Week 6: Anti-derivatives, Definite Integral

October 11, 2021

1. Antiderivatives

2. Area
   - Riemann Sums
   - Definite integral

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Antiderivatives

$f(x)$	Antiderivative $(F(x))$	$f(x)$	Antiderivative $(F(x))$
$cf(x)$	$cF(x)+C$	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$f(x)+g(x)$	$F(x)+G(x)+C$	e^x	$e^x + C$
x^n (where $n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$	b^x	$\frac{b^x}{\ln b} + C$
$\cos x$	$\sin x + C$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\sin x$	$-\cos x + C$	$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$\sec^2 x$	$\tan x + C$	$\cosh x$	$\sinh x + C$
$\sec x \tan x$	$\sec x + C$	$\sinh x$	$\cosh x + C$

Example

Find antiderivative of $g'(x) = 4\sin x + \dfrac{2x^5 - \sqrt{x}}{x}$

Example

Find antiderivative of $g'(x) = 4\sin x + \dfrac{2x^5 - \sqrt{x}}{x}$

Let's rewrite the equation first.

- $g'(x) = 4\sin x + \dfrac{2x^5 - \sqrt{x}}{x} = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$

Using the formula above, we get:

- $g(x) = 4(-\cos x) + \dfrac{2x^5}{5} - \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

Example

Find f if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$

Example

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Using the formula above, we get:

- $f(x) = e^x + 20\tan^{-1} x + C$

To find C, we use $f(0) = -2$

- $f(0) = e^0 + 20\tan^{-1} 0 + C = -2$
- $f(0) = 1 + 0 + C = -2$
- $C = -3$
- $f(x) = e^x + 20\tan^{-1} x - 3$

Example

A particle moves with **acceleration** given by $a(t) = 6t + 4$. Its initial **velocity** is $v(0) = -6$ cm/s and its initial **displacement** is $s(0) = 9$ cm. Find its position function $s(t)$.

Example

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Velocity is **acceleration**'s antiderivative - $v'(t) = a(t) = 6t + 4$

- $v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$

To find C, we use $v(0) = -6$

- $v(0) = 0 + 0 + C = -6$
- $v(t) = 3t^2 + 4t - 6$

Position is the antiderivative of **velocity** - $v(t) = s'(t)$

- $s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$
- $s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + 9$ (given $s(0) = 9$)

Example

Find f if $f''(x) = 12x^2 + 6x - 4$

Example

Find f if $f''(x) = 12x^2 + 6x - 4$

Using the formula above, we get:

- $f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$
- $f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D$

Exercise

1. Find the antiderivative:
   - $3\sqrt{x} - 2\sqrt[3]{x}$
   - $\dfrac{1+t+t^2}{\sqrt{t}}$
   - $2\cos v - \dfrac{3}{\sqrt{1-v^2}}$
2. Find the double antiderivative of $f''(x) = \dfrac{1}{x^2}$
3. Find the antiderivative of $f(x) = 5x^4 - 2x^5$ and $F(0) = 4$
4. A particle is moving with the given data. Find the position $s(t)$ of the particle. $v(t) = \sin t - \cos t$ and $s(0) = 0$

Exercise

Find the antiderivative of $3\sqrt{x} - 2\sqrt[3]{x}$

Using the formula, we get:

- $F(x) = 3\left(\frac{2}{3}x^{\frac{3}{2}}\right) - 2\left(\frac{3}{4}x^{\frac{4}{3}}\right) + C$

Exercise

Find the antiderivative of $\dfrac{1+t+t^2}{\sqrt{t}}$

Let's rewrite as:

- $t^{-\frac{1}{2}} + t^{\frac{1}{2}} + t^{\frac{3}{2}}$

Using the formula, we get:

- $2t^{\frac{1}{2}} + \dfrac{2}{3}t^{\frac{3}{2}} + \dfrac{2}{5}t^{\frac{5}{2}} + C$

Exercise

Find the antiderivative of $2\cos v - \dfrac{3}{\sqrt{1-v^2}}$

Using the formula, we get:

- $2\sin v - 3\sin^{-1} v + C$

Exercise

Find the double antiderivative of $f''(x) = \dfrac{1}{x^2}$

Using the formula, we get:

- $f''(x) = x^{-2}$
- $f'(x) = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x < 0 \\ -\frac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$
- $f(x) = \begin{cases} -\ln(-x) + C_1 x + D_1 & \text{if } x < 0 \\ -\ln(x) + C_2 x + D_2 & \text{if } x > 0 \end{cases}$

Exercise

Find the antiderivative of $f(x) = 5x^4 - 2x^5$ and $F(0) = 4$

Using the formula, we get:

- $F(x) = 5\dfrac{x^5}{5} - 2\dfrac{x^6}{6} + C = x^5 - \dfrac{1}{3}x^6 + C$

Given $F(0) = 4$:

- $F(0) = 0^5 - \dfrac{1}{3}0^6 + C = 4$
- $C = 4$
- $F(x) = x^5 - \dfrac{1}{3}x^6 + 4$

Exercise

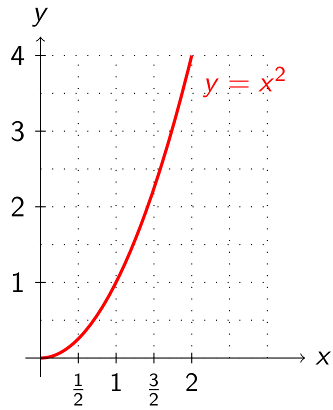
A particle is moving with the given data. Find the position $s(t)$ of the particle. $v(t) = \sin t - \cos t$ and $s(0) = 0$

- $v(t) = s'(t) = \sin t - \cos t$
- $s(t) = -\cos t - \sin t + C$
- $s(0) = -1 + C$
- $C = 1$
- $s(t) = -\cos t - \sin t + 1$

1. Antiderivatives

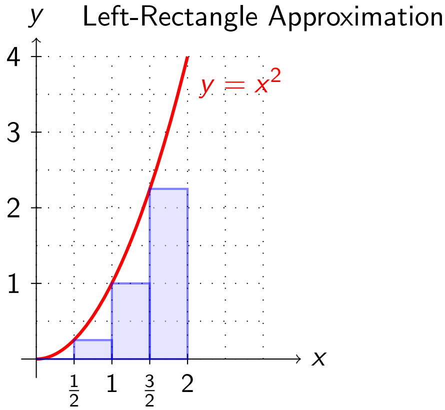
2. Area
   - Riemann Sums
   - Definite integral

Riemann Sums



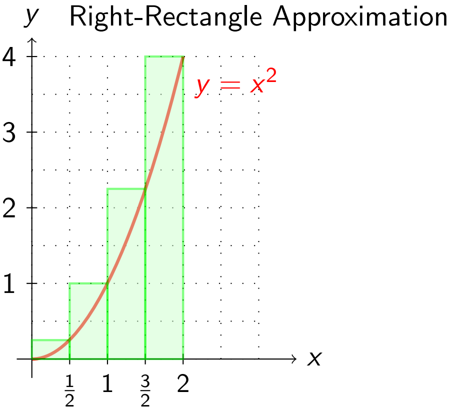
We want to find the area under the graph of $y = x^2$ in between $x = 0$ and $x = 2$.

Riemann Sums



Left Rectangle
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

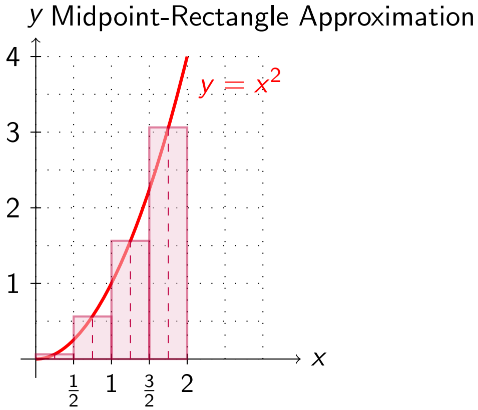
Riemann Sums



Left Rectangle
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

Right Rectangle
$= \frac{1}{2} \cdot [(\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2 + 2^2] = 3.75.$

Riemann Sums

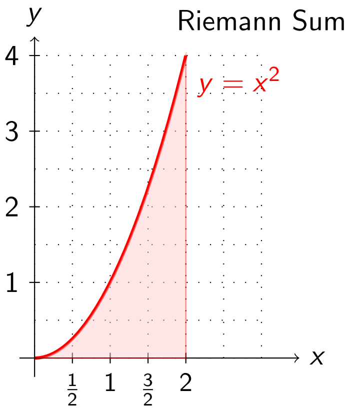


Left Rectangle
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

Right Rectangle
$= \frac{1}{2} \cdot [(\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2 + 2^2] = 3.75.$

Midpoint Rectangle
$= \frac{1}{2} \cdot [(\frac{1}{4})^2 + (\frac{3}{4})^2 + (\frac{5}{4})^2 + (\frac{7}{4})^2] = 2.625.$

Riemann Sums



Left Rectangle
$= \frac{1}{2} \cdot [0^2 + (\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2] = 1.75.$

Right Rectangle
$= \frac{1}{2} \cdot [(\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2 + 2^2] = 3.75.$

Midpoint Rectangle
$= \frac{1}{2} \cdot [(\frac{1}{4})^2 + (\frac{3}{4})^2 + (\frac{5}{4})^2 + (\frac{7}{4})^2] = 2.625.$

Actual area is $2\frac{2}{3}$ or 2.667

Definition

The **area** under the graph of the continuous function f to be the limit of the sum of areas of approximating rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\triangle x + f(x_2)\triangle x + \ldots + f(x_n)\triangle x].$$

or can be rewritten as

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x$$

Note: It can be shown that we get the same value if we use left endpoints rather than right endpoints. In fact, we can take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. These numbers $x_1^*, \ldots, x_n^*$ are called *sample points*.

Definite integral

Definition

If f is a function defined on $[a, b]$, we divide the intervals $[a, b]$ into n subintervals of equal width $\triangle x = (b - a)/n$. Let $x_0 = a$, $x_n = b$, and $x_i^* = a + i\triangle x$ (right endpoints), the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x_i$$

provided this limit exists. If it does exist, we say that f is *integrable* on $[a, b]$.

Definite integral

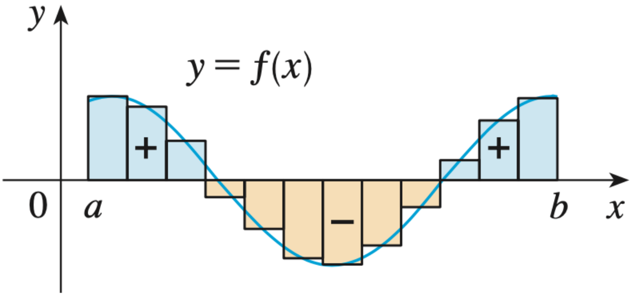


Figure 1: Integral is the net area

Useful sum of powers

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
- $\displaystyle\sum_{i=1}^{n} c = nc$
- $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
- $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Basic properties of the definite Integral

Suppose that a, b, d are real numbers and f is integrable on $[a, b]$, and c is a constant.

1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. $\displaystyle\int_a^a f(x)\,dx = 0$
3. $\displaystyle\int_a^b c\,dx = c(b-a)$
4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
5. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$

Comparison properties of the definite Integral

Suppose that $a \leq b$.

1. If $f(x) \geq 0$ for all x in $[a, b]$ then $\int_a^b f(x)\,dx \geq 0$.
2. If $f(x) \leq g(x)$ for all x in $[a, b]$ then
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$
3. If $m \leq f(x) \leq M$ for all x in $[a, b]$ then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$
4. If $|f|$ is integrable on $[a, b]$ then
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Example

Evaluate $\int_0^1 (4 + 3x^2)\, dx$

Example

Evaluate $\int_0^1 (4+3x^2)\,dx$

- $\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$
- $\int_0^1 4\,dx = 4(1-0) = 4$ (use prop. 3)
- $\triangle x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}, x_i^* = a + i\triangle x = 0 + i\frac{1}{n} = \frac{i}{n}$
- $\int_0^1 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\triangle x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} =$
  $\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{1}{n^3}(\frac{n(n+1)(2n+1)}{6}) = \lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3} =$
  $\lim_{n\to\infty}\frac{n^3(2+\frac{3}{n}+\frac{1}{n^2})}{6n^3} = \frac{1}{3}$
- $\int_0^1 4\,dx + 3\int_0^1 x^2\,dx = 4 + 3(\frac{1}{3}) = 5$

Exercises

Evaluate the following integrals:

- $\displaystyle\int_2^5 (4-2x)\,dx$
  - Ans: -9
- $\displaystyle\int_0^2 2x - x^3\,dx$
  - Ans: 0

Exercise

Evaluate $\int_2^5 (4 - 2x)\,dx$

Exercise

Evaluate $\int_2^5 (4-2x)\,dx$

- $\int_2^5 4\,dx - 2\int_2^5 x\,dx$
- $\int_2^5 4\,dx = 4(5-2) = 12$ (use prop. 3)
- $\triangle x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}, x_i^* = a + i\triangle x = 2 + i\frac{3}{n} = 2 + \frac{3i}{n}$
- $\int_2^5 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(2+\frac{3i}{n})\frac{3}{n} =$
  $\lim_{n\to\infty} \sum_{i=1}^{n} (2+\frac{3i}{n})\frac{3}{n} = \lim_{n\to\infty} \frac{3}{n}\left(\sum_{i=1}^{n} 2 + \frac{3}{n}\sum_{i=1}^{n} i\right) =$
  $\lim_{n\to\infty} \frac{3}{n}\left(2n + \frac{3}{n}(\frac{n(n+1)}{2})\right) = \lim_{n\to\infty} \frac{6n}{n} + \frac{9n+9}{2n} = 6 + \frac{9}{2} + 0 = \frac{21}{2}$
- $\int_2^5 4\,dx - 2\int_2^5 x\,dx = 12 - 2(\frac{21}{2}) = -9$

Exercise

Evaluate $\int_0^2 2x - x^3 \, dx$

Exercise

Evaluate $\int_0^2 2x - x^3\,dx$

- $\triangle x = \frac{2}{n}, x_i^* = \frac{2i}{n}$
- $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{2i}{n})\frac{2}{n} = \lim_{n\to\infty} \frac{2}{n}\left(\sum_{i=1}^{n} 2(\frac{2i}{n}) - (\frac{2i}{n})^3\right) =$
  $\lim_{n\to\infty} \frac{2}{n}\left(\sum_{i=1}^{n} \frac{4i}{n} - \frac{8i^3}{n^3}\right) = \lim_{n\to\infty} \frac{2}{n}\left(\frac{4}{n}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^3\right) =$
  $\lim_{n\to\infty} \frac{2}{n}\left(\frac{4}{n}\cdot\frac{n(n+1)}{2} - \frac{8}{n^3}\cdot\left[\frac{n(n+1)}{2}\right]^2\right) =$
  $\lim_{n\to\infty} \frac{8}{n^2}\cdot\frac{n^2+n}{2} - \frac{16}{n^4}\cdot\frac{n^4+n^3+n^3+n^2}{4} = 4 - 4 = 0$