# Midterm exam LA (with a solution) (100)

(1) (10) (a) Explain the geometrical[1] meaning of determinant. (b) Explain the geometrical meaning of eigenvectors.

*Solution.* Determinant of a matrix $A$ represents the factor by which the transformation $T_A$ changes the volume of objects. That is,

$$\text{volume(image of an object under } T) = \det(A) \text{ (volume of object)}$$

An eigenvector $\mathbf{v}$ of a linear transformation $T$ is a vector that is mapped by $T$ into the line spanned by $\mathbf{v}$. That is, the image of the eigenvector $(\mathrm{T}(\mathbf{v}))$ is a scalar multiple of the eigenvector $(\lambda\mathbf{v})$ and the scalar $(\lambda)$ is called the eigenvalue.

□

(2) (40) Find an invertible matrix $P$ and a diagonal matrix $D$ that diagonalize the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix}.$$

*Solution.* Let $\lambda$ denote an eigenvalue of $A$. We then calculate

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ -3 & -\lambda & 3 \\ -1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^2(\lambda - 2) = 0.$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.

When $\lambda = 0$,

$$\begin{bmatrix} 1 & 0 & -1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is eigenvector if $x_1 = x_3$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, the eigenspace $E_0 = \mathrm{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$.

[1] looking at matrices as linear transformations

We obtain the eigenvector $\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ for $\lambda = -2$ in a similar manner. Since there is enough linearly independent eigenvectors we can diagonalize $A$. We put $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ and thus $A$ is diagonalized as

$$P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

□

(3) (30) Let $a, b$ and $c$ be eigenvalues of a matrix $A$ with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

respectively. Compute the 555-th power of $A$.

*Solution.* Since $A$ has three linearly independent eigenvectors we can conclude from theory that $A$ is diagonalizable for any choice of $a, b$ and $c$. Moreover, for $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ we have that

$$P^{-1}AP = D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Therefore, $A = PDP^{-1}$ and

$$A^{555} = (PDP^{-1})^{555} = PD^{555}P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a^{555} & 0 & 0 \\ 0 & b^{555} & 0 \\ 0 & 0 & c^{555} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} a^{555} + b^{555} & 0 & a^{555} - b^{555} \\ 0 & 2c^{555} & 0 \\ a^{555} - b^{555} & 0 & a^{555} + b^{555} \end{bmatrix}.$$

□

(4) (10) Prove that if $A$ is a diagonalizable matrix with only one eigenvalue $\lambda$, then $A$ is of the form $A = \lambda I$. (Such a matrix is called a **scalar matrix**.)

*Solution.* If $A$ is diagonalizable with only one eigenvalue $\lambda$, then there exists an invertible matrix $P$ such that

$$P^{-1}AP = \lambda I$$

and thus we see that

$$A = P(\lambda I)P^{-1} = \lambda I.$$

□

(5) (10) Let $A$ be similar to $B$. Show that $\text{rank}(A) = \text{rank}(B)$.

(Hint: Show first the following statement: If $U$ is invertible, then rank(UV) $= \text{rank}(V)$. To do so, consider the relation of the null spaces of $U$ and $UV$.)

*Solution.* Let $U$ be an invertible matrix. We first compare the nullspaces as suggested.

If $\mathbf{u} \in \text{null}(V)$, then $V\mathbf{u} = \mathbf{0}$ and hence $(UV)\mathbf{u} = U(V\mathbf{u}) = U\mathbf{0} = \mathbf{0}$. We have just showed that

$$\text{null}(V) \subseteq \text{null}(UV).$$

On the other hand, when $\mathbf{u} \in \text{null}(UV)$, then $(UV)\mathbf{u} = \mathbf{0}$ and thanks to the invertibility of $U$ we have that $U^{-1}(UV)\mathbf{u} = U^{-1}\mathbf{0}$ and hence $V\mathbf{u} = \mathbf{0}$. This shows that

$$\text{null}(V) \supseteq \text{null}(UV).$$

We conclude that

$$\text{null}(V) = \text{null}(UV).$$

Using the Rank theorem this implies that

$$\text{rank(UV)} = \text{rank}(V),$$

whenever $U$ is invertible. Furthermore, as $(VU)^T = U^TV^T$ and rank(U) $= \text{rank}(U^T)$ we deduce that

$$\text{rank(VU)} = \text{rank}(V),$$

whenever $U$ is invertible.

We now apply this to $A \sim B$. Since there exists an invertible $P$ such that

$$AP = PB,$$

we can now easily conclude that

$$\text{rank}(A) = \text{rank}(B).$$

□