Lecture 10: Expectation of Discrete and Continuous Random Variables

Announcements and reminders

Practicum 1 posted, due Monday 4 March at 11:59 PM.

→ Monday after your midterm. Plan ahead!

Midterm:

- Tuesday 26 February, 7-8:30 PM, HUMN 1B50
- Special accommodations: 6-? PM, HUMN 335
- Tell me as soon as possible about conflicts
  - Include documentation
- Review in class on Monday 25 March (Q&A)
- Study from: old exams, lecture notes, homework, practicum, textbooks…

Previously, on CSCI 3022…

Definition: A **probability mass function** (pmf) is the map between the random variable's values and the probabilities of those values.

$$f(a) = P(X=a)$$

Definition: A random variable X is **continuous** if for some function $f : \mathbb{R} \to \mathbb{R}$ and for any numbers a and b with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

The **probability density function** (pdf) f must satisfy:

1) $f(x) \geq 0$ for all x, and

2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Homework planning

S'pose ***hypothetically*** that I write the homework questions as either: easy (takes 10 minutes), medium (60 mins), or hard (120 mins).

The probability that each question is easy/medium/hard is: 0.2, 0.3, 0.5, respectively

Question: If a homework consists of 5 questions, what's the average time it takes to do the homework?

Homework planning

S'pose ***hypothetically*** that I write the homework questions as either: easy (takes 10 minutes), medium (60 mins), or hard (120 mins).

The probability that each question is easy/medium/hard is: 0.2, 0.3, 0.5, respectively

Question: If a homework consists of 5 questions, what's the average time it takes to do the homework?

Answer: a *weighted* average

Guess = $0.2 \cdot 10 + 0.3 \cdot 60 + 0.5 \cdot 120 = 80$

(probably a little low?)

Expected value: discrete random variables

Definition: The **expectation** or **expected value** of a discrete random variable X that takes the values a_1, a_2, … and with pmf p is given by

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

Expected value: discrete random variables

Definition: The **expectation** or **expected value** of a discrete random variable *X* that takes the values a_1, a_2, … and with pmf *p* is given by

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

Intuition: Think of masses of weight $p(a_i)$ placed at the points a_i

→ E[X] is the balancing point

Expected value: discrete random variables

Example: Let X be a Bernoulli random variable with parameter *p*. What is E[X]?

Expected value: discrete random variables

Example: S'pose you and a friend are avoiding studying by each rolling a fair die. You decide that the first time that you roll the same number, you'll go back to work.

What is the expected number of times you'll roll the dice before getting a match?

Expected value: continuous random variables

Example: What if our boxes of mass $p(a_i)$ get smaller and smaller, and we have more and more of them? How does the center of gravity change from the discrete case to continuous?

Discrete:

Continuous:

Expected value: continuous random variables

Example: What if our boxes of mass $p(a_i)$ get smaller and smaller, and we have more and more of them? How does the center of gravity change from the discrete case to continuous?

Discrete: $$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

Continuous: $$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Definition: The **expectation**, **expected value**, or **mean**, of a continuous random variable X with probability density function *f* is

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Expected value: continuous random variables

Example: What if our boxes of mass $p(a_i)$ get smaller and smaller, and we have more and more of them? How does the center of gravity change from the discrete case to continuous?

Discrete: $$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

Continuous: $$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Definition: The **expectation**, **expected value**, or **mean**, of a continuous random variable X with probability density function *f* is

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Intuition: Think of a single big rock balancing on a fulcrum.

Expected value: continuous random variables

Example: The lifetime (in years) of a certain brand of battery is Exponentially distributed with parameter $\lambda = 0.25$.

1) How long, on average, will this battery last?
2) What are the units of λ?

Expected value: continuous random variables

Example: The lifetime (in years) of a certain brand of battery is Exponentially distributed with parameter λ = 0.25.

1) How long, on average, will this battery last?
2) What are the units of λ?

hits/times

$$\boxed{E[T] = \frac{1}{\lambda}}$$ ← for any r.v. $T \sim Exp(\lambda)$

$0.25 \frac{\text{deaths}}{\text{years}}$

$E[T] = \frac{1}{.25} = 4$ years

deaths/year

Expected value: continuous random variables

Example: The lifetime (in years) of a certain brand of battery is Exponentially distributed with parameter λ = 0.25.

1) How long, on average, will this battery last?
2) What are the units of λ?

X ~ Exp(λ) → $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ → $E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx$

Now, **integration by parts**: $\int_a^b u \; dv = uv\Big|_a^b - \int_a^b v \; du$

with u = λx and dv = $e^{-\lambda x} dx$

→ du = λ dx and v = $-\frac{1}{\lambda} e^{-\lambda x}$

Expected value: continuous random variables

Example: The lifetime (in years) of a certain brand of battery is Exponentially distributed with parameter λ = 0.25.

1) How long, on average, will this battery last?
2) What are the units of λ?

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} x\lambda e^{-\lambda x} dx$$
$$= (\lambda x)\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\Big|_0^{\infty} - \int_0^{\infty}\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\lambda dx$$
$$= xe^{-\lambda x}\Big|_{\infty}^{0} + \int_0^{\infty} e^{-\lambda x} dx$$
$$= \underbrace{0}_{\text{L'Hopital's Rule}} + -\frac{1}{\lambda}e^{-\lambda x}\Big|_0^{\infty}$$
$$= \frac{1}{\lambda} - 0 = \boxed{\frac{1}{\lambda}}$$

→ So we expect the battery to last 1/λ = **4 years**

Expected value: continuous random variables

Example: S'pose you have observed that, on average, 300 cars cross a particular bridge every hour. How much time do you expect to wait between two cars crossing the bridge?

Poisson process (r.v.): $\lambda = 300 \dfrac{\text{car (arrival)}}{\text{hours}}$

exponential distr. for time b/w car arrivals $\sim \text{Exp}(\lambda)$

$\rightarrow E[T] = 1/\lambda = \dfrac{1}{300}$ hours

Expected value: continuous random variables

Example: S’pose you have observed that, on average, 300 cars cross a particular bridge every hour. How much time do you expect to wait between two cars crossing the bridge?

Answer:

Counts/arrivals/hits: $X \sim \text{Pois}(\lambda = 300 \text{ hour}^{-1})$

→ Inter-arrival times: $T \sim \text{Exp}(\lambda = 300 \text{ hour}^{-1})$ and $E[T] = 1/\lambda = 1/300$ hours

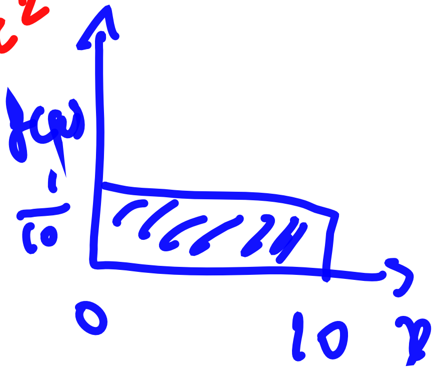
(= 12 seconds)

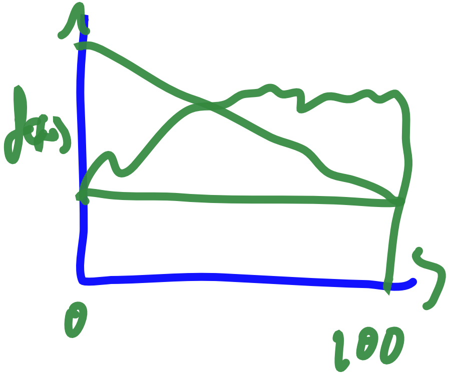
Expectation of functions of random variables

Often, we want to compute the expectation of a function of a random variable, instead of the random variable itself. For example, we might want to compute $E[X^2]$ instead of $E[X]$.

Example: S'pose an architect is designing a community and wants to maximize the diversity in the size of his square buildings that are of both width and depth X, but X is uniformly distributed by 0 and 10 meters. What is the distribution of the area X^2 of the building?

$Y = x^2$





$$E[X] = \int_0^{10} x f(x)\,dx$$

$$E[Y] = E[X^2] = \int_0^{10} x^2 f(x)\,dx$$

$y = x^2$

$$= \int_0^{10} x^2 \frac{1}{10}\,dx = \frac{1}{30} x^3 \Big|_0^{10}$$

$$= \frac{1000}{30}$$

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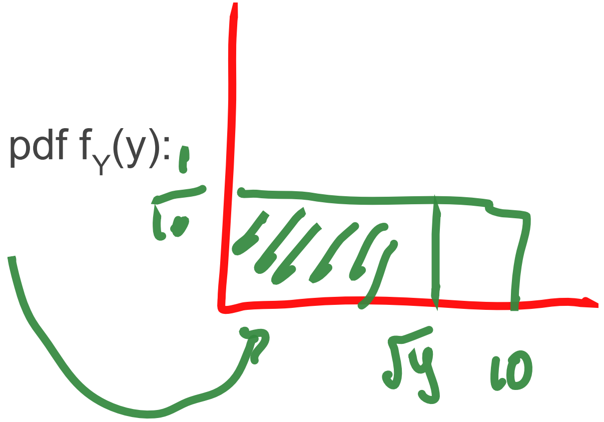
$X \sim U(0, 10)$, so we have $0 \le x \le 10$
$Y = X^2 =$ area of building, so we have $0 \le y \le 100$

Calculate the cdf of Y, then we'll differentiate to get the pdf $f_Y(y)$:

$\rightarrow F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = \dfrac{\sqrt{y}}{10}$

$$\rightarrow f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(\frac{\sqrt{y}}{10}\right) = \frac{1}{20\sqrt{y}}$$



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$$
\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_0^{10} x^2 \frac{1}{10} dx \\
&= \frac{1}{30} x^3 \Big|_0^{10} \\
&= \frac{100}{3} \approx 33.3 \text{ m}^2
\end{aligned}
$$

Expectation of functions of random variables

Often, we want to compute the expectation of a function of a random variable, instead of the random variable itself. For example, we might want to compute E[X^2] instead of E[X].

Change-of-variables formula: Let X be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a function.

If X is discrete and takes the values a_1, a_2, … then

$$E[g(x)] = \sum_i g(a_i) P(X = a_i)$$

$$E[X] = \sum_i a_i P(a_i)$$

$$E[g(x)] = \sum g(a_i) P(a_i)$$

If X is continuous, with probability density function *f*, then

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Linearity of expectation

Fun (and very useful) fact: Expectation is a **linear function.**

$$E[\underbrace{aX + b}_{\text{Linear tranformation}}] = aE[X] + b$$

$$E[ax] + E[b]$$

$$= aE[x] + b$$

$$E[ax+b] = \int_{-\infty}^{\infty} [ax+b] \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} [ax f(x) + b f(x)]\, dx$$

$$= \int_{-\infty}^{\infty}$$

Linearity of expectation

$$\underbrace{b\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

Fun (and very useful) fact: Expectation is a **linear function.**

$$E[aX + b] = aE[X] + b$$

Proof: almost as fun as the fact!

We'll do the continuous case, but **FYOG** do the discrete one!

$$\begin{aligned} E[aX+b] &= \int_{-\infty}^{\infty} (ax+b)f(x)dx \\ &= \int_{-\infty}^{\infty} axf(x)dx + \int_{-\infty}^{\infty} bf(x)dx \\ &= a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} f(x)dx \\ &= aE[X] + b(1) \\ &= aE[X] + b \quad \checkmark \end{aligned}$$

What just happened?

- We learned about **expected values**...

- … and their relationship to the pmf/pdf:

    - **Continuous RVs**: ____________________________

    - **Discrete RVs**: ____________________________

Next time:

- Great **variances**!