Finding Fair and Efficient Allocations When Valuations Don't Add Up*

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Abstract. In this paper, we present new results on the fair and efficient allocation of indivisible goods to agents whose preferences correspond to *matroid rank functions*. This is a versatile valuation class, with several desirable properties (monotonicity, submodularity) which naturally models several real-world domains. We use these properties to our advantage; first, we show that when agent valuations are matroid rank functions, a socially optimal (i.e. utilitarian social welfare-maximizing) allocation that achieves envy-freeness up to one item (EF1) exists and is computationally tractable. We also prove that the Nash welfare-maximizing and the leximin allocations both exhibit this fairness/efficiency combination, by showing that they can be achieved by minimizing any symmetric strictly convex function over utilitarian optimal outcomes. Moreover, for a subclass of these valuation functions based on maximum (unweighted) bipartite matching, we show that a leximin allocation can be computed in polynomial time.

Keywords: Fair Division · Envy-Freeness · Submodularity · Dichotomous preferences · Matroid rank functions · Optimal welfare

1 Introduction

Suppose that we are interested in allocating seats in courses to prospective students. How should this be done? On the one hand, courses offer limited seats and have scheduling conflicts; on the other, students have preferences over the classes that they take, which must be accounted for. Course allocation can be thought of as a problem of allocating a set of *indivisible goods* (course slots) to *agents* (students). How should we divide goods among agents with subjective valuations? Can we find a "good" allocation in polynomial time?

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These questions have been the focus of intense study in the CS/Econ community in recent years; several justice criteria as well as methods for computing allocations that satisfy them have been investigated. Generally speaking, there are two types of justice criteria: *efficiency* and *fairness*. *Efficiency* criteria are chiefly concerned with maximizing some welfare criterion, e.g. *Pareto optimality* (PO). *Fairness* criteria require that agents do not perceive the resulting allocation as mistreating them; for example, one might want to ensure that no agent wants another agent's assigned bundle [20]. This criterion is known as *envy-freeness* (EF); however, envy-freeness is not always achievable with indivisibilities: consider, for example, two students competing for a single course slot. Any student receiving this slot would envy the other (in our stylized example, there is just the one course with the one seat).
A simple solution ensuring envy-freeness would be to withhold the seat altogether, not assigning it to either student. This solution, however, violates most efficiency criteria. Indeed, as observed by Budish [12], envy-freeness is not always achievable, even with the weakest efficiency criterion of completeness requiring that each item is allocated to some agent. However, a less stringent fairness notion — *envy-freeness up to one good* (EF1) — can be attained. An allocation is EF1 if for any two agents i and j, there is some item in j's bundle whose removal results in i not envying j. EF1 complete allocations always exist, and in fact, can be found in polynomial time [32].
While trying to efficiently achieve individual criteria is challenging in itself, things get really interesting when trying to simultaneously achieve multiple justice criteria. Caragiannis et al. [14] show that when agent valuations are *additive* — i.e. every agent i values its allocated bundle as the sum of values of individual items — there exist allocations that are both PO and EF1. Specifically, these are allocations that maximize the product of agents' utilities — also known as the *max Nash welfare* (MNW). Further work [6] shows that such allocations can be found in pseudo-polynomial time. While encouraging, these results are limited to agents with additive valuations. In particular, they do not apply to settings such as the course allocation problem described above (e.g. being assigned two courses with conflicting schedules will not result in additive gain), or other settings we describe later on. In fact, Caragiannis et al. [14] left it open whether their result extends to other natural classes of valuation functions, such as the class of submodular valutions.[5] At present, little is known about other classes valuation functions; this is where our work comes in.

1.1 Our contributions

We focus on monotone submodular valuations with binary (or dichotomous) marginal gains, which we refer to as *matroid rank valuations*. In this setting, the added benefit of receiving another item is binary and obeys the law of diminishing marginal returns. This is equivalent to the class of valuations that can

[5] There is an instance of two agents with monotone supermodular/subadditive valuations where no allocation is PO and EF1 [14].

be captured by matroid constraints; namely, each agent has a different matroid constraint over the items, and the value of a bundle is determined by the size of a maximum independent set included in the bundle.

Matroids offer a highly versatile framework for describing a variety of domains [35]. This class of valuations naturally arises in many practical applications, beyond the course allocation problem described above (where students are limited to either approving/disapproving a class). For example, suppose that a government body wishes to fairly allocate public goods to individuals of different minority groups (say, in accordance with a diversity-promoting policy). This could apply to the assignment of kindergarten slots to children from different neighborhoods/socioeconomic classes[6] or of flats in public housing estates to applicants of different ethnicities [9, 8]. A possible way of achieving group fairness in this setting is to model each minority group as an agent consisting of many individuals: each agent's valuation function is based on *optimally matching* items to its constituent individuals; envy naturally captures the notion that no group should believe that other groups were offered better bundles (this is the fairness notion studied by Benabbou et al. [8]). Such assignment/matching-based valuations (known as OXS valuations [31]) are non-additive in general, and constitute an important subclass of submodular valuations. Matroid rank functions correspond to submodular valuations with binary (i.e. $\{0, 1\}$) marginal gains. The binary marginal gains assumption is best understood in context of matching-based valuations — in this scenario, it simply means that individuals either approve or disapprove of items, and do not distinguish between items they approve (we call OXS functions with binary individual preferences $(0, 1)$-OXS valuations). This is a reasonable assumption in kindergarten slot allocation (all approved/available slots are identical), and is implicitly made in some public housing mechanisms (e.g. Singapore housing applicants are required to effectively approve a subset of flats by selecting a block, and are precluded from expressing a more refined preference model).

In addition, imposing certain constraints on the underlying matching problem retains the submodularity of the agents' induced valuation functions: if there is a hard limit due to a *budget* or an exogenous *quota* (e.g. ethnicity-based quotas in Singapore public housing; socioeconomic status-based quotas in certain U.S. public school admission systems) on the number of items each group is able or allowed to receive, then agents' valuations are *truncated* matching-based valuations. Such valuation functions are not OXS, but are still matroid rank functions. Since agents still have binary/dichotomous preferences over items even with the quotas in place, our results apply to this broader class as well.

Using the matroid framework, we obtain a variety of positive existential and algorithmic results on the compatibility of (approximate) envy-freeness with welfare-based allocation concepts. The following is a summary of our main results (see also Table 1):

[6] see, e.g. https://www.ed.gov/diversity-opportunity.

(a) For matroid rank valuations, we show that an EF1 allocation that also maximizes the utilitarian social welfare or `USW` (hence is Pareto optimal) always exists and can be computed in polynomial time.
(b) For matroid rank valuations, we show that leximin[7] and MNW allocations both possess the EF1 property.
(c) For matroid rank valuations, we provide a characterization of the leximin allocations; we show that they are identical to the minimizers of *any* symmetric strictly convex function over utilitarian optimal allocations. We obtain the same characterization for MNW allocations.
(d) For $(0, 1)$-OXS valuations, we show that both leximin and MNW allocations can be computed efficiently.

	MNW	Leximin	max-`USW`+EF1
$(0, 1)$-OXS	poly-time (Th. 5)	poly-time (Th. 5)	poly-time (Th. 1)
matroid rank	?	?	poly-time (Th. 1)

Table 1. Summary of our computational complexity results.

All proofs omitted from the body of the proceedings version due to space constraints as well as clarifying examples, remarks, extensions, and additional references are available in this version.
Result (a) is remarkably positive: the EF1 and `USW` objectives are incompatible in general, even for additive valuations. Result (b) is reminiscent of Thm. 3.2 by Caragiannis et al. [14], showing that any MNW allocation is PO and EF1 under *additive* valuations. The PO+EF1 existence question beyond additive valuations, which they left open, has seen little progress. To our knowledge, the class of matroid rank valuations is the first valuation class not subsumed by additive valuations for which the EF1 property of the MNW allocation have been established.

1.2 Related work

Our paper is related to the vast literature on the fairness and efficiency issue in resource allocation. Early work on divisible resource allocation provides an elegant answer: an allocation that satisfies envy-freeness and Pareto optimality always exists under mild assumptions on valuations [41], and can be computed via convex programming of Eisenberg and Gale [19] for additive valuations. Four decades later, Caragiannis et al. [14] prove the discrete analogue of Eisenberg and Gale [19]: MNW allocation satisfies EF1 and Pareto optimality for additive

[7] Roughly speaking, a leximin allocation is one that maximizes the realized valuation of the worst-off agent and, subject to that, maximizes that of the second worst-off agent, and so on.

valuations. Subsequently, Barman et al. [6] provide a pseudo-polynomial-time algorithm for computing allocations satisfying EF1 and PO.

While computing leximin/MNW allocations of indivisible items is hard in general, several positive results are known when agents have binary additive valuations. Darmann and Schauer [16] and Barman et al. [7] show that the maximum Nash welfare can be computed efficiently for binary additive valuations. Further, the equivalence between leximin and MNW for binary additive valuations has been obtained in several recent papers. Aziz and Rey [3] show that the algorithm proposed by Darmann and Schauer outputs a leximin optimal allocation; in particular this implies that the leximin and MNW solutions coincide for binary additive valuations. This is implied by our results. Similar results are shown by Halpern et al. [25], who also show that the leximin/MNW optimal allocation is group-strategyproof for agents with binary additive valuations. In the context of *divisible* goods, Aziz and Ye [4] show the leximin and MNW solutions also coincide for dichotomous preferences.

From a technical perspective, our work makes extensive use of matroid theory; while some papers have explored the application of matroid theory to the fair division problem [10, 24], we believe that ours is the first to demonstrate its strong connection with fairness and efficiency guarantees.

One motivation for our paper is recent work by Benabbou et al. [8] on promoting diversity in assignment problems through efficient, EF1 allocations of items to groups in a population. Similar works study quota-based fairness/diversity [2, 9, 40, and references therein], or by the optimization of carefully constructed functions [1, 17, 29, and references therein] in allocation/subset selection.

Finally, Babaioff et al. [5] present a set of results similar to our own; they further explore strategyproof mechanisms for matroid rank valuations, showing that such mechanisms exist. Our work was developed independently, and is very different from a technical perspective.

2 Model and Definitions

Throughout the paper, given a positive integer r, let $[r]$ denote the set $\{1, 2, \ldots, r\}$. We are given a set $N = [n]$ of *agents*, and a set $O = \{o_1, \ldots, o_m\}$ of *items* or *goods*. Subsets of O are referred to as *bundles*, and each agent $i \in N$ has a *valuation function* $v_i : 2^O \to \mathbb{R}_+$ over bundles where $v_i(\emptyset) = 0$, i.e all valuations are *normalized*. We further assume polynomial-time oracle access to the valuation v_i of all agents. Given a valuation function $v_i : 2^O \to \mathbb{R}$, we define the *marginal gain* of an item $o \in O$ w.r.t. a bundle $S \subseteq O$, as $\Delta_i(S; o) \triangleq v_i(S \cup \{o\}) - v_i(S)$. A valuation function v_i is *monotone* if $v_i(S) \leq v_i(T)$ whenever $S \subseteq T$.

An *allocation* A of items to agents is a collection of n disjoint bundles $A_1, \ldots, A_n$, such that $\bigcup_{i \in N} A_i \subseteq O$; the bundle A_i is allocated to agent i. Given an allocation A, we denote by A_0 the set of unallocated items, also referred to as *withheld items*. We may refer to agent i's valuation of its bundle $v_i(A_i)$ under the allocation A as its *realized valuation* under A. An allocation is *complete* if every item is allocated to *some* agent, i.e. $A_0 = \emptyset$. We admit incomplete, but *clean* allocations: a bundle $S \subseteq O$ is *clean* for $i \in N$ if it contains no item $o \in S$ for which agent i has zero marginal gain (i.e., $\Delta_i(S \setminus \{o\}; o) = 0$, or equivalently $v_i(S \setminus \{o\}) = v_i(S)$); an allocation A is *clean* if each allocated bundle A_i is clean for the agent i that receives it. It is easy to 'clean' any allocation without changing any realized valuation by iteratively revoking items of zero marginal gain from respective agents and placing them in A_0 (see Example 1 in Appendix A).

2.1 Fairness and efficiency criteria

Our fairness criteria are based on the concept of *envy.* Agent i *envies* agent j under an allocation A if $v_i(A_i) < v_i(A_j)$. An allocation A is *envy-free* (EF) if no agent envies another. We will use the following relaxation of the EF property due to Budish [12]: we say that A is *envy-free up to one good* (EF1) if, for every $i, j \in N$, i does not envy j or there exists o in A_j such that $v_i(A_i) \geq v_i(A_j \setminus \{o\})$. The efficiency concept that we are primarily interested in is *Pareto optimality.* An allocation A' is said to *Pareto dominate* the allocation A if $v_i(A'_i) \geq v_i(A_i)$ for all agents $i \in N$ and $v_j(A'_j) > v_j(A_j)$ for some agent $j \in N$. An allocation is *Pareto optimal* (PO) if it is not Pareto dominated by any other allocation.
There are several ways of measuring the welfare of an allocation [38]. Specifically, given an allocation A, (i) its *utilitarian social welfare* is $\texttt{USW}(A) \triangleq \sum_{i=1}^{n} v_i(A_i)$; (ii) its *egalitarian social welfare* is $\texttt{ESW}(A) \triangleq \min_{i \in N} v_i(A_i)$; and (iii) its *Nash welfare* is $\texttt{NW}(A) \triangleq \prod_{i \in N} v_i(A_i)$. An allocation A is said to be *utilitarian optimal* (respectively, *egalitarian optimal*) if it maximizes $\texttt{USW}(A)$ (respectively, $\texttt{ESW}(A)$) among all allocations. Since it is possible that the maximum attainable Nash welfare is 0 (say, if there are less items than agents then one agent must have an empty bundle), we use the following refinement of the maximum Nash social welfare (MNW) used in [14]: we find a maximal subset of agents, say $N_{\max} \subseteq N$, to which we can allocate bundles of positive values, and compute an allocation to agents in $N_{\max}$ that maximizes the product of their realized valuations. If $N_{\max}$ is not unique, we choose the one that results in the highest product of realized valuations.
The *leximin* welfare is a lexicographic refinement of egalitarian optimality. Formally, for real n-dimensional vectors x and y, x is *lexicographically greater than or equal to* y (denoted by $x \geq_L y$) if and only if $x = y$, or $x \neq y$ and for the minimum index j such that $x_j \neq y_j$ we have $x_j > y_j$. For each allocation A, we denote by $\theta(A)$ the vector of the components $v_i(A_i)$ $(i \in N)$ arranged in non-decreasing order. A *leximin* allocation A is one that maximizes the egalitarian welfare in a lexicographic sense, i.e., $\theta(A) \geq_L \theta(A')$ for any other allocation A'.

2.2 Submodular Valuations

The main focus of this paper is on fair allocation when agent valuations are *submodular.* A valuation function v_i is *submodular* if single items contribute more to smaller sets than to larger ones, namely: for all $S \subseteq T \subseteq O$ and all $o \in O \setminus T$, $\Delta_i(S; o) \geq \Delta_i(T; o)$.

One important subclass of submodular valuations is *assignment valuations*, introduced by Shapley [39] and also called OXS valuations [30]. Fair allocation in this setting was explored by Benabbou et al. [8]. Here, each agent $h \in N$ represents a group of individuals N_h (such as ethnic groups and genders); each individual $i \in N_h$ (also called a *member*) has a fixed non-negative weight $u_{i,o}$ for each item o. An agent h values a bundle S via a *matching* of the items to its individuals (i.e. each item is assigned to at most one member and vice versa) that maximizes the sum of weights [33]; namely, $v_h(S) = \max\{\, \sum_{i \in N_h} u_{i,\pi(i)} \mid \pi \in \Pi(N_h, S)\,\}$, where $\Pi(N_h, S)$ is the set of matchings $\pi : N_h \to S$ in the complete bipartite graph with bipartition (N_h, S).
Our particular focus is on submodular functions with *binary marginal gains.* We say that v_i has *binary marginal gains* if $\Delta_i(S; o) \in \{0, 1\}$ for all $S \subseteq O$ and $o \in O \setminus S$. The class of submodular valuations with binary marginal gains includes the classes of binary additive valuations [7] and of assignment valuations where the weight is binary [8]. We say that v_i is a *matroid rank* valuation if it is a submodular function with binary marginal gains (these are equivalent definitions [35]), and $(0, 1)$-OXS if it is an assignment valuation with binary marginal gains.

3 Matroid Rank Valuations

The main theme of all results in this section is that, when all agents have matroid rank valuations, fairness and efficiency properties are compatible with one another, and there exist allocations that satisfy all three welfare criteria we consider. We start by introducing some notions from matroid theory. Formally, a *matroid* is an ordered pair $(E, \mathcal{I})$, where E is some finite set and $\mathcal{I}$ is a family of its subsets (referred to as the *independent sets* of the matroid), which satisfies the following three axioms:

(I1) $\emptyset \in \mathcal{I}$,
(I2) if $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$, and
(I3) if $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

The rank function $r : 2^E \to \mathbb{Z}$ of a matroid returns the *rank* of each set X, i.e. the maximum size of an independent subset of X. Another equivalent way to define a matroid is to use the axiom systems for a rank function. We require that (R1) $r(X) \leq |X|$, (R2) r is monotone, and (R3) r is submodular. Then, the pair $(E, \mathcal{I})$ where $\mathcal{I} = \{\, X \subseteq E \mid r(X) = |X|\,\}$ is a matroid [35]. In other words, if r satisfies properties (R1)–(R3) then it induces a matroid. In the fair allocation terminology, if an agent has a matroid rank valuation, then the set of *clean* bundles forms the set of independent sets of a matroid. Before proceeding further, we state some useful properties of the matroid rank valuation class.

Proposition 1. *A valuation function v_i with binary marginal gains is monotone and takes values in $[|S|]$ for any bundle S (hence $v_i(S) \leq |S|$).*

Proposition 2. *For matroid rank valuations, A is a clean allocation if and only if $v_i(A_i) = |A_i|$ for each $i \in N$.*

Even for binary additive valuations, EF and PO allocations may not exist (as a simple example of two agents and a single good valued at 1 by each of them demonstrates); thus, we turn our attention to EF1 and PO allocations.

3.1 Utilitarian optimal and EF1 allocation

For non-negative additive valuations, Caragiannis et al. [14] prove that every MNW allocation is Pareto optimal and EF1. However, the existence question of an allocation satisfying both the PO and EF1 properties remains open for submodular valuations. We show that the existence of a PO+EF1 allocation [14] extends to the class of matroid rank valuations. In fact, we provide a surprisingly strong relation between efficiency and fairness: utilitarian optimality (stronger than Pareto optimality) and EF1 turn out to be compatible under matroid rank valuations. Moreover, such an allocation can be computed in polynomial time!

Theorem 1. *For matroid rank valuations, a utilitarian optimal allocation that is also EF1 exists and can be computed in polynomial time.*

Our result is constructive: we provide a way of computing the above allocation in Algorithm 1. The proof of Theorem 1 and those of the latter theorems utilize Lemmas 1 and 2 which shed light on the interesting interaction between envy and matroid rank valuations.

Lemma 1 (Transferability property). *For monotone submodular valuation functions, if agent i envies agent j under an allocation A, then there is an item $o \in A_j$ for which i has a positive marginal gain with respect to A_i.*

Lemma 1 holds for submodular functions with arbitrary real-valued marginal gains, and is trivially true for (non-negative) additive valuations. However, there exist non-submodular valuation functions that violate the transferability property, even when they have binary marginal gains (see Example 2 in Appendix A). Below, we show that if i's envy towards j cannot be eliminated by removing one item, then the sizes of their *clean* bundles differ by at least two. Formally, we say that agent i envies j up to more than 1 item if $A_j \neq \emptyset$ and $v_i(A_i) < v_i(A_j \setminus \{o\})$ for every $o \in A_j$.

Lemma 2. *For matroid rank valuations, if agent i envies agent j up to more than* 1 *item under an allocation A and j's bundle A_j is clean, then $v_j(A_j) \geq v_i(A_i) + 2$.*

We are now ready to show that under matroid rank valuations, utilitarian social welfare maximization is polynomial-time solvable (2).

Theorem 2. *For matroid rank valuations, one can compute a clean utilitarian optimal allocation in polynomial time.*

Proof. We prove the claim by a reduction to the matroid intersection problem. Let E be the set of pairs of items and agents, i.e., $E = \{\, \{o, i\} \mid o \in O \wedge i \in N \,\}$.

For each $i \in N$ and $X \subseteq E$, we define X_i to be the set of edges incident to i, i.e., $X_i = \{ \{o, i\} \in X \mid o \in O \}$. Note that taking $E = X$, E_i is the set of all edges in E incident to $i \in N$. For each $i \in N$ and for each $X \subseteq E$, we define $r_i(X)$ to be the valuation of i, under function $v_i(\cdot)$, for the items $o \in O$ such that $\{o, i\} \in X_i$; namely,

$$r_i(X) = v_i(\{ o \in O \mid \{o, i\} \in X_i \}).$$

Clearly, r_i is also a submodular function with binary marginal gains; combining this with Proposition 1 and the fact that $r_i(\emptyset) = 0$, it is easy to see that each r_i is a rank function of a matroid. Thus, the set of clean bundles for i, i.e $\mathcal{I}_i = \{ X \subseteq E \mid r_i(X) = |X| \}$, is the set of independent sets of a matroid. Taking the union $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$, the pair $(E, \mathcal{I})$ is known to form a matroid [27], often referred to as a *union matroid*. By definition, $\mathcal{I} = \{ \bigcup_{i \in N} X_i \mid X_i \in \mathcal{I}_i \wedge i \in N \}$, so any independent set in $\mathcal{I}$ corresponds to a union of clean bundles for each $i \in N$ and vice versa. To ensure that each item is assigned at most once (i.e. bundles are disjoint), we will define another matroid $(E, \mathcal{O})$ where the set of independent sets is given by

$$\mathcal{O} = \{ X \subseteq E \mid |X \cap E_o| \leq 1, \forall o \in O \}.$$

Here, $E_o = \{ e = \{o, i\} \mid i \in N \}$ for $o \in O$. The pair $(E, \mathcal{O})$ is known as a *partition matroid* [27].

Now, observe that a common independent set of the two matroids $X \in \mathcal{O} \cap \mathcal{I}$ corresponds to a clean allocation A of our original instance where each agent i receives the items o with $\{o, i\} \in X$; indeed, each item o is allocated at most once because $|E_o \cap X| \leq 1$, and each A_i is clean because the realized valuation of agent i under A is exactly the size of the allocated bundle. Conversely, any clean allocation A of our instance corresponds to an independent set $X = \bigcup_{i \in N} X_i \in \mathcal{I} \cap \mathcal{O}$, where $X_i = \{ \{o, i\} \mid o \in A_i \}$: for each $i \in N$, $r_i(X_i) = |X_i|$ by Proposition 2, and hence $X_i \in \mathcal{I}_i$, which implies that $X \in \mathcal{I}$; also, $|X \cap E_o| \leq 1$ as A is an allocation, and hence $X \in \mathcal{O}$.

Thus, the maximum utilitarian social welfare is the same as the size of a maximum common independent set in $\mathcal{I} \cap \mathcal{O}$. It is well known that one can find a largest common independent set in two matroids in time $O(|E|^3 \theta)$ where θ is the maximum complexity of the two independence oracles [18]. Since the maximum complexity of checking independence in two matroids $(E, \mathcal{O})$ and $(E, \mathcal{I})$ is bounded by $O(mnF)$ where F is the maximum complexity of the value query oracle, we can find a set $X \in \mathcal{I} \cap \mathcal{O}$ with maximum $|X|$ in time $O(|E|^3 mnF)$. □

We are now ready to prove Theorem 1.

Proof (Proof of Theorem 1). Algorithm 1 maintains optimal `USW` as an invariant and terminates on an EF1 allocation. Specifically, we first compute a clean allocation that maximizes the utilitarian social welfare. The `EIT` subroutine in the algorithm iteratively diminishes envy by transferring an item from the envied bundle to the envious agent; Lemma 1 ensures that there is always an item in the envied bundle for which the envious agent has a positive marginal gain.

Algorithm 1: Algorithm for finding utilitarian optimal EF1 allocation

```
Compute a clean, utilitarian optimal allocation A.
/*Envy-Induced Transfers (EIT)*/
while there are two agents i, j such that i envies j more than 1 item. do
    Find item o ∈ A_j with Δ_i(A_i; o) = 1.
    A_j ← A_j \ {o}; A_i ← A_i ∪ {o}.
end
```

Correctness: Each EIT step maintains the optimal utilitarian social welfare as well as cleanness: an envied agent's valuation diminishes exactly by 1 while that of the envious agent increases by exactly 1. Thus, if it terminates, the EIT subroutine retains the initial (optimal) USW and, by the stopping criterion, induces the EF1 property. To show that the algorithm terminates in polynomial time, we define the potential function $\phi(A) \triangleq \sum_{i \in N} v_i(A_i)^2$. At each step of the algorithm, $\phi(A)$ strictly decreases by 2 or a larger integer. To see this, let A' denote the resulting allocation after reallocation of item o from agent j to i. Since A is clean, we have $v_i(A'_i) = v_i(A_i) + 1$ and $v_j(A'_j) = v_j(A_j) - 1$; since all other bundles are untouched, $v_k(A'_k) = v_k(A_k)$ for every $k \in N \setminus \{i, j\}$. Also, since i envies j up to more than one item under allocation A, $v_i(A_i) + 2 \leq v_j(A_j)$ by Lemma 2. Combining these, simple algebra gives us $\phi(A') - \phi(A) \leq -2$.
Complexity: By Theorem 2, computing a clean utilitarian optimal allocation can be done in polynomial time. The value of the non-negative potential function has a polynomial upper bound: $\sum_{i \in N} v_i(A_i)^2 \leq (\sum_{i \in N} v_i(A_i))^2 \leq m^2$. Thus, Algorithm 1 terminates in polynomial time. □

An interesting implication of the above analysis is that a utilitarian optimal allocation that minimizes $\sum_{i \in N} v_i(A_i)^2$ is always EF1.

Corollary 1. *For matroid rank valuations, any clean, utilitarian optimal allocation A that minimizes $\phi(A) \triangleq \sum_{i \in N} v_i(A_i)^2$ among all utilitarian optimal allocations is EF1.*

Despite its simplicity, Algorithm 1 significantly generalizes that of Benabbou et al. [8]'s Theorem 4 (which ensures the existence of a *non-wasteful* EF1 allocation for $(0, 1)$-OXS valuations) to matroid rank valuations. We note, however, that the resulting allocation may be neither MNW nor leximin even when agents have $(0, 1)$-OXS valuations: Example 3 in Appendix A illustrates this and also shows that the converse of Corollary 1 does not hold.

3.2 MNW and Leximin Allocations for Matroid Rank Functions

We characterize the set of leximin and MNW allocations under matroid rank valuations. We start by showing that Pareto optimal allocations coincide with utilitarian optimal allocations when agents have matroid rank valuations. Intuitively, if an allocation is not utilitarian optimal, one can find an 'augmenting' path that makes at least one agent happier but no other agent worse off. The full proof, which is more involved and relies on the concept of *circuits* of matrices, is available online in Appendix A.

Theorem 3. *For matroid rank valuations, PO allocations are utilitarian optimal.*

Since leximin and MNW allocations are Pareto optimal [14, 11], Theorem 3 implies that such allocations are utilitarian optimal as well. Next, we show that for the class of matroid rank valuations, leximin and MNW allocations are identical to each other; further, they can be characterized as the minimizers of any symmetric strictly convex function among all utilitarian optimal allocations.
A function $\Phi : \mathbb{Z}^n \to \mathbb{R}$ is *symmetric* if for any permutation $\pi : [n] \to [n]$,

$$\Phi(z_1, z_2, \dots, z_n) = \Phi(z_{\pi(1)}, z_{\pi(2)}, \dots, z_{\pi(n)}),$$

and is *strictly convex* if for any $x, y \in \mathbb{Z}^n$ with $x \neq y$ and $\lambda \in (0, 1)$ where $\lambda x + (1 - \lambda)y$ is an integral vector, $\lambda\Phi(x) + (1 - \lambda)\Phi(y) > \Phi(\lambda x + (1 - \lambda)y)$. Examples of symmetric, strictly convex functions include: $\Phi(z_1, z_2, \dots, z_n) \triangleq \sum_{i=1}^{n} z_i^2$ for $z_i \in \mathbb{Z} \quad \forall i$; $\Phi(z_1, z_2, \dots, z_n) \triangleq \sum_{i=1}^{n} z_i \ln z_i$ for $z_i \in \mathbb{Z}_{\geq 0}$ $\forall i$. For an allocation A, we define $\phi(A) \triangleq \phi(v_1(A_1), v_2(A_2), \dots, v_n(A_n))$.

Theorem 4. *Let $\Phi : \mathbb{Z}^n \to \mathbb{R}$ be a symmetric strictly convex function; let A be some allocation. For matroid rank valuations, the following are equivalent:*

1. *A is a minimizer of Φ over all the utilitarian optimal allocations; and*
2. *A is a leximin allocation; and*
3. *A maximizes Nash welfare.*

The proof is highly technical and is hence relegated to Appendix A online. To summarize, we first establish the equivalence of statements 1 and 2 by showing: (i) Lemma 4: given a non-leximin utilitarian optimal allocation A, there exists an "adjacent" utilitarian optimal allocation A which is the result of transferring one item from a 'happy' agent j to a less 'happy' agent i (the underlying submodularity guarantees the existence of such an allocation); (ii) Lemma 5: such an adjacent allocation A' has a strictly higher value of any symmetric strictly convex function than A. We complete the three-way equivalence by noting that maximizing Nash welfare is identical to minimizing the symmetric, strictly convex function $\phi(x) = -\sum_{i=1}^{n} \log x_i$ (carefully accounting for the possibility that some agents may realize zero valuations).
Theorem 4 does not generalize to the non-binary case: Example 5 in Appendix A presents an instance where the leximin and MNW allocation are not `USW` optimal. Combining the above characterization with the results of Section 3.1, we get the following fairness-efficiency guarantee for matroid rank valuations.

Corollary 2. *For matroid rank valuations, any clean leximin or MNW allocation is EF1.*

4 Assignment Valuations With Binary Gains

We now consider the practically important special case where valuations come from maximum matchings. For this valuation class, we show that invoking Theorem 3, one can find a leximin or MNW allocation in polynomial time, by a reduction to the network flow problem. We note that the complexity of the problem remains open for general matroid rank valuations.

Theorem 5. *For assignment valuations with binary marginal gains, one can find a leximin or MNW allocation in polynomial time.*

The proof, available in Appendix A, is based on the following key idea: given any instance with $(0, 1)$-OXS valuations, we construct a flow network such that the problem of finding a leximin allocation in the original instance reduces to that of finding a *increasingly-maximal integer-valued flow* on the induced network for which Frank and Murota [22] recently gave a polynomial-time algorithm.
In contrast with $(0, 1)$-OXS valuations, computing a leximin or MNW allocation becomes NP-hard for weighted assignment valuations, even for two agents.

Theorem 6. *Computing a leximin/MNW allocation for two agents with general assignment valuations is NP-hard.*

The proof is available in Appendix A. We give a Turing reduction from PARTITION. The reduction is similar to the hardness reduction for two agents with identical additive valuations [34, 37].

5 Discussion

We study allocations of indivisible goods under matroid rank valuations in terms of the interplay among envy, efficiency, and various welfare concepts. Since the class of matroid rank functions is rather broad, our results can be immediately applied to settings where agents' valuations are induced by a matroid structure. Beyond the domains described in this work, these include several others. For example, *partition matroids* model instances where agents' have access to different item types, but can only hold a limited number of each type (their utility is the total number of items they hold); a variety of other domains, such as spanning trees, independent sets of vectors, coverage problems and more admit a matroid structure (see Oxley [35] for an overview). Indeed, a well-known result in combinatorial optimization states that *any* agent valuation structure where the greedy algorithm can be used to find the (weighted) optimal bundle, is induced by some matroid [35, Theorem 1.8.5].
There are several known extensions to matroid structures, with deep connections to submodular optimization [35, Chapter 11]. Matroid rank functions are submodular functions with binary marginal gains; however, general submodular functions admit some matroid structure which may potentially be used to extend our results to more general settings. Finally, it would be interesting to explore other fairness criteria such as proportionality, the maximin share guarantee, equitability. etc. (see, e.g. [11] and references therein) for matroid rank valuations. We present some of our attempts along these lines in Appendices B through D.

Bibliography

[1] Ahmed, F., Dickerson, J.P., Fuge, M.: Diverse weighted bipartite b-matching. In: Proceedings of the 26th International Joint Conference on Artificial Intelligence (IJCAI). pp. 35–41 (2017)

[2] Aziz, H., Gaspers, S., Sun, Z., Walsh, T.: From matching with diversity constraints to matching with regional quotas. In: Proceedings of the 18th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS). pp. 377–385 (2019)

[3] Aziz, H., Rey, S.: Almost group envy-free allocation of indivisible goods and chores. CoRR abs/1907.09279 (2019)

[4] Aziz, H., Ye, C.: Cake cutting algorithms for piecewise constant and piecewise uniform valuations. In: Proceedings of the 10th Conference on Web and Internet Economics (WINE). pp. 1–14 (2014)

[5] Babaioff, M., Ezra, T., Feige, U.: Fair and truthful mechanisms for dichotomous valuations. CoRR abs/2002.10704 (2020)

[6] Barman, S., Krishnamurthy, S.K., Vaish, R.: Finding fair and efficient allocations. In: Proceedings of the 19th ACM Conference on Economics and Computation (EC). pp. 557–574. ACM (2018)

[7] Barman, S., Krishnamurthy, S.K., Vaish, R.: Greedy algorithms for maximizing Nash social welfare. In: Proceedings of the 17th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS). pp. 7–13 (2018)

[8] Benabbou, N., Chakraborty, M., Elkind, E., Zick, Y.: Fairness towards groups of agents in the allocation of indivisible items. In: Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI). pp. 95–101 (2019)

[9] Benabbou, N., Chakraborty, M., Xuan, V.H., Sliwinski, J., Zick, Y.: Diversity constraints in public housing allocation. Proceedings of the 17th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS) pp. 973–981 (2018)

[10] Biswas, A., Barman, S.: Fair division under cardinality constraints. In: Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI). pp. 91–97 (2018)

[11] Bouveret, S., Chevaleyre, Y., Maudet, N.: Fair allocation of indivisible goods. In: Brandt, Felix and Conitzer, Vincent and Endriss, Ulle and Lang, Jérôme and Procaccia, Ariel D. (ed.) Handbook of Computational Social Choice, chap. 12, pp. 284–310. Cambridge University Press (2016)

[12] Budish, E.: The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy 119(6), 1061–1103 (2011)

[13] Caragiannis, I., Gravin, N., Huang, X.: Envy-freeness up to any item with high Nash welfare: The virtue of donating items. In: Proceedings of the

19th ACM Conference on Economics and Computation (EC). pp. 527–545. ACM (2019)

[14] Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A.D., Shah, N., Wang, J.: The unreasonable fairness of maximum Nash welfare. In: Proceedings of the 17th ACM Conference on Economics and Computation (EC). pp. 305–322. ACM (2016)

[15] Chaudhury, B.R., Kavitha, T., Mehlhorn, K., Sgouritsa, A.: A little charity guarantees almost envy-freeness. arXiv preprint arXiv:1907.04596 (2019)

[16] Darmann, A., Schauer, J.: Maximizing Nash product social welfare in allocating indivisible goods. European Journal of Operational Research 247(2), 548–559 (2015)

[17] Dickerson, J.P., Sankararaman, K.A., Srinivasan, A., Xu, P.: Balancing relevance and diversity in online bipartite matching via submodularity. In: Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI). pp. 1877–1884 (2019)

[18] Edmonds, J.: Matroid intersection. Annals of Discrete Mathematics 4, 39–49 (1979), discrete Optimization I

[19] Eisenberg, E., Gale, D.: Consensus of subjective probabilities: The pari-mutuel method. The Annals of Mathematical Statistics 30(1), 165–168 (1959)

[20] Foley, D.: Resource allocation and the public sector. Yale Economics Essays 7 pp. 45–98 (1967)

[21] Frank, A., Murota, K.: Discrete Decreasing Minimization, Part I: Base-polyhedra with Applications in Network Optimization. arXiv e-prints arXiv:1808.07600v3 (Jul 2019)

[22] Frank, A., Murota, K.: Discrete Decreasing Minimization, Part III: Network Flows. arXiv e-prints arXiv:1907.02673v2 (Sep 2019)

[23] Freeman, R., Sikdar, S., Vaish, R., Xia, L.: Equitable allocations of indivisible goods. In: Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI). pp. 280–286 (2019)

[24] Gourvès, L., Monnot, J.: Approximate maximin share allocations in matroids. In: Fotakis, D., Pagourtzis, A., Paschos, V.T. (eds.) Algorithms and Complexity. pp. 310–321. Springer International Publishing (2017)

[25] Halpern, D., Procaccia, A.D., Psomas, A., Shah, N.: Fair division with binary valuations: One rule to rule them all (2020), unpublished Manuscript

[26] Harper, F.M., Konstan., J.A.: The movielens datasets: History and context. ACM Transactions on Interactive Intelligent Systems (TiiS) 5, 4, Article 19 (December 2015), 19 pages. (2015)

[27] Korte, B., Vygen, J.: Combinatorial Optimization: Polyhedra and Efficiency. Algorithms and Combinatorics, Springer (2006)

[28] Kyropoulou, M., Suksompong, W., Voudouris, A.A.: Almost envy-freeness in group resource allocation. arXiv preprint arXiv:1901.08463 (2019)

[29] Lang, J., Skowron, P.K.: Multi-attribute proportional representation. In: Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI). pp. 530–536 (2016)

[30] Lehmann, B., Lehmann, D., Nisan, N.: Combinatorial auctions with decreasing marginal utilities. Games and Economic Behavior 55(2), 270–296 (2006)

[31] Leme, R.P.: Gross substitutability: An algorithmic survey. Games and Economic Behavior 106, 294 – 316 (2017)

[32] Lipton, R.J., Markakis, E., Mossel, E., Saberi, A.: On approximately fair allocations of indivisible goods. In: Proceedings of the 5th ACM Conference on Electronic Commerce (EC). pp. 125–131. ACM (2004)

[33] Munkres, J.: Algorithms for the assignment and transportation problems. Journal of the Society for Industrial and Applied Mathematics 5(1), 32–38 (1957)

[34] Nguyen, T.T., Roos, M., Rothe, J.: A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation. Annals of Mathematics and Artificial Intelligence 68(1), 65–90 (Jul 2013)

[35] Oxley, J.: Matroid Theory. Oxford Graduate Texts in Mathematics, Oxford Univerity Press, 2nd edn. (2011)

[36] Plaut, B., Roughgarden, T.: Almost envy-freeness with general valuations. In: Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). pp. 2584–2603. Society for Industrial and Applied Mathematics (2018)

[37] Ramezani, S., Endriss, U.: Nash social welfare in multiagent resource allocation. In: Proceedings of the 11th International Workshop on Agent-Mediated Electronic Commerce and Trading Agents Design and Analysis (AMEC/TADA). pp. 117–131 (2010)

[38] Sen, A.: Collective choice and social welfare. Holden Day, San Francisco (1970)

[39] Shapley, L.S.: Complements and substitutes in the optimal assignment problem. ASTIA Document (1958)

[40] Suzuki, T., Tamura, A., Yokoo, M.: Efficient allocation mechanism with endowments and distributional constraints. In: Proceedings of the 17th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS). pp. 50–58 (2018)

[41] Varian, H.R.: Equity, envy, and efficiency. Journal of Economic Theory 9(1), 63 – 91 (1974)

Appendices to SAGT 2020 Paper 22

A Omitted proofs and examples

A.1 Example demonstrating the operation of cleaning an allocation

Example 1. For example, if for agent i, $v_i(\{1\}) = v_i(\{2\}) = v_i(\{1,2\}) = 1$, then the bundle $A_i = \{1,2\}$ is not clean for agent i (and neither is any allocation where i receives items 1 and 2) but it can be cleaned by moving item 1 (or item 2 but not both) to A_0.

A.2 Proof of Proposition 1

Proof. Consider subsets of items $T \subset S \subseteq O$ such that $S \setminus T = \{o_1, o_2, \ldots, o_r\}$ where $r = |S \setminus T|$. Define $S_0 = \emptyset$ and $S_t = \{o_1, o_2, \ldots, o_t\}$ for each $t \in [r]$. This gives us the following telescoping series:

$$\begin{aligned} v_i(S) - v_i(T) &= \sum_{t=1}^{r} (v_i(T \cup S_t) - v_i(T \cup S_{t-1})) \\ &= \sum_{t=1}^{r} (v_i(T \cup S_{t-1} \cup \{o_t\}) - v_i(T \cup S_{t-1})) \\ &= \sum_{t=1}^{r} \Delta_i(T \cup S_{t-1}; o_t). \end{aligned}$$

Since all marginal gains are binary, $\Delta_i(S \cup S_{t-1}; o_t) \geq 0$ for every $t \in [r]$, hence the above identity implies $v_i(S) - v_i(T) \geq 0$ for $S \supset T$, i.e. v_i is monotone.
Moreover, by setting $T = \emptyset$ and noting that $\Delta_i(S \cup S_{t-1}; o_t) \leq 1$ for every $t \in [r]$, we get $v_i(S) \leq v_i(\emptyset) + r = 0 + |S \setminus \emptyset| = |S|$. $\square$

A.3 Proof of Proposition 2

Proof. The "if" part: Suppose, there is an item $o \in S$ such that $\Delta_i(S \setminus \{o\}; o) = 0$. Now, by Proposition 1, $v_i(S \setminus \{o\}) \leq |S \setminus \{o\}| = |S| - 1$ since $o \in S$. This implies that $v_i(S) = v_i(S \setminus \{o\}) + \Delta_i(S \setminus \{o\}; o) < |S|$. Thus, by contraposition, if $v_i(S) = |S|$, then $\Delta_i(S \setminus \{o\}; o) = 1$ $\forall o \in S$ since the marginal gain can be either 0 or 1, i.e. S is a clean bundle for i.
The "only if" part: As in the proof of Proposition 1, let $S = \{o_1, o_2, \ldots, o_r\}$; define $S_0 = \emptyset$ and $S_t = \{o_1, o_2, \ldots, o_t\}$ for each $t \in [r]$. By the definition of cleanness, $\Delta_i(S \setminus \{o_t\}; o_t) = 1$ $\forall t \in [r]$. Since $S_{t-1} \subseteq S \setminus \{o_t\}$ for every $t \in [r]$, $\Delta_i(S_{t-1}; o_t) \geq \Delta_i(S \setminus \{o_t\}; o_t) = 1$; moreover due to marginal gains in $\{0,1\}$, we must have $\Delta_i(S_{t-1}; o_t) = 1$ for every $t \in [r]$. Hence, $v_i(S) = \sum_{t=1}^{r} \Delta_i(S_{t-1}; o_t) = r = |S|$. $\square$

A.4 Proof of Lemma 1

Proof. Assume that agent i envies agent j under an allocation A, i.e. $v_i(A_i) < v_i(A_j)$, but no item $o \in A_j$ has a positive marginal gain, i.e., $\Delta_i(A_i; o) = 0$ for each $o \in A_j$. Let $A_j = \{o_1, o_2, \dots, o_r\}$. As in the proof of Proposition 1, if we define $S_0 = \emptyset$ and $S_t = \{o_1, o_2, \dots, o_t\}$ for each $t \in [r]$, we can write the following telescoping series:

$$v_i(A_i \cup A_j) - v_i(A_i) = \sum_{t=1}^{r} \Delta_i(A_i \cup S_{t-1}; o_t).$$

However, submodularity implies that for each $t \in [r]$, $\Delta_i(A_i \cup S_{t-1}; o_t) \leq \Delta_i(A_i; o_t) = 0$, meaning that

$$v_i(A_i \cup A_j) - v_i(A_i) = \sum_{t=1}^{r} \Delta_i(A_i \cup S_{t-1}; o_t) = 0.$$

Together with monotonicity, this yields $v_i(A_j) \leq v_i(A_i \cup A_j) = v_i(A_i) < v_i(A_j)$, a contradiction. □

A.5 Example of non-submodular valuations that violate the transferability property

Example 2. Agent 1 wants to have a pair of matching shoes; her current allocated bundle is a single red shoe, whereas agent 2 has a matching pair of blue shoes. Agent 1 clearly envies agent 2, but cannot increase the value of her bundle by taking any one of agent 2's items. More formally, suppose $N = [2]$ and $O = \{r_L, b_L, b_R\}$; agent 1's valuation function is: $v_1(S) = 1$ only if $\{b_L, b_R\} \subseteq S$, $v_1(S) = 0$ otherwise. Under the allocation $A_1 = \{r_L\}$ and $A_2 = \{b_L, b_R\}$, $v_1(A_1) < v_1(A_2)$ but $\Delta_1(A_1; o) = 0$ for all $o \in A_2$. ■

A.6 Proof of Lemma 2

Proof. From the definition: $A_j \neq \emptyset$ and $v_i(A_i) < v_i(A_j \setminus \{o\})$ for every $o \in A_j$. Consider one such o. From Proposition 1, $v_i(A_j \setminus \{o\}) \leq |A_j \setminus \{o\}| = |A_j| - 1$. Since A_j is a clean bundle for j, Proposition 2 implies that $v_j(A_j) = |A_j|$. Combining these, we get

$$v_i(A_i) < v_i(A_j \setminus \{o\}) \leq |A_j| - 1 = v_j(A_j) - 1 \qquad \Rightarrow \qquad v_j(A_j) > v_i(A_i) + 1,$$

which proves the theorem statement since all valuations are integers. □

A.7 Proof of Corollary 1

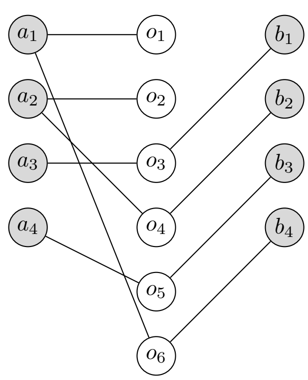
Proof. Take an allocation A' minimizes the sum of squares of the realized valuations among all utilitarian optimal allocations. Then, there is a clean, utilitarian optimal allocation A that has the same sum of squares as A'. We will show that A is EF1. Assume towards a contradiction that A is not EF1. Then, there is a pair of agents i, j such that i envies j up to more than 1 item. By Lemma 1, there is an item $o \in A_j$ such that $\Delta_i(A_i; o) = 1$. Let A^* be the allocation achieved by transferring o from j to i, everything else remaining the same. By Lemma 2 and the fact that A_j is clean, we have

$$v_i(A_i) \leq v_j(A_j) + 2,$$

which implies $\sum_{i \in N} v_i(A_i^*)^2 < \sum_{i \in N} v_i(A_i)^2$ proceeding exactly as in the proof of Theorem 1 — another contradiction. Hence, A must be EF1. □

A.8 Example that disproves the converse of Corollary 1

Example 3. The instance we use is Example 1 in Benabbou et al. [8]. There are two groups (i.e. agents with (0, 1)-OXS valuations) and six items $o_1, o_2, o_3, o_4, o_5, o_6$. The first group N_1 (identical to agent 1) contains four members a_1, a_2, a_3, a_4 and the second group N_2 (identical to agent 2) contains four members b_1, b_2, b_3, b_4; each individual has utility (weight) 1 for an item o if and only if she is adjacent to o in the adjoining graph:



The valuation function of each group for any bundle X is defined as the value (equivalently, the size) of a maximum-size matching of X to the group's members. The algorithm may initially compute a utilitarian optimal allocation A that assigns items o_1, o_2, o_3, o_5 to the group N_1 (with these items assigned to a_1, a_2, a_3, a_4 respectively), and the remaining items to group N_2 (with o_4, o_6 assigned to b_2, b_4 respectively). Then, $v_1(A_1) = 4 > 2 = v_1(A_2)$ and $v_2(A_2) =$

$2 = v_2(A_1)$, hence the allocation A is EF1 — in fact, envy-free! So, the `EIT` subroutine will not be invoked and the output of Algorithm 1 will be A. However, the (unique) leximin and MNW allocation assigns items o_1, o_2, o_3 to the first group, and the remaining items to the second group – this is also the (unique) utilitarian optimal allocation with the minimum sum of squares of the agents' valuations. ■

A.9 Remarks on Theorem 1 and Algorithm 1

Remark 1 (Choice of the potential function). In the proof of Theorem 1, we used the sum of squared valuations as the potential function to prove termination in polynomial time mainly for ease of exposition. However, any *symmetric, strictly convex,*[8] *polynomial* function of the realized valuations strictly decreases with each `EIT` step and, as such, it would be sufficient to use any such function as our potential function; moreover, Corollary 1 holds for any such function $\phi(\cdot)$ as well — we elaborate on this theme in Section 3.2.

Remark 2 (EFX allocation). It is worthwhile at this point to comment on the implications of our results for a stronger version of the EF1 property that has received considerable attention in recent literature: envy-freeness up to *any* item, often called the EFX condition. There are two definitions in the literature:

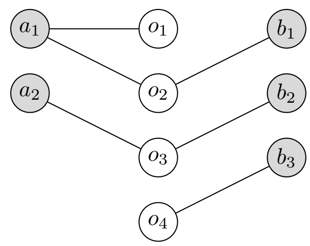
1. Caragiannis et al. [14] who introduced this concept (for additive valuations) called it *envy-freeness up to the least (positively) valued good*; we can naturally extend their definition to general valuations as follows: an allocation A is EFX if, for every pair of agents $i, j \in N$ such that i envies j, $v_i(A_i) \geq v_i(A_j \setminus \{o\})$ for every item $o \in A_j$ satisfying $\Delta_i(A_j \setminus \{o\}) > 0$. We will call this property EFX_+ for clarity.
2. Plaut and Roughgarden [36] defined an allocation A to be EFX if, for every pair of agents i, j, $v_i(A_i) \geq v_i(A_j \setminus \{o\})$ $\forall o \in A_j$ or equivalently $v_i(A_i) \geq \max_{o \in A_j} v_i(A_j \setminus \{o\})$ — this stronger definition favors allocations where more agents are envy-free of others since $v_i(A_j \setminus \{o\}) = v_i(A_j)$ whenever o is of zero marginal value to agent i with respect to the bundle A_j: the authors show that under this definition, no EFX allocation can be Pareto optimal even for two agents with additive valuations or general but identical valuations.[9] Caragiannis et al. [13] and Chaudhury et al. [15] use this definition as well. Following Kyropoulou et al. [28], who studied both the (above) weaker and (this) stronger variants of approximate envy-freeness under a different valuation model, we call this stronger property EFX_0.

[8] It suffices for the function to be strictly convex only over the non-negative orthant since valuations are always non-negative.

[9] However, both their examples establishing negative results for these sets of conditions on the valuation functions involve eliminating items with zero marginal value; their second example (for identical valuations) uses a non-submodular valuation function.

For matroid rank valuations, all items with non-zero marginal values for an agent are also valued identically at 1, hence EF1 trivially implies EFX$_+$; Theorem 1 and Corollary 1 further guarantee the existence of an EFX$_+$ and PO allocation for any instance under this valuation class. However, we demonstrate with Example 4 with $(0,1)$-OXS valuations shows that even an EF1 and utilitarian optimal (hence PO) allocation may not satisfy the EFX$_0$ condition. ♦

Example 4. There are two groups and four items o_1, o_2, o_3, o_4. The first group N_1 has two members a_1, a_2 and the second group N_2 has three b_1, b_2, b_3; each individual has utility 1 for an item if and only if she is adjacent to it in the adjoining graph:



The $(0,1)$-OXS valuation functions of groups N_1 and N_2 are denoted by $v_1(\cdot)$ and $v_2(\cdot)$ respectively. The allocation A where $A_1 = \{o_1\}$ and $A_2 = \{o_2, o_3, o_4\}$ is utilitarian optimal; it is also EF1 since $v_1(A_1) = 1 = v_1(A_2 \setminus \{o\})$ for $o \in \{o_2, o_3\}$, with $\Delta_1(A_2 \setminus \{o_4\}; o_4) = 0$, and $v_2(A_2) = 3 > 0 = v_2(A_1)$. A could be the output of Algorithm 1 and is clean and complete. However, A is not EFX$_0$ since $v_1(A_2 \setminus \{o_4\}) = 2 > 1 = v_1(A_1)$. ■

A.10 Proof of Theorem 3

In this proof, we will use the following notions and results from matroid theory: Given a matroid $(E, \mathcal{I})$, the sets in $2^E \setminus \mathcal{I}$ are called *dependent*, and a minimal dependent set of a matroid is called a *circuit*. The following is a crucial property of circuits.

Lemma 3 (Korte and Vygen [27]). *Let $(E, \mathcal{I})$ be a matroid, $X \in \mathcal{I}$, and $y \in E \setminus X$ such that $X \cup \{y\} \notin \mathcal{I}$. Then the set $X \cup \{y\}$ contains a unique circuit.*

Given a matroid $(E, \mathcal{I})$, we denote by $C(\mathcal{I}, X, y)$ the unique circuit contained in $X \cup \{y\}$ for any $X \in \mathcal{I}$ and $y \in E \setminus X$ such that $X \cup \{y\} \notin \mathcal{I}$.

Proof (Theorem 3). Define E, X_i, E_i, $\mathcal{I}_i$ for $i \in N$, $\mathcal{I}$, and $\mathcal{O}$ as in the proof of Theorem 2. We first observe that for each $X \in \mathcal{I}$ and each $y \in E \setminus X$, if $X \cup \{y\} \notin \mathcal{I}$, then there is agent $i \in N$ whose corresponding items in X_i

together with y is not clean, i.e., $X_i \cup \{y\} \notin \mathcal{I}_i$, which by Lemma 3 implies that the circuit $C(\mathcal{I}, X, y)$ is contained in E_i, i.e.,

$$C(\mathcal{I}, X, y) = C(\mathcal{I}_i, X, y). \tag{1}$$

Now to prove the claim, let A be a Pareto optimal allocation. Without loss of generality, we assume that A is clean. Then, as we have seen before, A corresponds to a common independent set X^* in $\mathcal{I} \cap \mathcal{O}$ given by

$$X^* = \bigcup_{i \in N} \{\, e = \{o, i\} \in E \mid o \in A_i \,\}.$$

Suppose towards a contradiction that A does not maximize the utilitarian social welfare. This means that X^* is not a largest common independent set of $\mathcal{I}$ and $\mathcal{O}$. It is known that given two matroids and their common independent set, if it is not a maximum-size common independent set, then there is an 'augmenting' path [18]. To formally define an augmenting path, we define an auxiliary graph $G_{X^*} = (E, B^{(1)}_{X^*} \cup B^{(2)}_{X^*})$ where the set of arcs is given by

$$B^{(1)}_{X^*} = \{\, (x, y) \mid y \in E \setminus X^* \wedge x \in C(\mathcal{O}, X^*, y) \setminus \{y\} \,\},$$
$$B^{(2)}_{X^*} = \{\, (y, x) \mid y \in E \setminus X^* \wedge x \in C(\mathcal{I}, X^*, y) \setminus \{y\} \,\}.$$

Since X^* is not a maximum common independent set of $\mathcal{O}$ and $\mathcal{I}$, the set X^* admits an *augmenting* path, which is an alternating path $P = (y_0, x_1, y_1, \dots, x_s, y_s)$ in G_{X^*} with $y_0, y_1, \dots, y_s \notin X^*$ and $x_1, x_2, \dots, x_s \in X^*$, where X^* can be augmented by one element along the path, i.e.,

$$X' = (X^* \setminus \{x_1, x_2, \dots, x_s\}) \cup \{y_0, y_1, \dots, y_s\} \in \mathcal{I} \cap \mathcal{O}.$$

Now let us write the pairs of agents and items that correspond to y_t and x_t as follows:

- $y_t = \{i(y_t), o(y_t)\}$ where $i(y_t) \in N$ and $o(y_t) \in O$ for $t = 0, 1, \dots, s$; and
- $x_t = \{i(x_t), o(x_t)\}$ where $i(x_t) \in N$ and $o(x_t) \in O$ for $t = 1, 2, \dots, s$.

Since each x_t ($t \in [s]$) belongs to the unique circuit $C(\mathcal{I}, X^*, y_{t-1})$, which is contained in the set of edges incident to $i(y_{t-1})$ by the observation made in (1), we have $i(x_t) = i(y_{t-1})$ for each $t \in [s]$. Thus, along the augmenting path P, each agent $i(x_t)$ receives a new item $o(y_{t-1})$ and discards the old item $o(x_t)$.
Now consider the reallocation corresponding to X' where agent $i(x_t)$ receives a new item $o(y_{t-1})$ but loses the item $o(x_t)$ for each $t = 1, 2, \dots, s$, and agent $i(y_s)$ receives the item $o(y_s)$. Such a reallocation increases the valuation of agent $i(y_s)$ by 1, while it does not decrease the valuations of all the intermediate agents, $i(x_1), i(x_2), \dots, i(x_s)$, as well as the other agents whose agents do not appear on P. We thus conclude that A is Pareto dominated by the new allocation, a contradiction. $\square$

A.11 Proof of Theorem 4

We will begin by formally stating and proving the two lemmas that we alluded to in Section 3.2. Let us denote by χ_i the n-dimensional incidence vector where the r^{th} component of χ_i is 1 if $r = i$, and 0 otherwise.

Lemma 4. *Suppose that agents have matroid rank valuations. Let A be a utilitarian optimal allocation. If A is not a leximin allocation, then there is another utilitarian optimal allocation A' such that, for some $i, j \in [n]$ with $\theta(A)_j \geq \theta(A)_i + 2$, $\theta(A') = \theta(A) + \chi_i - \chi_j$.*

Proof. Let A be an arbitrary utilitarian optimal allocation which is not leximin, and let A^* be a leximin allocation. Recall that A^* is utilitarian optimal by Theorem 3. Without loss of generality, we assume that both A and A^* are clean allocations. Now take a clean allocation A' that minimizes the symmetric difference $\sum_{i \in N} |A'_i \triangle A^*_i|$ over all clean allocation with $\theta(A') = \theta(A)$. Assume also w.l.o.g. that $v_1(A'_1) \leq v_2(A'_2) \leq \cdots \leq v_n(A'_n)$. We let $v_{j_1}(A^*_{j_1}) \leq v_{j_2}(A^*_{j_2}) \leq \cdots \leq v_{j_n}(A^*_{j_n})$. Since A^* lexicographically dominates A', for the minimum index k with $v_j(A'_k) \neq v_{j_k}(A^*_{j_k})$,

$$v_k(A'_k) < v_{j_k}(A^*_{j_k}). \tag{2}$$

We note that $v_h(A'_h) = v_{j_h}(A^*_{j_h})$ for all $1 \leq h \leq k-1$. By (2), there exists $i \in [k]$ with

$$v_i(A'_i) < v_i(A^*_i). \tag{3}$$

Indeed, if for all $i \in [k]$, $v_i(A'_i) \geq v_i(A^*_i)$, the k-th smallest value of realized valuations under A' is at least $v_{j_k}(A^*_{j_k})$, contradicting with (2). Take the minimum index i satisfying (3). Since both A' and A^* are clean allocations, we have

$$|A'_i| = v_i(A'_i) < v_i(A^*_i) = |A^*_i|. \tag{4}$$

By minimality, for all $h \in [i-1]$, $v_h(A'_h) \geq v_h(A^*_h)$. In fact, the equality

$$v_h(A'_h) = v_h(A^*_h) \tag{5}$$

holds for all $h \in [i-1]$. Indeed if $v_h(A'_h) > v_h(A^*_h)$ for some $h \in [i-1]$, then h-th smallest value of the realized valuations under A' would be strictly greater than that under A^*, yielding $\theta(A') >_L \theta(A^*)$, a contradiction.

Now, recall that the family of clean bundles $\mathcal{I}_h = \{ S \subseteq O \mid v_h(S) = |S| \}$ for $h \in N$ forms a family of independent sets of a matroid. By (I3) of the independent-set matroid axioms and by the inequality (4), there exists an item $o_1 \in A^*_i \setminus A'_i$ with positive contribution to A'_i, i.e., $v_i(A'_i \cup \{o_1\}) = v_i(A'_i) + 1$. By utilitarian optimality of A', o_1 is allocated to some agent, i.e., $o_1 \in A'_{i_1}$ for some $i_1 \neq i$. Consider the following three cases:

- Suppose $v_{i_1}(A'_{i_1}) \geq v_i(A'_i) + 2$. Then, we obtain a desired allocation by transferring o_1 from i_1 to i.

– Suppose $v_{i_1}(A'_{i_1}) = v_i(A'_i) + 1$. Then by transferring o_1 from i_1 to i, we get another utilitarian optimal allocation with the same vector as $\theta(A')$, which has a smaller symmetric difference than $\sum_{i \in N} |A'_i \triangle A^*_i|$, a contradiction.
– Suppose $v_{i_1}(A'_{i_1}) \leq v_i(A'_i)$. We will first show that $v_{i_1}(A'_{i_1}) \leq v_{i_1}(A^*_{i_1})$. By (5), this clearly holds if $i_1 \leq i$. Also, when $i_1 > i$, this means that $v_{i_1}(A'_{i_1}) = v_i(A'_i)$; thus $v_{i_1}(A'_{i_1}) \leq v_{i_1}(A^*_{i_1})$, as otherwise the i-th smallest value of realized valuations under A' would be greater than that under A^*, contradicting that A^* is leximin. Further by the facts that $|A'_{i_1} \setminus \{o_1\}| < |A^*_{i_1}|$ and that both $A'_{i_1} \setminus \{o_1\}$ and $A^*_{i_1}$ are clean (i.e., independent sets of a matroid), there exists an item $o_2 \in A^*_{i_1} \setminus A'_{i_1}$ such that $v_{i_1}(A'_{i_1} \cup \{o_2\} \setminus \{o_1\}) = v_i(A'_{i_1})$. Again by utilitarian optimality of A', o_2 is allocated to some agent, i.e., $o_2 \in A'_{i_2}$ for some $i_2 \neq i_1$.

Repeating the same argument and letting $i_0 = i$, we obtain a sequence of items and agents $(i_0, o_1, i_1, o_2, i_2, \ldots, o_t, i_t)$ such that

– $v_{i_h}(A'_{i_h}) = v_{i_h}(A'_{i_h} \cup \{o_{h+1}\} \setminus \{o_h\})$ for all $1 \leq h \leq t-1$; and
– $o_h \in A^*_{i_{h-1}} \setminus A'_{i_h}$ for all $1 \leq h \leq t$.

See Figure 1 for an illustration of the sequence. If the same agent appears again, i.e., $i_h = i_{h'}$ for some $h < h' \leq t$, then by transferring items along the cycle, we can decrease the symmetric difference with A^*, a contradiction. Thus, the sequence must terminate when we reach the agent i_t with $v_{i_t}(A'_{i_t}) \geq v_i(A'_i) + 2$. Exchanging items along the path, we get a desired allocation. □

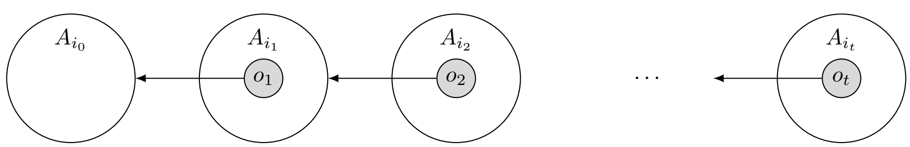


Fig. 1. The path $(i_0, o_1, i_1, o_2, i_2, \ldots, o_t, i_t)$

Lemma 5. *Let $\Phi : \mathbb{Z}^n \to \mathbb{Z}$ be a symmetric strictly convex function. Let A be a utilitarian optimal allocation. Let A' be another utilitarian optimal allocation such that $\theta(A') = \theta(A) + \chi_i - \chi_j$ for some $i, j \in [n]$ with $\theta(A)_j \geq \theta(A)_i + 2$. Then $\Phi(A) > \Phi(A')$.*

Proof. The proof is similar to that of Proposition 6.1 in Frank and Murota [21], which shows the analogous equivalence over the integral base-polyhedron. Let $\beta = \theta(A)_j - \theta(A)_i \geq 2$, and $y = \theta(A) + \beta(\chi_i - \chi_j)$. Thus $\Phi(\theta(A)) = \Phi(y)$ by symmetry of Φ. Define $\lambda = 1 - \frac{1}{\beta}$. We have $0 < \lambda < 1$ since $\beta \geq 2$. Observe that

$$\begin{aligned}\lambda\theta(A) + (1-\lambda)y &= (1 - \frac{1}{\beta})\theta(A) + \frac{1}{\beta}(\theta(A) + \beta(\chi_i - \chi_j)) \\ &= \theta(A) + \chi_i - \chi_j = \theta(A'),\end{aligned}$$

which gives us the following inequality (from the strict convexity of Φ): $\Phi(\theta(A)) = \lambda\Phi(\theta(A)) + (1-\lambda)\Phi(\theta(A)) > \Phi(\theta(A'))$. □

We are now ready to prove Theorem 4.

Proof (Theorem 4). To prove $1 \Leftrightarrow 2$, let A be a leximin allocation, and let A' be a minimizer of Φ over all the utilitarian optimal allocations. We will show that $\theta(A')$ is the same as $\theta(A)$, which, by the uniqueness of the leximin valuation vector and symmetry of Φ, proves the theorem statement.
Assume towards a contradiction that $\theta(A) \neq \theta(A')$. By Theorem 3, we have $\texttt{USW}(A) = \texttt{USW}(A')$. By Lemma 4, we can obtain another utilitarian optimal allocation A'' that is a lexicographic improvement of A' by decreasing the value of the j-th element of $\theta(A')$ by 1 and increasing the value of the i-th element of $\theta(A')$ by 1, where $\theta(A')_j \geq \theta(A')_i + 2$. Applying Lemma 5, we get $\Phi(\theta(A')) > \Phi(\theta(A''))$, which gives us the desired contradiction.
To prove $2 \Leftrightarrow 3$, let A be a leximin allocation, and let A' be an MNW allocation. Again, we will show that $\theta(A')$ is the same as $\theta(A)$, which by the uniqueness of the leximin valuation vector and symmetry of $\texttt{NW}$, proves the theorem statement. Let $N_{>0}(A)$ (respectively, $N_{>0}(A')$) be the agent subset to which we allocate bundles of positive values under leximin allocation A (respectively, MNW allocation A'). By definition, the number n' of agents who get positive values under leximin allocation A is the same as that of MNW allocation A'. Now we denote by $\bar{\theta}(A)$ (respectively, $\bar{\theta}(A')$) the vector of the non-zero components $v_i(A_i)$ (respectively, $v_i(A'_i)$) arranged in non-decreasing order. Assume towards a contradiction that $\bar{\theta}(A) >_L \bar{\theta}(A')$. Since A' maximizes the product $\texttt{NW}(A')$ when focusing on $N_{>0}(A')$ only, the value $\sum_{i \in N_{>0}(A')} \log v_i(A'_i)$ is maximized. However, $\phi(x) = -\sum_{i=1}^{n'} \log x_i$ is a symmetric, strictly convex function for $x \in \mathbb{Z}^n$ with each $x_i > 0$. Thus, by a similar argument as before, one can show that $\phi(\bar{\theta}(A')) < \phi(\bar{\theta}(A))$, a contradiction. This completes the proof. □

A.12 Example showing that neither leximin nor MNW allocation may be utilitarian optimal for general assignment valuations

Example 5. Consider an instance with assignment valuations given as follows. Suppose there are three groups, each of which contains a single agent, Alice, Bob, and Charlie, respectively, and three items with weights given in Table 2. The unique leximin and MNW allocation is the allocation that assigns Alice to the first item, Bob to the second item, and Charlie to the third item; each agent has positive utility at the allocation and the total utilitarian social welfare is 3.1. However, the utilitarian optimal allocation assigns Alice to nothing, Bob to the first item, and Charlie to the second item, which yields the total utilitarian social welfare 4.9. ■

	①	②	③
Alice:	2	1	0
Bob:	2	1	0
Charlie:	0	2.9	0.1

Table 2. An instance where neither leximin nor MNW allocation is utilitarian optimal.

A.13 Proof of Corollary 2

Proof. Since both leximin and MNW allocations are Pareto optimal, they maximize the utilitarian social welfare, by Theorem 3. By Theorem 4 and the fact that the function $\Phi(A) \triangleq \sum_{i \in N} v_i(A_i)^2$ is a symmetric strictly convex function, any leximin or MNW allocation is a utilitarian optimal allocation that minimizes $\Phi(A)$ among all utilitarian optimal allocations; hence, if such an allocation is clean, it must be EF1 by Corollary 1. □

A.14 Proof of Theorem 5

Proof. The problem of finding a leximin allocation can be reduced to that of finding an integral balanced flow (or increasingly-maximal integer-valued flow) in a network, which has been recently shown to be polynomial-time solvable [22]. Specifically, for a network $D = (V, A)$ with source s, sink t, and a capacity function $c : A \to \mathbb{Z}$, a *balanced flow* is a maximum integral feasible flow where the out-flow vector from the source s to the adjacent vertices h is lexicographically maximized among all maximum integral feasible flows; that is, the smallest flow-value on the edges (s, h) is as large as possible, the second smallest flow-value on the edges (s, h) is as large as possible, and so on. Frank and Murota [22] show that one can find a balanced flow in strongly polynomial time (see Section 7 in Frank and Murota [22]).
Now, given an instance of assignment valuations with binary marginal gains, we build the following instance (V, A) of a network flow problem. Let N_h denote the set of members in each group h. We first create a source s and a sink t. We create a vertex h for each group h, a vertex i for each member i of some group, and a vertex o for each item o. We construct the edges of the network as follows:

- for each group h, create an edge (s, h) with capacity m; and
- for each group h and member i in group h, create an edge (h, i) with unit capacity; and
- for each member i of some group and item o for which i has positive weight u_{io} (i.e. $u_{io} = 1$), create an edge (i, o) with unit capacity; and
- for each item o, create an edge (o, t) with unit capacity.

See Figure 2 for an illustration of the network. We will show that an integral balanced flow $f : A \to \mathbb{Z}$ of the constructed network corresponds to a leximin allocation. Consider an allocation A^f where each group receives the items o for which some member i of the group has positive flow $f(i,o) > 0$. It is easy to see that the allocation A^f maximizes the utilitarian social welfare since the flow f is a maximum integral feasible flow. Thus, by Theorem 3, A^f has the same utilitarian social welfare as any leximin allocation. To see balancedness, observe that the amount of flow from the source s to each group h is the valuation of h for bundle A^f_h, i.e., $f(s,h) = \sum_{i \in N_h} f(h,i) = v_h(A^f_h)$. Indeed if $v_h(A^f_h) > f(s,h)$, then it would contradict the optimality of the flow f; and if $v_h(A^f_h) < f(s,h)$, it would contradict the fact that $v_h(A^f_h)$ is the value of a maximum-size matching between A^f_h and N_h. Thus, among all utilitarian optimal allocations, A^f lexicographically maximizes the valuation of each group, and hence A_f is a leximin allocation. By Theorem 4, the leximin allocation A_f is also MNW. □

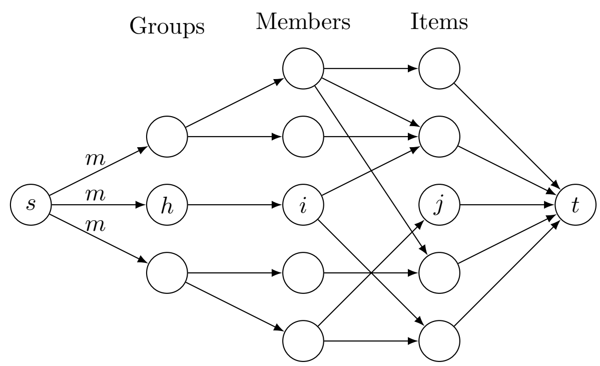


Fig. 2. An illustrative network flow instance constructed in the proof of Theorem 5: each edge is either labeled with its capacity or has unit capacity.

A.15 Proof of Theorem 6

Proof. The reduction is similar to the hardness reduction for two agents with identical additive valuations [34, 37]. We give a Turing reduction from PARTITION. Recall that an instance of PARTITION is given by a set of positive integers $W = \{w_1, w_2, \dots, w_m\}$; it is a 'yes'-instance if and only if it can be partitioned into two subsets S_1 and S_2 of W such that the sum of the numbers in S_1 equals the sum of the numbers in S_2.

Consider an instance of PARTITION $W = \{w_1, w_2, \dots, w_m\}$. We create m items $1, 2, \dots, m$, two groups 1 and 2, and m individuals for each group where every individual has a weight w_j for item j. Observe that fore each group, the value of each bundle X is the sum $\sum_{w_j \in X} w_j$: the number of members in the group exceeds the number of items in X, and thus one can fully assign each item to each member of the group.

Suppose we had an algorithm which finds a leximin allocation. Run the algorithm on the allocation problem constructed above to obtain a leximin allocation A. It can be easily verified that the instance of Partition has a solution if and only if $v_1(A_1) = v_2(A_2)$. Similarly, suppose we had an algorithm which finds an MNW allocation, and run the algorithm to find an MNW allocation A'. Since the valuations are identical, the utilitarian social welfare of the MNW allocation is the sum $\sum_{w_j \in W} w_j$, which means that the product of the valuations is maximized when both groups have the same realized valuation. Thus, the instance of Partition has a solution if and only if $v_1(A'_1) = v_2(A'_2)$. □

B Appendix: Other fairness criteria under matroid rank valuations

In the main paper, we have focused on Pareto optimal and EF1 allocations for the matroid rank valuation class. However, many other concepts have been defined and studied in the literature that formalize different intuitive ideas for what it means for an allocation of indivisible goods to be fair. In this section, we will investigate the implications of our results from the main paper for some alternative fairness notions.

An allocation A is said to be *equitable* or EQ if the realized valuations of all agents are equal under it, i.e. for every pair of agents $i, j \in N$, $v_i(A_i) = v_j(A_j)$; an allocation A is *equitable up to one item* or EQ1 if, for every pair of agents $i, j \in N$ such that $A_j \neq \emptyset$, there exists some item $o \in A_j$ such that $v_i(A_i) \geq v_j(A_j \setminus \{o\})$ [23].[10] We can further relax the equitability criterion up to an arbitrary number of items: an allocation A is said to be *equitable up to c items* or EQc if, for every pair of agents $i, j \in N$ such that $|A_j| > c$, there exists some subset $S \in A_j$ of size $|S| = c$ such that $v_i(A_i) \geq v_j(A_j \setminus S)$.[11]

Freeman et al. [23] show[12] that, even for binary additive valuations (which is a subclass of the $(0, 1)$-OXS valuation class), an allocation that is both EQ1 and PO may not exist; however, in Theorem 4, they establish that it can be verified in polynomial time whether an EQ1, EF1 and PO allocation exists and, whenever it does exist, it can also be computed in polynomial time — under binary additive valuations. We will show that the above positive result about computational

[10] Note that if $A_j = \emptyset$ for some j, $v_i(A_i) \geq v_j(A_j)$ trivially. Hence the ordered pair (i, j) for any $i \in N \setminus \{j\}$ could never prevent the allocation from being EQ1.

[11] Again, if $|A_j| \leq c$ for some j, no ordered pair (i, j) for any $i \in N \setminus \{j\}$ could get in the way of the allocation being EQc.

[12] Freeman et al. [23] use an example with 3 agents having binary additive valuations (Example 1). But it is easy to construct a fair allocation instance with only two agents having binary additive valuations that does not admit an EQ1 and PO allocation: $N = [2]$; $O = \{o_1, o_2, o_3, o_4\}$; $v_1(o) = 1$ for every $o \in O$; $v_2(o_1) = 1$ and $v_2(o) = o$ for every $o \in O \setminus \{o_1\}$. Obviously, any PO allocation must give $\{o_2, o_3, o_4\}$ to agent 1 so that this agent's realized valuation is at least 2 even after dropping one of its items; even if agent 2 receives o_1, her realized valuation of 1 will always be less than the above.

tractability extends to the $(0,1)$-OXS valuation class. We will begin by proving that under matroid rank valuations, an EQ1 and PO allocation, if it exists, is also EF1 — we achieve this by combining Theorem 7 below with Corollary 1. This simplifies the problem of finding an EQ1, EF1 and PO allocation to that of finding an EQ1 and PO allocation.

Theorem 7. *For submodular valuations with binary marginal gains, any EQ1 and PO allocation, if it exists, is a leximin allocation.*

Hence, from Theorem 4, we further obtain that if an EQ1 and PO allocation exists under matroid rank valuations, it is also MNW and a minimizer of any symmetric strictly convex function of agents' realized valuations among all utilitarian optimal allocations. Moreover, Theorem 7, together with Corollary 2, implies that if an EQ1 and PO allocation exists under matroid rank valuations, it must be EF1.

Proof. Let the optimal $\texttt{USW}$ for a problem instance under this valuation class be U^*; also, suppose this instance admits an EQ1 and PO allocation A. The EQ1 property implies that for every pair of agents $i, j \in N$ such that $A_j \neq \emptyset$,

$$\begin{aligned} v_i(A_i) &\geq v_j(A_j \setminus \{o\}) \qquad \text{for some } o \in A_j \\ &= v_j(A_j) - \Delta_j(A_j \setminus \{o\}; o) \\ &\geq v_j(A_j) - 1, \qquad \text{since } \Delta_j(A_j \setminus \{o\}; o) \in \{0, 1\}. \end{aligned}$$

This inequality holds trivially and strictly if $A_j = \emptyset$. Thus, $\max_{i \in N} v_i(A_i) \leq \min_{i \in N} v_i(A_i) + 1$. In other words, there exist a non-negative integer $\alpha \leq U^*$ and a positive integer $n_0 \in [n]$ such that n_0 agents have valuations α each and the remaining agents, if any, have valuations $\alpha + 1$ each under allocation A, with $U^* = n_0\alpha + (n - n_0)(\alpha + 1) = n\alpha + n - n_0$. We can write the agents' realized valuations under A (with arbitrary tie-breaking) as the n-dimensional vector

$$\theta(A) = \left(\underbrace{\alpha, \alpha, \ldots, \alpha}_{n_0 \text{ entries}}, \underbrace{\alpha + 1, \alpha + 1, \ldots, \alpha + 1}_{n - n_0 \text{ entries}} \right).$$

If A were not leximin, there would be another allocation A' for which the corresponding valuation vector $\theta(A')$ would have an entry strictly higher than that of A at the same position, say $n' \in [n]$. If $n' \leq n_0$, then every entry of $\theta(A')$ from position n' is at least $\alpha + 1$, so the $\texttt{USW}$ under A' is

$$\begin{aligned} U' &\geq (n' - 1)\alpha + (n - n' + 1)(\alpha + 1) \\ &= n\alpha + n - n' + 1 \\ &\geq n\alpha + n - n_0 + 1 \\ &= U^* + 1. \end{aligned}$$

If $n_0 < n' \leq n$, then similarly,

$$\begin{aligned} U' &\geq n_0\alpha + (n' - n_0 - 1)(\alpha + 1) + (n - n' + 1)(\alpha + 2) \\ &= n\alpha + 2n - n' - n_0 + 1 \\ &\geq n\alpha + 2n - n - n_0 + 1 \\ &= n\alpha + n - n_0 + 1 \\ &= U^* + 1. \end{aligned}$$

In either case, we have a contradiction since U^* is the optimal utilitarian social welfare for this instance. Hence, A must be leximin. □

We conjecture that a stronger result holds: under matroid rank valuations, the leximin allocation is optimally EQc for $c \in \{0, 1, \ldots, m\}$ among all PO allocations. A proof or a counterexample remains elusive. We present this more formally as follows:

Conjecture 1. For a problem instance where all agents have submodular valuations with binary marginal gains, any EQ1 and PO allocation, if it exists, if c^* is the smallest $c \in [m]$ for which a leximin allocation is EQc, then the instance admits no PO allocation that is EQc for any $c < c^*$.

C Appendix: Submodularity with subjective binary gains

An obvious generalization of the matroid rank valuation function class is the class of submodular valuation functions with *subjective* binary marginal gains: agent i's bundle-valuation function $v_i(\cdot)$ is said to have subjective binary marginal gains if $\Delta_i(S; o) \in \{0, \lambda_i\}$ for some agent-specific constant $\lambda_i > 0$, for every $i \in N$. We define clean bundles and clean allocations for this function class exactly as we did for matroid rank valuations in Section 2.
Understandably, most of the properties of allocations under matroid rank valuations do not extend to this more general setting. It is obvious that Pareto optimality does not imply utilitarian optimality (e.g. consider an instance with two agents and one item which the agents value at 1 and 2 respectively: assigning the item to agent 1 is PO but not utilitarian optimal). Moreover, the leximin allocation may not be EF1, as shown by the following example where both agents have additive valuations.

Example 6. Suppose $N = [2]$; $O = \{o_1, o_2, o_3, o_4\}$; the valuations are additive with $v_1(\{o_1\}) = 0$, $v_1(\{o\}) = 1\ \forall o \in O \setminus \{o_1\}$, and $v_2(\{o\}) = 3\ \forall o \in O$. It is straightforward to check that the unique leximin allocation is $A_1 = \{o_1, o_2, o_3\}$, $A_2 = \{o_4\}$. Under this allocation, $v_1(A_2) = 0 < 3 = v_1(A_1)$, but $v_2(A_1 \setminus \{o\}) = 6 > 3 = v_2(A_2)$ for every $o \in A_1$ — in fact, at least two (any two) items must be removed from A_1 for agent 2 to stop envying agent 1. ■

Note another difference of this valuation class from matroid rank valuations that is also evidenced by Example 6: the leximin and MNW allocations may not coincide. In this example, any allocation A that gives two of the items $\{o_2, o_3, o_4\}$ to agent 1 and the rest to agent 2 is MNW, with $v_1(A_1) = 2$ and $v_2(A_2) = 6$, so that $\texttt{NW}(A) = 12$; such an allocation is also EF1 (in fact, envy-free) since $v_1(A_2) = 1 < 2 = v_1(A_1)$ and $v_2(A_1) = 6 = v_2(A_2)$. This is not an accident, as the following theorem shows.

Theorem 8. *For agents having submodular valuation functions with subjective binary marginal gains, any clean, MNW allocation is EF1.*

Since our valuation functions are still submodular, the transferability property (Lemma 1) still holds. Two other components of the proof of Theorem 8 are natural extensions of Propositions 2 and Lemma 2 — Proposition 3 and Lemma 6 below, respectively:

Proposition 3. *For submodular valuations with subjective binary marginal gains defined by agent-specific positive constants λ_i $\forall i \in N$, A is a clean allocation if and only if $v_i(A_i) = \lambda_i|A_i|$ for each $i \in N$.*

Proof. Consider an arbitrary bundle $S \subseteq O$ such that $S = \{o_1, o_2, \ldots, o_r\}$ for some $r \in [m]$ w.l.o.g. Let $S_0 = \emptyset$ and $S_t = S_{t-1} \cup \{o_t\}$ for every $t \in [r]$. Then, an arbitrary agent i's valuation of bundle S under marginal gains in $\{0, \lambda_i\}$ is

$$v_i(S) = \sum_{t=1}^{r} \Delta_i(S_{t-1}; o_t) \leq \sum_{t=1}^{r} \lambda_i = \lambda_i r = \lambda_i|S|. \tag{6}$$

Now, if agent i's allocated bundle under an allocation A has a valuation $v_i(A_i) = \lambda_i|A_i|$, then her marginal gain for any item in $o \in A_i$ is given by

$$\begin{aligned} v_i(A_i) - v_i(A_i \setminus \{o\}) &= \lambda_i|A_i| - v_i(A_i \setminus \{o\}) \\ &\geq \lambda_i|A_i| - \lambda_i(|A_i| - 1) \\ &= \lambda_i > 0, \end{aligned}$$

where the first inequality follows from Inequality (6) and the fact that $|A_i \setminus \{o\}| = |A_i| - 1$. This means that the bundle A_i is clean and, since this holds for every i, the allocation is clean. This completes the proof of the "if" part.
If allocation A is clean, then we must have $\Delta_i(A_i \setminus \{o\}; o) > 0$ for every $o \in A_i$ for every $i \in N$. Let us define an arbitrary agent i's bundle A_i as S above, so that $|A_i| = r$. Then, since $S_{t-1} \subseteq A_i \setminus \{o_t\}$ for every $t \in [r]$, submodularity dictates that

$$\Delta_i(S_{t-1}; o_t) \geq \Delta_i(A_i \setminus \{o_t\}; o_t) > 0 \quad \forall t \in [r].$$

Since $\Delta_i(S_{t-1}; o_t) \in \{0, \lambda_i\}$ with $\lambda_i > 0$, the above inequality implies that $\Delta_i(S_{t-1}; o_t) = \lambda_i$ $\forall t \in [r]$. Hence,

$$v_i(A_i) = \sum_{t=1}^{r} \Delta_i(S_{t-1}; o_t) = \sum_{t=1}^{r} \lambda_i = \lambda_i r = \lambda_i|A_i|.$$

This completes the proof of the "only if" part. □

Lemma 6. *For submodular functions with subjective binary marginal gains, if agent i envies agent j up to more than 1 item under clean allocation A, then $|A_j| \geq |A_i| + 2$.*

Proof. Since i envies j under A up to more than 1 item, we must have $A_j \neq \emptyset$ and $v_i(A_i) < v_i(A_j \setminus \{o\})$ for every $o \in A_j$. Consider one such o. From Inequality (6) in the proof of Proposition 3, $v_i(A_j \setminus \{o\}) \leq \lambda_i |A_j \setminus \{o\}| = \lambda_i(|A_j| - 1)$. Since A is clean, $v_i(A_i) = \lambda_i |A_i|$. Combining these, we get $\lambda_i |A_i| = v_i(A_i) < v_i(A_j \setminus \{o\}) \leq \lambda_i(|A_j| - 1)$. Since $\lambda_i > 0$, we have $|A_i| < |A_j| - 1$, i.e. $|A_i| \leq |A_j| - 2$ because $|A_i|$ and $|A_j|$ are integers. □

We are now ready to prove Theorem 8.

Proof (Proof of Theorem 8). Our proof non-trivially extends that of Theorem 3.2 of Caragiannis et al. [14]. We will first address the case when it is possible to allocate items in such a way that each agent has a positive realized valuation for its bundle, i.e. $N_{\max} = N$ in the definition of an MNW allocation, and then tackle the scenario $N_{\max} \subsetneq N$.

Consider a pair of agents $1, 2 \in N$ w.l.o.g. such that 1 envies 2 up to two or more items, if possible, under an MNW allocation A. Since every agent has a positive realized valuation under A, we have $v_i(A_i) = \lambda_i |A_i| > 0$, i.e. $|A_i| > 0$ for each $i \in \{1, 2\}$. From Lemma 1, we know that there is an item in A_2 for which agent 1 has positive marginal utility – consider any one such item $o \in A_2$. Thus, $\Delta_1(A_1; o) > 0$, i.e. $\Delta_1(A_1; o) = \lambda_1$; also, since A_2 is a clean bundle, $\Delta_2(A_2 \setminus \{o\}; o) > 0$, i.e. $\Delta_2(A_2 \setminus \{o\}; o) = \lambda_2$.

Let us convert A to a new allocation A' by only transferring this item o from agent 2 to agent 1. Hence, $v_1(A'_1) = v_1(A_1) + \Delta_1(A_1; o) = v_1(A_1) + \lambda_1$, $v_2(A'_2) = v_2(A_2) - \Delta_2(A_2 \setminus \{o\}; o) = v_2(A_2) - \lambda_2$, $v_i(A'_i) = v_i(A_i)$ for each $i \in N \setminus \{1, 2\}$. $\mathtt{NW}(A)$ is positive since A is MNW and $N_{\max} = N$. Hence,

$$\begin{aligned}
\frac{\mathtt{NW}(A')}{\mathtt{NW}(A)} &= \left[\frac{v_1(A_1) + \lambda_1}{v_1(A_1)}\right] \left[\frac{v_2(A_2) - \lambda_2}{v_2(A_2)}\right] \\
&= \left[1 + \frac{\lambda_1}{v_1(A_1)}\right] \left[1 - \frac{\lambda_2}{v_2(A_2))}\right] \\
&= \left[1 + \frac{\lambda_1}{\lambda_1 |A_1|}\right] \left[1 - \frac{\lambda_2}{\lambda_2 |A_2|}\right] \\
&= \left[1 + \frac{1}{|A_1|}\right] \left[1 - \frac{1}{|A_2|}\right] \\
&= 1 + \frac{|A_2| - |A_1| - 1}{|A_1||A_2|}, \\
&\geq 1 + \frac{(|A_1| + 2) - |A_1| - 1}{|A_1||A_2|}, \\
&\geq 1 + \frac{1}{|A_1||A_2|}, \\
&> 1.
\end{aligned}$$

Here, the third equality comes from Proposition 3 since A is clean, and the first inequality from Lemma 6 due to our assumption. But $\mathtt{NW}(A') > \mathtt{NW}(A)$ contradicts the optimality of A, implying that any agent can envy another up to at most 1 item under A.
This completes the proof for the $N_{\max} = N$ case. The rest of the proof mirrors the corresponding part of the proof of Caragiannis et al. [14]'s Theorem 3.2. If $N_{\max} \subsetneq N$, it is easy to see that there can be no envy towards any $i \notin N_{\max}$: this is because we must have $v_i(A_i) = 0$ for any such i from the definition of $N_{\max}$, which in turn implies that $A_i = \emptyset$ since A is clean; hence, $v_j(A_i) = 0$ for every $j \in N$. Also, for any $i, j \in N_{\max}$, we can show exactly as in the proof for the $N_{\max} = N$ case above that there cannot be envy up to more than one item between them, since A maximizes the Nash welfare over this subset of agents $N_{\max}$. Suppose for contradiction that an agent $i \in N \backslash N_{\max}$ envies some $j \in N_{\max}$ up to more than one item under A. Then, from Lemma 1, there is one item $o_1 \in A_j$ w.l.o.g. such that $v_i(\{o_1\}) = \Delta_i(\emptyset; o_1) = \Delta_i(A_i; o_1) > 0$. Moreover, since A is clean,

$$\begin{aligned}
v_j(A_j \setminus \{o_1\}) &= v_j(A_j) - \Delta_j(A_j \setminus \{o_1\}; o_1) \\
&= \lambda_j |A_j| - \lambda_j \\
&= \lambda_j (|A_j| - 1) \\
&\geq \lambda_j (|A_i| + 1) \\
&= \lambda_j > 0,
\end{aligned}$$

where the first inequality comes from Lemma 6. Thus, if we transfer o_1 from j to i and leave all other bundles unchanged, then every agent in $N_{\max} \cup \{i\}$ will have a positive valuation under the new allocation. This contradicts the maximality of $N_{\max}$. Hence, any $i \in N \backslash N_{\max}$ must be envy-free up to one item towards any $j \in N_{\max}$. □

D Appendix: General assignment valuations

In this section, we address the fair and efficient allocation of items to agents who have general assignment or OXS valuations, as defined in Section 2.2. Recall that an agent with such a valuation function is equivalent to a group with multiple members each having an arbitrary non-negative weight for each item. As such, we will henceforth use the terms "group" and "agent" interchangeably.
We know that, for arbitrary non-negative monotone valuations, the classic *envy graph algorithm* due to Lipton et al. [32] produces a complete, EF1 allocation that does not, however, come with any efficiency guarantee (except completeness, of course). The trick is to iterate over the items and allocate each to an agent that is currently not envied by any other agent (the existence of such an unenvied agent can be guaranteed by *de-cycling*, if necessary, the graph induced by a directed edge from every envious agent to every agent that it envies: see Lipton et al. [32] for details).

Benabbou et al. [8] focus on fair allocation to *types* that are, in fact, agents/groups with OXS valuations; they use a natural extension of this procedure that they denote by **H**. In an iteration of **H**, we do not give an arbitrary unallocated item to an arbitrary unenvied agent; instead, we find an item-agent pair having the maximum marginal utility among all currently unenvied agents and all unallocated items (breaking further ties uniformly at random, say), and allocate that item to that agent. Although this modification should, intuitively, improve efficiency, Benabbou et al. [8] provide no formal guarantee in this regard; they evaluate the performance of **H** in experiments where all agents have OXS valuations in terms of *waste* which they define as follows: under a complete allocation A, an item o is said to be *wasted* if it has positive marginal utility for some group h under A (i.e. $v_h(A_h \cup \{o\}) > v_h(A_h)$) but is allocated to another group h' (i.e. $o \in A_{h'}$) where it is either unassigned or assigned to a member $i \in N_{h'}$ with zero weight for it (i.e. $u_{i,o} = 0$), under the particular optimal matching of $A_{h'}$ to $N_{h'}$. The waste of a run of Algorithm **H** is defined as the percentage of the total number of items that are wasted under the complete allocation produced by **H**.

Here, we ask whether the concept of *envy-induced transfers* (`EIT`) presented in Algorithm 1 for matroid rank valuations (Section 3.1) can be used to compute fair and efficient allocations (perhaps in some approximate sense) under more general monotone submodular valuation functions. This is motivated in part by the fact that the transferability property (Lemma 1), on which the `EIT` concept relies, characterizes any monotone submodular function and not just matroid rank valuations. In Algorithm 2, we delineate our work in progress in this vein: a heuristic scheme that extends Algorithm 1 to general OXS valuations.

Algorithm 2 retains the general principle of starting with a(n arbitrary) clean, utilitarian optimal allocation[13] and iteratively eliminating envy by transferring an item from an envied bundle to an envious agent. For matroid rank valuations, the "donor" and the recipient of the transferred item have their valuations decreased and increased respectively by exactly 1 for any envy-induced transfer; this is no longer the case when we remove the binary marginal utilities restriction. Hence, such a transfer does not, in general, keep the utilitarian social welfare unchanged; the welfare is only constrained to never exceed its starting (optimal) value computed in line 1. As an approach to minimizing the loss in welfare/efficiency due to such transfers, we employ various heuristics in Algorithm 2:

[13] Maximizing the utilitarian social welfare is NP-hard when agents have general monotone submodular valuations but can be accomplished in polynomial time under the subclass gross substitutes valuations, assuming oracle access to each valuation function [30]. In particular, under OXS valuations (assuming that such a valuation function is specified in terms of the weights of each member of the group for all items), computing a utilitarian optimal allocation reduces to the polynomial-time solvable *assignment problem* or maximum sum-of-weights matching on a bipartite graph [33]; the result is automatically clean if we make sure that no item is assigned to an individual with zero weight for it.

Algorithm 2: Envy-Induced Transfers for general OXS valuations

1 Compute a clean, utilitarian optimal allocation.
2 **/*Envy-Induced Transfers (`EIT`)*/**
3 **while** $\exists i, j \in N$ *such that i envies j up to more than* 1 *item* **do**
4   Pick i, j, o maximizing $\Delta_i(A_i; o) + \Delta(A_j \setminus \{o\}; o)$ over all $i, j \in N$ and all $o \in O$ such that i envies j more than 1 item and $\Delta_i(A_i; o) > 0$.
5     $A_j \leftarrow A_j \backslash \{o\}$; $A_i \leftarrow A_i \cup \{o\}$.
6   **if** $\exists o \in A_0$ *such that* $\Delta_j(A_j; o) > 0$ **then**
7   Pick $o \in A_0$ that maximizes $\Delta_j(A_j; o)$.
8         $A_j \leftarrow A_j \cup \{o\}$.
9   **end**
10   **if** $\exists o^* \in A_i$ *that is unused* **then**
11         $A_i \leftarrow A_i \backslash \{o^*\}$; revoked = **true**.
12   **while** *revoked* = ***true and*** $\exists k$ *s.t.* $\Delta(A_k; o^*) > 0$ **do**
13   Allocate o^* to agent k maximizing $\Delta(A_k; o^*)$.
14   **if** $\exists o \in A_k$ *that is unused* **then**
15                 $A_k \leftarrow A_k \backslash \{o\}$; $o^* \leftarrow o$.
16   **else** revoked = **false**.
17   **end**
18   **if** *revoked* = ***true*** **then** $A_0 \leftarrow A_0 \cup \{o^*\}$.
19   **end**
20 **end**

- First, in each `EIT` step, we transfer the item that induces the minimal decrease in — or, equivalently, the maximal increase — in the welfare (see lines 3-6).
- Next, as the donor agent loses one of its items, it may develop a positive marginal utility for a currently withheld item; in that case, the item in A_0 for which she has maximal marginal utility is given to it (see lines 6-8).
- Finally, if an agent (group) i acquires a new item o due to an envy-induced transfer, at most one of its previous items, say o^*, may become *unused*, i.e. it is no longer assigned to a member of the group under the new matching. This happens, for example, if i's positive marginal utility for o with its previous bundle A_i was due to the fact that the member who was assigned item o^* has a higher weight for o than for o^* and no other member prefers o^* to its assigned item. In such a case, item o^* is revoked from agent i and allocated to the agent with maximal and strictly positive marginal utility for it (see lines 10-13). If this creates another unused item, we repeat the process until there are no unused items or the unused item has zero marginal utility for all agents – in the latter case, the unused item is added to the withheld set (see lines 14-18).

We do not yet have theoretical guarantees for Algorithm 2; but, if the `EIT` subroutine terminates, then the final allocation is EF1 and has zero waste (as defined above) by construction. To estimate the efficiency properties of our scheme, we ran numerical tests with it on a set of fair allocation instances based on a real-world data set.

In our experiments, we measure and compare the performances of Algorithm 2 and the procedure **H** as described above in terms of waste (as defined above) as well as the *price of fairness* (PoF) which we formally define as follows

$$PoF(P) = \frac{\max\{\mathtt{USW}(A) \mid A \text{ is an allocation}\}}{\mathtt{USW}(A(P))}$$

where $A(P)$ is the allocation returned by a given procedure P (Algorithm 2 or **H**) on a problem instance. Obviously, PoF is bounded below by 1 for any instance and lower values are better.

The data set we use is *MovieLens-ml-1m* [26] which contains approximately 1,000,000 ratings (from 0 to 5) of 4,000 movies made by 6,000 users. To generate an instance of our allocation problem, we select 200 movies uniformly at random ($|O| = 200$) and then we only consider the users that rated at least one of these movies. Each such sample of 200 movies defines one run of our experiments. The users are our group-members and the movies our items. We generate agents/groups by partitioning users based on a demographic attribute; in fact, we use two attributes recorded in the data set, giving us two sets of allocation problem instances for each run:

- Gender: 2 agents (male or female, as recorded in the data set);
- Age: 7 agents representing the 7 age-groups recognized in the data set.

Moreover, for each such set (with 2 and 7 agents respectively), we adopt two models for the member-item weights or, equivalently, agents' valuation functions (raw and normalized ratings), giving us a 2×2 experimental design:

- Ratings:
$$v_h(S) = \max\left\{ \sum_{u \in N_h} r_{u,\pi(u)} \mid \pi \in \Pi(N_h, S) \right\}$$
  where r_{uo} is the user u's rating of movie o;
- Norm: $v'_h(S) = v_h(S)/v_h(O)$,

for every agent h (group N_h) and for any bundle of movies $S \subseteq O$. We provide the results, averaged over 50 runs, in Table 3.

	Attribute (#groups)	**H** [32, 8] Ratings	**H** [32, 8] Norm	Algorithm 2 Ratings	Algorithm 2 Norm
PoF	Age (7)	1.01	1.15	1.05	1.19
Waste		1.25%	0.20%	0%	0%
PoF	Gender (2)	1.00	1.02	1.00	1.03
Waste		0.00%	0.00%	0%	0%

Table 3. Experimental assessment of allocation procedures under OXS valuations.

We observe that **H** has no guarantees on waste but, in practice, has negligible waste; and the waste appears to be lower for a lower number of agents in our experiments. In comparison, Algorithm 2 (which always terminated on its own for all our instances) is waste-free by design but has at least as much average PoF as **H** in all our experiments.