# Appendices to SAGT 2020 Paper 22

## A Omitted proofs and examples

### A.1 Example demonstrating the operation of cleaning an allocation

*Example 1.* For example, if for agent $i$, $v_i(\{1\}) = v_i(\{2\}) = v_i(\{1,2\}) = 1$, then the bundle $A_i = \{1,2\}$ is not clean for agent $i$ (and neither is any allocation where $i$ receives items 1 and 2) but it can be cleaned by moving item 1 (or item 2 but not both) to $A_0$.

### A.2 Proof of Proposition 1

*Proof.* Consider subsets of items $T \subset S \subseteq O$ such that $S \setminus T = \{o_1, o_2, \ldots, o_r\}$ where $r = |S \setminus T|$. Define $S_0 = \emptyset$ and $S_t = \{o_1, o_2, \ldots, o_t\}$ for each $t \in [r]$. This gives us the following telescoping series:

$$\begin{aligned} v_i(S) - v_i(T) &= \sum_{t=1}^{r} (v_i(T \cup S_t) - v_i(T \cup S_{t-1})) \\ &= \sum_{t=1}^{r} (v_i(T \cup S_{t-1} \cup \{o_t\}) - v_i(T \cup S_{t-1})) \\ &= \sum_{t=1}^{r} \Delta_i(T \cup S_{t-1}; o_t). \end{aligned}$$

Since all marginal gains are binary, $\Delta_i(S \cup S_{t-1}; o_t) \geq 0$ for every $t \in [r]$, hence the above identity implies $v_i(S) - v_i(T) \geq 0$ for $S \supset T$, i.e. $v_i$ is monotone.
Moreover, by setting $T = \emptyset$ and noting that $\Delta_i(S \cup S_{t-1}; o_t) \leq 1$ for every $t \in [r]$, we get $v_i(S) \leq v_i(\emptyset) + r = 0 + |S \setminus \emptyset| = |S|$. □

### A.3 Proof of Proposition 2

*Proof.* The "if" part: Suppose, there is an item $o \in S$ such that $\Delta_i(S \setminus \{o\}; o) = 0$. Now, by Proposition 1, $v_i(S \setminus \{o\}) \leq |S \setminus \{o\}| = |S| - 1$ since $o \in S$. This implies that $v_i(S) = v_i(S \setminus \{o\}) + \Delta_i(S \setminus \{o\}; o) < |S|$. Thus, by contraposition, if $v_i(S) = |S|$, then $\Delta_i(S \setminus \{o\}; o) = 1 \; \forall o \in S$ since the marginal gain can be either 0 or 1, i.e. $S$ is a clean bundle for $i$.
The "only if" part: As in the proof of Proposition 1, let $S = \{o_1, o_2, \ldots, o_r\}$; define $S_0 = \emptyset$ and $S_t = \{o_1, o_2, \ldots, o_t\}$ for each $t \in [r]$. By the definition of cleanness, $\Delta_i(S \setminus \{o_t\}; o_t) = 1 \; \forall t \in [r]$. Since $S_{t-1} \subseteq S \setminus \{o_t\}$ for every $t \in [r]$, $\Delta_i(S_{t-1}; o_t) \geq \Delta_i(S \setminus \{o_t\}; o_t) = 1$; moreover due to marginal gains in $\{0,1\}$, we must have $\Delta_i(S_{t-1}; o_t) = 1$ for every $t \in [r]$. Hence, $v_i(S) = \sum_{t=1}^{r} \Delta_i(S_{t-1}; o_t) = r = |S|$. □

### A.4 Proof of Lemma 1

*Proof.* Assume that agent $i$ envies agent $j$ under an allocation $A$, i.e. $v_i(A_i) < v_i(A_j)$, but no item $o \in A_j$ has a positive marginal gain, i.e., $\Delta_i(A_i; o) = 0$ for each $o \in A_j$. Let $A_j = \{o_1, o_2, \dots, o_r\}$. As in the proof of Proposition 1, if we define $S_0 = \emptyset$ and $S_t = \{o_1, o_2, \dots, o_t\}$ for each $t \in [r]$, we can write the following telescoping series:

$$v_i(A_i \cup A_j) - v_i(A_i) = \sum_{t=1}^{r} \Delta_i(A_i \cup S_{t-1}; o_t).$$

However, submodularity implies that for each $t \in [r]$, $\Delta_i(A_i \cup S_{t-1}; o_t) \leq \Delta_i(A_i; o_t) = 0$, meaning that

$$v_i(A_i \cup A_j) - v_i(A_i) = \sum_{t=1}^{r} \Delta_i(A_i \cup S_{t-1}; o_t) = 0.$$

Together with monotonicity, this yields $v_i(A_j) \leq v_i(A_i \cup A_j) = v_i(A_i) < v_i(A_j)$, a contradiction. □

### A.5 Example of non-submodular valuations that violate the transferability property

*Example 2.* Agent 1 wants to have a pair of matching shoes; her current allocated bundle is a single red shoe, whereas agent 2 has a matching pair of blue shoes. Agent 1 clearly envies agent 2, but cannot increase the value of her bundle by taking any one of agent 2's items. More formally, suppose $N = [2]$ and $O = \{r_L, b_L, b_R\}$; agent 1's valuation function is: $v_1(S) = 1$ only if $\{b_L, b_R\} \subseteq S$, $v_1(S) = 0$ otherwise. Under the allocation $A_1 = \{r_L\}$ and $A_2 = \{b_L, b_R\}$, $v_1(A_1) < v_1(A_2)$ but $\Delta_1(A_1; o) = 0$ for all $o \in A_2$. ■

### A.6 Proof of Lemma 2

*Proof.* From the definition: $A_j \neq \emptyset$ and $v_i(A_i) < v_i(A_j \setminus \{o\})$ for every $o \in A_j$. Consider one such $o$. From Proposition 1, $v_i(A_j \setminus \{o\}) \leq |A_j \setminus \{o\}| = |A_j| - 1$. Since $A_j$ is a clean bundle for $j$, Proposition 2 implies that $v_j(A_j) = |A_j|$. Combining these, we get

$$v_i(A_i) < v_i(A_j \setminus \{o\}) \leq |A_j| - 1 = v_j(A_j) - 1 \qquad \Rightarrow \qquad v_j(A_j) > v_i(A_i) + 1,$$

which proves the theorem statement since all valuations are integers. □

### A.7 Proof of Corollary 1

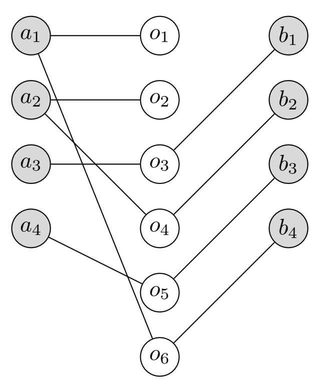
*Proof.* Take an allocation $A'$ minimizes the sum of squares of the realized valuations among all utilitarian optimal allocations. Then, there is a clean, utilitarian optimal allocation $A$ that has the same sum of squares as $A'$. We will show that $A$ is EF1. Assume towards a contradiction that $A$ is not EF1. Then, there is a pair of agents $i, j$ such that $i$ envies $j$ up to more than 1 item. By Lemma 1, there is an item $o \in A_j$ such that $\Delta_i(A_i; o) = 1$. Let $A^*$ be the allocation achieved by transferring $o$ from $j$ to $i$, everything else remaining the same. By Lemma 2 and the fact that $A_j$ is clean, we have

$$v_i(A_i) \leq v_j(A_j) + 2,$$

which implies $\sum_{i \in N} v_i(A_i^*)^2 < \sum_{i \in N} v_i(A_i)^2$ proceeding exactly as in the proof of Theorem 1 — another contradiction. Hence, $A$ must be EF1. $\square$

### A.8 Example that disproves the converse of Corollary 1

*Example 3.* The instance we use is Example 1 in Benabbou et al. [8]. There are two groups (i.e. agents with $(0, 1)$-OXS valuations) and six items $o_1, o_2, o_3, o_4, o_5, o_6$. The first group $N_1$ (identical to agent 1) contains four members $a_1, a_2, a_3, a_4$ and the second group $N_2$ (identical to agent 2) contains four members $b_1, b_2, b_3, b_4$; each individual has utility (weight) 1 for an item $o$ if and only if she is adjacent to $o$ in the adjoining graph:




The valuation function of each group for any bundle $X$ is defined as the value (equivalently, the size) of a maximum-size matching of $X$ to the group's members. The algorithm may initially compute a utilitarian optimal allocation $A$ that assigns items $o_1, o_2, o_3, o_5$ to the group $N_1$ (with these items assigned to $a_1, a_2, a_3, a_4$ respectively), and the remaining items to group $N_2$ (with $o_4, o_6$ assigned to $b_2, b_4$ respectively). Then, $v_1(A_1) = 4 > 2 = v_1(A_2)$ and $v_2(A_2) =$

$2 = v_2(A_1)$, hence the allocation $A$ is EF1 — in fact, envy-free! So, the `EIT` subroutine will not be invoked and the output of Algorithm 1 will be $A$. However, the (unique) leximin and MNW allocation assigns items $o_1, o_2, o_3$ to the first group, and the remaining items to the second group – this is also the (unique) utilitarian optimal allocation with the minimum sum of squares of the agents' valuations. ■

### A.9 Remarks on Theorem 1 and Algorithm 1

*Remark 1 (Choice of the potential function).* In the proof of Theorem 1, we used the sum of squared valuations as the potential function to prove termination in polynomial time mainly for ease of exposition. However, any *symmetric, strictly convex,*[8] *polynomial* function of the realized valuations strictly decreases with each `EIT` step and, as such, it would be sufficient to use any such function as our potential function; moreover, Corollary 1 holds for any such function $\phi(\cdot)$ as well — we elaborate on this theme in Section 3.2.

*Remark 2 (EFX allocation).* It is worthwhile at this point to comment on the implications of our results for a stronger version of the EF1 property that has received considerable attention in recent literature: envy-freeness up to *any* item, often called the EFX condition. There are two definitions in the literature:

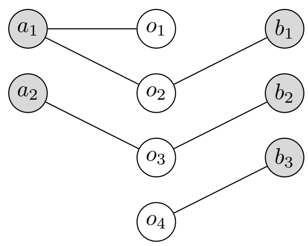
1. Caragiannis et al. [14] who introduced this concept (for additive valuations) called it *envy-freeness up to the least (positively) valued good*; we can naturally extend their definition to general valuations as follows: an allocation $A$ is EFX if, for every pair of agents $i, j \in N$ such that $i$ envies $j$, $v_i(A_i) \geq v_i(A_j \setminus \{o\})$ for every item $o \in A_j$ satisfying $\Delta_i(A_j \setminus \{o\}) > 0$. We will call this property $\text{EFX}_+$ for clarity.
2. Plaut and Roughgarden [36] defined an allocation $A$ to be EFX if, for every pair of agents $i, j$, $v_i(A_i) \geq v_i(A_j \setminus \{o\})$ $\forall o \in A_j$ or equivalently $v_i(A_i) \geq \max_{o \in A_j} v_i(A_j \setminus \{o\})$ — this stronger definition favors allocations where more agents are envy-free of others since $v_i(A_j \setminus \{o\}) = v_i(A_j)$ whenever $o$ is of zero marginal value to agent $i$ with respect to the bundle $A_j$: the authors show that under this definition, no EFX allocation can be Pareto optimal even for two agents with additive valuations or general but identical valuations.[9] Caragiannis et al. [13] and Chaudhury et al. [15] use this definition as well. Following Kyropoulou et al. [28], who studied both the (above) weaker and (this) stronger variants of approximate envy-freeness under a different valuation model, we call this stronger property $\text{EFX}_0$.

[8] It suffices for the function to be strictly convex only over the non-negative orthant since valuations are always non-negative.

[9] However, both their examples establishing negative results for these sets of conditions on the valuation functions involve eliminating items with zero marginal value; their second example (for identical valuations) uses a non-submodular valuation function.

For matroid rank valuations, all items with non-zero marginal values for an agent are also valued identically at 1, hence EF1 trivially implies $\text{EFX}_+$; Theorem 1 and Corollary 1 further guarantee the existence of an $\text{EFX}_+$ and PO allocation for any instance under this valuation class. However, we demonstrate with Example 4 with $(0,1)$-OXS valuations shows that even an EF1 and utilitarian optimal (hence PO) allocation may not satisfy the $\text{EFX}_0$ condition. ⧫

*Example 4.* There are two groups and four items $o_1, o_2, o_3, o_4$. The first group $N_1$ has two members $a_1, a_2$ and the second group $N_2$ has three $b_1, b_2, b_3$; each individual has utility 1 for an item if and only if she is adjacent to it in the adjoining graph:




The $(0,1)$-OXS valuation functions of groups $N_1$ and $N_2$ are denoted by $v_1(\cdot)$ and $v_2(\cdot)$ respectively. The allocation $A$ where $A_1 = \{o_1\}$ and $A_2 = \{o_2, o_3, o_4\}$ is utilitarian optimal; it is also EF1 since $v_1(A_1) = 1 = v_1(A_2 \setminus \{o\})$ for $o \in \{o_2, o_3\}$, with $\Delta_1(A_2 \setminus \{o_4\}; o_4) = 0$, and $v_2(A_2) = 3 > 0 = v_2(A_1)$. $A$ could be the output of Algorithm 1 and is clean and complete. However, $A$ is not $\text{EFX}_0$ since $v_1(A_2 \setminus \{o_4\}) = 2 > 1 = v_1(A_1)$. ■

### A.10 Proof of Theorem 3

In this proof, we will use the following notions and results from matroid theory: Given a matroid $(E, \mathcal{I})$, the sets in $2^E \setminus \mathcal{I}$ are called *dependent*, and a minimal dependent set of a matroid is called a *circuit*. The following is a crucial property of circuits.

**Lemma 3 (Korte and Vygen [27]).** *Let $(E, \mathcal{I})$ be a matroid, $X \in \mathcal{I}$, and $y \in E \setminus X$ such that $X \cup \{y\} \notin \mathcal{I}$. Then the set $X \cup \{y\}$ contains a unique circuit.*

Given a matroid $(E, \mathcal{I})$, we denote by $C(\mathcal{I}, X, y)$ the unique circuit contained in $X \cup \{y\}$ for any $X \in \mathcal{I}$ and $y \in E \setminus X$ such that $X \cup \{y\} \notin \mathcal{I}$.

*Proof (Theorem 3).* Define $E$, $X_i$, $E_i$, $\mathcal{I}_i$ for $i \in N$, $\mathcal{I}$, and $\mathcal{O}$ as in the proof of Theorem 2. We first observe that for each $X \in \mathcal{I}$ and each $y \in E \setminus X$, if $X \cup \{y\} \notin \mathcal{I}$, then there is agent $i \in N$ whose corresponding items in $X_i$

together with $y$ is not clean, i.e., $X_i \cup \{y\} \notin \mathcal{I}_i$, which by Lemma 3 implies that the circuit $C(\mathcal{I}, X, y)$ is contained in $E_i$, i.e.,

$$C(\mathcal{I}, X, y) = C(\mathcal{I}_i, X, y). \tag{1}$$

Now to prove the claim, let $A$ be a Pareto optimal allocation. Without loss of generality, we assume that $A$ is clean. Then, as we have seen before, $A$ corresponds to a common independent set $X^*$ in $\mathcal{I} \cap \mathcal{O}$ given by

$$X^* = \bigcup_{i \in N} \{\, e = \{o, i\} \in E \mid o \in A_i \,\}.$$

Suppose towards a contradiction that $A$ does not maximize the utilitarian social welfare. This means that $X^*$ is not a largest common independent set of $\mathcal{I}$ and $\mathcal{O}$. It is known that given two matroids and their common independent set, if it is not a maximum-size common independent set, then there is an 'augmenting' path [18]. To formally define an augmenting path, we define an auxiliary graph $G_{X^*} = (E, B^{(1)}_{X^*} \cup B^{(2)}_{X^*})$ where the set of arcs is given by

$$B^{(1)}_{X^*} = \{\, (x, y) \mid y \in E \setminus X^* \wedge x \in C(\mathcal{O}, X^*, y) \setminus \{y\} \,\},$$
$$B^{(2)}_{X^*} = \{\, (y, x) \mid y \in E \setminus X^* \wedge x \in C(\mathcal{I}, X^*, y) \setminus \{y\} \,\}.$$

Since $X^*$ is not a maximum common independent set of $\mathcal{O}$ and $\mathcal{I}$, the set $X^*$ admits an *augmenting* path, which is an alternating path $P = (y_0, x_1, y_1, \ldots, x_s, y_s)$ in $G_{X^*}$ with $y_0, y_1, \ldots, y_s \notin X^*$ and $x_1, x_2, \ldots, x_s \in X^*$, where $X^*$ can be augmented by one element along the path, i.e.,

$$X' = (X^* \setminus \{x_1, x_2, \ldots, x_s\}) \cup \{y_0, y_1, \ldots, y_s\} \in \mathcal{I} \cap \mathcal{O}.$$

Now let us write the pairs of agents and items that correspond to $y_t$ and $x_t$ as follows:

- $y_t = \{i(y_t), o(y_t)\}$ where $i(y_t) \in N$ and $o(y_t) \in O$ for $t = 0, 1, \ldots, s$; and
- $x_t = \{i(x_t), o(x_t)\}$ where $i(x_t) \in N$ and $o(x_t) \in O$ for $t = 1, 2, \ldots, s$.

Since each $x_t$ ($t \in [s]$) belongs to the unique circuit $C(\mathcal{I}, X^*, y_{t-1})$, which is contained in the set of edges incident to $i(y_{t-1})$ by the observation made in (1), we have $i(x_t) = i(y_{t-1})$ for each $t \in [s]$. Thus, along the augmenting path $P$, each agent $i(x_t)$ receives a new item $o(y_{t-1})$ and discards the old item $o(x_t)$. Now consider the reallocation corresponding to $X'$ where agent $i(x_t)$ receives a new item $o(y_{t-1})$ but loses the item $o(x_t)$ for each $t = 1, 2, \ldots, s$, and agent $i(y_s)$ receives the item $o(y_s)$. Such a reallocation increases the valuation of agent $i(y_s)$ by 1, while it does not decrease the valuations of all the intermediate agents, $i(x_1), i(x_2), \ldots, i(x_s)$, as well as the other agents whose agents do not appear on $P$. We thus conclude that $A$ is Pareto dominated by the new allocation, a contradiction. □

### A.11 Proof of Theorem 4

We will begin by formally stating and proving the two lemmas that we alluded to in Section 3.2. Let us denote by $\chi_i$ the $n$-dimensional incidence vector where the $r^{\text{th}}$ component of $\chi_i$ is 1 if $r = i$, and 0 otherwise.

**Lemma 4.** *Suppose that agents have matroid rank valuations. Let $A$ be a utilitarian optimal allocation. If $A$ is not a leximin allocation, then there is another utilitarian optimal allocation $A'$ such that, for some $i, j \in [n]$ with $\theta(A)_j \geq \theta(A)_i + 2$, $\theta(A') = \theta(A) + \chi_i - \chi_j$.*

*Proof.* Let $A$ be an arbitrary utilitarian optimal allocation which is not leximin, and let $A^*$ be a leximin allocation. Recall that $A^*$ is utilitarian optimal by Theorem 3. Without loss of generality, we assume that both $A$ and $A^*$ are clean allocations. Now take a clean allocation $A'$ that minimizes the symmetric difference $\sum_{i \in N} |A'_i \triangle A^*_i|$ over all clean allocation with $\theta(A') = \theta(A)$. Assume also w.l.o.g. that $v_1(A'_1) \leq v_2(A'_2) \leq \cdots \leq v_n(A'_n)$. We let $v_{j_1}(A^*_{j_1}) \leq v_{j_2}(A^*_{j_2}) \leq \cdots \leq v_{j_n}(A^*_{j_n})$. Since $A^*$ lexicographically dominates $A'$, for the minimum index $k$ with $v_j(A'_k) \neq v_{j_k}(A^*_{j_k})$,

$$v_k(A'_k) < v_{j_k}(A^*_{j_k}). \tag{2}$$

We note that $v_h(A'_h) = v_{j_h}(A^*_{j_h})$ for all $1 \leq h \leq k-1$. By (2), there exists $i \in [k]$ with

$$v_i(A'_i) < v_i(A^*_i). \tag{3}$$

Indeed, if for all $i \in [k]$, $v_i(A'_i) \geq v_i(A^*_i)$, the $k$-th smallest value of realized valuations under $A'$ is at least $v_{j_k}(A^*_{j_k})$, contradicting with (2). Take the minimum index $i$ satisfying (3). Since both $A'$ and $A^*$ are clean allocations, we have

$$|A'_i| = v_i(A'_i) < v_i(A^*_i) = |A^*_i|. \tag{4}$$

By minimality, for all $h \in [i-1]$, $v_h(A'_h) \geq v_h(A^*_h)$. In fact, the equality

$$v_h(A'_h) = v_h(A^*_h) \tag{5}$$

holds for all $h \in [i-1]$. Indeed if $v_h(A'_h) > v_h(A^*_h)$ for some $h \in [i-1]$, then $h$-th smallest value of the realized valuations under $A'$ would be strictly greater than that under $A^*$, yielding $\theta(A') >_L \theta(A^*)$, a contradiction.
Now, recall that the family of clean bundles $\mathcal{I}_h = \{\, S \subseteq O \mid v_h(S) = |S| \,\}$ for $h \in N$ forms a family of independent sets of a matroid. By (I3) of the independent-set matroid axioms and by the inequality (4), there exists an item $o_1 \in A^*_i \setminus A'_i$ with positive contribution to $A'_i$, i.e., $v_i(A'_i \cup \{o_1\}) = v_i(A'_i) + 1$. By utilitarian optimality of $A'$, $o_1$ is allocated to some agent, i.e., $o_1 \in A'_{i_1}$ for some $i_1 \neq i$. Consider the following three cases:

- Suppose $v_{i_1}(A'_{i_1}) \geq v_i(A'_i) + 2$. Then, we obtain a desired allocation by transferring $o_1$ from $i_1$ to $i$.

- Suppose $v_{i_1}(A'_{i_1}) = v_i(A'_i) + 1$. Then by transferring $o_1$ from $i_1$ to $i$, we get another utilitarian optimal allocation with the same vector as $\theta(A')$, which has a smaller symmetric difference than $\sum_{i \in N} |A'_i \triangle A^*_i|$, a contradiction.
- Suppose $v_{i_1}(A'_{i_1}) \leq v_i(A'_i)$. We will first show that $v_{i_1}(A'_{i_1}) \leq v_{i_1}(A^*_{i_1})$. By (5), this clearly holds if $i_1 \leq i$. Also, when $i_1 > i$, this means that $v_{i_1}(A'_{i_1}) = v_i(A'_i)$; thus $v_{i_1}(A'_{i_1}) \leq v_{i_1}(A^*_{i_1})$, as otherwise the $i$-th smallest value of realized valuations under $A'$ would be greater than that under $A^*$, contradicting that $A^*$ is leximin. Further by the facts that $|A'_{i_1} \setminus \{o_1\}| < |A^*_{i_1}|$ and that both $A'_{i_1} \setminus \{o_1\}$ and $A^*_{i_1}$ are clean (i.e., independent sets of a matroid), there exists an item $o_2 \in A^*_{i_1} \setminus A'_{i_1}$ such that $v_{i_1}(A'_{i_1} \cup \{o_2\} \setminus \{o_1\}) = v_i(A'_{i_1})$. Again by utilitarian optimality of $A'$, $o_2$ is allocated to some agent, i.e., $o_2 \in A'_{i_2}$ for some $i_2 \neq i_1$.

Repeating the same argument and letting $i_0 = i$, we obtain a sequence of items and agents $(i_0, o_1, i_1, o_2, i_2, \dots, o_t, i_t)$ such that

- $v_{i_h}(A'_{i_h}) = v_{i_h}(A'_{i_h} \cup \{o_{h+1}\} \setminus \{o_h\})$ for all $1 \leq h \leq t-1$; and
- $o_h \in A^*_{i_{h-1}} \setminus A'_{i_h}$ for all $1 \leq h \leq t$.

See Figure 1 for an illustration of the sequence. If the same agent appears again, i.e., $i_h = i_{h'}$ for some $h < h' \leq t$, then by transferring items along the cycle, we can decrease the symmetric difference with $A^*$, a contradiction. Thus, the sequence must terminate when we reach the agent $i_t$ with $v_{i_t}(A'_{i_t}) \geq v_i(A'_i) + 2$. Exchanging items along the path, we get a desired allocation. □

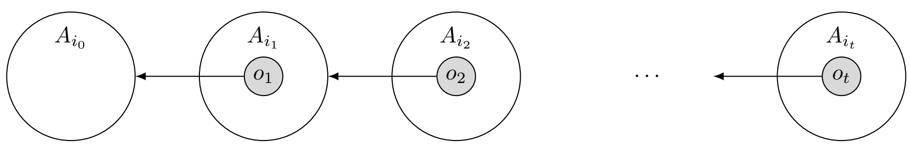



**Fig. 1.** The path $(i_0, o_1, i_1, o_2, i_2, \dots, o_t, i_t)$

**Lemma 5.** *Let $\Phi : \mathbb{Z}^n \to \mathbb{Z}$ be a symmetric strictly convex function. Let $A$ be a utilitarian optimal allocation. Let $A'$ be another utilitarian optimal allocation such that $\theta(A') = \theta(A) + \chi_i - \chi_j$ for some $i, j \in [n]$ with $\theta(A)_j \geq \theta(A)_i + 2$. Then $\Phi(A) > \Phi(A')$.*

*Proof.* The proof is similar to that of Proposition 6.1 in Frank and Murota [21], which shows the analogous equivalence over the integral base-polyhedron. Let $\beta = \theta(A)_j - \theta(A)_i \geq 2$, and $y = \theta(A) + \beta(\chi_i - \chi_j)$. Thus $\Phi(\theta(A)) = \Phi(y)$ by symmetry of $\Phi$. Define $\lambda = 1 - \frac{1}{\beta}$. We have $0 < \lambda < 1$ since $\beta \geq 2$. Observe that

$$\begin{aligned} \lambda\theta(A) + (1-\lambda)y &= (1 - \frac{1}{\beta})\theta(A) + \frac{1}{\beta}(\theta(A) + \beta(\chi_i - \chi_j)) \\ &= \theta(A) + \chi_i - \chi_j = \theta(A'), \end{aligned}$$

which gives us the following inequality (from the strict convexity of $\Phi$): $\Phi(\theta(A)) = \lambda\Phi(\theta(A)) + (1-\lambda)\Phi(\theta(A)) > \Phi(\theta(A'))$. □

We are now ready to prove Theorem 4.

*Proof (Theorem 4).* To prove $1 \Leftrightarrow 2$, let $A$ be a leximin allocation, and let $A'$ be a minimizer of $\Phi$ over all the utilitarian optimal allocations. We will show that $\theta(A')$ is the same as $\theta(A)$, which, by the uniqueness of the leximin valuation vector and symmetry of $\Phi$, proves the theorem statement.

Assume towards a contradiction that $\theta(A) \neq \theta(A')$. By Theorem 3, we have $\texttt{USW}(A) = \texttt{USW}(A')$. By Lemma 4, we can obtain another utilitarian optimal allocation $A''$ that is a lexicographic improvement of $A'$ by decreasing the value of the $j$-th element of $\theta(A')$ by 1 and increasing the value of the $i$-th element of $\theta(A')$ by 1, where $\theta(A')_j \geq \theta(A')_i + 2$. Applying Lemma 5, we get $\Phi(\theta(A')) > \Phi(\theta(A''))$, which gives us the desired contradiction.

To prove $2 \Leftrightarrow 3$, let $A$ be a leximin allocation, and let $A'$ be an MNW allocation. Again, we will show that $\theta(A')$ is the same as $\theta(A)$, which by the uniqueness of the leximin valuation vector and symmetry of $\texttt{NW}$, proves the theorem statement. Let $N_{>0}(A)$ (respectively, $N_{>0}(A')$) be the agent subset to which we allocate bundles of positive values under leximin allocation $A$ (respectively, MNW allocation $A'$). By definition, the number $n'$ of agents who get positive values under leximin allocation $A$ is the same as that of MNW allocation $A'$. Now we denote by $\bar{\theta}(A)$ (respectively, $\bar{\theta}(A')$) the vector of the non-zero components $v_i(A_i)$ (respectively, $v_i(A'_i)$) arranged in non-decreasing order. Assume towards a contradiction that $\bar{\theta}(A) >_L \bar{\theta}(A')$. Since $A'$ maximizes the product $\texttt{NW}(A')$ when focusing on $N_{>0}(A')$ only, the value $\sum_{i \in N_{>0}(A')} \log v_i(A'_i)$ is maximized. However, $\phi(x) = -\sum_{i=1}^{n'} \log x_i$ is a symmetric, strictly convex function for $x \in \mathbb{Z}^n$ with each $x_i > 0$. Thus, by a similar argument as before, one can show that $\phi(\bar{\theta}(A')) < \phi(\bar{\theta}(A))$, a contradiction. This completes the proof. □

### A.12 Example showing that neither leximin nor MNW allocation may be utilitarian optimal for general assignment valuations

*Example 5.* Consider an instance with assignment valuations given as follows. Suppose there are three groups, each of which contains a single agent, Alice, Bob, and Charlie, respectively, and three items with weights given in Table 2. The unique leximin and MNW allocation is the allocation that assigns Alice to the first item, Bob to the second item, and Charlie to the third item; each agent has positive utility at the allocation and the total utilitarian social welfare is 3.1. However, the utilitarian optimal allocation assigns Alice to nothing, Bob to the first item, and Charlie to the second item, which yields the total utilitarian social welfare 4.9. ■

| | ① | ② | ③ |
|---|---|---|---|
| Alice: | 2 | 1 | 0 |
| Bob: | 2 | 1 | 0 |
| Charlie: | 0 | 2.9 | 0.1 |

**Table 2.** An instance where neither leximin nor MNW allocation is utilitarian optimal.

### A.13 Proof of Corollary 2

*Proof.* Since both leximin and MNW allocations are Pareto optimal, they maximize the utilitarian social welfare, by Theorem 3. By Theorem 4 and the fact that the function $\Phi(A) \triangleq \sum_{i \in N} v_i(A_i)^2$ is a symmetric strictly convex function, any leximin or MNW allocation is a utilitarian optimal allocation that minimizes $\Phi(A)$ among all utilitarian optimal allocations; hence, if such an allocation is clean, it must be EF1 by Corollary 1. □

### A.14 Proof of Theorem 5

*Proof.* The problem of finding a leximin allocation can be reduced to that of finding an integral balanced flow (or increasingly-maximal integer-valued flow) in a network, which has been recently shown to be polynomial-time solvable [22]. Specifically, for a network $D = (V, A)$ with source $s$, sink $t$, and a capacity function $c : A \to \mathbb{Z}$, a *balanced flow* is a maximum integral feasible flow where the out-flow vector from the source $s$ to the adjacent vertices $h$ is lexicographically maximized among all maximum integral feasible flows; that is, the smallest flow-value on the edges $(s, h)$ is as large as possible, the second smallest flow-value on the edges $(s, h)$ is as large as possible, and so on. Frank and Murota [22] show that one can find a balanced flow in strongly polynomial time (see Section 7 in Frank and Murota [22]).
Now, given an instance of assignment valuations with binary marginal gains, we build the following instance $(V, A)$ of a network flow problem. Let $N_h$ denote the set of members in each group $h$. We first create a source $s$ and a sink $t$. We create a vertex $h$ for each group $h$, a vertex $i$ for each member $i$ of some group, and a vertex $o$ for each item $o$. We construct the edges of the network as follows:

- for each group $h$, create an edge $(s, h)$ with capacity $m$; and
- for each group $h$ and member $i$ in group $h$, create an edge $(h, i)$ with unit capacity; and
- for each member $i$ of some group and item $o$ for which $i$ has positive weight $u_{io}$ (i.e. $u_{io} = 1$), create an edge $(i, o)$ with unit capacity; and
- for each item $o$, create an edge $(o, t)$ with unit capacity.

See Figure 2 for an illustration of the network. We will show that an integral balanced flow $f : A \to \mathbb{Z}$ of the constructed network corresponds to a leximin allocation. Consider an allocation $A^f$ where each group receives the items $o$ for which some member $i$ of the group has positive flow $f(i,o) > 0$. It is easy to see that the allocation $A^f$ maximizes the utilitarian social welfare since the flow $f$ is a maximum integral feasible flow. Thus, by Theorem 3, $A^f$ has the same utilitarian social welfare as any leximin allocation. To see balancedness, observe that the amount of flow from the source $s$ to each group $h$ is the valuation of $h$ for bundle $A^f_h$, i.e., $f(s,h) = \sum_{i \in N_h} f(h,i) = v_h(A^f_h)$. Indeed if $v_h(A^f_h) > f(s,h)$, then it would contradict the optimality of the flow $f$; and if $v_h(A^f_h) < f(s,h)$, it would contradict the fact that $v_h(A^f_h)$ is the value of a maximum-size matching between $A^f_h$ and $N_h$. Thus, among all utilitarian optimal allocations, $A^f$ lexicographically maximizes the valuation of each group, and hence $A_f$ is a leximin allocation. By Theorem 4, the leximin allocation $A_f$ is also MNW. □

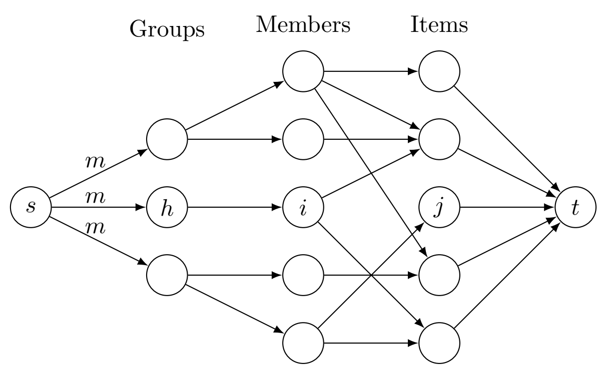



**Fig. 2.** An illustrative network flow instance constructed in the proof of Theorem 5: each edge is either labeled with its capacity or has unit capacity.

### A.15 Proof of Theorem 6

*Proof.* The reduction is similar to the hardness reduction for two agents with identical additive valuations [34, 37]. We give a Turing reduction from PARTITION. Recall that an instance of PARTITION is given by a set of positive integers $W = \{w_1, w_2, \ldots, w_m\}$; it is a 'yes'-instance if and only if it can be partitioned into two subsets $S_1$ and $S_2$ of $W$ such that the sum of the numbers in $S_1$ equals the sum of the numbers in $S_2$.

Consider an instance of PARTITION $W = \{w_1, w_2, \ldots, w_m\}$. We create $m$ items $1, 2, \ldots, m$, two groups 1 and 2, and $m$ individuals for each group where every individual has a weight $w_j$ for item $j$. Observe that fore each group, the value of each bundle $X$ is the sum $\sum_{w_j \in X} w_j$: the number of members in the group exceeds the number of items in $X$, and thus one can fully assign each item to each member of the group.

Suppose we had an algorithm which finds a leximin allocation. Run the algorithm on the allocation problem constructed above to obtain a leximin allocation $A$. It can be easily verified that the instance of PARTITION has a solution if and only if $v_1(A_1) = v_2(A_2)$. Similarly, suppose we had an algorithm which finds an MNW allocation, and run the algorithm to find an MNW allocation $A'$. Since the valuations are identical, the utilitarian social welfare of the MNW allocation is the sum $\sum_{w_j \in W} w_j$, which means that the product of the valuations is maximized when both groups have the same realized valuation. Thus, the instance of PARTITION has a solution if and only if $v_1(A'_1) = v_2(A'_2)$. □

# B Appendix: Other fairness criteria under matroid rank valuations

In the main paper, we have focused on Pareto optimal and EF1 allocations for the matroid rank valuation class. However, many other concepts have been defined and studied in the literature that formalize different intuitive ideas for what it means for an allocation of indivisible goods to be fair. In this section, we will investigate the implications of our results from the main paper for some alternative fairness notions.

An allocation $A$ is said to be *equitable* or EQ if the realized valuations of all agents are equal under it, i.e. for every pair of agents $i, j \in N$, $v_i(A_i) = v_j(A_j)$; an allocation $A$ is *equitable up to one item* or EQ1 if, for every pair of agents $i, j \in N$ such that $A_j \neq \emptyset$, there exists some item $o \in A_j$ such that $v_i(A_i) \geq v_j(A_j \setminus \{o\})$ [23].[10] We can further relax the equitability criterion up to an arbitrary number of items: an allocation $A$ is said to be *equitable up to $c$ items* or EQ$c$ if, for every pair of agents $i, j \in N$ such that $|A_j| > c$, there exists some subset $S \in A_j$ of size $|S| = c$ such that $v_i(A_i) \geq v_j(A_j \setminus S)$.[11]

Freeman et al. [23] show[12] that, even for binary additive valuations (which is a subclass of the $(0,1)$-OXS valuation class), an allocation that is both EQ1 and PO may not exist; however, in Theorem 4, they establish that it can be verified in polynomial time whether an EQ1, EF1 and PO allocation exists and, whenever it does exist, it can also be computed in polynomial time — under binary additive valuations. We will show that the above positive result about computational

[10] Note that if $A_j = \emptyset$ for some $j$, $v_i(A_i) \geq v_j(A_j)$ trivially. Hence the ordered pair $(i, j)$ for any $i \in N \setminus \{j\}$ could never prevent the allocation from being EQ1.

[11] Again, if $|A_j| \leq c$ for some $j$, no ordered pair $(i, j)$ for any $i \in N \setminus \{j\}$ could get in the way of the allocation being EQ$c$.

[12] Freeman et al. [23] use an example with 3 agents having binary additive valuations (Example 1). But it is easy to construct a fair allocation instance with only two agents having binary additive valuations that does not admit an EQ1 and PO allocation: $N = [2]$; $O = \{o_1, o_2, o_3, o_4\}$; $v_1(o) = 1$ for every $o \in O$; $v_2(o_1) = 1$ and $v_2(o) = o$ for every $o \in O \setminus \{o_1\}$. Obviously, any PO allocation must give $\{o_2, o_3, o_4\}$ to agent 1 so that this agent's realized valuation is at least 2 even after dropping one of its items; even if agent 2 receives $o_1$, her realized valuation of 1 will always be less than the above.

tractability extends to the $(0,1)$-OXS valuation class. We will begin by proving that under matroid rank valuations, an EQ1 and PO allocation, if it exists, is also EF1 — we achieve this by combining Theorem 7 below with Corollary 1. This simplifies the problem of finding an EQ1, EF1 and PO allocation to that of finding an EQ1 and PO allocation.

**Theorem 7.** *For submodular valuations with binary marginal gains, any EQ1 and PO allocation, if it exists, is a leximin allocation.*

Hence, from Theorem 4, we further obtain that if an EQ1 and PO allocation exists under matroid rank valuations, it is also MNW and a minimizer of any symmetric strictly convex function of agents' realized valuations among all utilitarian optimal allocations. Moreover, Theorem 7, together with Corollary 2, implies that if an EQ1 and PO allocation exists under matroid rank valuations, it must be EF1.

*Proof.* Let the optimal `USW` for a problem instance under this valuation class be $U^*$; also, suppose this instance admits an EQ1 and PO allocation $A$. The EQ1 property implies that for every pair of agents $i, j \in N$ such that $A_j \neq \emptyset$,

$$\begin{aligned} v_i(A_i) &\geq v_j(A_j \setminus \{o\}) \qquad \text{for some } o \in A_j \\ &= v_j(A_j) - \Delta_j(A_j \setminus \{o\}; o) \\ &\geq v_j(A_j) - 1, \qquad \text{since } \Delta_j(A_j \setminus \{o\}; o) \in \{0, 1\}. \end{aligned}$$

This inequality holds trivially and strictly if $A_j = \emptyset$. Thus, $\max_{i \in N} v_i(A_i) \leq \min_{i \in N} v_i(A_i) + 1$. In other words, there exist a non-negative integer $\alpha \leq U^*$ and a positive integer $n_0 \in [n]$ such that $n_0$ agents have valuations $\alpha$ each and the remaining agents, if any, have valuations $\alpha + 1$ each under allocation $A$, with $U^* = n_0\alpha + (n - n_0)(\alpha + 1) = n\alpha + n - n_0$. We can write the agents' realized valuations under $A$ (with arbitrary tie-breaking) as the $n$-dimensional vector

$$\theta(A) = \left( \underbrace{\alpha, \alpha, \ldots, \alpha}_{n_0 \text{ entries}}, \underbrace{\alpha + 1, \alpha + 1, \ldots, \alpha + 1}_{n - n_0 \text{ entries}} \right).$$

If $A$ were not leximin, there would be another allocation $A'$ for which the corresponding valuation vector $\theta(A')$ would have an entry strictly higher than that of $A$ at the same position, say $n' \in [n]$. If $n' \leq n_0$, then every entry of $\theta(A')$ from position $n'$ is at least $\alpha + 1$, so the `USW` under $A'$ is

$$\begin{aligned} U' &\geq (n' - 1)\alpha + (n - n' + 1)(\alpha + 1) \\ &= n\alpha + n - n' + 1 \\ &\geq n\alpha + n - n_0 + 1 \\ &= U^* + 1. \end{aligned}$$

If $n_0 < n' \leq n$, then similarly,

$$\begin{aligned}
U' &\geq n_0\alpha + (n' - n_0 - 1)(\alpha + 1) + (n - n' + 1)(\alpha + 2) \\
&= n\alpha + 2n - n' - n_0 + 1 \\
&\geq n\alpha + 2n - n - n_0 + 1 \\
&= n\alpha + n - n_0 + 1 \\
&= U^* + 1.
\end{aligned}$$

In either case, we have a contradiction since $U^*$ is the optimal utilitarian social welfare for this instance. Hence, $A$ must be leximin. □

We conjecture that a stronger result holds: under matroid rank valuations, the leximin allocation is optimally EQ$c$ for $c \in \{0, 1, \ldots, m\}$ among all PO allocations. A proof or a counterexample remains elusive. We present this more formally as follows:

*Conjecture 1.* For a problem instance where all agents have submodular valuations with binary marginal gains, any EQ1 and PO allocation, if it exists, if $c^*$ is the smallest $c \in [m]$ for which a leximin allocation is EQ$c$, then the instance admits no PO allocation that is EQ$c$ for any $c < c^*$.

## C Appendix: Submodularity with subjective binary gains

An obvious generalization of the matroid rank valuation function class is the class of submodular valuation functions with *subjective* binary marginal gains: agent $i$'s bundle-valuation function $v_i(\cdot)$ is said to have subjective binary marginal gains if $\Delta_i(S; o) \in \{0, \lambda_i\}$ for some agent-specific constant $\lambda_i > 0$, for every $i \in N$. We define clean bundles and clean allocations for this function class exactly as we did for matroid rank valuations in Section 2.

Understandably, most of the properties of allocations under matroid rank valuations do not extend to this more general setting. It is obvious that Pareto optimality does not imply utilitarian optimality (e.g. consider an instance with two agents and one item which the agents value at 1 and 2 respectively: assigning the item to agent 1 is PO but not utilitarian optimal). Moreover, the leximin allocation may not be EF1, as shown by the following example where both agents have additive valuations.

*Example 6.* Suppose $N = [2]$; $O = \{o_1, o_2, o_3, o_4\}$; the valuations are additive with $v_1(\{o_1\}) = 0$, $v_1(\{o\}) = 1\ \forall o \in O \setminus \{o_1\}$, and $v_2(\{o\}) = 3\ \forall o \in O$. It is straightforward to check that the unique leximin allocation is $A_1 = \{o_1, o_2, o_3\}$, $A_2 = \{o_4\}$. Under this allocation, $v_1(A_2) = 0 < 3 = v_1(A_1)$, but $v_2(A_1 \setminus \{o\}) = 6 > 3 = v_2(A_2)$ for every $o \in A_1$ — in fact, at least two (any two) items must be removed from $A_1$ for agent 2 to stop envying agent 1. ■

Note another difference of this valuation class from matroid rank valuations that is also evidenced by Example 6: the leximin and MNW allocations may not coincide. In this example, any allocation $A$ that gives two of the items $\{o_2, o_3, o_4\}$ to agent 1 and the rest to agent 2 is MNW, with $v_1(A_1) = 2$ and $v_2(A_2) = 6$, so that $\mathtt{NW}(A) = 12$; such an allocation is also EF1 (in fact, envy-free) since $v_1(A_2) = 1 < 2 = v_1(A_1)$ and $v_2(A_1) = 6 = v_2(A_2)$. This is not an accident, as the following theorem shows.

**Theorem 8.** *For agents having submodular valuation functions with subjective binary marginal gains, any clean, MNW allocation is EF1.*

Since our valuation functions are still submodular, the transferability property (Lemma 1) still holds. Two other components of the proof of Theorem 8 are natural extensions of Propositions 2 and Lemma 2 — Proposition 3 and Lemma 6 below, respectively:

**Proposition 3.** *For submodular valuations with subjective binary marginal gains defined by agent-specific positive constants $\lambda_i$ $\forall i \in N$, $A$ is a clean allocation if and only if $v_i(A_i) = \lambda_i |A_i|$ for each $i \in N$.*

*Proof.* Consider an arbitrary bundle $S \subseteq O$ such that $S = \{o_1, o_2, \ldots, o_r\}$ for some $r \in [m]$ w.l.o.g. Let $S_0 = \emptyset$ and $S_t = S_{t-1} \cup \{o_t\}$ for every $t \in [r]$. Then, an arbitrary agent $i$'s valuation of bundle $S$ under marginal gains in $\{0, \lambda_i\}$ is

$$v_i(S) = \sum_{t=1}^{r} \Delta_i(S_{t-1}; o_t) \leq \sum_{t=1}^{r} \lambda_i = \lambda_i r = \lambda_i |S|. \tag{6}$$

Now, if agent $i$'s allocated bundle under an allocation $A$ has a valuation $v_i(A_i) = \lambda_i |A_i|$, then her marginal gain for any item in $o \in A_i$ is given by

$$\begin{aligned} v_i(A_i) - v_i(A_i \setminus \{o\}) &= \lambda_i |A_i| - v_i(A_i \setminus \{o\}) \\ &\geq \lambda_i |A_i| - \lambda_i (|A_i| - 1) \\ &= \lambda_i > 0, \end{aligned}$$

where the first inequality follows from Inequality (6) and the fact that $|A_i \setminus \{o\}| = |A_i| - 1$. This means that the bundle $A_i$ is clean and, since this holds for every $i$, the allocation is clean. This completes the proof of the "if" part.

If allocation $A$ is clean, then we must have $\Delta_i(A_i \setminus \{o\}; o) > 0$ for every $o \in A_i$ for every $i \in N$. Let us define an arbitrary agent $i$'s bundle $A_i$ as $S$ above, so that $|A_i| = r$. Then, since $S_{t-1} \subseteq A_i \setminus \{o_t\}$ for every $t \in [r]$, submodularity dictates that

$$\Delta_i(S_{t-1}; o_t) \geq \Delta_i(A_i \setminus \{o_t\}; o_t) > 0 \quad \forall t \in [r].$$

Since $\Delta_i(S_{t-1}; o_t) \in \{0, \lambda_i\}$ with $\lambda_i > 0$, the above inequality implies that $\Delta_i(S_{t-1}; o_t) = \lambda_i$ $\forall t \in [r]$. Hence,

$$v_i(A_i) = \sum_{t=1}^{r} \Delta_i(S_{t-1}; o_t) = \sum_{t=1}^{r} \lambda_i = \lambda_i r = \lambda_i |A_i|.$$

This completes the proof of the "only if" part. $\square$

**Lemma 6.** *For submodular functions with subjective binary marginal gains, if agent $i$ envies agent $j$ up to more than 1 item under clean allocation $A$, then $|A_j| \geq |A_i| + 2$.*

*Proof.* Since $i$ envies $j$ under $A$ up to more than 1 item, we must have $A_j \neq \emptyset$ and $v_i(A_i) < v_i(A_j \setminus \{o\})$ for every $o \in A_j$. Consider one such $o$. From Inequality (6) in the proof of Proposition 3, $v_i(A_j \setminus \{o\}) \leq \lambda_i |A_j \setminus \{o\}| = \lambda_i(|A_j| - 1)$. Since $A$ is clean, $v_i(A_i) = \lambda_i |A_i|$. Combining these, we get $\lambda_i |A_i| = v_i(A_i) < v_i(A_j \setminus \{o\}) \leq \lambda_i(|A_j| - 1)$. Since $\lambda_i > 0$, we have $|A_i| < |A_j| - 1$, i.e. $|A_i| \leq |A_j| - 2$ because $|A_i|$ and $|A_j|$ are integers. □

We are now ready to prove Theorem 8.

*Proof (Proof of Theorem 8).* Our proof non-trivially extends that of Theorem 3.2 of Caragiannis et al. [14]. We will first address the case when it is possible to allocate items in such a way that each agent has a positive realized valuation for its bundle, i.e. $N_{\max} = N$ in the definition of an MNW allocation, and then tackle the scenario $N_{\max} \subsetneq N$.

Consider a pair of agents $1, 2 \in N$ w.l.o.g. such that 1 envies 2 up to two or more items, if possible, under an MNW allocation $A$. Since every agent has a positive realized valuation under $A$, we have $v_i(A_i) = \lambda_i |A_i| > 0$, i.e. $|A_i| > 0$ for each $i \in \{1, 2\}$. From Lemma 1, we know that there is an item in $A_2$ for which agent 1 has positive marginal utility – consider any one such item $o \in A_2$. Thus, $\Delta_1(A_1; o) > 0$, i.e. $\Delta_1(A_1; o) = \lambda_1$; also, since $A_2$ is a clean bundle, $\Delta_2(A_2 \setminus \{o\}; o) > 0$, i.e. $\Delta_2(A_2 \setminus \{o\}; o) = \lambda_2$.

Let us convert $A$ to a new allocation $A'$ by only transferring this item $o$ from agent 2 to agent 1. Hence, $v_1(A'_1) = v_1(A_1) + \Delta_1(A_1; o) = v_1(A_1) + \lambda_1$, $v_2(A'_2) = v_2(A_2) - \Delta_2(A_2 \setminus \{o\}; o) = v_2(A_2) - \lambda_2$, $v_i(A'_i) = v_i(A_i)$ for each $i \in N \setminus \{1, 2\}$. $\texttt{NW}(A)$ is positive since $A$ is MNW and $N_{\max} = N$. Hence,

$$\begin{aligned}
\frac{\texttt{NW}(A')}{\texttt{NW}(A)} &= \left[\frac{v_1(A_1) + \lambda_1}{v_1(A_1)}\right] \left[\frac{v_2(A_2) - \lambda_2}{v_2(A_2)}\right] \\
&= \left[1 + \frac{\lambda_1}{v_1(A_1)}\right] \left[1 - \frac{\lambda_2}{v_2(A_2))}\right] \\
&= \left[1 + \frac{\lambda_1}{\lambda_1 |A_1|}\right] \left[1 - \frac{\lambda_2}{\lambda_2 |A_2|}\right] \\
&= \left[1 + \frac{1}{|A_1|}\right] \left[1 - \frac{1}{|A_2|}\right] \\
&= 1 + \frac{|A_2| - |A_1| - 1}{|A_1||A_2|}, \\
&\geq 1 + \frac{(|A_1| + 2) - |A_1| - 1}{|A_1||A_2|}, \\
&\geq 1 + \frac{1}{|A_1||A_2|}, \\
&> 1.
\end{aligned}$$

Here, the third equality comes from Proposition 3 since $A$ is clean, and the first inequality from Lemma 6 due to our assumption. But $\texttt{NW}(A') > \texttt{NW}(A)$ contradicts the optimality of $A$, implying that any agent can envy another up to at most 1 item under $A$.

This completes the proof for the $N_{\max} = N$ case. The rest of the proof mirrors the corresponding part of the proof of Caragiannis et al. [14]'s Theorem 3.2. If $N_{\max} \subsetneq N$, it is easy to see that there can be no envy towards any $i \notin N_{\max}$: this is because we must have $v_i(A_i) = 0$ for any such $i$ from the definition of $N_{\max}$, which in turn implies that $A_i = \emptyset$ since $A$ is clean; hence, $v_j(A_i) = 0$ for every $j \in N$. Also, for any $i, j \in N_{\max}$, we can show exactly as in the proof for the $N_{\max} = N$ case above that there cannot be envy up to more than one item between them, since $A$ maximizes the Nash welfare over this subset of agents $N_{\max}$. Suppose for contradiction that an agent $i \in N \backslash N_{\max}$ envies some $j \in N_{\max}$ up to more than one item under $A$. Then, from Lemma 1, there is one item $o_1 \in A_j$ w.l.o.g. such that $v_i(\{o_1\}) = \Delta_i(\emptyset; o_1) = \Delta_i(A_i; o_1) > 0$. Moreover, since $A$ is clean,

$$\begin{aligned} v_j(A_j \setminus \{o_1\}) &= v_j(A_j) - \Delta_j(A_j \setminus \{o_1\}; o_1) \\ &= \lambda_j |A_j| - \lambda_j \\ &= \lambda_j (|A_j| - 1) \\ &\geq \lambda_j (|A_i| + 1) \\ &= \lambda_j > 0, \end{aligned}$$

where the first inequality comes from Lemma 6. Thus, if we transfer $o_1$ from $j$ to $i$ and leave all other bundles unchanged, then every agent in $N_{\max} \cup \{i\}$ will have a positive valuation under the new allocation. This contradicts the maximality of $N_{\max}$. Hence, any $i \in N \backslash N_{\max}$ must be envy-free up to one item towards any $j \in N_{\max}$. □

## D Appendix: General assignment valuations

In this section, we address the fair and efficient allocation of items to agents who have general assignment or OXS valuations, as defined in Section 2.2. Recall that an agent with such a valuation function is equivalent to a group with multiple members each having an arbitrary non-negative weight for each item. As such, we will henceforth use the terms "group" and "agent" interchangeably.

We know that, for arbitrary non-negative monotone valuations, the classic *envy graph algorithm* due to Lipton et al. [32] produces a complete, EF1 allocation that does not, however, come with any efficiency guarantee (except completeness, of course). The trick is to iterate over the items and allocate each to an agent that is currently not envied by any other agent (the existence of such an unenvied agent can be guaranteed by *de-cycling*, if necessary, the graph induced by a directed edge from every envious agent to every agent that it envies: see Lipton et al. [32] for details).

Benabbou et al. [8] focus on fair allocation to *types* that are, in fact, agents/groups with OXS valuations; they use a natural extension of this procedure that they denote by **H**. In an iteration of **H**, we do not give an arbitrary unallocated item to an arbitrary unenvied agent; instead, we find an item-agent pair having the maximum marginal utility among all currently unenvied agents and all unallocated items (breaking further ties uniformly at random, say), and allocate that item to that agent. Although this modification should, intuitively, improve efficiency, Benabbou et al. [8] provide no formal guarantee in this regard; they evaluate the performance of **H** in experiments where all agents have OXS valuations in terms of *waste* which they define as follows: under a complete allocation $A$, an item $o$ is said to be *wasted* if it has positive marginal utility for some group $h$ under $A$ (i.e. $v_h(A_h \cup \{o\}) > v_h(A_h)$) but is allocated to another group $h'$ (i.e. $o \in A_{h'}$) where it is either unassigned or assigned to a member $i \in N_{h'}$ with zero weight for it (i.e. $u_{i,o} = 0$), under the particular optimal matching of $A_{h'}$ to $N_{h'}$. The waste of a run of Algorithm **H** is defined as the percentage of the total number of items that are wasted under the complete allocation produced by **H**.

Here, we ask whether the concept of *envy-induced transfers* (`EIT`) presented in Algorithm 1 for matroid rank valuations (Section 3.1) can be used to compute fair and efficient allocations (perhaps in some approximate sense) under more general monotone submodular valuation functions. This is motivated in part by the fact that the transferability property (Lemma 1), on which the `EIT` concept relies, characterizes any monotone submodular function and not just matroid rank valuations. In Algorithm 2, we delineate our work in progress in this vein: a heuristic scheme that extends Algorithm 1 to general OXS valuations.

Algorithm 2 retains the general principle of starting with a(n arbitrary) clean, utilitarian optimal allocation[13] and iteratively eliminating envy by transferring an item from an envied bundle to an envious agent. For matroid rank valuations, the "donor" and the recipient of the transferred item have their valuations decreased and increased respectively by exactly 1 for any envy-induced transfer; this is no longer the case when we remove the binary marginal utilities restriction. Hence, such a transfer does not, in general, keep the utilitarian social welfare unchanged; the welfare is only constrained to never exceed its starting (optimal) value computed in line 1. As an approach to minimizing the loss in welfare/efficiency due to such transfers, we employ various heuristics in Algorithm 2:

[13] Maximizing the utilitarian social welfare is NP-hard when agents have general monotone submodular valuations but can be accomplished in polynomial time under the subclass gross substitutes valuations, assuming oracle access to each valuation function [30]. In particular, under OXS valuations (assuming that such a valuation function is specified in terms of the weights of each member of the group for all items), computing a utilitarian optimal allocation reduces to the polynomial-time solvable *assignment problem* or maximum sum-of-weights matching on a bipartite graph [33]; the result is automatically clean if we make sure that no item is assigned to an individual with zero weight for it.

**Algorithm 2:** Envy-Induced Transfers for general OXS valuations

1 Compute a clean, utilitarian optimal allocation.
2 **/*Envy-Induced Transfers (EIT)*/**
3 **while** $\exists i, j \in N$ *such that* $i$ *envies* $j$ *up to more than* 1 *item* **do**
4     Pick $i$, $j$, $o$ maximizing $\Delta_i(A_i; o) + \Delta(A_j \setminus \{o\}; o)$ over all $i, j \in N$ and all $o \in O$ such that $i$ envies $j$ more than 1 item and $\Delta_i(A_i; o) > 0$.
5     $A_j \leftarrow A_j \setminus \{o\}$; $A_i \leftarrow A_i \cup \{o\}$.
6     **if** $\exists o \in A_0$ *such that* $\Delta_j(A_j; o) > 0$ **then**
7         Pick $o \in A_0$ that maximizes $\Delta_j(A_j; o)$.
8         $A_j \leftarrow A_j \cup \{o\}$.
9     **end**
10     **if** $\exists o^* \in A_i$ *that is unused* **then**
11         $A_i \leftarrow A_i \setminus \{o^*\}$; revoked $=$ **true**.
12         **while** *revoked* $=$ ***true and*** $\exists k$ *s.t.* $\Delta(A_k; o^*) > 0$ **do**
13             Allocate $o^*$ to agent $k$ maximizing $\Delta(A_k; o^*)$.
14             **if** $\exists o \in A_k$ *that is unused* **then**
15                 $A_k \leftarrow A_k \setminus \{o\}$; $o^* \leftarrow o$.
16             **else** revoked $=$ **false**.
17         **end**
18         **if** *revoked* $=$ ***true*** **then** $A_0 \leftarrow A_0 \cup \{o^*\}$.
19     **end**
20 **end**

- First, in each `EIT` step, we transfer the item that induces the minimal decrease in — or, equivalently, the maximal increase — in the welfare (see lines 3-6).
- Next, as the donor agent loses one of its items, it may develop a positive marginal utility for a currently withheld item; in that case, the item in $A_0$ for which she has maximal marginal utility is given to it (see lines 6-8).
- Finally, if an agent (group) $i$ acquires a new item $o$ due to an envy-induced transfer, at most one of its previous items, say $o^*$, may become *unused*, i.e. it is no longer assigned to a member of the group under the new matching. This happens, for example, if $i$'s positive marginal utility for $o$ with its previous bundle $A_i$ was due to the fact that the member who was assigned item $o^*$ has a higher weight for $o$ than for $o^*$ and no other member prefers $o^*$ to its assigned item. In such a case, item $o^*$ is revoked from agent $i$ and allocated to the agent with maximal and strictly positive marginal utility for it (see lines 10-13). If this creates another unused item, we repeat the process until there are no unused items or the unused item has zero marginal utility for all agents – in the latter case, the unused item is added to the withheld set (see lines 14-18).

We do not yet have theoretical guarantees for Algorithm 2; but, if the `EIT` subroutine terminates, then the final allocation is EF1 and has zero waste (as defined above) by construction. To estimate the efficiency properties of our scheme, we ran numerical tests with it on a set of fair allocation instances based on a real-world data set.

In our experiments, we measure and compare the performances of Algorithm 2 and the procedure **H** as described above in terms of waste (as defined above) as well as the *price of fairness* (PoF) which we formally define as follows

$$PoF(P) = \frac{\max\{\mathtt{USW}(A) \mid A \text{ is an allocation}\}}{\mathtt{USW}(A(P))}$$

where $A(P)$ is the allocation returned by a given procedure $P$ (Algorithm 2 or **H**) on a problem instance. Obviously, PoF is bounded below by 1 for any instance and lower values are better.

The data set we use is *MovieLens-ml-1m* [26] which contains approximately 1,000,000 ratings (from 0 to 5) of 4,000 movies made by 6,000 users. To generate an instance of our allocation problem, we select 200 movies uniformly at random ($|O| = 200$) and then we only consider the users that rated at least one of these movies. Each such sample of 200 movies defines one run of our experiments. The users are our group-members and the movies our items. We generate agents/groups by partitioning users based on a demographic attribute; in fact, we use two attributes recorded in the data set, giving us two sets of allocation problem instances for each run:

- Gender: 2 agents (male or female, as recorded in the data set);
- Age: 7 agents representing the 7 age-groups recognized in the data set.

Moreover, for each such set (with 2 and 7 agents respectively), we adopt two models for the member-item weights or, equivalently, agents' valuation functions (raw and normalized ratings), giving us a $2 \times 2$ experimental design:

- Ratings:
$$v_h(S) = \max\left\{ \sum_{u \in N_h} r_{u,\pi(u)} \mid \pi \in \Pi(N_h, S) \right\}$$
  where $r_{uo}$ is the user $u$'s rating of movie $o$;
- Norm: $v'_h(S) = v_h(S)/v_h(O)$,

for every agent $h$ (group $N_h$) and for any bundle of movies $S \subseteq O$. We provide the results, averaged over 50 runs, in Table 3.

| | | **H** [32, 8] | | Algorithm 2 | |
|---|---|---|---|---|---|
| | Attribute (#groups) | Ratings | Norm | Ratings | Norm |
| PoF | Age (7) | 1.01 | 1.15 | 1.05 | 1.19 |
| Waste | | 1.25% | 0.20% | 0% | 0% |
| PoF | Gender (2) | 1.00 | 1.02 | 1.00 | 1.03 |
| Waste | | 0.00% | 0.00% | 0% | 0% |

**Table 3.** Experimental assessment of allocation procedures under OXS valuations.

We observe that **H** has no guarantees on waste but, in practice, has negligible waste; and the waste appears to be lower for a lower number of agents in our experiments. In comparison, Algorithm 2 (which always terminated on its own for all our instances) is waste-free by design but has at least as much average PoF as **H** in all our experiments.