Week 2: Limit and Continuity

October 11, 2021

1. Limit
   - Limits of functions at a point
   - Limits at infinity
   - Special Trigonometric Limits

2. Continuity
   - Continuous functions
   - The intermediate value theorem

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   - Limits of functions at a point
   - Limits at infinity
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2. Continuity
   - Continuous functions
   - The intermediate value theorem

Meaning of limits

A function may or may not be defined at a particular point, but it can have a limiting value at that point.

Let us consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$.

x	0.9	0.999	1.01	1.0001
$\dfrac{x^2 - 1}{x - 1}$	1.9	1.999	2.01	2.0001

$f(x)$ is not defined at $x = 1$, but as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2. We say that *the limit of $f(x)$ as x approaches to 1, is 2.*

Definition of limits

Definition

The limit of $f(x)$, as x approaches a, equals L is written as

$$\lim_{x \to a} f(x) = L \text{ iff } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

This limit *exists* only if the left sided limit (a^-) and right sided limit (a^+) both exist and are equaled. Another possible way in which limit does not exist is when $L = \pm\infty$. Lastly, this limit does not depend whether $f(x)$ is defined or what $f(x)$ is.

Rules for calculating limits

Basic limit rules. Suppose that $a \in \mathbb{R}$ and that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and are finite real numbers. Then

(i) $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(iii) $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.

A similar set of rules hold for left- and right-hand limits.

Rules for calculating limits

Limit rule for compositions.

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to L} g(x) = g(L)$ then

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

Rules for calculating limits

The squeeze theorem. Suppose that $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

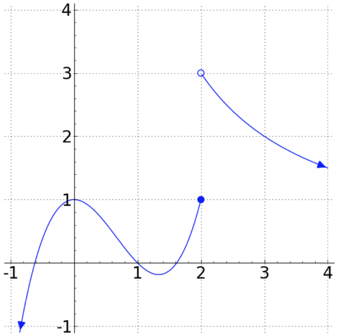
Then

$$\lim_{x \to a} g(x) = L.$$

(Similar versions of the squeeze theorem exist for left- and right-hand limits.)

Example

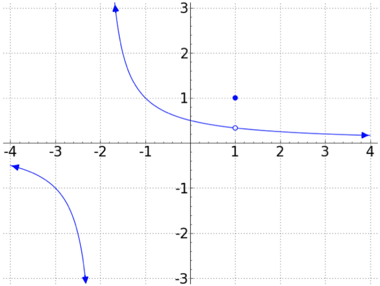
Describe the limit of this graph when x is near 2



$$\lim_{x \to 2^+} f(x) = 3 \qquad \lim_{x \to 2^-} f(x) = 1 \qquad \lim_{x \to 2} f(x) = \text{DNE}$$

Example

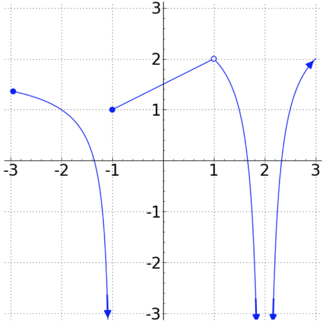
Describe the limit of this graph when x is near -2



$$\lim_{x\to -2^+} f(x) = \infty \qquad \lim_{x\to -2^-} f(x) = -\infty \qquad \lim_{x\to -2} f(x) = \text{DNE}$$

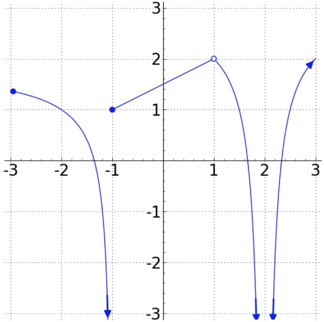
Exercise

Describe the limit of this graph when x is near -1, 1 and 2



Exercise

Describe the limit of this graph when x is near -1, 1 and 2



$\lim_{x\to -1^+} f(x) = 1$ $\lim_{x\to -1^-} f(x) = -\infty$ $\lim_{x\to -1} f(x) =$DNE

$\lim_{x\to 1^+} f(x) = 2$ $\lim_{x\to 1^-} f(x) = 2$ $\lim_{x\to 1} f(x) = 2$

$\lim_{x\to 2^+} f(x) = -\infty$ $\lim_{x\to 2^-} f(x) = -\infty$ $\lim_{x\to 2} f(x) = -\infty$

Example

Find $\lim_{x \to 2} \frac{x^2 + 3x + 6}{x + 9}$

We can directly plug in value as long as the value DOES NOT make the function undefined (based on the Limit Law). Thus the limit is simply $\frac{16}{11}$

Example

Find $\lim_{x \to 0} \frac{|x|}{x}$

As $\lim_{x \to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x \to 0^-} \frac{|x|}{x} = -1$, $\lim_{x \to 0} \frac{|x|}{x}$ doesn't exist.

Example

Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. We cannot also apply the Quotient Law, because the limit of the denominator is 0. Instead, we can factor the numerator: $\dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x - 1} = x + 1$. Thus $\lim_{x \to 1} (x + 1) = 2$.

Example

Find $\lim_{x \to 0} \frac{1}{x^2}$

As x approaches 0 but not 0, f becomes arbitrarily large and positive, thus the limit is ∞ or does not exist (DNE).

Example

Find $\lim_{x \to 0} f(x) = \begin{cases} \sin(x) & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}$

As x approaches 0, both $\lim_{x \to 0^-}$ and $\lim_{x \to 0^+}$ are approaching 0 because $\sin(0) = 0$, thus the limit is 0 even though $f(0) = 5$.

Example

Find $\lim_{x\to 4} f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x = 0 \end{cases}$

we have $\lim_{x\to 4^+} \sqrt{x-4} = 0$ (just substitute)

we also have $\lim_{x\to 4^-} 8-2x = 0$ (just substitute)

Thus the limit is 0.

Example

Find $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$ using squeeze theorem

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$
$$-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$$

Since $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} -x^2 = 0$, thus $\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = 0$

Example

Find $\lim_{x \to 3} \frac{2x}{x-3}$

$$\lim_{x \to 3+} \frac{2x}{x-3} = \infty \text{ (try 3.000001)}$$
$$\lim_{x \to 3-} \frac{2x}{x-3} = -\infty \text{ (try 2.999999)}$$
$$\lim_{x \to 3} \frac{2x}{x-3} = \text{DNE}$$

Self-Exercise

1. $\lim_{x \to -1} \frac{2x+2}{x+1}$
   - Ans: 2
2. $\lim_{x \to 0} |x|$
   - Ans: 0
3. $\lim_{x \to -5} \frac{2x+10}{|x+5|}$
   - Ans: DNE because left limit is -2 and right limit is 2
4. $\lim_{x \to 1} \frac{\sqrt{x+3}-2}{x-1}$ [1]
   - Ans: $\frac{1}{4}$
5. $4\sqrt{x} \leq g(x) \leq 3x^2 - 4x + 5$. Find limit of $g(x)$ approaching 1.
   - Ans: 4

[1](Hint: multiply top and bottom with $\sqrt{x+3}+2$)

Self-Exercise

1. $\lim\limits_{x\to 6} \dfrac{x+2}{x-6}$
   - Ans: DNE because left and right limits are not the same, and also of infinities (positive and negative)
2. $\lim\limits_{x\to -4} \dfrac{5x}{|x+4|}$
   - Ans: ∞. Even both limits are of same sign but it is still infinity, thus DNE

Example

Find $\lim_{x \to \infty} \frac{1}{x}$

As 1 divides by a large number approaches 0, thus the limit is 0

Example

Find $\lim_{x\to\infty} \dfrac{5x^2 - 4x}{2x^3 - 11x^2 + 12x}$

We cannot just put in since we will get $\dfrac{\infty}{\infty}$. Let's factor.

$$= \lim_{x\to\infty} \frac{x^2(5 - \frac{4}{x})}{x^3(2 - \frac{11}{x} + \frac{12}{x^2})}$$

$$= \lim_{x\to\infty} \frac{1}{x} * \frac{5 - \frac{4}{x}}{2 - \frac{11}{x} + \frac{12}{x^2}}$$

$$= \lim_{x\to\infty} 0 * \frac{5 - 0}{2 - 0 + 0}$$

$$= 0$$

Example

Find $\lim\limits_{x\to-\infty} \dfrac{\sin^2 x}{x^3}$ using squeeze theorem

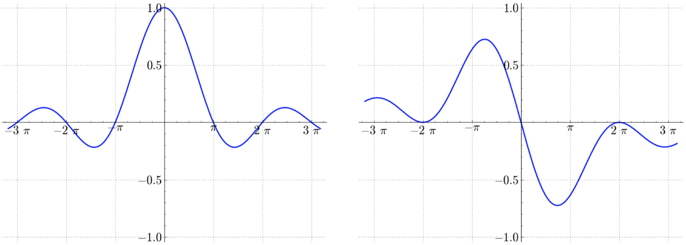
$$0 \leq \sin^2 x \leq 1$$
$$0 \leq \frac{\sin^2 x}{x^3} \leq \frac{1}{x^3}$$

Since $\lim\limits_{x\to-\infty} 0 = 0$ and $\lim\limits_{x\to-\infty} \dfrac{1}{x^3} = 0$, thus the limit is 0.

Self-Exercise

1. $\lim\limits_{x\to-\infty} \dfrac{3x^3+6x^2+10x+2}{2x^3+x^2+5}$
   - Ans: $\dfrac{3}{2}$
2. $\lim\limits_{x\to-\infty} \dfrac{x^4-3x^2+6}{-5x^2+x+2}$
   - Ans: $-\infty$
3. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$ (use squeeze theorem)
   - Ans: 0
4. $\lim\limits_{x\to\infty} e^x$
   - Ans: ∞ or DNE
5. $\lim\limits_{x\to\infty} \sqrt{3+x^2}-x$ (multiply top and bottom with $\sqrt{3+x^2}+x$)
   - Ans: 0

Special Trigonometric Limits



$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

Note: Given the ratio of 1, it means $\sin \theta \approx \theta$ when θ is near 0

Example

Find $\lim_{x \to 0} \frac{\sin 4x}{\sin 6x}$

$$
\begin{aligned}
&= \lim_{x \to 0} \frac{\sin 4x}{\sin 6x} \\
&\approx \lim_{x \to 0} \frac{4x}{6x} \\
&\approx \lim_{x \to 0} \frac{4}{6}
\end{aligned}
$$

Example

Find $\lim_{x\to 0} \dfrac{\tan 7x}{\sin 4x}$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\frac{\sin 7x}{\cos 7x}}{\sin 4x} \\
&= \lim_{x\to 0} \frac{\sin 7x}{\cos 7x} * \frac{1}{\sin 4x} \\
&\approx \lim_{x\to 0} \frac{7x}{\cos 7x} * \frac{1}{4x} \\
&\approx \frac{7}{4} \lim_{x\to 0} \frac{1}{\cos 7x} \\
&\approx \frac{7}{4}
\end{aligned}
$$

Continuous functions

Definition

Suppose that f is defined on some open interval containing the point a. If

$$\lim_{x \to a} f(x) = f(a)$$

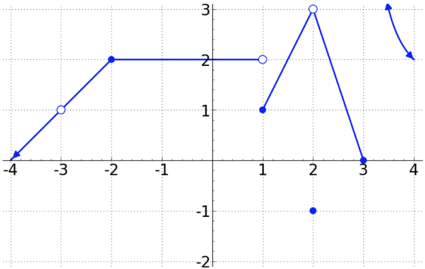
then we say that f is *continuous* at a; otherwise, we say that f is *discontinuous* at a. Continuity can also be described from only one side as follows:

$$\lim_{x \to a-} f(x) = f(a)$$

$$\lim_{x \to a+} f(x) = f(a)$$

Example

What are places where f is not continuous and why?



-3 because $f(-3)$ DNE

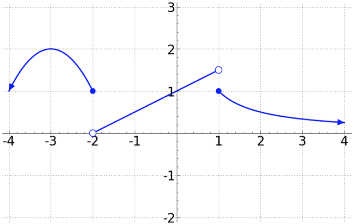
1 because $\lim_{x \to 1} f(x)$ DNE

2 because $\lim_{x \to 2} f(x) \neq f(2)$

3 because $\lim_{x \to 3} f(x)$ DNE

Example

Is it continuous at $x = -2$ and $x = 1$? How about one-sided continuity?



not cont. at $x = -2$ because $\lim_{x \to -2^+} \neq \lim_{x \to -2^-}$ thus limit DNE

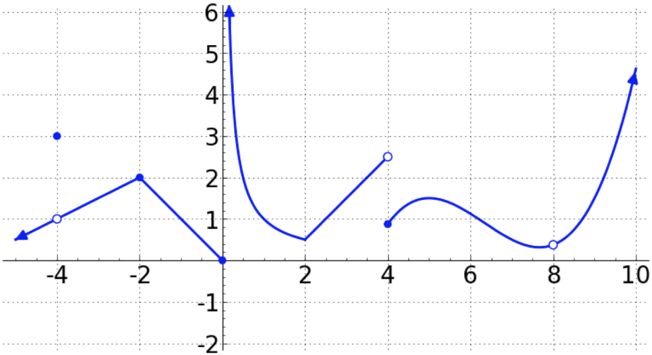
cont at $x = -2^-$ because $\lim_{x \to -2^-} = f(-2) = 1$

not cont. at $x = 1$ because $\lim_{x \to 2^+} \neq \lim_{x \to 1^-}$ thus limit DNE

cont at $x = 1^+$ because $\lim_{x \to 2^+} = f(1) = 1$

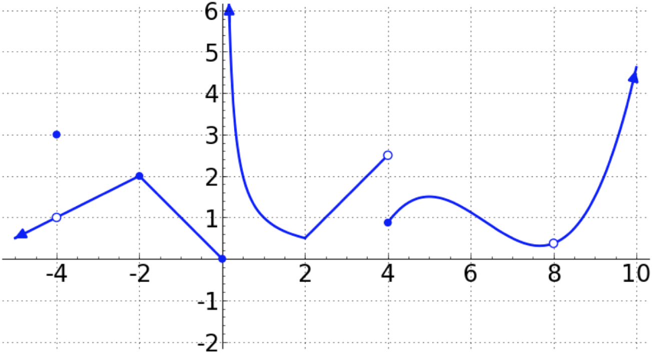
Exercise

Which point is NOT continuous? (both side)



Exercise

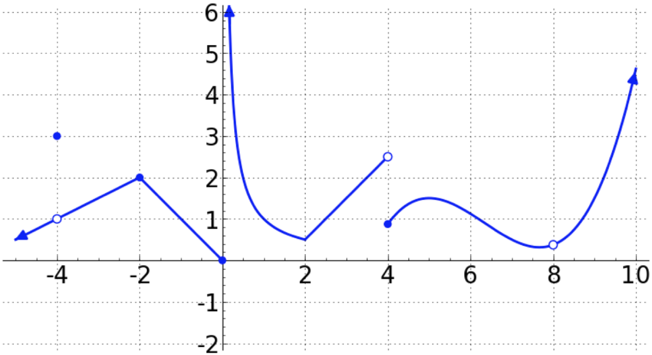
Which point is NOT continuous? (both side)



Ans: -4 (limit $\neq f$), 0 (limit DNE), 4 (limit DNE), 8 (f(8) DNE)

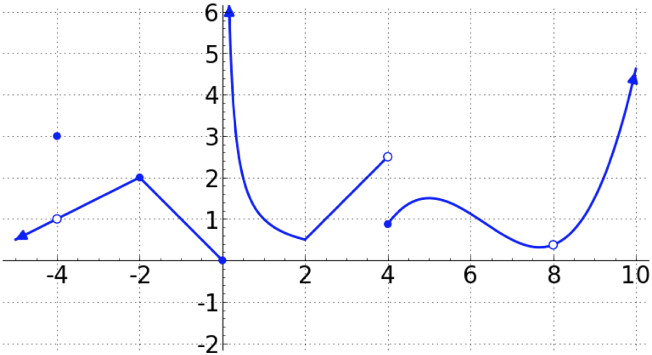
Exercise

Which point is continuous from the (1) right and (2) left?



Exercise

Which point is continuous from the (1) right and (2) left?



Right: 4^+ ; Left: 0^-

Continuous functions at intervals

Definition

Suppose that f is a real-valued function defined on an **open** interval (a, b). We say that f is a *continuous on* (a, b) if f is continuous at every point in the interval (a, b).

Definition

Suppose that f is a real-valued function defined on a **closed** interval $[a, b]$. We say that

(a) f is continuous at the endpoint a if $\lim_{x \to a^+} f(x) = f(a)$,

(b) f is continuous at the endpoint b if $\lim_{x \to b^-} f(x) = f(b)$,

(c) f is continuous on the closed interval $[a, b]$ if f is continuous on the open interval (a, b) and at each of the endpoints a and b.

Combining continuous functions

Proposition

Suppose that the functions f and g are continuous at a point a. Then $f+g$, $f-g$ and fg are continuous at a. If $g(a) \neq 0$ then f/g is also continuous at a.

Proposition

Suppose that f is continuous at a and that g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Corollary

Suppose that f and g are continuous on their domains and that $\lim_{x \to a} f(x)$ belongs to $\mathrm{Dom}(g)$. Then

$$\lim_{x \to a} g(f(x)) = g\big(\lim_{x \to a} f(x)\big).$$

("you can move limits inside continuous functions")

The intermediate value theorem (IVT)

Theorem (The intermediate value theorem)

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one real number c in $[a, b]$ such that $f(c) = z$.

The intermediate value theorem (IVT)

Theorem (The intermediate value theorem)

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one real number c in $[a, b]$ such that $f(c) = z$.

Example. Show that if $f(x)$ is continuous on $[0, 4]$ and $f(0) = 5$ and $f(4) = 1$, then $f(a) = \sqrt{2}$ for some number a

Since $5 < \sqrt{2} < 1$, thus $f(a) = \sqrt{2}$ for some number a, according to the IVT theorem.