Week 12: Sequence

November 23, 2021

Sequence

- **Definition**: A **sequence** is an ordered list of numbers
- A sequence is often denoted as $\{a_1, a_2, a_3, \cdots, a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$
- **Example**: write out the first three terms of $\left\{\frac{3n+1}{(n+2)!}\right\}_{n=1}^{\infty}$

$$a_1 = \frac{3 \cdot 1 + 1}{(1+2)!} = \frac{4}{3!} = \frac{4}{3 \cdot 2 \cdot 1} = \frac{2}{3}$$
$$a_2 = \frac{3 \cdot 2 + 1}{(2+2)!} = \frac{7}{4!} = \frac{7}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7}{24}$$
$$a_3 = \frac{10}{5!} = \frac{1}{12}$$

Arithmetic sequence

- **Definition**: An **arithmetic sequence** is a sequence for which consecutive terms have the same common differences. If a is the first term and d is the common difference, then the arithmetic sequence has the form:

$$\{d \cdot k + a\}_{k=0}^{\infty}$$

- **Example**: Write a formula for the general term a_n, starting with $n = 0$, of $\{7, 10, 13, 16, 19, \cdots\}$
  - Ans: $\{3n + 7\}_{n=0}^{\infty}$

Geometric sequence

- **Definition**: A **geometric** sequence is a sequence for which consecutive terms have the same common ratio. If a is the first term and r is the common ratio, then a geometric sequence has the form:

$$\{a \cdot r^n\}_{n=0}^{\infty}$$

- **Example**: Write a formula for for the general term a_n (start with $n = 0$) of $\{3, 0.3, 0.03, 0.003, 0.003, \cdots\}$
  - Ans: $\{3 \cdot (0.1)^n\}_{n=0}^{\infty}$

Exercise

For each sequence, write a formula for the general term a_n (start with $n = 0$)

- $\{\frac{15}{2}, \frac{75}{4}, \frac{375}{8}, \frac{1875}{16}, \cdots\}$
- $\{3, -2, \frac{4}{3}, -\frac{8}{9}, \cdots\}$

For each sequence, write a formula for the general term a_n (start with $n = 1$)

- $\{-\frac{2}{9}, \frac{4}{16}, -\frac{8}{25}, \frac{16}{36}, \cdots\}$
- $\{-6, 5, -1, 4, 3, 7, 10, 17, \cdots\}$

Exercise

For each sequence, write a formula for the general term a_n (start with $n = 0$)

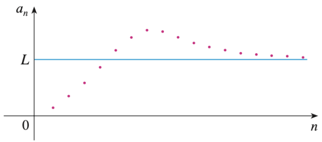
- $\left\{\frac{15}{2}, \frac{75}{4}, \frac{375}{8}, \frac{1875}{16}, \cdots\right\}$
  - Ans: $\left\{\frac{15}{2} \cdot \left(\frac{5}{2}\right)^n\right\}_{n=0}^{\infty}$
- $\{3, -2, \frac{4}{3}, -\frac{8}{9}, \cdots\}$
  - Ans: $\left\{3 \cdot \left(-\frac{2}{3}\right)^n\right\}_{n=0}^{\infty}$

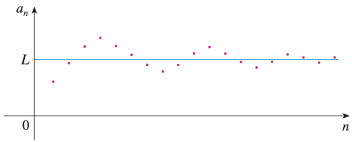
For each sequence, write a formula for the general term a_n (start with $n = 1$)

- $\left\{-\frac{2}{9}, \frac{4}{16}, -\frac{8}{25}, \frac{16}{36}, \cdots\right\}$
  - Ans: $\left\{(-1)^n \frac{2n}{(n+2)^2}\right\}_{n=1}^{\infty}$
- $\{-6, 5, -1, 4, 3, 7, 10, 17, \cdots\}$
  - Ans: $a_n = a_{n-1} + a_{n-2}$

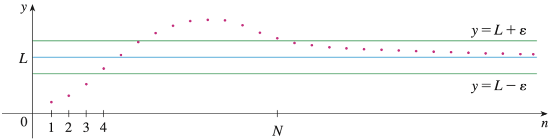
Convergence

- A sequence $\{a_n\}$ converges if $\lim_{n\to\infty} a_n$ exists as a finite number
- That is $\lim_{n\to\infty} a_n = L$ converges if L is ot DNE, $-\infty$ or ∞





- We can also say that $\{a_n\}$ converges if L is trapped within certain error bound ϵ if N is big enough.



Convergence

Examples.

- Converges
  - $\left\{\frac{1}{n}\right\}_{n}^{\infty}$ since $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$
- Diverges
  - $\{2^n\}_{n=1}^{\infty}$ since $\displaystyle\lim_{n\to\infty}2^n=\infty$
- Bounded bu still diverges
  - $\{(-1)^n\}_{n=1}^{\infty}$ since it alternates between -1 and 1

Example

Does $\{a_n\} = \left\{\frac{\ln(1+2e^n)}{n}\right\}_{n=1}^{\infty}$ converges?

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Suppose $a_n = f(n)$ for some function f, where $n = 1, 2, 3, \cdots$. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

We can first look at the sequence as a function and use L'Hospital's Rule or other tricks we learn before to show that the limit exists.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\ln(1+2e^x)}{x} = \frac{\infty}{\infty} \quad \text{indeterminate form, use L'Hospital Rule} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+2e^x}\cdot 2e^x}{1} = \frac{\infty}{\infty} \quad \text{indeterminate form, use L'Hospital Rule}\textit{again} \\
&= \lim_{x\to\infty} \frac{2e^x}{2e^x} = 1
\end{aligned}$$

Thus this sequence converges to 1.

Example

Does $\{a_n\} = \left\{ \frac{\cos n + \sin n}{n^{\frac{2}{3}}} \right\}$ converges?

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Does $\{a_n\} = \left\{ \frac{\cos n + \sin n}{n^{\frac{2}{3}}} \right\}$ converges?

Here we can use the Squeeze Theorem.

- We know $-2 \leq \cos(n) + \sin(n) \leq 2$, hence
- $\frac{-2}{n^{\frac{2}{3}}} \leq \frac{\cos(n)+\sin(n)}{n^{\frac{2}{3}}} \leq \frac{-2}{n^{\frac{2}{3}}}$, hence
- Since $\lim\limits_{n\to\infty} \dfrac{-2}{n^{\frac{2}{3}}} = \lim\limits_{n\to\infty} \dfrac{-2}{n^{\frac{2}{3}}} = 0$, thus $\lim\limits_{n\to\infty} \dfrac{\cos(n) + \sin(n)}{n^{\frac{2}{3}}} = 0$

Example

Does $\{0.1, 0.12, 0.123, 0.1234, \cdots, 0.12345\}$ converges?

Example

Does $\{0.1, 0.12, 0.123, 0.1234, \cdots, 0.12345\}$ converges?

If $\{a_n\}$ is bounded (not smaller or larger than some values) and monotonic (either increasing or decreasing), then it converges.

But what number it converges can be mysterious. (Actually, it is *champernowne* constant.)

Example

Does $\{r^n\}$ converges?

Example

Does $\{r^n\}$ converges?

- If $r > 1$, it diverges to ∞
- If $r = 1$, it converges to 1
- If $0 \leq r < 1$, it converges to 0
- If $-1 < r < 0$, it converges to 0
- If $r = -1$, it alternates between -1 and 1, thus it diverges
- If $r < -1$, it diverges in two direction, thus DNE

If we multiply a with the limit, this is also true.

- $\{ar^n\}$ converges to 0 when $-1 < r < 1$
- $\{ar^n\}$ converges to a when $r = 1$
- $\{ar^n\}$ diverges when $r < -1$ or $r > 1$

Example

Does $\left\{ \frac{(-1)^t e^{t-1}}{3^{t+2}} \right\}_{t=3}^{\infty}$ converges?

Example

Does $\left\{\frac{(-1)^t e^{t-1}}{3^{t+2}}\right\}_{t=3}^{\infty}$ converges?

- Let's simplify to $\frac{(-1)^t e^t e^{-1}}{3^t 3^2} = (\frac{-e}{3})^t \cdot \frac{1}{3^2 e}$
- Here we can see that $a = \frac{1}{3^2 e}$ and $r = \frac{-e}{3}$
- Here $-1 < r < 0$, thus it converges to 0

Example

Does $\left\{\frac{x-5}{x^2} - \frac{3 \cdot 4^x}{5^x}\right\}_{x=1}^{\infty}$ converge?

$$= \lim_{x\to\infty} \frac{x-5}{x^2} - \lim_{x\to\infty} \frac{3 \cdot 4^x}{5^x} \qquad \text{r in second term is } \frac{4}{5}$$

$$= 0 - 0 = 0 \qquad \text{using limit law}$$

Thus it converges to 0.

Exercise

Is these sequences convergent or divergent?

- $\frac{n}{\sqrt{10+n}}$
  - Diverges to ∞
- $\frac{\ln n}{n}$
  - Converges to 0 using L'Hospital Rule
- $\frac{3+5n^2}{n+n^2}$
  - Converges to 5
- $\frac{4^n}{1+9^n}$
  - Converges to 0