Math Document Template

C ANISH

***Abstract*—This is a document explaining question about the concept of Linear algebra.**

Download all python codes from

```
svn co https://github.com/chakki1234/summer
    −2020/trunk/linearalg/codes
```

and latex-tikz codes from

```
svn co https://github.com/chakki1234/summer
    −2020/trunk/linearalg/figs
```

1 Triangle

1.1 Problem

In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

1.2 Construction

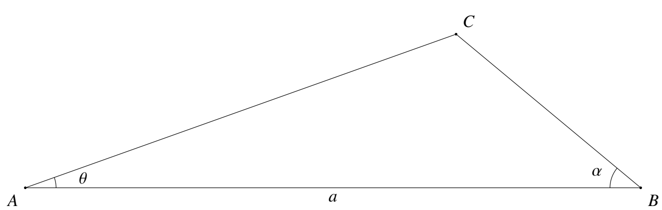

Fig. 1.2.0: Triangle by Latex-Tikz

1.2.1. The figure obtained looks like Fig. 1.2.0.

1.2.2. The design parameters used for construction See Table. 1.2.2.

Design Parameters	
Parameters	Value
a	10

TABLE 1.2.2: Triangle *ABC*

1.2.3. To find the coordinates of $\mathbf{C}$ in Fig 1.2.0
Solution:

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.2.3.1}$$

$$\mathbf{B} = \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{1.2.3.2}$$

$$\mathbf{C} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.2.3.3}$$

Finding the Scalar Products:

$$\begin{aligned} (\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) &= \\ \|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\| \cos\theta & \end{aligned} \tag{1.2.3.4}$$

$$\begin{aligned} (\mathbf{C} - \mathbf{B})^T (\mathbf{A} - \mathbf{B}) &= \\ \|\mathbf{C} - \mathbf{B}\| \, \|\mathbf{A} - \mathbf{B}\| \cos\alpha & \end{aligned} \tag{1.2.3.5}$$

On simplifying equation 1.2.3.4 and 1.2.3.5:

$$x^2 \tan\theta^2 = y^2 \tag{1.2.3.6}$$

$$(x - a)^2 = \left((x - a)^2 + y^2\right)\cos\alpha^2 \tag{1.2.3.7}$$

Substituting 1.2.3.6 in 1.2.3.7:

$$\begin{aligned} &x^2\left(1 - \cos\alpha^2 - \tan\theta^2\cos\alpha^2\right) \\ &+x\left(2a\cos\alpha^2 - 2a\right) + a^2\sin\alpha^2 \end{aligned} \tag{1.2.3.8}$$

If θ and α are accute angles:

$$x = \frac{\left(-b - \sqrt{b^2 - 4ac}\right)}{2a} \tag{1.2.3.9}$$

else:

$$x = \frac{\left(-b + \sqrt{b^2 - 4ac}\right)}{2a} \tag{1.2.3.10}$$

The value of x can then be substituted in 1.2.3.6 to find the coordinates of $\mathbf{C}$

1.2.4. From the given information, The values are listed in 1.2.4

Output values	
Parameter	Value
C	$\begin{pmatrix} 7 \\ 2.5 \end{pmatrix}$

TABLE 1.2.4: Value of **C**

1.2.5. Draw Fig. 1.2.5.

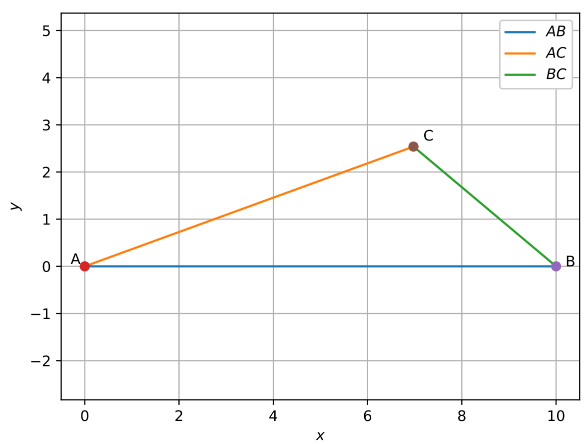

Fig. 1.2.5: Triangle generated using python

Solution: The following Python code generates Fig. 1.2.5

```
codes/triangle_ex/triangle_linearalg.py
```

and the equivalent latex-tikz code generating Fig. 1.2.5 is

```
figs/triangle_ex/triangle_fig.tex
```

The above latex code can be compiled as a standalone document as

```
figs/triangle_ex/triangle_final.tex
```

1.3 Solution

Theorem 1.1. *Sum of all angles in a triangle equals* 180°.

Solution: From theorem 1.1

$$\angle A + \angle B + \angle C = 180° \tag{5.1}$$

From the given information:

$$\frac{\angle C}{6} + \frac{\angle C}{3} + \angle C = 180° \tag{5.2}$$
$$\therefore \angle C = 120° \ \angle A = 20° \ \angle B = 40°$$

2 Quadilateral

2.1 Problem

In a $ABCD$ is a cyclic quadilateral with

$$\angle A = 4y + 20 \tag{5.1}$$
$$\angle B = 3y - 5 \tag{5.2}$$
$$\angle C = -4x \tag{5.3}$$
$$\angle D = -7x + 5 \tag{5.4}$$

Find its angles.

2.2 Construction

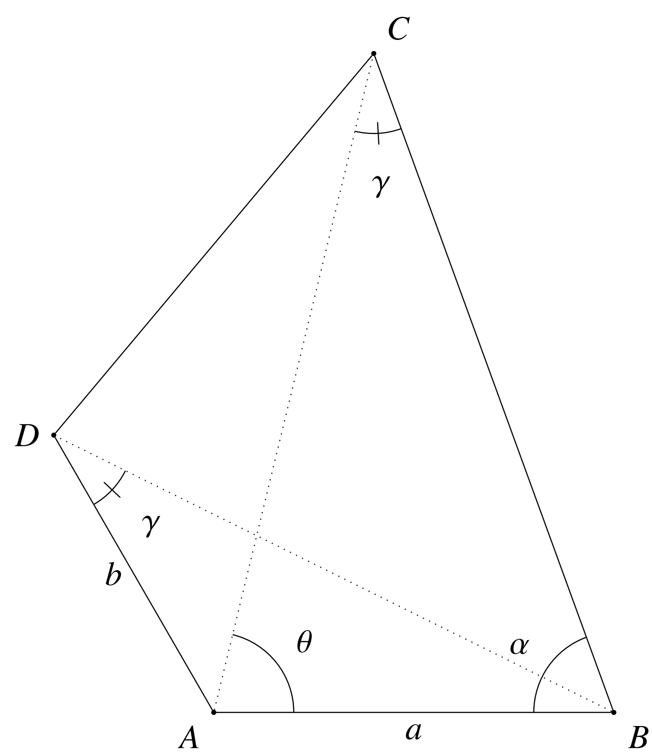

Fig. 2.2.0: Cyclic quadilateral by Latex-Tikz

2.2.1. The figure obtained looks like Fig. 2.2.0.

2.2.2. The design parameters used for construction See Table. 2.2.2.

Design Parameters	
Parameters	Value
a	10
b	8

TABLE 2.2.2: Quadilateral $ABCD$

2.2.3. Coordinates of cyclic quadilateral Fig2.2.0.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.3.1}$$

$$\mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{2.2.3.2}$$

$$\mathbf{C} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.2.3.3}$$

$$\mathbf{D} = \begin{pmatrix} b\cos\theta \\ b\sin\theta \end{pmatrix} \tag{2.2.3.4}$$

2.2.4. To find the coordinates of $\mathbf{C}$.

Theorem 2.1. *Angles formed in the same segment of a circle are always equal in measure.*

$$\cos\gamma = \frac{(A-D)^T (B-D)}{\|A-D\| \, \|B-D\|} \tag{2.2.4.1}$$

$$\theta = 180^\circ - \gamma - \angle B \tag{2.2.4.2}$$

In $\triangle ACB$. Finding the Scalar Products:

$$\begin{aligned}(\mathbf{B}-\mathbf{A})^T (\mathbf{C}-\mathbf{A}) &= \\ \|\mathbf{B}-\mathbf{A}\| \, \|\mathbf{C}-\mathbf{A}\| \cos\theta\end{aligned} \tag{2.2.4.3}$$

$$\begin{aligned}(\mathbf{C}-\mathbf{B})^T (\mathbf{A}-\mathbf{B}) &= \\ \|\mathbf{C}-\mathbf{B}\| \, \|\mathbf{A}-\mathbf{B}\| \cos\alpha\end{aligned} \tag{2.2.4.4}$$

On simplifying equation 2.2.4.3 and 2.2.4.4:

$$x^2 \tan\theta^2 = y^2 \tag{2.2.4.5}$$

$$(x-a)^2 = \left((x-a)^2 + y^2\right)\cos\alpha^2 \tag{2.2.4.6}$$

Substituting 2.2.4.5 in 2.2.4.6:

$$\begin{aligned}&x^2\left(1 - \cos\alpha^2 - \tan\theta^2\cos\alpha^2\right) \\ &+x\left(2a\cos\alpha^2 - 2a\right) + a^2\sin\alpha^2\end{aligned} \tag{2.2.4.7}$$

If θ and α are accute angles:

$$x = \frac{\left(-b - \sqrt{b^2 - 4ac}\right)}{2a} \tag{2.2.4.8}$$

else:

$$x = \frac{\left(-b + \sqrt{b^2 - 4ac}\right)}{2a} \tag{2.2.4.9}$$

The value of x can then be substituted in 2.2.4.5 to find the coordinates of $\mathbf{C}$

2.2.5. From the given information, The values are listed in 2.2.5

Output values	
Parameter	Value
C	$\begin{pmatrix} 4 \\ 16.47 \end{pmatrix}$
D	$\begin{pmatrix} -4 \\ 6.93 \end{pmatrix}$

TABLE 2.2.5: Values of $\mathbf{C}$ and $\mathbf{D}$

2.2.6. Draw Fig. 2.2.6.

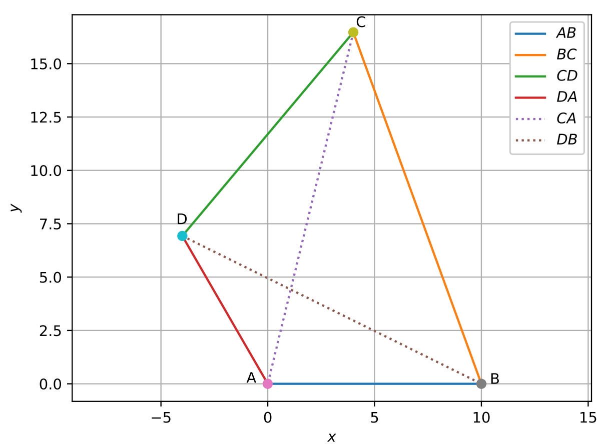

Fig. 2.2.6: Triangle generated using python

Solution: The following Python code generates Fig. 2.2.6

```
codes/quadilateral_ex/cyclic_quad.py
```

and the equivalent latex-tikz code generating Fig. 2.2.6 is

```
figs/quadilateral_ex/cyclic_quad_fig.tex
```

The above latex code can be compiled as a standalone document as

```
figs/quadilateral_ex/cyclic_quad_final.tex
```

2.3 Solution

Theorem 2.2. *Sum of opposite angles in a cyclic quadilateral equals* 180°.

Solution: From theorem 2.2

$$\angle A + \angle C = 180^\circ \tag{6.1}$$
$$\angle B + \angle D = 180^\circ \tag{6.2}$$

From the given information:

$$4y + 20 - 4x = 180^\circ \tag{6.3}$$
$$3y - 5 - 7x + 5 = 180^\circ \tag{6.4}$$

Solving equations 6.3 and 6.4:

$$x = -15 \tag{6.5}$$
$$y = 25 \tag{6.6}$$
$$\implies \angle A = 120^\circ \tag{6.7}$$
$$\implies \angle B = 70^\circ \tag{6.8}$$
$$\implies \angle C = 60^\circ \tag{6.9}$$
$$\implies \angle D = 110^\circ \tag{6.10}$$

3 Line

3.1 Points and vectors

3.1.1 Problem:

Find the distance between the points $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}36\\15\end{pmatrix}$.

3.1.2 Construction:

1) The figure obtained looks like Fig. 3.

2) The coordinates are:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.1}$$
$$\mathbf{B} = \begin{pmatrix}36\\15\end{pmatrix} \tag{2.2}$$

3) Draw Fig. 3.
Solution: The following Python code generates Fig. 3

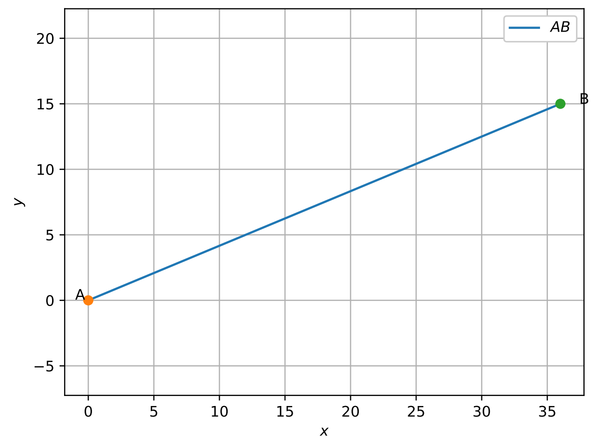

Fig. 3: AB generated using python

```
codes/line_ex/pts_and_vectors/
    dist_btw_pts.py
```

and the equivalent latex-tikz code generating Fig. 3 is

```
figs/line_ex/pts_and_vectors/dist_bt_pts.
    eps_fig.tex
```

The above latex code can be compiled as a standalone document as

```
figs/line_ex/pts_and_vectors/dist_bt_pts.
    eps_fig_final.tex
```

3.1.3 Solution:

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix} \tag{3.1}$$
$$\mathbf{B} = \begin{pmatrix}36\\15\end{pmatrix} \tag{3.2}$$

Distance between **A** and **B** is:

$$\|\mathbf{A} - \mathbf{B}\| \tag{3.3}$$

From the given information:

$$\|\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}36\\15\end{pmatrix}\| = 39 \tag{3.4}$$

3.2 Points on a line

3.2.1 Problem:

Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix}4\\-1\end{pmatrix}$ and $\begin{pmatrix}-2\\-3\end{pmatrix}$.

3.2.2 Construction:

1)
$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1}$$
$$\mathbf{B} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{1.2}$$

2) To find the coordinates of **C** and **D**.
Solution: Let **E** be a point which divides line segment AB in the ratio $k : 1$:
$$\mathbf{E} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{2.1}$$
C divides the line in the ratio $\frac{1}{2} : 1$ and **D** divides the line in the raio $\frac{2}{1} : 1$
$$\mathbf{C} = \frac{0.5\mathbf{A} + \mathbf{B}}{0.5+1} \tag{2.2}$$
$$\mathbf{D} = \frac{2\mathbf{A} + \mathbf{B}}{2+1} \tag{2.3}$$
$$\therefore \mathbf{C} = \begin{pmatrix} 0 \\ -2.33 \end{pmatrix} \tag{2.4}$$
$$\therefore \mathbf{D} = \begin{pmatrix} 2 \\ -1.66 \end{pmatrix} \tag{2.5}$$

3) From the given information, The values are listed in 3

Output values	
Parameter	Value
C	$\begin{pmatrix} 0 \\ -2.33 \end{pmatrix}$
D	$\begin{pmatrix} 2 \\ -1.66 \end{pmatrix}$

TABLE 3: Values of **C** and **D**

4) Draw Fig. 4.
Solution: The following Python code generates Fig. 4

```
codes/line_ex/pts_on_a_line/trisection.py
```

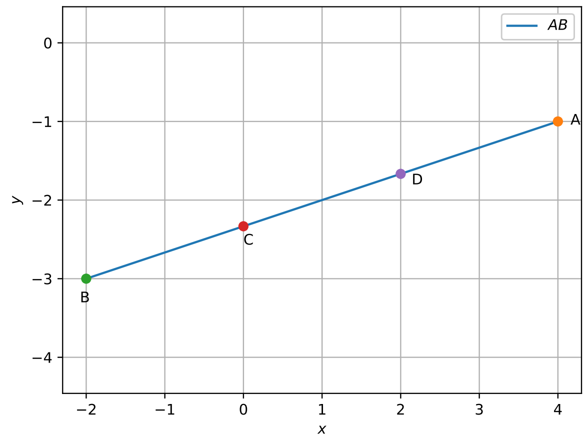

Fig. 4: Triangle generated using python

3.2.3 Solution:

Let **E** be a point which divides line segment AB in the ratio $k : 1$:
$$\mathbf{E} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{4.1}$$
C divides the line in the ratio $\frac{1}{2} : 1$ and **D** divides the line in the raio $\frac{2}{1} : 1$
$$\mathbf{C} = \frac{0.5\mathbf{A} + \mathbf{B}}{0.5+1} \tag{4.2}$$
$$\mathbf{D} = \frac{2\mathbf{A} + \mathbf{B}}{2+1} \tag{4.3}$$
$$\therefore \mathbf{C} = \begin{pmatrix} 0 \\ -2.33 \end{pmatrix} \tag{4.4}$$
$$\therefore \mathbf{D} = \begin{pmatrix} 2 \\ -1.66 \end{pmatrix} \tag{4.5}$$

3.3 Lines and planes

3.3.1 Problem:

Find the zero of the polynomial in each of the following cases:
$$p(x) = x + 5$$
$$p(x) = x - 5$$
$$p(x) = 2x + 5$$
$$p(x) = 3x - 2$$
$$p(x) = 3x$$

3.3.2 Construction:

1) Draw Fig. 1, 2, 3, 4, 5 .

   Solution: The following Python code generates all the figures.

   ```
   codes/line_ex/lines_and_planes/
       linear_eq_roots.py
   ```

3.3.3 Solution:

1) **Solution:** For $p(x) = x + 5$
   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} x + 5 = 0 \tag{1.1}$$

   To find the roots $y = 0$:

$$x + 5 = 0 \tag{1.2}$$
$$x = -5 \tag{1.3}$$

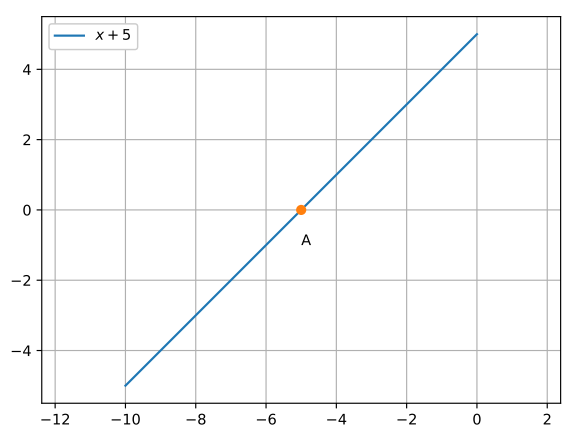

Fig. 1: $x + 5$ generated using python

2) **Solution:** For $p(x) = x - 5$
   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} x - 5 = 0 \tag{2.1}$$

   To find the roots $y = 0$:

$$x - 5 = 0 \tag{2.2}$$
$$x = 5 \tag{2.3}$$

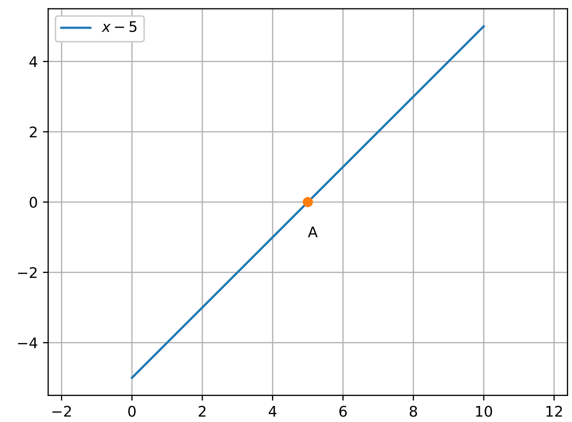

Fig. 2: $x - 5$ generated using python

3) **Solution:** For $p(x) = 2x + 5$
   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 2 & -1 \end{pmatrix} x + 5 = 0 \tag{3.1}$$

   To find the roots $y = 0$:

$$2x + 5 = 0 \tag{3.2}$$
$$x = \frac{-5}{2} \tag{3.3}$$

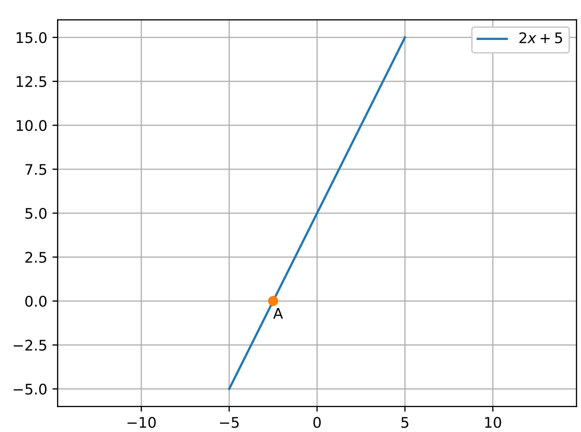

Fig. 3: $2x + 5$ generated using python

4) **Solution:** For $p(x) = 3x - 2$
   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} x - 2 = 0 \tag{4.1}$$

To find the roots $y = 0$:

$$3x - 2 = 0 \tag{4.2}$$

$$x = \frac{2}{3} \tag{4.3}$$

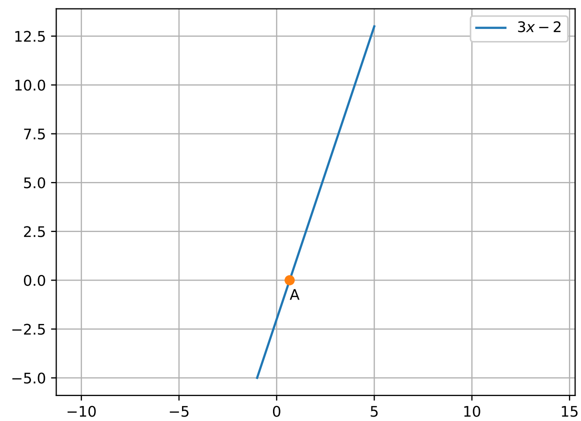

Fig. 4: $3x - 2$ generated using python

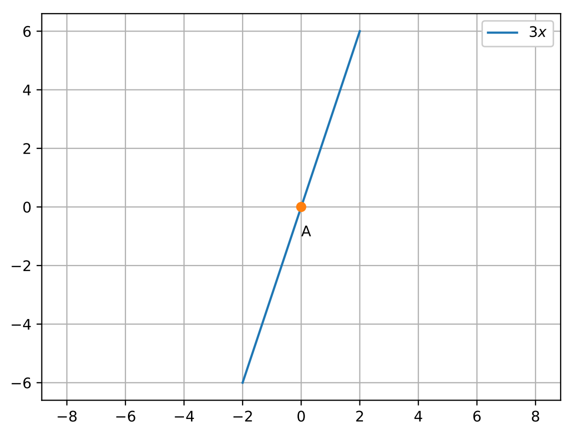

Fig. 5: $3x$ generated using python

5) **Solution:** For $p(x) = 3x$

The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} x = 0 \tag{5.1}$$

To find the roots $y = 0$:

$$3x = 0 \tag{5.2}$$

$$x = 0 \tag{5.3}$$

3.4 Motion in a plane

3.4.1 Problem:

A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

3.4.2 Construction:

1) Draw Fig. 1.

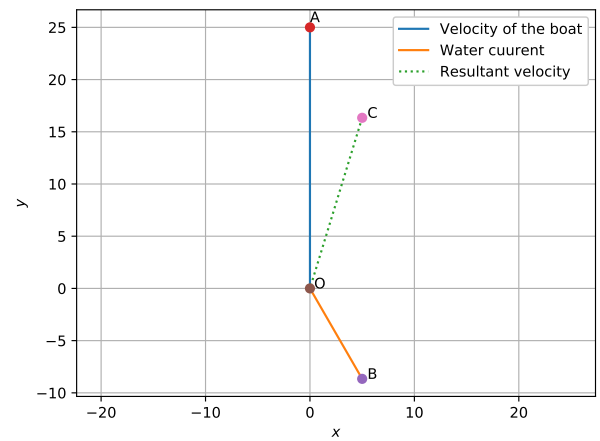

Fig. 1: Vectorial representation of velocities generated using python

Solution: The following Python code generates Fig. 1

```
codes/line_ex/motion_in_a_plane/
    motion_plane.py
```

3.4.3 Solution:

A denotes the velocity of the boat and **B** denotes the water current and **C** represents the resultant velocity.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 25 \end{pmatrix} \tag{1.1}$$

$$\mathbf{B} = \begin{pmatrix} 5 \\ -8.67 \end{pmatrix} \tag{1.2}$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.3}$$

$$\mathbf{C} = \begin{pmatrix} 5 \\ 16.34 \end{pmatrix} \tag{1.4}$$

Magnitude of resulant velocity:

$$\|\mathbf{C}\| = 17.08 \tag{1.5}$$

Direction of resultant velocity:

$$\cos\theta = \frac{(\mathbf{A})^T (\mathbf{C})}{\|\mathbf{A}\| \, \|\mathbf{C}\|} \tag{1.6}$$
$$\theta = 17.01° \tag{1.7}$$

∴ The resulant velocity is 17.08 km/h at an angle of 17.01° east of north.

3.5 Matrix

3.5.1 Problem:

If a matrix has 24 elements, what are the possible orders it can have ? What, if it has 13 elements.

3.5.2 Solution: The total number of elements in a matrix is $m\ x\ n$.

1) If the total number of elements is 24. The possible orders are:

$$1x24 = 24 \tag{1.1}$$
$$24x1 = 24 \tag{1.2}$$
$$2x12 = 24 \tag{1.3}$$
$$12x2 = 24 \tag{1.4}$$
$$3x8 = 24 \tag{1.5}$$
$$8x3 = 24 \tag{1.6}$$
$$4x6 = 24 \tag{1.7}$$
$$6x4 = 24 \tag{1.8}$$

2) If the total number of elements is 13. The possible orders are:

$$1x13 = 13 \tag{2.1}$$
$$13x1 = 13 \tag{2.2}$$

3.6 Determinants

3.6.1 Problem:

Find the determinant of

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3.6.2 Solution:

Determinant of a $2x2$ matrix is obtained as follows

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$detA = a_{11}a_{22} - a_{12}a_{21} \tag{2.1}$$

From 2.1:

$$(i)\ det = \cos\theta^2 + \sin\theta^2 = 1 \tag{2.2}$$
$$(ii)\ det = x^3 - x^2 + 2 \tag{2.3}$$

3.7 Linear inequalities

3.7.1 Problem:

Solve $7x + 3\ <\ 5x + 9$.Show the graph of the solutions on number line.

3.7.2 Solution:

$$7x + 3\ <\ 5x + 9 \tag{2.1}$$
$$2x - 6\ <\ 0 \tag{2.2}$$
$$x < 3 \tag{2.3}$$
$$\therefore\ \ x \in \{3, -\infty\} \tag{2.4}$$

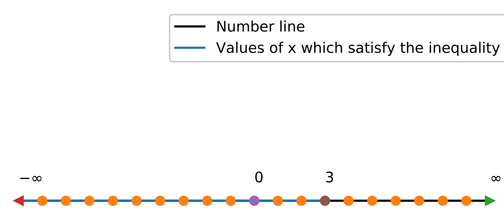

Fig. 2: Values of x satisfying the inequality in the number line generated using python

The following Python code generates Fig. 2

```
codes/line_ex/lin_ineq/dist_btw_pts.py
```

4 Circle

4.1 Problem

FInf the center of a circle passing through the points $\begin{pmatrix} 6 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

4.2 Construction

4.2.1. The figure obtained looks like Fig. 4.2.0.

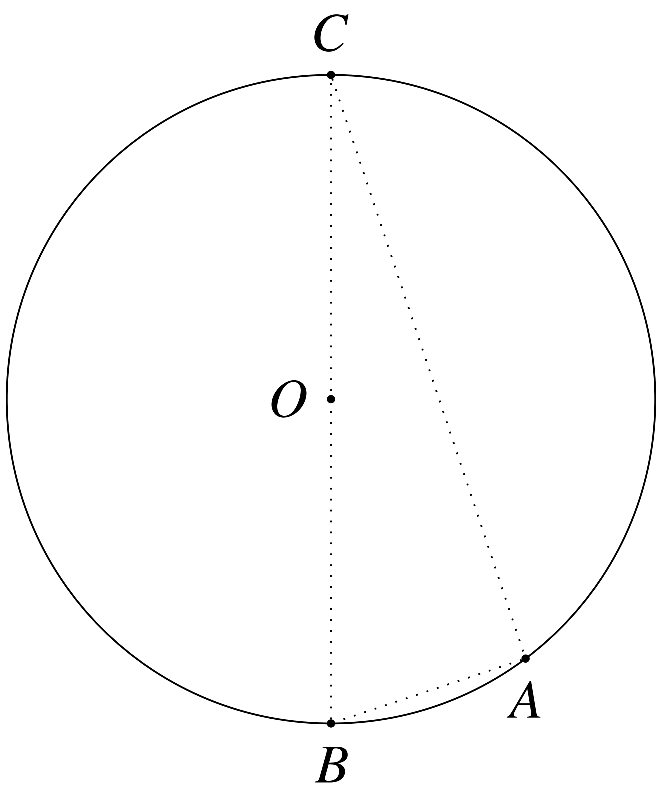

Fig. 4.2.0: Circumcircle by Latex-Tikz

4.2.2. Coordinates of $\triangle ABC$ Fig4.2.0.

$$\mathbf{A} = \begin{pmatrix} 6 \\ -6 \end{pmatrix} \tag{4.2.2.1}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -7 \end{pmatrix} \tag{4.2.2.2}$$

$$\mathbf{C} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{4.2.2.3}$$

4.2.3. To find the coordinates of **O**.
Solution: A circle passing through three non-collinear points is the circumcircle and the center is the circumcenter.

$$\mathbf{O} = \frac{A \sin \angle 2A + B \sin \angle 2B + C \sin \angle 2C}{\sin \angle 2A + \sin \angle 2B + \sin \angle 2C} \tag{4.2.3.1}$$

4.2.4. From the given information, The values are listed in 4.2.4

Output values	
Parameter	Value
O	$\begin{pmatrix} 3 \\ -2 \end{pmatrix}$
radius	5

TABLE 4.2.4: Value of **O**

4.2.5. Draw Fig. 4.2.5.

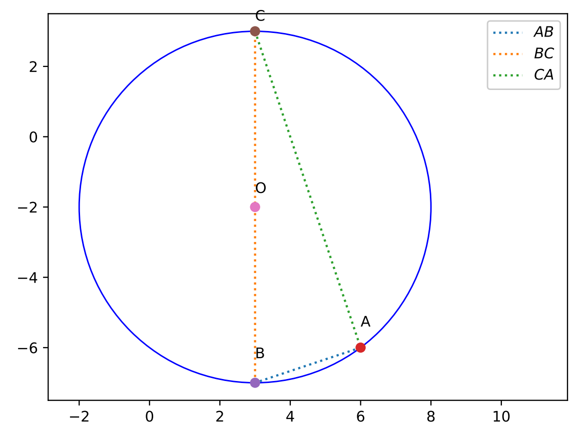

Fig. 4.2.5: Circumcircle generated using python

Solution: The following Python code generates Fig. 4.2.5

```
codes/circle_ex/circumcircle.py
```

and the equivalent latex-tikz code generating Fig. 4.2.5 is

```
figs/circle_ex/circumcircle_fig.tex
```

The above latex code can be compiled as a standalone document as

```
figs/circle_ex/circumcircle_fig_final.tex
```

4.3 Solution

Solution: A circle passing through three non-collinear points is the circumcircle and the center is the circumcenter.

$$\mathbf{O} = \frac{A \sin \angle 2A + B \sin \angle 2B + C \sin \angle 2C}{\sin \angle 2A + \sin \angle 2B + \sin \angle 2C} \tag{5.1}$$

Substituting values from 4.2.2.1, 4.2.2.2 and 4.2.2.3

$$\therefore \mathbf{O} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{5.2}$$

5 Conics

5.1 Problem

Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

5.2 Construction

5.2.1. Draw Fig. 5.2.1.

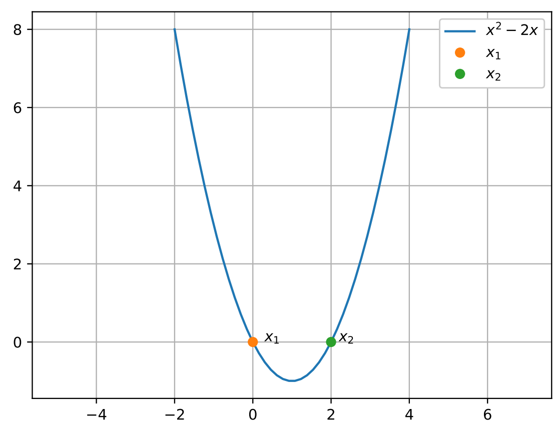

Fig. 5.2.1: $x^2 - 2x$ generated using python

Solution: The following Python code generates Fig. 5.2.1

codes/conics_example/conics.py

5.3 Solution

Solution: $p(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ can be represented as follow in the vector form:

$$x^T \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} x + \begin{pmatrix} D & E \end{pmatrix} x + F = 0 \tag{1.1}$$

The given equation can be represented as follows in the vector form:

$$x^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} -2 & 0 \end{pmatrix} x + 0 = 0 \tag{1.2}$$

To find the roots $y = 0$:

$$X^2 - 2x = 0 \tag{1.3}$$
$$x(x-2) = 0 \tag{1.4}$$
$$x = 0, 2 \tag{1.5}$$