CS 170: Algorithms

CS 170: Algorithms

Hello and ...

H . . . H . S . H H

CS 170: Algorithms

Hello and ...

H . . . H .   . H H

S

CS 170: Algorithms

Hello and ...

H . . . H .   . H

S H

CS 170: Algorithms

Hello and ...

H . . . . . H

S H H

CS 170: Algorithms

Hello and ...

H . . . . .

S H H H

CS 170: Algorithms

Hello and ...

S H H H H

CS 170: Algorithms

Hello and ...

S H H H H .

CS 170: Algorithms

Hello and ...

S H H H H . .

CS 170: Algorithms

Hello and ...

S H H H H . . .

CS 170: Algorithms

Hello and ...

S H H H H . . . .

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please,

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides),

CS 170: Algorithms

Hello and ...

S H H H H . . . . .

Please, no laptops (unless lecture draft slides), ...

CS 170: Algorithms

Hello and ...

S H H H H . . . . .

Please, no laptops (unless lecture draft slides), ...

Story:

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: "Shut ...."

If you must leave early, please sit by exit.

Please stay still until the end of class.

CS 170: Algorithms

Hello and ...

S H H H H . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: "Shut ...."

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you

CS 170: Algorithms

Hello and ...

S H H H H . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you !

CS 170: Algorithms

Hello and ...

S H H H H . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! !

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! ! !

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! ! ! !

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: "Shut ...."

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! ! ! ! !

CS 170: Algorithms

Hello and ...

S H H H H . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! ! ! ! ! !

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! ! ! ! ! ! !

CS 170: Algorithms

Hello and ...

S H H H H . . . . . .

Please, no laptops (unless lecture draft slides), ...

Story: “Shut ....”

If you must leave early, please sit by exit.

Please stay still until the end of class. Distracting!

Thank you ! ! ! ! ! ! ! !

Today.

...Complex numbers, polynomials today. FFT.

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6

(B) 5

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6
(B) 5

(A) 6

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6
(B) 5

(A) 6 of course!

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6
(B) 5

(A) 6 of course!

Coeefficient of x^2 in result?

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Coefficient of x^4 in result?

(A) 6
(B) 5

(A) 6 of course!

Coeefficient of x^2 in result?

Uh oh...

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$x^0$$

Multiplying polynomials.

$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$

$$x^0 \quad ((1)(4))$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$x^0 \quad ((1)(4)) \qquad = 4$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & & \end{array}$$

Multiplying polynomials.

$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$

$$\begin{array}{ll} x^0 & ((1)(4)) \qquad\qquad\qquad\qquad = 4 \\ x^1 & ((1)(3) \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & & \end{array}$$

Multiplying polynomials.

$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3) + (2)(4)) & = 11 \\ x^2 & ((1)(2) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4)) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4)) & = 20 \\ x^3 & & \end{array}$$

Multiplying polynomials.

$(1 + 2x + 3x^2)(4 + 3x + 2x^2)$

$$\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3) + (2)(4)) & = 11 \\
x^2 & ((1)(2) + (2)(3) + (3)(4))) & = 20 \\
x^3 & ((2)(2) &
\end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3)+(2)(4)) & = 11 \\
x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\
x^3 & ((2)(2)+(3)(3)) & = 13 \\
x^4 & &
\end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & \end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3)+(2)(4)) & = 11 \\
x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\
x^3 & ((2)(2)+(3)(3)) & = 13 \\
x^4 & ((3)(2)) & = 6
\end{array}$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0 + a_1 x + \cdots a_d x^d$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0+a_1x+\cdots a_dx^d$ In example: $a_0=1, a_1=2, a_2=3$

$b_0+b_1x+\cdots b_dx^d$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$
\begin{array}{lll}
x^0 & ((1)(4)) & = 4 \\
x^1 & ((1)(3)+(2)(4)) & = 11 \\
x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\
x^3 & ((2)(2)+(3)(3)) & = 13 \\
x^4 & ((3)(2)) & = 6
\end{array}
$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$

$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:
$a_0+a_1x+\cdots a_dx^d$ In example: $a_0=1, a_1=2, a_2=3$
$b_0+b_1x+\cdots b_dx^d$ In example: $b_0=4, b_1=3, b_2=2$
Product: $c_0+c_1x+\cdots c_{2d}x^{2d}$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & =4 \\ x^1 & ((1)(3)+(2)(4)) & =11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & =20 \\ x^3 & ((2)(2)+(3)(3)) & =13 \\ x^4 & ((3)(2)) & =6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:

$a_0+a_1x+\cdots a_dx^d$ In example: $a_0=1, a_1=2, a_2=3$

$b_0+b_1x+\cdots b_dx^d$ In example: $b_0=4, b_1=3, b_2=2$

Product: $c_0+c_1x+\cdots c_{2d}x^{2d}$

$$c_k = \sum_{0\le i\le k} a_i * b_{k-i}.$$

Multiplying polynomials.

$(1+2x+3x^2)(4+3x+2x^2)$

$$\begin{array}{lll} x^0 & ((1)(4)) & = 4 \\ x^1 & ((1)(3)+(2)(4)) & = 11 \\ x^2 & ((1)(2)+(2)(3)+(3)(4))) & = 20 \\ x^3 & ((2)(2)+(3)(3)) & = 13 \\ x^4 & ((3)(2)) & = 6 \end{array}$$

$4+11x+20x^2+13x^3+6x^4$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \leq i \leq k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.

or (D)

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:
$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.
or (D) ..will use n as parameter shortly.

Multiplying polynomials.

Multiply: $(1+2x+3x^2)(4+3x+2x^2)$

Given:

$a_0 + a_1 x + \cdots a_d x^d$ In example: $a_0 = 1, a_1 = 2, a_2 = 3$
$b_0 + b_1 x + \cdots b_d x^d$ In example: $b_0 = 4, b_1 = 3, b_2 = 2$
Product: $c_0 + c_1 x + \cdots c_{2d} x^{2d}$

$$c_k = \sum_{0 \le i \le k} a_i * b_{k-i}.$$

E.g.: $c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$.
Runtime?

(A) $O(d)$

(B) $O(d \log d)$

(C) $O(n^2)$

(D) $O(d^2)$

Time: $O(k)$ multiplications for each k up to $k = 2d$.
$\implies O(d^2)$.
or (D) ..will use n as parameter shortly. so (C) also.

Hmmm...

$O(d^2)$ time!

Hmmm...

$O(d^2)$ time!
Quadratic Time!

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes? No?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes? No?
How?

Hmmm...

$O(d^2)$ time!
Quadratic Time!
Can we do better?
Yes? No?
How?
Use different representation.

Another representation.

Represent a line?

Another representation.

Represent a line?
Slope and intercept!

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d + 1$

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d+1$

How to find points on function?

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d+1$

How to find points on function?
plug in x-values...

Another representation.

Represent a line?
Slope and intercept! a_0, a_1

How many points determine a line? 2
Represent line as two points on line instead of coefficients!

How many points determine a parabola (a quadratic polynomial)? 3

How many points determine a a degree d polynomial?

$d+1$

How to find points on function?
plug in x-values...and evaluate.

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How to find polynomial from points?
Solve $d+1$ variable system of equations!

Point-value representation.

$A(x_0), \cdots, A(x_{2d})$
$B(x_0), \cdots, B(x_{2d})$

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Point-value representation.

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$O(d)$ multiplications!

Given: $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$.
Evaluate: $A(x), B(x)$ on $2d+1$ points: $x_0, \cdots, x_{2d}$.

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Recall(from CS70): unique representation of polynomial.

Point-value representation.

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Multiply: $A(x)B(x)$ on points to get points for $C(x)$.

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Interpolate: find $c_0 + c_1 x + c_2 x^2 + \cdots c_{2d} x^{2d}$.

Polynomial Evaluation.

Evaluate $A(x) = a_0 + a_1 x + \cdots a_{n-1} x^{n-1}$ on n points: $x_0, \cdots, x_{n-1}$.

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On one point at at a time:

Example: $4 + 3x + 5x^2 + 4x^3$ on 2.

Polynomial Evaluation.

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Example: $4 + 3x + 5x^2 + 4x^3$ on 2.

Horners Rule: $4 + x(3 + x(5 + 4x))$

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$5 + 4x = 13,$

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$5 + 4x = 13$, then $3 + 2(13) = 29$,

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In general: $a_0 + x(a_1 + x(a_2 + x(...)))$.

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n multiplications/additions to evaluate one point.

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Evaluate on n points ——— $O(n^2)$ time.

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n multiplications/additions to evaluate one point.

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Could have just multiplied polynomials!

Evaluation of polynomials: Recursive.

$$A(x) = A_e(x^2) + x(A_o(x^2))$$

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Example:

$$A(x) = 4 + 12x + 20x^2 + 13x^3 + 6x^4 + 7x^5$$

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$$\begin{aligned} A(x) &= 4 + 12x + 20x^2 + 13x^3 + 6x^4 + 7x^5 \\ &= (4 + 20x^2 + 6x^4) + (12x + 13x^3 + 7x^5) \end{aligned}$$

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Plug in x^2 into A_e and A_o

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$$A(x) = A_e(x^2) + xA_o(x^2)$$

Plug in x^2 into A_e and A_o use results to find $A(x)$.

Recursive Evaluation.

$$A(x) = A_e(x^2) + x(A_o(x^2))$$

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Even coefficient polynomial.

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Evaluate recursively:

Recursive Evaluation.

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Evaluate recursively:
For a point x:

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Evaluate recursively:
For a point x:

Compute $A_e(x^2)$ and $A_o(x^2)$.

Recursive Evaluation.

$$A(x) = A_e(x^2) + x(A_o(x^2))$$

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Evaluate recursively:
For a point x:

Compute $A_e(x^2)$ and $A_o(x^2)$.

$$T(n) = 2T(n/2) + 1$$

Recursive Evaluation.

$$A(x) = A_e(x^2) + x(A_o(x^2))$$

where

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Odd coefficient polynomial.

$$A_o(x) = a_1 + a_3x + a_5x^2 ....$$

Evaluate recursively:
For a point x:

Compute $A_e(x^2)$ and $A_o(x^2)$.

$$T(n) = 2T(n/2) + 1 = O(n).$$

Recursive Evaluation.

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For a point x:

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$$T(n) = 2T(n/2) + 1 = O(n).$$

$O(n)$ for 1 point!

Recursive Evaluation.

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n points – $O(n^2)$ time to evaluate on n points.

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n points – $O(n^2)$ time to evaluate on n points.

No better than polynomial multiplication!

Recursive Evaluation.

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Evaluate recursively:
For a point x:

Compute $A_e(x^2)$ and $A_o(x^2)$.

$$T(n) = 2T(n/2) + 1 = O(n).$$

$O(n)$ for 1 point!

n points – $O(n^2)$ time to evaluate on n points.

No better than polynomial multiplication! Bummer.

Recursive on more than one point.

$A(x) = A_e(x^2) + x(A_o(x^2))$
Reuse computations.

Recursive on more than one point.

$A(x) = A_e(x^2) + x(A_o(x^2))$
Reuse computations.

n points: $\pm x_0, \pm x_1, \ldots, \pm x_{(n-1)/2}$.

Recursive on more than one point.

$A(x) = A_e(x^2) + x(A_o(x^2))$
Reuse computations.

n points: $\pm x_0, \pm x_1, \ldots, \pm x_{(n-1)/2}$. Also $n = d + 1$: number of corefficients.

Recursive on more than one point.

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Reuse computations.

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$\{x_0^{\log n}, \ldots, x_{n-1}^{\log n}\}| = 1$.

Each recursive level evaluates:
polynomials of half the degree on half as many points.
n represents both degree and number of points.

In reverse: start with a number 1
Take square roots: 1, -1.
Take square roots: 1,-1, i, $-i$.
Uh oh.

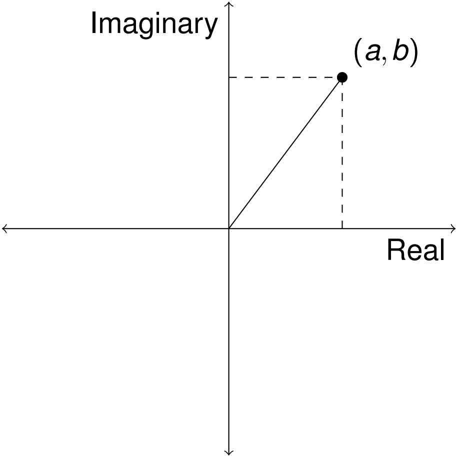
Actually: ± 1, $\pm i$, $\pm\frac{1}{\sqrt{2}}(1+i)$, $\pm\frac{1}{\sqrt{2}}(-1+i)$,

Complex numbers!

Uh oh? Can we get a pattern?

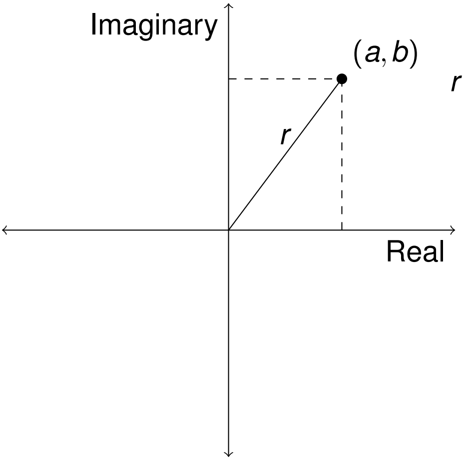
Complex plane

$$z = a + bi$$



Complex plane

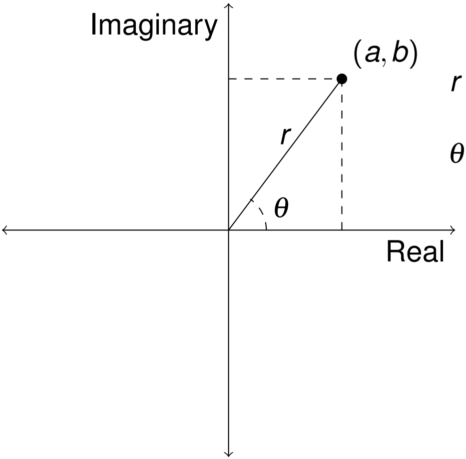
$$z = a + bi$$



$$r = \sqrt{a^2 + b^2}$$

Complex plane

$$z = a + bi$$

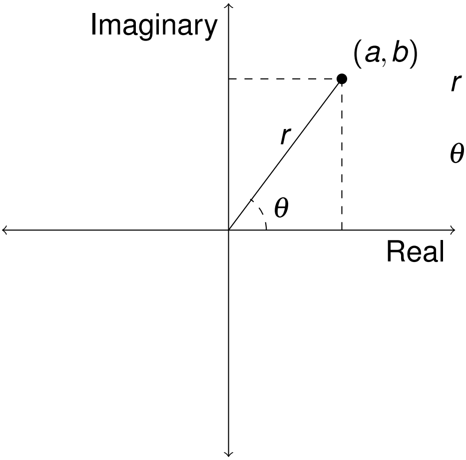


$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}(\tfrac{b}{a})$$

Complex plane

$$z = a + bi$$

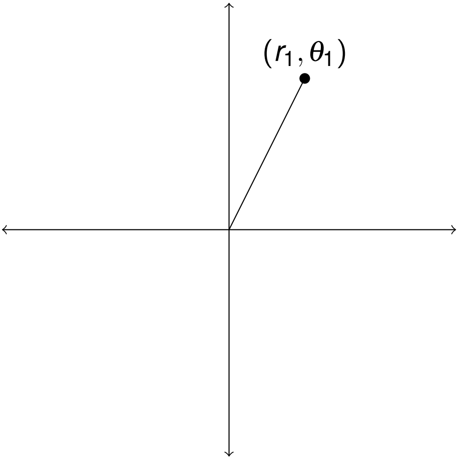


$$r = \sqrt{a^2 + b^2}$$

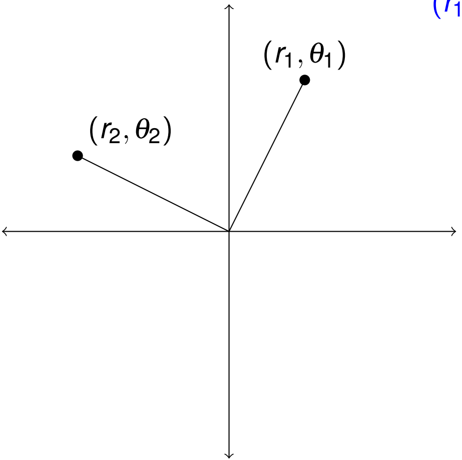
$$\theta = \tan^{-1}(\tfrac{b}{a})$$

Polar coordinate: $r(\cos\theta + i\sin\theta) = re^{i\theta}$ or (r, θ)

Multiplying Complex Numbers

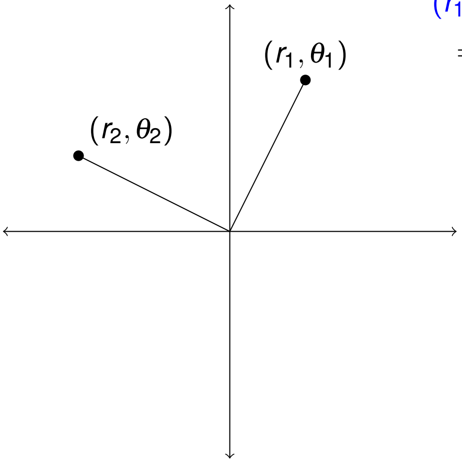


Multiplying Complex Numbers



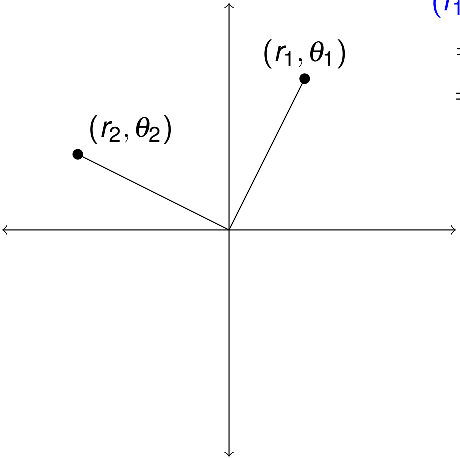
$(r_1, \theta_1) \times (r_2, \theta_2)$

Multiplying Complex Numbers



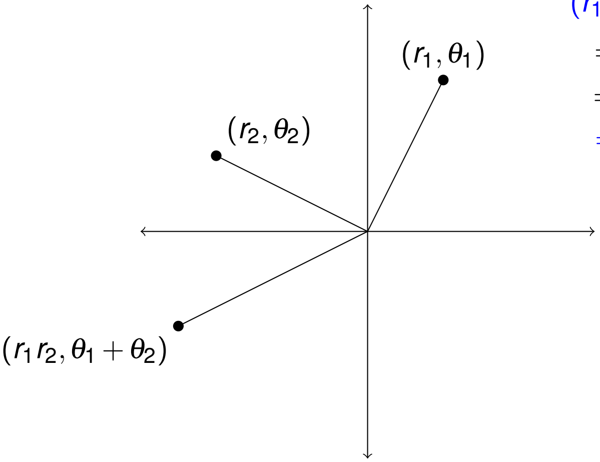
$$(r_1, \theta_1) \times (r_2, \theta_2)$$
$$= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$

Multiplying Complex Numbers



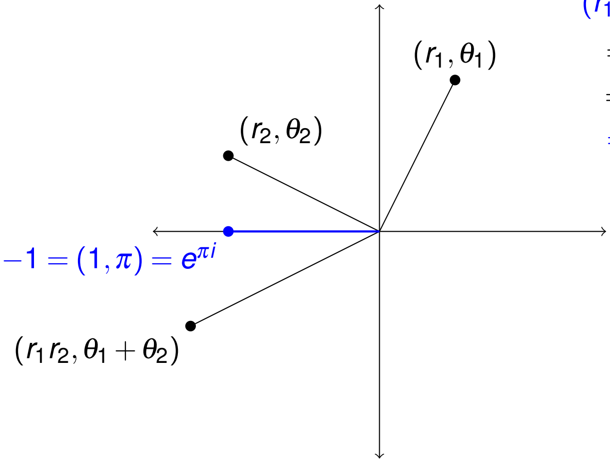
$$(r_1, \theta_1) \times (r_2, \theta_2)$$
$$= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$
$$= (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

Multiplying Complex Numbers



$$
\begin{aligned}
&(r_1, \theta_1) \times (r_2, \theta_2) \\
&\quad = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \\
&\quad = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\
&\quad = (r_1 r_2, \theta_1 + \theta_2)
\end{aligned}
$$

Multiplying Complex Numbers



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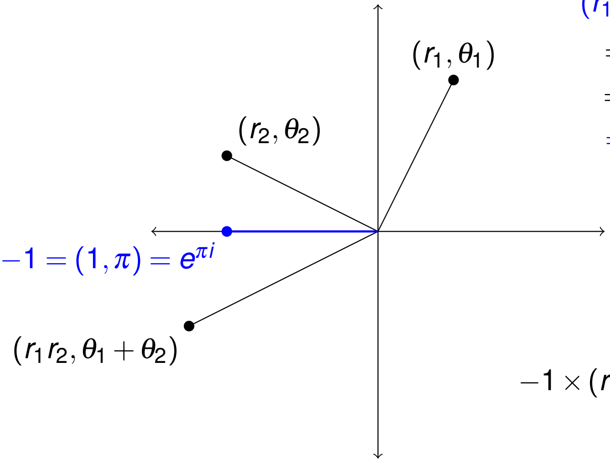
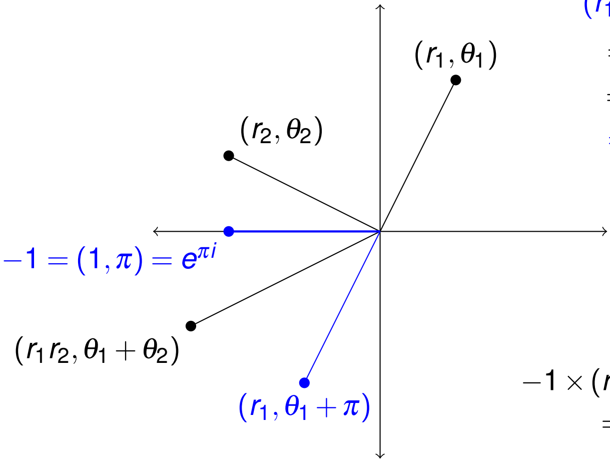
Multiplying Complex Numbers



$$
\begin{aligned}
(r_1, \theta_1) \times (r_2, \theta_2) & \\
&= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \\
&= (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\
&= (r_1 r_2, \theta_1 + \theta_2)
\end{aligned}
$$

$$-1 \times (r_1, \theta_1) = (r_1, \theta_1 + \pi)$$

Multiplying Complex Numbers

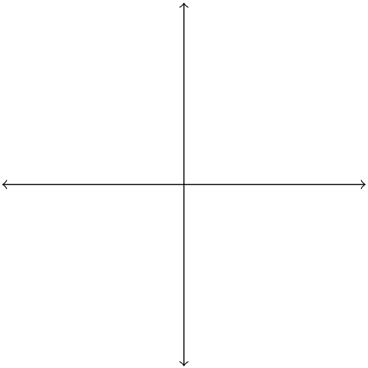


$$\begin{aligned}(r_1, \theta_1) &\times (r_2, \theta_2) \\ &= r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} \\ &= (r_1 r_2) e^{i(\theta_1 + \theta_2)} \\ &= (r_1 r_2, \theta_1 + \theta_2)\end{aligned}$$

$$\begin{aligned}-1 \times (r_1, \theta_1) &= (r_1, \theta_1 + \pi) \\ &= r_1 e^{r_1(\theta_1 + \pi)i}\end{aligned}$$

The nth complex roots of unity.

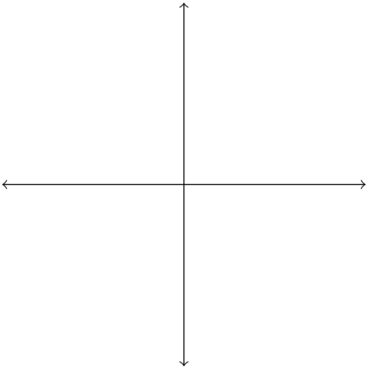
Solutions to $z^n = 1$



The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \frac{2\pi}{n})^n = (1, \frac{2\pi}{n} \times n) = (1, 2\pi) = 1!$$



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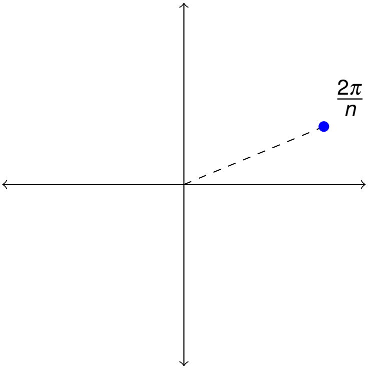
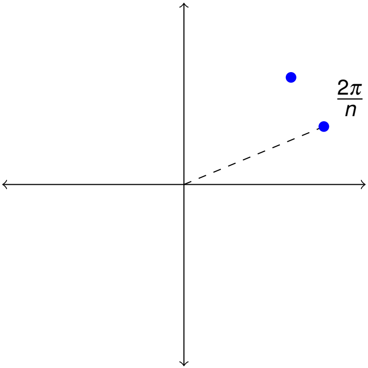


The nth complex roots of unity.

Solutions to $z^n = 1$

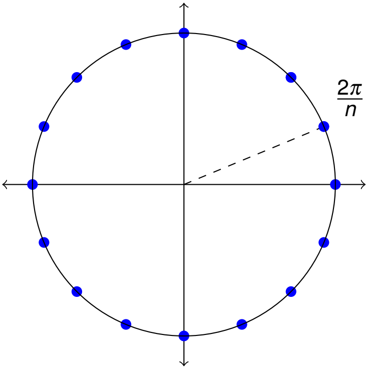
$$(1, \tfrac{4\pi}{n})^n = (1, \tfrac{4\pi}{n} \times n) = (1, 4\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

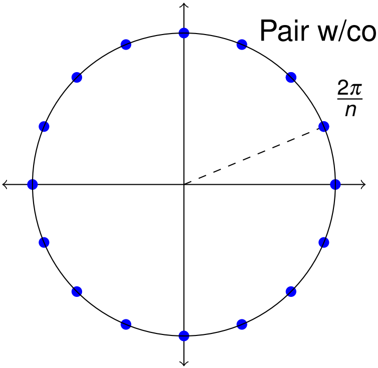
$$(1, \tfrac{2k\pi}{n})^n = (1, \tfrac{2k\pi}{n} \times n) = (1, 2k\pi) = 1!$$



The nth complex roots of unity.

Solutions to $z^n = 1$

$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$



Pair w/common square!

The nth complex roots of unity.

Solutions to $z^n = 1$

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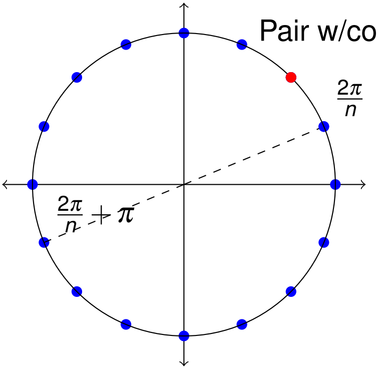
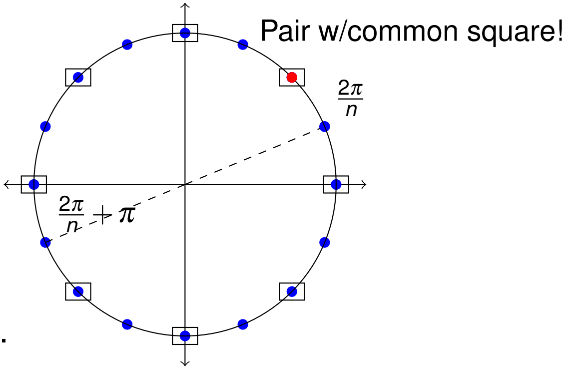


The *n*th complex roots of unity.

Solutions to $z^n = 1$

$$(1, \theta + \pi)^2 = (1, 2\theta + 2\pi) = \boxed{(1, 2\theta)} = (1, \theta)^2.$$



Squares: $\frac{n}{2}$th roots.

The FFT!

Defn: $\omega = (1, \frac{2\pi}{n}) = e^{\frac{2\pi}{n}i}$, nth root of unity.

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Evaluate $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$

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Procedure:

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Recursively compute A_e and A_o on $\frac{n}{2}$ roots of unity:

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For each $i \leq \frac{n}{2}$.

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For each $i \leq \frac{n}{2}$.

$$A(\omega^i) = A_e(\omega^{2i}) + \omega^i A_o(\omega^{2i})$$

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Runtime Recurrence:

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Runtime Recurrence:
A_e and A_o are degree $\frac{n}{2}$, $\frac{n}{2}$ points in recursion.

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$T(n) = 2T(\frac{n}{2}) + O(n)$

The FFT!

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Procedure:
Recursively compute A_e and A_o on $\frac{n}{2}$ roots of unity:
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For each $i \leq \frac{n}{2}$.

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Runtime Recurrence:
A_e and A_o are degree $\frac{n}{2}$, $\frac{n}{2}$ points in recursion.
$T(n) = 2T(\frac{n}{2}) + O(n) = O(n \log n)$!

More FFT ..

...on Friday..