What is the probability that you will get head for the first time at the r^{th} toss where $r \leq n$?
- ▶ ANS: $(1 - p)^{r-1} p$.

Example 8

- ▶ Suppose you roll a dice n times, what is probability that half of them show 1 and remaining half show 6? (n is even)
- ▶ ANS: $\binom{n}{n/2} \left(\frac{1}{6}\right)^n$
- ▶ What is the probability that n_1 of them show 1 and n_2 show 6?
- ▶ $\frac{n!}{n_1!n_2!(n-n_1-n_2)!} \left(\frac{1}{6}\right)^{n_1} \left(\frac{1}{6}\right)^{n_2} \left(\frac{4}{6}\right)^{(n-n_1-n_2)}$

Example: Monty Hall

- ▶ There are 3 doors, 2 goats and 1 car.
- ▶ Please choose one door! Suppose you choose Door 1.
- ▶ As a presenter, I open a door which has a goat (say door 2) and give you an option to change your choice. Will you ?
- ▶ Sol: Let C_i denote the event that door i conceals a car and G denote the event that a goat is shown at door 2.
- ▶ $P(G) = 1$ since the presenter has shown a goat behind door 2.
- ▶ What is $P(C_3/G)$? Verify if the following is correct.
- ▶
$$P(C_3/G) = \frac{P(C_3 \cap G|C_1)P(C_1) + P(C_3 \cap G|C_1^c)P(C_1^c)}{P(G)} = 2/3 > P(C_3)$$