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EE16BTECH11022

Concentration Inequalities

Assignment 1

① ⓐ Almost sure convergence

A random variable X_n is said to be almost surely converging to another random variable X given

$$Pr\left\{\lim_{n\to\infty} X_n = X\right\} = 1$$

ⓑ X_n is said to probabilistically converge to X. given

$$Pr\left\{\lim_{n\to\infty} |X_n - X| \geq \varepsilon\right\} = 0 \quad \text{for any } \varepsilon > 0$$

ⓒ $X_n \xrightarrow{ms} X$ given

$$\lim_{n\to\infty} E\left\{(X_n - X)^2\right\} = 0.$$

ⓓ $X_n \xrightarrow{d} X$ given

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

② $\theta = U[0, 2\pi]$ $\quad U$ = uniform distr

ⓐ $(X_n : n > 1)$ $\quad X_n = \cos(n\theta)$

$$F_{X_n}(x) = Pr\left\{\cos(n\theta) > x\right\}$$

$$= Pr\left\{\theta > \frac{\cos^{-1}(x)}{n}\right\}$$

$$F_{X_n}(x) = \left[\frac{\cos^{-1}(x)}{n}\right]\left(\frac{1}{2\pi}\right)$$

$$\Rightarrow f_{X_n}(x) = \frac{d}{dx}\left(\frac{1}{2\pi n}\cdot \cos^{-1}(x)\right)$$

$$f_{X_n}(x) = \frac{1}{2\pi n} \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$\boxed{\lim_{n\to\infty} f_{X_n}(x) = 0} \; ??$$

(a) $\cos(n\theta)$ where $\theta = U[0, 2\pi]$

doesnt converge to any sensible distribution

as Cos is a periodic function

(b) $\theta \in [0, 2\pi]$

$$\frac{\theta}{\pi} \in [0, 2]$$

$$1 - \frac{\theta}{\pi} \in [-1, 1]$$

$$\left|1 - \frac{\theta}{\pi}\right| \in [0, 1]$$

$$\left|1 - \frac{\theta}{\pi}\right|^n \in [0, 1] \quad \forall n \in \mathbb{R}.$$

$$\Rightarrow \lim_{n\to\infty} \left|1 - \frac{\theta}{\pi}\right|^n = 0 \quad \text{if } [\theta \neq 2\pi \text{ (or) } \theta \neq 0]$$

$\therefore$ Distribution converges to 0

except at one point

$\theta = 1$

③ MGF = $E(e^{sx})$ $\quad X \sim N(0, \sigma^2)$

$$MGF = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2} + sx} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2 + 2\sigma^2 s x - (\sigma^2 s)^2 + (\sigma^2 s)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\sigma\sqrt{s})^2}{2\sigma^2} + \frac{\sqrt{s}}{2\sigma}} dx$$

$$= \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}}\, e^{\frac{(\sigma^2 s)^2}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\sigma\sqrt{s})^2}{2\sigma^2}} dx \rightarrow \cancel{\sigma\sqrt{s}}\ N(\sigma^2 s, \sigma^2)$$

$$= e^{(\sigma^2 s)^2 / 2\sigma^2} \left[1 \right]$$

$$\boxed{MGF = e^{\frac{\sigma^2 s^2}{2}}}$$

4. Markov Inequality :-

$$Pr\{X \geq \varepsilon\} \leq \frac{E(X)}{\varepsilon}$$

(a) Chebisev's Inequality

Consider $Y = |X - \mu|$

$$Pr\{|X-\mu| > \varepsilon\} = Pr\{|X-\mu|^2 > \varepsilon^2\}$$

by monotonicity property

and

$$Pr\{|X-\mu|^2 > \varepsilon^2\} \leq \frac{E(|X-\mu|^2)}{\varepsilon^2}$$

$$Pr\{|X-\mu|^2 > \varepsilon^2\} \leq \frac{Var(X)}{\varepsilon^2}$$

Hence proved

(b) Chernoff Bound

$$Pr\{X \geq \varepsilon\} \leq \frac{E(X)}{\varepsilon}$$

Let t be a constant, then by monotonicity

$$P_r(X > \varepsilon) = P_r\{Xt > \varepsilon t\} = P_r\{e^{Xt} \geq e^{\varepsilon t}\}.$$

now, let $e^{Xt} = Y$

$$P_r\{Y > e^{\varepsilon t}\} \leq \frac{E(Y)}{e^{\varepsilon t}}$$

$$\therefore \boxed{P_r\{e^{Xt} > e^{\varepsilon t}\} \leq \frac{E(e^{Xt})}{e^{\varepsilon t}}}$$

(6) (a) LLN mathematical statement

$$\lim_{n\to\infty} P_r\left\{\left|\frac{S}{n} - \mu\right| > \varepsilon\right\} = 0 \qquad S = \sum_i X_i \qquad X_i \sim iid.$$

$$X_i \sim N(\mu, \sigma)$$

now $E\left(\frac{S}{n}\right) = \mu$

$$Var\left(\frac{S}{n}\right) = \frac{\sigma^2}{n}$$

Consider

$$P_r\left\{\left|\frac{S}{n} - \mu\right| > \varepsilon\right\} \cancel{P_r\left\{\left|\frac{S}{n} - \mu\right|^2 > \varepsilon^2\right\}}$$

by Chebyshev's Inequality

$$P_r\left\{\left|\frac{S}{n} - \mu\right| > \varepsilon\right\} \leq \frac{Var\left(\frac{S}{n} - \mu\right)}{\varepsilon^2}$$

$$\leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\boxed{\lim_{n\to\infty} Pr\left\{ \left|\frac{S}{n} - \mu\right| > \varepsilon \right\} = 0}$$

(5)

(a) Markov Inequality is tight

Let $X = 1$

$X = -1$

with probabilities 0.5

then $P\left(|X| \geq 1\right) \leq \dfrac{E(|X|)}{1}$

$$P\left(|X| \geq 1\right) \leq 1$$

this inequality holds with all equalities

$$P\left(|X| = 1\right) = 1$$

$\therefore$ Bound is tight

(b) Chernoff Bound tighter than Chebysev's bound

$$Pr\left\{ |X - \mu| > \varepsilon \right\} \leq \frac{Var(X)}{\varepsilon^2}$$

$$Pr\left\{ e^{Xt} > e^{\varepsilon t} \right\} \leq \frac{E\left(e^{Xt}\right)}{e^{\varepsilon t}}$$

Consider gaussian $X \sim (0, \sigma^2)$

$$Pr\{|X-\mu| > \varepsilon\} \le \frac{\sigma^2}{\varepsilon^2} \quad \{\text{for } \mu = 0\} \quad Pr\{|X| > \varepsilon\} \le \frac{\sigma^2}{\varepsilon^2}$$

$$Pr\{e^{Xt} > e^{\varepsilon t}\} \le \frac{E(e^{Xt})}{e^{\varepsilon t}}$$

$$Pr\{e^{Xt} > e^{\varepsilon t}\} \le e^{\frac{\sigma^2 t^2}{2} - \varepsilon t}$$

now technically

Chernoff bound is for $Pr\{X > \varepsilon\}$

And Chebyshev is for $Pr\{|X| > \varepsilon\}$

then considering only positive half gaussian, we have

$$\frac{e^{\frac{\sigma^2 t^2 - 2\varepsilon t}{2}}}{2} \le \frac{\sigma^2}{\varepsilon^2}$$

$$e^{\frac{\sigma^2 t^2}{2}} \le \frac{2 e^{\varepsilon t} \sigma^2}{\varepsilon^2}$$

$$\frac{e^{\sigma^2 t^2/2}}{\sigma^2} \le \frac{2 e^{\varepsilon t}}{\varepsilon^2}$$

$$\frac{e^{\alpha}}{\alpha} = \frac{\alpha e^{\alpha} - e^{\alpha}}{\alpha^2}$$

$$e \quad \cancel{\frac{e^{\alpha}(\alpha - 1)}{\alpha^2}}$$

$$e^{t^2/2} \le \frac{2 e^{\varepsilon t}}{\varepsilon^2} \quad \cancel{\sigma^2 \frac{t^2}{2} e^{\sigma^2 t^2/2}} \; \cancel{e^{\sigma^2 t^2/2}}$$

$$\boxed{\begin{array}{c} \sigma^2 = 1 \\ \sigma = 1 \end{array}}$$

$\therefore$ for standard normals

∴ for standard normal positive gauss..

Chernoff is tighter than chebisev

(c) ~~Pr{X < 3000}~~

$$\Pr\{|X-\mu| > 3000+\mu\} \le \frac{Var(X)}{(3000+\mu)^2}$$

$$\le \frac{168.75}{(3027.50)^2}$$

$$\boxed{\Pr\{|X-\mu| > 3000+\mu\} \le 1.84\times 10^{-5}}$$ → Chebysev's Bound

By CLT and assuming Gaussian we have

$$\Pr\{X > 3000\}$$

$$\Pr\left\{\frac{X-\mu}{\sigma} > \frac{3000-27.50}{168.75}\right\}$$

$$= Q\left(\frac{3000-27.50}{168.75}\right) = Q(17.61)$$

$$\approx 7.16\times 10^{-69}$$

⇒ CLT straight forward says the likely hood of it occuring is zero

~~tends~~

Whereas Chebysev gives a more realistic view of the situation.

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(7) Let X be a random variable

(b) $X \sim N(0, \sigma)$

then $E\left[e^{\lambda X}\right] \le e^{\lambda^2 \sigma^2 / 2}$

RHS ~~this~~ is the MGF of the RV so always true

(a) Rademacher distro

$$X = \begin{cases} 1 & Pr = 1/2 \\ -1 & Pr = 1/2 \\ 0 & \text{else} \end{cases}$$

$$e^{\lambda X} = \begin{cases} e^{\lambda} & Pr = 1/2 \\ e^{-\lambda} & Pr = 1/2 \\ 1 & \text{else} \end{cases}$$

$$E(e^{\lambda X}) = \frac{e^{\lambda} + e^{-\lambda}}{2} \ge$$

$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots$$

$$e^{-\lambda} = 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots$$

$$\frac{e^{\lambda} + e^{-\lambda}}{2} = \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots\right)$$

$$e^{\lambda^2} = 1 + \frac{\lambda^2}{1!} + \frac{\lambda^4}{2!} + \dots$$

$$\therefore \quad \frac{e^{\lambda} + e^{-\lambda}}{2} \leq e^{\lambda^2}$$

$$\therefore \boxed{\sigma = \sqrt{2}}$$

$\therefore$ Randemarcher is sub gaussian with $\sigma \geq \sqrt{2}$

(c) $$e^{\lambda x} \in \left[e^{a\lambda}, e^{b\lambda}\right]$$

$$\mathbb{E}\left(e^{\lambda X}\right) = \int_a^b e^{\lambda x} P_X(x)\,dx$$

but $$\int_a^b x P_X(x)\,dx = 0$$

and by Hoeffding's Lemma

$$E\left[e^{\lambda X}\right] \leq e^{\frac{\lambda^2 (b-a)^2}{8}}$$

$$\boxed{\therefore \sigma^2 = (b-a)^2/4}$$

(8) $P[|X-\mu|>t] \leq \frac{Var(X)}{t^2}$ by Chebyshev

but $Pr[|X-\mu|>t] = Pr[e^{\lambda|X-\mu|} > e^{\lambda t}] \leq \frac{E(e^{\lambda|X-\mu|})}{e^{\lambda t}}$

$\therefore Pr[|X-\mu|>t] \leq \frac{E(e^{\lambda|X-\mu|})}{e^{\lambda t}}$

Also

$Pr[|X-\mu|>t] = Pr[X-\mu>t] + Pr[X-\mu<-t]$

$Pr[X-\mu>t] \leq \frac{E(e^{\lambda(X-\mu)})}{e^{\lambda t}} \leq \left(e^{\lambda^2\sigma^2/2 - \lambda t}\right)$

for the minimum most

$\lambda = t/\sigma^2$ (by differentiation)

$P[X-\mu>t] \leq e^{\frac{t^2}{2\sigma^2} - t^2/\sigma^2}$

$P[X-\mu>t] \leq e^{-t^2/2\sigma^2}$ ang ll^y for $P[X-\mu<-t]$

$$\boxed{Pr[|X-\mu|>t] \leq 2e^{-t^2/2\sigma^2}}$$

(9) $E(e^{\lambda x_i}) \leq e^{\sigma^2\lambda^2/2}$ by subgaussian definition

$$E(e^{\lambda \Sigma x_i}) = E(e^{\lambda x_1} e^{\lambda x_2} \cdots e^{\lambda x_n})$$

Since x_i are independent

$$= E(e^{\lambda x_1}) E(e^{\lambda x_2}) \cdots$$

$$= \prod_i E(e^{\lambda x_i})$$

$$\leq \prod_i e^{\sigma_i^2 \lambda^2/2}$$

$$\leq e^{\frac{\lambda^2 \Sigma \sigma_i^2}{2}}$$

Let $\Sigma \sigma_i^2 = \alpha^2$

$$\therefore E(e^{\lambda \Sigma x_i}) \leq e^{\lambda^2 \alpha^2/2}$$

$\therefore \exists \alpha$

$\therefore \Sigma X_i$ is an α sub gaussian

(10) Let $Y = \frac{1}{n} \sum_{i=1}^{n} (X_i - E(X_i))$

$Pr\{Y \geq t\}$ is what we need

$$Pr\{Y \geq t\} = Pr\{e^{\lambda Y} > e^{\lambda t}\} \leq e^{-\lambda t} E(e^{\lambda Y})$$

$$Pr\{Y \geq t\} \leq e^{-\lambda t} E\left(e^{\lambda/n [\Sigma(X_i - E(X_i))]}\right)$$

if assumed independent then

$$Pr\{Y \geq t\} \leq \prod_{i=1}^{n} e^{-\lambda t} E\left(e^{\lambda/n (X_i - E(X_i))}\right)$$

Consider $E\left(e^{\frac{\lambda(X_i - E(X_i))}{n}}\right)$

by Hoeffding's Inequality

$$E(e^{\lambda Y_i}) \leq e^{\frac{\lambda^2 (b_i - a_i)^2}{8}}$$

$$\therefore Pr\{Y \geq t\} \leq \prod_{i=1}^{n} e^{-\lambda t} e^{\frac{\lambda^2 (b_i - a_i)^2}{8n^2}}$$

to tighten the bound, derivate WRT λ

$$\frac{d}{d\lambda}\left(\frac{\lambda^2 \Sigma (b_i - a_i)^2}{8n^2} - \lambda t\right) = 0 \Rightarrow \lambda = \frac{4n^2 t}{\lambda(\Sigma(b_i - a_i)^2)}$$

$$\therefore \Pr\left(\left\{\frac{\Sigma(X_i - E(X_i))}{n}\right\} \geq t\right) \leq e^{\frac{-2n^2t^2}{\Sigma(b_i-a_i)^2}}$$

for the other part, consider

$$\Pr\left\{Y = \frac{\Sigma(E(X_i) - X_i)}{n} > t\right\}.$$

the results still stay similar due to hoefding's inequality

$$\therefore \Pr\left\{\frac{\Sigma X_i - E(X_i)}{n} \leq -t\right\} \leq \exp\left(-\frac{2n^2t^2}{\Sigma(b_i-a_i)^2}\right)$$

Reason :-

$$Y = \frac{E(X_i) - X_i}{n}$$

$$Y \in \left[\frac{E(X_i) - b}{n}, \frac{E(X_i) - a}{n}\right]$$

$$\left(\begin{array}{l}\text{interval}\\ \text{size}\end{array}\right) = \left(\frac{b-a}{n}\right)$$

interval is all that matters for Hoeffding's Ineq.

(12) $\Pr[X > \mu + t] = \Pr[e^{\lambda X} > e^{\lambda(\mu+t)}]$

$$\Pr[e^{\lambda(X-\mu)}] \le e^{v^2\lambda^2/2} \quad \text{for } |\lambda| \le \frac{1}{b}$$

Now let $X - \mu = Y$

$$\Pr[e^{\lambda Y}] \le e^{\lambda^2 v^2/2} \quad \text{for } |\lambda| \le \frac{1}{b}$$

$$\therefore \Pr[X > \mu + t] = \Pr[X - \mu > t] = \Pr[Y > t] \le \frac{E(e^{\lambda Y})}{e^{\lambda t}}$$

$$\therefore \Pr[X > \mu + t] \le e^{\frac{v^2\lambda^2}{2} - \lambda t}$$

for $\lambda = t/v^2$, we have a tighter bound

$$\Pr[X > \mu + t] \le e^{-t^2/2v^2}$$

$$|\lambda| \le \frac{1}{b}$$

$$\frac{|t|}{v^2} < \frac{1}{b} \qquad \boxed{|t| < v^2/b}$$

(13) X_i is sub exponential

$$\mathbb{E}\left(e^{\lambda(x_i - \mu_i)}\right) \le e^{v^2\lambda^2/2}$$

$$\mathbb{E}\left(e^{\Sigma\lambda(x_i-\mu)}\right) = \prod_{i=1}^{n} \mathbb{E}\left(e^{\lambda(x_i-\mu)}\right)$$

$$\le \prod_{i=1}^{n} e^{v_i^2\lambda^2/2}$$

$$\le e^{\frac{\lambda^2 \Sigma v_i^2}{2}}$$

for this to hold

$$|\lambda| \leq \frac{1}{b_i} \quad \forall\, b_i \text{ corresponding to } x_i$$

worst case take

$$b^* = \max b_i$$

$$\boxed{|\lambda| \leq \frac{1}{b^*}}$$

Also

$$\mathbb{E}\left\{ e^{\Sigma\lambda\left(\Sigma X_i - E(X_i)\right)} \right\} \leq e^{(\Sigma v_i^2)\lambda^2/2}$$

is sub exponential with $(\Sigma v_i^2, b_*)$

(14) $\mathbb{E}\left((X-\mu)^k \right) \leq \frac{1}{2} k!\, \sigma^2 b^{k-2}$ with param b

$$E\left[e^{\lambda(X-\mu)} \right] = \mathbb{E}\left(1 + \lambda(x-\mu) + \frac{\lambda^2 (x-\mu)^2}{2!} + \dots \right)$$

$$\leq 1 + \lambda \underbrace{\cancel{\mathbb{E}(X-\mu)}}_{0} + \frac{\lambda^2}{2!} \mathbb{E}\left((X-\mu)^2\right) + \dots$$

$$E\left(e^{\lambda(X-\mu)}\right) \leq 1 + \frac{\lambda^2}{2!} E\left((X-\mu)^2\right) + \frac{\lambda^3}{3!} E\left((X-\mu)^3\right) + \cdots$$

~~as $E\left((X-\mu)^{2n+1}\right)$ is 0 $n \in N$~~

$$E\left(e^{\lambda(X-\mu)}\right) \leq 1 + \frac{\lambda^2}{\cancel{2!}}\left[\frac{1}{2}\cancel{2!}\,\sigma^2 b^0\right] + \frac{\lambda^3}{\cancel{3!}}\left[\frac{1}{2}\cdot\cancel{3!}\,\sigma^2 b^1\right]$$

$$+ \cdots$$

$$E\left(e^{\lambda(X-\mu)}\right) \leq 1 + \frac{\lambda^2\sigma^2 b^0}{2} + \frac{\lambda^3\sigma^2 b^1}{2} + \cdots$$

$|\lambda| b < 1$

$|\lambda| < \frac{1}{b}$

$$\leq 1 + \frac{\sigma^2}{2} \sum_{k=2,4..} |\lambda|^k b^{k-2}$$

$$E\left(e^{\lambda(X-\mu)}\right) \leq 1 + \left(\frac{\sigma^2}{2}\right)\left(\frac{\lambda^2}{1-|\lambda| b}\right) \leq e^{\frac{\sigma^2\lambda^2}{2(1-|\lambda| b)}}$$

as $1 + x \leq e^x$

$$\boxed{\therefore\ E\left(e^{\lambda(X-\mu)}\right) \leq e^{\frac{\sigma^2\lambda^2}{2(1-|\lambda| b)}}}$$

(11)

(a) $p_X(x) = \lambda e^{-\lambda x}$

$E(X) = \frac{1}{\lambda}$

$E\left(e^{s(X-\frac{1}{\lambda})}\right) =$ MGF of exponential RV

$$= \int_0^\infty \left[e^{s(x-\frac{1}{\lambda})}\right] \lambda e^{-\lambda x} \, dx$$

$$= \int_0^\infty \frac{e^{sx}}{e^{s/\lambda}} \lambda e^{-\lambda x} \, dx$$

$$= \lambda e^{-s/\lambda} \int_0^\infty e^{x(s-\lambda)} \, dx$$

$$= \left. \frac{\lambda e^{-s/\lambda} e^{x(s-\lambda)}}{(s-\lambda)} \right]_0^\infty \Rightarrow \text{finite iff } \underline{\underline{s < \lambda}}$$

also $\lambda = \frac{1}{b}$ where <u>b is the mean</u>

$$\therefore\ \mathbb{E}\left(e^{s\left(X-\frac{1}{\lambda}\right)}\right) = \frac{\lambda e^{-s/\lambda}}{(s-\lambda)} = \frac{e^{-s/\lambda}}{\left(\frac{s}{\lambda}-1\right)} \qquad e^{\frac{v^2 s^2}{2}}$$

$$= \cancel{e^{-s/\lambda}\left\{1+\left(\frac{s}{\lambda}\right)+\left(\frac{s}{\lambda}\right)^2+\cdots\right\}}$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots$$

$$= \frac{e^{-sb}}{sb-1} \quad \text{— (1)}$$

also it is known that

$$\frac{e^{-s}}{\sqrt{1-2s}} \le e^{2s^2} \quad \rightsquigarrow \quad |1-2s| > e^{-2s-4s^2} \quad |s| \le \frac{1}{4}$$

$$\frac{e^{-sb/2}}{\sqrt{1-sb}} \le e^{\frac{s^2b^2}{2}} \quad \text{— (2)}$$

$\Rightarrow$

from (1) & (2)

exponential distribution is a sub-exponential with some params for (ν, b)

(b) MGF of a χ^2 distribution is

$$(1-2t)^{-k/2} \; ; \; t < 1/2$$

k is the degree of freedom

Lets say degree 1 and 0 means

$$MGF(X) = \cancel{(1-2t)^{-1}} \quad \frac{1}{\sqrt{1-2t}} \le e^{t} \cdot e^{t^2} \quad \text{as } t < \frac{1}{2}$$

$$\le e^{(t+\frac{1}{2})^2}$$

$$\therefore \; E(e^{\lambda X}) \le e^{\lambda^2 v^2/2} \quad \text{in some sense}$$