Week 7: Fundamental Theorem of Calculus

October 11, 2021

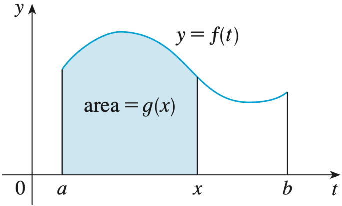
1. Fundamental Theorem of Calculus
   - Part 1
   - Part 2
   - Indefinite Integral

2. Substitution Rule

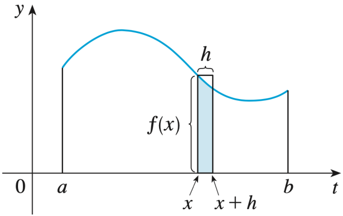
Theorem (Fundamental Theorem of Calculus)

Suppose a function f is continuous on $[a, b]$, then

1. $g(x) = \int_a^x f(t)\,dt$ *for* $x \in [a, b]$, *and* $g'(x) = f(x)$
2. $\int_a^b f(x)\,dx = F(b) - F(a)$
    - *where F is an antiderivative of f, i.e.,* $F' = f$
    - *Other common notation is* $F(x)\Big]_a^b = F(b) - F(a)$



Why



- $g(x+h) - g(x) \approx h \cdot f(x)$
- $\dfrac{g(x+h) - g(x)}{h} \approx f(x)$
- $g'(x) = \lim\limits_{h \to 0} \dfrac{g(x+h) - g(x)}{h} = f(x)$
- Thus, $g' = f$

Example

Find these derivatives using Theorem Part 1.

1. $$\frac{d}{dx}\int_5^x \sqrt{t^2+3}\,dt = \sqrt{x^2+3}$$
2. $$\frac{d}{dx}\int_4^x \sqrt{t^2+3}\,dt = \sqrt{x^2+3}$$
3. $$\frac{d}{dx}\int_x^4 \sqrt{t^2+3}\,dt = \frac{d}{dx}\left(-\int_4^x \sqrt{t^2+3}\,dt\right) = -\sqrt{x^2+3}$$
4. $$\frac{d}{dx}\int_4^{\sin x} \sqrt{t^2+3}\,dt$$
   - Let u be $\sin x$
   - $$\frac{d}{dx}\int_4^{\sin x} \sqrt{t^2+3}\,dt = \frac{d}{du}\int_4^{u} \sqrt{t^2+3}\,dt \cdot \frac{du}{dx} = \sqrt{u^2+3}\cdot\cos x = \sqrt{(\sin x)^2+3}\cdot\cos x$$

Exercise

Find these derivatives using Theorem Part 1.

- $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{t+t^3}\,dt$
  - Ans: $\sqrt{x+x^3}$
- $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\,dt$
  - Ans: $\sec x^4 \cdot 4x^3$
- $\dfrac{d}{dx}\displaystyle\int_1^{e^x} \ln t\,dt$
  - $\ln e^x \cdot e^x = xe^x$

Example

Recall the part 2 theorem as follows:

If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Evaluate the integral $\displaystyle\int_1^3 e^x\,dx$

Example

Recall the part 2 theorem as follows:

If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

Evaluate the integral $\displaystyle\int_1^3 e^x\,dx$

- $F = e^x$
- $F(3) - F(1)$
- $e^3 - e$

Example

Evaluate the integral $\int_0^1 x^2\,dx$

Example

Evaluate the integral $\int_0^1 x^2\,dx$

- $F = \dfrac{x^3}{3}$
- $F(1) - F(0)$
- $\dfrac{1^3}{3} - \dfrac{0^3}{3} = \dfrac{1}{3}$

Example (caution!)

Evaluate the integral $\int_{-1}^{3} \frac{1}{x^2}\, dx$

Example (caution!)

Evaluate the integral $\int_{-1}^{3} \frac{1}{x^2}\, dx$

- $F = \frac{-1}{x}$
- $F(3) - F(-1)$
- $\frac{-1}{3} - \frac{-1}{-1} = \frac{-1}{3} - 1 = \frac{-4}{3}$

This violates the comparison property, recall this property:

$$\text{If } f(x) \geq 0 \text{ for all } x \text{ in } [a, b] \text{ then } \int_a^b f(x)\, dx \geq 0$$

Looking more deeply, $f(x) = \frac{1}{x^2}$ has an infinite discontinuity at $x = 0$, thus the integral does not exist.

Exercise

- $\int_1^3 (x^2 + 2x - 4)\, dx$
  - Ans: $\frac{26}{3}$
- $\int_0^4 (4 - t) \cdot \sqrt{t}\, dt$
  - Ans: $\frac{128}{15}$
- $\int_0^3 (2 \sin x - e^x)\, dx$
  - Ans: $3 - 2\cos 3 - e^3$
- $\int_{-2}^2 f(x) = \begin{cases} 2 & \text{if } -2 \leq x \leq 0 \\ 4 - x^2 & \text{if } 0 < x \leq 2 \end{cases}$
  - Ans: $\dfrac{28}{3}$

Definition

If $F'(x) = f(x)$, we write $\int f(x)\,dx = F(x) + C$ and call the symbol $\int f(x)\,dx$ the **indefinite integral** of f, and C is called the **constant of integration**.

Note that the **definite** integral is a *number*, whereas the **indefinite** integral is a *function*, or a *family of functions*.

Example:

$$\int (10x^4 - 2\sec x^2)\,dx = 10 \cdot \frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$$

Substitution Rule

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

That is, *it is permissible to work with dx and du after integral signs as if they are differentials.*

Example

Find $\int 2x \sin x^2 \, dx$

Example

Find $\int 2x \sin x^2 \, dx$

Let $u = x^2$ and $du = 2x \, dx$

Thus, $\int \sin u \, du = -\cos u + C = -\cos x^2 + C$

Example

Find $\displaystyle\int \frac{x}{1+3x^2}\,dx$

Example

Find $\displaystyle\int \frac{x}{1+3x^2}\,dx$

Let $u = 1+3x^2$ and $du = 6x\,dx$ or $\frac{1}{6}du = x\,dx$

Thus, $\displaystyle\int \frac{x}{1+3x^2}\,dx = \int \frac{\frac{1}{6}}{u}\,du = \frac{1}{6}\int \frac{1}{u}\,du$

The antiderivative is $\displaystyle\frac{1}{6}(\ln|u| + C) = \frac{1}{6}(\ln|1+3x^2| + C)$

Example

Find $\int e^{7x}\,dx$

Example

Find $\displaystyle\int e^{7x}\,dx$

Let $u = 7x$ and $du = 7\,dx$ or $\frac{1}{7}du = dx$

Thus, $\displaystyle\int e^u \cdot \frac{1}{7}\,du = \frac{1}{7}e^u + C = \frac{1}{7}e^{7x} + C$

Example

Find $\int \sqrt{1+x^2} \cdot x^5 \, dx$

Example

Find $\int \sqrt{1+x^2} \cdot x^5 \, dx$

Let $u = 1 + x^2$ and $du = 2x \, dx$ so $x \, dx = \frac{1}{2} \, du$. Also, $x^2 = u - 1$, so $x^4 = (u-1)^2$

$$\begin{aligned}
\int \sqrt{1+x^2} \cdot x^5 \, dx &= \int \sqrt{1+x^2} \cdot x^4 \cdot x \, dx \\
&= \int \sqrt{u} \cdot (u-1)^2 \cdot \frac{1}{2} \, du \\
&= \frac{1}{2} \int \sqrt{u}(u^2 - 2u + 1) \, du \\
&= \frac{1}{2} \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) \, du \\
&= \frac{1}{2}(\frac{2}{7} u^{\frac{7}{2}} - 2 \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}}) + C \\
&= \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C
\end{aligned}$$

Example

Find $\int_0^4 \sqrt{2x+1}\,dx$

Example

Find $\int_0^4 \sqrt{2x+1}\,dx$

Let $u = 2x+1$ and $dx = \frac{1}{2}\,du$

Next, we need to change the bound of x to u, i.e., when $x = 0, u = 2(0)+1 = 1$; when $x = 4, u = 2(4)+1 = 9$

$$\begin{aligned}\int_0^4 \sqrt{2x+1}\,dx &= \int_1^9 \frac{1}{2}\sqrt{u}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big]_1^9 \\ &= \frac{1}{3}(9^{\frac{3}{2}} - 1^{\frac{3}{2}}) \\ &= \frac{26}{3}\end{aligned}$$

Exercise

- $\int \tan x \, dx$
  - $-\ln|\cos x| + C$
- $\int x^3 \cos(x^4 + 2) \, dx$
  - $\frac{1}{4}\sin(x^4 + 2) + C$
- $\int \sqrt{2x + 1} \, dx$
  - $\frac{1}{3}(2x + 1)^{\frac{3}{2}} + C$
- $\int_1^2 \frac{1}{(3 - 5x)^2} \, dx$
  - $1/14$
- $\int_1^e \frac{\ln x}{x} \, dx$
  - $1/2$