Sri Lanka Institute of Information Technology
B.Sc. Special Honours Degree/Diploma
in
Information Technology

Mathematics for computing
Graph Theory

Basic Definition

- A graph consists of finite set of vertices V and edges E.
- Often we denote a graph by G and with the two sets of vertices and edges it is represented by V(G) and E(G).

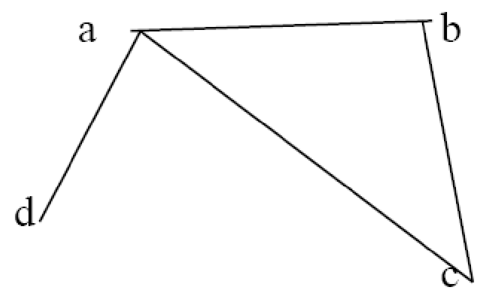
  Example:

  Let V(G) = { a,b,c,d} and E(G) = {{a,b},{a,c}, {b,c},{a,d}}
- If {a,b}ε V(G) and a and b are the end points of the edge{a,b}
- A vertex 'a' and the edge {a, b} are said to be incident since the vertex is an end point of the edge.
- Two vertices u, v of a graph G are said to be adjacent if they are joined by an edge. When u and v are adjacent, we say they are neighbors.

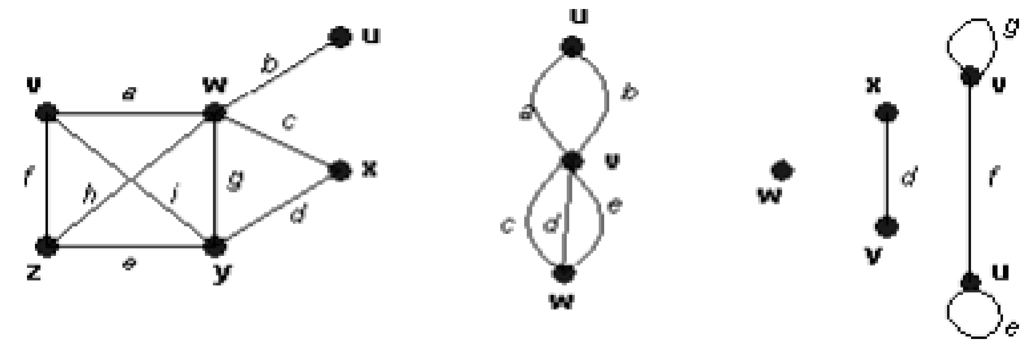
Definition Contd…

- The above graph can be shown pictorially as follows.

- V(G) = { a, b, c, d} and E(G) = {{a, b},{a, c), {b c},{a, d}}



Features of Graphs



(a) G_1 (b) G_2 (c) G_3

•In the graph G2, the edges a and b have the same endpoints, and so do the edges c, d and e.

•Such sets of edges are called multiple edges or Parallel edges.

•The edge e of the graph G3 in Fig.7.l(c) is an example for a loop (two endpoints of an edge are coincide)

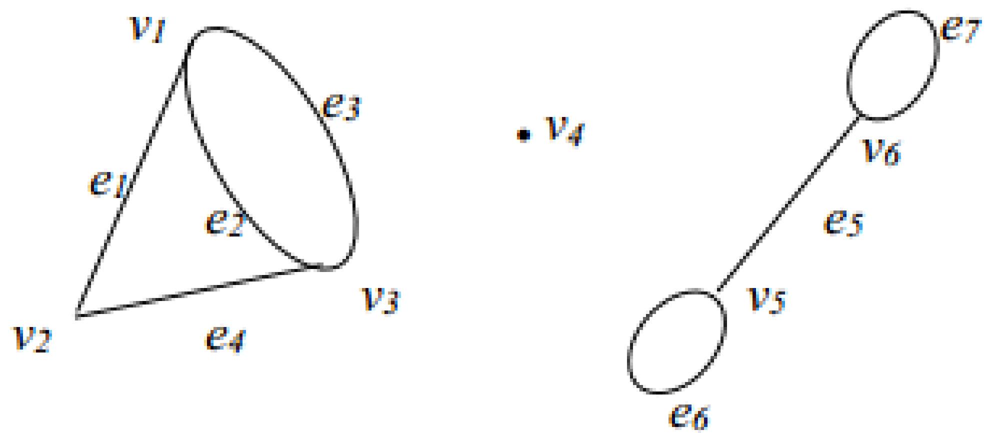
Multi-graphs and Simple graphs

- A graph that has loops or multiple edges is known as a multi-graph.
- A graph with neither loops nor multiple edges is called a simple graph.
- Therefore G_2 and G_3 are multi-graphs and G_1 is a simple graph.

Degree of a Vertex

- The degree of a vertex deg(v) is the number of edges incident with it.
- A loop will contribute 2 towards the degree.
- Thus in the graph G_1 deg(z) = 3, deg(w)= 5 and in G_2 , deg(v)=5 and in G_3, deg(u) =3 and deg(w)=0.
- W in G_3, is an isolated vertex.

Example



Write the vertex set and the edge set, and give a table showing the edge-endpoint function;

vertex set = {v1, v2, v3, v4, v5, v6}

edge set = {e1, e2, e3, e4, e5, e6, e7}

The end-point function table

Edge	End points
e_1	$\{V_1, V_3\}$
e_2	$\{V_2, V_4\}$
e_3	$\{V_1, V_2\}$
e_4	$\{V_1, V_2\}$
e_5	$\{V_3, V_4\}$
e_6	$\{V_7, V_6\}$
e_7	$\{V_6, V_5\}$
e_8	$\{V_7, V_5\}$
e_9	$\{V_6\}$

Q: Find all edges that are incident on v_1, all loops, all parallel edges, and all isolated vertices.

Theorem

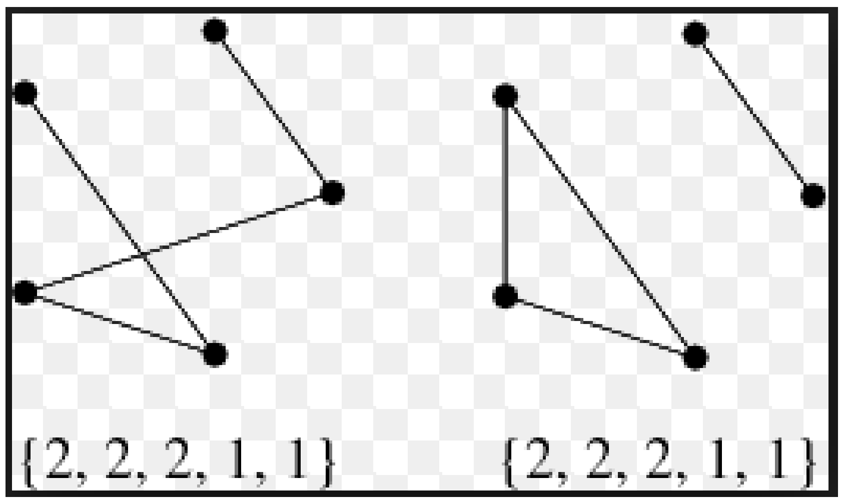
- Let g be a graph . Then the sum of degrees of the vertices of G is equal to twice the number of edges of V.

- Show that the theorem is valid for the above example.

Degree Sequence of a Graph

- The degree sequence of a graph G is the sequence of the degrees of its vertices in descending order of size.

  Example :



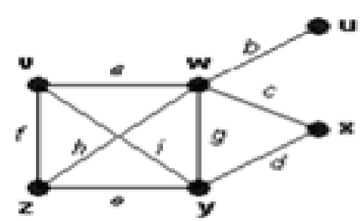
Paths, cycles and connectivity

- Path is an alternating sequence of vertices and edges of the form $v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k$.

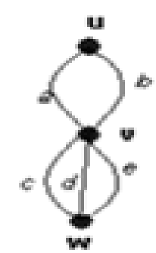
  (Edges can repeat but vertices cannot repeat)

- If G is a simple graph, path can be specified just by a sequence of distinct vertices $v_1 v_2 v_3 \ldots v_k$.

Q: Find the paths from u to w in G_1 and G_2 ?



(a) G_1



(b) G_2

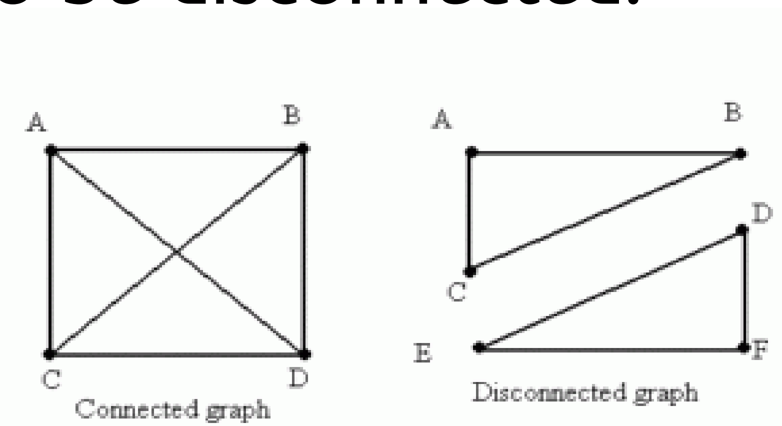
Paths, cycles and connectivity Contd…

- The length of a path is the number of EDGES in it. The path vfzey in G_1 has length 2.
- A cycle is a sequence of distinct vertices and edges that begins and ends at the same vertex.

  (xywx or wyxw are not 2 different cycles in G_1)
- Two cycles are different if they differ in at least one edge.

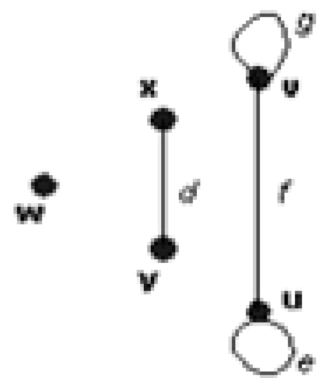
Paths, cycles and connectivity Contd…

- Two vertices u, v in a graph G are said to be connected if there is a path in G from u to v.
- The graph G is connected if every pair of vertices are connected; otherwise, G is said to be disconnected.



Components

- The separate parts of a disconnected graph are called its components. Thus the graph G3 has 3 components.
- every connected graph has just one component, the graph itself



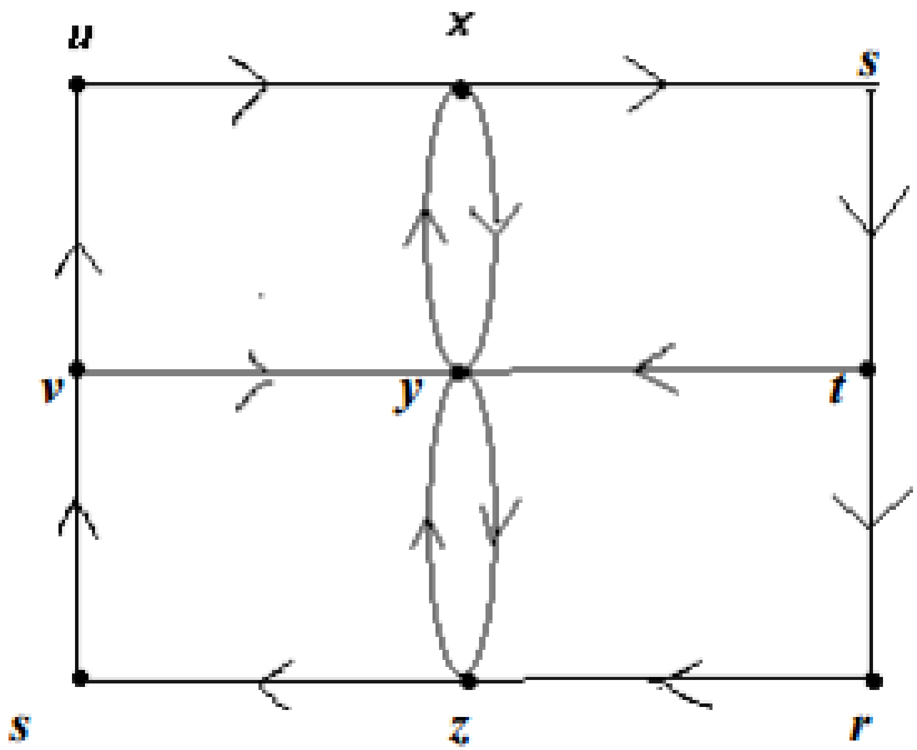
(c) G_3

Digraphs and relationship graphs

- A graph in which every edge has a direction assigned to it is called a digraph (an abbreviation of directed graph).
- The directed edges are often called arcs.
- In a digraph, we define the outdegree of vertex u, denoted by outdeg(u), as the number of arcs directed out of (away from) the vertex u.
- The indegree of vertex u, denoted by indeg(u), as the number of arcs directed into (towards) the vertex u.

Example

Q: Find the sum of the indegrees and the sum of the outdegrees of the vertices of the digraph shown in the following figure.



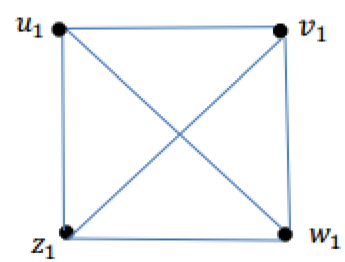
Example Contd…

Answer:

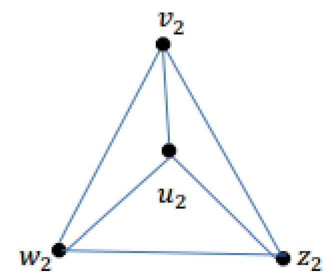
vertex	indegrees	outdegrees
u	1	1
v	1	2
w	1	1
x	2	2
y	4	2
z	2	2
s	1	1
r	1	1
t	1	2
Total	14	14

Isomorphism of Graphs

- A graph G is determined by its vertex set V(G) and its edge set E(G).
- Given this information, two people might draw the graph differently.
- Example:



G



H

$V(G)=\{u_1, v_1, w_1, z_1\}$

$E(G)=\{u_1v_1, u_1z_1, v_1w_1, z_1w_1, u_1w_1, z_1v_1\}$

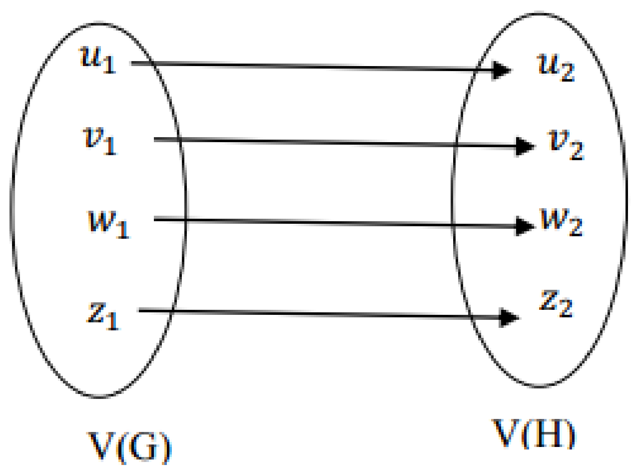
Isomorphism of Graphs Contd…

- G and H are isomorphic if we can label their vertices with the same set of labels in such a way that any pair of vertices u_1, v_1 are joined by the same number of edges in G as they are in H.

- In other words ,G and H are isomorphic if there is a one-to-one correspondence between V(G) and V(H).

Isomorphism of Graphs Contd…

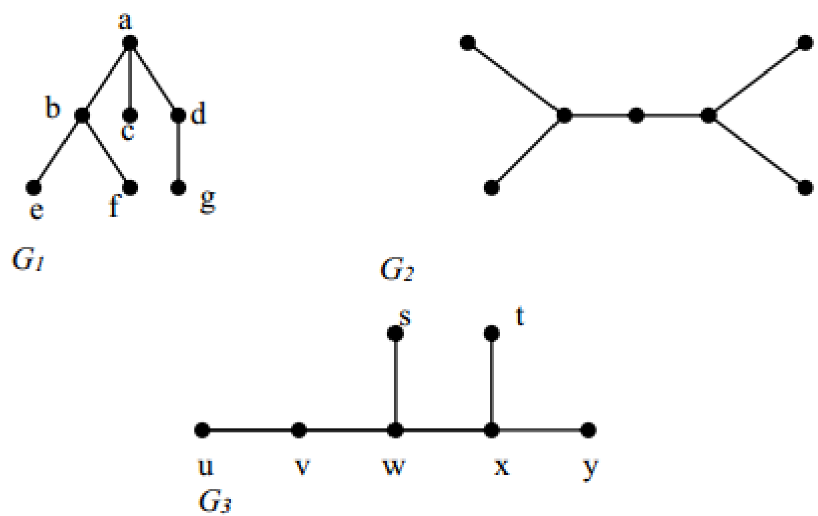
- G and H in the previous example are Isomorphic.



There is a 1-1 correspondence between V(G) and V(H)

Isomorphism of Graphs Contd…

- Show that G_1 and G_3 are Isomorphic and G_2 is not isomorphic to either G_1 or G_3.



note that in G_1 and G_3, the two vertices of degree 3 are adjacent, whereas in G_2 they are not

Isomorphism of Graphs Contd…

- When two graphs are isomorphic, their structural properties must be exactly the same.

- Let G and H be isomorphic graphs. Then G and H have

  1. the same number of components;
  2. the same number of vertices;
  3. the same number of edges;
  4. the same degree sequence;
  5. the same number of paths of any given length k;
  6. the same number of cycles of any given length k.

Note: The converse is not true.

Adjacency Matrix

- If we want to use a computer to analyze the properties of a graph, we must have some way of representing it in the computer.
- One method is using an Adjacency Matrix.

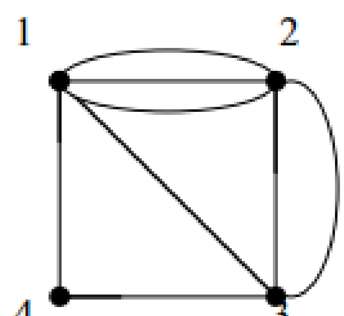


Fig 7.7

G

$$A(G) = \begin{bmatrix} 0 & 3 & 1 & 1 \\ 3 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Euler Path and Circuit

- An **Euler circuit** in a graph G is a simple circuit containing every edge of G. **Euler path** in G is a simple path containing every edge of G. (Edges cannot repeat)

  Theorem 1:

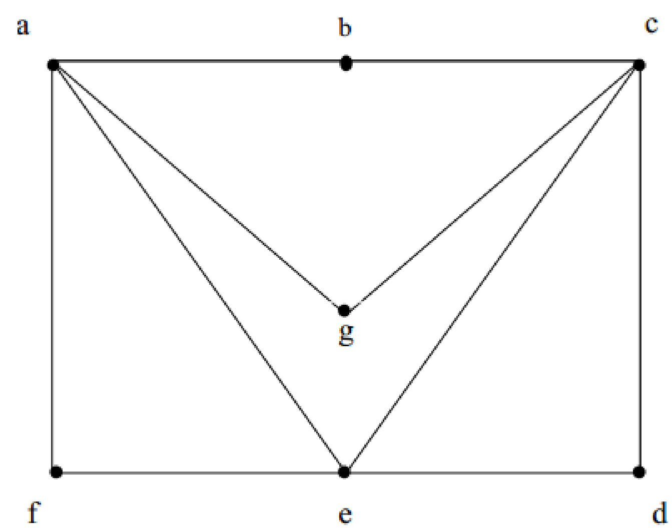
- A connected multi-graph has an Euler circuit if and only if each of its vertices has even degree.

  Theorem 2 :

- A connected multi-graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Euler Path and Circuit Contd…

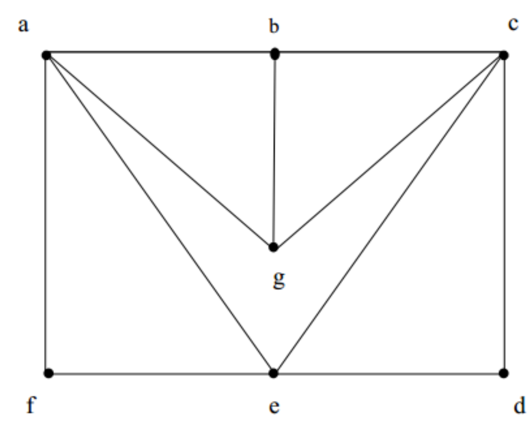
- Example:
  - Find an Euler Circuit of the following graph.



Euler Circuit: abcgaecdefa

Hamilton Path and Circuit

- a **Hamiltonian path** is a **path** in a graph that visits each vertex exactly once. (Edges and vertices cannot repeat)
- A **Hamiltonian** circuit is a **Hamiltonian path** that is a **circuit**.
- Example:
  - Find a Hamilton Circuit of the following graph.



Hamilton Circuit: **abgcdefa**

Hamilton Path and Circuit Contd…

- Dirac's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

- Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

The End