Week 5: Approximation

October 11, 2021

1. Linear Approximation
2. Differential Approximation
3. L'Hospital's Rule
   - Quotient
   - Products
   - Differences
   - Powers

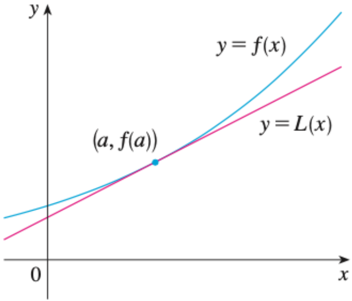
1. Linear Approximation
2. Differential Approximation
3. L'Hospital's Rule
   - Quotient
   - Products
   - Differences
   - Powers

Linear Approximation

Definition

We could use tangent line at $x = a$ to predict $f(x)$.

$$L(x) = f(x) \approx f(a) + f'(a)(x - a)$$



Example

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Example

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

- $f'(x) = \dfrac{1}{2\sqrt{x+3}}$
- So we have $f(1) = 2$ and $f'(1) = \dfrac{1}{4}$
- Putting into the linearization equation, we get $L(x) = f(1) + f'(1)(x-1) = 2 + \dfrac{1}{4}(x-1) = \dfrac{7}{4} + \dfrac{x}{4}$
- Thus $\sqrt{x+3} \approx \dfrac{7}{4} + \dfrac{x}{4}$
- For $\sqrt{3.98}$, $\sqrt{3.98} \approx \dfrac{7}{4} + \dfrac{0.98}{4} = 1.995$
- For $\sqrt{4.05}$, $\sqrt{4.05} \approx \dfrac{7}{4} + \dfrac{1.05}{4} = 2.0125$

Example

Use linear approximation to estimate $\sqrt{59}$ without a calculator.

Example

Use linear approximation to estimate $\sqrt{59}$ without a calculator.

- Let $f(x) = \sqrt{x}$
- Let $a = 64$ since $\sqrt{64}$ is easy
- $f'(x) = \dfrac{1}{2\sqrt{x}}$
- So we have $f(64) = 8$ and $f'(64) = \dfrac{1}{16}$
- Putting into the linearization equation, we get
  $L(x) = f(64) + f'(64)(x - 64) = 8 + \dfrac{1}{16}(x - 64) = 4 + \dfrac{x}{16}$
- Thus $\sqrt{x} \approx 4 + \dfrac{x}{16}$
- For $\sqrt{59}$, $\sqrt{59} \approx 4 + \dfrac{59}{16} = 7.6875$

Exercise

Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$. Then use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

Exercise

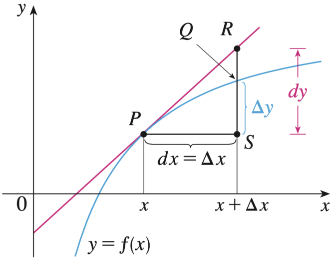
Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$. Then use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

- $f'(x) = -\dfrac{1}{2\sqrt{1-x}}$
- So we have $f(0) = 1$ and $f'(0) = -\dfrac{1}{2}$
- Putting into the linearization equation, we get $L(x) = f(0) + f'(0)(x-0) = 1 + -\dfrac{1}{2}(x-0) = 1 - \dfrac{1}{2}x$
- Thus $\sqrt{0.9} = \sqrt{1-0.1} \approx 1 - \dfrac{1}{2}(0.1) = 0.95$
- Thus $\sqrt{0.99} = \sqrt{1-0.01} \approx 1 - \dfrac{1}{2}(0.01) = 0.995$

Differential Approximation

Definition

Given $\dfrac{dy}{dx} = f'(x)$, we know $dy = f'(x)dx$. On a given graph below, let $dx = \Delta x$. The corresponding change in y is $\Delta y = f(x + \Delta x) - f(x)$. The dy is the amount in which the tangent line rises, while Δy is the amount in which $y = f(x)$ falls. Many times, dy can be used to approximate Δy.



Example

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05

Example

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05

- $f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$
- $f(2.05) = 2.05^3 + 2.05^2 - 2(2.05) + 1 = 9.717625$
- $\Delta y = f(2.05) - f(2) = 0.717625$
- $dy = f'(x)dx = (3x^2 + 2x - 2)dx$
- Given $x = 2$ and $dx = 0.05$,
  $dy = (3(2)^2 + 2(2) - 2)0.05 = 0.7$

We can see that dy can be used to approximate Δy.

Exercise

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.01

Exercise

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.01

- $f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$
- $f(2.01) = 2.01^3 + 2.01^2 - 2(2.01) + 1 = 9.140701$
- $\Delta y = f(2.01) - f(2) = 0.140701$
- $dy = f'(x)dx = (3x^2 + 2x - 2)dx$
- Given $x = 2$ and $dx = 0.01$,
  $dy = (3(2)^2 + 2(2) - 2)0.01 = 0.14$

We can see that dy is closer to Δy as dx gets smaller. For more complicated functions, it may not be possible to compute Δy thus such approximation is useful.

Indeterminant Forms

The limits of the form $\frac{\infty}{\infty}$ (also called '*indeterminate forms*') that we studied so far can be calculated using algebraic trick. What about the following limits?

$$\lim_{x\to\infty} \frac{e^x}{x} \qquad \lim_{x\to\infty} \frac{\ln x}{x}$$

One way to solve this problem is to use the derivative.

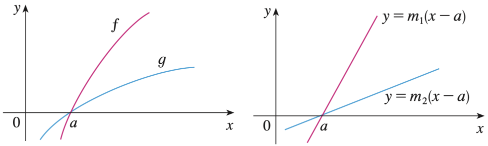


Figure 1: If we zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. If the graph is actually linear, their ratio would be
$\dfrac{m_1(x-a)}{m_2(x-a)} = \dfrac{m_1}{m_2}$ which is the ratio of their derivatives. Thus
$\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

L'Hôpital's rule

Suppose that f and g are differentiable functions, $a \in \mathbb{R}$, and $g'(x) \neq 0$, except possibly at a. Suppose also that either one of the two following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
- $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$.

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists or is $\pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Remarks.

(i) The theorem also holds for limits at infinity or one-sided limits, That is, as $x \to \infty$ or $x \to -\infty$, or $x \to a^+$ or as $x \to a^-$.

(ii) When using L'Hospital's Rule, we **do not** use the quotient rule. We differentiate the numerator and the denominator **separately**.

(iii) Be sure to verify that the hypotheses in L'Hospital's rule are satisfied before applying it!

(iv) The rule can be applied multiple times.

(v) Can convert the rule for indeterminate form in products, differences, or powers.

Example

Solve $\lim_{x \to 1} \frac{\ln x}{x - 1}$

This is an indeterminate form of type $\frac{0}{0}$

Example

Solve $\lim_{x\to 1} \frac{\ln x}{x-1}$

This is an indeterminate form of type $\frac{0}{0}$

- $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(x-1)}$
- $\lim_{x\to 1} \frac{\frac{1}{x}}{1}$
- $\lim_{x\to 1} \frac{1}{x} = 1$

Example

Solve $\lim_{x \to \infty} \frac{e^x}{x^2}$

This is an indeterminate form of type $\frac{\infty}{\infty}$

Example

Solve $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2}$

This is an indeterminate form of type $\dfrac{\infty}{\infty}$

- $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \lim\limits_{x\to 1} \dfrac{\frac{d}{dx}e^x}{\frac{d}{dx}x^2} = \lim\limits_{x\to\infty} \dfrac{e^x}{2x}$

We still get indeterminate form of $\dfrac{\infty}{\infty}$. Apply the rule again!

- $\lim\limits_{x\to\infty} \dfrac{e^x}{2x} = \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \infty$

Exercise

Solve $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

This is an indeterminate form of type $\dfrac{\infty}{\infty}$

Exercise

Solve $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

This is an indeterminate form of type $\dfrac{\infty}{\infty}$

- $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$

We still get indeterminate form of $\dfrac{0}{0}$. Let's just simplify it and not apply the rule.

- $\lim_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

Indeterminate forms with products. Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then what is $\lim_{x \to a} f(x)g(x)$? This is called an indeterminate form of type $0 \cdot \infty$. We apply L'Hospital's rule after first writing $fg = \frac{f}{1/g}$ or $fg = \frac{g}{1/f}$.

Example

Solve $\lim_{x \to 0^+} x \ln x$

This is an indeterminate form of type $0^+ * -\infty$

Example

Solve $\lim_{x \to 0^+} x \ln x$

This is an indeterminate form of type $0^+ * -\infty$

- $$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$$

Exercise

Solve $\lim\limits_{x\to\infty} \sqrt{x}e^{\frac{-x}{2}}$
This is an indeterminate form of type $\infty * 0$

Exercise

Solve $\lim\limits_{x\to\infty} \sqrt{x}e^{\frac{-x}{2}}$

This is an indeterminate form of type $\infty * 0$

- $\lim\limits_{x\to\infty} \sqrt{x}e^{\frac{-x}{2}} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{e^{\frac{x}{2}}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}x^{\frac{-1}{2}}}{\frac{1}{2}e^{\frac{x}{2}}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}e^{\frac{x}{2}}} = 0$

Indeterminate forms with differences. Now consider what happens if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, and we are looking at $\lim_{x \to a} [f(x) - g(x)]$. This is an indeterminate form of type $\infty - \infty$. We examine these by **converting them into a quotient** and using L'Hospital's rule.

Example

Solve $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$
This is an indeterminate form of type $\infty - \infty$

Example

Solve $\lim\limits_{x\to 1^+}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$
This is an indeterminate form of type $\infty - \infty$

First we can make common denominator:

- $\lim\limits_{x\to 1^+}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right) = \lim\limits_{x\to 1^+}\dfrac{x-1-\ln x}{(x-1)\ln x}$

We will get the form $\dfrac{0}{0}$, so let's apply the rule!

- $\lim\limits_{x\to 1^+}\dfrac{x-1-\ln x}{(x-1)\ln x} = \lim\limits_{x\to 1^+}\dfrac{1-\frac{1}{x}}{(x-1)\frac{1}{x}+\ln x} = \lim\limits_{x\to 1^+}\dfrac{x-1}{x-1+x\ln x}$

We will still get the form $\dfrac{0}{0}$, so apply the rule again!

- $\lim\limits_{x\to 1^+}\dfrac{x-1}{x-1+x\ln x} = \lim\limits_{x\to 1^+}\dfrac{1}{1+x\frac{1}{x}+\ln x} = \lim\limits_{x\to 1^+}\dfrac{1}{2+\ln x} = \dfrac{1}{2}$

Exercise

Solve $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$
This is an indeterminate form of type $\infty - \infty$

Exercise

Solve $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$
This is an indeterminate form of type $\infty - \infty$

First we can make common denominator:

- $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x \to 1} \left(\frac{x \ln x - (x-1)}{(x-1)\ln x}\right)$

We will get the form $\frac{0}{0}$, so let's apply the rule!

- $\lim_{x \to 1} \left(\frac{x \ln x - (x-1)}{(x-1)\ln x}\right) = \lim_{x \to 1} \left(\frac{x(\frac{1}{x}) + \ln x - 1}{(x-1)(\frac{1}{x}) + \ln x}\right) = \lim_{x \to 1} \left(\frac{\ln x}{1 - \frac{1}{x} + \ln x}\right)$

We will still get the form $\frac{0}{0}$, so apply the rule again!

- $\lim_{x \to 1} \left(\frac{\ln x}{1 - \frac{1}{x} + \ln x}\right) = \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$

Indetermininate forms with powers.

Some limits involving powers are difficult to calculate because the variable is in both the base and the index. By taking the **natural logarithm**, i.e., ln, the power is transformed into a product, and the problem becomes manageable.

Example

Solve $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$
This is an indeterminate form of type 1^∞

Example

Solve $\lim_{x\to\infty} (1+\frac{1}{x})^x$
This is an indeterminate form of type 1^∞

First we can apply ln both sides:

- $\ln y = \ln(1+\frac{1}{x})^x$
- $\ln y = x\ln(1+\frac{1}{x})$

We will get the form $\infty * 0$, so let's first make it into quotient.

- $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \dfrac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$

Then apply the rule:

- $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \dfrac{\frac{1}{1+\frac{1}{x}}(-\frac{1}{x^2})}{\frac{-1}{x^2}} = \lim_{x\to\infty} \dfrac{1}{1+\frac{1}{x}} = 1$
- $\lim_{x\to\infty} \ln y = 1$
- $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = \lim_{x\to\infty} e^1 = e$

Exercise

Solve $\lim_{x \to \infty} x^{e^{-x}}$

This is an indeterminate form of type ∞^0

Exercise

Solve $\lim_{x\to\infty} x^{e^{-x}}$

This is an indeterminate form of type ∞^0

First we can apply ln both sides:

- $y = x^{e^{-x}}$
- $\ln y = e^{-x} \ln x$

We will get the form $0 * \infty$, so let's first make it into quotient.

- $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{e^x}$

Then apply the rule:

- $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$
- $\lim_{x\to\infty} \ln y = 0$
- $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$