Solving Linear Equations

MATHEMATICS FOR COMPUTING (IT1030)

Linear System

In mathematics a system of linear equations (or linear system) is a collection of linear equations involving the same set of variables.

For example,

$$x_1 + 2x_2 + x_3 = 1$$

$$-2x_1 + 3x_2 - x_3 = -7$$

$$x_1 + 4x_2 - 2x_3 = -7$$

Different Types of Solutions

A solution of a linear system is an assignment of values to the variables x_1, x_2, ... x_n such that each of the equations is satisfied. The set of all possible solutions is called the solution set.

A linear system may behave in any one of three possible ways:

1. The system has infinitely many solutions.
2. The system has a single unique solution.
3. The system has no solution.

Solving Linear Equations – Method I

We can convert the linear system in to a matrix form that is $AX = b$.

i.e

$$AX = b.$$

$$A^{-1}AX = A^{-1}b$$

We know $A^{-1}A = I$

$$I\,X = A^{-1}b$$

$$X = A^{-1}b$$

Where X means unknown variables, A^{-1} is inverse matrix and d is a constant matrix.

Consider now the case in which A is an arbitrary square matrix. If the inverse of A exists, then multiplication of on the left by A^{-1} leads to the solution vector

$$X = A^{-1}b = \frac{adjA}{\det A}b$$

Example

Solve the linear system.

$$x_1 + 2x_2 - x_3 = 1$$

$$x_2 - x_3 = -7$$

$$x_1 - x_2 - 2x_3 = -7$$

So

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix} x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -7 \\ -7 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{4}\begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{adjA}{\det A} b = A^{-1}\begin{bmatrix} 1 \\ -7 \\ -7 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -7 \\ -7 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -31 \\ -1 \\ -29 \end{bmatrix}$$

Therefore $x_1 = \frac{31}{4}, x_2 = \frac{1}{4}, x_3 = \frac{29}{4}$

Solving Linear Equations – Method II (Cramer's Rule)

Suppose $AX = b$ is a square linear system in the variables $X = x_1, x_2, x_3 \ \ldots x_n$

With the property that $\det A \neq 0$. Then the (unique) solution to the system is given by

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1,2,3,\ldots,n$$

where A_i is the matrix formed by replacing the i^{th} column of A by the column vector b.

Example

Solve the following system using Cramer's rule.

$$x_1 + x_2 - x_3 = 6$$
$$x_1 - x_2 + x_3 = 2$$
$$x_1 \qquad - 2x_3 = 0$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 6 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 6 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Example

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 1(-1)^{1+1}M_{11} + 1(-1)^{1+2}M_{12} + (-1)(-1)^{1+3}M_{13}$$
$$= \begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$$
$$= (2-0) - (-2-1) - (0+1)$$
$$= 4$$

$$\det A_1 = 16$$
$$\det A_2 = 16$$
$$\det A_3 = 8$$

Thus by Cramer's Rule

$$x_1 = \frac{\det A_1}{\det A} = \frac{16}{4} = 4$$
$$x_2 = \frac{\det A_2}{\det A} = \frac{16}{4} = 4$$
$$x_3 = \frac{\det A_3}{\det A} = \frac{8}{4} = 2$$

Solving Linear Equations – Method III (Gaussian Elimination Method)

$x + 2y + z = 1$

-2$x + 3y - z = -7$

$x + 4y - 2z = -7$

Obtain the Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ -2 & 3 & -1 & -7 \\ 1 & 4 & -2 & -7 \end{bmatrix},$$

which is known as the **augmented matrix** for the system of equations. The elementary operations referred to previously become **elementary row operations on the matrix**. We can reproduce the steps above by the following more compact procedure:

Example

$$\begin{bmatrix} \underline{1} & 2 & 1 & 1 \\ -2 & 3 & -1 & -7 \\ 1 & 4 & -2 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & \underline{7} & 1 & -5 \\ 0 & 2 & -3 & -8 \end{bmatrix} \quad \begin{pmatrix} r_2' = r_2 + 2r_1 \\ r_3' = r_3 - r_1 \end{pmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 7 & 1 & -5 \\ 0 & 0 & \frac{-23}{7} & \frac{-46}{7} \end{bmatrix} \left(r_3' = r_3 - \frac{2}{7} r_2 \right)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{7} & \frac{-5}{7} \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} r_3' = -\frac{7}{23} r_3 \\ r_2' = \frac{1}{7} r_2 \end{pmatrix}$$

$x_1 = 1$, $\quad x_2 = -1$, $\quad x_3 = 2$.

Where the arrow '$\rightarrow$' means 'is transformed into'. The final matrix is said to be in **echelon** form, that is, it has zeros below the diagonal elements starting from the top left. We can now solve the equations by back substitution as before.

Incompatible set of Equations (No Solution)

$x + y - z = 3$
$3x - y + 3z = 5$
$x - y + 2z = 2$

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 3 & -1 & 3 & 5 \\ 1 & -1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 6 & -4 \\ 0 & -2 & 3 & -1 \end{bmatrix} \quad \begin{pmatrix} r_2' = r_2 - 3r_1 \\ r_3' = r_3 - r_1 \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \left(r_3' = r_3 - \frac{1}{2} r_2 \right)$$

Which is the echelon form for this set of equations. However, row 3 is inconsistent since $0 \neq 1$. Hence these equations can have no solutions.

Compatible set of equations (Infinite Number of Solutions)

$$x + y - z = 1,$$

$$3x - y + 3z = 5,$$

$$x - y + 2z = 2,$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & -1 & 3 & 5 \\ 1 & -1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -4 & 6 & 2 \\ 0 & -2 & 3 & 1 \end{bmatrix} \quad \begin{pmatrix} r_2' = r_2 - 3r_1 \\ r_3' = r_3 - r_1 \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left(r_3' = r_3 - \frac{1}{2} r_2 \right)$$

Compatible set of equations (Infinite Number of Solutions)

Row 3 is now consistent, and row 2 is $-4y + 6z = 2$. Hence

$$y = -\frac{1}{4}(2-6z)$$

and, from row 1 ,

$$x = 1 - y + z = \frac{3}{2} - \frac{1}{2}z.$$

Thus z can take any value, say λ , so the full solution set is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}\lambda \\ -\frac{1}{2} + \frac{3}{2}\lambda \\ \lambda \end{bmatrix}$$

for any value of λ . It can be seen in this case that there exists an infinite number of solutions, a different one for each different value of λ .

The End