5 Test (with a solution) (40)

1. (20) Determine whether the set of all **skew-symmetric**[1] $n \times n$ matrices with the usual matrix addition and scalar multiplication is a vector space. If it is not, give a counter example for the axiom that fails.

   Solution. Since we are considering a subset of the vector space of all $n \times n$ matrices it is enough to verify three conditions thanks to the subspace test.

   (i) It is nonempty: The zero matrix is a skew-symmetric matrix in any dimension and hence the space is nonempty.

   (ii) Closed under addition: Assume that A, B are skew-symmetric then

   $$(A+B)^T = A^T + B^T = -A + (-B) = -(A+B)$$

   and hence the sum is also skew-symmetric.

   (iii) Closed under scalar multiplication: For any scalar c we have

   $$(cA)^T = cA^T = c(-A) = -(cA)$$

   and hence the scalar multiple is also skew-symmetric.

   Since we did not use the dimension of a matrix in any of the above calculations, we can conclude that the set of all skew-symmetric matrices is actually a vector space for any n. □

2. (20) Determine the dimension of $V = \{A \text{ in } M_{22} : \ A \text{ is skew-symmetric}\}$ and give a basis for V.

   Solution. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a skew-symmetric matrix from M_{22} then by definition it has to satisfy $A^T = -A$, i.e.,

   $$\begin{bmatrix} \mathbf{a} & \mathbf{c} \\ \mathbf{b} & \mathbf{d} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\mathbf{a} & -\mathbf{b} \\ -\mathbf{c} & -\mathbf{d} \end{bmatrix}.$$

   In other words, $a = d = 0$ and $c = -b$. We thus conclude that

   $$V = \{A \text{ in } M_{22} : A \text{ has the form } \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, b \in \mathbb{R}\}$$

   is a one-dimensional vector space and the vector $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ forms its basis. □

[1] An $n \times n$ matrix A is called *skew-symmetric* if $A^T = -A$, i.e., transpose of A equals the negative of A.