Week 9: Integration Techniques Part 1

October 26, 2021

1. Integration by parts

2. Trigonometric integrals ($\sin^m x \cos^n x$)

Integration by parts

Let's start with the product rule:

$$\begin{aligned}(f(x)g(x))' &= f'(x)g(x) + f(x)g'(x)\\ f(x)g'(x) &= (f(x)g(x))' - f'(x)g(x)\end{aligned}$$

Integrating both sides, we get:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

If we let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we obtain an equivalent (perhaps easier to memorize) formula:

$$\int u\,dv = uv - \int v\,du.$$

Example

Evaluate $\int xe^x \, dx$.

Example

Evaluate $\int xe^x \, dx$.

Let $u = x$, $dv = e^x \, dx$, $du = dx$, and $v = e^x$

$$\int u \, dv = uv - \int v \, du$$
$$\int xe^x \, dx = xe^x - \int e^x \, dx$$
$$= xe^x - e^x + C$$

Example

Evaluate $\int x \sin x \, dx$.

Example

Evaluate $\int x \sin x \, dx$.

Let $u = x$, $dv = \sin x \, dx$, $du = dx$, and $v = -\cos x$

$$\int u \, dv = uv - \int v \, du$$
$$\int xe^x \, dx = x(-\cos x) - \int (-\cos x) \, dx$$
$$= -x \cos x + \sin x + C$$

Exercise

1. $\int \ln x \, dx$
   - Ans: $x \ln x - x + C$
2. $\int t^2 e^t \, dt$
   - Ans: $t^2 e^t - 2te^t + 2e^t + C$
3. $\int (x^2 + 2x) \cos x \, dx$
   - $(x^2 + 2x) \sin x + (2x + 2) \cos x - 2 \sin x + C$

$$\int \sin^m x \cos^n x \, dx.$$

There are two cases to consider. The first case is if **at least one** of m or n is odd, and the second case is if **both** m and n are even.

Case 1: Integrals with an odd power of $\sin x$ or an odd power of $\cos x$.

- If there is an odd power of $\sin x$ (i.e. if m above is odd), we save one factor of $\sin x$ and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$. Then substitute $u = \cos x$.
- If there is an odd power of $\cos x$ (i.e. if n above is odd), we save one factor of $\cos x$ and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of $\sin x$. Then substitute $u = \sin x$.
- If the powers of both $\sin x$ and $\cos x$ are odd, we may use either of the above strategies

Example

Evaluate $\displaystyle\int \cos^3 x \, dx$.

Example

Evaluate $\displaystyle\int \cos^3 x\, dx$.

- $\displaystyle\int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x\, dx = \int 1 - \sin^2 x \cdot \cos x\, dx$
- Let $u = \sin x$ and $du = \cos x\, dx$
- $\displaystyle\int (1 - u^2)\, du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$

Example

Evaluate $\int \sin^5 x \cos^2 x \, dx$.

Example

Evaluate $\int \sin^5 x \cos^2 x \, dx$.

- $\int \sin^5 x \cos^2 x \, dx = \int (\sin^2 x)^2 \cos^2 x \sin x \, dx = (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$
- Let $u = \cos x$ and $du = -\sin x \, dx$
- $\int (1 - u^2)^2 u^2 (-\, du) = -\int (u^2 - 2u^4 + u^6) \, du = -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

Exercise

Evaluate $\int \sin^2 x \cos^3 x \, dx$.

Exercise

Evaluate $\int \sin^2 x \cos^3 x\, dx$.

- $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x (1 - \sin^2 x) \cos x\, dx$.
- Let $u = \sin x$ and $du = \cos x\, dx$
- $\int u^2(1 - u^2)\, du = \int (u^2 - u^4)\, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

Case 2: Integrals where both powers of $\sin x$ and $\cos x$ are even.

We use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is also useful to use the identity:

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Example

Evaluate $\displaystyle\int \sin^2 x \, dx$.

Example

Evaluate $\int \sin^2 x \, dx$.

$$\begin{aligned}\int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2}(x - \frac{1}{2} \sin 2x)\end{aligned}$$

Example

Evaluate $\int \sin^2 t \cos^4 t \, dt$.

Example

Evaluate $\int \sin^2 t \cos^4 t \, dt$.

$$\begin{aligned}
\int \sin^2 t \cos^4 t \, dt &= \frac{1}{4} \int (4 \sin^2 t \cos^2 t) \cos^2 t \, dt \\
&= \frac{1}{4} \int (2 \sin t \cos t)^2 \frac{1}{2}(1 + \cos 2t) \, dt && \text{[half-angle identity]} \\
&= \frac{1}{8} \int (\sin 2t)^2 (1 + \cos 2t) \, dt && \text{[identity]} \\
&= \frac{1}{8} \int (\sin^2 2t + \sin^2 2t \cos 2t) \, dt \\
&= \frac{1}{8} \int \sin^2 2t \, dt + \frac{1}{8} \int \sin^2 2t \cos 2t \, dt \\
&= \frac{1}{8} \int \frac{1}{2}(1 - \cos 4t) \, dt + \frac{1}{8} \left(\frac{\sin^3 2t}{6} \right) && \text{[Sub. u} = \sin 2t\text{]} \\
&= \frac{1}{16} \left(t - \frac{1}{4} \sin 4t \right) + \frac{1}{8} \left(\frac{\sin^3 2t}{6} \right)
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \cos^4 2t \, dt$.

Exercise

Evaluate $\int \cos^4 2t\, dt$.

$$\begin{aligned}
\int \cos^4 2t\, dt &= \int (\cos^2 2t)^2 \\
&= \int \left(\frac{1}{2}(1+\cos 4t)\right)^2 && \text{[half-angle identity]} \\
&= \int \frac{1}{4}(1 + 2\cos 4t + \cos^2 4t)\, dt \\
&= \frac{1}{4}\int (1 + 2\cos 4t + \frac{1}{2}(1+\cos 8t))\, dt && \text{[half-angle identity]} \\
&= \frac{1}{4}\int (\frac{3}{2} + 2\cos 4t + \frac{1}{2}\cos 8t)\, dt \\
&= \frac{1}{4}\left(\frac{3}{2}t + \frac{1}{2}\sin 4t + \frac{1}{16}\sin 8t\right)
\end{aligned}$$