Week 10: Integration Techniques Part 2

November 3, 2021

1. Trigonometric integrals ($\tan^m x \sec^n x$)

2. Trigonometric substitution

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2. Trigonometric substitution

$$\int \tan^m x \sec^n x \, dx.$$

We consider, as above, two cases.

Case 1: The integral involves an even power of $\sec x$**.** We save one factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$. Then substitute $u = \tan x$.

Case 2: The integral involves an odd power of $\tan x$**.** We save one factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then substitute $u = \sec x$.

- Recall that $\int \tan x \, dx = \ln|\sec x| + C$ and $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Example

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

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- $\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx = \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx$.
- Let $u = \tan x$ and $du = \sec^2 x \, dx$
- $\int u^6 (1 + u^2) \, du = \int u^6 + u^8 \, du$
- $\frac{u^7}{7} + \frac{u^9}{9} + C = \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C$

Example

Evaluate $\int \tan^5 x \sec^7 x \, dx$.

Example

Evaluate $\displaystyle\int \tan^5 x \sec^7 x \, dx$.

- $\displaystyle\int \tan^5 x \sec^7 x \, dx = \int \tan^4 x \sec^6 x \sec x \tan x \, dx$
- $\displaystyle\int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x \, dx$.
- Let $u = \sec x$ and $du = \sec x \tan x \, dx$
- $\displaystyle\int (u^2 - 1)^2 u^6 \, du = \int (u^{10} - 2u^8 + u^6) \, du$
- $\displaystyle\frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C = \frac{\sec^{11} x}{11} - 2\frac{\sec^9 x}{9} + \frac{\sec^7 x}{7} + C$

Example

Evaluate $\displaystyle\int \tan^3 x\,dx$.

Example

Evaluate $\int \tan^3 x \, dx$.

- $\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx = \int \tan x (\sec^2 x - 1) \, dx$
- $\int \tan x \sec^2 x \, dx - \int \tan x \, dx$.
- Let $u = \tan x$ and $du = \sec^2 x \, dx$ for the left integral
- $\dfrac{\tan^2 x}{2} - \ln|\sec x| + C$

Example

Evaluate $\int \sec^3 x \, dx$.

Example

Evaluate $\int \sec^3 x \, dx$.

- Use integration by parts where $u = \sec x$, $du = \sec x \tan x \, dx$, $dv = \sec^2 x \, dx$, and $v = \tan x$, and recall the formula $\int u \, dv = uv - \int v \, du$.
- $\int \sec^3 x = \sec x \tan x - \int \sec x \tan^2 x \, dx$
- $= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$
- $= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$
- $\int \sec^3 x + \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx + \int \sec^3 x \, dx$
- $\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$

Exercise

$$\int \tan x \sec^3 x \, dx.$$

- Ans: $\frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$

$$\int \tan^3 x \sec^6 x \, dx.$$

- Ans: $\frac{1}{8}\tan^8 x + \frac{1}{3}\tan^6 x + \frac{1}{4}\tan^4 x + C$

$$\int \tan^5 x \, dx.$$

- Ans: $\frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C$

$$\int \tan^2 x \sec x \, dx.$$

- Ans: $\frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$

Trigonometric substitution

We now look at integrals of the form $\int \sqrt{a^2 - x^2}\, dx$, where $a > 0$. A good strategy is to change variables from x to θ by using the substitution $x = a \sin\theta$. The identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a|\cos\theta|$$

If we assume that the range θ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then it becomes $a\cos\theta$

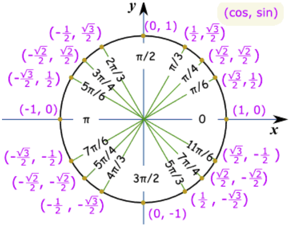
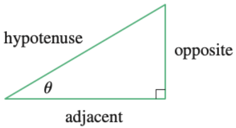


Table of substitutions and trigonometric functions.

Expression	Substitution	Range for θ	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$\sec^2\theta - 1 = \tan^2\theta$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Example

Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

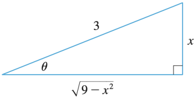


FIGURE 1

$\sin\theta = \dfrac{x}{3}$

Example

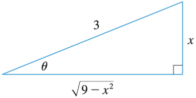


FIGURE 1
$\sin\theta = \frac{x}{3}$

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Let $x = 3\sin\theta$, then $dx = 3\cos\theta d\theta$, hence
$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta} d\theta = \int \cot^2\theta d\theta \\ &= \int (\csc^2\theta - 1) d\theta = -\cot\theta - \theta + C\end{aligned}$$

See figure 1, we know that $\cot\theta = \frac{\sqrt{9-x^2}}{x}$, we also know that $\theta = sin^{-1}(\frac{x}{3})$ thus

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx. = -\frac{\sqrt{9-x^2}}{x} - sin^{-1}(\frac{x}{3}) + C$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

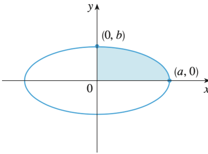


FIGURE 2

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Example

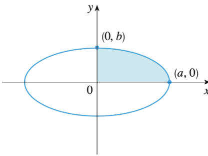


FIGURE 2

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$ or
- $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2). The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \text{ where } 0 \leq x \leq a \text{ thus}$$

$$\frac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

Let $x = a\sin\theta$, then $dx = a\cos\theta d\theta$. To change the limits of integration, when $x = 0$, $a\sin\theta = 0$, so $\theta = 0$. When $x = a$, $a\sin\theta = a$, thus $\theta = \frac{\pi}{2}$.

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$$
\begin{aligned}
A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta d\theta \\
&= 4ab\int_0^{\pi/2}\cos^2\theta d\theta = 4ab\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)d\theta \\
&= 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) \\
&= \pi ab
\end{aligned}
$$

Example

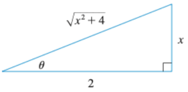

FIGURE 3

$\tan\theta = \frac{x}{2}$

Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Example

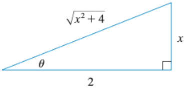

FIGURE 3

$\tan\theta = \frac{x}{2}$

Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$, thus

$$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

So we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2sec^2\theta d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int \frac{sec\theta}{\tan^2\theta}d\theta$$

We put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

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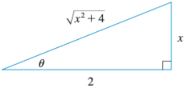

FIGURE 3

$\tan\theta = \frac{x}{2}$

Continued....

Here make the substitution $u = \sin\theta$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{cos\theta}{sin^2\theta}d\theta = \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C$$

$$= -\frac{csc\theta}{4} + C$$

Using figure 3, we determine that $\csc\theta = \frac{\sqrt{x^2+4}}{x}$, so

$$\frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

Example

Find $\int \frac{x}{\sqrt{x^2+4}}\, dx$.

Example

Find $\int \frac{x}{\sqrt{x^2+4}}\,dx$.

It would be possible to use the trigonometric substitution $x = 2\tan\theta$ here. But the direct substitution $u = x^2 + 4$ is simpler, because then $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2+4}}dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C$$

Thus, note that even when trigonometric substitutions are possible, they may not give the easiest solution.

Exercise

- $\int_0^3 \frac{x}{\sqrt{36-x^2}}$
  - Ans: $6-3\sqrt{3}$
- $\int \frac{\sqrt{1+x^2}}{x}$
  - Ans: $\ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $a > 0$
  - Ans: $\ln\left|x+\sqrt{x^2-a^2}\right| - \ln a + C$