3 Test (with a solution) (40)

1. (20)

   (a) Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x} \in \mathbb{R}^3$ the easiest way you know.

   (b) Calculate the coordinate vector $[\mathbf{x}]_{\mathcal{C}}$ from $[\mathbf{x}]_{\mathcal{B}}$ using change of basis matrix.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \ \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Solution. (a) Since $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ we immediately see that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(b) To calculate $[\mathbf{x}]_{\mathcal{C}}$ we use the change of basis $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}}$. First let us find the change of basis matrix.

Since neither $\mathcal{B}$ or $\mathcal{C}$ are standard basis, we use the Gauss-Jordan method.

To calculate $P_{\mathcal{C}\leftarrow\mathcal{B}}$ we use the fact that for the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ it is straightforward to see that

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

From theory we know that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}} P_{\mathcal{E}\leftarrow\mathcal{B}} = (P_{\mathcal{E}\leftarrow\mathcal{C}})^{-1} P_{\mathcal{E}\leftarrow\mathcal{B}},$$

and thus using row reduction we get

$$[P_{\mathcal{E}\leftarrow\mathcal{C}} | P_{\mathcal{E}\leftarrow\mathcal{B}}] \to [I | P_{\mathcal{C}\leftarrow\mathcal{B}}].$$

We calculate

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

and hence

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Finally, using the relation $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ we conclude that

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$\square$

2. (20) Determine, whether $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution. Since the matrix is upper triangular it is easy to see that it has eigenvalues $\lambda_{1,2} = 1$ and $\lambda_3 = 2$. Next we look for the bases of the eigenspaces.

For $\lambda_{1,2}$ we have that $E_{\lambda_{1,2}}$ is the null space of $A - I$, i.e., the null space of

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $E_{\lambda_{1,2}} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. Since the algebraic multiplicity and the geometric multiplicity of $\lambda_{1,2}$ differ, we conclude that A is not diagonalizable. $\square$