from P6 eq 44

$$R Y_j^p \cdot R Y_l^m = -\left(\frac{1}{2}\left[C_{j,p,1} C_{l,m,2} Y_j^{p+1} Y_l^{m-1} + C_{j,p,2} C_{l,m,1} Y_j^{p-1} Y_l^{m+1}\right] + mp\, Y_j^p Y_l^m\right)$$

$$= (-1)^{m+p} \sum_{k=|l-j|}^{l+j} Y_k^{m+p} \left[\frac{1}{2}\left(C_{j,p,1} C_{l,m,2}\, \alpha_{l(m-1)j(p+1)k} + C_{jp2} C_{lm1}\, \alpha_{l(m+1)j(p-1)k}\right) + mp\, \alpha_{lmjpk}\right]$$

$(-1)\cdot$

where $\alpha_{lmjpk} = \int_{\|\vec{m}\|=1} Y_j^p Y_l^m Y_k^{-m-p}\, d\vec{m} = \left[\frac{(2l+1)(2j+1)(2k+1)}{4\pi}\right]^{1/2} \begin{pmatrix} j & l & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j & l & k \\ p & m & -(m+p) \end{pmatrix}$

Since the recursive relation in P760, 34.3.14, NIST Handbook, 2010.

we have $\frac{1}{2}\left(C_{j,p,1} C_{l,m,2}\, \alpha_{l(m-1)j(p+1)k} + C_{jp2} C_{lm1}\, \alpha_{l(m+1)j(p-1)k} + 2mp\, \alpha_{lmjpk}\right)$

$$= \frac{1}{2}\left[\frac{(2l+1)(2j+1)(2k+1)}{4\pi}\right]^{1/2} \begin{pmatrix} j & l & k \\ 0 & 0 & 0 \end{pmatrix} ① \cdot$$

$$\left[C_{jp1} C_{lm2} \begin{pmatrix} j & l & k \\ p+1 & m-1 & -(m+p) \end{pmatrix} + C_{jp2} C_{lm1} \begin{pmatrix} j & l & k \\ p-1 & m+1 & -(m+p) \end{pmatrix} + 2mp \begin{pmatrix} j & l & k \\ p & m & -(m+p) \end{pmatrix}\right]$$

$$= ① \cdot \begin{pmatrix} k & j & l \\ -(m+p) & p & m \end{pmatrix} \left(k^2 - j^2 - l^2 + k - j - l - 2pm + 2mp\right)$$

$$= \left[\frac{(2l+1)(2j+1)(2k+1)}{4\pi}\right]^{1/2} \begin{pmatrix} j & l & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k & j & l \\ -(m+p) & p & m \end{pmatrix} \frac{1}{2}\left(k^2 - j^2 - l^2 + k - j - l\right)$$

$$= \alpha_{lmjpk} \cdot \frac{1}{2}\left(k^2 - j^2 - l^2 + k - j - l\right)$$

$$\therefore R Y_j^p \cdot R Y_l^m = (-1)^{m+p} \sum_{k=|l-j|}^{l+j} Y_k^{m+p} \cdot \alpha_{lmjpk} \cdot \frac{1}{2}\left(j^2 + l^2 - k^2 + j + l - k\right)$$