CS 170: Algorithms

CS 170: Algorithms

Good Morning!

CS 170: Algorithms

Good Morning!

CS 170: Algorithms

Good Morning!

S

CS 170: Algorithms

Good Morning!

Sh

CS 170: Algorithms

Good Morning!

Shh

CS 170: Algorithms

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Shhh

CS 170: Algorithms

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Shhhh

CS 170: Algorithms

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CS 170: Algorithms

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Shhhh..

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Math Facts

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and since $S \subseteq T$ xare the same!

Testing primality.

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Primality or Carmichael (N) :

Do 100 times:
  Choose random $0 < a < N$:
    if $a^{N-1} \neq 1 \mod N$ return "Not prime."
Return "Prime or Carmichael"

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The probability that the algorithm is wrong $\leq \frac{1}{2^{100}}$.

One bad, many bad.

Primes/Carmichael number's:

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There is a with $\gcd(a,N) = 1$: $a^{N-1} \neq 1 \pmod N$.

Consider b: $b^{N-1} = 1 \pmod N$.

Then $(ab)^{N-1} = a^{N-1}b^{N-1} = a^{N-1} \neq 1 \pmod N$.
$(ab)^{N-1} \neq 1 \mod N$!

For every bad b (passes test), ab fails test!

How many fail test? (are good).
...ab fails test if b passes!

a has inverse $\pmod N$ since $gcd(a,N) = 1 \pmod N$. $ax \mod N$ is 1-to-1

For each bad b, ab is different and good.

At least as many good (fail test) as bad! At least half are good. Q.E.D.
Test your following: are exactly half good?
Yes? No? No.

One bad, half bad.

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Primality Testing

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def primalityOrCarmichael(N):
   for i in xrange(100):
     a = random_int(1,N-1)
     if  not( exp(a,N-1,N) == 1):
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Public Key Cryptography

Amazon wants to speak privately with

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and you!

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A single key for everyone! No privately shared keys!!

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Decrypt: $x^d \mod N$, modular exponentiation, $O(n^3)$.

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For each IP, x, place x in list at position $h(x)$ in array.

Hash Functions.

Hash Function: $h : U \to [0, \ldots, n-1]$.

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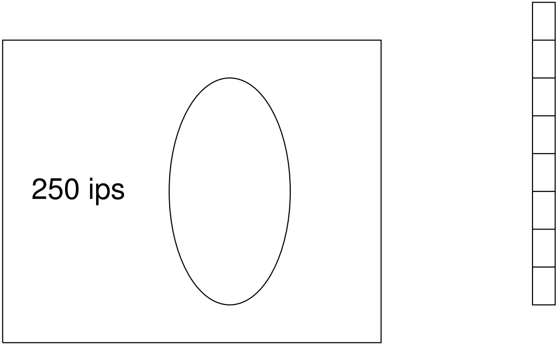

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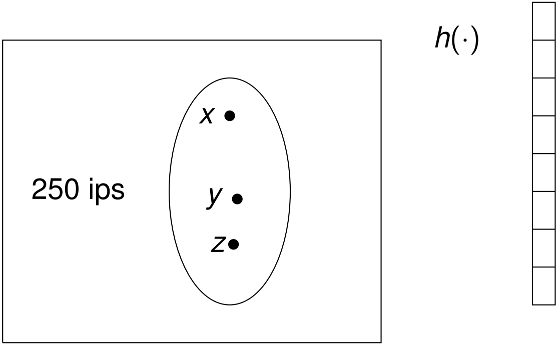

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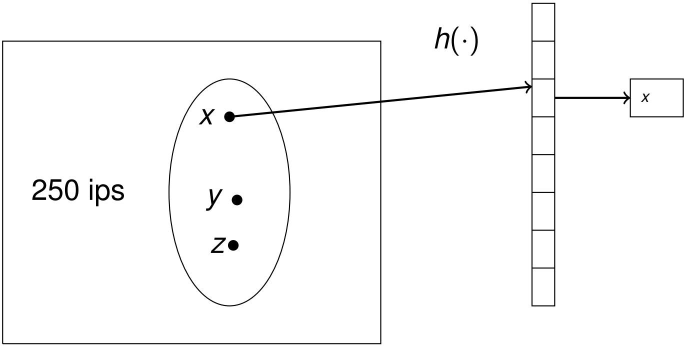

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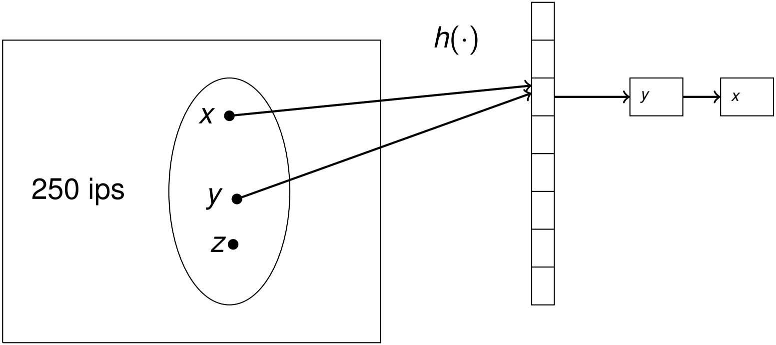

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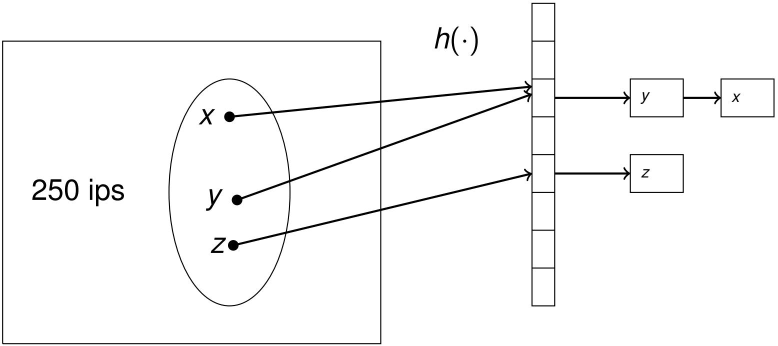

Picking a hash function

Last byte?

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Can this hash function be bad?

Is there a good hash function?

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there are 160,000,000 ips that map to the same entry!

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The first one goes to some entry.
The second has n choices, one of which entails a collision.

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"A bunch of hash functions" $\equiv$ A class of hash functions.

Class of hash functions.

Ip addresses consist of four bytes: x_1, x_2, x_3, x_4

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EX: $h_a(192, 168, 1, 10)$

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Specify hash function: $a = (a_1, a_2, a_3, a_4)$ where $a_i \in [0, \ldots, 256]$.

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Class of functions, indexed by four-tuple from $\{0, \ldots, 256\}$.

EX: $a = (1,1,1,1)$ has $h_a = x_1 + x_2 + x_3 + x_4 \pmod{257}$.
EX: $h_a(192, 168, 1, 10) = 192 + 168 + 1 + 10 = 114 \pmod{257}$
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(C) $1/n$ (as if x and y were placed randomly.)

Let's see.

For x and y, $h_a(x) = h_a(y)$, only if

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For x and y, $h_a(x) = h_a(y)$, only if

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How probable for random a?

Assume $x_4 \neq y_4$ (wlog). Let a_1, a_2, a_3 be chosen first.

Then $h_a(x) = h_a(y)$ only if

$$X + a_4x_4 = Y + a_4y_4 \pmod N.(*)$$

for $X = a_1x_1 + a_2x_2 + a_3x_3$, and analagous Y.

How many values of a_4 satisfy equation (*)?

$$a_4(x_4 - y_4) = Y - X \pmod N$$

N is prime. So $x_4 - y_4$ has a multiplicative inverse!

So, there is 1 value of a_4 that works..

1 out of n ... $\frac{1}{n}$ is probability of collision.

Wrapping up.

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$$Pr[h_a(x) = h_a(y)] = ???$$

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(B) $1/n^2$

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Just as if the keys were placed at random!

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Design pattern.

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Choose a hash function uniformly at random from family $\mathscr{H}$.

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$$h_a(x_1,\ldots,x_k) = a_1x_1 + \cdots a_kx_k \mod n.$$

This is universal by the same argument as above if n is prime.

Other applications...

..from polynomials.

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Error Correcting Codes: Reed-Solomon....

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Complexity theory: error correcting proofs!