CS 170 Algorithms
Fall 2013 Satish Rao

HW 10

Due November 15, 6:00pm

1. (10 pts.) Pizza Predicament

The pizza business in Little Town is split between two rivals, Tony and Joey. They are each investigating strategies to steal business away from the other. Joey is considering either lowering prices or cutting bigger slices. Tony is looking into starting up a line of gourmet pizzas, or offering outdoor seating, or giving free sodas at lunchtime. The effects of these various strategies are summarized in the following payoff matrix (entries are dozens of pizzas, Joey's gain and Tony's loss)

			Tony	
		Gourmet	Seating	Free soda
Joey	Lower price	+2	0	-3
	Bigger slices	-1	-2	+1

For instance, if Joey reduces prices and Tony goes with the gourmet option, then Tony will lose 2 dozen pizzas worth of business to Joey.

What is the value of this game, and what are the optimal strategies for Tony and Joey?

2. (15 pts.) Hollywood Hiring

A film producer is seeking actors and investors for his new movie. There are n available actors; actor i charges s_i dollars. For funding, there are m available investors. Investor j will provide p_j dollars, but only on the condition that certain actors $L_j \subseteq \{1,2,\ldots,n\}$ are included in the cast (*all* of these actors L_j must be chosen in order to receive funding from investor j).

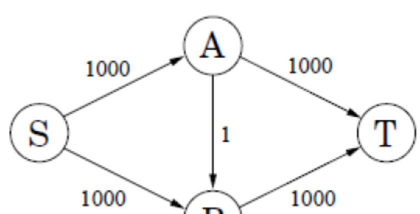
The producer's profit is the sum of the payments from investors minus the payments to actors. The goal is to maximize this profit.

a) Express this problem as an integer linear program in which the variables take on values $\{0,1\}$

b) Now relax this to a linear program, and show that there must in fact be an *integral* optimal solution (as is the case, for example, with maximum flow and bipartite matching).

3. (20 pts.) Fast Flow

Consider the following simple network with edge capacities as shown



(a) Show that, if Ford-Fulkerson algorithm is run on this graph, a careless choice of updates might cause it to take 1000 iterations. Imagine if the capacities were a million instead of 1000!

We will now find a strategy for choosing paths under which the algorithm is guaranteed to terminate in a reasonable number of iterations.
Consider an arbitrary directed network $(G = (V, E), s, t, \{c_e\})$ in which we want to find the maximum flow. Assume for simplicity that all edge capacities are at least 1, and define the capacity of an $s-t$ path to be the smallest capacity of its constituent edges. The *fattest path* from s to t is the path with the most capacity.

(b) Show that the fattest $s-t$ path in a graph can be computed by a variant of Dijkstra's algorithm.

(c) Show that the maximum flow in G is the sum of individual flows along at most $|E|$ paths from s to t.

(d) Now show that if we always increase flow along the fattest path in the residual graph, then the Ford-Fulkerson algorithm will terminate in at most $O(|E| \log F)$ iterations, where F is the size of the maximum flow. (Hint: It might help to recall the proof for the greedy set cover algorithm in Section 5.4)

In fact, an even simpler rule – finding a path in the residual graph using breadth-first search – guarantees that at most $O(|V| \cdot |E|)$ iterations will be needed.

4. (15 pts.) Evacuation Emergency

You are in charge of determining whether buildings have quick exit plans. You wish to determine the maximum number of people that can evacuate the building given a particular start room in at most T seconds. To do this, you model the interior plan of the building as follows:

- There are $|V|$ rooms in the building.
- The start room s. You wish to know the maximum number of people who can leave this room safely in T seconds.
- The single room t which represents the exit. Once people reach this point, they are considered safe.
- Rooms are connected to each other by one-way hallways. You have $|E|$ hallways total, with the hallway (u, v, c_{uv}, t_{uv}) connecting room u to room v having a maximum capacity of c_{uv} people and transit time t_{uv} seconds. At every second, this hallway can only send out c_{uv} from room u, and it takes those people t_{uv} seconds to make it to room v. Notice, this means that a hallway may contain up to $c_{uv} \times t_{uv}$ people in it at any moment in time. You may also assume the rooms can hold an arbitrarily large number of people who can wait in the room without using a hallway.

You may assume that the capacities, transit times, and T are all integers. You want to know the maximum value of N such that if N people begin at s at time 0, they can all reach t in at most time T using the hallways. Given the rooms V, hallways E, and start and end rooms, show how maximum flow in a modified graph can be used to solve this problem
Hint: Consider creating T copies of each room.

5. (20 pts.) Directed Decomposition

Consider a directed acylcic graph. We define a path to be starting at any node u and traversing 0 or more edges of the DAG and ending at an arbitrary node v (if we travel through 0 edges, we have $u = v$). A path P consists of the set of nodes that we touch between u and v (inclusive). We wish to find the minimum number of paths such that every vertex is in *exactly* one of these paths. Show how maximum matching in an appropriate bipartite graph can be used to solve the problem of determining the minimum number of paths needed to cover a DAG.

Week 11 Fun Fact

In their paper, Ford and Fulkerson originally credit their model to T.E. Harris who modeled a railway network as a maximum flow problem. Regions were represented as vertices, and each railway had a number assigned to it representing the rate at which material could be shipped from one region to the next, and they wanted to know the maximum material that could be shipped from one region to another.

However, contrary to Ford and Fulkerson's papers, the interest of the Air Force researcher Harris was not only to find the maximum flow, but also the minimum cut. In fact, the particular railway network was the Soviet's. The network was modeled as a graph with 44 verticies and 105 edges. In what was essentially trial and error, Harris and his colleague Ross determined the the maximum amount of material that could be shipped from Russia to Europe as well as the the cheapest way to disrupt the network.