Math Document Template

C ANISH

Abstract—**This is a document explaining questions about the concept of Linear algebra.**

Download all python codes from

```
svn co https://github.com/chakki1234/summer
    -2020/trunk/linearalg/codes
```

and latex-tikz codes from

```
svn co https://github.com/chakki1234/summer
    -2020/trunk/linearalg/figs
```

1 Triangle

1.1 Problem

In a $\triangle ABC$, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find the three angles.

1.2 Solution

Theorem 1.1. *Sum of all angles in a triangle equals* 180°.

1.2.1. **Solution:** From theorem 1.1

$$\angle A + \angle B + \angle C = 180^\circ \tag{1.2.1.1}$$

From the given information:

$$\angle A = \angle C \tag{1.2.1.2}$$
$$\angle B = \angle C \tag{1.2.1.3}$$
$$\angle A + \angle B + \angle C = 180^\circ \tag{1.2.1.4}$$

In vector form:

$$\begin{pmatrix} 6 & 0 & -1 \\ 0 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 180 \end{pmatrix} \tag{1.2.1.5}$$

To find the angles:

$$\begin{pmatrix} 6 & 0 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 180 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{6}} \begin{pmatrix} 1 & 0 & \frac{-1}{6} & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 180 \end{pmatrix} \tag{1.2.1.6}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & \frac{-1}{6} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & \frac{7}{6} & 180 \end{pmatrix} \tag{1.2.1.7}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{7}{6} & 180 \end{pmatrix} \tag{1.2.1.8}$$

$$\xleftrightarrow{R_3 \leftarrow \ R_3 - R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{3}{2} & 180 \end{pmatrix} \tag{1.2.1.9}$$

$$\xleftrightarrow{R_3 \leftarrow \ \frac{2R_3}{3}} \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 120 \end{pmatrix} \tag{1.2.1.10}$$

$$\xleftrightarrow[R_2 \leftarrow \ R_2 + \frac{R_3}{3}]{R_1 \leftarrow \ R_1 + \frac{R_3}{6}} \begin{pmatrix} 1 & 0 & 1 & 20 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 120 \end{pmatrix} \tag{1.2.1.11}$$

1.2.2.

$$\therefore \angle C = 120^\circ \ \angle A = 20^\circ \ \angle B = 40^\circ$$

1.2.3. The following Python code generates Fig. 1.2.3

```
codes/triangle_ex/triangle_linearalg.py
```

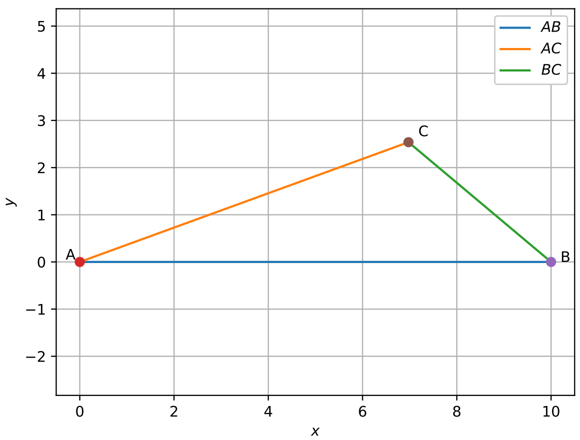

Fig. 1.2.3: Triangle generated using python

2 Quadilateral

2.1 Problem

In a $ABCD$ is a cyclic quadilateral with

$$\angle A = 4y + 20 \tag{3.1}$$
$$\angle B = 3y - 5 \tag{3.2}$$
$$\angle C = -4x \tag{3.3}$$
$$\angle D = -7x + 5 \tag{3.4}$$

Find its angles.

2.2 Solution

Theorem 2.1. *Sum of opposite angles in a cyclic quadilateral equals* 180°.

2.2.1. **Solution:** From theorem 2.1

$$\angle A + \angle C = 180^\circ \tag{2.2.1.1}$$
$$\angle B + \angle D = 180^\circ \tag{2.2.1.2}$$

2.2.2. From the given information:

$$\begin{pmatrix} -4 & 4 \\ -7 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 160 \\ 180 \end{pmatrix} \tag{2.2.2.1}$$

To find the angles:

$$\begin{pmatrix} -4 & 4 & 160 \\ -7 & 3 & 180 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{-R_1}{4}} \begin{pmatrix} 1 & -1 & -40 \\ -7 & 3 & 180 \end{pmatrix} \tag{2.2.2.2}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 7R_1} \begin{pmatrix} 1 & -1 & -40 \\ 0 & -4 & -100 \end{pmatrix} \tag{2.2.2.3}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-R_2}{4}} \begin{pmatrix} 1 & -1 & -40 \\ 0 & 1 & 25 \end{pmatrix} \tag{2.2.2.4}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & -15 \\ 0 & 1 & 25 \end{pmatrix} \tag{2.2.2.5}$$

2.2.3.

$$x = -15 \tag{2.2.3.1}$$
$$y = 25 \tag{2.2.3.2}$$
$$\implies \angle A = 120^\circ \tag{2.2.3.3}$$
$$\implies \angle B = 70^\circ \tag{2.2.3.4}$$
$$\implies \angle C = 60^\circ \tag{2.2.3.5}$$
$$\implies \angle D = 110^\circ \tag{2.2.3.6}$$

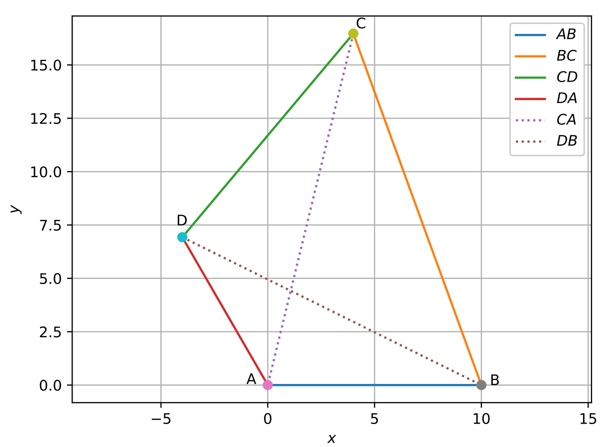

Fig. 2.2.4: Quadilateral generated using python

2.2.4. The following Python code generates Fig. 2.2.4

```
codes/quadilateral_ex/cyclic_quad.py
```

3 Line

3.1 Comlex Numbers

3.1.1 Problem:

Find $\begin{pmatrix} -\sqrt{3} \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} \\ -1 \end{pmatrix}$

3.1.2 Solution:

1) A complex number $\begin{pmatrix} a \\ b \end{pmatrix}$ can be represented as a 2 x 2 matrix:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{1.1}$$

2) Multiplying the given matrices after converting them to a 2 x 2 matrix:

$$\begin{pmatrix} -\sqrt{3} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{3} \end{pmatrix} \begin{pmatrix} 2\sqrt{3} & 1 \\ -1 & 2\sqrt{3} \end{pmatrix} \tag{2.1}$$

$$\implies \begin{pmatrix} \sqrt{2}-6 & -\sqrt{3}-2\sqrt{6} \\ \sqrt{3}+2\sqrt{6} & \sqrt{2}-6 \end{pmatrix} \tag{2.2}$$

3) Matrix (2.2) can be represented as a vector:

$$\therefore \begin{pmatrix} \sqrt{2}-6 \\ \sqrt{3}+2\sqrt{6} \end{pmatrix} \tag{3.1}$$

4) Python code to multiply two complex numbers:

```
codes/line_ex/complex_ex/complex_ex.py
```

3.2 Points and vectors

3.2.1 Problem:

Find the distance between the points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 36 \\ 15 \end{pmatrix}$.

3.2.2 Solution:

1)

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.1}$$

$$\mathbf{B} = \begin{pmatrix} 36 \\ 15 \end{pmatrix} \tag{1.2}$$

2) Distance between $\mathbf{A}$ and $\mathbf{B}$ is:

$$\|\mathbf{A} - \mathbf{B}\| \tag{2.1}$$

3) From the given information:

$$\left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 36 \\ 15 \end{pmatrix} \right\| = 39 \tag{3.1}$$

4) The following Python code generates Fig. 4

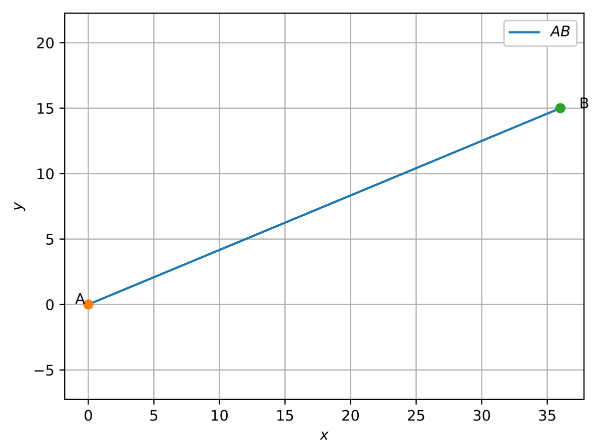

Fig. 4: Line AB generated using python

```
codes/line_ex/pts_and_vectors/
    dist_btw_pts.py
```

3.3 Points on a line

3.3.1 Problem:

Find the coordinates of the points of trisection of the line segment joining $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

3.3.2 Solution:

1) Let $\mathbf{E}$ be a point which divides line segment AB in the ratio $k : 1$:
2)

$$\mathbf{E} = \frac{k\mathbf{A} + \mathbf{B}}{k+1} \tag{2.1}$$

3) $\mathbf{C}$ divides the line in the ratio $\frac{1}{2} : 1$ and $\mathbf{D}$ divides the line in the raio $\frac{2}{1} : 1$
4)

$$\mathbf{C} = \frac{0.5\mathbf{A} + \mathbf{B}}{0.5+1} \tag{4.1}$$

$$\mathbf{D} = \frac{2\mathbf{A} + \mathbf{B}}{2+1} \tag{4.2}$$

$$\therefore \mathbf{C} = \begin{pmatrix} 0 \\ -2.33 \end{pmatrix} \tag{4.3}$$

$$\therefore \mathbf{D} = \begin{pmatrix} 2 \\ -1.66 \end{pmatrix} \tag{4.4}$$

5) The following Python code generates Fig. 5

```
codes/line_ex/pts_on_a_line/trisection.py
```

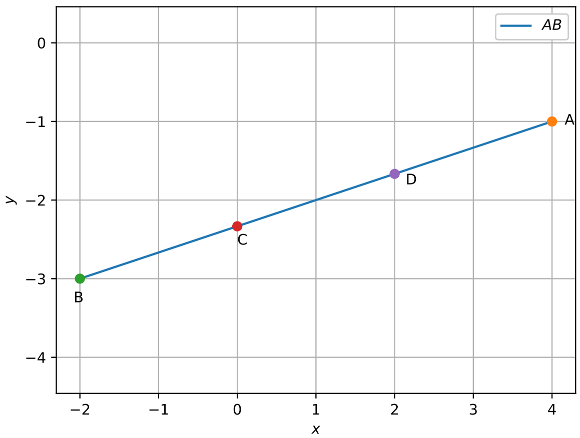

Fig. 5: Line AB trisected - generated using python

3.4 Lines and planes

3.4.1 Problem:

Find the zero of the polynomial in each of the following cases:

$$p(x) = x + 5$$
$$p(x) = x - 5$$
$$p(x) = 2x + 5$$
$$p(x) = 3x - 2$$
$$p(x) = 3x$$

3.4.2 Solution:

1) **Solution:** For $p(x) = x + 5$

   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{1.1}$$

   To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{1.2}$$
$$x_1 + 5 = 0 \tag{1.3}$$
$$x_1 = -5 \tag{1.4}$$

2) **Solution:** For $p(x) = x - 5$

   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} - 5 = 0 \tag{2.1}$$

   To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{2.2}$$
$$x_1 - 5 = 0 \tag{2.3}$$
$$x_1 = 5 \tag{2.4}$$

3) **Solution:** For $p(x) = 2x + 5$

   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 5 = 0 \tag{3.1}$$

   To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{3.2}$$
$$2x_1 + 5 = 0 \tag{3.3}$$
$$x_1 = \frac{-5}{2} \tag{3.4}$$

4) **Solution:** For $p(x) = 3x - 2$

   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} - 2 = 0 \tag{4.1}$$

   To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{4.2}$$
$$3x_1 - 2 = 0 \tag{4.3}$$
$$x_1 = \frac{2}{3} \tag{4.4}$$

5) **Solution:** For $p(x) = 3x$

   The given equation can be represented as follows in the vector form:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{5.1}$$

   To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \tag{5.2}$$
$$3x_1 = 0 \tag{5.3}$$
$$x_1 = 0 \tag{5.4}$$

The following Python code generates Fig 5

```
codes/line_ex/lines_and_planes/
    linear_eq_roots.py
```

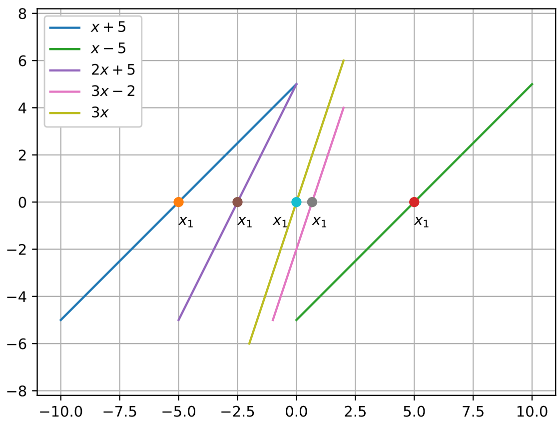

Fig. 5: Linear Equations generated using python

3.5 *Motion in a plane*

3.5.1 *Problem:*

A motorboat is racing towards north at 25 km/h and the water current in that region is 10 km/h in the direction of 60° east of south. Find the resultant velocity of the boat.

3.5.2 *Solution:*

1) **A** denotes the velocity of the boat and **B** denotes the water current and **C** represents the resultant velocity.
2)

$$\mathbf{A} = \begin{pmatrix} 0 \\ 25 \end{pmatrix} \tag{2.1}$$

$$\mathbf{B} = \begin{pmatrix} 8.6 \\ -5 \end{pmatrix} \tag{2.2}$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{2.3}$$

$$\mathbf{C} = \begin{pmatrix} 8.6 \\ 20 \end{pmatrix} \tag{2.4}$$

3) Magnitude of resulant velocity:

$$\|\mathbf{C}\| = 21.8 \tag{3.1}$$

4) Direction of resultant velocity:

$$\cos\theta = \frac{(\mathbf{A})^T (\mathbf{C})}{\|\mathbf{A}\| \ \|\mathbf{C}\|} \tag{4.1}$$

$$\theta = 23.41° \tag{4.2}$$

5) ∴ The resulant velocity is 21.8 km/h at an angle of 23.41° east of north.
6) The following Python code generates Fig. 6

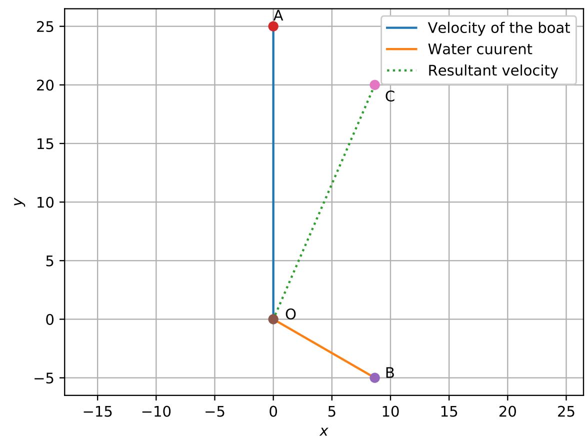

Fig. 6: Vectorial representation of velocities generated using python

```
codes/line_ex/motion_in_a_plane/
    motion_plane.py
```

3.6 *Matrix*

3.6.1 *Problem:*

If a matrix has 24 elements, what are the possible orders it can have ? What, if it has 13 elements.

3.6.2 *Solution:*

The total number of elements in a matrix is $m\ x\ n$.

1) If the total number of elements is 24. The possible orders are:

$$1x24 = 24 \tag{1.1}$$
$$24x1 = 24 \tag{1.2}$$
$$2x12 = 24 \tag{1.3}$$
$$12x2 = 24 \tag{1.4}$$
$$3x8 = 24 \tag{1.5}$$
$$8x3 = 24 \tag{1.6}$$
$$4x6 = 24 \tag{1.7}$$
$$6x4 = 24 \tag{1.8}$$

2) If the total number of elements is 13. The possible orders are:

$$1x13 = 13 \tag{2.1}$$
$$13x1 = 13 \tag{2.2}$$

The following Python code generates all possible dimensions for any matrix size:

```
codes/line_ex/matrix/matrix.py
```

3.7 Determinants

3.7.1 Problem:

Find the determinant of

$$(i) \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} (ii) \begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$$

3.7.2 Solution:

1) Determinant of a $2x2$ matrix is obtained as follows
2) $$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$ $$detA = a_{11}a_{22} - a_{12}a_{21} \tag{2.1}$$
3) From (2.1): $$(i)\ det = \cos\theta^2 + \sin\theta^2 = 1 \tag{3.1}$$ $$(ii)\ det = x^3 - x^2 + 2 \tag{3.2}$$ Python code to calculate the deteminant of a matrix:

```
codes/line_ex/determinants/det.py
```

3.8 Linear inequalities

3.8.1 Problem:

Solve $7x+3 \ < \ 5x+9$.Show the graph of the solutions on number line.

3.8.2 Solution:

1) $$7x+3 \ < \ 5x+9 \tag{1.1}$$ $$2x-6 \ < \ 0 \tag{1.2}$$ $$x < 3 \tag{1.3}$$ $$\therefore \ x \in \{3, -\infty\} \tag{1.4}$$
2) The following Python code to generate Fig 2 :

```
codes/line_ex/lin_ineq/dist_btw_pts.py
```

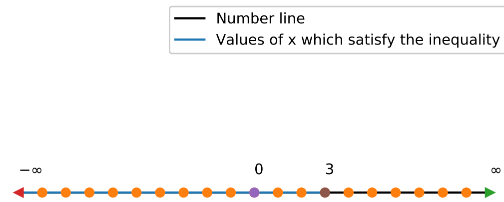

Fig. 2: Values of x satisfying the inequality in the number line generated using python

3.9 Miscellaneous

3.9.1 Problem:

Solve the following pair of equations:

$$\begin{aligned} \begin{pmatrix} a-b & a+b \end{pmatrix} \mathbf{x} &= a^2 - 2ab - b^2 \\ \begin{pmatrix} a+b & a+b \end{pmatrix} \mathbf{x} &= a^2 + b^2 \end{aligned} \tag{2.1}$$

3.9.2 Solution:

3.9.1. Vector form of the given equations:

$$\begin{pmatrix} a-b & a+b \\ a+b & a+b \end{pmatrix} \mathbf{x} = \begin{pmatrix} a^2-2ab-b^2 \\ a^2+b^2 \end{pmatrix} \tag{3.9.1.1}$$

3.9.2. To find $\mathbf{x}$:

$$\begin{pmatrix} a-b & a+b & a^2-2ab-b^2 \\ a+b & a+b & a^2+b^2 \end{pmatrix}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{a+b}]{R_1 \leftarrow \frac{R_1}{a-b}} \begin{pmatrix} 1 & \frac{a+b}{a-b} & \frac{a^2-2ab-b^2}{a-b} \\ 1 & 1 & \frac{a^2+b^2}{a+b} \end{pmatrix} \tag{3.9.2.1}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & \frac{a+b}{a-b} & \frac{a^2-2ab-b^2}{a-b} \\ 0 & \frac{-2b}{a-b} & \frac{4ab^2}{a^2-b^2} \end{pmatrix} \tag{3.9.2.2}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-(a-b)R_2}{2b}} \begin{pmatrix} 1 & \frac{a+b}{a-b} & \frac{a^2-2ab-b^2}{a-b} \\ 0 & 1 & \frac{-2ab}{a+b} \end{pmatrix} \tag{3.9.2.3}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & a+b \\ 0 & 1 & \frac{-2ab}{a+b} \end{pmatrix} \tag{3.9.2.4}$$

$$\therefore \mathbf{x} = \begin{pmatrix} a+b \\ \frac{2ab}{a-b} \end{pmatrix} \tag{3.9.2.5}$$

4 Circle

4.1 Problem

Find the center of a circle passing through the points $\begin{pmatrix}6\\-6\end{pmatrix}, \begin{pmatrix}3\\-7\end{pmatrix}$ and $\begin{pmatrix}3\\3\end{pmatrix}$.

4.2 Solution

4.2.1.

$$\mathbf{P_1} = \begin{pmatrix}6\\-6\end{pmatrix} \tag{4.2.1.1}$$

$$\mathbf{P_2} = \begin{pmatrix}3\\-7\end{pmatrix} \tag{4.2.1.2}$$

$$\mathbf{P_3} = \begin{pmatrix}3\\3\end{pmatrix} \tag{4.2.1.3}$$

4.2.2. The general of a circle equation is:

$$\|\mathbf{x} - \mathbf{O}\| = r \tag{4.2.2.1}$$

Substituting the given coordinates:

$$\left\|\begin{pmatrix}6\\-6\end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{4.2.2.2}$$

$$\left\|\begin{pmatrix}3\\-7\end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{4.2.2.3}$$

$$\left\|\begin{pmatrix}3\\3\end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{4.2.2.4}$$

From (4.2.2.2), (4.2.2.3), (4.2.2.4):

$$\left\|\begin{pmatrix}3\\-7\end{pmatrix} - \mathbf{O}\right\|^2 - \left\|\begin{pmatrix}6\\-6\end{pmatrix} - \mathbf{O}\right\|^2 = 0 \tag{4.2.2.5}$$

$$\left\|\begin{pmatrix}3\\3\end{pmatrix} - \mathbf{O}\right\|^2 - \left\|\begin{pmatrix}6\\-6\end{pmatrix} - \mathbf{O}\right\|^2 = 0 \tag{4.2.2.6}$$

Simplifying equations (4.2.2.5) and (4.2.2.6):

$$\begin{pmatrix}3 & 1\\1 & -3\end{pmatrix}\mathbf{O} = \begin{pmatrix}7\\9\end{pmatrix} \tag{4.2.2.7}$$

$$\begin{pmatrix}3 & 1 & 7\\1 & -3 & 9\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{3}} \begin{pmatrix}1 & \frac{1}{3} & \frac{7}{3}\\1 & -3 & 9\end{pmatrix} \tag{4.2.2.8}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}1 & \frac{1}{3} & \frac{7}{3}\\1 & -\frac{10}{3} & \frac{20}{3}\end{pmatrix} \tag{4.2.2.9}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-3R_2}{10}} \begin{pmatrix}1 & \frac{1}{3} & \frac{7}{3}\\1 & 1 & -2\end{pmatrix} \tag{4.2.2.10}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{3}} \begin{pmatrix}1 & 0 & 3\\0 & 1 & -2\end{pmatrix} \tag{4.2.2.11}$$

$$\therefore \mathbf{O} = \begin{pmatrix}3\\-2\end{pmatrix} \tag{4.2.2.12}$$

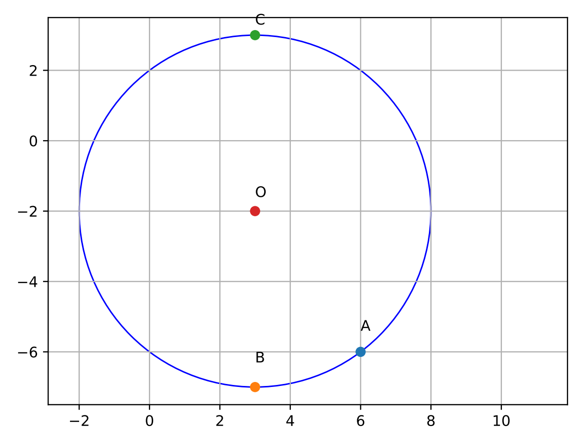

Fig. 4.2.3: Circumcircle generated using python

4.2.3. The following Python code generates Fig. 4.2.3

```
codes/circle_ex/circumcircle.py
```

5 Circle-example

5.1 Problem

Find the center and radius of the circle

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix}8\\10\end{pmatrix}\mathbf{x} - 8 = 0 \tag{3.1}$$

5.2 Solution

5.2.1. The general of a circle equation is $\|\mathbf{x}-\mathbf{O}\| = r$.

$$\|\mathbf{x}-\mathbf{O}\|^2 = r^2 \quad (5.2.1.1)$$
$$\implies (\mathbf{x}-\mathbf{O})^T(\mathbf{x}-\mathbf{O}) = r^2 \quad (5.2.1.2)$$
$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \mathbf{O}^T\mathbf{O} - r^2 = 0 \quad (5.2.1.3)$$
$$\implies \mathbf{x}^T\mathbf{x} - 2\mathbf{O}^T\mathbf{x} + \|\mathbf{O}\|^2 - r^2 = 0 \quad (5.2.1.4)$$

Comparing equation (5.2.1.4) with the given circle equation:

$$\mathbf{O} = \begin{pmatrix}-4\\-5\end{pmatrix} \quad (5.2.1.5)$$
$$\|\mathbf{O}\|^2 = 41 \quad (5.2.1.6)$$
$$r^2 = 41 + 8 \quad (5.2.1.7)$$
$$\therefore r = 7 \quad (5.2.1.8)$$

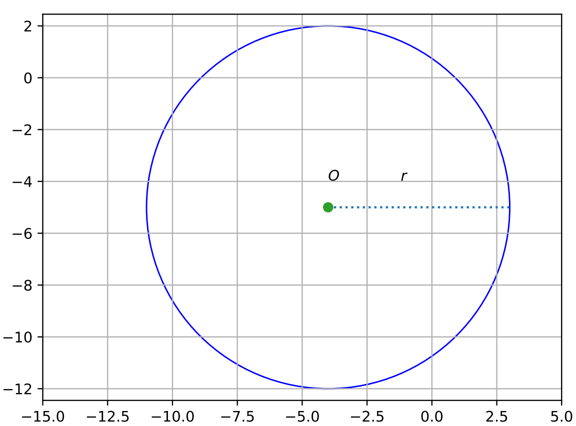

Fig. 5.2.2: Circle generated using python

5.2.2. The following Python code generates Fig. 5.2.2

```
codes/circle_exam.py
```

6 Conics example

6.1 Problem

Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

6.2 Solution

6.2.1. $p(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ can be represented as follow in the vector form:

$$\mathbf{x}^T\begin{pmatrix}A & \frac{B}{2}\\ \frac{B}{2} & C\end{pmatrix}\mathbf{x} + \begin{pmatrix}D & E\end{pmatrix}\mathbf{x} + F = 0 \quad (6.2.1.1)$$

6.2.2. The given equation can be represented as follows in the vector form:

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}-2 & 0\end{pmatrix}\mathbf{x} + 0 = 0 \quad (6.2.2.1)$$

6.2.3. To find the roots $y = 0$:

$$x^2 - 2x = 0 \quad (6.2.3.1)$$
$$x(x-2) = 0 \quad (6.2.3.2)$$
$$x = 0, 2 \quad (6.2.3.3)$$

6.2.4. To verify:

a) Substitute $\mathbf{x} = \begin{pmatrix}0\\0\end{pmatrix}$ in (7.2.11.1)

$$\begin{pmatrix}0\\0\end{pmatrix}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-2 & 0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + 0 \quad (6.2.4.1)$$
$$\implies 0 \quad (6.2.4.2)$$

b) Substitute $\mathbf{x} = \begin{pmatrix}2\\0\end{pmatrix}$ in (7.2.11.1)

$$\begin{pmatrix}2\\0\end{pmatrix}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}-2 & 0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} + 0 \quad (6.2.4.3)$$
$$\implies 0 \quad (6.2.4.4)$$

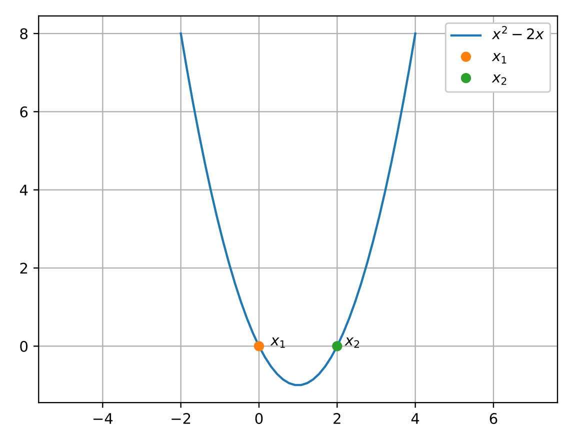

Fig. 6.2.5: $x^2 - 2x$ generated using python

6.2.5. The following Python code generates Fig. 6.2.5

```
codes/conics_example/conics.py
```

7 Conics excercise

7.1 Problem

Solve each of the following equations

1) $3x^2 - 4x + \frac{20}{3} = 0$
2) $x^2 - 2x + \frac{3}{2} = 0$
3) $27x^2 - 10x + 1 = 0$
4) $21x^2 - 28x + 10 = 0$

7.2 Solution

7.2.1. $p(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ can be represented as follow in the vector form:

$$\mathbf{x}^T \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \mathbf{x} + \begin{pmatrix} D & E \end{pmatrix} \mathbf{x} + F = 0 \qquad (7.2.1.1)$$

7.2.2. To solve the equation - $3x^2 - 4x + \frac{20}{3} = 0$:
The given equation can be represented as follows in the vector form:

$$\mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + \frac{20}{3} = 0 \qquad (7.2.2.1)$$

7.2.3. To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad (7.2.3.1)$$

$$3x^2 - 4x + \frac{20}{3} = 0 \qquad (7.2.3.2)$$

$$\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{2\sqrt{14}}{3} \end{pmatrix}\right)\left(x - \begin{pmatrix} \frac{2}{3} \\ \frac{-2\sqrt{14}}{3} \end{pmatrix}\right) = 0 \qquad (7.2.3.3)$$

$$x = \begin{pmatrix} \frac{2}{3} \\ \frac{2\sqrt{14}}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2\sqrt{14}}{3} \end{pmatrix} \qquad (7.2.3.4)$$

7.2.4. To verify:
Figure 7.2.4 show that the equation does not intersect the x-axis hence there are no real roots.

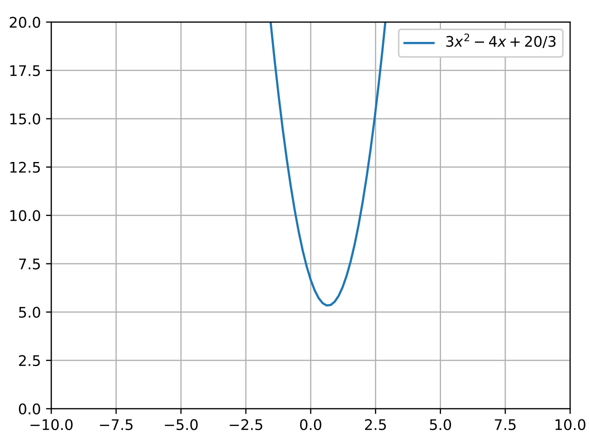

Fig. 7.2.4: $3x^2 - 4x + \frac{20}{3}$ generated using python

7.2.5. To solve the equation - $x^2 - 2x + \frac{3}{2} = 0$:

The given equation can be represented as follows in the vector form:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + \frac{3}{2} = 0 \qquad (7.2.5.1)$$

7.2.6. To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad (7.2.6.1)$$

$$x^2 - 2x + \frac{3}{2} = 0 \qquad (7.2.6.2)$$

$$\left(x - \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}\right)\left(x - \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\right) = 0 \qquad (7.2.6.3)$$

$$x = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad (7.2.6.4)$$

7.2.7. To verify:
Figure 7.2.7 show that the equation does not intersect the x-axis hence there are no real roots.

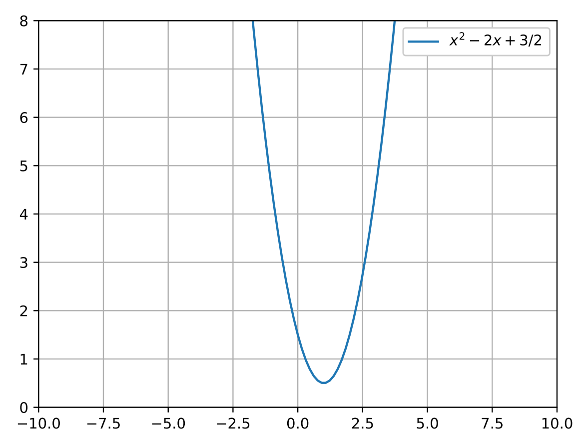

Fig. 7.2.7: $x^2 - 2x + \frac{3}{2}$ generated using python

7.2.8. To solve the equation - $27x^2 - 10x + 1 = 0$:

The given equation can be represented as follows in the vector form:

$$\mathbf{x}^T \begin{pmatrix} 27 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -10 & 0 \end{pmatrix} \mathbf{x} + 1 = 0 \qquad (7.2.8.1)$$

7.2.9. To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{7.2.9.1}$$

$$27x^2 - 10x + 1 = 0 \tag{7.2.9.2}$$

$$\left(x - \left(\frac{\frac{5}{27}}{\frac{\sqrt{2}}{27}}\right)\right)\left(x - \left(\frac{\frac{5}{27}}{\frac{-\sqrt{2}}{27}}\right)\right) = 0 \tag{7.2.9.3}$$

$$x = \left(\frac{\frac{5}{27}}{\frac{\sqrt{2}}{27}}\right), \left(\frac{\frac{5}{27}}{\frac{-\sqrt{2}}{27}}\right) \tag{7.2.9.4}$$

7.2.10. To verify:
Figure 7.2.10 show that the equation does not intersect the x-axis hence there are no real roots.

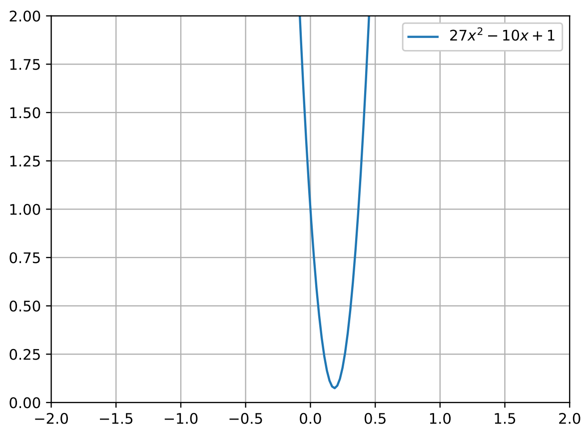

Fig. 7.2.10: $27x^2 - 10x + 1$ generated using python

7.2.11. To solve the equation - $21x^2 - 28x + 10 = 0$:
The given equation can be represented as follows in the vector form:

$$\mathbf{x}^T \begin{pmatrix} 21 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -28 & 0 \end{pmatrix} \mathbf{x} + 10 = 0 \tag{7.2.11.1}$$

7.2.12. To find the roots $y = 0$:

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{7.2.12.1}$$

$$21x^2 - 28x + 10 = 0 \tag{7.2.12.2}$$

$$\left(x - \left(\frac{\frac{2}{3}}{\frac{\sqrt{14}}{21}}\right)\right)\left(x - \left(\frac{\frac{2}{3}}{\frac{-\sqrt{14}}{21}}\right)\right) = 0 \tag{7.2.12.3}$$

$$x = \left(\frac{\frac{2}{3}}{\frac{\sqrt{14}}{21}}\right), \left(\frac{\frac{2}{3}}{\frac{-\sqrt{14}}{21}}\right) \tag{7.2.12.4}$$

7.2.13. To verify:
Figure 7.2.13 show that the equation does not intersect the x-axis hence there are no real roots.

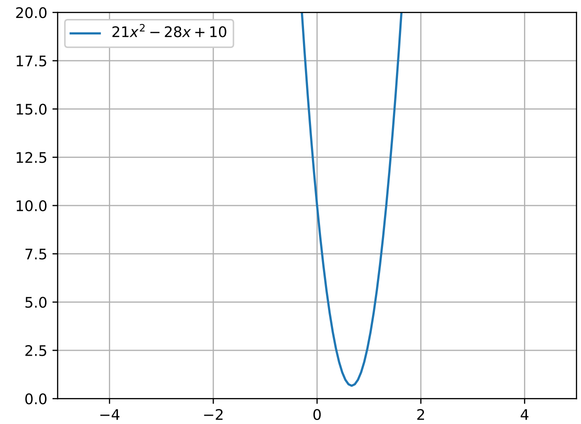

Fig. 7.2.13: $21x^2 - 28x + 10$ generated using python

The following Python code generates Fig.7.2.4, 7.2.7, 7.2.10 and 7.2.13

```
codes/conics_ex/conics_ex.py
```