2 Test (with a solution) (40)

1. (30) Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$. Compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each of the eigenspaces of A, and (d) the algebraic and geometric multiplicity of each eigenvalue.

   Solution. (a)
   $$\det(A - \lambda I) = (3 - \lambda)^3.$$

   (b) The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 3$.

   (c) A basis for the associated eigenspace is
   $$\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right).$$

   (d) The algebraic multiplicity equals to 3 and the geometric multiplicity equals to 2. $\square$

2. (10) If a square matrix A has two equal rows, why must A have 0 as one of its eigenvalues?

   Solution. From the Fundamental theorem of invertible matrices, since the rows of A are linearly dependent, the matrix A is not invertible and $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. In other words $(A - 0I)\mathbf{x} = A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and thus 0 is an eigenvalue of A.

   Another solution: Since A has two equal rows, we know from Theorem 4.3 (c) that its determinant is zero. Thus by Theorem 4.6 the matrix A is not invertible and $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. In other words $(A - 0I)\mathbf{x} = A\mathbf{x} = \mathbf{0}$ has a nontrivial solution and thus 0 is an eigenvalue of A. $\square$