Week 8: Applications: Area and Average Value

October 15, 2021

1. Area

2. Average Value

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2. Average Value

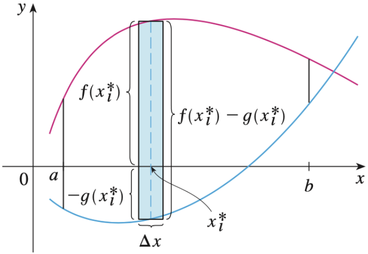
Definition

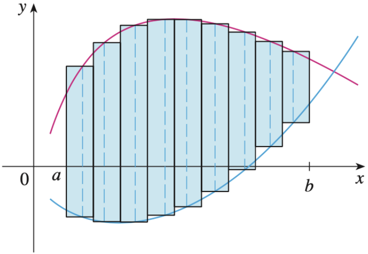
Area between two curves $f(x)$ and $g(x)$, in the interval $[a, b]$ is:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x$$

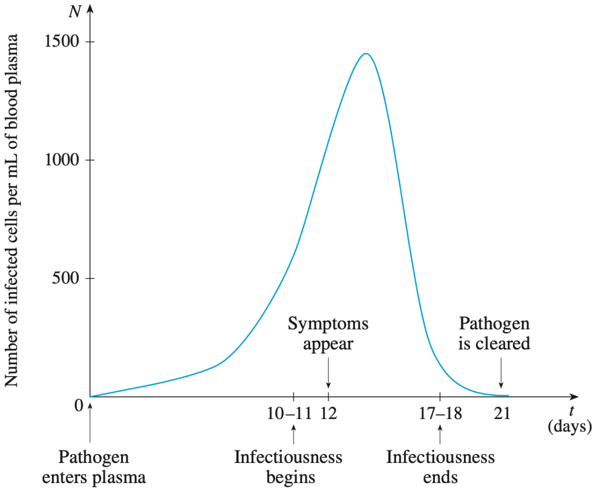
or in a integral form as:

$$A = \int_a^b [f(x) - g(x)]\; dx$$

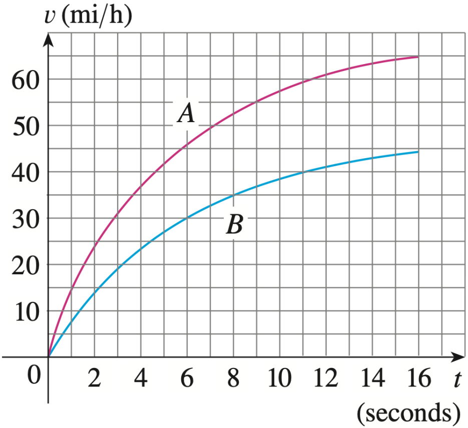




Use Case: Level of infected cells

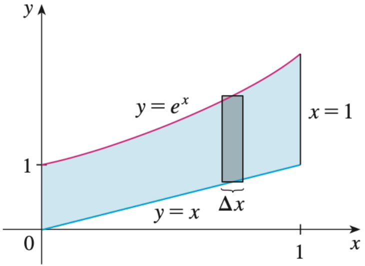


Use Case: Distances between cars



Example

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$

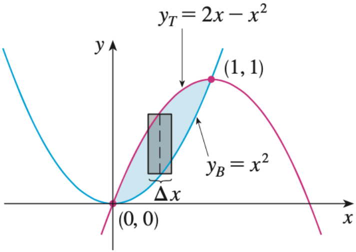


$$A = \int_0^1 (e^x - x)\,dx = e^x - \frac{1}{2}x^2 \Big]_0^1$$

$$= e - \frac{1}{2} - 1 = e - 1.5$$

Example

Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x^2 - x^2$



First we need to find the intersection point by solving both equations: $x^2 = 2x - x^2; 2x^2 - 2x = 0; 2x(x-1) = 0; x = 0, 1.$ Thus the intersections points are (0, 0) and (1, 1).

$$A = \int_0^1 (2x - x^2 - x^2)\, dx = 2\int_0^1 (x - x^2)\, dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Exercise

Find the area of the region enclosed by $y = e^x$ and $y = x^2 - 1$, with the interval of $x = -1$ and $x = 1$

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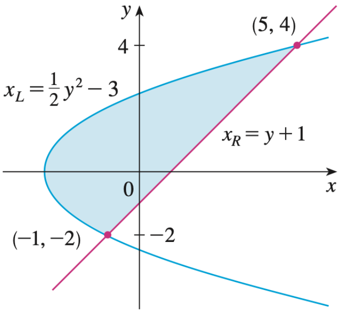
$$
\begin{aligned}
A &= \int_{-1}^{1} \left(e^x - (x^2 - 1)\right) dx \\
&= e^x - \frac{1}{3}x^3 + x\Big]_{-1}^{1} \\
&= e - \frac{1}{e} + \frac{4}{3}
\end{aligned}
$$

Example

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$

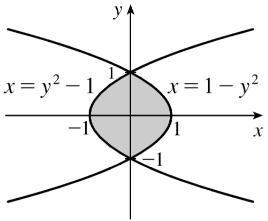


$$A = \int_{-2}^{4} ((y+1) - (\frac{1}{2}y^2 - 3))\,dy$$

$$= \int_{-2}^{4} (-\frac{1}{2}y^2 + y + 4)\,dy = -\frac{1}{2}(\frac{y^3}{3}) + \frac{y^2}{2} + 4y\Big]_{-2}^{4} = 18$$

Exercise

Find the area of the region enclosed by $x = 1 - y^2$ and $x = y^2 - 1$ with respect to y.



Exercise

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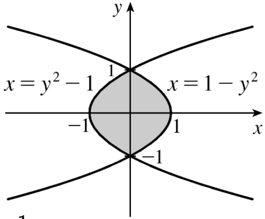

$$
\begin{aligned}
A &= \int_{-1}^{1} ((1 - y^2) - (y^2 - 1))\, dy \\
&= \int_{-1}^{1} 2(1 - y^2)\, dy \\
&= 2 \cdot \int_{0}^{1} 2(1 - y^2)\, dy = \frac{8}{3}
\end{aligned}
$$

1. Area

2. Average Value

The average value of a function

The average value of finitely many numbers $y_1, y_2, \ldots, y_n$ is given by

$$y_{\text{ave}} = \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

In general, how do we calculate the average value of a function $y = f(x)$, where $a \leq x \leq b$?

Definition

Suppose that f is continuous on $[a, b]$. Then the *average value* f_{ave} of f on $[a, b]$ is defined by the formula

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The average value is sometimes denoted by $\overline{f}$.

Example

Find the average value of the function f over $[-1, 2]$, where $f(x) = 1 + x^2$

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$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)\,dx \\ &= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} \\ &= 2 \end{aligned}$$

Exercise

Find the average value of the function f over $[-1, 2]$, where $f(x) = 3x^2 + 8x$

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$$
\begin{aligned}
f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{2-(-1)}\int_{-1}^{2}(3x^2+8x)\,dx \\
&= \frac{1}{3}\left[x^3+4x^2\right]_{-1}^{2} \\
&= \frac{1}{3}\left[(8+16)-(-1+4)\right] \\
&= 7
\end{aligned}
$$

Suppose that $T(t)$ is the temperature (in $^\circ$C) at time t (in hours) and that T_{ave} is the average temperature on the time interval $[0, 24]$. Is there a specific time t_0 in $[0, 24]$ when the temperature $T(t_0)$ is equal to the average temperature T_{ave}? More generally, given a function f, is there a specific value c for which $f(c) = f_{\text{ave}}$? The answer is yes! This is called the mean value theorem for integrals.

Theorem (The Mean Value Theorem for integrals)

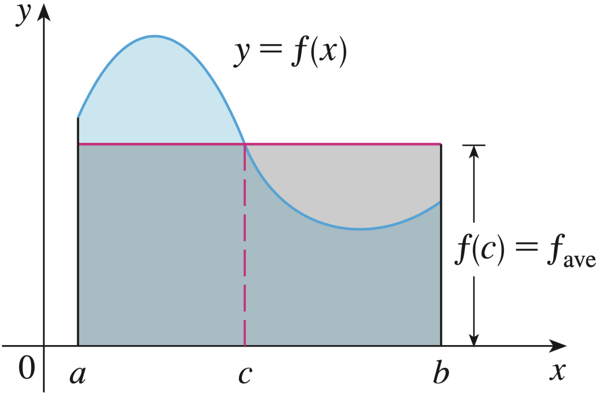
If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Mean Value Theorem for Integrals



Example

Find all numbers c that satisfy the conclusion of the MVT for integrals when $f(x) = 1 + x^2$ and $[a, b] = [-1, 2]$.

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First, we will find the average value:

$$f_{\text{ave}} = \frac{1}{2-(-1)} \int_{-1}^{2} (1 + x^2)\, dx = 2$$

Thus, $f(c) = f_{\text{ave}} = 2$

Then, we set the $f(c) = 2$, where $f(c) = 1 + c^2$, and solve for c:

$$\begin{aligned} 1 + c^2 &= 2 \\ c &\pm 1 \end{aligned}$$

Exercise

Find all numbers c that satisfy the conclusion of the MVT for integrals when $f(x) = (x-3)^2$ and $[a, b] = [2, 5]$.

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Find all numbers c that satisfy the conclusion of the MVT for integrals when $f(x) = (x-3)^2$ and $[a,b] = [2,5]$.

First, we will find the average value:

$$f_{\text{ave}} = \frac{1}{5-2}\int_2^5 (x-3)^2\,dx$$
$$= \frac{1}{3}\left[\frac{1}{3}(x-3)^3\right]_2^5 = 1$$

Thus, $f(c) = f_{\text{ave}} = 1$

Then, we set the $f(c) = 1$, where $f(c) = (c-3)^2$, and solve for c:

$$(c-3)^2 = 1$$
$$c - 3 = \pm 1$$
$$c = 2 \text{ or } 4$$