4 Test (with a solution) (40)

1. (20) Let W be the subspace spanned by $\mathbf{w}_1, \mathbf{w}_2$. First, calculate the dimensions of W and $W^\perp$. Second, find a basis for $W^\perp$.

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

Solution. Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $W = \mathrm{col}(A)$. Since the columns are linearly independent we see that $\dim(W) = 2$. Using theorem from lecture we conclude that $\dim(W^\perp) = 4 - \dim(W) = 2$. To find a basis of $W^\perp$ we use orthogonality of matrix subspaces. Namely,

$$W^\perp = (\mathrm{col}(A))^\perp = \mathrm{null}(A^T) = \mathrm{null}\left(\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}\right)$$

and hence we calculate $\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$. In other words,

$$x_1 = x_2, \quad x_3 = x_4,$$

and the solutions have the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence we got a basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

□

2. (20) Find the orthogonal decomposition of $\mathbf{v}$ with respect to W.

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad W = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Solution. The orthogonal decomposition of $\mathbf{v}$ with respect to W means to find $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$ in $\mathbb{R}^3$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$. From the proof of orthogonal decomposition theorem we know that

$$\mathbf{w} = \mathrm{proj}_W(\mathbf{v}) \text{ and } \mathbf{w}^\perp = \mathrm{perp}_W(\mathbf{v}).$$

Taking $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as an orthogonal basis of W we calculate

$$\mathbf{w} = \mathrm{proj}_W(\mathbf{v}) = \mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathbf{w}^{\perp} = \mathrm{perp}_W(\mathbf{v}) = \mathbf{v} - \mathrm{proj}_W(\mathbf{v}) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Ps: The result can be easily verified by calculating $\mathbf{w} \cdot \mathbf{w}^{\perp} = 0$!

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