CS 170: Algorithms

CS 170: Algorithms

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CS 170: Algorithms

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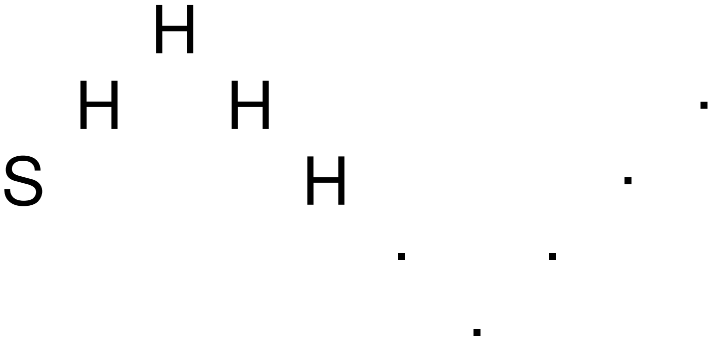
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CS 170: Algorithms

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No laptops please.

CS 170: Algorithms

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No laptops please.

Thank you

CS 170: Algorithms

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No laptops please.

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No laptops please.

Thank you ! ! ! ! ! ! ! !

Convolution = polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Convolution $=$ polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

Convolution $=$ polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

Convolution $=$ polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

Convolution = polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

At time 1:

Convolution = polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

At time 1:

$b_0 a_1 + b_1 a_0$

Convolution = polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

At time 1:

$b_0 a_1 + b_1 a_0$

At time i:

Convolution = polynomial multiplication.

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Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

At time 1:

$b_0 a_1 + b_1 a_0$

At time i:

$b_0 a_i + b_1 a_{i-1} + \cdots b_i a_0$

Convolution = polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

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At time 1:

$b_0 a_1 + b_1 a_0$

At time i:

$b_0 a_i + b_1 a_{i-1} + \cdots b_i a_0$

Response: $c_0, \ldots, c_{2n-1}$

Convolution = polynomial multiplication.

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Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

At time 1:

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Response: $c_0, \ldots, c_{2n-1}$

$c_i = \sum_j b_j a_{i-j}$

Convolution = polynomial multiplication.

System: $a_0, \ldots, a_{n-1}$

Signal: $b_0, \ldots, b_{n-1}$

At time 0:

$b_0 a_0$

At time 1:

$b_0 a_1 + b_1 a_0$

At time i:

$b_0 a_i + b_1 a_{i-1} + \cdots b_i a_0$

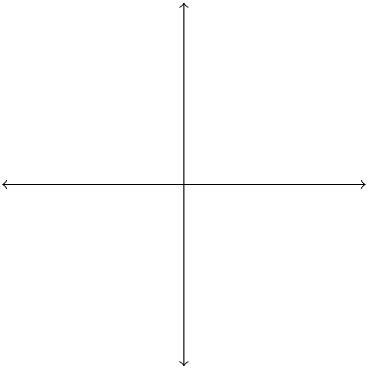
Response: $c_0, \ldots, c_{2n-1}$

$c_i = \sum_j b_j a_{i-j}$

Same as multiplying polynomials $A(\cdot)$ and $B(\cdot)$!

The nth complex roots of unity.

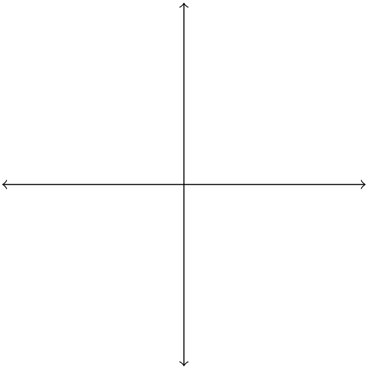
Solutions to $z^n = 1$



The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \tfrac{2\pi}{n})^n = (1, \tfrac{2\pi}{n} \times n) = (1, 2\pi) = 1!$$



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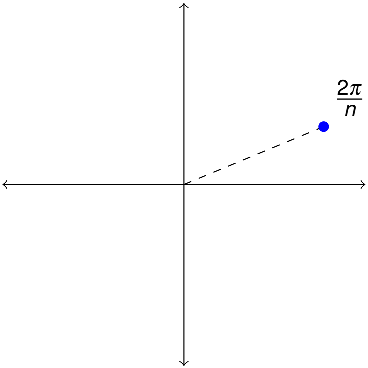
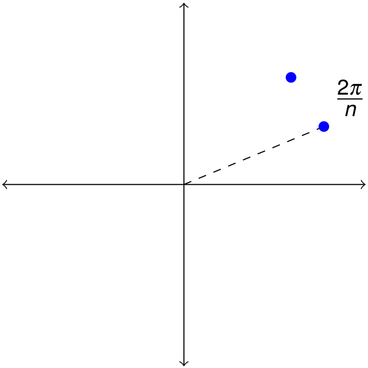


The nth complex roots of unity.

Solutions to $z^n = 1$

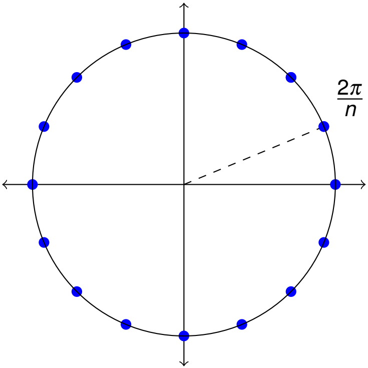
$$(1, \tfrac{4\pi}{n})^n = (1, \tfrac{4\pi}{n} \times n) = (1, 4\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

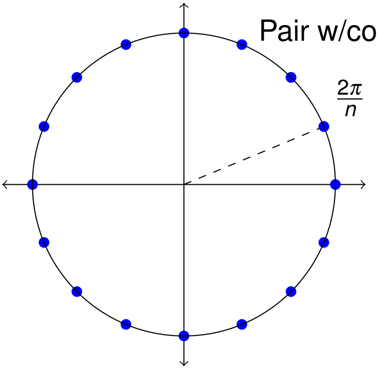
$$(1, \tfrac{2k\pi}{n})^n = (1, \tfrac{2k\pi}{n} \times n) = (1, 2k\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$

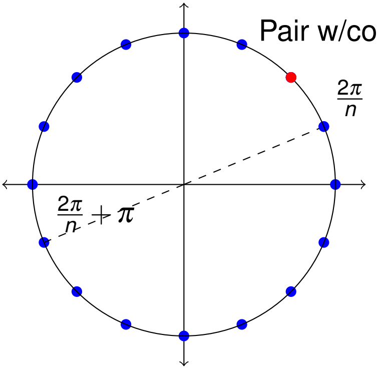


Pair w/common square!

The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$



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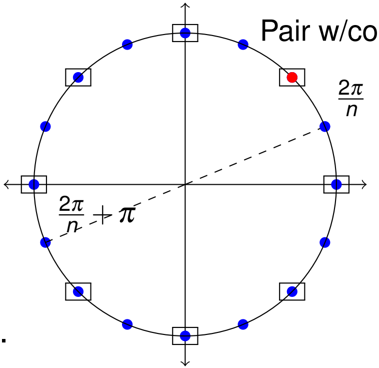

Squares: $\frac{n}{2}$th roots.

The Fast Fourier Transform!

Defn: $\omega = (1, \frac{2\pi}{n}) = e^{\frac{2\pi}{n} i}$, nth root of unity.

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Pairs: ω^i and $\omega^{i+n/2}$

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Common Squares: are $\frac{n}{2}$ root of unity.

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Evaluate $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$

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Procedure:

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Procedure:
Recursively compute A_e and A_o on $\frac{n}{2}$ roots of unity:

The Fast Fourier Transform!

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For each $i \leq n/2$.

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$$A(\omega^i) = A_e(\omega^{2i}) + \omega^i A_o(\omega^{2i})$$

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Runtime Recurrence:

The Fast Fourier Transform!

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Runtime Recurrence:
$T(n) = 2T(n/2) + O(n)$

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Runtime Recurrence:
$T(n) = 2T(n/2) + O(n) = O(n \log n)$!

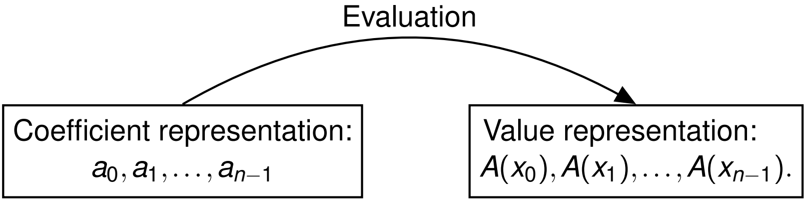
Multiplying polynomials?

Coefficient representation:
$a_0, a_1, \ldots, a_{n-1}$

Value representation:
$A(x_0), A(x_1), \ldots, A(x_{n-1})$.

Multiplying polynomials?

Evaluation: $O(n \log n)$ if choose $1, \omega, \omega^2, \ldots, \omega^{n-1}$.



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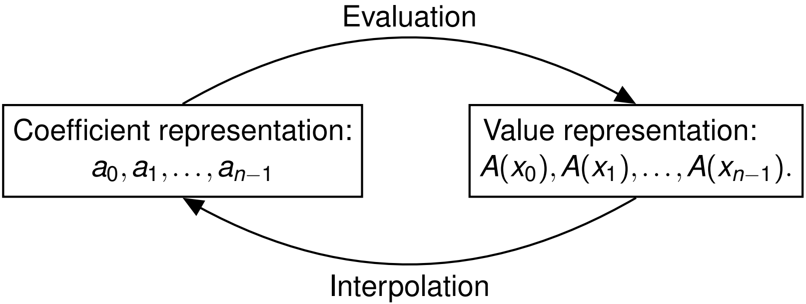

Interpolation: From points $A(x_0), \ldots, A(x_{n-1})$ to "function".

Multiplying polynomials?

Evaluation: $O(n \log n)$ if choose $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Coefficient representation:}\\ a_0, a_1, \ldots, a_{n-1}\end{array}} \arrow[r, bend left, "\text{Evaluation}"] & \boxed{\begin{array}{c}\text{Value representation:}\\ A(x_0), A(x_1), \ldots, A(x_{n-1}).\end{array}} \arrow[l, bend left, "\text{Interpolation}"]
\end{tikzcd}
$$

Interpolation: From points $A(x_0), \ldots, A(x_{n-1})$ to “function”.
How?

Polynomial Evaluation and Matrices

Compute $A(\cdot)$ from a_i's:

$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots x_1^{n-1} \\ & & \vdots & \cdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Compute inverse of matrix above.

Polynomial Evaluation and Matrices

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Compute inverse of matrix above.
Multiply.

Polynomial Evaluation and Matrices

Compute $A(\cdot)$ from a_i's:

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Doh!!

Polynomial Evaluation and Matrices

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Doh!!

Also, computing inverse not even easy.

Getting done?

FFT: ω is complex nth root of unity

Getting done?

FFT: ω is complex nth root of unity
and matrix is ...

$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(n-1)} \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

Getting done?

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and matrix is ...

$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

Getting done?

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Compute inverse of $M_n(\omega)$?

Geometry and FFT.

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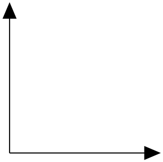

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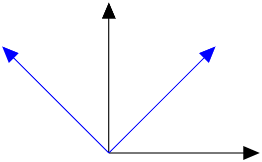

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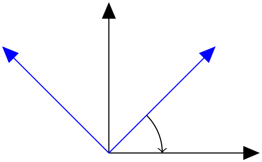

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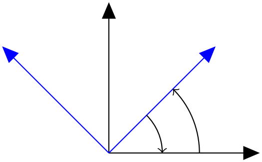

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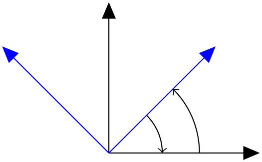

Reverse Rotation is inverse operation.
Scaling: for rotation, axis should have length 1, FFT length n.

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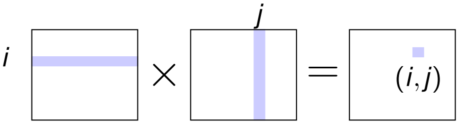

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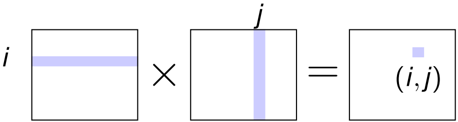

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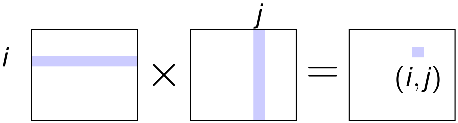

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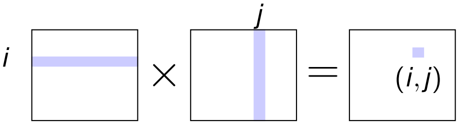

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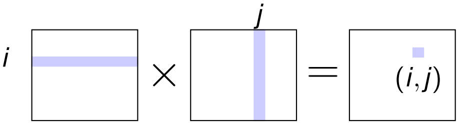

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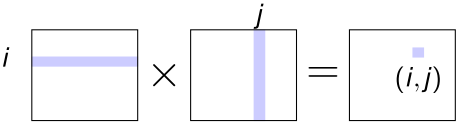

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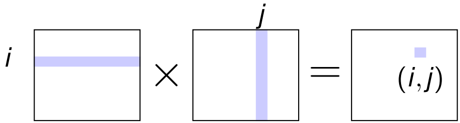

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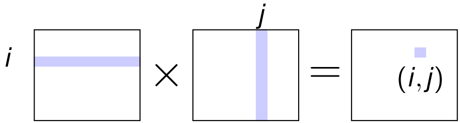

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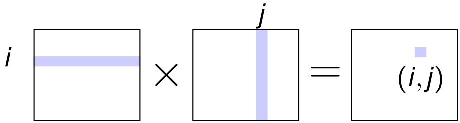

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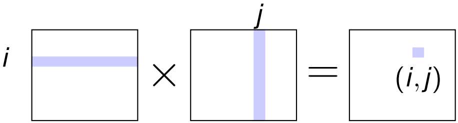

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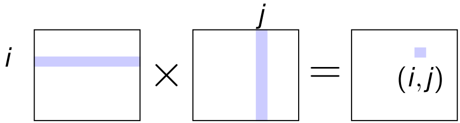

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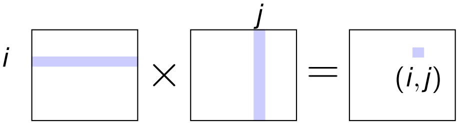

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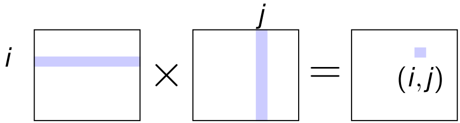

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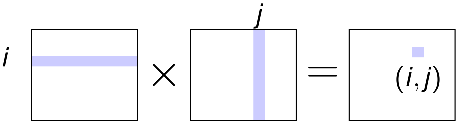

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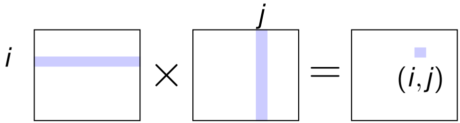

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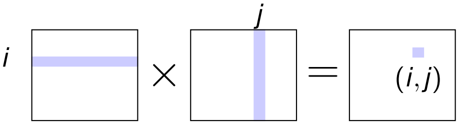

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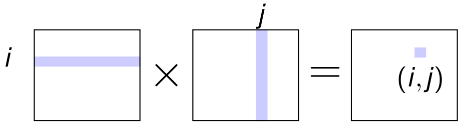

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Cool!!