Week 3: Derivatives

October 11, 2021

1. Derivatives
   - Meaning and Computation of derivatives
   - Differentiable functions
2. Rules for differentiation
   - Sum, Product, Quotient, Exponential, Chain
   - Inverse Trigonometric Functions
3. Implicit differentiation

1. Derivatives
   - Meaning and Computation of derivatives
   - Differentiable functions

2. Rules for differentiation
   - Sum, Product, Quotient, Exponential, Chain
   - Inverse Trigonometric Functions

3. Implicit differentiation

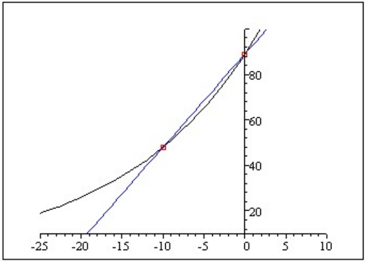
Average rate of change

Definition

Average rate of change of $f(x)$ between $x = a$ and $x = b$

$$= \frac{f(b) - f(a)}{b - a}$$

Average rate of change



Average rate of change of $f(x)$ between $x = -10$ and $x = 0$
$=$ Slope of the **secant line** joining $(-10, 48)$ and $(0, 88)$.

$$= \frac{f(0) - f(-10)}{(0 - (-10))} = \frac{88 - 48}{10} = \frac{40}{10} = 4.$$

Instantaneous rate of change

Definition

Instantaneous rate of change of a function $f(x)$ *at* $x = a$ is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

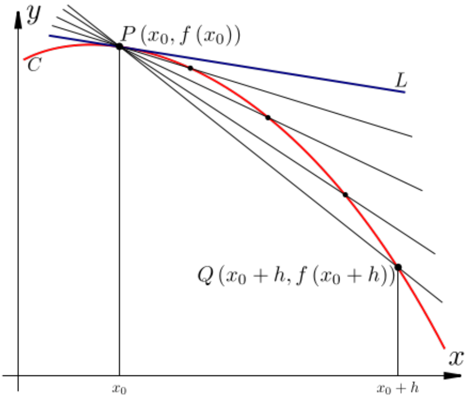
Taking $x = a + h$, $h \to 0$ as $x \to a$. So, we can rewrite as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The second definition is also known as **derivative**. Also denoted as $f'(x)$ or $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$. We can also have **higher derivative** (e.g., 2) which is denoted as $f''(x)$ or $\frac{d^2y}{dx^2}$ or $\frac{d}{dx}(\frac{dy}{dx})$.

Instantaneous rate of change

Instantaneous rate of change of $f(x)$ at $x = a$
$=$ Slope of the **tangent** line to $f(x)$ at the point $x = a$.



Example

Find the equation of the tangent line to the graph $f(x) = x^2$ at $(1, 1)$.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\
&= \lim_{h \to 0} \frac{2h + h^2}{h} \\
&= \lim_{h \to 0} (2 + h) = 2
\end{aligned}$$

Using the point-slope form $(y - y_1 = m(x - x_1))$, we can find the tangent line at $(1, 1)$ as $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

Example

Find the equation of the tangent line to the graph $f(x) = \dfrac{3}{x}$ at $(3, 1)$.

$$\begin{aligned}\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0}\frac{\dfrac{3}{3+h}-1}{h}\\ &= \lim_{h\to 0}-\frac{1}{3+h} = -\frac{1}{3}\end{aligned}$$

Using the point-slope form $(y - y_1 = m(x - x_1))$, we can find the tangent line at $(3, 1)$ as $y - 1 = -\dfrac{1}{3}(x - 3)$ which becomes $x + 3y - 6 = 0$.

Example

Find f' of $f(x) = x^3 - x$.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-[x^3-x]}{h}\\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}\\
&= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h}\\
&= \lim_{h\to 0} 3x^2+3xh+h^2-1\\
&= 3x^2-1
\end{aligned}$$

Self-Exercise

1. Find the derivative of $y = x^2 - 3x$ at $x = 2$
   - Ans: 1
2. Find f' of $\dfrac{1-x}{2+x}$
   - Ans: $-\dfrac{3}{(2+x)^2}$
3. Find f' of $\sqrt{x}$ (Hint: Multiply with $\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$)
   - Ans: $\dfrac{1}{2\sqrt{x}}$
4. Find equation of the tangent line of $y = x^2 - 8x + 9$ at point $(3, -6)$
   - Ans: $y = -2x$

Definition

A function is differentiable at $x = a$ if $f'(a)$ exists.

Theorem

If f is differentiable at a then it is continuous at a.

Theorem

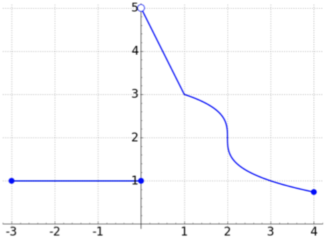
If f is not continuous at $x = a$ then f is not differentiable at $x = a$.

Theorem

If f is continuous at a f may or may not be differentiable at $x = a$.

e.g., $f(x) = |x|$ is cont. at $x = 0$ but is not differentiable at $x = 0$

What are places where this graph is **NOT differentiable** and why?



- -3: left limit DNE thus discontinuous
- 0: discontinuous
- 1: sharp corner
- 2: the slope is ∞
- 4: right limit DNE thus discontinuous

Rules for differentiation

1. **Constant Rule** $f(x) = c \Rightarrow f'(x) = 0$.
2. **Power Rule** $f(x) = x^n \Rightarrow f'(x) = nx^{(n-1)}$, for any $n \in \mathbb{R}$.
3. **Scalar Multiplication Rule** $(kf(x))' = kf'(x)$ for any $k \in \mathbb{R}$.
4. **Sum/Difference Rule** $[u(x) \pm v(x)]' = u'(x) \pm v'(x)$.
5. **Product Rule** $[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$.
6. **Quotient Rule** $\left[\dfrac{u(x)}{v(x)}\right]' = \dfrac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$, when $v(x) \neq 0$.
7. **Exponential Rule** If $f(x) = e^{ax}$, $f'(x) = ae^{ax}$.
8. **Logarithm Rule** if $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, for $x \neq 0$.
9. **Chain Rule** If $h(x) = f \circ g(x)$ is the composition of two functions f and g, then
$$h'(x) = f'[g(x)] \cdot g'(x).$$
10. **Trigonometric Rule**
    - $\sin x \Rightarrow \cos x$; $\cos x \Rightarrow -\sin x$; $\tan x \Rightarrow \sec^2 x$
    - $\cot x \Rightarrow -\csc^2 x$; $\sec x \Rightarrow \sec x \tan x$; $\csc x \Rightarrow -\csc x \cot x$

Example

Find derivative of $f(x) = \sqrt{x} + \sqrt{\pi}$.

$$
\begin{aligned}
&= \frac{d}{dx}\sqrt{x} + \frac{d}{dx}\sqrt{\pi} \\
&= \frac{d}{dx}x^{\frac{1}{2}} + \frac{d}{dx}\sqrt{\pi} \\
&= \frac{1}{2}x^{\frac{1}{2}-1} + 0 \\
&= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Example

Find derivative of $\pi t\sqrt{t}e^t$.

$$
\begin{aligned}
&= \pi \frac{d}{dt}(t^{\frac{3}{2}} e^t) \\
&= \pi(e^t * \frac{d}{dt} t^{\frac{3}{2}} + t^{\frac{3}{2}} * \frac{d}{dt} e^t) \\
&= \pi(e^t * \frac{3}{2} * t^{\frac{1}{2}} + t^{\frac{3}{2}} * e^t)
\end{aligned}
$$

Example

Find derivative of $g(x) = ex^2 + 2e^x + xe^2 + x^{e^2}$.

$$= e * \frac{d}{dx}(x^2) + 2 * \frac{d}{dx}(e^x) + e^2 * \frac{d}{dx}(x) + \frac{d}{dx}(x^{e^2})$$
$$= 2ex + 2e^x + e^2(1) + e^2 x^{e^2-1}$$

Example

Find derivative of $\dfrac{z^2}{z^3+1}$.

$$= \frac{(z^3+1)*2z - z^2(3z^2+0)}{(z^3+1)^2}$$

$$= \frac{2z^4+2z-3z^4}{(z^3+1)^2}$$

$$= \frac{2z-z^4}{(z^3+1)^2}$$

Example

Find derivative of $\sqrt{sinx}$.

We can use the chain rule. Let $g(x) = \sin x$ and $f(u) = u^{\frac{1}{2}}$

$$\begin{aligned} f' &= f'(g(x)) * g'(x) \\ &= \frac{1}{2}(\sin x)^{-\frac{1}{2}} * \cos x \end{aligned}$$

Example

Find derivative of $e^{\sin x^2}$. (Hint: 3 chains)

Let $g(x) = e^x$ and $f(u) = \sin u$ and $h(t) = t^2$

$$= e^{\sin x^2} * \frac{d}{dx} \sin x^2$$

$$= e^{\sin x^2} * \cos x^2 * \frac{d}{dx} x^2$$

$$= e^{\sin x^2} * \cos x^2 * 2x$$

Example

Find derivative of 5^x.

Since $5 = e^{\ln 5}$, thus $5^x = (e^{\ln 5})^x$

$$
\begin{aligned}
&= \frac{d}{dx} e^{\ln 5x} \\
&= e^{\ln 5x} * \ln 5 \\
&= 5^x * \ln 5
\end{aligned}
$$

Note: $\dfrac{d}{dx} a^x = a^x \ln a$

Self-Exercise

1. Find the derivative of $g(t) = 4t^2 + \dfrac{1}{4t^2}$
   - Ans: $8t - \dfrac{1}{2t^3}$
2. Find the derivative of $y = e^3$
   - Ans: 0
3. Find f'' of $(t^2 - 1) * e^t$
   - Ans: $e^t(t^2 + 4t + 1)$
4. Find derivative of $\dfrac{xe^x}{x^2 + \pi e^x}$
   - Ans: $\dfrac{x^2 + \pi e^x(xe^x + e^x) - xe^x(2x + \pi e^x)}{(\pi e^x + x^2)^2}$
5. Find derivative of $5(\tan x + \sec x)^3$
   - Ans: $15(\tan x + \sec x)^2 * \sec^2 x + \sec x \tan x$

Derivatives of inverse trig functions

Note: $y = \sin^{-1} x$ means $\sin y = x$. Another notation is $y = \arcsin x$

1. $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}$
2. $\dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}$
3. $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1 + x^2}$
4. $\dfrac{d}{dx} \cot^{-1} x = -\dfrac{1}{1 + x^2}$
5. $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x(\sqrt{x^2 - 1})}$
6. $\dfrac{d}{dx} \csc^{-1} x = -\dfrac{1}{x(\sqrt{x^2 - 1})}$

Example

Find derivative of $y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$

$$\frac{dy}{dx} = \frac{1}{1+(\frac{a+x}{a-x})^2} * \frac{d}{dx}(\frac{a+x}{a-x})$$

$$\frac{dy}{dx} = \frac{1}{1+(\frac{a+x}{a-x})^2} * \frac{(a-x)*1-(a+x)(-1)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{a-x+a+x}{(1+\frac{(a+x)^2}{(a-x)^2})(a-x)^2}$$

$$\frac{dy}{dx} = \frac{a}{a^2+x^2}$$

Implicit differentiation

Find derivative of $9x^2 + 4y^2 = 25$ at point $(1, 2)$

$$\begin{aligned}
\frac{d}{dx}(9x^2 + 4y^2) &= \frac{d}{dx}(25) \\
9\frac{d}{dx}x^2 + 4\frac{d}{dx}y^2 &= 0 \\
9 * 2x + 4 * 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-9}{4} * \frac{x}{y}
\end{aligned}$$

At point $(1, 2)$, $\dfrac{dy}{dx} = \dfrac{-9}{8}$

Implicit differentiation

Find derivative of $y = x^x$

$$\begin{aligned}
\ln y &= \ln x^x \\
\ln y &= x \ln x \\
\frac{d}{dx} \ln y &= \frac{d}{dx} x \ln x \\
\frac{1}{y} \frac{dy}{dx} &= x * \frac{1}{x} + 1 * \ln x \\
\frac{dy}{dx} &= y(1 + \ln x) \\
\frac{dy}{dx} &= x^x(1 + \ln x)
\end{aligned}$$

Implicit differentiation

Find derivative of $y = \log_a x$

$$
\begin{aligned}
a^y &= x \\
\frac{d}{dx} a^y &= \frac{d}{dx} x \\
\ln a * a^y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\ln a * a^y} \\
\frac{dy}{dx} &= \frac{1}{\ln a * x}
\end{aligned}
$$

Note: $\frac{d}{dx} \log_a x = \frac{1}{\ln a * x}$

Self-Exercise

1. Find the derivative of $x^2 + y^2 = 25$ at point $(3, 4)$
   - Ans: $-\dfrac{3}{4}$
2. Find the derivative of $x^3 + y^3 = 6xy$ at point $(3, 3)$
   - Ans: -1
3. Find the derivative of $\sin(x + y) = y^2 \cos x$
   - Ans: $\dfrac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$
4. Find the derivative of $x^4 + y^4 = 16$
   - Ans: $-\dfrac{x^3}{y^3}$