## Solution of C9 - Orthogonality I

1. Determine which sets of vectors are orthogonal.

a) $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

By taking the dot products we get that

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0, \quad \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 0.$$

Hence the set is orthogonal.

b) $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

By taking the dot products we get that

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 0, \quad \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \neq 0.$$

Hence the set is NOT orthogonal.

c) $\begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Here, we do not need to calculate anything. Since 4 vectors are linearly dependent in $\mathbb{R}^3$ we immediately conclude that the set is NOT orthogonal.

2. Show that the given vectors form an orthogonal basis for $\mathbb{R}^3$. Find the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ of $\mathbf{w}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$. (Hint: use the orthogonality!)

a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since we have 3 vectors in $\mathbb{R}^3$, to show that they form a basis it is enough to show that they either span $\mathbb{R}^3$ or that they are linearly independent. However, as we also need to show that they form an orthogonal set, it is easiest to verify only that. Indeed, from theory we know that an orthogonal set of nonzero vectors is automatically linearly independent. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ we conclude that $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$.

To find $[\mathbf{w}]_{\mathcal{B}}$ we could solve the system of equations as usual" however we can use the orthogonality of the basis to save ourselves lots of work. Since $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ and hence $\mathbf{w} \cdot \mathbf{v}_i = c_i\mathbf{v}_i \cdot \mathbf{v}_i$ for every $i = 1, 2, 3$ we calculate

$$\mathbf{w} \cdot \mathbf{v}_1 = 0, \quad \mathbf{w} \cdot \mathbf{v}_2 = 4, \quad \mathbf{w} \cdot \mathbf{u}_3 = 1, \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = 2, \quad \mathbf{v}_2 \cdot \mathbf{v}_2 = 6, \quad \mathbf{v}_3 \cdot \mathbf{v}_3 = 3$$

to get $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. The result is easy to verify by substituting into the linear combination.

b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Similarly to above we verify that $\mathcal{B}$ is an orthogonal basis and since

$$\mathbf{w} \cdot \mathbf{v}_1 = 6, \quad \mathbf{w} \cdot \mathbf{v}_2 = -1, \quad \mathbf{w} \cdot \mathbf{u}_3 = -3, \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = 3, \quad \mathbf{v}_2 \cdot \mathbf{v}_2 = 2, \quad \mathbf{v}_3 \cdot \mathbf{v}_3 = 6$$

we conclude that $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

3. Determine whether the given orthogonal set of vectors is orthonormal. If it is not, normalize the vectors to form an orthonormal set.

a) $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$

Since $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3}$, $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = \frac{3}{2}$ we see that the set is not orthonormal and we have to normalize it as

$$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}.$$

b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Similarly to above, since $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$ we see that the set is not orthonormal and we have to normalize it as

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

4. Let $W$ be the subspace spanned by the vectors given below. Find a basis for $W^{\perp}$. (Hint: use the orthogonality of fundamental spaces)

a)

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then $W = \mathrm{col}(A)$.

$$W^{\perp} = (\mathrm{col}(A))^{\perp} = \mathrm{null}(A^T) = \mathrm{null}\left( \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \right).$$

Since $\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$, setting

$$x_1 = -3x_3 - 4x_4, \quad x_2 = -2x_3 - 3x_4,$$

we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 - 4x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence we got a basis

$$\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

b)

$$\mathbf{w}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 3 \\ -2 \\ 6 \\ -2 \\ 5 \end{bmatrix}.$$

Follows the same ideas as above and hence

$$W^{\perp} = \mathrm{null}\left( \begin{bmatrix} 3 & 2 & 0 & -1 & 4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix} \right).$$

Since

$$\begin{bmatrix} 3 & 2 & 0 & -1 & 4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & -4 & 6 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we get a basis

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

5. Find the orthogonal complement $W^\perp$ of $W$ and give a basis for $W^\perp$.

a)

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \; 3x + 2y = 0 \right\}$$

We see that $W \subset \mathbb{R}^2$ and $W$ is a straight line containing $\mathbf{0}$. From the equation $3x+2y=0$ we see that

$$W = \left\{ \begin{bmatrix} -\frac{2}{3}y \\ y \end{bmatrix}, \; y \in \mathbb{R} \right\} = \operatorname{span}\left( \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \right).$$

Since $\dim W + \dim W^\perp = 2$ (from Theorem 5.13) and $\dim W = 1$ we see that it is enough to find one vector orthogonal to $W$ and the span of this vector will be $W^\perp$. Immediately, we see that

$$\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} = 0.$$

Hence, $W^\perp = \operatorname{span}\left( \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right)$.

b)

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \; x = t, y = -t, z = 3t \right\}$$

It is easy to see that

$$W = \operatorname{span}\left( \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right).$$

Hence, $\dim(W^\perp) = 2$ and $W^\perp$ can be expressed as

$$W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = x - y + 3z = 0 \right\}.$$

To find a basis of such space is a standard problem, e.g, we get

$$W^\perp = \operatorname{span}\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right).$$

c)

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Using the orthogonality of fundamental spaces we get that

$$W^{\perp} = \left( \operatorname{col} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right) \right)^{\perp} = \operatorname{null} \left( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right).$$

From the form of the transposed matrix, we deduce that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

A basis for $W^{\perp}$ is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

6. Show that $W \cap W^{\perp} = \{\mathbf{0}\}$ for any subspace $W$ of $\mathbb{R}^n$.

   Let $\mathbf{u} \in W \cap W^{\perp}$. Since $\mathbf{u} \in W^{\perp}$ there holds $\mathbf{u} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$, but $\mathbf{u} \in W$ as well and thus $\mathbf{u} \cdot \mathbf{u} = 0$. Thus we get $\mathbf{u} = \mathbf{0}$. Hence, $W \cap W^{\perp} = \{\mathbf{0}\}$.

7. Prove the following statement: Let $W$ be a subspace of $\mathbb{R}^n$. If $W = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then $\mathbf{v}$ is in $W^{\perp}$ if and only if $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$.

   If $\mathbf{v} \in W^{\perp}$ then from definition $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$, in particular, $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$. Conversely, let $\mathbf{w} \in W$ is arbitrary. Since $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ span $W$, there exist scalars $c_1, \ldots, c_k$ such that $\mathbf{w} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k$. Since $\mathbf{v} \cdot \mathbf{w}_i = 0$ for all $i = 1, \ldots, k$ we obtain

   $$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \cdot (c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k) = 0.$$

   Hence, $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in W$.