HW 1

EE16BTECH11022
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(1) From the plots generated by the python code, it is evident that Chi-Squared distribution is a non-Sub Gaussian distribution

Except when the degrees of freedom tend to ∞, then Central Limit Theorem enacts and results in a Gaussian

(3) Cramer's Transform

$$G(X) = f(t) = \sup_{\lambda} \left(\lambda t - \log \left(\mathbb{E}\left(e^{\lambda X} \right) \right) \right)$$

X = Centred Bernoulli RV

Let $Z = \begin{cases} 1 & Pr = p \\ 0 & Pr = 1-p \end{cases}$

$$\mathbb{E}(Z) = p$$

$$\therefore X = Z - p$$

$$\mathbb{E}(X) = \mathbb{E}(Z-p) = p - p = 0$$

$\therefore X$ is centred.

$$X = \begin{cases} 1-p & Pr = p \\ -p & Pr = 1-p \end{cases}$$

$$MGF(X) = \mathbb{E}(e^{\lambda x})$$

$$= p e^{\lambda(1-p)} + (1-p) e^{-\lambda p}$$

$$E(e^{\lambda x}) = e^{-\lambda p}(p e^{\lambda} + 1 - p)$$

$$\log E(e^{\lambda x}) = -\lambda p + \log(1-p+pe^{\lambda})$$

$$f(t) = \sup_{\lambda} \Big[\underbrace{\lambda t + \lambda p - \log(1-p+pe^{\lambda})}_{h(\lambda)} \Big]$$

for $f(t)$, find λ such that

$$h'(\lambda) = 0$$

$$t + p - \frac{pe^{\lambda}}{1-p+pe^{\lambda}} = 0$$

$$t+p = \frac{pe^{\lambda}}{1-p+pe^{\lambda}}$$

$$(t+p)(1-p) + (t+p)pe^{\lambda} = pe^{\lambda}$$

$$(t+p-1)pe^{\lambda} = (p-1)(t+p)$$

$$pe^{\lambda} = \frac{(p-1)(t+p)}{(t+p-1)p} \Rightarrow \lambda = \ln\left(\frac{(1-p)(t+p)}{p(1-t-p)}\right)$$

wherever it is defined

$\therefore$

$$f(t) = \ln\left(\frac{(1-p)(t+p)}{p(1-t-p)}\right)(t+p) - \ln\left(1-p+\cancel{p}\left[\frac{(1-p)(t+p)}{\cancel{p}(1-t-p)}\right]\right)$$

$$= \ln\left(\frac{(1-p)(t+p)}{p(1-t-p)}\right)(t+p) - \ln\left((1-p)\left[1+\frac{t+p}{1-t-p}\right]\right)$$

$$= \ln\left(\frac{(1-p)(t+p)}{(p)(1-t-p)}\right)(t+p) - \ln\left(\frac{(1-p)(1-\cancel{t}-\cancel{p}+\cancel{t}+\cancel{p})}{(1-t-p)}\right)$$

$$\boxed{f(t) = \ln\left(\frac{(1-p)(t+p)}{(p)(1-t-p)}\right)(t+p) - \ln\left(\frac{1-p}{1-t-p}\right)}$$

$\forall t$ $\cancel{\frac{t+p}{1-t-p}}$ $\cancel{0 < p < 1-t-p}$

wherever $\ln\left(\frac{(1-p)(t+p)}{p(1-t-p)}\right)$ is defined.

④ **Bennett's Inequality** :-

$X_1, \ldots X_n$ are centred RNs

$X_i \leq b \quad b > 0$

$$v = \sum_{i=1}^{n} E(X_i^2)$$

$$S = \sum_{i=1}^{n} X_i$$

$$E(e^{\lambda S}) \leq e^{\frac{v}{b^2}\phi(\lambda b)}$$

more precisely

$$E(e^{\lambda S}) \leq e^{\left[1 + \frac{v}{nb^2}\phi(\lambda b)\right]^n} \leq e^{\frac{v}{b^2}\phi(\lambda b)}$$

Now

$$P(S \geq t) = P(e^{\lambda S} \geq e^{\lambda t}) \leq \frac{E(e^{\lambda S})}{e^{\lambda t}}$$

$$\leq \frac{e^{\frac{v}{b^2}\phi(\lambda b)}}{e^{\lambda t}}$$

$$\leq e^{\left[\frac{v}{b^2}\phi(\lambda b) - \lambda t\right]}$$

$$\leq e^{\frac{v}{b^2}\left[\phi(\lambda b) - \frac{b^2 \lambda t}{v}\right]}$$

Concentrate on

$$\frac{v}{b^2}\left[\phi_1(\lambda b) - \frac{b^2\lambda t}{v}\right]$$

$$\cancel{= e^{\lambda b} - \lambda b - 1 - \frac{b^2 \lambda t}{v}}$$

We need max of this func. wrt λ

$$\frac{d}{d\lambda}\left[\frac{v}{b^2}\left[e^{\lambda b} - \lambda b - 1 - \frac{b^2\lambda t}{v}\right]\right] = 0$$

$$b e^{\lambda b} - b - \frac{b^2 t}{v} = 0$$

$$e^{\lambda b} = 1 + \frac{bt}{v}$$

$$\lambda = \frac{1}{b}\ln\left(1+\frac{bt}{v}\right)$$

Subst. back we have

$$\frac{v}{b^2}\left[e^{\ln\left(1+\frac{bt}{v}\right)} - \ln\left(1+\frac{bt}{v}\right) - 1 - \frac{b\ln\left(1+\frac{bt}{v}\right)t}{v}\right]$$

$$= \frac{v}{b^2}\left[\left(1+\frac{bt}{v}\right) - 1 - \ln\left(1+\frac{bt}{v}\right)\left(1+\frac{bt}{v}\right)\right]$$

$$\cancel{= \frac{v}{b^2}\left[\left(1+\frac{bt}{v}\right)\left(-1-\ln\left(1+\frac{bt}{v}\right)\right)-1\right]}$$

$$= \frac{v}{b^2}\left[-\left[\ln\left(1+\frac{bt}{v}\right)\right]\left(1+\frac{bt}{v}\right) + \frac{bt}{v}\right]$$

Let $\frac{bt}{v} = u$

$$= \frac{-v}{b^2}\left[(1+u)(\log(1+u)) - u\right]$$

$$\therefore \quad P(S \geq t) \leq e^{-\frac{v}{b^2} h\left(\frac{bt}{v}\right)}$$

$$\boxed{\text{where } h(u) = (1+u)\log(1+u) - u}$$