Week 1: Functions

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1. Introduction
   - Sets
   - Inequalities

2. Functions
   - Representation of functions
   - A catalogue of functions

Administration

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Sets

A *set* is a collection of well defined and distinct objects.

Sets of Numbers.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots, 100, \ldots\}$
- $\mathbb{Z} = \{\ldots, -100, \ldots, -12, \ldots, 0, 1, 2, 3, \ldots, 100, \ldots\}$
- $\mathbb{Q} = \{1/3, -4/1, \ldots, 1/12345, \ldots\}$ (rational numbers represented by a fraction a/b with a belonging to $\mathbb{Z}$ and b belonging to $\mathbb{Z}*$ (except zero))
- $\mathbb{R} = \{\pi, \sqrt{2}, \sqrt{3}, \ldots\}$ (rational $+$ irrational numbers; irrational numbers have infinite, non-periodic decimal part)

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Set notation. If A is a set of numbers and the number x is a member of the set A, then we write $x \in A$. If x is not a member of A then we write $x \notin A$

Example.

$$\frac{1}{2} \notin \mathbb{Z}; \qquad \sqrt{2} \in \mathbb{R}; \qquad 3 \in \mathbb{N}; \qquad \pi \notin \mathbb{Q}.$$

Intervals. Suppose that $a, b \in \mathbb{R}$ and $a < b$.

- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

  (The colon ':' is read as *such that*.)
  a and b are called *endpoints* of the interval.
  b is included in the interval; a is not.

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

Definition

Suppose that A and B are two sets. We say that A is a *subset* of B if $x \in A$ implies that $x \in B$, and we will denote this by $A \subseteq B$. If A is a subset of B then we also say that B *contains* the set A.

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Example. True or false?

- $\{3,4,5,6\}$ is a subset of $\{2,3,4,5,6\}$ (T)
- $\mathbb{Z}$ is a subset of $\mathbb{N}$ (F)
- $\mathbb{N}$ is a subset of $\mathbb{Z}$ (T)
- $(2,8]$ is a subset of $[2,8]$ (T)
- $[2,8)$ is a subset of $(2,8]$ (F)
- $[3,4]$ is a subset of $\mathbb{Q}$ (T)
- $\mathbb{Q}$ is a subset of $\mathbb{R}$ (T)

Inequalities

Solving Inequalities.

- If $a < b$ and $c < 0$, then $ac > bc$: Multiply or divide by a negative quantity reverse the inequality sign.
- If $0 < a < b$, then $1/a > 1/b$: Taking reciprocals reverse the inequality sign.

Inequalities

Example. Solve $1 + x < 7x + 5$

$$x < 7x + 4 \qquad \text{Minus 1 from both sides}$$

$$-6x < 4$$

$$x > -\frac{4}{6} = -\frac{2}{3} \qquad \text{Reverse the sign}$$

Inequalities

Example. Solve $x^2 - 5x + 6 \leq 0$

$$\begin{aligned}(x-2)(x-3) &\leq 0 \\ (x-2)(x-3) &= 0 \qquad \text{Set the equation to zero to find the interval} \\ x &= 2,3 \qquad \text{This gives us three intervals}\end{aligned}$$

Trying some values under the three intervals $(-\infty, 2), (2, 3), (3, \infty)$
We will get that the solution is $\{x|2 \leq x \leq 3\} = [2, 3]$

Self-Exercise

1. Solve $x^2 < 2x + 8$
   - Ans: $(-2, 4)$
2. Solve $x^3 + 3x^2 > 4x$
   - Ans: $(-4, 0) \cup (1, \infty)$
3. Solve $\frac{1}{x} < 4$
   - Ans: $(-\infty, 0) \cup (\frac{1}{4}, \infty)$
4. Solve $-3 < \frac{1}{x} \leq 1$ (Hint: solve separately)
   - Ans: $(-\infty, -\frac{1}{3}) \cup [1, \infty)$

Absolute values.

1. *Definition.* If $x \in \mathbb{R}$ then $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

2. *Properties.*
    1. $|-x| = |x|$
    2. $|xy| = |x||y|$
    3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ if y is nonzero
    4. The triangle inequality: $|x+y| \leq |x| + |y|$
    5. $|x| = \sqrt{x^2}$ and $|x|^n = x^n$
3. *Inequalities.*
    1. $|y| = a$ iff $y = \pm a$
    2. $|y| < a$ iff $-a < y < a$
    3. $|y| > a$ iff $y < -a$ or $y > a$

Example.

Solve $|2x - 5| = 3$

$$2x - 5 = 3 \text{ or } 2x - 5 = -3 \qquad \text{Using property 3.1}$$
$$x = 4 \text{ or } x = 1$$

Example.

Solve $|3x+2| \geq 4$

$$3x+2 \geq 4 \text{ or } 3x+2 \leq -4 \qquad \text{Using property 3.1 and 3.3}$$

In the first case $3x \geq 2$, which gives $x > \frac{2}{3}$. In the second case $3x \leq -6$, which gives $x \leq -2$. So the solution set is $(-\infty, -2] \cup [\frac{2}{3}, \infty)$

Example.

If $|x-4| < 0.1$ and $|y-7| < 0.2$, use the triangle inequality (property 2.4) to estimate $|(x+y)-11|$

$$\begin{aligned} |(x+y)-11| &= |(x-4)+(y-7)| \\ &\leq |x-4| + |y-7| \\ &< 0.1 + 0.2 = 0.3 \end{aligned}$$

Thus $|(x+y)-11| < 0.3$

Self-Exercise

1. Solve $|x-5|<2$
   - Ans: $(3,7)$
2. Solve $|x|<3$
   - Ans: $(-3,3)$
3. Solve $1 \leq |x| \leq 4$
   - Ans: $[-4,-1] \cup [1,4]$
4. Solve $0<|x-5|<\frac{1}{2}$
   - Ans: $(4.5,5) \cup (5,5.5)$

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Functions

Many naturally occurring quantities that vary with time can be modelled using *functions*.

Example

The volume of water stored in Lake Burley-Griffin is a variable that depends on time. For any particular time t, we could denote the corresponding volume by $f(t)$. In this situation

- f is called a function,
- if t is an input for the function f, then $f(t)$ is the corresponding output.

Specifying a function.

A function f has *two* parts to its definition:

1. the rule which explains how to get outputs from inputs,
2. the specification of the function's *domain* (i.e. set of inputs).

A key point is that any input must give *exactly one* output.

Example and terminology. A function f with domain $[0, \infty)$ is given by the rule

$$f(x) = x^2 \qquad \forall x \in [0, \infty).$$

The maximal domain. If the domain of a function is not specified, but the function rule is, then the default domain, known as the *maximal* or *natural domain*, is the largest possible domain for which the rule makes sense.

Example. Find the domain of $f(x) = \sqrt{x+2}$

- Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geq 0$. This is equivalent to $x \geq -2$, so the domain is the interval $[-2, \infty)$

The maximal domain. If the domain of a function is not specified, but the function rule is, then the default domain, known as the *maximal* or *natural domain*, is the largest possible domain for which the rule makes sense.

Example. Find the domain of $g(x) = \frac{1}{x^2 - x}$

- Since
$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$
and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Self-Exercise

Find the domain of these functions:

1. $f(x) = \frac{x+4}{x^2-9}$
   - Ans: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
2. $f(x) = \frac{2x^3-5}{x^2+x-6}$
   - Ans: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
3. $f(t) = \sqrt[3]{2t-1}$
   - Ans: $\mathbb{R}$
4. $g(t) = \sqrt{3-t} - \sqrt{2+t}$
   - Ans: $[-2, 3]$

The range of a function. Suppose that f is a function. The *range* of f, denoted by $\mathrm{Ran}(f)$, is defined by

$$\mathrm{Ran}(f) = \{f(x) \in B : x \in \mathrm{Dom}(f)\}.$$

Note that $\mathrm{Ran}(f)$ is the set of all output values for f. Note also that the range of f depends on the domain of f. If you are asked to give the range of a function f where the domain is not specified, you may assume that the domain is maximal.

Example. Suppose $f(x) = x^2$.

- If the domain of f is $\mathbb{R}$, the range of f is $[0, \infty)$.
- If the domain of f is $[0, \infty)$, the range of f is still $[0, \infty)$.
- If the domain of f is $[2, 4]$, the range of f is $[4, 16]$.

Special classes of functions

In this section we list some familar classes of functions.

Polynomials. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *polynomial* if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$. The a_j are called the *coefficients* of the polynomial and n is the *degree* of the polynomial (as long as a_n is nonzero). If $n = 1$ the polynomial is called linear, $n = 2$ quadratic, $n = 3$ cubic, etc.

Rational functions. Suppose that p and q are polynomials. A function f is called a *rational function* if

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : q(x) \neq 0\}$$

and

$$f(x) = \frac{p(x)}{q(x)} \qquad \forall x \in \mathrm{Dom}(f)$$

A piecewise defined function. is given by different formulas on different pieces of its domain.

Example.

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1. \end{cases}$$

Trigonometric functions. You should be familar with the functions sin, cos, tan, sec, cosec and cot.

Important note: In calculus, it is essential that angles are given in radian measure. Recall that 360 degrees is equal to 2π radians.

In particular, you should know (this list is not exhaustive!)

- That sine and cosine are periodic with a period of 2π.
- That the range of sine and cosine is $[-1, 1]$.
- That $sin(x) = 0$ if $x \in \{0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$ and $cos(x) = 0$ if $x \in \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\}$.
- The values of x for which $sin(x)$ equals 1 or -1 and the values of x for which $cos(x)$ equals 1 or -1.
- How to draw the graphs of sin, cos, tan (and to a lesser extent how to draw the graphs of sec, cosec, and cot).
- How to compute sin, cos, tan, sec, cosec, and cot of the values $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, etc.

Important – make sure you know the standard trig formulae:

- complementary identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

- Pythagorean identities

$$\cos^2 x + \sin^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$\cot^2 x + 1 = \mathrm{cosec}^2 x$$

- the sum and difference formulae

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Important – make sure you know the standard trig formulae:

- double-angle formulae

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}.$$

The exponential and logarithm functions.

Definition

Functions given as $f(x) = a^x, a > 0, a \in \mathbb{R}$ are called exponential functions. $\mathrm{Dom}(f) = \mathbb{R}$, $\mathrm{Ran}(f) = (0, \infty)$.

Definition

Logarithmic functions, denoted $\log_a$, is the inverse of a^x for $a > 0, a \in \mathbb{R}$:

- $\log_a(a^x) = x$, $a^{\log_a x} = x$.
- $\mathrm{Dom}(\log_a) = (0, \infty)$ and $\mathrm{Ran}(\log_a) = \mathbb{R}$.

Roots.

Functions of the form $x^{\frac{1}{n}}$. e.g. $x^{\frac{1}{3}} = \sqrt[3]{x}$.

Constructing new functions from known ones

Combining functions. If two functions f and g have the **same** domain A, we can construct new functions $f+g$, $f-g$ and $f \cdot g$ each with domain A. These are defined *pointwise* by the following formulae:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{provided that } g(x) \neq 0.$$

Combining functions. If two functions f and g have the **different** domain A

and B, we can construct new functions $f+g$, $f-g$ and $f \cdot g$ each with domain $A \cap B$. For f/g, the domain will be $A \cap B$ and $g(x) \neq 0$.

Constructing new functions from known ones

Example. The domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$. What is the domain of $(f + g)(x)$?

- $A \cap B = [0, 2]$

Example. $f(x) = x^2$ is $A = [0, \infty)$ and $g(x) = x - 1$. What is the domain of $(f/g)(x)$?

- $A \cap B$ and $g(x) \neq 0 \rightarrow x \neq 1$ or $(-\infty, 1) \cup (1, \infty)$

Constructing new functions from known ones

Example. If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

- $(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)^2$
- $(g \circ f)(x) = g(f(x)) = g(x^2) = (x^2 - 3)$

Self-Exercise

$f(x) = \sqrt{x}; g(x) = \sqrt{2-x}$. Find $f \circ g$, $g \circ f$ and their domains:

1. $f \circ g$: $f(\sqrt{2-x}) = \sqrt[4]{2-x}$
   - Domain: $2 - x \geq 0$ or $(-\infty, 2]$
2. $g \circ f$: $g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$
   - Domain: For $\sqrt{x}$ to be defined we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

Function transformations. Once the shapes of the graphs of basic functions are known, we can alter the shape of the graph by altering the function in simple ways.

Suppose that $a > 0$ and $c > 1$.

New graph	Obtained from $y = f(x)$ by
$y = f(x) + a$	translating the graph upwards by a units
$y = f(x) - a$	translating the graph downwards by a units
$y = f(x + a)$	translating the graph to the left by a units
$y = f(x - a)$	translating the graph to the right by a units
$y = cf(x)$	stretching the graph vertically by factor c
$y = (1/c)f(x)$	compressing the graph vertically by factor c
$y = f(cx)$	compressing the graph horizontally by factor c
$y = f(x/c)$	stretching the graph horizontally by factor c
$y = -f(x)$	reflecting the graph about the x-axis
$y = f(-x)$	reflecting the graph about the y-axis

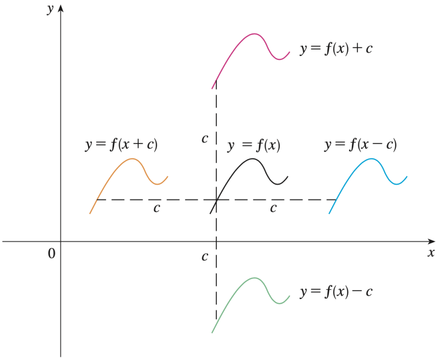

FIGURE 1 Translating the graph of f

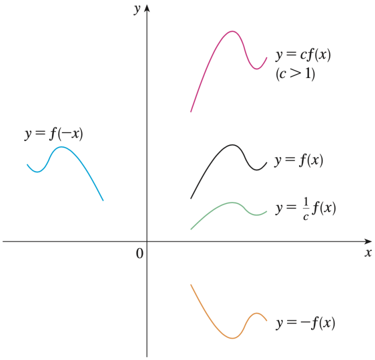

FIGURE 2 Stretching and reflecting the graph of f