Week 4: Related Rates, Maxima, Derivative Tests

October 11, 2021

Related rates

Related rates concern **real-world problems** related to **derivatives**.

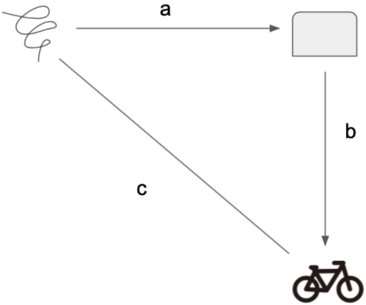
Strategy for solving related rates problems.

1. Draw a diagram
2. Assign variables
3. Express the problem in terms of derivatives

They are not easy, but the key is at least know that derivatives can be used to solve such problem.

Example

A tornado is 20 km west of us, heading due east towards your house at a rate of 40 km/h. You take your bicycle and ride due south at a speed of 12 km/h. How fast is the distance between you and the tornado changing after 15 minutes?



Example

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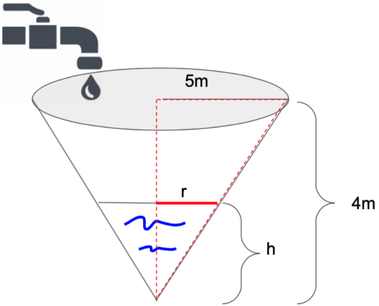
$$a^2 + b^2 = c^2$$
$$\frac{d}{dt}(a^2 + b^2) = \frac{d}{dt}(c^2)$$
$$2a * \frac{da}{dt} + 2b * \frac{db}{dt} = 2c * \frac{dc}{dt}$$

$\frac{da}{dt} = -40; \frac{db}{dt} = 12$, when $t = 0.25$ hrs, $a = 10, b = 3, c^2 = 10^2 + 3^2 = \sqrt{109}$

$$2(10) * (-40) + 2(3) * 2 = 2 * \sqrt{109} * \frac{dc}{dt}$$
$$\frac{dc}{dt} = -35\text{km/h}$$

Example

Water flows into a tank at a rate of 3 cubic meters per minute. The tank is shaped like a cone with a height of 4 meters and a radius of 5 meters at the top. Find the rate at which the water level is rising in the tank when the water height is 2 meters. Find $\dfrac{dh}{dt}$.



Example

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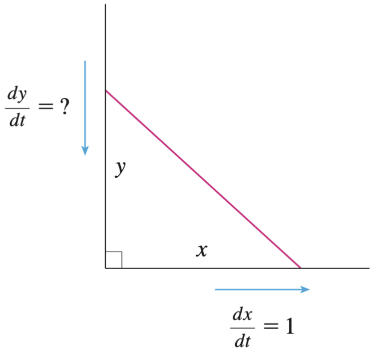
$$\begin{aligned}
V &= \frac{1}{3}(\text{area})(\text{height}) = \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\frac{25}{16}h^3 = \frac{25}{48}\pi h^3 && \left(\frac{r}{h} = \frac{5}{4} : \text{Properties of similar triangles}\right) \\
\frac{dV}{dt} &= \frac{25}{48}\pi 3h^2\frac{dh}{dt} \\
3 &= \frac{25}{48}\pi 3(2)^2\frac{dh}{dt} \\
\frac{12}{25}\pi &= \frac{dh}{dt} && (\text{meter per minute})
\end{aligned}$$

Self-Exercise

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Self-Exercise

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$$
\begin{aligned}
x^2 + y^2 &= 100 \\
2x\frac{dx}{dt} = 2y\frac{dy}{dt} &= 0 \\
\frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt}
\end{aligned}
$$

When $x = 6$, $y = 8$. Also, we know $\frac{dx}{dt} = 1$, thus

$$\frac{dy}{dt} = -\frac{6}{8} * 1 = -\frac{3}{4}\text{ft/s}$$

Definition

A function $f(x)$ has an **absolute maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in the **domain** of f.

Definition

A function $f(x)$ has an **absolute minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in the **domain** of f.

Definition

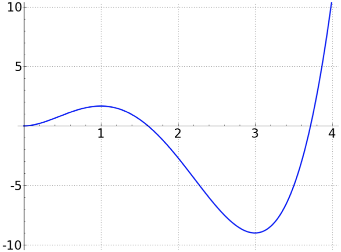
A function $f(x)$ has an **local (relative) maximum** at $x = c$ if $f(c) \geq f(x)$ for all x **near** c.

Definition

A function $f(x)$ has an **local (relative) minimum** at $x = c$ if $f(c) \leq f(x)$ for all x **near** c.

Example

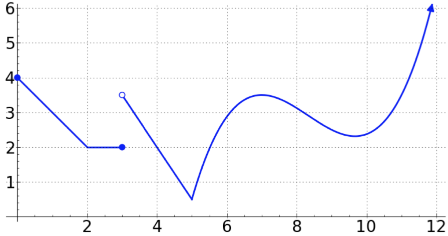
Where is the absolute minimum, maximum; local minimum, maximum?



Abs. min: (3, -8) Abs. max: (4, 10)
Local min: (3,-8) Local max: (1, 2)
Note: (4,10) is NOT a local max because f(x) is not defined on an open interval around 4

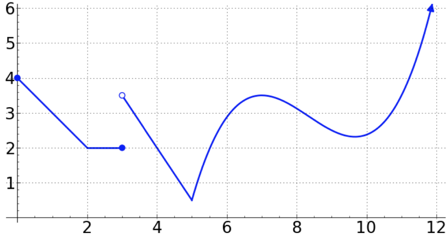
Exercise

Where is the absolute minimum, maximum; local minimum, maximum?



Exercise

Where is the absolute minimum, maximum; local minimum, maximum?



Abs. min: (5, 0.5)
Abs. max: None because the graph keeps going up
Local min: (2, 2) to (3, 2); (5, 0.5); (9.8, 2,2)
Local max: (7, 3.5)
Note: (0,4) is NOT a local max because f(x) is not defined on an open interval
Note: (3,3.5) is NOT a local max because is not defined, and also, there is always a higher value if you keep close in on (3, 3.5)

Critical Number

Definition

A number c is critical number for a function c if $f'(c)$ DNE or $f'(c) = 0$.

Proposition

If f has a local max or min at $x = c$, then c must be a critical number for f.

If c is a critical number, $f(c)$ may or may not be a local max or min (e.g. $f(x) = x^3$ at $c = 0$).

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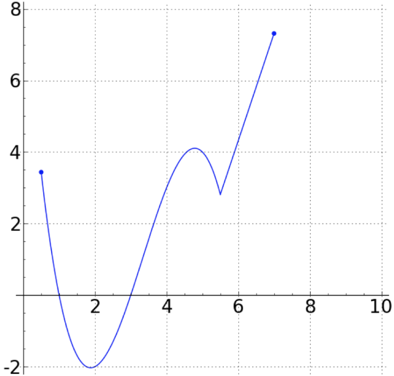

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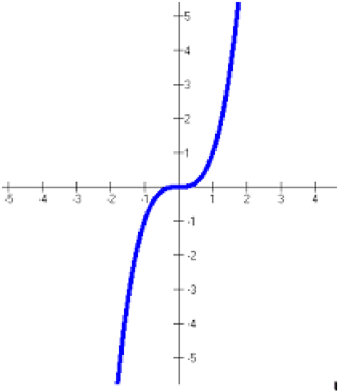

Example

Find critical numbers of $f(x) = x^{3/5}(4 - x)$

Example

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$$f'(x) = \frac{12-8x}{5x^{2/5}}$$

Find $f'(x) = 0$ or $f'(x) =$DNE.

$$f'(x) = 0 \text{ when } x = \frac{3}{2}$$

$$f'(x) = \text{DNE when } x = 0$$

Exercise

Find critical numbers of $g(t) = |3t - 4|$

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Find critical numbers of $g(t) = |3t - 4|$

$$g(t) = \begin{cases} 3t - 4 & \text{if } 3t - 4 \geq 0 \\ -(3t - 4) & \text{if } 3t - 4 < 0 \end{cases}$$

$$g(t) = \begin{cases} 3t - 4 & \text{if } t \geq \frac{4}{3} \\ -(3t - 4) & \text{if } t < \frac{4}{3} \end{cases}$$

$$g'(t) = \begin{cases} 3 & \text{if } t \geq \frac{4}{3} \\ -3 & \text{if } t < \frac{4}{3} \end{cases}$$

Since the derivative is a sharp corner at $\frac{4}{3}$, thus $f'(\frac{4}{3})$ =DNE, thus $\frac{4}{3}$ is the critical number.

Absolute Maximum/Minimum

Proposition (The Closed Interval Method)

To find the absolute maximum and minimum values (also called extreme values) of a continuous function f on a closed interval [a,b]:

1. *Find the values of f at the critical numbers of f in $[a, b]$.*
2. *Find the values of f at the endpoints of the interval.*
3. *The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example

Find the absolute minimum and maximum of $f(x) = x^3 - 3x^2 + 1$ in $[-\frac{1}{2}, 4]$

Example

Find the absolute minimum and maximum of $f(x) = x^3 - 3x^2 + 1$ in $[-\frac{1}{2}, 4]$

- $f'(x) = 3x(x-2)$
- The critical numbers are 0 and 2
- The values at critical numbers *inside the interval* are $f(0) = 1$ and $f(2) = -3$
- The values at endpoints are $f(-\frac{1}{2}) = \frac{1}{8}$ and $f(4) = 17$
- Comparing these four numbers, $f(4) = 17$ is the absolute max, and $f(2) = -3$ is the absolute min.

Exercise

Find the absolute minimum and maximum of $f(x) = \dfrac{x-1}{x^2+x+2}$ in $[0, 4]$

Exercise

Find the absolute minimum and maximum of $f(x) = \dfrac{x-1}{x^2+x+2}$ in $[0,4]$

- $f'(x) = \dfrac{-x^2+2x+3}{(x^2+x+2)^2}$
- The critical numbers are 3 and -1
- The values at critical numbers *inside the interval* is $f(3) = \dfrac{1}{7}$
- The values at endpoints are $f(0) = -\dfrac{1}{2}$ and $f(4) = \dfrac{3}{22}$
- Comparing these three numbers, $f(3) = \dfrac{1}{7}$ is the absolute max, and $f(0) = -\dfrac{1}{2}$ is the absolute min.

The Mean Value Theorem

Theorem (The Mean Value Theorem)

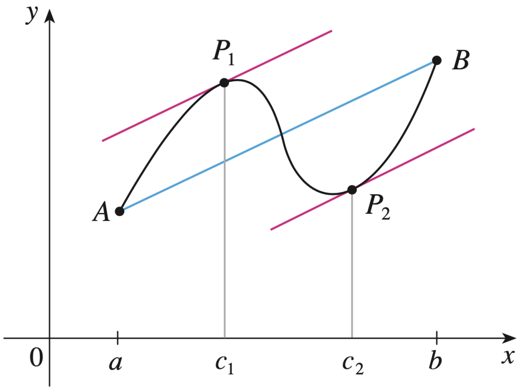
Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \textit{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

Example



Example

Verify the mean value theorem for $f(x) = 2x^3 - 8x + 1$ on the interval $[1, 3]$.

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Verify the mean value theorem for $f(x) = 2x^3 - 8x + 1$ on the interval $[1, 3]$.

- f is cont on $[1, 3]$
- f is differentiable on $(1, 3)$

Find c in [1, 3] such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$.

$$\begin{aligned} f'(c) &= \frac{f(3) - f(1)}{3 - 1} \\ 6c^2 - 8 &= \frac{31 - (-5)}{2} \\ 6c^2 - 8 &= 18 \\ 6c^2 &= 24 \\ c^2 &= 4 = \pm 2 \end{aligned}$$

$c = 2$ is the answer because it falls inside the interval $[1, 3]$.

Example

If f is a differentiable function and $f(1) = 7$ and $-3 \leq f'(x) \leq -2$ on the interval $[1, 6]$, then what is the biggest and smallest values that are possible for $f(6)$?

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If f is a differentiable function and $f(1) = 7$ and $-3 \leq f'(x) \leq -2$ on the interval $[1, 6]$, then what is the biggest and smallest values that are possible for $f(6)$?

The MVT tells us that $\dfrac{f(6) - f(1)}{6 - 1} = f'(c)$ for some c in $[1, 6]$.

Thus we know that $-3 \leq \dfrac{f(6) - f(1)}{6 - 1} \leq -2$.

We can replace $f(1) = 7$ and get $-3 \leq \dfrac{f(6) - 7}{6 - 1} \leq -2$.

Solving this, we find that $-8 \leq f(6) \leq -3$.

Rolle's Theorem

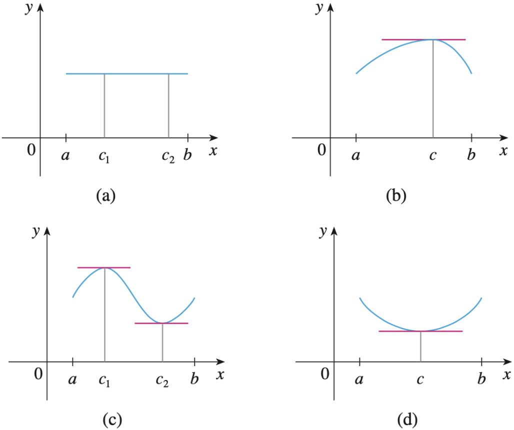
Theorem (Rolle's Theorem)

Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there is a number c in $[a, b]$ such that $f'(c) = 0$

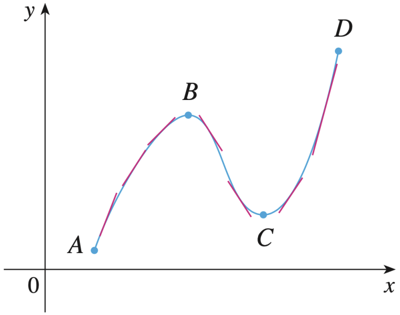
Example



Increasing/Decreasing Test

Test (Increasing/Decreasing Test)

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

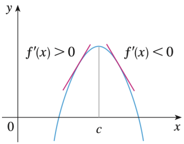


First Derivative Test

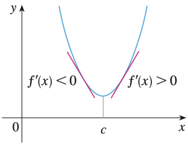
Test (First Derivative Test)

Suppose that c is a critical number of a continuous function f.

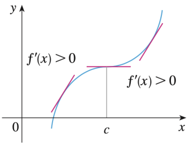
- If f'changes from positive to negative at c, then f has a local max at c.
- If f'changes from negative to positive at c, then f has a local min at c.
- If f' is positive or negative to the left and right of c, f has no local max or min at c.



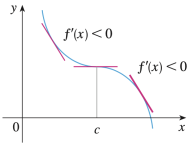
(a) Local maximum



(b) Local minimum



(c) No maximum or minimum

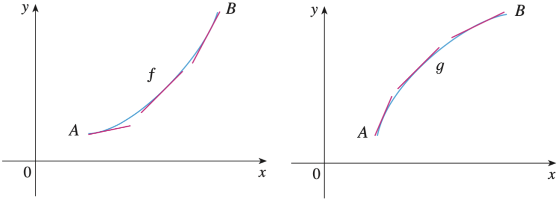


(d) No maximum or minimum

Concavity Test

Test (Concavity Test)

- If $f''(x) > 0$ for all x in an interval, then the graph of f is concave upward on the interval
- If $f''(x) < 0$ for all x in an interval, then the graph of f is concave download on the interval



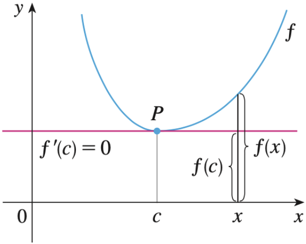
(a) Concave upward (b) Concave downward

Second Derivative Test

Test (Second Derivative Test)

Suppose f'' is continuous near c

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.



Example

Given $f(x) = x^3 - 3x^2 - 9x + 4$, (1) Find intervals on which f is increasing or decreasing. (2) Find the local max and min. (3) Find the intervals of concavity.

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Given $f(x) = x^3 - 3x^2 - 9x + 4$, (1) Find intervals on which f is increasing or decreasing. (2) Find the local max and min. (3) Find the intervals of concavity.

$f'(x) = 3(x+1)(x-3)$ and $f''(x) = 6(x-1)$

1. After trying some values on $f'(x)$ in the interval $(-\infty, -1), (-1, 3), (3, \infty)$, we will find that f is increasing on $(-\infty, -1), (3, \infty)$ and decreasing on $(-1, 3)$
2. f changes from increasing to decreasing at $x = -1$, and from decreasing to increasing at $x = 3$. Thus $f(-1) = 9$ is a local max, and $f(3) = -23$ is a local min.
3. Since $f''(x) > 0$ when $x > 1$ and $f''(x) < 0$ when $x < 1$, thus f is concave upward on $(1, \infty)$ and downward on $(-\infty, 1)$.

Exercise

Given $f(x) = 2x^3 - 9x^2 + 12x - 3$, (1) Find intervals on which f is increasing or decreasing. (2) Find the local max and min. (3) Find the intervals of concavity.

Exercise

Given $f(x) = 2x^3 - 9x^2 + 12x - 3$, (1) Find intervals on which f is increasing or decreasing. (2) Find the local max and min. (3) Find the intervals of concavity.

$f'(x) = 6(x-1)(x-2)$ and $f''(x) = 12(x - \frac{3}{2})$

1. f is increasing on $(-\infty, 1), (2, \infty)$ and decreasing on $(1, 2)$
2. f changes from increasing to decreasing at $x = 1$, and from decreasing to increasing at $x = 2$. Thus $f(1) = 2$ is a local max, and $f(2) = 1$ is a local min.
3. Since $f''(x) > 0$ when $x > \frac{3}{2}$ and $f''(x) < 0$ when $x < \frac{3}{2}$, thus f is concave upward on $(\frac{3}{2}, \infty)$ and downward on $(-\infty, \frac{3}{2})$.

Example

Given $f(x) = 1 + 3x^2 - 2x^3$, find local max and min using (1) First Derivative Test, and (2) Second Derivative Test.

Example

Given $f(x) = 1 + 3x^2 - 2x^3$, find local max and min using (1) First Derivative Test, and (2) Second Derivative Test.

$f'(x) = 6x(1 - x)$ and $f''(x) = 6 - 12x$

1. $f'(x) > 0$ when $0 < x < 1$ and $f'(x) < 0$ when $x < 0$ or $x > 1$. Since f' changes from negative to positive at $x = 0$, $f(0) = 1$ is a local min. Since f' changes from positive to negative at $x = 1$, $f(1) = 2$ is a local max.
2. $f'(x) = 0$ when $x = 0, 1$. Since $f''(0) = 6 > 0$, $f(0) = 1$ is a local min. Since $f''(1) = -6 < 0$, $f(1) = 2$ is a local max.

Exercise

Given $f(x) = \dfrac{x^2}{x-1}$, find local max and min using (1) First Derivative Test, and (2) Second Derivative Test.

Exercise

Given $f(x) = \dfrac{x^2}{x-1}$, find local max and min using (1) First Derivative Test, and (2) Second Derivative Test.

$f'(x) = \dfrac{x(x-2)}{(x-1)^2}$ and $f''(x) = \dfrac{2}{(x-1)^3}$

1. $f'(x) > 0$ when $x < 0$ or $x > 2$ and $f'(x) < 0$ when $0 < x < 1$ or $1 < x < 2$. Since f' changes from positive to negative at $x = 0$, $f(0) = 0$ is a local max. Since f' changes from negative to positive at $x = 2$, $f(2) = 4$ is a local min.
2. $f'(x) = 0$ when $x = 0, 2$. Since $f''(0) = -2 < 0$, $f(0) = 0$ is a local max. Since $f''(2) = 2 > 0$, $f(2) = 4$ is a local min.