Week 13: Series

November 24, 2021

1. Geometric Series

2. Power Series

Series

Definition: Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \text{ or } \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Thus $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$

Geometric series

An important example of an infinite series is the **geometric series**.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

- If $r = 1$, the limit diverges to infinity because it becomes $n \cdot a$
- If $r \neq 1$, then

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n && \text{multiply both sides by r} \\ s_n - rs_n &= a - ar^n && \text{substracting these two equations} \\ s_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

- If $-1 < r < 1$ (same as $|r| < 1$,) then $\lim_{n\to\infty} s_n = \dfrac{a}{1-r}$ since $r^n \to 0$
- If $r \leq 1$ or $r > 1$ (same as $|r| \geq 1$), then $\{r^n\}$ is divergent since $r^n \to$ DNE

Example

Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent?

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Let's rewrite the nth term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty}(2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\frac{4}{3}^{n-1}$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges.

Example

A drug is administered to a patient at the same time every day. Suppose the concentration of the drug is C_n (measured in mg/mL) after the injection on the nth day. Before the injection the next day, only 30% of the drug remains in the bloodstream and the daily dose raises the concentration by 0.2 mg/mL. (a) Find the concentration after three days, (b) what is the concentration after the nth does? (c) what is the limiting concentration?

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(a). The concentration after the next day is

$$\begin{aligned} C_{n+1} &= 0.2 + 0.3C_n \\ C_1 &= 0.2 + 0.3C_0 = 0.2 \\ C_2 &= 0.2 + 0.3(C_1 = 0.2) = 0.26 \\ C_3 &= 0.2 + 0.3(C_2 = 0.26) = 0.278 \end{aligned}$$

continued...

(b) After the nth does the concentration is

$$C_n = 0.2 + 0.2(0.3) + 0.2(0.3)^2 + \cdots + 0.2(0.3)^{n-1}$$

Given $a = 0.2$ and $r = 0.3$, so we have

$$C_n = \frac{a(1-r^n)}{1-r} = \frac{0.2[1-0.3^n]}{1-0.3} = \frac{2}{7}[1-(0.3)^n] \text{ mg/mL}$$

(c) Since $0.3 < 1$, thus $r^n \to 0$, thus

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{2}{7}(1-0) = \frac{2}{7} \text{ mg/mL}$$

Exercise

Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

- $3 - 4 + \frac{16}{3} - \frac{64}{9}$
  - Diverges
- $4 + 3 + \frac{9}{3} + \frac{27}{16}$
  - Converges at 16
- $\displaystyle\sum_{n=1}^{\infty} 12(0.73)^{n-1}$
  - Converges at $\frac{400}{9}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\pi^n}$
  - Converges at $\frac{5}{\pi - 1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$
  - Diverges

Power Series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series. For each fixed x, the series is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of x and diverge for other values of x. For example, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$
A **power series centered at** a can be written in the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

Ratio Test

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent
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- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the series $\sum_{n=1}^{\infty} a_n$ is inconclusive

Example

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$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{(-1)^n\frac{n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is convergent.

Example

Let's try it on power series. Is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

Thus, the series diverges when $x \neq 0$ and converges only when $x = 0$

Example

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$
$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \qquad \text{as } n\to\infty$$

By the Ratio Test, the given series is convergent when $|x-3|<1$ and divergent when $|x-3|>1$. Now $|x-3|<1 \Leftrightarrow -1<x-3<1 \Leftrightarrow 2<x<4$, thus the series converges when $2<x<4$ and diverges when $x<2$ or $x>4$.

Radius of convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

- The series converges only when $x = a$
- The series converges for all x
- There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

The number R is called **radius of convergence**.

Example

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}|x| \to 3|x| \qquad \text{as } n\to\infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

Example

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

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$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$
$$= \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \qquad \text{as } n\to\infty$$

By the Ratio Test, the given series converges if $\frac{|x+2|}{3} < 1$ and diverges if $\frac{|x+2|}{3} > 1$. Thus it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. This means that the radius of convergence is $R = 3$.

Exercise

Find the radius of convergence.

- $\displaystyle\sum_{n=1}^{\infty}(-1)^n n x^n$
  - Ans: 1
- $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{2n-1}$
  - Ans: 1
- $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{\sqrt{n}}x^n$
  - Ans: $\frac{1}{4}$
- $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$
  - Ans: 1