Notes on differential geometry

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Prerequisites: manifolds, metric spaces, topology.
Literature: [1, 2, 3, 4, 5]. See also my (Russian) notes on undergraduate math course.

0.1 Tensor field transmission

Here I follow pp.[VI]31-49 and [VII]14-15 from my math lecture notes. Consider two manifolds: X and Y: $\dim X = m$, $\dim Y = n$. Let us imagine that we defined covariant tensor field $S \in \mathbb{T}^{(p,0)}Y$.

We can *transfer* the field S from Y to X using some smooth function $f : X \to Y$ according to the following rule:

$$\begin{gathered} f : X \to Y \Rightarrow \forall p, q \; df : \mathbb{T}^{(p,q)}X \to \mathbb{T}^{(p,q)}Y, \\ M \in X, \quad f^*S = \tilde{S} \in \mathbb{T}^{(p,0)}X : \quad \tilde{S}_M\left(\vec{T}_1 \ldots \vec{T}_q\right) \stackrel{\mathrm{df}}{=} S_{f(M)}\left[df_M\left(\vec{T}_1\right) \ldots df_M\left(\vec{T}_q\right)\right]. \end{gathered} \tag{1}$$

For example, consider a situation when $X \equiv S^2 \in \mathbb{R}^3$ is a 2D-sphere, and $f = i : S^2 \to \mathbb{R}^3$ is an insertion function:

$$\begin{gathered} M \equiv (\theta, \varphi) \in S^2, \quad f(M) \equiv i(\theta, \varphi) = \vec{r}(\theta, \varphi) = \left[x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi)\right], \\ x = \sin\theta\cos\varphi, \; y = \sin\theta\sin\varphi, \; z = \cos\theta, \quad 0 \le \theta \le \pi, \quad -\pi \le \varphi < \pi. \end{gathered} \tag{2}$$

The metric tensor from $\mathbb{R}^3$ induces [through the transmission mechanism (1)] metric tensor on S^2:

$$p = 2, \; g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^{\mathrm{j}} \in \mathbb{T}^{(2,0)}, \quad g \in \mathbb{T}^{(2,0)}\mathbb{R}^3, \; i^*g \equiv \tilde{g} \in \mathbb{T}^{(2,0)}S^2. \tag{3}$$

The differential df is determined by the Jacobi matrix of f:

$$df = \left(\frac{\partial y_i}{\partial x_j}\right) dx^j, \quad dx^j \frac{\partial}{\partial x^i} = \delta_i^j, \tag{4}$$

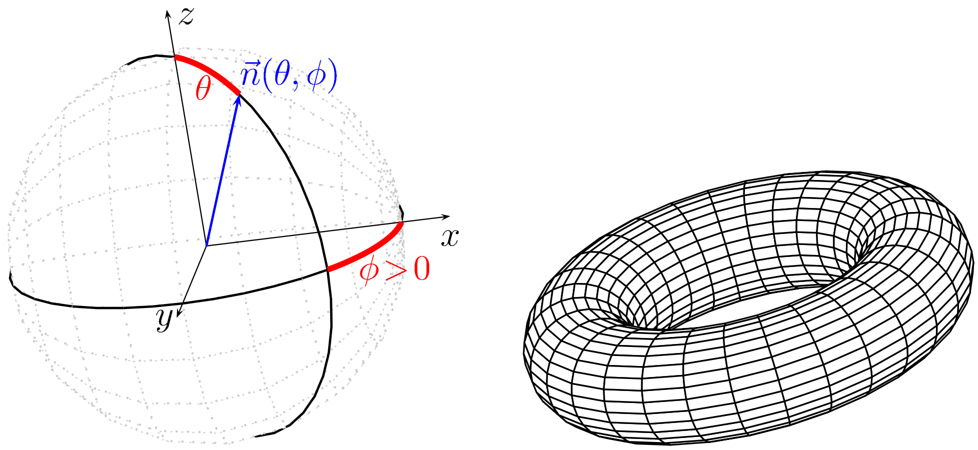
and we obtain the induced metric tensor $\tilde{g}$ on a sphere, see (8) below.

0.2 Connection = covariant derivativation

A usual derivative operator applied to a tensor field on a manifold does not generally produce a tensor field. One has to use the s.c. *covariant derivativation* operator instead, which always produces a tensor field out of any tensor field; The most general definition is written on p. [2]259. We will study covariant derivative of a vector and a $\mathbb{T}^{(2,0)}$ (metric) tensor:

$$T \in \mathbb{T}^{(0,1)}, \quad \nabla_l T^k = \frac{\partial T^k}{\partial x^l} + \Gamma^k_{sl}T^{\mathrm{s}}, \quad g \in \mathbb{T}^{(2,0)}, \quad \nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^l_{ik}g_{lj} - \Gamma^l_{jk}g_{il}. \tag{5}$$

From (5) we see that covariant derivativation operator is determined by the s.c. Christoffel symbol Γ. There is an obvious similarity between Cristoffel symbol Γ and vector potential [also non-abelian one due to spin-orbit interaction (SOI)]. Someone should have already exploited it to build geometrical interpretation of magnetic field and/or of the SOI.



(a) Coordinates (7) on a sphere.   (b) Torus – a 2D periodic manifold.

Figure 1: A sphere and a torus. Note that momentum space of a 2D electron gas with periodic boundary conditions is a torus.

Definition 1 *Connection is called* ***symmetrical*** *if there exists*[1] *a coordinate system (i.e., a map) where* $\Gamma^l_{jk} = \Gamma^l_{kj}$.

Definition 2 *Suppose metric tensor is covariantly constant:* $\nabla_k g_{ij} = 0$. *Then the corresponding connection is called* ***pseudoriemannian.***

Theorem 1 *On any pseudoriemannian manifold* $\exists!$ *pseudoriemannian connection:*

$$\Gamma^k_{sl} = g^{ki}\Gamma_{i,sl}, \quad \Gamma_{i,sl} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^s} + \frac{\partial g_{si}}{\partial x^l} - \frac{\partial g_{ls}}{\partial x^i}\right). \tag{6}$$

0.3 Sphere S^2

From the Rank Theorem we realize that an arbitrary map on the manifold $\mathbb{R}^3$ induces a map (7) on a two-dimensional sub-manifold $S^2 = \{x, y, z | x^2 + y^2 + z^2 = 1\}$.

We parametrize a sphere (or, roughly speaking,[2] define a map):

$$S^2 = \{\vec{r} \equiv (x, y, z) \in \mathbb{R}^3 | x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta\}. \tag{7}$$

This parametrization defines a tangent 2D space $\mathbb{T}sS \equiv \mathbb{R}^2 \subset \mathbb{R}^3$ in every point of $s \in S$. Parallels (circular lines for which $\theta = \text{const}$) are perpendicular (in the standard $\mathbb{R}^3$ metric) to meridians (circular lines for which $\varphi = \text{const}$):

$$\frac{\partial\vec{r}}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \quad \frac{\partial\vec{r}}{\partial\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0), \quad \frac{\partial\vec{r}}{\partial\theta}\perp\frac{\partial\vec{r}}{\partial\varphi},$$

$$\text{so we define} \quad \frac{\partial\vec{r}}{\partial\theta} \overset{\text{df}}{=} \text{id}\frac{\partial}{\partial\theta} \quad \text{and} \quad \frac{\partial\vec{r}}{\partial\varphi} \overset{\text{df}}{=} \text{id}\frac{\partial}{\partial\varphi}, \quad \text{so that } \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} \in \mathbb{T}_{\theta,\varphi}S \text{ and } \frac{\partial}{\partial\theta}\perp\frac{\partial}{\partial\varphi},$$

where id is the insertion operator.[3] Then we see that the corresponding 2D metric tensor (tensor field defined on a sphere) is φ-independent:

$$\tilde{g}(\varphi,\theta) = \begin{pmatrix} \frac{\partial\vec{r}}{\partial\theta}\cdot\frac{\partial\vec{r}}{\partial\theta} & \frac{\partial\vec{r}}{\partial\varphi}\cdot\frac{\partial\vec{r}}{\partial\theta} \\ \frac{\partial\vec{r}}{\partial\theta}\cdot\frac{\partial\vec{r}}{\partial\varphi} & \frac{\partial\vec{r}}{\partial\varphi}\cdot\frac{\partial\vec{r}}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} = g(\theta) \equiv \{g_{ij}\}^2_{i,j=1} \in \mathbb{T}^{(2,0)}S^2. \tag{8}$$

[1]Since Γ is not a tensor, the statement $\Gamma^l_{jk} = \Gamma^l_{kj}$ is map-dependent.

[2]In reality one can not cover a sphere with a single map, see the end of this section.

[3]$\forall\xi \in S \subset \mathbb{R}^3$ the insertion operator id inserts the *two-component* object $\xi = (\theta, \varphi)$ into $\mathbb{R}^3$ according to (7): $\text{id}\xi \in \mathbb{R}^3$.

and the corresponding pseudo-Riemannian connection[4] is defined by

$$\Gamma^k_{sl} = g^{kk}\Gamma_{k,sl}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta, \quad \Gamma^1_{22} = -\frac{\sin 2\theta}{2}, \tag{9}$$

and all other components of Γ are zero. The choice (9) is the only one that conserves scalar product during *parallel vector transport*.

As I have mentioned before, we can not describe the whole sphere with one map (7): the lines (in the parameter space) $\theta = 0$ and $\theta = \pi$ (which correspond to north and south poles of the sphere) are problematic so that the map (7) is valid only for the *open* parameter set $0 < \theta < \pi$ and $0 < \varphi < 2\pi$. In order to cover the poles and the "Greenwich meridian" we have to introduce another map, e.g.,

$$\begin{gathered} S = \{R_y^{\pi/2} R_z^{\pi}\vec{r} \in \mathbb{R}^3 | x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta\}, \\ \frac{\partial\vec{r}}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \quad \frac{\partial\vec{r}}{\partial\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0), \end{gathered} \tag{10}$$

where $R_y^{\pi/2}$ and R_z^{π} are the usual rotation matrices in $\mathbb{R}^3$. The expression for the metric tensor and for the connection will, of course, be different in coordinates (10).

1 Parallel transport on a sphere

A 3D vector $\vec{r} = (1, 0, 0) \in \mathbb{R}^3$ corresponds to $T = (1, 0) \in \mathbb{T}^{(0,1)}_{\theta=0} S^2$. I am going to apply parallel shift to it using the connection (9). First I will move this vector from the "north pole" of the sphere to its "equator" along the path $\varphi = 0$:

$$\begin{gathered} \theta(t) = \frac{\pi t}{2}, \quad x(t) = \sin[\theta(t)], \quad y(t) \equiv 0, \quad z(t) = \cos[\theta(t)], \quad 0 \le t \le 1, \quad \vec{v}_{\rm 3D}(t) = \frac{\partial\vec{r}}{\partial\theta}\Big|_{\varphi=0}\dot{\theta}(t), \\ \vec{v}_{\rm 2D} \equiv v = (v_\theta, v_\varphi), \quad v_\theta = \vec{v}_{\rm 3D}\cdot\frac{\partial\vec{r}}{\partial\theta} = \dot{\theta}(t) = \frac{\pi}{2}, \quad v_\varphi = \vec{v}_{\rm 3D}\cdot\frac{\partial\vec{r}}{\partial\varphi}\Big|_{\varphi=0} = 0. \end{gathered}$$

Parallel transport of a vector along the path is defined according to first-order differential equations:

$$\frac{\partial T^k}{\partial t} + v^l\Gamma^k_{sl}T^s = 0,$$

which have the only solution when the initial condition $\vec{T}\Big|_{t=0} = (1, 0)$ is set: $\vec{T}$ remains unchanged. The next step is to move the vector to the West:

$$\begin{gathered} \theta(t) = \frac{\pi}{2}, \quad \varphi(t) = \frac{\pi t}{2}, \quad x(t) = \cos[\varphi(t)], \quad y(t) \equiv \sin[\varphi(t)], \quad z(t) = 0, \quad 0 \le t \le 1, \\ \vec{v}_{\rm 3D}(t) = \frac{\partial\vec{r}}{\partial\varphi}\Big|_{\substack{\theta=\pi/2 \\ \varphi=\varphi(t)}}\dot{\varphi}(t), \quad \vec{v}_{\rm 2D} \equiv v = (v_\theta, v_\varphi), \quad v_\theta = 0, \quad v_\varphi = \vec{v}_{\rm 3D}\cdot\frac{\partial\vec{r}}{\partial\varphi} = \frac{\pi}{2}. \end{gathered} \tag{11}$$

Again we see that the vector is unchanged. Now let us make a final move – back to the North pole along the curve $\varphi = \pi/2$:

$$\begin{gathered} \theta(t) = \frac{\pi(1-t)}{2}, \quad x(t) = 0, \quad y(t) \equiv \sin[\theta(t)], \quad z(t) = \cos[\theta(t)], \quad 0 \le t \le 1, \\ \vec{v}_{\rm 3D}(t) = \frac{\partial\vec{r}}{\partial\theta}\Big|_{\varphi=0}\dot{\theta}(t) = \frac{\pi}{2}(0, 0, \sin\theta), \quad v_\theta = \vec{v}_{\rm 3D}\cdot\frac{\partial\vec{r}}{\partial\theta} = \dot{\theta}(t) = -\frac{\pi}{2}, \quad v_\varphi = 0. \end{gathered}$$

[4] The choice of the Christoffel symbol Γ^k_{sl} defines the connection $\vec{\nabla}$ which, in its turn, defines the way of how tangent space at some point of the considered manifold is *connected* to the tangent space at another point of the same manifold.

We see that our 2D vector is unchanged, and this is connected with the fact that we can not cover a sphere with a single map. In particular, our map (7) is ill-defined at the "North pole", and this is manifested in that, e.g., our metrical tensor becomes degenerate at this point.[5]

But what happens if we avoid the North pole, namely, use the following path:

1. move the vector $(1, 0)$ from the point $(\theta, \varphi) = (\pi/4, 0)$ to the point $(\pi/2, 0)$, then
2. move the (unchanged during previous move) vector $(1, 0)$ to the point $(\pi/2, \pi/2)$, then
3. move the (unchanged during previous move) vector $(1, 0)$ to the point $(\pi/4, \pi/2)$, and, finally,
4. move the (unchanged during all previous steps) vector $(1, 0)$ to the (initial) point $(\pi/4, 0)$.

During the last move (to the East) the vector is *also unchanged*:

$$
\begin{gathered}
\theta(t) = \frac{\pi}{4}, \quad \varphi(t) = \frac{\pi(1-t)}{2}, \quad x(t) = \frac{\cos[\varphi(t)]}{\sqrt{2}}, \quad y(t) \equiv \frac{\sin[\varphi(t)]}{\sqrt{2}}, \quad z(t) = \frac{1}{\sqrt{2}}, \\
\vec{v}_{3\mathrm{D}}(t) = \frac{\partial \vec{r}}{\partial \varphi}\bigg|_{\substack{\theta=\pi/4 \\ \varphi=\varphi(t)}} \dot{\varphi}(t), \quad \vec{v}_{2\mathrm{D}} \equiv v = (v_\theta, v_\varphi), \quad v_\theta = 0, \quad v_\varphi = \vec{v}_{3\mathrm{D}} \cdot \frac{\partial \vec{r}}{\partial \varphi} = \frac{\pi}{2}.
\end{gathered}
\tag{12}
$$

So the claim (citation needed) that the angle shift is equal to the solid angle of the closed path is wrong.

Let me see what happens if I shift a tangential vector *not along the geodesic lines,* but along the circle (drawn on a sphere):

$$
\begin{gathered}
0 \le t \le 1, \quad \begin{pmatrix} \theta(t) \\ \varphi(t) \end{pmatrix} = \frac{\pi}{2}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + r_0 \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}\right], \quad \begin{pmatrix} v_\theta(t) \\ v_\varphi(t) \end{pmatrix} = \frac{\pi^2 r_0}{2}\begin{pmatrix} -\sin 2\pi t \\ \cos 2\pi t \end{pmatrix}, \\
\dot{T}^\theta + T^\varphi \frac{\pi^2 r_0}{2}\Gamma^1_{22}\cos 2\pi t = 0, \quad \dot{T}^\varphi + \frac{\pi^2 r_0}{2}\Gamma^1_{12}\left[T^\theta \cos 2\pi t - T^\varphi \sin 2\pi t\right] = 0,
\end{gathered}
\tag{13}
$$

where Γ^1_{22} and $\Gamma^1_{12} = \Gamma^1_{21}$ are derived in (9).

$$
\dot{T}^\theta = \frac{\pi^2 r_0}{4} T^\varphi \cdot \sin\left[2\theta(t)\right] \cdot \cos 2\pi t, \quad \dot{T}^\varphi = -\frac{\pi^2 r_0}{2} \cdot \cot\left[\theta(t)\right] \cdot \left[T^\theta \cos 2\pi t - T^\varphi \sin 2\pi t\right]. \tag{14}
$$

I solve (14) numerically with the initial condition $T = (T^\theta, T^\varphi) = (1, 0)$ at $t = 0$. The norm of the vector [calculated using my non-Euclidean metric tensor (8)] is conserved during its entire voyage, but the direction is changed. I see that two vectors $T(t = 0)$ and $T(t = 1)$ are different. The arc-cosine of their scalar product is plotted in Fig. 2 for different values of the "radius" r_0 in (13). It is clear that when r_0 is very small, the zone occupied by the rotation contour is almost flat (Euclidean), so the parallel vector shift is almost path-independent; this is why the phase shift disappears for $r_0 \to 0$. As I've mentioned before, the "North pole" is a singular point[6] [see, e.g., (9)] so one should not overtrust Fig. 2 at $r_0 \to 1$.

2 Fiber bundles

2.1 Definition

A 2D plane can be imagined as a direct set product of two lines: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Seemingly, a torus [see Fig. 1(b)] is a direct product of two circles, and a cylinder is given by the direct product of circle and a line.

[5]This is connected with the fact that when you stand at the North pole, you always look to the south, no matter where you turn. In other words, all tangential vectors have only one non-zero component at $\theta = 0$.

[6]Note that also at $r_0 = 1$ the path defined in (13) is not a circle; it contains the "North pole" but it is not geodesic.

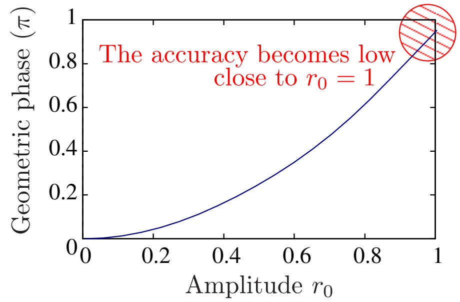


Figure 2: The geometric phase (angle difference) gained by a vector during parallel shift along the *non-geodesic* curve depends on the "radius" r_0 in (13).

A fiber bundle is a generalization of direct product of two sets: namely, a fiber bundle E is a manifold which *locally* can be represented as a direct product of the so-called base M and fiber F. I follow the definition from the Bible [2] (see the 6th Chapter).

Definition 3 *A fiber bundle is a compound object composed of*

1. *manifold E called* bundle,
2. *another manifold M called* fiber base,
3. *smooth projector $p : E \to M$, $p \circ p = p$.*
4. *third manifold F called fiber space,*
5. *group G containing operators which transform F; it is called a* structure group, *and, finally,*
6. *bundle structure: the base M is covered with maps $\{U_\alpha, \varphi_\alpha\}_\alpha$ so that $\varphi_\alpha : F \times U_\alpha \to p^{-1}(U_\alpha) \subset E$ and a "coordinate change" $\varphi_\alpha \to \varphi_\beta$ corresponds to the action of some operator $g_{\alpha\beta} \in G$: $\forall m \in U_\alpha \cap U_\beta$ $\varphi_\alpha(f, m) = \varphi_\beta(g_{\alpha\beta} f, m)$.*

A fiber bundle where $G = F$ (the action of a group onto another group element is just a multiplication) are called ***principal****.*

Simple direct products discussed above (plane, torus, and cylinder) are trivial (and thus non-interesting) fiber bundles. In each of these examples it is enough to define only one map which contains the entire base M so that the group G contains only one (unity) element.

2.2 Möbius stripe

Perhaps the simplest example of a non-trivial fiber bundle is a Möbius stripe: locally it is a direct product of a line and a circle (just like a cylinder), but globally it is not. One has to consider at least two maps [(U_1, φ_1) and (U_2, φ_2) – see (15)] for a proper description of a Möbius stripe [I use cylindrical coordinates (r, α, z)]:

$$\begin{gathered} U_1 = 3(-\pi, \pi)/4, \quad U_2 = [-\pi, -\pi/4) \cup (\pi/4, \pi), \quad M \equiv S_1 = \{r = r_0, \alpha \in [-\pi, \pi)\} = U_1 \cup U_2, \\ \varphi_1(z, \alpha) = \left(r_0 + \frac{w_0 z}{2}\sqrt{1 - s_\alpha^2}, \alpha, \frac{w_0 z s_\alpha}{2}\right), \quad \varphi_2(z, \alpha) = \left(r_0 + \frac{w_0 z}{2}\sqrt{1 - s_\alpha^2}, \alpha, \frac{w_0 z |s_\alpha|}{2}\right), \\ s_\alpha \stackrel{\text{df}}{=} \tanh\left[\tan\frac{\alpha}{2}\right], \quad z \in F = (-1, 1), \quad G = \mathbb{Z}_2 \equiv \{1, -1\}, \quad \varphi_1, \varphi_2 \in \mathbb{C}^{(1)}, \end{gathered} \tag{15}$$

where S_1 denotes a 1D-sphere (that is, a circle) in polar coordinates (r, α); w_0 and r_0 are the width and the radius of the stripe. We see that $\varphi_1 : F \times U_1 \to \mathbb{R}^3$ and $\varphi_2 : F \times U_2 \to \mathbb{R}^3$. Strictly speaking, Eq. (15) does not fulfill all the requirements of the fiber bundle definition, and this happens because $U_1 \cap U_2$ consists of two disconnected open sets (connection[7] components) which correspond to different group elements.

This defect is easily healed, though: I can split any of two maps $[(U_1, \varphi_1)$ or $(U_2, \varphi_2)]$ in two trivial ones, say, instead of (U_2, φ_2) consider two intersecting maps: (U_{2a}, φ_{2a}) and (U_{2b}, φ_{2b}):

$$\begin{aligned} U_{2a} \cap U_{2b} \neq \emptyset, \quad U_2 = U_{2a} \cup U_{2b}, \quad \varphi_{2a} \equiv \varphi_{2b} \equiv \varphi_2, \\ \forall \alpha \in U_1 \cap U_{2a} \ \varphi_{2a}(z, \alpha) = \varphi_1(z, \alpha), \quad \forall \alpha \in U_1 \cap U_{2b} \ \varphi_{2b}(z, \alpha) = \varphi_1(-z, \alpha) \\ \Rightarrow G = \{g_{2a,1} = 1, g_{2b,1} = -1\} = \mathbb{Z}_2. \end{aligned} \tag{16}$$

Note that $F \neq G$ in (15), so that torus *is not* a principal fiber bundle.

2.3 Horizontal and vertical tangential spaces

See p. [2]601:

$$\forall x \in M \quad \exists! F_x = p^{-1}(x) \subset E \quad \text{and } E = \cup_{x \in M} F_x, \tag{17}$$

where one should *not confuse* F_x with F (which are totally unrelated). The newly introduced object F_x is called a *layer*. A layer helps me to define horizontal H_z and vertical V_z subspaces of any tangential space $\mathbb{T}_z E$:

$$\forall z \in E \quad \mathbb{T}_z E = H_z \oplus V_z, \quad H_z = \mathbb{T}_z E \cap \mathbb{T}_{p(z)} M. \tag{18}$$

A line (or path) in E is called "horizontal" if all its velocities are horizontal (i.e., entirely lie in horizontal tangential subspaces). See p. [2]601-602: when the fiber space F is compact[8], for an arbitrary path γ in the base M a "covering" horizontal path $\tilde{\gamma}$ in E (such that $\tilde{\gamma} = \gamma$) is uniquely determined by the starting point $z_0 \in E$ (of course, $p(z_0) \in \gamma \subset M$).

3 Berry phase

In this section I mostly follow Ref. [3]. I disregard spin. I will use different from [3] notations because the ones from Ref. [3] are misleading.

Our fiber bundle $E = \{\psi | \psi : \mathbb{R}^{3N} \to \mathbb{C}\}$ is a space of all *normalized* complex-valued functions with integrable square. Let us consider a Schrödinger equation (SE) with some Hamiltonian, which may include interactions between the particles (electrons). Let N be the total number of particles in the considered quantum system. The space E is large enough to contain all possible solutions of SE with any Hamiltonian[9] and arbitrary energy.

If I apply a gauge transformation to any solution of an SE, it will still solve the SE with the *same* energy. This means that $G = U(1) \equiv F = \{e^{i\theta} | -\pi \leq \theta < \pi\}$. Extended explanation: every element of M belongs to infinitely many maps $(U_\theta, \varphi_\theta)$, where coordinate functions are defined as products $\varphi_\theta(e^{i\theta'}, \Phi) \stackrel{\text{df}}{=} e^{i(\theta+\theta')}\Phi$ and we assumed that the group $G \equiv F$ is Abelian (which is not true in case when we have SOI). The coordinate change is clear on the example of two maps, $(U_{\theta_1}, \varphi_{\theta_1})$ and $(U_{\theta_2}, \varphi_{\theta_2})$:

$$\forall m \in U_{\theta_1} \cap U_{\theta_2}, \varphi_{\theta_1}(f, m) = \varphi_{\theta_2}(g_{\theta_1\theta_2} f, m), \quad g_{\theta_1\theta_2} = e^{i(\theta_1 - \theta_2)} \in G \equiv F. \tag{19}$$

[7]Here the word "connection" is used in its topological meaning – a mathematical concept of a "entire piece" (a topological subspace is called "connected" if it can not be separated in two non-empty open sets). This should not be confused with the "connection" used in differential geometry in the context of a parallel shift of a vector (or tensor or layer – see Sec.2.3) along a path on a manifold (or on a fiber bundle).

[8]A set (topological space) is called compact if its any open coverage has a finite subcoverage. Equivalent definition: compact set is bounded and closed.

[9]For the time being I do not consider spin.

In order to define the projection operator p uniquely I have to "fix the gauge". Namely I choose some point $\vec{r}_0 \in \mathbb{R}^{3N}$ and require that the value $(p\psi)(\vec{r}_0)$ is real and positive:

$$p : E \to M, \quad \psi \in E, \quad p\psi \stackrel{\text{df}}{=} \psi / C_\psi = \Phi \in M, \quad C_\psi = \lim_{\vec{r} \to \vec{r}_0} \frac{\psi(\vec{r})}{|\psi(\vec{r})|} = \langle \Phi | \psi \rangle \Rightarrow p\psi = \langle \psi | \Phi \rangle \Phi. \tag{20}$$

A path in a manifold E is a continuous isomorphism which maps some finite segment (usually $[0, 1] \subset \mathbb{R}$) onto E. In some Berry-phase articles it is assumed that a path in E corresponds to an adiabatic time evolution[10] of a wave function; if this is the case, then a closed path in the base M should correspond to adiabatic (slowly varying in time) Hamiltonian so that the studied quantum system remains in the ground state during the whole evolution.[11]

I will consider most general closed path $\{\psi(t)|0 \leq t \leq 1\} \subset E$: the path variable t must not necessarily correspond to the evolution in time. For example, one can choose a closed path in the parameter space [e.g., Brillouin zone or torus in Fig. 1(b)]:[12]

$$\gamma = \{\psi(t) \in E | t \in [0, 1]\} \subset E, \quad p\gamma = \tilde{\gamma} = \{p(\psi(t)) = \Phi(t) \in M | t \in [0, 1]\} \subset M, \quad \Phi(0) = \Phi(1). \tag{21}$$

The path "velocity" (or tangent vector) $\dot{\psi}(t) \in \mathbb{T}_{\psi(t)} E$ can be separated into horizontal and vertical parts (see Sec. 2.3):

$$\dot{\psi}(t) = \dot{\psi}_v(t) + \dot{\psi}_h(t), \quad \dot{\psi}_v(t) \in V_{\psi(t)}, \quad \dot{\psi}_h(t) \in H_{\psi(t)}, \quad \dot{\psi}_v(t) \perp \dot{\psi}_h(t) \Leftrightarrow (\dot{\psi}_v(t), \dot{\psi}_h(t)) = 0, \tag{22}$$

where the scalar product is the standard L^2 (quantum-mechanical) one. From (20) we obtain the corrected Eq. ([3]1):

$$\mathbb{T}M \equiv M, \quad \dot{\psi} = \dot{\psi}_v + \dot{\psi}_h, \quad \dot{\psi}_v = p\dot{\psi} = \langle \dot{\psi} | \Phi \rangle \Phi, \quad \langle \dot{\psi}_v | \dot{\psi}_h \rangle = 0. \tag{23}$$

The equivalence $\mathbb{T}M \equiv M$ is implicitly assumed in Ref. [3], and I will try to explain it. Both bundle E and the base M are vector (functional) spaces. On p.[3]1206 it is mentioned that $M \equiv P(H)$ *is a linear space* which is obviously incorrect: a linear combination of *normalized* functions is generally not normalized. In order to fix that I could waive the normalization requirement, and E would become linear space then. In order to claim that M is a linear space too I have to request that linear combinations coefficients are *real*. So you see that I have to do lots of reverse engineering while studying Ref. [3], and I do not like this. This makes me suspect that the Ref. [3] is not trustworthy.

The next step would be to choose a better author – among the articles from my bookshelf. Otherwise I could still try to follow up (probably incorrect) liogic of [3].

Unfortunately, the lack of time does not allow me to finish this project so I have to postpone it to some time in the future...

4 Topological insulators

Literature: [4],...

[10]In the inverse-scattering approach arXiv/1002.1510: in case of a symmetric single well the parameter space (base) M might be given by the values of energy levels inside the well.

[11]A non-adiabatic evolution will excite the system and change its energy.

[12]Moreover: the path might be unphysical in the sense that some of the intermediate functions might solve SE with a *different* energy or even not solve SE with any energy.

Appendix

A Russian-English Dictionary

(Not sure if all translations are incorrect.)
вложение = insertion, гладкий = smooth, горизонтальные направления связности = horizontal subspace (connection), окрестность = local patch = local neighborhood, область = region, параллельный перенос (в расслоениях) = horizontal lift, перенос тензора = tensor transmission, покрытие = coverage, расслоение = fiber bundle, связность = connection, слой = fiber space/layer

Список литературы

[1] Theodore Frankel. *The Geometry of Physics: An Introduction*. Cambridge University Press, 2 edition, 2006. 1

[2] Борис А. Дубровин, Сергей Петрович Новиков, and Анатолий Т. Фоменко. *Современная геометрия: методы и приложения*. Наука, 2 edition, 1988. Novikov, Dubrovin, Fomenko: Modern geometry. The English translation is divided in three volumes: first, second, and third. 1, 5, 6

[3] Arno Bohm, Luis J. Boya, and Brian Kendrick. Derivation of the geometrical phase. *Phys. Rev. A*, 43:1206–1210, Feb 1991. 1, 6, 7

[4] J. E. Avron, R. Seiler, and B. Simon. Homotopy and quantization in condensed matter physics. *Phys. Rev. Lett.*, 51:51–53, Jul 1983. 1, 7

[5] Y. Ben-Aryeh. Berry and pancharatnam topological phases of atomic and optical systems. cond-mat/0402003v1, 2004. 1