CS 170	Algorithms	
Fall 2013	Satish Rao	HW 1

1. (10 pts) Getting Started

Students receive full credit for writing "I understand the course policies" under problem 1.

2. (20 pts) Compare Growth Rates

(a) $f = \Omega(g)$; both are polynomials and $3.75 > 2.72$

(b) $f = \Theta(g)$; both are linear up to an additive polylog term

(c) $f = \Theta(g)$; both are linear in n^3

(d) $f = \Theta(g)$; both are linear in $\log n$

(e) $f = \Theta(g)$; both are linear in $\log n$

(f) $f = \Theta(g)$; $\log 5n = \log 5 + \log n$, so both are $\Theta(n \log n)$

(g) $f = O(g)$; $\frac{n^3}{\log n} \in O(n^3) \in O(n^4) \in O(n^4(\log n)^3)$

(h) $f = O(g)$; n to any power greater than 0 grows much faster than $\log n$ to any constant power

(i) $f = O(g)$; same reason as above

(j) $f = \Omega(g)$; same reason as above

(k) $f = O(g)$; $f = O(n^2)$ but $g = \Omega(n^k)$ for any constant k

(l) $f = O(g)$; g is bounded below by $3^{\log_3 n} = n$

(m) $f = \Omega(g)$; $(\frac{5}{4})^n \in \Omega(n) \implies \lim_{n\to\infty} \frac{(5/4)^n}{n} = \infty \implies \lim_{n\to\infty} \frac{5^n}{n4^n} = \infty \implies 5^n \in \Omega(n4^n)$

(n) $f = O(g)$; $2^n/2^{3n} = 1/2^{2n}$ approaches zero as n increases

(o) $f = \Theta(g)$; they differ by a factor of $1/7$

(p) $f = O(g)$; $g(n) = 7^{(\log n)^2} = (2^{\log 7})^{(\log n)^2} = ((2^{\log n})^{\log 7})^{\log n} = n^{(\log 7)(\log n)} \in \Omega((\log n)^{\log n})$

(q) $f = O(g)$; $f(n) = 2^{3n}$ and $g(n) = 2^{n \log n}$. The exponent of $g(n)$ grows faster than the exponent of $f(n)$.

3. (12 pts) Prove Order of Growth

1. We will follow the hint. To show $\sum_{i=1}^n i^k = O(n^{k+1})$:

$$\sum_{i=1}^n i^k \le \sum_{i=1}^n n^k = n^{k+1}.$$

To show $\sum_{i=1}^n i^k = \Omega(n^{k+1})$, only consider $n/2 \le i \le n$:

$$\sum_{i=1}^n i^k \ge \sum_{i=n/2}^n (n/2)^k = 2^{-k-1} n^{k+1}.$$

Note. This problem can also be solved by integrating: $\sum_{i=1}^n i^k = \int_1^{n+1} \lceil x^k \rceil \, dx = \Theta(\log n^{k+1})$.

2. We can lower bound $n!$ as:

$$\underbrace{\left(\frac{n}{2}\right)\cdot\dots\cdot\left(\frac{n}{2}\right)}_{\frac{n}{2}\text{ terms}} \le 1\cdot 2\cdot 3\cdot\dots\cdot\left(\frac{n}{2}\right)\cdot\underbrace{\left(\frac{n}{2}+1\right)\cdot\dots\cdot n}_{\frac{n}{2}\text{ terms}}$$

and upper bound it as:

$$\underbrace{1\cdot 2\cdot 3\cdot\dots\cdot n}_{n\text{ terms}} \le \underbrace{n\cdot\dots\cdot n}_{n\text{ terms}}$$

Hence,

$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \le n! \le n^n$$
$$\frac{n}{2}\log\left(\frac{n}{2}\right) \le \log(n!) \le n\log n$$
$$\forall n \ge 4, \log n \ge 2 \implies \log n - \frac{1}{2}\log n \ge 1 \implies \frac{1}{2}\log n \le \log n - 1 = \log n - \log 2 = \log\left(\frac{n}{2}\right)$$
$$\forall n \ge 4, \frac{n}{2}\left(\frac{1}{2}\log n\right) \le \frac{n}{2}\log\left(\frac{n}{2}\right) \le \log(n!) \le n\log n$$
$$\forall n \ge 4, \frac{1}{4}n\log n \le \log(n!) \le n\log n$$

3. We will follow the hint. To show an upper bound, we'll replace $1/i$ with $1/t$, where t is the power of 2 just smaller than i. For simplicity, assume n is a power of 2. Then:

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{i} &\le 1 + \sum_{i=2}^{3}\tfrac{1}{2} + \sum_{i=4}^{7}\tfrac{1}{4} + \dots + \sum_{i=n/2}^{n}\frac{1}{n/2}\\ &= 1 + 1 + \dots + 1\\ &= \log n\end{aligned}$$

The lower bound is similar.

Note. This problem can also be solved by integrating: $\sum_{i=1}^{n}\frac{1}{i} = \int_1^{n+1}\left\lceil\frac{1}{x}\right\rceil dx = \Theta(\log n)$.

4. (10 pts) Problem 0.2

By the formula for the sum of a partial geometric series, for $c \neq 1$: $g(n) = \frac{1-c^{n+1}}{1-c} = \frac{c^{n+1}-1}{c-1}$.

a) $1 > 1 - c^{n+1} > 1 - c$. So: $\frac{1}{1-c} > g(n) > 1$.

b) For $c = 1$, $g(n) = \underbrace{1+1+\dots+1}_{n+1\text{ times}} = n+1$.

c) For sufficiently large n, $c^{n+1} > c^{n+1} - 1 > c^n$. So: $\frac{c}{c-1}c^n > g(n) > \frac{1}{c-1}c^n$.

5. (10 pts) Problem 1.19

We can show this by induction on n. For $n = 1$, $\gcd(F_1, F_2) = \gcd(1,1) = 1$. Now assume that $\gcd(F_{n+1}, F_n) = 1$ for all $n \le k$. This implies that for $n = k+1$:

$$\gcd(F_{k+1}, F_{k+2}) = \gcd(F_{k+1}, F_{k+2} - F_{k+1}) = \gcd(F_{k+1}, F_k) = 1$$

Hence, the statement is true for all $n \ge 1$.

6. (5 pts) Problem 1.22

Since a has an inverse mod b, we know that a, b are coprime. Thus, b also has an inverse mod a.

Alternative solution: Let $x \equiv a^{-1} \pmod{b}$. Then there exists an integer y such that $ax + by = 1$, therefore $b^{-1} \equiv y \pmod{a}$.