6 Vector Spaces

6.0 Introduction: Fibonacci in (Vector) Space

Algebra is generous; she often gives more than is asked of her.
—Jean le Rond d'Alembert (1717–1783)
In Carl B. Boyer
A History of Mathematics
Wiley, 1968, p. 481

The Fibonacci sequence was introduced in Section 4.6. It is the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

of nonnegative integers with the property that after the first two terms, each term is the sum of the two terms preceding it. Thus $0 + 1 = 1$, $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on.

If we denote the terms of the Fibonacci sequence by $f_0, f_1, f_2, \ldots$, then the entire sequence is completely determined by specifying that

$$f_0 = 0, f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

By analogy with vector notation, let's write a sequence $x_0, x_1, x_2, x_3, \ldots$ as

$$\mathbf{x} = [x_0, x_1, x_2, x_3, \ldots)$$

The Fibonacci sequence then becomes

$$\mathbf{f} = [f_0, f_1, f_2, f_3, \ldots) = [0, 1, 1, 2, \ldots)$$

We now generalize this notion.

Definition A ***Fibonacci-type sequence*** is *any* sequence $\mathbf{x} = [x_0, x_1, x_2, x_3, \ldots)$ such that x_0 and x_1 are real numbers and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$.

For example, $[1, \sqrt{2}, 1 + \sqrt{2}, 1 + 2\sqrt{2}, 2 + 3\sqrt{2}, \ldots)$ is a Fibonacci-type sequence.

Problem 1 Write down the first five terms of three more Fibonacci-type sequences.

By analogy with vectors again, let's define the *sum* of two sequences $\mathbf{x} = [x_0, x_1, x_2, \ldots)$ and $\mathbf{y} = [y_0, y_1, y_2, \ldots)$ to be the sequence

$$\mathbf{x} + \mathbf{y} = [x_0 + y_0, x_1 + y_1, x_2 + y_2, \ldots)$$

If c is a scalar, we can likewise define the scalar multiple of a sequence by

$$c\mathbf{x} = [cx_0, cx_1, cx_2, \ldots)$$

Problem 2 (a) Using your examples from Problem 1 or other examples, compute the sums of various pairs of Fibonacci-type sequences. Do the resulting sequences appear to be Fibonacci-type?

(b) Compute various scalar multiples of your Fibonacci-type sequences from Problem 1. Do the resulting sequences appear to be Fibonacci-type?

Problem 3 (a) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are Fibonacci-type sequences, then so is $\mathbf{x} + \mathbf{y}$.

(b) Prove that if $\mathbf{x}$ is a Fibonacci-type sequence and c is a scalar, then $c\mathbf{x}$ is also a Fibonacci-type sequence.

Let's denote the set of all Fibonacci-type sequences by *Fib*. Problem 3 shows that, like $\mathbb{R}^n$, *Fib* is closed under addition and scalar multiplication. The next exercises show that *Fib* has much more in common with $\mathbb{R}^n$.

Problem 4 Review the algebraic properties of vectors in Theorem 1.1. Does *Fib* satisfy all of these properties? What Fibonacci-type sequence plays the role of $\mathbf{0}$? For a Fibonacci-type sequence $\mathbf{x}$, what is $-\mathbf{x}$? Is $-\mathbf{x}$ also a Fibonacci-type sequence?

Problem 5 In $\mathbb{R}^n$, we have the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The Fibonacci sequence $\mathbf{f} = [0, 1, 1, 2, \ldots)$ can be thought of as the analogue of $\mathbf{e}_2$ because its first two terms are 0 and 1. What sequence $\mathbf{e}$ in *Fib* plays the role of $\mathbf{e}_1$?

What about $\mathbf{e}_3, \mathbf{e}_4, \ldots$? Do these vectors have analogues in *Fib*?

Problem 6 Let $\mathbf{x} = [x_0, x_1, x_2, \ldots)$ be a Fibonacci-type sequence. Show that $\mathbf{x}$ is a linear combination of $\mathbf{e}$ and $\mathbf{f}$.

Problem 7 Show that $\mathbf{e}$ and $\mathbf{f}$ are linearly independent. (That is, show that if $c\mathbf{e} + d\mathbf{f} = \mathbf{0}$, then $c = d = 0$.)

Problem 8 Given your answers to Problems 6 and 7, what would be a sensible value to assign to the "dimension" of *Fib*? Why?

Problem 9 Are there any geometric sequences in *Fib*? That is, if

$$[1, r, r^2, r^3, \ldots)$$

is a Fibonacci-type sequence, what are the possible values of r?

Problem 10 Find a "basis" for *Fib* consisting of geometric Fibonacci-type sequences.

Problem 11 Using your answer to Problem 10, give an alternative derivation of *Binet's formula* [formula (5) in Section 4.6]:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for the terms of the Fibonacci sequence $\mathbf{f} = [f_0, f_1, f_2, \ldots)$. [*Hint:* Express $\mathbf{f}$ in terms of the basis from Problem 10.]

The Lucas sequence is named after Edouard Lucas (see page 336).

The ***Lucas sequence*** is the Fibonacci-type sequence

$$\mathbf{l} = [l_0, l_1, l_2, l_3, \ldots) = [2, 1, 3, 4, \ldots)$$

Problem 12 Use the basis from Problem 10 to find an analogue of Binet's formula for the nth term l_n of the Lucas sequence.

Problem 13 Prove that the Fibonacci and Lucas sequences are related by the identity

$$f_{n-1} + f_{n+1} = l_n \quad \text{for } n \geq 1$$

[*Hint:* The Fibonacci-type sequences $\mathbf{f}^- = [1, 1, 2, 3, \ldots)$ and $\mathbf{f}^+ = [1, 0, 1, 1, \ldots)$ form a basis for *Fib*. (Why?)]

In this Introduction, we have seen that the collection *Fib* of all Fibonacci-type sequences behaves in many respects like $\mathbb{R}^2$, even though the "vectors" are actually infinite sequences. This useful analogy leads to the general notion of a *vector space* that is the subject of this chapter.

6.1 Vector Spaces and Subspaces

In Chapters 1 and 3, we saw that the algebra of vectors and the algebra of matrices are similar in many respects. In particular, we can add both vectors and matrices, and we can multiply both by scalars. The properties that result from these two operations (Theorem 1.1 and Theorem 3.2) are identical in both settings. In this section, we use these properties to define generalized "vectors" that arise in a wide variety of examples. By proving general theorems about these "vectors," we will therefore simultaneously be proving results about all of these examples. This is the real power of algebra: its ability to take properties from a concrete setting, like $\mathbb{R}^n$, and *abstract* them into a general setting.

Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V. — Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associativity
4. There exists an element $\mathbf{0}$ in V, called a ***zero vector***, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. — Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

The German mathematician Hermann Grassmann (1809–1877) is generally credited with first introducing the idea of a vector space (although he did not call it that) in 1844. Unfortunately, his work was very difficult to read and did not receive the attention it deserved. One person who did study it was the Italian mathematician Giuseppe Peano (1858–1932). In his 1888 book *Calcolo Geometrico*, Peano clarified Grassmann's earlier work and laid down the axioms for a vector space as we know them today. Peano's book is also remarkable for introducing operations on sets. His notations $\cup$, $\cap$, and $\in$ (for "union," "intersection," and "is an element of") are the ones we still use, although they were not immediately accepted by other mathematicians. Peano's axiomatic definition of a vector space also had very little influence for many years. Acceptance came in 1918, after Hermann Weyl (1885–1955) repeated it in his book *Space, Time, Matter*, an introduction to Einstein's general theory of relativity.

Remarks

- By "scalars" we will usually mean the real numbers. Accordingly, we should refer to V as a *real vector space* (or a *vector space over the real numbers*). It is also possible for scalars to be complex numbers or to belong to $\mathbb{Z}_p$, where p is prime. In these cases, V is called a *complex vector space* or a *vector space over* $\mathbb{Z}_p$, respectively. Most of our examples will be real vector spaces, so we will usually omit the adjective "real." If something is referred to as a "vector space," assume that we are working over the real number system.

  In fact, the scalars can be chosen from any number system in which, roughly speaking, we can add, subtract, multiply, and divide according to the usual laws of arithmetic. In abstract algebra, such a number system is called a ***field***.

- The definition of a vector space does not specify what the set V consists of. Neither does it specify what the operations called "addition" and "scalar multiplication" look like. Often, they will be familiar, but they need not be. See Example 6.6 and Exercises 5–7.

We will now look at several examples of vector spaces. In each case, we need to specify the set V and the operations of addition and scalar multiplication and to verify axioms 1 through 10. We need to pay particular attention to axioms 1 and 6 (closure),

axiom 4 (the existence of a zero vector in V), and axiom 5 (each vector in V must have a negative in V).

Example 6.1

For any $n \geq 1$, $\mathbb{R}^n$ is a vector space with the usual operations of addition and scalar multiplication. Axioms 1 and 6 follow from the definitions of these operations, and the remaining axioms follow from Theorem 1.1.

Example 6.2

The set of all 2×3 matrices is a vector space with the usual operations of matrix addition and matrix scalar multiplication. Here the "vectors" are actually matrices. We know that the sum of two 2×3 matrices is also a 2×3 matrix and that multiplying a 2×3 matrix by a scalar gives another 2×3 matrix; hence, we have closure. The remaining axioms follow from Theorem 3.2. In particular, the zero vector $\mathbf{0}$ is the 2×3 zero matrix, and the negative of a 2×3 matrix A is just the 2×3 matrix $-A$.

There is nothing special about 2×3 matrices. For any positive integers m and n, the set of all $m \times n$ matrices forms a vector space with the usual operations of matrix addition and matrix scalar multiplication. This vector space is denoted M_{mn}.

Example 6.3

Let $\mathcal{P}_2$ denote the set of all polynomials of degree 2 or less with real coefficients. Define addition and scalar multiplication in the usual way. (See Appendix D.) If

$$p(x) = a_0 + a_1x + a_2x^2 \quad \text{and} \quad q(x) = b_0 + b_1x + b_2x^2$$

are in $\mathcal{P}_2$, then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

has degree at most 2 and so is in $\mathcal{P}_2$. If c is a scalar, then

$$cp(x) = ca_0 + ca_1x + ca_2x^2$$

is also in $\mathcal{P}_2$. This verifies axioms 1 and 6.

The zero vector $\mathbf{0}$ is the zero polynomial—that is, the polynomial all of whose coefficients are zero. The negative of a polynomial $p(x) = a_0 + a_1x + a_2x^2$ is the polynomial $-p(x) = -a_0 - a_1x - a_2x^2$. It is now easy to verify the remaining axioms. We will check axiom 2 and leave the others for Exercise 12. With $p(x)$ and $q(x)$ as above, we have

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \\ &= (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2 \\ &= (b_0 + b_1x + b_2x^2) + (a_0 + a_1x + a_2x^2) \\ &= q(x) + p(x) \end{aligned}$$

where the third equality follows from the fact that addition of real numbers is commutative.

In general, for any fixed $n \geq 0$, the set $\mathcal{P}_n$ of all polynomials of degree less than or equal to n is a vector space, as is the set $\mathcal{P}$ of *all* polynomials.

Example 6.4

Let $\mathcal{F}$ denote the set of all real-valued functions defined on the real line. If f and g are two such functions and c is a scalar, then $f + g$ and cf are defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = cf(x)$$

In other words, the *value* of $f + g$ at x is obtained by adding together the values of f and g at x [Figure 6.1(a)]. Similarly, the value of cf at x is just the value of f at x multiplied by the scalar c [Figure 6.1(b)]. The zero vector in $\mathcal{F}$ is the constant function f_0 that is identically zero; that is, $f_0(x) = 0$ for all x. The negative of a function f is the function $-f$ defined by $(-f)(x) = -f(x)$ [Figure 6.1(c)].

Axioms 1 and 6 are obviously true. Verification of the remaining axioms is left as Exercise 13. Thus, $\mathcal{F}$ is a vector space.

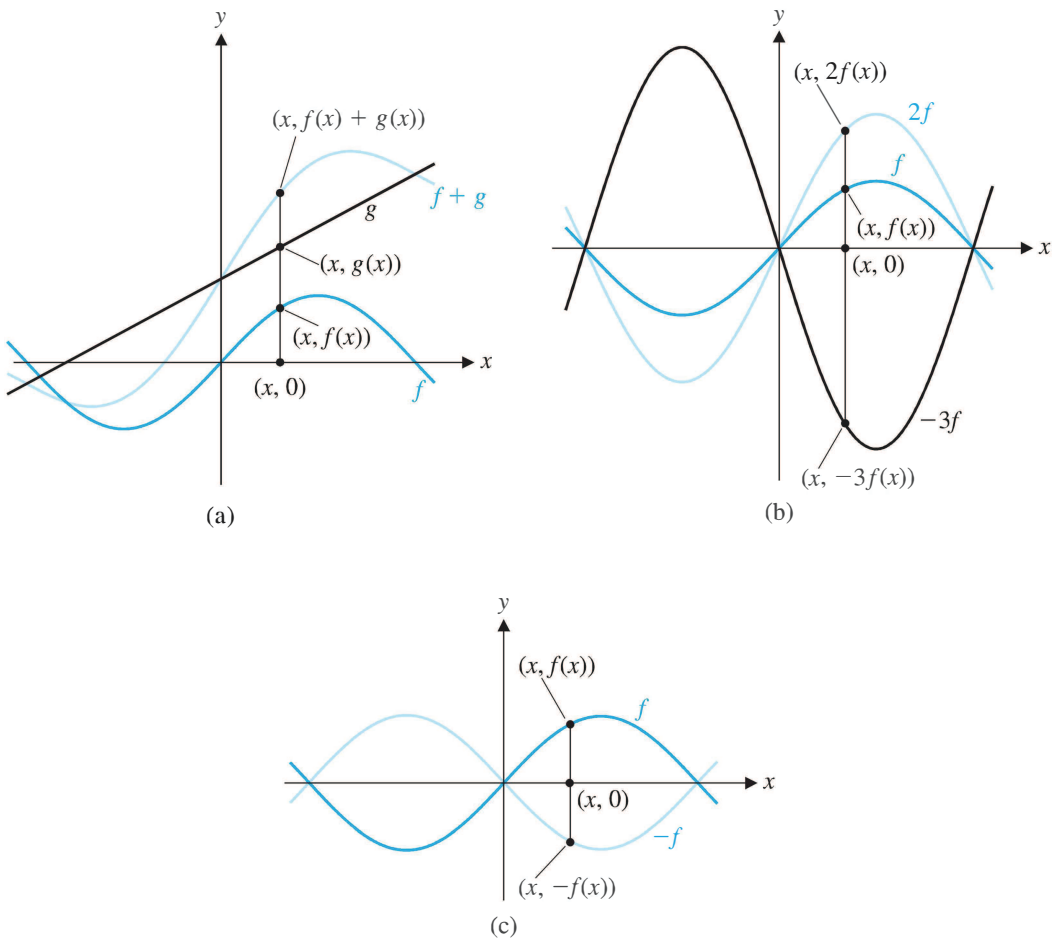

Figure 6.1
The graphs of (a) f, g, and $f + g$, (b) f, $2f$, and $-3f$, and (c) f and $-f$

In Example 6.4, we could also have considered only those functions defined on some *closed interval* $[a, b]$ of the real line. This approach also produces a vector space, denoted by $\mathcal{F}[a, b]$.

Example 6.5

The set $\mathbb{Z}$ of integers with the usual operations is *not* a vector space. To demonstrate this, it is enough to find that *one* of the ten axioms fails and to give a specific instance in which it fails (a *counterexample*). In this case, we find that we do not have closure under scalar multiplication. For example, the multiple of the integer 2 by the scalar $\frac{1}{3}$ is $(\frac{1}{3})(2) = \frac{2}{3}$, which is not an integer. Thus, it is not true that cx is in $\mathbb{Z}$ for *every* x in $\mathbb{Z}$ and *every* scalar c (i.e., axiom 6 fails).

Example 6.6

Let $V = \mathbb{R}^2$ with the usual definition of addition but the following definition of scalar multiplication:

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix}$$

Then, for example,

$$1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so axiom 10 fails. [In fact, the other nine axioms are all true (check this), but we do not need to look into them, because V has already failed to be a vector space. This example shows the value of looking ahead, rather than working through the list of axioms in the order in which they have been given.]

a + bi

Example 6.7

Let $\mathbb{C}^2$ denote the set of all ordered pairs of complex numbers. Define addition and scalar multiplication as in $\mathbb{R}^2$, except here the scalars are complex numbers. For example,

$$\begin{bmatrix} 1 + i \\ 2 - 3i \end{bmatrix} + \begin{bmatrix} -3 + 2i \\ 4 \end{bmatrix} = \begin{bmatrix} -2 + 3i \\ 6 - 3i \end{bmatrix}$$

and

$$(1 - i)\begin{bmatrix} 1 + i \\ 2 - 3i \end{bmatrix} = \begin{bmatrix} (1 - i)(1 + i) \\ (1 - i)(2 - 3i) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 - 5i \end{bmatrix}$$

Using properties of the complex numbers, it is straightforward to check that all ten axioms hold. Therefore, $\mathbb{C}^2$ is a complex vector space.

In general, $\mathbb{C}^n$ is a complex vector space for all $n \geq 1$.

Example 6.8

If p is prime, the set $\mathbb{Z}_p^n$ (with the usual definitions of addition and multiplication by scalars from $\mathbb{Z}_p$) is a vector space over $\mathbb{Z}_p$ for all $n \geq 1$.

Before we consider further examples, we state a theorem that contains some useful properties of vector spaces. It is important to note that, by proving this theorem for vector spaces in *general*, we are actually proving it for *every specific* vector space.

Theorem 6.1 Let V be a vector space, $\mathbf{u}$ a vector in V, and c a scalar.

a. $0\mathbf{u} = \mathbf{0}$
b. $c\mathbf{0} = \mathbf{0}$
c. $(-1)\mathbf{u} = -\mathbf{u}$
d. If $c\mathbf{u} = \mathbf{0}$, then $c = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof We prove properties (b) and (d) and leave the proofs of the remaining properties as exercises.

(b) We have

$$c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$$

by vector space axioms 4 and 7. Adding the negative of $c\mathbf{0}$ to both sides produces

$$c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$$

which implies

$$\begin{aligned} \mathbf{0} &= c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0})) && \text{by axioms 5 and 3} \\ &= c\mathbf{0} + \mathbf{0} && \text{by axiom 5} \\ &= c\mathbf{0} && \text{by axiom 4} \end{aligned}$$

(d) Suppose $c\mathbf{u} = \mathbf{0}$. To show that either $c = 0$ or $\mathbf{u} = \mathbf{0}$, let's assume that $c \neq 0$. (If $c = 0$, there is nothing to prove.) Then, since $c \neq 0$, its reciprocal $1/c$ is defined, and

$$\begin{aligned} \mathbf{u} &= 1\mathbf{u} && \text{by axiom 10} \\ &= \left(\frac{1}{c}c\right)\mathbf{u} \\ &= \frac{1}{c}(c\mathbf{u}) && \text{by axiom 9} \\ &= \frac{1}{c}\mathbf{0} \\ &= \mathbf{0} && \text{by property (b)} \end{aligned}$$

We will write $\mathbf{u} - \mathbf{v}$ for $\mathbf{u} + (-\mathbf{v})$, thereby defining ***subtraction*** of vectors. We will also exploit the associativity property of addition to unambiguously write $\mathbf{u} + \mathbf{v} + \mathbf{w}$ for the sum of three vectors and, more generally,

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$$

for a ***linear combination*** of vectors.

Subspaces

We have seen that, in $\mathbb{R}^n$, it is possible for one vector space to sit inside another one, giving rise to the notion of a subspace. For example, a plane through the origin is a subspace of $\mathbb{R}^3$. We now extend this concept to general vector spaces.

Definition A subset W of a vector space V is called a ***subspace*** of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V.

As in $\mathbb{R}^n$, checking to see whether a subset W of a vector space V is a subspace of V involves testing only two of the ten vector space axioms. We prove this observation as a theorem.

Theorem 6.2 Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

a. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
b. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Proof Assume that W is a subspace of V. Then W satisfies vector space axioms 1 to 10. In particular, axiom 1 is condition (a) and axiom 6 is condition (b).

Conversely, assume that W is a subset of a vector space V, satisfying conditions (a) and (b). By hypothesis, axioms 1 and 6 hold. Axioms 2, 3, 7, 8, 9, and 10 hold in W because they are true for *all* vectors in V and thus are true in particular for those vectors in W. (We say that W *inherits* these properties from V.) This leaves axioms 4 and 5 to be checked.

Since W is nonempty, it contains at least one vector $\mathbf{u}$. Then condition (b) and Theorem 6.1(a) imply that $0\mathbf{u} = \mathbf{0}$ is also in W. This is axiom 4.

If $\mathbf{u}$ is in V, then, by taking $c = -1$ in condition (b), we have that $-\mathbf{u} = (-1)\mathbf{u}$ is also in W, using Theorem 6.1(c).

Remark Since Theorem 6.2 generalizes the notion of a subspace from the context of $\mathbb{R}^n$ to general vector spaces, all of the subspaces of $\mathbb{R}^n$ that we encountered in Chapter 3 are subspaces of $\mathbb{R}^n$ in the current context. In particular, lines and planes through the origin are subspaces of $\mathbb{R}^3$.

Example 6.9 We have already shown that the set $\mathcal{P}_n$ of all polynomials with degree at most n is a vector space. Hence, $\mathcal{P}_n$ is a subspace of the vector space $\mathcal{P}$ of *all* polynomials.

Example 6.10 Let W be the set of symmetric $n \times n$ matrices. Show that W is a subspace of M_{nn}.

Solution Clearly, W is nonempty, so we need only check conditions (a) and (b) in Theorem 6.2. Let A and B be in W and let c be a scalar. Then $A^T = A$ and $B^T = B$, from which it follows that

$$(A + B)^T = A^T + B^T = A + B$$

Therefore, $A + B$ is symmetric and, hence, is in W. Similarly,

$$(cA)^T = cA^T = cA$$

so cA is symmetric and, thus, is in W. We have shown that W is closed under addition and scalar multiplication. Therefore, it is a subspace of M_{nn}, by Theorem 6.2.

Example 6.11

Let $\mathscr{C}$ be the set of all continuous real-valued functions defined on $\mathbb{R}$ and let $\mathscr{D}$ be the set of all differentiable real-valued functions defined on $\mathbb{R}$. Show that $\mathscr{C}$ and $\mathscr{D}$ are subspaces of $\mathscr{F}$, the vector space of all real-valued functions defined on $\mathbb{R}$.

Solution From calculus, we know that if f and g are continuous functions and c is a scalar, then $f + g$ and cf are also continuous. Hence, $\mathscr{C}$ is closed under addition and scalar multiplication and so is a subspace of $\mathscr{F}$. If f and g are differentiable, then so are $f + g$ and cf. Indeed,

$$(f + g)' = f' + g' \quad \text{and} \quad (cf)' = c(f')$$

So $\mathscr{D}$ is also closed under addition and scalar multiplication, making it a subspace of $\mathscr{F}$.

It is a theorem of calculus that every differentiable function is continuous. Consequently, $\mathscr{D}$ is contained in $\mathscr{C}$ (denoted by $\mathscr{D} \subset \mathscr{C}$), making $\mathscr{D}$ a subspace of $\mathscr{C}$. It is also the case that every polynomial function is differentiable, so $\mathscr{P} \subset \mathscr{D}$, and thus $\mathscr{P}$ is a subspace of $\mathscr{D}$. We therefore have a *hierarchy* of subspaces of $\mathscr{F}$, one inside the other:

$$\mathscr{P} \subset \mathscr{D} \subset \mathscr{C} \subset \mathscr{F}$$

This hierarchy is depicted in Figure 6.2.

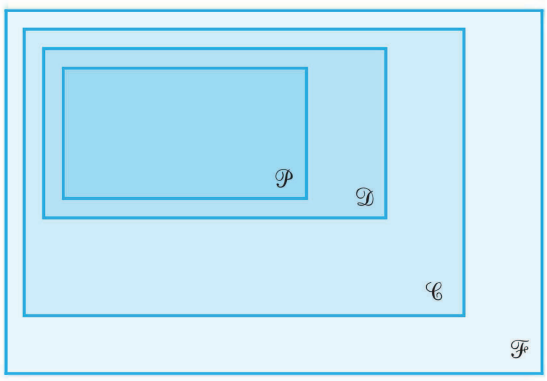

Figure 6.2
The hierarchy of subspaces of $\mathscr{F}$

There are other subspaces of $\mathscr{F}$ that can be placed into this hierarchy. Some of these are explored in the exercises.

In the preceding discussion, we could have restricted our attention to functions defined on a closed interval $[a, b]$. Then the corresponding subspaces of $\mathscr{F}[a, b]$ would be $\mathscr{C}[a, b]$, $\mathscr{D}[a, b]$, and $\mathscr{P}[a, b]$.

Example 6.12

Let S be the set of all functions that satisfy the differential equation

$$f'' + f = 0 \tag{1}$$

That is, S is the solution set of Equation (1). Show that S is a subspace of $\mathscr{F}$.

Solution S is nonempty, since the zero function clearly satisfies Equation (1). Let f and g be in S and let c be a scalar. Then

$$\begin{aligned}(f+g)'' + (f+g) &= (f'' + g'') + (f+g)\\ &= (f'' + f) + (g'' + g)\\ &= 0 + 0\\ &= 0\end{aligned}$$

which shows that $f + g$ is in S. Similarly,

$$\begin{aligned}(cf)'' + cf &= cf'' + cf\\ &= c(f'' + f)\\ &= c0\\ &= 0\end{aligned}$$

so cf is also in S.

Therefore, S is closed under addition and scalar multiplication and is a subspace of $\mathcal{F}$.

The differential Equation (1) is an example of a ***homogeneous linear differential equation.*** The solution sets of such equations are always subspaces of $\mathcal{F}$. Note that in Example 6.12 we did not actually solve Equation (1) (i.e., we did not find any specific solutions, other than the zero function). We will discuss techniques for finding solutions to this type of equation in Section 6.7.

As you gain experience working with vector spaces and subspaces, you will notice that certain examples tend to resemble one another. For example, consider the vector spaces $\mathbb{R}^4$, $\mathcal{P}_3$, and M_{22}. Typical elements of these vector spaces are, respectively,

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad p(x) = a + bx + cx^2 + dx^3, \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In the words of Yogi Berra, "It's déjà vu all over again."

Any calculations involving the vector space operations of addition and scalar multiplication are essentially the same in all three settings. To highlight the similarities, in the next example we will perform the necessary steps in the three vector spaces side by side.

Example 6.13

(a) Show that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

(b) Show that the set W of all polynomials of the form $a + bx - bx^2 + ax^3$ is a subspace of $\mathcal{P}_3$.

(c) Show that the set W of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of M_{22}.

Solution

(a) W is nonempty because it contains the zero vector $\mathbf{0}$. (Take $a = b = 0$.) Let $\mathbf{u}$ and $\mathbf{v}$ be in W—say,

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + c \\ b + d \\ -b - d \\ a + c \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \\ -(b + d) \\ a + c \end{bmatrix}$$

so $\mathbf{u} + \mathbf{v}$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$k\mathbf{u} = \begin{bmatrix} ka \\ kb \\ -kb \\ ka \end{bmatrix}$$

so $k\mathbf{u}$ is in W.

Thus, W is a nonempty subset of $\mathbb{R}^4$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 6.2.

(b) W is nonempty because it contains the zero polynomial. (Take $a = b = 0$.) Let $p(x)$ and $q(x)$ be in W—say,

$$p(x) = a + bx - bx^2 + ax^3$$

and

$$q(x) = c + dx - dx^2 + cx^3$$

Then

$$\begin{aligned} p(x) + q(x) = (a + c) &\\ + (b + d)x &\\ - (b + d)x^2 &\\ + (a + c)x^3 & \end{aligned}$$

so $p(x) + q(x)$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kp(x) = ka + kbx - kbx^2 + kax^3$$

so $kp(x)$ is in W.

Thus, W is a nonempty subset of $\mathcal{P}_3$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathcal{P}_3$ by Theorem 6.2.

(c) W is nonempty because it contains the zero matrix O. (Take $a = b = 0$.) Let A and B be in W—say,

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and

$$B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a + c & b + d \\ -(b + d) & a + c \end{bmatrix}$$

so $A + B$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kA = \begin{bmatrix} ka & kb \\ -kb & ka \end{bmatrix}$$

so kA is in W.

Thus, W is a nonempty subset of M_{22} that is closed under addition and scalar multiplication. Therefore, W is a subspace of M_{22}, by Theorem 6.2.

Example 6.13 shows that it is often possible to relate examples that, on the surface, appear to have nothing in common. Consequently, we can apply our knowledge of $\mathbb{R}^n$ to polynomials, matrices, and other examples. We will encounter this idea several times in this chapter and will make it precise in Section 6.5.

Example 6.14

If V is a vector space, then V is clearly a subspace of itself. The set $\{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace of V, called the ***zero subspace.*** To show this, we simply note that the two closure conditions of Theorem 6.2 are satisfied:

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad c\mathbf{0} = \mathbf{0} \quad \text{for any scalar } c$$

The subspaces $\{\mathbf{0}\}$ and V are called the ***trivial subspaces*** of V.

An examination of the proof of Theorem 6.2 reveals the following useful fact:

If W is a subspace of a vector space V, then W contains the zero vector $\mathbf{0}$ of V.

This fact is consistent with, and analogous to, the fact that lines and planes are subspaces of $\mathbb{R}^3$ if and only if they contain the origin. The requirement that every subspace must contain $\mathbf{0}$ is sometimes useful in showing that a set is *not* a subspace.

Example 6.15

Let W be the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix}$$

Is W a subspace of M_{22}?

Solution Each matrix in W has the property that its $(1, 2)$ entry is one more than its $(1, 1)$ entry. Since the zero matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

does not have this property, it is not in W. Hence, W is not a subspace of M_{22}.

Example 6.16

Let W be the set of all 2×2 matrices with determinant equal to 0. Is W a subspace of M_{22}? (Since $\det O = 0$, the zero matrix is in W, so the method of Example 6.15 is of no use to us.)

Solution Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $\det A = \det B = 0$, so A and B are in W. But

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\det(A + B) = 1 \neq 0$, and therefore $A + B$ is not in W. Thus, W is not closed under addition and so is not a subspace of M_{22}.

Spanning Sets

The notion of a spanning set of vectors carries over easily from $\mathbb{R}^n$ to general vector spaces.

Definition If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the ***span*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\operatorname{span}(S)$. If $V = \operatorname{span}(S)$, then S is called a ***spanning set*** for V and V is said to be ***spanned*** by S.

Example 6.17

Show that the polynomials $1, x$, and x^2 span $\mathcal{P}_2$.

Solution By its very definition, a polynomial $p(x) = a + bx + cx^2$ is a linear combination of $1, x$, and x^2. Therefore, $\mathcal{P}_2 = \text{span}(1, x, x^2)$.

Example 6.17 can clearly be generalized to show that $\mathcal{P}_n = \text{span}(1, x, x^2, \ldots, x^n)$. However, no finite set of polynomials can possibly span $\mathcal{P}$, the vector space of all polynomials. (See Exercise 44 in Section 6.2.) But, if we allow a spanning set to be infinite, then clearly the set of *all* nonnegative powers of x will do. That is, $\mathcal{P} = \text{span}(1, x, x^2, \ldots)$.

Example 6.18

Show that $M_{23} = \text{span}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$, where

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(That is, E_{ij} is the matrix with a 1 in row i, column j and zeros elsewhere.)

Solution We need only observe that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{21}E_{21} + a_{22}E_{22} + a_{23}E_{23}$$

Extending this example, we see that, in general, M_{mn} is spanned by the mn matrices E_{ij}, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Example 6.19

In $\mathcal{P}_2$, determine whether $r(x) = 1 - 4x + 6x^2$ is in $\text{span}(p(x), q(x))$, where

$$p(x) = 1 - x + x^2 \quad \text{and} \quad q(x) = 2 + x - 3x^2$$

Solution We are looking for scalars c and d such that $cp(x) + dq(x) = r(x)$. This means that

$$c(1 - x + x^2) + d(2 + x - 3x^2) = 1 - 4x + 6x^2$$

Regrouping according powers of x, we have

$$(c + 2d) + (-c + d)x + (c - 3d)x^2 = 1 - 4x + 6x^2$$

Equating the coefficients of like powers of x gives

$$\begin{aligned} c + 2d &= 1 \\ -c + d &= -4 \\ c - 3d &= 6 \end{aligned}$$

which is easily solved to give $c = 3$ and $d = -1$. Therefore, $r(x) = 3p(x) - q(x)$, so $r(x)$ is in $\text{span}(p(x), q(x))$. (Check this.)

Example 6.20

In $\mathcal{F}$, determine whether $\sin 2x$ is in $\text{span}(\sin x, \cos x)$.

Solution We set $c \sin x + d \cos x = \sin 2x$ and try to determine c and d so that this equation is true. Since these are functions, the equation must be true for *all* values of x. Setting $x = 0$, we have

$$c \sin 0 + d \cos 0 = \sin 0 \quad \text{or} \quad c(0) + d(1) = 0$$

from which we see that $d = 0$. Setting $x = \pi/2$, we get

$$c \sin(\pi/2) + d \cos(\pi/2) = \sin(\pi) \quad \text{or} \quad c(1) + d(0) = 0$$

giving $c = 0$. But this implies that $\sin 2x = 0(\sin x) + 0(\cos x) = 0$ for all x, which is absurd, since $\sin 2x$ is not the zero function. We conclude that $\sin 2x$ is not in $\text{span}(\sin x, \cos x)$.

Remark It *is* true that $\sin 2x$ can be written in terms of $\sin x$ and $\cos x$. For example, we have the double angle formula $\sin 2x = 2 \sin x \cos x$. However, this is not a *linear* combination.

Example 6.21

In M_{22}, describe the span of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution Every linear combination of A, B, and C is of the form

$$\begin{aligned} cA + dB + eC &= c\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c + d & c + e \\ c + e & d \end{bmatrix} \end{aligned}$$

This matrix is symmetric, so $\text{span}(A, B, C)$ is contained within the subspace of symmetric 2×2 matrices. In fact, we have equality; that is, *every* symmetric 2×2 matrix is in $\text{span}(A, B, C)$. To show this, we let $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$ be a symmetric 2×2 matrix. Setting

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} c + d & c + e \\ c + e & d \end{bmatrix}$$

and solving for c and d, we find that $c = x - z$, $d = z$, and $e = -x + y + z$. Therefore,

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = (x - z)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-x + y + z)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(Check this.) It follows that $\text{span}(A, B, C)$ is the subspace of symmetric 2×2 matrices.

As was the case in $\mathbb{R}^n$, the span of a set of vectors is always a subspace of the vector space that contains them. The next theorem makes this result precise. It generalizes Theorem 3.19.

Theorem 6.3 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in a vector space V.

a. $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of V.
b. $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is the smallest subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Proof (a) The proof of property (a) is identical to the proof of Theorem 3.19, with $\mathbb{R}^n$ replaced by V.

(b) To establish property (b), we need to show that any subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ also contains $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$. Accordingly, let W be a subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then, since W is closed under addition and scalar multiplication, it contains every linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Therefore, $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is contained in W.

Exercises 6.1

In Exercises 1–11, determine whether the given set, together with the specified operations of addition and scalar multiplication, is a vector space. If it is not, list all of the axioms that fail to hold.

1. The set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} x \\ x \end{bmatrix}$, with the usual vector addition and scalar multiplication

2. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 0, y \geq 0$ (i.e., the first quadrant), with the usual vector addition and scalar multiplication

3. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $xy \geq 0$ (i.e., the union of the first and third quadrants), with the usual vector addition and scalar multiplication

4. The set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq y$, with the usual vector addition and scalar multiplication

5. $\mathbb{R}^2$, with the usual addition but scalar multiplication defined by

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ y \end{bmatrix}$$

6. $\mathbb{R}^2$, with the usual scalar multiplication but addition defined by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \end{bmatrix}$$

7. The set of all positive real numbers, with addition $\oplus$ defined by $x \oplus y = xy$ and scalar multiplication $\odot$ defined by $c \odot x = x^c$

8. The set of all rational numbers, with the usual addition and multiplication

9. The set of all upper triangular 2×2 matrices, with the usual matrix addition and scalar multiplication

10. The set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad = 0$, with the usual matrix addition and scalar multiplication

11. The set of all skew-symmetric $n \times n$ matrices, with the usual matrix addition and scalar multiplication (see page 162).

12. Finish verifying that $\mathcal{P}_2$ is a vector space (see Example 6.3).

13. Finish verifying that $\mathcal{F}$ is a vector space (see Example 6.4).

a + bi *In Exercises 14–17, determine whether the given set, together with the specified operations of addition and scalar multiplication, is a complex vector space. If it is not, list all of the axioms that fail to hold.*

14. The set of all vectors in $\mathbb{C}^2$ of the form $\begin{bmatrix} z \\ \bar{z} \end{bmatrix}$, with the usual vector addition and scalar multiplication

15. The set $M_{mn}(\mathbb{C})$ of all $m \times n$ complex matrices, with the usual matrix addition and scalar multiplication

16. The set $\mathbb{C}^2$, with the usual vector addition but scalar multiplication defined by $c\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{c}z_1 \\ \bar{c}z_2 \end{bmatrix}$

17. $\mathbb{R}^n$, with the usual vector addition and scalar multiplication

In Exercises 18–21, determine whether the given set, together with the specified operations of addition and scalar multiplication, is a vector space over the indicated $\mathbb{Z}_p$. If it is not, list all of the axioms that fail to hold.

18. The set of all vectors in $\mathbb{Z}_2^n$ with an even number of 1s, over $\mathbb{Z}_2$ with the usual vector addition and scalar multiplication

19. The set of all vectors in $\mathbb{Z}_2^n$ with an odd number of 1s, over $\mathbb{Z}_2$ with the usual vector addition and scalar multiplication

20. The set $M_{mn}(\mathbb{Z}_p)$ of all $m \times n$ matrices with entries from $\mathbb{Z}_p$, over $\mathbb{Z}_p$ with the usual matrix addition and scalar multiplication

21. $\mathbb{Z}_6$, over $\mathbb{Z}_3$ with the usual addition and multiplication (Think this one through carefully!)

22. Prove Theorem 6.1(a).

23. Prove Theorem 6.1(c).

In Exercises 24–45, use Theorem 6.2 to determine whether W is a subspace of V.

24. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ 0 \\ a \end{bmatrix}\right\}$

25. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ -a \\ 2a \end{bmatrix}\right\}$

26. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix}\right\}$

27. $V = \mathbb{R}^3$, $W = \left\{\begin{bmatrix} a \\ b \\ |a| \end{bmatrix}\right\}$

28. $V = M_{22}$, $W = \left\{\begin{bmatrix} a & b \\ b & 2a \end{bmatrix}\right\}$

29. $V = M_{22}$, $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad \geq bc\right\}$

30. $V = M_{nn}$, $W = \{A \text{ in } M_{nn} : \det A = 1\}$

31. $V = M_{nn}$, W is the set of diagonal $n \times n$ matrices

32. $V = M_{nn}$, W is the set of idempotent $n \times n$ matrices

33. $V = M_{nn}$, $W = \{A \text{ in } M_{nn} : AB = BA\}$, where B is a given (fixed) matrix

34. $V = \mathcal{P}_2$, $W = \{bx + cx^2\}$

35. $V = \mathcal{P}_2$, $W = \{a + bx + cx^2 : a + b + c = 0\}$

36. $V = \mathcal{P}_2$, $W = \{a + bx + cx^2 : abc = 0\}$

37. $V = \mathcal{P}$, W is the set of all polynomials of degree 3

38. $V = \mathcal{F}$, $W = \{f \text{ in } \mathcal{F} : f(-x) = f(x)\}$

39. $V = \mathcal{F}$, $W = \{f \text{ in } \mathcal{F} : f(-x) = -f(x)\}$

40. $V = \mathcal{F}$, $W = \{f \text{ in } \mathcal{F} : f(0) = 1\}$

41. $V = \mathcal{F}$, $W = \{f \text{ in } \mathcal{F} : f(0) = 0\}$

42. $V = \mathcal{F}$, W is the set of all integrable functions

43. $V = \mathcal{D}$, $W = \{f \text{ in } \mathcal{D} : f'(x) \geq 0 \text{ for all } x\}$

44. $V = \mathcal{F}$, $W = \mathcal{C}^{(2)}$, the set of all functions with continuous second derivatives

45. $V = \mathcal{F}$, $W = \{f \text{ in } \mathcal{F} : \; = \lim_{x \to 0} f(x) = \infty\}$

46. Let V be a vector space with subspaces U and W. Prove that $U \cap W$ is a subspace of V.

47. Let V be a vector space with subspaces U and W. Give an example with $V = \mathbb{R}^2$ to show that $U \cup W$ need not be a subspace of V.

48. Let V be a vector space with subspaces U and W. Define the ***sum of U and W*** to be

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \text{ is in } U, \mathbf{w} \text{ is in } W\}$$

(a) If $V = \mathbb{R}^3$, U is the x-axis, and W is the y-axis, what is $U + W$?

(b) If U and W are subspaces of a vector space V, prove that $U + W$ is a subspace of V.

49. If U and V are vector spaces, define the ***Cartesian product*** of U and V to be

$$U \times V = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \text{ is in } U \text{ and } \mathbf{v} \text{ is in } V\}$$

Prove that $U \times V$ is a vector space.

50. Let W be a subspace of a vector space V. Prove that $\Delta = \{(\mathbf{w}, \mathbf{w}) : \mathbf{w} \text{ is in } W\}$ is a subspace of $V \times V$.

In Exercises 51 and 52, let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. *Determine whether C is in span(A, B).*

51. $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

52. $C = \begin{bmatrix} 3 & -5 \\ 5 & -1 \end{bmatrix}$