Matrices – Part II

MATHEMATICS FOR COMPUTING (IT1030)

Determinant of a Square Matrix

The determinant of a square matrix A is a real number denoted by

det A or $|A|$

The determinant of a 2×2 matrix

Given the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

Then the determinant of A is denoted and defined by

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Minor

Minor

Minor of A is the determinant of a smaller matrix formed from deleting its rows and columns. M_{ij} is the minor of matrix A formed by eliminating row i and column j from A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

M_{11} is the minor of A formed by eliminating row 1 and column 1 .

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = a_{22}a_{33} - a_{23}a_{32}$$

Minor

M_{21} is the minor of A formed by eliminating row 2 and column1.

$$\begin{bmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{21} = a_{12}a_{33} - a_{13}a_{32}$$

Cofactor

Cofactor

Cofactor of a matrix is denoted as C_{ij} and is equal to $(-1)^{i+j} M_{ij}$, Where M_{ij} is minor of matrix.

$$\boldsymbol{C_{ij} = (-1)^{i+j} M_{ij}}$$

So
$$C_{11} = (-1)^{1+1} M_{11}$$
$$= M_{11}$$

$$C_{12} = (-1)^{1+2} M_{12}$$
$$= -M_{12}$$

Expanding to find the Determinant

This is the i^{th} row expansion

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

This is the j $^{\text{th}}$ column expansion

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Here are the steps to go through to find the determinant.

1. Pick any row or column in the matrix. It does not matter which row or which column you use, the answer will be the same for any row. There are some rows or columns that are easier than others, but we'll get to that later.
2. Multiply every element in that row or column by its cofactor and add. The result is the determinant.

Determinant by Expanding 1^{st} Row

Determinant of the matrix A by expanding 1^{st} row,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \det A &= \sum_{j=1}^{3} a_{1j} C_{1j} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} + a_{13}(-1)^{1+3} M_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

$$\det \mathrm{A} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Determinant by Expanding 1st Column

$$\det A = \sum_{i=1}^{3} a_{i1} C_{i1}$$

$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= a_{11}(-1)^{1+1} M_{11} + a_{21}(-1)^{2+1} M_{21} + a_{31}(-1)^{3+1} M_{31}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}).$$

Determinant ctd..

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution

$$\det A = 1 \times \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} - (-1) \times \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + 0 \times \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= (3\text{-}2\ ) + (\ 2 + 2\ ) = 5$$

Determinant ctd…

$$\det A = \begin{vmatrix} 1 & 2 & k \\ 2 & -1 & 3 \\ -1 & 4 & -2 \end{vmatrix}$$

for any k. Find the value of k for which the determinant is zero.

Determinant ctd…

Expanding by the first row gives

$$\det A = \begin{vmatrix} 1 & 2 & k \\ 2 & -1 & 3 \\ -1 & 4 & -2 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$= 1 \times (2 - 12) - 2 \times (-4 + 3) + k(8 - 1)$$
$$= -10 + 2 + 7k = -8 + 7k.$$

Hence det A = 0 if $k = \dfrac{8}{7}$

Properties of Determinants

***1. det A^T = det A**, where A^T is the transpose of A.* The determinant of a square matrix and its transpose are equal.

Eg:

Evaluate

$$\det A = \begin{vmatrix} 1 & 28 & -29 \\ 0 & 1 & -4 \\ 0 & -2 & 5 \end{vmatrix}.$$

Properties of Determinants

Solution

Since the determinant has two zeros in the first column, it is advantageous to use Rule 1. The determinant of the transpose of A is given by

$$\det A^{T} = \begin{vmatrix} 1 & 0 & 0 \\ 28 & 1 & -2 \\ -29 & -4 & 5 \end{vmatrix}$$

Which now has two zeros in the first row. Hence the expansion by the top row becomes particularly easy:

$$\det \mathrm{A}^{T} = 1 \times \begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 5 - 8 = -3$$

Properties of Determinants

2. If every element of any single row or column of the matrix A is multiplied by a scalar k, then the determinant of this matrix is k det A.

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}ka_{22}a_{33} - a_{11}ka_{32}a_{23} - a_{12}ka_{21}a_{33} + a_{12}ka_{31}a_{23} + a_{13}ka_{21}a_{32} - a_{13}ka_{31}a_{22}$$

$$= k \det A$$

By putting $k = 0$ in this result, note that any determinant must have zero value if all the elements of any row or column are zero.

Properties of Determinants

3. If B is obtained from A by interchanging two rows (or columns) then

det B = -det A.

5. If two rows (or columns) of A are identical, then det $A = 0$.

6. If the matrix B is constructed from A by adding k times one row (or column) to another row (column) then det B = det A*: in other words, any number of such operation on rows and on columns has no effect on the value of det A.*

Properties of Determinants

$$\det B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{11} & a_{32}+ka_{12} & a_{33}+ka_{13} \end{vmatrix}$$

$$= (a_{31}+ka_{11})C_{31}+(a_{32}+ka_{12})C_{32}+(a_{33}+ka_{13})C_{33} \quad \text{(Expanding by row 3)}$$

$$= a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}+k(a_{11}C_{31}+a_{12}C_{32}+a_{13}C_{33})$$

$$= \det A + \text{k} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

$$= \det A$$

Properties of Determinants

7. *If the matrix A and B are of equal size,* $\det AB = \det A \times \det B$

Inverse Matrix

$A \rightarrow A^{-1}$

Requirements to have an inverse

1. Matrix should be a square matrix
2. Determinant of the matrix should be a non-zero value

Inverse Matrix

There are 2 methods to find the inverse matrix

- Method I

$$A^{-1} = \frac{adjoint\ A}{|A|}$$

- Method II

  Gaussian Elimination Method

The Adjoint

The matrix formed by taking the transpose of the cofactor matrix of a given original matrix. The adjoint of matrix A is often written adj A.

Elements of the cofactor matrix are cofactors.

$$\text{Cofactor matrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Adjoint of a matrix A is transpose of cofactor matrix.

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

The Adjoint

Find the adjoint of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

The Adjoint

$$C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5$$

$$C_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4 \qquad C_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12$$

$$C_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad C_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad C_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad C_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

The Adjoint

As a result the cofactor matrix of A is

$$\text{Cofactor matrix} = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Finally the adjoint of A is the transpose of the cofactor matrix:

$$adjA = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

Inverse Matrix – Method I

$$A^{-1} = \frac{adjA}{\det A} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \frac{1}{\det A}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix}.$$

We evaluate det A first. Thus

$$\det A = 1 \times (-2-1) - 2 \times (0+1) - 1 \times (0-1) = -4$$

$$C_{11} = -3 \quad C_{12} = -1 \quad C_{13} = -1$$
$$C_{21} = 5 \quad C_{22} = -1 \quad C_{23} = 3$$
$$C_{31} = -1 \quad C_{32} = 1 \quad C_{33} = 1$$

$$\text{Hence } A^{-1} = -\frac{1}{4}\begin{bmatrix} -3 & 5 & -1 \\ -1 & -1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Inverse Matrix – Method II

The inverse matrix by Gaussian elimination

Use following elementary row operations to transform A into the identity I, and use the same operations to transform I into A^{-1}.

1. Any equation can be multiplied by a nonzero constant,
1. Any two equations can be interchanged,
2. Any equation can be replaced by the sum of, itself and any multiple of another equation.

Inverse Matrix – Method II

Write down the entries of the matrix A in a double - wide matrix.

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & & & & \\ 1 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 1 & & & & \\ 1 & 0 & 2 & 0 & & & & \end{array}\right].$$

In the other half of the double-wide matrix, write the identity matrix.

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Inverse Matrix – Method II

Then do matrix row operations to convert the left-hand side of the double wide in to the identity matrix. The right - hand side of the double wide is Inverse matrix of A.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 0 & 1 & \end{array}\right].$$

Inverse Matrix – Method II

Suppose that we require the inverse of

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}.$$

We reduce A to I_4 and perform the same row operations on I_4. Thus, we can write down the steps **in parallel** as follows:

Inverse Matrix – Method II

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad (r_1 \leftrightarrow r_2) \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(r_4' = r_4 - r_1)} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{array}\right] (r_3' = r_3 - r_2$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{array}\right] \quad (r_4' = -r_4) \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right] (r_2' = r_2 - 2r_4) \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$(r_1' = r - r_{31}) \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$= \left[I_4 \middle| A^{-1}\right]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

Summary

Students Should be able to,

- Find the Determinant of a matrix
- Apply the properties of determinants
- Find the inverse matrix

The End