Week 11: Integration Techniques Part 3

November 12, 2021

Integrating rational functions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate. To illustrate the method, observe that:

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the fnction:

$$\int \frac{x+5}{x^2+x-2} dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx$$
$$= 2\ln|x-1| - \ln|x+2| + C$$

Example

Find $\displaystyle\int \frac{x^3 + x}{x - 1}$

$$
\begin{array}{r}
x^2 + x \;\; + 2 \\
x-1 \,\big)\overline{x^3 \qquad\quad + x} \\
\underline{x^3 - x^2} \qquad\;\; \\
x^2 + x \\
\underline{x^2 - x} \\
2x \\
\underline{2x - 2} \\
2
\end{array}
$$

Example

$$\begin{array}{r l}
 & x^2 + x + 2 \\
x-1 \,\big)\!\!\!\! & \overline{x^3 \qquad + x} \\
 & \underline{x^3 - x^2} \\
 & \qquad x^2 + x \\
 & \qquad \underline{x^2 - x} \\
 & \qquad\qquad 2x \\
 & \qquad\qquad \underline{2x - 2} \\
 & \qquad\qquad\quad 2
\end{array}$$

Find $\displaystyle\int \frac{x^3 + x}{x - 1}$

$$\begin{aligned}
\frac{x^3 + x}{x - 1} dx &= \int \left(x^2 + x + 2 + \frac{2}{x - 1} \right) dx \\
&= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln |x - 1| + C
\end{aligned}$$

Case 1: The denominator splits into distinct linear factors.

Examples of two such rational functions and the form of their partial fractions decompositions are given below:

$$\frac{x-3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$
$$\frac{x^2-x+7}{x(2x+1)(x-3)} = \frac{A}{x} + \frac{B}{2x+1} + \frac{C}{x-3}.$$

Example

Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

Example

Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$

Since the degree of the numerator (top one) is less than the degree of the denominator, we don't divide. Instead, we factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

We can do the partial fraction as:

$$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To find A, B, C, we multiply both sides:

$$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

We can rewrite as:

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

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Coefficients on both sides of the equation must be equal, thus:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

Solving this, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, $C = -\frac{1}{10}$, and so

$$\begin{aligned} \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx &= \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2} \right) dx \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C \end{aligned}$$

Note that when we integrate the middle term, we made the substitution $u = 2x - 1$ which gives $du = 2dx$, and $dx = \frac{1}{2}du$

Case 2: The denominator has a repeated linear factor.

Examples of two such rational functions and the form of their partial fractions decompositions are given below:

$$\frac{x^2+1}{(x+4)^3} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3}$$
$$\frac{x^2-2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

Example

Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$

Example

Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx$

The first step is to divide. The result of long division is

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factor the denominator. Since $Q(1) = 0$, we know that $x - 1$ is a factor and we obtain:

$$x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$$

Since the linear factor $x - 1$ occurs twice, the partial fraction decomposition is

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

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Multiplying by the least common denominator, $(x-1)^2(x+1)$, we get

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Equating coefficients, we get:

$$\begin{aligned} A + C &= 0 \\ B - 2C &= 4 \\ -A + B + C &= 0 \end{aligned}$$

Solving this, we get $A = 1, B = 2, C = -1$, so

$$\begin{aligned} \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx &= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C \\ &= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C \end{aligned}$$

Exercise

- $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}dx$
    - Ans: $\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$
- $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$
    - Ans: $\frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$
- $\displaystyle\int \frac{dx}{x^2-a^2}$
    - Ans: $\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$
- $\displaystyle\int \frac{x^3+4x^2+x-1}{x^3+x^2}dx$
    - Ans: $\frac{1}{2} + \ln 6$

Case 3: The denominator has an irreducible quadratic factor.

Examples of two such rational functions and the form of their partial fractions decomposition are given below:

$$\frac{x^2+x}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$
$$\frac{x^3-2x+4}{(x^2+5)(x^2+x+1)} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{x^2+x+1}.$$

Note that irreducible quadratic factor of ax^2+bx+c has determinants of $b^2-4ac<0$

Example

Find $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$
Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying both sides:

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

Equating coefficients:

$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

Thus $A = 1, B = 1, C = -1,$ *(continued next slide)*

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$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \frac{1}{x} + \left(\frac{x-1}{x^2+4} \right) dx$$
$$= \int \frac{1}{x} dx + \int \frac{x}{x^2+4} - \int \frac{1}{x^2+4} dx$$
$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}(\frac{x}{2}) + C$$

Note that in the second step, we let $u = x^2 + 4$ so $du = 2xdx$. Also note that $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Case 4: The denominator has a repeated irreducible quadratic factor.

It has the form as follows:

$$\frac{x^2+x}{(x^2+9)^3} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2} + \frac{Ex+F}{(x^2+9)^3}$$

$$\frac{x^3-2x+4}{(x-2)(x^2+x+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}.$$

Example

Find $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx$

Example

Find $\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} dx$

The form of the decomposition is

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$$

Multiplying both sides:

$$\begin{aligned} -x^3 + 2x^2 - x + 1 &= A(x^2+1)^2 + (Bx + C)x(x^2+1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

Equating the coefficients, we get:

$$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$$

$$A = 1, B = -1, C = -1, D = 1, E = 0$$

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$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx \\ &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx - \int \frac{dx}{x^2+1} + \int \frac{xdx}{(x^2+1)^2} \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C\end{aligned}$$

Note that we made a mental substitution of $u = x^2+1$ in second and fourth terms.

Exercise

- $\displaystyle\int \frac{x^2 - x + 6}{x^3 + 3x} dx$
  - Ans: $2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$
- $\displaystyle\int \frac{x^3 + 6x - 2}{x^4 + 6x^2} dx$
  - Ans: $\ln|x| + \frac{1}{3x} + \frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$