Midterm 1

Name:

TA:

Section Time:

Course Login:

Answer all questions. Read them car efully first. Be precise and concise. Write in the space provided, and use the back of the page for scratch. Good luck!

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total	

1 True/False (28 points)

1. There are more 10 digit phone (decimal) numbers than 5 character usernames (using lowercase letters and digits.)

2. Karatsuba's (the faster recursive multiplication) algorithm for multiplication is $O(n^2)$ for two n-bit numbers.

3. An undirected graph with n vertices and n edges must have a cycle.

4. A directed graph with n vertices and n edges must have a cycle.

5. Depth first search of a strongly connected directed graph has at least one back edge.

6. Recall that a bridge in an undirected graph is an edge whose removal disconnects the graph. Also recall that a separating vertex was a node whose removal disconnects the graph. Any edge between two separating vertices is a bridge for a graph on a connected graph with 3 or more nodes.

7. One of the endpoints of any bridge is a separating vertex for a connected graph with 3 or more nodes. (See above for definition of bridge.)

8. It's possible for a DFS to set pre and post values of u and v such that $pre(v) < post(v) < pre(u) < post(u)$ even though there is a directed edge from u to v.

9. Edges not contained in strongly connected components are always either forward or tree edges of any depth first search tree.

10. Let d be the distance labelling result from a breadth first search for an undirected graph G. All edges (u, v) in G have $d(u) \neq d(v)$, i.e., are between nodes with a different distance label.

11. Again, let d be the distance labelling result from a breadth first search for an undirected graph G. There is **no** edge (u, v) between two nodes at different odd distances, i.e., there is no edge (u, v) where $d(u) \neq d(v)$, and both $d(u)$ and $d(v)$ are odd.

12. For any weighted graph with only one negative edge weight and no negative cycles, Bellman Ford finds the shortest paths in one iteration.

13. There is no positive integer less than 35 that is divisible by 5 and 7.

14. There is no positive integer less than 54 that is divisible by 9 and 12.

2 Short answer. (36 points)

1. What is $4^{34} \pmod{35}$?

2. How many elements in $\{1, \ldots, N\}$, are relatively prime to N if $N = p^2q$ and p and q are distinct primes?

3. Give a tight asymptotic upper bound on the recurrence defined by $T(n) = 6T(n/2) + O(n^3)$. (As usual you may assume $T(1)$ is $O(1)$.)

4. Give a tight asymptotic upper bound on the recurrence defined by $T(n) = 8T(n/2) + O(n^3)$. (As usual you may assume $T(1)$ is $O(1)$.)

5. Give a tight asymptotic upper bound on the recurrence defined by $T(n) = 2T(\sqrt{n}) + O(\log n)$. (As usual you may assume $T(2)$ is $O(1)$.)

6. Give a method to multiply two complex numbers with three multiplications of real numbers. (Let $a + bi$ and $c + di$ be the complex numbers.)

7. Show how to use the selection algorithm to find out whether an array has a majority element. Recall that a majority element of an n element array is an element that appears more than $n/2$ times. (Max two sentences.)

8. What is the average number of times one must choose a random element from a set S of k distinct numbers to get an element in the first i numbers in S ?

9. Consider a strongly connected n-node m-edge directed graph, and a node v with pre-order number 0. Give an expression for the post order number of v or show that the post-order number is not uniquely determined by giving appropriate examples.

10. Consider a n-node m-edge directed acyclic graph, and a source node v with pre-order number 0. Give an expression for the post order number of v or show that the post-order number is not uniquely determined by giving appropriate examples.

11. Draw a three node graph whose minimum spanning tree differs from the shortest path tree (specify your starting node for the shortest path tree.)

3 Finding blocking paths. (12 points)

Given a directed acyclic graph, $G = (V, E)$, and nodes s and t, we wish to find a set of $s - t$ paths whose removal disconnects s and t.

1. Give an $O(|V| + |E|)$ time method to find one path from s to t if one exists. (No need to justify the runtime.)

2. Give an $O(|V|(|V| + |E|))$ time algorithm to remove a set of $s - t$ paths from G so that there is no path from s to t. (Justify the running time.)

3. (*) Give an $O(|V| + |E|)$ time algorithm to find a set of paths whose removal disconnects leaves no paths from s to t. (Probably a hint: carefully examine your implementation for parts (1) and (2).)

(*) Difficult.

4 Divide and Conquer. (12 points)

We consider a sorted array that has been cyclicly shifted by an arbitrary unknown amount. For example, $[3, 5, 7, 23, 28]$ could be circularly shifted to $[23, 28, 3, 5, 7]$. Give an efficient algorithm to find the maximum element of a sorted array which has been cyclicly by an unknown amount. (Let n be the size of the array.)

5 Fast Fourier Transform. (12 points.)

1. If $a(\cdot)$ and $b(\cdot)$ are degree d polynomials. Let $c(x) = a(x) \times b(x)$.

   (a) What is the maximum possible degree of the polynomial $c(x)$?

   (b) What is the value of $c(0)$ if $a(0) = 5$ and $b(0) = 6$?

   (c) For $d = 2$, if $a(0) = 5$, can $a(1) = 7$? If yes, must it be?

   (d) For $d = 1$, if $b(0) = 6$ and $b(1) = 7$, can $b(2) = 8$? If yes, must it be?

2. (a) If one wishes to do polynomial multiplication for two degree $d = 2^k - 1$ polynomials using the FFT, what should the value of n be for the FFT algorithm?

   (b) What is the value of entry $2, 3$ for the FFT matrix of dimension 8? (The matrix is "0 indexed", for example entry $0, 0$ is 1 and entry $1, 1$ is $\frac{1}{\sqrt{2}}(1 + i)$.)

   (c) What is the value of entry $2, 3$ for the inverse FFT matrix of dimension 8? (The matrix is "0 indexed" as above, i.e., entry 0,0 is $\frac{1}{8}$.)

   (d) Given that the values of $FFT([u, x]) = [1, 2]$ and $FFT([v, y]) = [1, 1]$, what is $FFT([u, v, x, y])$?