Assignment 4 - Probability

Due: Dec 4 by 11.59pm

Submit your assignment by the due date to Gradescope. Recall that the rules of academic integrity for this class allow you to work with one other person.

You must list your partner on Gradescope and only one of you submits the assignment.

*You **may not use any other sources** such as the internet or other textbooks to find solutions to these questions. Your answers will be graded both on their final solution as well as the work you show to achieve that solution. **Note that a subset of the questions will be graded.** By submitting your assignment you agree to abide by these rules.*

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1. (a) Suppose that 10 cookies are distributed among 5 distinct children, what is the probability that every student get at least one cookie?

    Let S represent the sample size,
    Let E represent the event that each child gets at least one cookie,

    For S: find out total number of ways to distribute 10 cookies to 5 children,
    10 cookies (objects)
    5 children (types)
    $C(10 + 5 - 1, 5 - 1)$
    $\therefore S = C(14, 4)$

    For E: find out total number of ways to distribute 10 cookies to 5 children if each get at least one,
    10 cookies (objects)
    5 children (types)
    Let $1 \geq i \geq 5$ and $x_i \geq 1$, where x_i represents the number of cookies child i has.
    $x_1 + x_2 + x_3 + x_4 + x_5 = 10$
    Get rid of the restriction for x_i gives us:
    $x_1 + x_2 + x_3 + x_4 + x_5 = 5$
    5 cookies (objects)
    5 children (types)
    $C(5 + 5 - 1, 5 - 1)$
    $\therefore E = C(9, 4)$

    $P(E) = \frac{E}{S} = \frac{C(9,4)}{C(14,4)} = \frac{\frac{9!}{5!4!}}{\frac{14!}{10!4!}} \approx 0.13 = 13\%$

    (b) If n cookies are distributed among 32 distinct children, what is the probability in terms of n that at least one student gets none?

    Let S represent the sample size,
    Let E represent the event that each child gets at least one cookie,

    For S: find out total number of ways to distribute n cookies to 32 children,
    n cookies (objects)
    32 children (types)

$$C(n + 32 - 1, n)$$
$$\therefore S = C(n + 31, n)$$

For E: find out total number of ways to distribute n cookies to 32 children if at least one student gets none,
Since we can guarantee one child gets no cookies in every scenario, there are 32 choices for who that one child will be and we can count the rest normally as if we have 31 children.
I.E., $32 \cdot C(n + 31 - 1, n) = 32 \cdot C(n + 30, n) = E$

$P(E) = \frac{E}{S} = \frac{32 \cdot C(n+30,n)}{C(n+31,n)} = \frac{32 \cdot (n+30)!}{n!30!} \cdot \frac{31!n!}{(n+31)!} = \frac{32 \cdot 31}{n+1}$

2. What is the probability of rolling a six-sided die six times and having all numbers from 1 through 6 appear once (in any order)?

$S = 6^6$
$E = 6!$
$P(E) = \frac{6!}{6^6} = \frac{720}{46,656} \approx 0.02 = 2\%$

3. Suppose that A and B are events in a sample space S and that $P(A)$, $P(B)$ and $P(A|B)$ are known. Show how to find a formula for $P(A|B^c)$.
If $P(A \wedge B) = P(A \mid B) \cdot P(B)$ and $P(A \wedge B^c) = P(A \mid B^c) \cdot P(B^c)$, then:

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c) \tag{1}$$
$$P(A) - P(A \mid B)P(B) = P(A \mid B^c)P(B^c) \tag{2}$$
$$P(A) - P(A \mid B)P(B) = P(A \mid B^c)(1 - P(B)) \tag{3}$$
$$\frac{P(A) - P(A \mid B)P(B)}{1 - P(B)} = P(A \mid B^c) \tag{4}$$

4. A bag contains 25 red balls and 15 blue balls. Two are chosen at random one after the other, without replacement (ie, once you take the first one out, there are now 39 balls in the bag).

(a) Determine each of the following probabilities:

- the probability that both balls are red

Let S_1 represent original size of bag, $S_1 = 40$
Let S_2 represent size of bag after choosing a ball, $S_2 = 39$
Let E_1 represent number of red balls, $E_1 = 25$
Let E_2 represent number of red balls after choosing a red ball, $E_2 = 24$

$$P(E_1 \wedge E_2) = P(E_1) \cdot P(E_2 \mid E_1) \tag{5}$$
$$= \frac{E_1}{S_1} \cdot \frac{E_2}{S_2} \tag{6}$$
$$= \frac{25}{40} \cdot \frac{24}{39} \tag{7}$$
$$= \frac{5}{13} \approx 0.38 = 38\% \tag{8}$$

- the probability that the first ball is red and second is not

Let S_1 represent original size of bag, $S_1 = 40$
Let S_2 represent size of bag after choosing a ball, $S_2 = 39$
Let E_1 represent number of red balls, $E_1 = 25$
Let E_2 represent number of blue balls after choosing a red ball, $E_2 = 15$

$$P(E_1 \wedge E_2) = P(E_1) \cdot P(E_2 \mid E_1) \tag{9}$$

$$= \frac{E_1}{S_1} \cdot \frac{E_2}{S_2} \tag{10}$$

$$= \frac{25}{40} \cdot \frac{15}{39} \tag{11}$$

$$= \frac{25}{104} \approx 0.24 = 24\% \tag{12}$$

- the probability that the first ball is not red and the second is red.

Let S_1 represent original size of bag, $S_1 = 40$
Let S_2 represent size of bag after choosing a ball, $S_2 = 39$
Let E_1 represent number of blue balls, $E_1 = 15$
Let E_2 represent number of red balls after choosing a blue ball, $E_2 = 25$

$$P(E_1 \wedge E_2) = P(E_1) \cdot P(E_2 \mid E_1) \tag{13}$$

$$= \frac{E_1}{S_1} \cdot \frac{E_2}{S_2} \tag{14}$$

$$= \frac{15}{40} \cdot \frac{25}{39} \tag{15}$$

$$= \frac{25}{104} \approx 0.24 = 24\% \tag{16}$$

- the probability that neither ball is red

Let S_1 represent original size of bag, $S_1 = 40$
Let S_2 represent size of bag after choosing a ball, $S_2 = 39$
Let E_1 represent number of blue balls, $E_1 = 15$
Let E_2 represent number of blue balls after choosing a blue ball, $E_2 = 14$

$$P(E_1 \wedge E_2) = P(E_1) \cdot P(E_2 \mid E_1) \tag{17}$$

$$= \frac{E_1}{S_1} \cdot \frac{E_2}{S_2} \tag{18}$$

$$= \frac{15}{40} \cdot \frac{44}{39} \tag{19}$$

$$= \frac{7}{52} \approx 0.13 = 13\% \tag{20}$$

(b) What is the probability that the second ball is red?

Let R_1 represent that the first ball is red.
Let R_2 represent that the second ball is red.
Let B_1 represent that the first ball is blue.
Let B_2 represent that the second ball is blue.

$$P(R_2) = P((R_1 \wedge R_2) \vee (B_1 \wedge R_2)) \tag{21}$$

$$= P(R_1 \wedge R_2) + P(B_1 \wedge R_2) - P((R_1 \wedge R_2) \wedge (B_1 \wedge R_2)) \tag{22}$$

$$\approx 0.38 + 0.24 - 0 \qquad\qquad found\ in\ part\ a) \tag{23}$$

$$= 0.62 = 62\% \tag{24}$$

(c) What is the probability that at least one ball is red?

Find probability of both balls being blue, and subtract from 1 (total probability)

Let B_1 represent that the first ball is blue.
Let B_2 represent that the second ball is blue.
Let R represent that at least one ball is red.

$$P(R) = 1 - P(B_1 \wedge B_2) \tag{25}$$
$$\approx 1 - 0.13 \qquad\qquad found\ in\ part\ a) \tag{26}$$
$$= 0.87 = 87\% \tag{27}$$

(d) What is the probability that exactly one ball is red?

Let R represent that exactly one ball is red.
Let R_1 represent that the first ball is red.
Let R_2 represent that the second ball is red.
Let B_1 represent that the first ball is blue.
Let B_2 represent that the second ball is blue.

$$P(R) = P((R_1 \wedge B_2) \vee (B_1 \wedge R_2)) \tag{28}$$
$$= P(R_1 \wedge B_2) + P(B_1 \wedge R_2) - P((R_1 \wedge B_2) \wedge (B_1 \wedge R_2)) \tag{29}$$
$$\approx 0.24 + 0.24 - 0 \qquad\qquad found\ in\ part\ a) \tag{30}$$
$$= 0.48 = 48\% \tag{31}$$

5. In a lottery game, there is a bag of 15 numbered balls (from 1 to 15). You select 5 balls without replacement. You win if no two balls are consecutive numbers. For example, if the balls are numbered 1, 4, 6, 9, 15 you would win but if they were numbered 3, 7, 8, 10, 12 you would loose. What is the probability that you win the game?

6. In order to work for a public transit company, drivers need to be drug free before starting their shifts. Assume that the percentage of drivers who actually use drugs is 4%, the same as the general population. The transit company randomly screens drivers before their shifts start. The drug-screening test has a false positive rate of 3% and a false negative rate of 2%. This means that a driver who uses drugs tests positive for them 98% of the time and a driver who does not use drugs tests negative for them 97% of the time.

(a) What is the probability that a randomly chosen driver who tests positive for drugs actually uses drugs?

Let D represent the event that the driver is a drug user.
Let T represent the event that the driver tests positive.
$\therefore P(D) = 0.04$
$P(D^c) = 0.96$
$P(T \mid D) = 0.98$
We want to find $P(D \mid T)$:

$$P(D \mid T) = \frac{P(T \mid D)P(D)}{P(T)} \tag{32}$$

$$= \frac{P(T \mid D)P(D)}{P(T \mid D)P(D) + P(T \mid D^c)P(D^c)} \tag{33}$$

$P(T \mid D^c)$ is asking for the probability of a positive test given that the driver does not use drugs. We know that a driver who does not use drugs tests negative for them 97% of the time.

∴ the probability of a driver testing positive for drugs if they do not use drugs is $1 - 0.97 = 0.03 = P(T \mid D^c)$

$$= \frac{0.98 \cdot 0.04}{(0.98 \cdot 0.04) + (0.03 \cdot 0.96)} \tag{34}$$

$$= \frac{0.0392}{0.068} \tag{35}$$

$$\approx 0.58 = 58\% \tag{36}$$

(b) What is the probability that a randomly chosen driver who tests negative for drugs does not use drugs?

Let D represent the event that the driver is a drug user.
Let N represent the event that the driver tests negative.
∴ $P(D) = 0.04$
$P(D^c) = 0.96$
$P(N \mid D^c) = 0.97$
We want to find $P(D^c \mid N)$:

$$P(D^c \mid N) = \frac{P(N \mid D^c)P(D^c)}{P(N)} \tag{37}$$

$$= \frac{P(N \mid D^c)P(D^c)}{P(N \mid D^c)P(D^c) + P(N \mid D)P(D)} \tag{38}$$

$P(N \mid D)$ is asking for the probability of a negative test given that the driver does use drugs. We know that a driver who does use drugs tests positive for them 98% of the time.
∴ the probability of a driver testing negative for drugs if they do use drugs is $1 - 0.98 = 0.02 = P(N \mid D)$

$$= \frac{0.97 \cdot 0.96}{(0.97 \cdot 0.96) + (0.02 \cdot 0.04)} \tag{39}$$

$$= \frac{0.9312}{0.932} \tag{40}$$

$$\approx 0.999 = 99.9\% \tag{41}$$

7. A team of computer science students are working on a large programming project. Two students A and B are selected to do quality assurance and check the application for bugs. Assume that A and B work independently and that A misses 12% of the bugs and B misses 15% of the bugs.

(a) What is the probability that a randomly chosen bug will be missed by both quality assurance checkers?

Let A represent student A misses a bug.
Let B represent student B misses a bug.
$P(A) = 0.12$
$P(B) = 0.15$
Want: $P(A \wedge B)$, since A and B are independent events, $P(A \wedge B) = P(A) \cdot P(B)$

$$P(A \wedge B) = P(A) \cdot P(B) \tag{42}$$

$$= 0.12 \cdot 0.15 \tag{43}$$

$$= 0.018 = 1.8\% \tag{44}$$

(b) If the program contains 1000 bugs, how many bugs can be expected to be missed?

Since there is a 1.8% chance to miss a bug, then we can expect $(1000 \cdot 0.018)$ bugs to be missed.
$1000 \cdot 0.018 = 18$
∴ we can expect 18 bugs to be missed.

8. In a lottery game, a player pays \$2 to play and gets to roll five fair dice. The player wins

- $0 for a roll where all five die display different values
- $1 for a roll containing two of the same value
- $2 for a roll containing three of the same value
- $4 for a roll containing four of the same value
- $8 for a roll containing five of the same value

(a) What is the expected winnings on average for the player each time the game is played?

(b) If 10,000 people play the game each day, how much money will the lottery commission make in one (365 day) year from this game?

(c) How big a payoff could the lottery commission pay for five of a kind and still make money (ie, instead of $8 for five of a kind, how much could they pay out to the a winner)?

9. **Challenge for fun (not graded).** Describe a sample space and events A, B and C where $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ but A, B and C are not pairwise independent. Your solution should not be the trivial one so none of $P(A), P(B)$ or $P(C)$ should be 0.