Problem 1

a) $D = \{(x^i, y^i) \mid 1 \leq i \leq m\}$

Feature mapping function ϕ

$K(x, z) = \langle \phi(x), \phi(z) \rangle$

Uni- variance radial basis kernel

$K(x, z) = \exp\left(\frac{-1}{2} \|x - z\|^2\right)$

$= \|\phi(x) - \phi(z)\|^2$

$= \langle \phi(x), \phi(x) \rangle + \langle \phi(z), \phi(z) \rangle - 2 \langle \phi(x) \cdot \phi(z) \rangle$

$= K(x, x) + K(z, z) - 2 K(x, z)$

$= 1 + 1 \cancel{0} - 2 \exp\left(\frac{1}{2} \|x - z\|^2\right)$

$\|\phi(x) - \phi(z)\|^2 < 2 \Rightarrow \sqrt{\|\phi(x) - \phi(z)\|^2} < \sqrt{2}$

Hence the

the distance between the ~~dist~~ feature mapping of x & z i.e $\phi(x)$ & $\phi(z)$ is at most $\sqrt{2}$

A function is a valid kernel if it is Real-value positive semi definite and is symmetric

Let us define a kernel matrix for a dataset $\{x_i\}_{i=1}^{n} \in \mathbb{R}^n$

gram matrix

$$K = \begin{bmatrix} K(x_1, x_1) & \cdots & K(x_1, x_n) \\ \vdots & & \\ K(x_n, x_1) & \cdots & K(x_n, x_n) \end{bmatrix}$$

K is a valid kernel if for any set of points $\{x_i\}_{i=1}^{n} \in \mathbb{R}^n$

it gives rise to matrix K that is positive semidefinite i.e if

$$\boxed{\forall a \in \mathbb{R}^n, \quad a^T K a \geq 0} \quad \text{PSD.}$$

A matrix is PSD if all its eigen values are non-negative

b) $K(x,z) = K_1(x,z) + K_2(x,z)$

K_1 & K_2 are both valid kernels.

By gram matrix

$$K = \sum_{j=1}^{2} K^j$$

Now for positive semidefinite

$$\forall a \in \mathbb{R}^n, \; a^T K a$$

$$= \sum_{j=1}^{2} a^T K^j a \geq 0$$

Now if K^j is valid

$$\boxed{K(x,z) = K_1(x,z) + K_2(x,z)}$$

Hence it is a valid kernel

c) $K(x,z) = K_1(x,z) - K_2(x,z)$

Σ - K_1 & K_2 are valid kernels

$$K_1 = \sum_{i=1}^{n} \lambda_i v_i v_i^T \qquad K_2 = \sum_{j=1}^{n} \alpha_j v_j v_j^T$$

$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T - \sum_{j=1}^{n} \alpha_j v_j v_j^T$$

which cannot be broken as

$K(x,z)$ ~~as~~ , $a^T K a$

$\boxed{\text{Hence it is not a valid kernel}}$

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d) $K(x,z) = a K_1(x,z)$ $(a>0)$

$K = \alpha K_1$ $\alpha > 0$

$\forall \, a \in R^n$, $a^T K a = \alpha a^T K_1 a \geq 0$

due to $\alpha > 0$ & validity of K_1

$\boxed{K = \alpha K_1(x,z) \text{ is a valid kernel}}$

e) $K(x,z) = b K_1(x,z)$

$K = b K_1$ $b < 0$

$\forall \, a \in R^n$, $a^T K a = a^T b K_1 a < 0$

Now b is negative & K_1 is ≥ 0. and valid.

Hence

$\boxed{K(x,y) = b K_1(x,y)}$ is not valid

f) $K(x,z) = K_1(x,z)\, K_2(x,z)$

Gram matrix is given by.

$$\boxed{K = K_1 \cdot K_2}$$

K_1 & K_2 positive semi definite

their eigen decomposition

$$K_1 = \sum_{i=1}^{n} \lambda_i u_i u_i^T \qquad K_2 = \sum_{j=1}^{n} \mu_j v_j v_j^T$$

$$\lambda_i \geq 0 \qquad \mu_j \geq 0$$

$$K = \sum_{i=1}^{n} \sum_{j=1}^{n} \underbrace{\lambda_i \mu_j}_{\gamma_K} \underbrace{(u_i v_i^T)}\, \underbrace{(v_j v_j^T)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_K (u_i \cdot v_j)(u_i \cdot v_j)^T$$

$$\boxed{K = \sum_{K=1}^{n^2} \gamma_K w_K w_K^T}$$

with $\gamma_K = \lambda_i \mu_j \geq 0$ $\quad w_k = u \cdot v$

$$\forall a \in \mathbb{R}^n \quad a^T K a = \sum_{K=1}^{n^2} \gamma_K a^T w_K w_K^T a$$

$$K = \sum_{K=1}^{n^2} \gamma_K (w_K^T a)^2$$

Hence valid kernel

g) $K(x,z) = f(x) f(z)$ ($f: \mathbb{R}^n \to \mathbb{R}$, real valued func^n)

$\phi : \mathcal{X} \to \mathcal{X}^\phi$

$x \to \phi(x)$

A valid kernel is
$k(x,z) = \langle \phi(x), \phi(z) \rangle_{\mathcal{X}^\phi}$

for any real ~~fu~~ valued func^n

$$\boxed{K(x,z) = f(x) f(z)}$$

Hence a valid kernel

h) $K(x,z) = K_3(\phi(x), \phi(z))$ (K_3: another valid kernel over $\mathbb{R}^d \to \mathbb{R}^d$)

Now $\phi(x), \phi(z)$ are transformation to higher dimensional feature space

& K_3 is a valid kernel in that space

Hence $\boxed{K(x,z) = k(\phi(x), \phi(z)) \text{ is a valid kernel}}$

i) $K(x,z) = P(K_1(x,z))$

$p(x)$ a polynomial function with positive coefficients

$P: R \to R$ with (+ve) cofficients.

P is a linear combination of power of kernel K_1 ~~witch~~ which is valid with +ve coefficien

since the power of K_1 of K_1 by itsely.

this makes

$$\boxed{K(x,y) = p(K_1(x,z)) \text{ a valid kernel}}$$