Matrices

MATHEMATICS FOR COMPUTING (IT 1030)

Matrix Definition and Notations

- A matrix is an array, which satisfies certain algebraic rules of operation.
- Capital letters are usually used to denote matrices

Eg: $\begin{bmatrix} -2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [a_{ij}],\ (1 \le i \le m,\ 1 \le j \le n)$$

Vectors

- Matrices, which have either one row or one column, are known as **vectors**

- **Row vectors**

Eg: $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$

- **Column Vectors**

Eg: $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

Square Matrix

- A matrix in which the number of rows equals the number of columns is called a **square matrix**

Eg: $\begin{bmatrix} \mathbf{1} & \mathbf{8} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$

Rectangular Matrix

- A matrix in which the number of rows not equals the number of columns is called a **rectangular matrix**

Eg: $\begin{bmatrix} \mathbf{8} & \mathbf{-4} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Rules of Matrix Algebra

1. ***Equality***. Two matrices can only be equated if they are of the same *order,* that is, if they each have the same number of rows and the same number of columns. They are then said to be **equal** if the corresponding elements are equal. Thus if

   $A = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ and $B = \begin{bmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{bmatrix}$

   then ***A = B*** if and only if *a = e, b = f,* c = *g,* and *d = h*

Rules of Matrix Algebra

2. Multiplication by a constant. Let *k* be a constant or scalar. By the product *kA* we mean the matrix in which every element of *A* is **multiplied** by *k.* Thus, if

A = $\begin{bmatrix} -2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}$ and k = 10

k*A = $\begin{bmatrix} -2 * 10 & 1 * 10 & 0 * 10 \\ -1 * 10 & 3 * 10 & 4 * 10 \end{bmatrix} = \begin{bmatrix} -20 & 10 & 0 \\ -10 & 30 & 40 \end{bmatrix}$

Rules of Matrix Algebra

3. Zero matrix*.* Any matrix in which every element is zero is called a **zero** or **null matrix**. If *A* is a zero matrix, we can simply write *A* = 0.

4. ***Matrix sums and differences****.* The sum of two matrices *A* and *B* is defined if *A* and *B* are of the same order, in which case *A* + *B* is defined as the matrix C whose elements are the sums of the corresponding elements in *A* and *B.* We write C = *A* + *B.*

Thus, if

$A = [a_{ij}]$ and $B = [b_{ij}]$ are both *m* x *n* matrices, then

$C = A + B = [a_{ij} + b_{ij}]$.

Rules of Matrix Algebra

5. Commutative property of matrix addition, that is,

$A + B = B + A$

6. **Associative law of additio**n

$A + (B+C) = (A+B) + C$

the order of addition of matrices is immaterial

7. Distributive law of addition

$A(B + \mathrm{C}) = AB + \mathrm{AC}$

8. Associative law of multiplication

$A(BC) = (\mathrm{AB})\mathrm{C}$

Rules of Matrix Algebra

9. $k(A+B) = kA + kB$

$(k+l)A = kA + lA.$ $\quad\quad$ *k, l are real numbers*

Rules of Matrix Algebra

10. Matrix multiplication. Matrices can be multiplied only if the number of columns in *A* equals the number of rows in *B*. Let us look at the case where *A* is a 2x3 matrix, and *B* is a 3x2 matrix, which are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Rules of Matrix Algebra

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}} & \boxed{a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \end{bmatrix}$$

$$= C.$$

Rules of Matrix Algebra

- One conclusion which can be inferred from the previous example is that **matrix multiplication does not commute**; that is, in general,

$$AB \neq BA.$$

- *AB* can be a zero matrix without either *A* or *B* or *BA* being zero. Also, as a consequence, $A(B - C) = 0$ does not necessarily imply $B = C$.

Special Matrices

1. The transpose of any matrix is one in which the rows and columns are interchanged. Then, the first row becomes the first column, the second row the second column, and so on. We denote the transpose of *A* by A^T. Hence if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

The 3 x 2 matrix A becomes the 2 x 3 matrix A^T.

Properties of the transpose. Provided that the sum $A + B$ and product AB are defined for two matrices A and B, then

(a) $(A + B)^T = A^T + B^T$;

(b) $(AB)^T = B^T A^T$.

(c) $(A^T)^T = A$.

Special Matrices

2. Symmetric matrices. A *square* matrix is said to be **symmetric** if $A = A^T$. Since rows and columns are interchanged in the transpose, this is equivalent to $a_{ij} = a_{ji}$ for all elements if $A = [a_{ij}]$. Symmetric matrices are easy to recognize since their elements are reflected in the **leading diagonal.**

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 2 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$ is a 3 x 3 *symmetric matrix.*

Special Matrices

A square matrix A for which $A = -A^T$ is said to be **skew- symmetric**. The elements along the leading diagonal of a skew-symmetric matrix must all be zero.

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \text{ is skew-symmetric.}$$

Note: If A is any square matrix, then $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

Special Matrices

3. Row and column vectors. As we defined them in Section 4.1, a row vector is a matrix with one row, and a column vector is one with one column. For vectors, we usually use bold-faced small letters and write, for example,

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \qquad \boldsymbol{b}^{\mathrm{T}} = [b_1 \;\; b_2 \;\; \ldots \;\; b_n]$$

The transpose of a row vector is a column vector and vice versa. If A is an m x n matrix, then $A\boldsymbol{a}$ is a column vector with m rows.

Special Matrices

4. Diagonal matrices. A square matrix all of whose elements off the leading diagonal are zero is called a **diagonal matrix**. Thus, if $A = [a_{ij}]$ is an n x n matrix, then A is diagonal if $a_{ij} = 0$ for all $i \neq j$.

Hence

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ is an example of a 3 x 3 diagonal matrix.

A diagonal matrix is obviously symmetric. If A and B are diagonal matrices of the same order then $A + B$ and AB are also both diagonal

Special Matrices

5. ***Identity matrix***. The diagonal matrix with all diagonal elements 1 is called the **Identity** or **unit matrix** I_n. Hence, the 3 x 3 identity is

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. ***Powers of matrices.*** If A is a square matrix of order n x n, then we write AA as A^2, AA^2 as A^3 and so on.

7. ***Inverse matrix***, for a square matrix A, the inverse is written A^{-1}. When A is multiplied by A^{-1} the result is the identity matrix.

$$A\,A^{-1} = A^{-1}A = I$$