Generally, for the normalized complex spherical harmonic functions, we have

$$\begin{aligned}
&(\mathcal{R}Y_j^p)\cdot(\mathcal{R}Y_l^m) \\
= \ &(\mathcal{R}_x Y_j^p)\,(\mathcal{R}_x Y_l^m) + (\mathcal{R}_y Y_j^p)\,(\mathcal{R}_y Y_l^m) + (\mathcal{R}_z Y_j^p)\,(\mathcal{R}_z Y_l^m)
\end{aligned} \tag{43}$$

Since

$$\begin{aligned}
(\mathcal{R}_x Y_j^p)\,(\mathcal{R}_x Y_l^m) &= -\frac{1}{4}(L_+Y_j^p + L_-Y_j^p)(L_+Y_l^m + L_-Y_l^m) \\
(\mathcal{R}_y Y_j^p)\,(\mathcal{R}_y Y_l^m) &= -\frac{1}{4}(L_+Y_j^p - L_-Y_j^p)(L_+Y_l^m - L_-Y_l^m)
\end{aligned}$$

So,

$R_z = i L_z =$

$$\begin{aligned}
&(\mathcal{R}Y_j^p)\cdot(\mathcal{R}Y_l^m) \\
= \ &-\frac{1}{2}\left((L_+Y_j^p)(L_-Y_l^m) + (L_-Y_j^p)(L_+Y_l^m)\right) - (L_z Y_j^p)\,(L_z Y_l^m) \\
= \ &-\frac{1}{2}\left(c_{j,p,1}\,c_{l,m,2} Y_j^{p+1} Y_l^{m-1} + c_{j,p,2}\,c_{l,m,1}\, Y_j^{p-1} Y_l^{m+1}\right) - m\,p\,Y_j^p Y_l^m
\end{aligned} \tag{44}$$

$a l$ $\quad \theta\phi)$

Therefore

$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^m Y_l^m \quad \sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_l^m Y_l^m \quad \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_l^m Y_l^m,$

$$\int_{||\mathbf{m}||=1} Y_n^k\,(\mathcal{R}Y_j^p)\cdot(\mathcal{R}Y_l^m)\,d\mathbf{m} \quad = \sum\sum\left(\sim a_l^m + \sim b_l^m + \sim c_l^m\right) Y_{l'}^{m'}$$

$$\begin{aligned}
= \ -&\left[\frac{1}{2}\left(c_{l,m,2}\,c_{j,p,1}\,|n,k;j,p+1;l,m-1\rangle + c_{l,m,1}\,c_{j,p,2}\,|n,k;j,p-1;l,m+1\rangle\right) + \right. \\
&\left. m\,p\,|n,k;j,p;l,m\rangle\right], \qquad \text{if } k=-m-p \text{ and } |l-j|\le n\le l+j;
\end{aligned} \tag{45}$$

otherwise, it is zero. Here, the symbol $|n,k;j,p;l,m\rangle$ denotes the integral

$$\int_{||\mathbf{m}||=1} Y_n^k\, Y_j^p\, Y_l^m\, d\mathbf{m} \tag{46}$$

Appendix C

Denote $\mathbf{m}$, $\mathbf{s}$ and $\mathbf{t}$ the unit vectors in the spherical coordinate. We can compute

$$\begin{aligned}
\Omega\cdot\mathbf{m} &= \frac{1}{2}\mu(\sin\theta\,\sin\phi,\ -\sin\theta\cos\phi,\ 0) \\
&= (-\frac{1}{2}\mu\sin\theta)\,\mathbf{t}
\end{aligned} \tag{47}$$

$$\begin{aligned}
\mathbf{D}\cdot\mathbf{m} &= \frac{1}{2}\mu(\sin\theta\,\sin\phi,\ \sin\theta\cos\phi,\ 0) \\
&= \frac{1}{2}\mu\sin\theta\left[(\sin\theta\sin 2\phi)\,\mathbf{m} + (\cos\theta\sin 2\phi)\,\mathbf{s} + (\cos 2\phi)\,\mathbf{t}\right].
\end{aligned} \tag{48}$$

$$\mathbf{D}:\mathbf{mm} = \mu\,\sin^2\theta\,\sin\phi\,\cos\phi \tag{49}$$

$$\mathbf{D}:\mathbf{mmm} = \frac{1}{2}\mu\,\sin^2\theta\,\sin 2\phi\,\mathbf{m} \tag{50}$$