Solution of 2 × 2 systems of first order linear equations

Consider a system of 2 simultaneous first order linear equations

$$x_1' = a x_1 + b x_2$$
$$x_2' = c x_1 + d x_2$$

It has the alternate matrix-vector representation

$$\boldsymbol{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{x}.$$

Or, in shorthand $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, if $\boldsymbol{A}$ is already known from context.

We know that the above system is equivalent to a second order homogeneous linear differential equation. As a result, we know that the general solution contains two linearly independent parts. As well, the solution will be consisted of some type of exponential functions. Therefore, assume that $\boldsymbol{x} = \boldsymbol{k} e^{rt}$ is a solution of the system, where $\boldsymbol{k}$ is a vector of coefficients (of x_1 and x_2). Substitute $\boldsymbol{x}$ and $\boldsymbol{x}' = r\boldsymbol{k} e^{rt}$ into the equation $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, and we have

$$r\boldsymbol{k} e^{rt} = \boldsymbol{A}\boldsymbol{k} e^{rt}.$$

Since e^{rt} is never zero, we can always divide both sides by e^{rt} and get

$$r\boldsymbol{k} = \boldsymbol{A}\boldsymbol{k}.$$

We see that this new equation is exactly the relation that defines eigenvalues and eigenvectors of the coefficient matrix $\boldsymbol{A}$. In other words, in order for a function $\boldsymbol{x} = \boldsymbol{k} e^{rt}$ to satisfy our system of differential equations, the number r must be an eigenvalue of $\boldsymbol{A}$, and the vector $\boldsymbol{k}$ must be an eigenvector of $\boldsymbol{A}$ corresponding to r. Just like the solution of a second order homogeneous linear equation, there are three possibilities, depending on the number of distinct, and the type of, eigenvalues the coefficient matrix $\boldsymbol{A}$ has.

The possibilities are that $\boldsymbol{A}$ has

I. Two distinct real eigenvalues
II. Complex conjugate eigenvalues
III. A repeated eigenvalue

A related note, (from linear algebra,) we know that eigenvectors that each corresponds to a different eigenvalue are always linearly independent from each others. Consequently, if r_1 and r_2 are two different eigenvalues, then their respective eigenvectors $\boldsymbol{k}_1$ anf $\boldsymbol{k}_2$, and therefore the corresponding solutions, are always linearly independent.

Case I Distinct real eigenvalues

If the coefficient matrix ***A*** has two distinct real eigenvalues r_1 and r_2, and their respective eigenvectors are $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the 2 × 2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.}$$

Example:
$$\boldsymbol{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \boldsymbol{x}.$$

We have already found that the coefficient matrix has eigenvalues $r = -1$ and 6. And they each respectively has an eigenvector

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Therefore, a general solution of this system of differential equations is

$$x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t}$$

Example: $$\boldsymbol{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The characteristic equation is $r^2 - r - 2 = (r + 1)(r - 2) = 0$. The eigenvalues are $r = -1$ and 2. They have, respectively, eigenvectors

For $r = -1$, the system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} x = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the bottom equation of the system: $2x_1 - x_2 = 0$, we get the relation $x_2 = 2x_1$. Hence,

$$k_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

For $r = 2$, the system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} x = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $x_1 - 2x_2 = 0$, we get the relation $x_1 = 2x_2$. Hence,

$$k_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} .$$

Apply the initial values,

$$x(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0} = \begin{bmatrix} C_1 + 2C_2 \\ 2C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

That is

$$\begin{array}{ccccc} C_1 & + & 2C_2 & = & 1 \\ 2C_1 & + & C_2 & = & -1 \end{array} .$$

We find $C_1 = -1$ and $C_2 = 1$, hence we have the particular solution

$$x = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{-t} + 2e^{2t} \\ -2e^{-t} + e^{2t} \end{bmatrix} .$$

Case II Complex conjugate eigenvalues

If the coefficient matrix $\boldsymbol{A}$ has two distinct complex conjugate eigenvalues $\lambda \pm \mu i$. Also suppose $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\, i$ is an eigenvector (necessarily has complex-valued entries) of the eigenvalue $\lambda + \mu i$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a <u>real-valued</u> general solution

$$\boxed{x = C_1 e^{\lambda t}\left(a\cos(\mu t) - b\sin(\mu t)\right) + C_2 e^{\lambda t}\left(a\sin(\mu t) + b\cos(\mu t)\right)}$$

A little detail: Similar to what we have done before, first there was the complex-valued general solution in the form

$$x = C_1 k_1 e^{(\lambda + \mu i)t} + C_2 k_2 e^{(\lambda - \mu i)t}.$$

We "filter out" the imaginary parts by carefully choosing two sets of coefficients to obtain two corresponding real-valued solutions that are also linearly independent:

$$u = e^{\lambda t}\left(a\cos(\mu t) - b\sin(\mu t)\right)$$
$$v = e^{\lambda t}\left(a\sin(\mu t) + b\cos(\mu t)\right)$$

The real-valued general solution above is just $\boldsymbol{x} = C_1\, \boldsymbol{u} + C_2\, \boldsymbol{v}$. In particular, it might be useful to know how $\boldsymbol{u}$ and $\boldsymbol{v}$ could be derived by expanding the following complex-valued expression (the front half of the complex-valued general solution):

$$k_1 e^{(\lambda + \mu i)t} = (a + bi)e^{\lambda t} e^{(\mu t)i} = e^{\lambda t}(a + bi)(\cos(\mu t) + i\sin(\mu t))$$
$$= e^{\lambda t}(a\cos(\mu t) + ia\sin(\mu t) + ib\cos(\mu t) + i^2 b\sin(\mu t))$$
$$= e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t)) + i\, e^{\lambda t}(a\sin(\mu t) + b\cos(\mu t))$$

Then, $\boldsymbol{u}$ is just the real part of this complex-valued function, and $\boldsymbol{v}$ is its imaginary part.

Example: $$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$$

The characteristic equation is $r^2 + 1 = 0$, giving eigenvalues $r = \pm\, i$. That is, $\lambda = 0$ and $\mu = 1$.

Take the first (the one with positive imaginary part) eigenvalue $r = i$, and find one of its eigenvectors:

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $(2 - i)x_1 - 5x_2 = 0$, we get the relation $(2 - i)x_1 = 5x_2$. Hence,

$$k = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \underset{\boldsymbol{a}}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}} + \underset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} i = a + bi$$

Therefore, a general solution is

$$x = C_1 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin(t) \right) + C_2 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos(t) \right)$$

$$= C_1 \begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + C_2 \begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

Example: $\boldsymbol{x}' = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix}\boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The characteristic equation is $r^2 - 4r + 13 = 0$, giving eigenvalues $r = 2 \pm 3i$. Thus, $\lambda = 2$ and $\mu = 3$.

Take $r = 2 + 3i$ and find one of its eigenvectors:

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} -1-(2+3i) & -6 \\ 3 & 5-(2+3i) \end{bmatrix} x = \begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the second equation of the system: $3x_1 + (3 - 3i)x_2 = 0$, we get the relation $x_1 = (-1 + i)x_2$. Hence,

$$k = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = a + bi$$

The general solution is

$$x = C_1 e^{2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\cos(3t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(3t)\right) + C_2 e^{2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin(3t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(3t)\right)$$

$$= C_1 e^{2t}\begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix}$$

Apply the initial values to find C_1 and C_2:

$$x(0) = C_1 e^0 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(0) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(0) \right) + C_2 e^0 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(0) \right)$$

$$= C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -C_1 + C_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Therefore, $C_1 = 2$ and $C_2 = 2$. Consequently, the particular solution is

$$x = 2e^{2t} \begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + 2e^{2t} \begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} -4\sin(3t) \\ 2\cos(3t) + 2\sin(3t) \end{pmatrix}$$

Case III Repeated real eigenvalue

Suppose the coefficient matrix $\boldsymbol{A}$ has a repeated real eigenvalues r, there are 2 sub-cases.

(i) If r has two linearly independent eigenvectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.}$$

Note: For 2×2 matrices, this possibility only occurs when the coefficient matrix $\boldsymbol{A}$ is a scalar multiple of the identity matrix. That is, $\boldsymbol{A}$ has the form

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \text{for any constant } \alpha.$$

Example:

$$\boldsymbol{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}.$$

The eigenvalue is $r = 2$ (repeated). There are 2 sets of linearly independent eigenvectors, which could be represented by any 2 nonzero vectors that are not constant multiples of each other. For example

$$k_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}.$$

(ii) If r, as it usually does, only has one linearly independent eigenvector $\boldsymbol{k}$. Then the 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{\boldsymbol{x} = C_1 \boldsymbol{k} e^{rt} + C_2(\boldsymbol{k} t e^{rt} + \boldsymbol{\eta} e^{rt}).}$$

Where the second vector $\boldsymbol{\eta}$ is any solution of the nonhomogeneous linear system of algebraic equations

$$\boxed{(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{k}.}$$

Example: $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The eigenvalue is $r = -3$ (repeated). The corresponding system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} x = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations of the system are $4x_1 - 4x_2 = 0$, we get the same relation $x_1 = x_2$. Hence, there is only one linearly independent eigenvector:

$$k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, solve for $\boldsymbol{\eta}$:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It has solution in the form $\boldsymbol{\eta} = \begin{bmatrix} \frac{1}{4} + \eta_2 \\ \eta_2 \end{bmatrix}$.

Choose $\eta_2 = 0$, we get $\boldsymbol{\eta} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$.

A general solution is, therefore,

$$x = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + C_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t\, e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right)$$

Apply the initial values to find $C_1 = 1$ and $C_2 = -12$. The particular solution is

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} - 12 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t\, e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right) = \begin{bmatrix} -12t - 2 \\ -12t + 1 \end{bmatrix} e^{-3t}$$

<u>Summary</u>: Solving a Homogeneous System of Two Linear First Order Equations in Two Unknowns

Given:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}.$$

First find the two eigenvalues, r, and their respective corresponding eigenvectors, $\boldsymbol{k}$, of the coefficient matrix $\boldsymbol{A}$. Depending on the eigenvalues and eigenvectors, the general solution is:

I. Two distinct real eigenvalues r_1 and r_2:

$$x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.$$

II. Two complex conjugate eigenvalues $\lambda \pm \mu i$, where $\lambda + \mu i$ has as an eigenvector $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\,i$:

$$x = C_1 e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big) + C_2 e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)$$

III. A repeated real eigenvalue r:

(i) When two linearly independent eigenvectors exist –

$$\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.$$

(ii) When only one linearly independent eigenvector exist –

$$\boldsymbol{x} = C_1 \boldsymbol{k} e^{rt} + C_2 (\boldsymbol{k}\, t e^{rt} + \boldsymbol{\eta}\, e^{rt}).$$

Note: Solve the system $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{k}$ to find the vector $\boldsymbol{\eta}$.