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Complex Variables and Functions

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Contents

Preface for students

You are the creators. These notes are a guide.

The notes will not show you how to solve all the problems that are presented, but they should *enable* you to find solutions, on your own and working together. They will also provide historical and cultural background about the context in which some of these ideas were conceived and developed. You will see that the material you are about to study did not come together fully formed at a single moment in history. It was composed gradually over the course of centuries, with various mathematicians building on the work of others, improving the subject while increasing its breadth and depth.

Mathematics is essentially a human endeavor. Whatever you may believe about the true nature of mathematics—does it exist eternally in a transcendent Platonic realm, or is it contingent upon our shared human consciousness? is math "invented" or "discovered"?—our *experience* of mathematics is temporal, personal, and communal. Like music, mathematics that is encountered only as symbols on a page remains inert. Like music, mathematics must be created in the moment, and it takes time and practice to master each piece. The creation of mathematics takes place in writing, in conversations, in explanations, and most profoundly in our mental construction of its edifices on the basis of reason and observation.

To continue the musical analogy, you might think of these notes like a performer's score. Much is included to direct you towards particular ideas, but much is missing that can only be supplied by you: participation in the creative process that will make those ideas come alive. Moreover, the success of the class will depend on the pursuit of both *individual* excellence and *collective* achievement. Like a musician in an orchestra, you should bring your best work and be prepared to blend it with others' contributions.

In any act of creation, there must be room for experimentation, and thus allowance for mistakes, even failure. A key goal of our community is that we support each other—sharpening each other's thinking but also bolstering each other's confidence—so that we can make failure a *productive* experience. Mistakes are inevitable, and they should not be an obstacle to further progress. It's normal to struggle and be confused as you work through new material. Accepting that means you can keep working even while feeling stuck, until you overcome and reach even greater accomplishments.

These notes are a guide. You are the creators.

What's this subject about anyway?

The shortest possible answer might be "calculus with complex numbers." But that brief phrase hides much of the power and beauty of the subject.

In the complex numbers, arithmetic (addition, subtraction, multiplication, division) and geometry (lengths, angles) are inextricably intertwined. We will begin our study, after a brief review of some properties of real numbers, by examining these connections. Then we will try to develop an understanding of how some common functions behave when their inputs and outputs are allowed to be complex numbers.

Most of the course will be devoted to adapting the notions of calculus to functions of a complex variable. In doing so, we will revisit familiar ideas (such as limits, derivatives, and integrals), but we shall also find a whole new collection of tools and methods at our disposal. Because of the connections between complex operations and geometry, the simple assumption that a complex-valued function has a derivative turns out to have far-reaching consequences.

Starting in the 16th century CE, mathematicians discovered that complex numbers have certain advantages over real numbers. At first, complex numbers were not considered worth studying on their own; they were introduced merely as an aid to solving real cubic equations. As it turns out, however, complex numbers can be used to solve not only cubic and quadratic equations—most famously, the number i is a solution to the equation $x^2 + 1 = 0$, which has no real solutions—they can solve any polynomial equation of any degree! (This fact goes by the name of "the Fundamental Theorem of Algebra".) By 1900, complex numbers had become sufficiently indispensable to solving other kinds of problems where they did not initially appear that the mathematician Paul Painlevé was able to claim that "between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain."

Finally, the study of complex variables is a gateway to a wide variety of topics that range throughout mathematics, physics, and other sciences. These notes will not touch on most of these applications, so we mention a few here: algebraic geometry, analytic number theory, biological analysis, computational geometry, dynamical systems, electromagnetism, non-Euclidean geometry, quantum mechanics, random processes, signal analysis, thermodynamics, ... the list is constantly growing.

Some practical matters

This is not a textbook, at least not in the traditional sense. Although definitions are provided, along with an occasional example or bit of exposition, most of the material is contained in a sequence of tasks for you to complete. You will develop techniques, look for patterns, make conjectures, prove major and minor results, and apply your discoveries. The tasks are sorted into three categories, labeled C, D, and E. The distinctions among these are somewhat arbitrary, but the purpose of the labels is to indicate what sorts of tools I consider appropriate to each task.

(C) — "calculation"
These can be completed by straightforward computation, based on either previous knowledge or material already developed in the class.

(D) — "description"
These may require computation, but more importantly some sort of detailed verbal depiction of a mathematical process or phenomenon should be provided.

(E) — "explanation"
These call for logical justification, usually a formal proof. Take extra care in considering the assumptions and all possible cases.

The label on an task is not any indication of its difficulty; there are hard (C) tasks and easy (E) tasks. Many of the (E) tasks rely exclusively on algebraic manipulation. Some use more advanced proof techniques. Some tasks carry more than one label; in the process of finding a solution, you should be able to recognize which parts expect different types of responses.

You will also find two kinds of footnotes, which are distinguished by the type of label they bear (star or dagger):

*, **, etc. — These provide historical or cultural information.

†, ††, etc. — These provide hints for the tasks.

The footnotes are intended to be helpful, but any or all of them can be safely ignored without compromising the main content of the course.

Statements labeled as "Theorems" are presented without proof, usually because the methods required stray too far from the main course content. They may be applied in solving other tasks without further justification. I have tried to keep these to a minimum, and you should certainly try to prove them if you feel like it. (I'm happy to discuss the details of these statements with you.)

Many of the tasks produce results that are theorems in their own right. Keep an eye out for these! As you're working through the tasks, especially those labeled (E), try to judge the relative importance of each result, and determine how you would label it. When you reach the end of each unit, take time to compile for yourself a summary of the major ideas, definitions, and theorems.

Finally, be prepared to struggle! I have tried to provide tasks that lie within what psychologists call the "zone of proximal development": that region of knowledge beyond what you currently can do on your own, but not so far from what is familiar that progress is impossible. Solutions to many of the tasks may not be immediately evident to you, but I am confident that you can uncover them through persistence and assistance, from either me or your classmates. Such is the way of doing mathematics. I hope you will experience both satisfaction and delight as you uncover the beautiful subject of complex variables and functions.

Preface for instructors

In the study of complex variables, the seemingly disparate fields of geometry, algebra, analysis, and topology come together in spectacular ways. One goal of this course is to illustrate this confluence without becoming tedious or making the prerequisites too heavy. No previous formal exposure to real analysis, abstract algebra, or topology is assumed. Students using these notes are expected to have studied multivariable calculus and to have some familiarity with conventional proofs, however. I have included a brief appendix that reviews some concepts, vocabulary, and notation from logic and set theory.

Although I hope these notes will be useful generally, while composing them I have kept in mind the particular position that a course in complex variables occupies within the math program at Pepperdine. The students in this course are mainly math majors, with a few math minors and physics or engineering majors. The prerequisites are minimal: students must have successfully completed a course in multivariable calculus, and they should either have taken an introduction to proofs or be concurrently enrolled in such a course. Some will have studied real analysis, but many have not. The goal, then, is to create an environment that will challenge students with this diverse set of backgrounds while also establishing something close to a level playing field in which all can work together to their mutual benefit.

Because much of the material itself will be new to all of the students in the class, the main thing to decide is which results the students should be expected to justify, and at what level of rigor and generality. I have chosen, when asking students to provide proofs, to emphasize those areas that illuminate the common background of the students. Thus the tools of calculus are assumed (unlike, say, in a real analysis class, which often seeks to question the fundamental principles that undergird the theory of calculus), but students will revisit certain major ideas of calculus with the goal of deepening their understanding of those ideas while extending them to the complex realm.

These principles are exemplified in the approach to Cauchy's Theorem on contour integrals. When proving this theorem, one can (and often does) simply invoke Green's Theorem. (For a survey of pedagogical approaches to Cauchy's Theorem, see [4].) My suspicion, however, is that most students who have taken multivariable calculus and learned to apply Green's Theorem did not learn, or did not appreciate, how Green's Theorem is derived as a consequence of the Fundamental Theorem of Calculus. Therefore I have structured the proof of Cauchy's Theorem as a sequence of exercises that mimics the proof of Green's Theorem and treats the Cauchy–Riemann equations as a set of integrability conditions. This approach also reflects the generalization to higher dimensions provided by the Poincaré Lemma, which states that every closed differential form is locally exact. (The more general Cauchy–Goursat Theorem, which removes the assumption that the derivative of a holomorphic function be continuous, is discussed in these notes, but it is not proved because the techniques rely on a greater experience with analysis.)

The only form of Green's Theorem that is required is the one in which the region is rectangular, where it can be proved directly; see Task 185. After this case is established, the rest of our proof of Cauchy's Theorem follows the path laid out in the article [15], which I learned about from the book [1]; see Tasks 186, 187, and 189. This method treats Cauchy's Theorem as a corollary of the existence of antiderivatives, which recalls the Fundamental Theorem of Calculus.

Selection and organization of topics

For the most part, the material covered in these notes is standard for an undergraduate course on complex variables. The vast majority of exercises ("tasks") are original. However, a few tasks are borrowed or adapted from exercises or exposition in the references.

Chapters 0 ("The real line") and 1 ("The complex plane") set up the basic arithmetic and geometric properties of $\mathbb{R}$ and $\mathbb{C}$ that are needed. Some of this material may be familiar to some students already, but after completing these two chapters the students should have a fairly common knowledge base regarding operations with complex numbers. (A notable exception is Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, which may be known to some students but is introduced later in the notes.) These two chapters, covering the first 43 tasks, also give students a chance to get settled into the course structure before the material becomes more challenging and unfamiliar.

Chapter 2 ("A few important functions") presents a core collection of functions that illuminate how complex-valued functions behave in ways different from their real-valued analogues. The emphasis is on visualizing and describing complex-valued functions as transformations. While I know this is not the only way to visualize such functions (a beautiful and substantially different approach is described in the article [14], for instance), it fits well with the calculus-spirit of this class, and does not require any specialized software, although it can of course be supported by technology such as graphing calculators.

Chapter 3 introduces a small amount of topology. I believe that topology is the correct language for expressing much of the theory of complex variables, but also that this course is not the place to examine the general notion of topology. The instructor may choose to include some of the standard early exercises in point-set topology, such as the fact that the intersection or union of two open sets is open; I have not. This chapter is focused on what will be needed to state and understand later principles.

The main content is explored in chapters 4 ("Derivatives") through 9 ("Singularities"). These chapters cover complex differentiability, the Cauchy–Riemann equations, basic properties of power series, holomorphic and analytic functions (including why these two notions coincide for complex-valued functions), contour integrals, Cauchy's theorem, integral formulas and other applications of Cauchy's theorem, the residue theorem, Laurent series, and the classification of isolated singularities.

The final chapter ("The Riemann sphere") introduces stereographic projection, limits involving infinity, and Möbius transformations. The main goals of this chapter are to clarify the behavior of holomorphic functions near poles and to provide somewhat less taxing material as the course draws to a close.

One organizational decision that must be made in any treatment of complex variables is whether to study power series or integrals first, after the basic notions of complex differentiability have been covered. I have chosen to start with power series, for three reasons. First, doing so presents an opportunity to expand on the collection of functions introduced in unit 2. The exponential function is revisited and the notation e^z is justified, along with Euler's formula. The complex trigonometric and hyperbolic functions are introduced. Second, because the formal operations on power series are so simple and lead directly to the differentiability of functions defined by power series, it can seem that analytic functions form a "special" class of holomorphic functions. I hope that encountering power series first leads students to properly appreciate the later revelation that all holomorphic functions are analytic! Third, perhaps least importantly, the integral formulas involving higher derivatives are easier to justify when Taylor series have already been introduced.

My goal has been to cover as much of the standard undergraduate material as possible while also avoiding "black boxes" as much as possible. Even though this is not intended to be a "proofs-heavy" course, students should provide at least some justification for each of the major results, not just learn to apply them in examples. The result is sometimes an unevenness of difficulty in the tasks; my hope is that a few students will be drawn to the more challenging tasks.

Notation and vocabulary

In choosing what symbols and terminology to use in this course, sometimes I felt pulled by dueling impulses. On the one hand, I believe students should encounter and use standard forms of expression as much as possible, so that they will be prepared to read outside materials and to converse with other members of the mathematical community. On the other hand, small shifts in expression can highlight for students that different conventions are available and encourage them to think in fresh ways about familiar notions. Like most mathematicians, I also have some idiosyncratic beliefs about what constitutes the "right" expression in certain circumstances. Here are a few places where the instructor should be aware that I have followed my idiosyncratic tendencies:

- I include an axiom of Dedekind completeness to give a proper definition of the real numbers. Not wanting to introduce the notion of least upper bound, I avoid the usual formulation ("every non-empty set that is bounded from above has a supremum") and state an equivalent version (Definition 0.2.3) that nicely captures how Dedekind completeness eliminates the possibility of "holes" in the real line. In [8], a similar formulation is called the "cut property".
- My preference is to define $\mathbb{C}$ as a quadratic extension of $\mathbb{R}$ rather than to start with ordered pairs of real numbers on which addition and multiplication are defined abstractly. In the notes, however, I do not use the language of field extensions, but simply adjoin to $\mathbb{R}$ an element i such that $i^2 + 1 = 0$. I believe this is the fastest way to get to the more useful and interesting properties of complex arithmetic. In my experience, students are much less concerned with where i comes from than professional mathematicians are, and many will have worked with complex numbers before, at least in solving quadratic equations.
- I chose to define limits of functions in terms of sequences rather than the more common δ's and ε's. Sequential limits fit well with the emphasis throughout on functions as transformations, and they more closely reflect the intuition for limits that is developed in calculus. Moreover, most math students will encounter the δ-ε definition of limits in an analysis class, and so the sequential definition provides an alternative focus. In practice, which definition is used makes little difference.
- The need to distinguish between a (real) one-dimensional subset of $\mathbb{C}$ and a parametrization of such a subset is a persistent and sometimes frustrating

one. The same problem arises in multivariable calculus and differential geometry. Fortunately, in the theory of complex variables integrals over subsets of $\mathbb{C}$ that are parameterized by one real variable are commonly called "contour integrals", and I have taken advantage of this vocabulary to distinguish between a "curve" (a mapping from an interval in $\mathbb{R}$ to $\mathbb{C}$) and a "contour" (the image of such a mapping). More accurately, a contour maintains an orientation, but it is indifferent to the particular parametrization.

- I use the term "C^1-holomorphic" to describe a holomorphic function whose derivative is continuous. Of course, one consequence of Cauchy's Theorem is that any holomorphic function is automatically analytic, and thus *a fortiori* C^1-holomorphic, so this terminology is formally unnecessary. However, it provides a useful shorthand since the proof of Cauchy's Theorem we present relies on the continuity of the derivative. The term C^1-holomorphic is employed only in §7.
- I use the French notation of reverse brackets for open intervals, so what is usually written as (a,b) becomes $]a,b[$. This is an example of notation I consider superior to the more standard notation, but others may disagree. If necessary, I can produce a version of these notes that reverts to the more common interval notation.
- I use the symbolic expression $U \overset{\circ}{\subseteq} \mathbb{C}$ to mean "U is an open subset of $\mathbb{C}$". This piece of notation is entirely my own invention; I find it compact, transparent, and just a touch whimsical. I would like to see it more widely adopted, but instructors can certainly dispense with it if they prefer. (On rare occasions I use $U \overset{\circ}{\subset} \mathbb{C}$ when it is clear that $U \neq \mathbb{C}$.)

Here is a somewhat subtle typesetting principle in the notes that will make little difference to most readers: I have tried to reserve the use of *italics* for terms that are being defined and for titles of books or articles. For emphasis, I use <u>underlining</u>. In cases where informal language is being used or the focus is on a word or phrase as a signifier, I use "quotation marks".

Logistical matters

I have taught two classes using these notes, once in Spring 2019 and once in Spring 2021. The structures of these classes were quite different from each other.

- When I taught the class in Spring 2019, I had 21 students enrolled.
  - I wanted to get through about 220 tasks, so one requirement of the class was that each student should present solutions to 10 tasks during the semester, or slightly less than one per week during our 14-week semester. The vast majority of students met this requirement. A few carried out additional presentations after reaching 10 tasks.
  - Students signed up through an online form to present solutions each day. A few minutes before class, if more than one student had signed up for a task, I selected which student would present that solution. I gave priority to students with fewer presentations. If no one had a presentation ready for a certain task, we would either postpone that task or spend part of the class period working in small groups until someone had a solution to share.
  - After a solution was presented, the class discussed the work. I partici-

pated minimally in these discussions.

- When I taught the class in Spring 2021, I had 25 students enrolled. Due to the COVID-19 pandemic, all instruction was done remotely, and students were in different time zones.
  - Because the class enrollment was so large, I arranged the students into teams of 6–7 students that remained together through the entire semester. During class periods, they met in Zoom Breakout Rooms. Students who were unable to attend the class synchronously formed teams that met at different times.
  - The teams were largely self-monitored. The members took turns presenting solutions while I rotated among the breakout rooms. They kept a record of each day's work in a spreadsheet. I could view these spreadsheets to see how quickly each team was progressing, which students had made presentations, and what questions remained after a discussion.
  - Based on my experience with these teams, it is very likely I will use them even for in-person classes in the future when enrollment surpasses 15. The team size of six or seven students allowed everyone to participate actively without placing too much of a work burden on the individuals.

In both classes, the only written homework was to submit a weekly report on the tasks that had been covered the previous week. From the tasks that had been presented, each student selected four—ideally, two that they found most interesting, and two that were the most challenging—and submitted written solutions to these tasks.

I have also shared these notes with many math educators and interested students for individual study. In short, the notes should be flexible enough to work in a variety of settings.

First day activity

On the first day of class in 2019, I used the following activity to get students engaged with the material right away. As described, this activity can also provide a chance to meet classmates.

> Transformations: Students stand along a large number line placed on the floor. Make sure to include both positive and negative numbers. Each student should think of the number where they're standing as "x".
>
> - Everyone moves to $x + 1$. Now discuss: What is the overall effect of this transformation? Does everyone move the same way? What numbers get covered?
> - Return to starting position, then everyone moves to $2x$. Discuss the same questions.
> - Repeat with $2x + 1$, x^2. (This last one will introduce you to a new person!)
>
> Return to desks, start working on Tasks 1–3.

In 2021, I adapted this activity to the remote instructional modality by creating a

Desmos graph that illustrated the transformations and sharing it with the class to generate discussion.

Because the transformational view of functions is likely to be new to many students, and it plays a key role in several parts of the course, I encourage instructors to use a version of the above activity or otherwise to incorporate transformations into the first meeting with a class.

Some background and acknowledgements

I began composing these notes at an IBL workshop run by the Academy of Inquiry Based Learning in July 2018. The workshop instructors were Danielle Champney (Cal Poly San Luis Obispo), Jane Cushman (SUNY Buffalo State), TJ Hitchman (University of Northern Iowa), Matt Jones (CSU Dominguez Hills), Elizabeth Thoren (Pepperdine University), and Xiao Xiao (Utica College). I am grateful for their guidance and help, and for the support of my fellow workshop members.

An anonymous reviewer of these notes provided a detailed set of helpful comments and corrections, for which I am also immensely grateful.

My own first experience teaching an IBL class was as a Visiting Assistant Professor at Smith College, where a set of real analysis notes written by David Cohen (with contributions by Christophe Golé) had long been in use. I especially appreciated two features of those notes. First, they led students through standard material by non-standard routes. Doing so allowed certain technical arguments to be broken up over several exercises, so that students could prove deep results by their own work. Second, they used well-chosen and recurring examples to explore key definitions as well as to establish the conceptual understanding needed to complete more theoretical exercises. I have tried to emulate these features in the present notes. Hopefully this goal will justify some of the quirkier choices I have made.

Finally, I learned complex analysis from two excellent teachers, Ted Vessey at St. Olaf College and Cliff Earle at Cornell University.

Images

All images were created by me using MetaPost, except the level curves in section 4.5 (made by me using the Desmos graphing calculator), the figure in Task 165 (made by Karthik Dondeti, used with permission), the surface in section 9.3 (made by me using Grapher for Mac), and the stereographic map projection in section 10.1 (made by Lars H. Rohwedder, obtained from Wikimedia Commons, used under the Creative Commons Attribution-Share Alike 3.0 Unported license).

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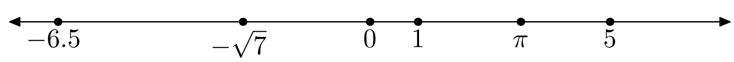
Chapter 0

The real line

The set of real numbers is denoted by $\mathbb{R}$. We will soon discuss exactly what we mean by "the set of real numbers," but for now use your intuition. We also call $\mathbb{R}$ *the real line*, because it can be represented visually as a one-dimensional line:

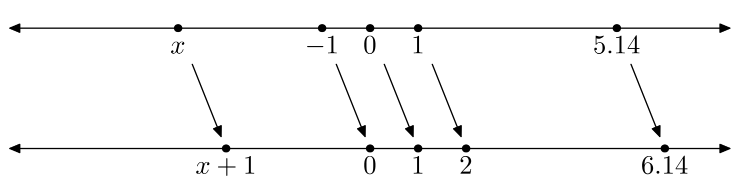


0.1 Functions as transformations

Definition 0.1.1. Let A be a subset of $\mathbb{R}$. A *function* from A to $\mathbb{R}$ is a rule f for assigning, to each number $x \in A$, exactly one number $f(x) \in \mathbb{R}$. The expression $f : A \to \mathbb{R}$ is interpreted to mean "f is a function from A to $\mathbb{R}$." The expression $f : x \mapsto f(x)$ is read "f maps the element x in A to the element $f(x)$ in $\mathbb{R}$," or more simply "f maps x to $f(x)$."[*] We call A the *domain* of f. If $B \subseteq A$, then the set $f(B) = \{f(x) : x \in B\}$ is called the *image* of B by f.

You are probably used to representing a function f by its *graph*, which is the set of points (x, y) in the plane $\mathbb{R}^2$ such that $y = f(x)$. In order to transfer our understanding to the context of complex variables, however, we'll need other ways of visualizing functions.

One method is to think of a function $\mathbb{R} \to \mathbb{R}$ as "moving points from one place to another on the real line." This is a *transformational* view of functions.[**] For example, the function $x \mapsto x + 1$ sends 0 to 1, 1 to 2, -1 to 0, 5.14 to 6.14, and so on.



The overall effect of $x \mapsto x + 1$ is that points are translated one unit to the right.

[*]Notice the difference in usage between the arrows $\to$ and $\mapsto$. The first is used for sets, the second for individual elements.

[**]Indeed, in some contexts functions are called transformations.

Task 1 (D)**.** What is the effect of each of the following functions on points of the real line?

- $x \mapsto 2x$
- $x \mapsto 2x+1$
- $x \mapsto -x$
- $x \mapsto -x+4$
- $x \mapsto -3x+6$

Task 2 (D)**.** What is the effect of each of the following functions on points of the real line?

- $x \mapsto x^2$
- $x \mapsto x^3$
- $x \mapsto 1/x$

Task 3 (D)**.** What is the effect of each of the following functions on points of the real line?

- $x \mapsto 10^x$
- $x \mapsto e^x$
- $x \mapsto e^{-x}$
- $x \mapsto -e^x$
- $x \mapsto -e^{-x}$
- $x \mapsto \sin x$
- $x \mapsto \cos x$

0.2 Definition of real numbers

It does not make sense to define what is a single real number; something can be recognized as a real number only by virtue of how it relates to other real numbers.* Therefore, to be precise about what real numbers are requires that we define the entire system of real numbers. When we craft this definition, we make certain assumptions about how the system behaves—assumptions that will be the foundation for our later reasoning and understanding.

Most of the properties of real numbers that will be listed in this section are probably already familiar to you. We make them explicit for later reference. The collection of properties contained in Definition 0.2.1 apply to the real number system but also to other number systems, so we state them separately.

Definition 0.2.1. A *field* is a set F together with two operations $+$ and $\cdot$, called addition and multiplication, respectively, such that the following are true:

(F1) If $a \in F$ and $b \in F$, then $a+b=b+a$ and $a \cdot b = b \cdot a$. (commutativity)

(F2) If $a,b,c \in F$, then $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associativity)

(F3) If $a,b,c \in F$, then $a \cdot (b+c) = a \cdot b + a \cdot c$. (distributivity)

(F4) F has two distinct elements labeled 0 and 1 such that $a+0=a$ and $a \cdot 1 = a$ for any $a \in F$. (existence of additive and multiplicative identities)

*Possibly lurking around here is a metaphor about what it means to be human, but I don't want to stretch it.

(F5) If $a \in F$, then there exists an element $-a \in F$ such that $a + (-a) = 0$. (existence of additive inverses)

(F6) If $a \in F$ and $a \neq 0$, then there exists an element $a^{-1} \in F$ such that $a \cdot (a^{-1}) = 1$. (existence of multiplicative inverses)

Certain number systems with which you are familiar have some, but not all, of these properties. For example, the set of *natural numbers** $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ has addition and multiplication defined, and with these operations (F1)–(F4) are true for $\mathbb{N}$.

Task 4 (D)**.** Why are properties (F5) and (F6) not true for $\mathbb{N}$?

Task 5 (D)**.**

- Which of the properties (F1)–(F6) are true for the set of *integers* $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$?
- Which of the properties (F1)–(F6) are true for the set of *rational numbers* $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$?

Definition 0.2.2. A field F is *ordered* if it has a relation $>$ such that the following are true:

(OF1) Given any two elements $a, b \in F$, exactly one of these three holds: $a = b$, $a > b$, or $b > a$. (trichotomy)

(OF2) If $a > 0$ and $b > 0$, then $a + b > 0$ and $a \cdot b > 0$. (positive numbers are closed under addition and multiplication)

We also use $a < b$ to mean $b > a$. The expression $a \geq b$ means that either $a > b$ or $a = b$ is true.

Task 6 (D)**.** Is $\mathbb{Q}$ an ordered field? How did you decide?

Definition 0.2.3. An ordered field F is *Dedekind complete* if the following is true:

(DC) Whenever A and B are nonempty subsets of F with the property that $a < b$ for all $a \in A$ and all $b \in B$, there exists $c \in F$ such that $c \geq a$ for all $a \in A$ and $c \leq b$ for all $b \in B$.

The notion of Dedekind completeness is named after Richard Dedekind, who introduced it** as one way to formalize the idea of a line "having no holes." Dedekind completeness allows us to be certain that a sequence of numbers has a limit when some reasonable set of hypotheses is satisfied (such as you learned in calculus). In contrast, the field of rational numbers $\mathbb{Q}$ <u>does</u> have holes, as you will show in the next task.

Task 7 (E)**.** Let A and B be the subsets of $\mathbb{Q}$ defined by

$$A = \{p/q \in \mathbb{Q} : p > 0,\ q > 0,\ p^2 < 2q^2\} \qquad \text{and}$$
$$B = \{p/q \in \mathbb{Q} : p > 0,\ q > 0,\ p^2 > 2q^2\}.$$

Show that A and B are nonempty (each contains at least one element), and that every element of A is less than every element of B. Explain why, nevertheless, there is no number $c \in \mathbb{Q}$ that is greater than or equal to every element of A and less than or equal to every element of B, and thus $\mathbb{Q}$ is <u>not</u> Dedekind complete.

*Some sources do not include 0 as a natural number.

**In an essay entitled *Stetigkeit und irrationale Zahlen*, published in 1872.

We have now introduced everything necessary to say precisely what we mean by "real numbers".

Definition 0.2.4. The *set of real numbers* $\mathbb{R}$ is a Dedekind complete ordered field.*

This definition, loosely speaking, says that everything you know about the real numbers so far (and perhaps a bit more) we will assume to be true. Thus you are free to carry out arithmetic and algebraic operations with real numbers as you always have, and also to compute ordinary limits as in calculus.

Each of the assumptions we have stated—(F1)–(F6), (OF1)–(OF2), and (DC)—is an *axiom* of the real number system. The word "axiom" comes from the Greek word ἄξιος ("axios"), meaning "worthy".** An axiom is a starting assumption that we accept without proof. Indeed, we have not even shown that the real number system, as defined above, is guaranteed to exist! However, for purposes of this course we shall rely on our experience with real numbers and proceed from this set of axioms as a starting point.

Another approach is available: in many analysis courses, the existence of a real number system $\mathbb{R}$ as in Definition 0.2.4 is proved as a theorem, based on other foundational axioms. More than one construction is possible, but all must satisfy our properties. The choice of a suitable set of axioms is context-dependent, and a system that realizes a certain set of axioms is called a *model* of that axiom set. In effect, we are deliberately not choosing a particular model of the real numbers for this course.

0.3 Subsets of $\mathbb{R}$

We have already discussed some important subsets of $\mathbb{R}$. Let us consider their geometric interpretation.

Task 8 (D)**.** How are $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ represented visually as subsets of the real line?

Intervals form another collection of important subsets of $\mathbb{R}$. By convention, $\mathbb{R}$ itself is an interval. There are eight types of intervals that are proper subsets of $\mathbb{R}$, defined by the following eight expressions:

$$\begin{array}{cccc} a \leq x & a < x & x \leq b & x < b \\ a \leq x \leq b & a \leq x < b & a < x \leq b & a < x < b \end{array}$$

The four types of interval on the second row are called *bounded*; all other intervals are *unbounded*. A bounded interval has two endpoints, whereas an unbounded interval has only one (if the interval is not all of $\mathbb{R}$) or zero (if the interval equals $\mathbb{R}$ itself). An interval that contains all of its endpoints (whether it has zero, one,

*You might be bothered by the use of the indefinite article "a" in this definition. You might wonder whether it's possible to have multiple different real number systems. If that question doesn't concern you, then you can ignore the rest of this footnote. If it does, rest assured that there is only one number system $\mathbb{R}$ that satisfies this definition. That is, any two Dedekind complete ordered fields are "the same", in the sense that the elements of one field can be put into one-to-one correspondence with the elements of the other field in a way that preserves all the relevant structures (i.e., addition, multiplication, and order). If you're still nervous, think about how you could start establishing such a correspondence (hint: 0 and 1 are distinguished in both fields).

**"πιστὸς ὁ λόγος καὶ πάσες ἀποδοχης ἄξιος…", "This is a faithful saying and worthy of all acceptance…" (1 Timothy 1:15) or two) is called *closed*. An interval that does not contain any endpoints is called *open*.*

In order to avoid confusion between open intervals and ordered pairs, in these notes we will follow the French notation and use a reverse bracket to indicate when an endpoint is left out of an interval. For example, $[0,1]$ is closed, $]0,1[$ is open, and $[0,1[$ includes 0 but not 1. $\mathbb{R}$ is equal to the interval $]-\infty,\infty[$.

Task 9 (C)**.** What is the image of $]0,1[$ by each of the functions in Tasks 1–3? What about $[2,5]$? Write your answers in interval notation.

*Sometimes a bounded interval that contains only one of its endpoints is called *half open* or *half closed*, but we will not need this notion or terminology. Notice that, as an interval, $\mathbb{R}$ is both closed and open! It is the only interval with this property, however (unless one considers the empty set $\varnothing$ to be an interval, a question on which we will remain agnostic).

Interlude: filling gaps in number systems

As we learn about numbers throughout our lives, we discover at various times that certain number systems seem to be lacking desirable properties, usually with regard to ordinary operations. When working with the set of natural numbers $\mathbb{N}$, for instance, we find that we can add two numbers in either order, but we can only subtract them in one order (unless they are the same, and the difference is zero). By introducing negative numbers, we allow for subtraction in either order, and we obtain the set of integers $\mathbb{Z}$. But then another unsatisfactory situation persists: we can, for instance, divide some numbers by 2 (the evens), but not all (no integer is equal to 3 divided by 2). So we introduce fractions (both positive and negative) and the set of rational numbers $\mathbb{Q}$ appears. With this number system, we have made it possible to add, subtract, multiply, and divide to the fullest extent possible.*

However, when we consider the next simplest operation, powers, we quickly find a deficit once again: it is possible to take integer powers of any non-zero rational number, but fractional powers only exist for certain rational numbers ($4^{1/2}$ makes sense in $\mathbb{Q}$, but $2^{1/2}$ does not). This situation might have persisted through more of human history had it not been realized that the square root of 2 has a clear geometric representation, as the length of the diagonal in a square of side length 1. The motivation for representing lengths led to the introduction of a number system, $\mathbb{R}$, which includes not only a value for $2^{1/2}$, but also $2^{1/3}$, π, e, and a vast array of other irrational numbers whose geometric meaning was, or seemed to be, apparent. The lack of an immediate geometric interpretation meant that mathematicians for centuries were largely unbothered by the fact that only non-negative numbers were assigned square roots.

We will not describe here the historical progression that led mathematicians to deliberately introduce square roots of negative numbers.** Instead, we will simply treat the lack of such square roots in $\mathbb{R}$ as a deficit analogous to the deficits we have noted in other number systems, and in order to fill this gap we will assume the existence of such square roots. More precisely, we introduce a number i such that $i^2 = -1$,*** and consider what kind of number system is created by this new arrival. Like the real numbers we have already studied, the number i does not have an independent existence; it is understood by its relationship to -1 and the knowledge that it cannot be an ordinary real number.

At the same time, we provide the new number system with a geometric interpretation, which aided the full acceptance of this new number system to the mathematical scene. The question of what deficiencies remain, if any, will be addressed later.

*As you saw in section 0.2, $\mathbb{Q}$ is a field. It may seem unsatisfactory that we still cannot divide by 0 in $\mathbb{Q}$. For reasons explained in abstract algebra courses, it is impossible to divide by 0 in any field.

**Many good histories of complex numbers are freely available. They typically start with Cardano's work on solving cubic equations in the 16th century and progress to the full acceptance of "imaginary" numbers in the mathematical community by the mid-19th century.

***In her book entitled $x + y$, Eugenia Cheng writes about this number i, "You might think we just made it up, and moreover, we even acknowledge this by calling it imaginary. But we've made up everything in abstract math, really. We've made up the ordinary numbers 1, 2, 3, and so on—what in fact are those? They're just ideas. So we can make up the idea i too. We can make up any idea as long as it doesn't cause a logical contradiction." That it is logically possible simply to create such a number and adjoin it to the existing real number system is proved in abstract algebra courses.

Chapter 1

The complex plane

1.1 Definition and representation of complex numbers

Definition 1.1.1. The *imaginary unit* is a number i that satisfies the equation $i^2 = -1$.* The *set of complex numbers* is

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\}.$$

If $z = x + yi$, where x and y are real numbers, we call x the *real part* of z and y the *imaginary part* of z, and we write

$$x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

In these notes, x and y will henceforth always represent real numbers.

If either the real or imaginary part of a complex number is zero, we usually omit that term when writing the number: for example, $3i = 0 + 3i$ and $5 = 5 + 0i$. If both the real and imaginary parts of z are zero, we simply write $z = 0$.

To represent complex numbers visually, we take advantage of the fact that each complex number is determined by a pair of real numbers (its real and imaginary parts) and treat those as coordinates in $\mathbb{R}^2$. The horizontal axis is called the *real axis*, and the vertical axis is called the *imaginary axis*.

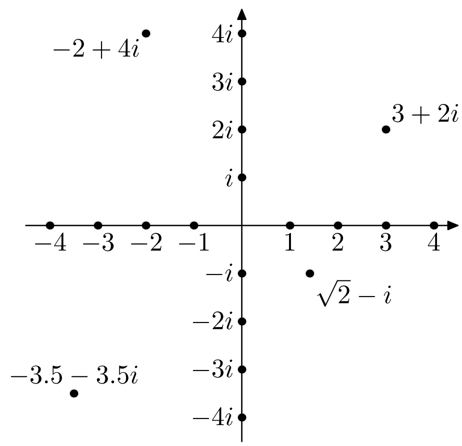
This way of presenting complex numbers is called an *Argand diagram*, after Jean-Robert Argand, an amateur mathematician who devised this geometrical interpretation of complex numbers and wrote a pamphlet** justifying its use. Because each element of $\mathbb{C}$ is determined by two real numbers, the Argand diagram is often called the *complex plane*. However, it is important to think of a complex number as a <u>unified</u> object; each point of the plane is determined by a <u>single</u> complex number.* The word "complex" means, etymologically, "braided together"**—the

*Again that indefinite article "a"! Can there be more than one imaginary unit, then? In this case, the answer is <u>yes</u>: if u is an imaginary unit, so that $u^2 = -1$, then it is also true that $(-u)^2 = -1$ (check this, assuming that multiplication of real numbers with u is commutative), and so $-u$ is also an imaginary unit. There's honestly no way to distinguish between u and $-u$, so we pick one to be "the" imaginary unit, call it i, and stick with that choice forever.

***Essai sur une manière de representer les quantités imaginaires dans les constructions géométriques*, published privately in 1806.

*If you're familiar with linear algebra, this may help: in terms of vector spaces, $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$, but it is a one-dimensional vector space over itself (this statement assumes that $\mathbb{C}$ is a field, which you will later prove).

**The Latin root "-plex" is related to the words "plait" (which means "braid") and "pleat" (which



An Argand diagram.

two (real numbers) have become one (complex number). If this idea seems strange, remember that in a similar way a single rational number is determined by two integers, its numerator and its denominator.

Task 10 (D)**.** Find four complex numbers such that, when placed on an Argand diagram, they are located at the vertices of a square. (Try to pick numbers that you believe will be different from everyone else's.)

1.2 Operations with complex numbers

Definition 1.2.1. If $z = x + yi$ and $w = u + vi$, with $x, y, u, v \in \mathbb{R}$, then the sum and the product of z and w are defined by

$$z + w = (x + u) + (y + v)i \qquad \text{and} \qquad z \cdot w = (xu - yv) + (xv + yu)i.$$

We may also write the product of z and w simply as zw.

Task 11 (E)**.** Show that the expressions for $z + w$ and $z \cdot w$ in the previous definition match the results you get by treating each complex number as a binomial (with i as an indeterminate, like "X") and adding or multiplying them according to the rules of polynomials, replacing i^2 with -1 wherever it appears.

Task 12 (C)**.** For each pair of complex numbers z and w given below, compute $z + w$ and zw. Then plot all four numbers z, w, $z + w$, and zw on an Argand diagram.

- $z = 4,\ w = i$
- $z = 3 + 5i,\ w = -i$
- $z = 1 + i,\ w = -1 - i$
- $z = -1 + i\sqrt{3},\ w = -1 - i\sqrt{3}$ (see footnote* below)

means "fold").

*Here I have made the aesthetic decision to write z and w in the form $x + iy$ instead of $x + yi$. Both forms are acceptable; there are no firm rules about when to use one form or the other. As always, notation should strive for clarity. In this case, the presence of the square root symbol suggests that it's better to put i in front of its coefficient. It is harder to read $\sqrt{3}i$ than $i\sqrt{3}$.

- $z = 2 + i,\ w = 3 + i$

Task 13 (E). Suppose $z, w \in \mathbb{C}$. Show that $z + w = w + z$ and $z \cdot w = w \cdot z$.

Task 14 (E). Suppose $z_1, z_2, w \in \mathbb{C}$. Show that $w(z_1 + z_2) = wz_1 + wz_2$.

Task 15 (CD). Using the usual interpretation of natural number powers ($z^2 = zz$, $z^3 = zzz$, and so on), compute $(1+i)^2$, $(1+i)^3$, and $(1+i)^4$. Plot $(1+i)^n$ for $n = 1,2,3,4$ on an Argand diagram. Do the same for $(1 - i\sqrt{3})^n$, $n = 1,2,3,4$. What do you notice?

Task 16 (CD). Consider the sequence of powers of the imaginary unit: i, i^2, i^3, and so on. Simplify the first few of these using the equation $i^2 = -1$. What pattern do you notice? Formulate a general rule for finding powers of i. Use it to compute i^{99}, i^{1001}, and i^{57368}.

Task 17 (E). Suppose that $z = x + yi$ is a nonzero complex number. Define

$$z^{-1} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}\, i.$$

Show that $z \cdot z^{-1} = 1$.

We define division in $\mathbb{C}$ by $w/z = wz^{-1}$. In particular, $1/z = z^{-1}$.

Task 18 (C). Write the following numbers in the form $x + yi$.

$$i^{-1}, \qquad (3+4i)^{-1}, \qquad \left(\frac{1}{2} - \frac{1}{2}i\right)^{-1}, \qquad \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-1}$$

Task 19 (E). Show that $\mathbb{C}$ is a field.†

Task 20 (C). Solve the following equations in $\mathbb{C}$. Write the solutions in the form $x + yi$.

- $4z + 8i = 0$
- $4iz + 8i = 0$
- $4iz + 8 = 0$
- $(1 - i)z - 4 = 0$
- $z^2 + 4 = 0$

Task 21 (E). Show that $\mathbb{C}$ cannot be an ordered field. That is, there is no relation $>$ on $\mathbb{C}$ that satisfies both properties (OF1) and (OF2).†

Definition 1.2.2. If $z = x + yi$, with $x, y \in \mathbb{R}$, then the *complex conjugate* of z is $\overline{z} = \overline{x + yi} = x - yi$.

†Which of the properties (F1)–(F6) have already been shown to be true in earlier tasks? What remains to be shown?

†The properties of an ordered field imply that either $a > 0$ or $-a > 0$ for all nonzero elements a of the field. What happens if you apply these two possibilities to i?

Task 22 (C). Write the following numbers in the form $x + yi$.

- $\overline{5 + 3i}$
- $\overline{-5 - 3i}$
- $3i(\overline{1 + 2i})$
- $\overline{3i}(1 + 2i)$
- $\overline{i^{99}}$

Task 24 (D). Describe what geometric shape, as a subset of $\mathbb{C}$, is defined by each of these equations.[††]

- $z = \overline{z}$
- $z = i\overline{z}$
- $z + \overline{z} = 1$
- $z^2 + \overline{z}^2 = 2$
- $z^{-1} = \overline{z}$

1.3 Modulus and distance

Task 25 (DE). Show that $z\overline{z} \geq 0$ for all $z \in \mathbb{C}$.[†††] What geometric quantity does $\sqrt{z\overline{z}}$ represent?[*]

Definition 1.3.1. The *modulus*, or *absolute value*, of a complex number $z \in \mathbb{C}$ is $|z| = \sqrt{z\overline{z}}$.

Task 26 (C). Determine the modulus of each of the following numbers.

- $z = i$
- $z = 1 + i$
- $z = -3$
- $z = 3 - i\sqrt{3}$
- $z = -5 + 12i$

Task 27 (E). Show that if $z \neq 0$, then $z^{-1} = \overline{z}/|z|^2$.[**]

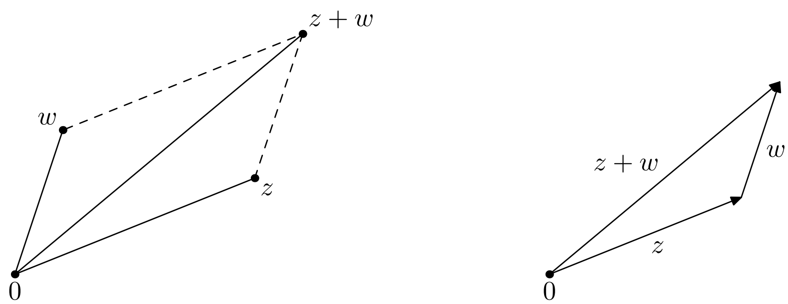
[††]One approach is to write each equation in terms of the real and imaginary parts of z.

[†††]You should first explain why $z\overline{z}$ must be in $\mathbb{R}$, because as you showed in Task 21, the relation $\geq$ does not make sense for general complex numbers.

[*]By convention, if $a > 0$, then the symbol $\sqrt{a}$ designates the positive square root of a. The negative square root of a is then $-\sqrt{a}$. Both $\sqrt{a}$ and $-\sqrt{a}$ are solutions to the equation $x^2 = a$.

[**]Notice the similarity to "rationalizing the denominator." In effect, we find $1/z$ by multiplying the numerator and denominator by the (complex) conjugate of z.

The way we add complex numbers, by adding their real parts and imaginary parts separately, should be familiar to you as vector addition in $\mathbb{R}^2$, in which we add corresponding components. You may recall that there are two common ways of representing vector addition visually, both illustrated in the next figure.

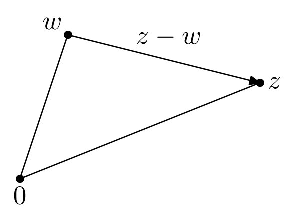


Task 28 (E). Show that the following inequalities are true.

- $|\mathrm{Re}\, z| \leq |z|$
- $|\mathrm{Im}\, z| \leq |z|$
- $\mathrm{Re}(z\overline{w}) \leq |z||w|$
- $|z+w| \leq |z| + |w|$

The last result of Task 28 is called the *Triangle Inequality*.

Definition 1.3.2. The *distance* between two complex numbers z and w is $|z-w|$.



Task 29 (DE). Show the general triangle inequality: if $a, b, c \in \mathbb{C}$, then

$$|a-c| \leq |a-b| + |b-c|.$$

Draw a picture that illustrates what this inequality means. When does equality hold?

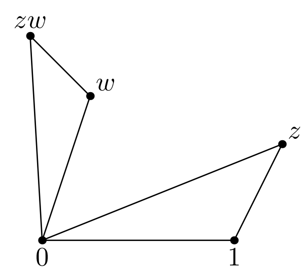
Task 30 (E). Show that if $z_1, z_2, \ldots, z_n$ is any finite collection of complex numbers, then[†]

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

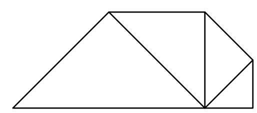
Task 31 (E). Suppose $z, w \in \mathbb{C}$. Show that $|z \cdot w| = |z| \cdot |w|$.

Task 32 (E). Suppose $z, w \in \mathbb{C}$. Let T_1 be the triangle with vertices 0, 1, and z, and let T_2 be the triangle with vertices 0, w, and zw. Show that T_1 and T_2 are similar triangles. (See illustration below.)

[†]If you are familiar with proof by induction, you may find it useful here.



Task 33 (D)**.** Using the triangle visualization of multiplication from Task 32, explain how the figure below relates to the sequence $(1+i)^n$, $1 \le n \le 4$. Draw the corresponding picture for $(1-i\sqrt{3})^n$, $1 \le n \le 4$.



Use these pictures to explain your observations from Task 15.

Task 34 (D)**.** Describe what geometric shape is defined by each of these equations.

- $|z| = 1$
- $|z-1| = 2$
- $|z+4i| = 3$
- $|z| = |z-1|$
- $|z-i| = |z+i|$
- $|z-2| = \operatorname{Re} z$

1.4 Arguments and polar form

Task 35 (DE)**.** Show that if z is a nonzero complex number, then $z/|z|$ has modulus 1. Describe geometrically how $z/|z|$ relates to z.

Task 36 (E)**.** Show that, if θ is any real number, then $\cos\theta + i\sin\theta$ has modulus 1.

Definition 1.4.1. An *argument* of a nonzero complex number z, written $\arg z$, is an angle $\theta \in \mathbb{R}$ chosen so that $\cos\theta + i\sin\theta = z/|z|$. The *principal argument* of z, written $\operatorname{Arg} z$, is the argument that lies in $]-\pi, \pi]$.*

As the above definition makes clear, for any $z \neq 0$ the expression $\arg z$ has infinitely many different values, any two of which differ by an integer multiple of 2π. When we write an equation like $\theta = \arg z$, it is understood that θ can be any of these values.

Task 37 (C)**.** Find the principal argument and one other argument for each number in Task 26.

Definition 1.4.2. The *polar form* of a complex number $z \neq 0$ is $r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.

*In case it's not clear, we will always measure angles in radians.

Task 38 (C). Write each of the numbers from Task 26 in polar form.

Task 39 (DE). Find general formulas that convert between the rectangular form $x + yi$ and the polar form $r(\cos\theta + i\sin\theta)$ of a complex number, and explain why they work.

From trigonometry, we have the *angle sum formulas*: if $\alpha, \beta \in \mathbb{R}$, then

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Task 40 (E). Use the angle sum formulas to show that for all $\alpha, \beta \in \mathbb{R}$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha + \beta) + i\sin(\alpha + \beta).$$

Task 41 (D, Optional). How could the formula in Task 40 have been derived from the result of Task 32?

Task 42 (E). Show that *de Moivre's formula*** is true: if $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$, then†

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Task 43 (C). Find $(1 + i)^n$ and $(1 - i\sqrt{3})^n$ for $n = 10, 25, 99$. Write your answers in the form $x + yi$.

**Named after Abraham de Moivre, who published it in a 1722 paper entitled *De sectione anguli*. (More precisely, he published a system of equations that contain the formula implicitly; Euler was the first to write the formula as it appears in this task.)

†This is another occasion where proof by induction may be useful.

Chapter 2

A few important functions

2.1 Complex-valued functions of a complex variable

We define functions of a complex variable just as we do for a real variable.

Definition 2.1.1. Let A be a subset of $\mathbb{C}$. A *function* from A to $\mathbb{C}$ is a rule f that assigns, to each number $z \in A$, exactly one number $f(z) \in \mathbb{C}$. A is called the *domain* of f. If $B \subseteq A$, then the set $f(B) = \{f(z) : z \in B\}$ is called the *image* of B by f.

Before delving into the theory of complex functions, we will spend some time studying several important examples. Consider each of these functions from a transformational perspective, as you did with the real-valued functions in Tasks 1–3. A graphing program such as Desmos may be helpful.

2.2 Affine functions

Definition 2.2.1. An *affine function** has the form $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Task 44 (D)**.** What is the effect of each of these functions on points of the complex plane?

- $z \mapsto z + i$
- $z \mapsto iz$
- $z \mapsto iz + 4$
- $z \mapsto (1+i)z$
- $z \mapsto 3z + 6i$
- $z \mapsto 2 + 4i - z$

Here are some possible kinds of answers: "translation in such-and-such a direction by such-and-such an amount," "rotation around such-and-such a point by such-and-such an angle," "scaling distances from such-and-such a point by such-and-such an amount." You might think about the following questions:

- Do all points of $\mathbb{C}$ move in the same way?
- Are there any points that do not move? (These are called *fixed points*.)

*In early math classes, it is common to call this kind of function "linear," because its graph is a line. However, in more advanced math the adjective "linear" has a narrower meaning. In this narrower sense, a function $f : \mathbb{C} \to \mathbb{C}$ is linear if $f(z + w) = f(z) + f(w)$ and $f(\lambda z) = \lambda f(z)$ for all $z, w, \lambda \in \mathbb{C}$. Equivalently, a linear function is an affine function with $b = 0$.

It might also be useful to determine how a specific figure (such as the unit square, having vertices 0, 1, i, and $1+i$) is transformed by the function.

Task 45 (D)**.** Return to the four points you chose in Task 10.

- Find an affine function that rotates your square by $\pi/2$ around its center.
- Find an affine function that rotates your square by π around one of its vertices.

Definition 2.2.2. Suppose A and B are sets and $f : A \to B$ is a one-to-one and onto function. Then f has an *inverse*, which is the function $f^{-1} : B \to A$ such that $f^{-1}(f(z)) = z$ for all $z \in A$ and $f(f^{-1}(w)) = w$ for all $w \in B$.

Task 46 (E)**.** Show that if $f(z) = az + b$ is an affine function, then f has an inverse if and only if $a \neq 0$.

Task 47 (C)**.** Find the inverse of each function in Task 44. Describe its effect on $\mathbb{C}$.†

2.3 Complex conjugation

Task 48 (D)**.** What is the effect of each of these functions on points of the complex plane?

- $z \mapsto \bar{z}$
- $z \mapsto i\bar{z}$
- $z \mapsto \bar{z} + 2i$
- $z \mapsto -\bar{z} + 4$
- $z \mapsto \bar{z} + 1$

Task 49 (E)**.** Show that $f(z) = a\bar{z} + b$ has an inverse if and only if $a \neq 0$.

Task 50 (C)**.** Find the inverse of each function in Task 48. Describe its effect on $\mathbb{C}$.

2.4 The function $1/z$ and circle inversion

Task 51 (DE)**.**

- Show that $|1/z| = 1/|z|$. What does this equation mean geometrically?
- Show that $\arg(1/z) = -\arg z$.† What does this equation mean geometrically?
- What is the effect of $z \mapsto 1/z$ on points of the unit circle $|z| = 1$?
- What is the effect of $z \mapsto 1/z$ on points inside the unit circle (excluding 0)?
- What is the effect of $z \mapsto 1/z$ on points outside the unit circle?

Task 52 (E)**.** The domain of $f : z \mapsto 1/z$ is $\mathbb{C} \setminus \{0\}$. Show that f is one-to-one, and find f^{-1}.

When we combine the function $1/z$ with complex conjugation, we get a kind of transformation called *circle inversion*. The next task asks you to describe this transformation.

†See Task 44 for some suggestions about what kinds of description might apply.

†That is, if θ is any argument of z, then $-\theta$ is an argument of $1/z$.

Task 53 (D)**.**

- Show that $\arg(1/\bar{z}) = \arg z$. What does this equation mean geometrically?
- Show that $z = 1/\bar{z}$ if and only if z is on the unit circle.
- Show that $z \mapsto 1/\bar{z}$ is its own inverse.
- Describe the function $z \mapsto 1/\bar{z}$ as a transformation of $\mathbb{C} \setminus \{0\}$.

2.5 Squaring

It is worth considering this function in both its rectangular and polar forms.

Task 54 (D)**.**

- Express the real and imaginary parts of z^2 in terms of the real and imaginary parts of z.
- What is the effect of $z \mapsto z^2$ on vertical lines, of the form $\operatorname{Re} z = \text{constant}$?
- What is the effect of $z \mapsto z^2$ on horizontal lines, of the form $\operatorname{Im} z = \text{constant}$?

Task 55 (D)**.**

- Express the modulus and argument of z^2 in terms of the modulus and argument of z.
- What is the effect of $z \mapsto z^2$ on circles centered at the origin?
- What is the effect of $z \mapsto z^2$ on lines through the origin?

Task 56 (D)**.** Using your work from the previous two tasks, give as complete a description as you can of the function $z \mapsto z^2$ as a transformation.

Task 57 (D)**.** Let R_+ be the *right half-plane*

$$R_+ = \{z : \operatorname{Re} z > 0\}.$$

- Show that the image of R_+ by $z \mapsto z^2$ covers all of $\mathbb{C}$ except 0 and the negative real axis.
- Find another subset of $\mathbb{C}$, different from R_+, whose image by $z \mapsto z^2$ also covers all of $\mathbb{C}$ except 0 and the negative real axis.

2.6 Square roots

Task 58 (DE)**.** Show that if a is any complex number besides 0, then there are two numbers $z \in \mathbb{C}$ such that $z^2 = a$.[†] How are these two square roots of a related to each other algebraically and geometrically?

Task 59 (C)**.** What are the square roots of i? Write them in the form $x + yi$.

Definition 2.6.1. Given $z \in \mathbb{C} \setminus \{0\}$, set $r = |z|$ and $\theta = \operatorname{Arg} z$. Then the *principal square root* of z is $\sqrt{z} = \sqrt{r}\big(\cos(\theta/2) + i\sin(\theta/2)\big)$ where, as usual, $\sqrt{r}$ denotes the positive square root of r. Also $\sqrt{0} = 0$.

[†] Is this easier to show using rectangular or polar coordinates?

Task 60 (D)**.** Explain why the convention of Definition 2.6.1 means that we have firmly decided $\sqrt{-1} = i$, not $-i$. (Note that this is not the same as saying that $-i$ is not a square root of -1, only that it is not the principal square root.)

Task 61 (D)**.** What is the effect of the function $z \mapsto \sqrt{z}$ on points of $\mathbb{C}$?

Task 62 (E)**.** Show that if a, b, c are any complex numbers with $a \neq 0$, and $\sqrt{b^2 - 4ac}$ is the principal square root of $b^2 - 4ac$, then

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad z = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are the solutions to $az^2 + bz + c = 0$.

Task 63 (C)**.** Solve the following equations in $\mathbb{C}$.

- $z^2 + z + 1 = 0$
- $z^2 - 2z + 2 = 0$
- $z^2 + iz + 1 = 0$
- $z^2 + 2iz - 1 = 0$
- $z^2 - (1 + i)z + i = 0$

2.7 Other powers and roots

Task 64 (C)**.**

- Find three numbers z such that $z^3 = 1$.
- Find three numbers z such that $z^3 = -1$.
- Find three numbers z such that $z^3 = 8$.
- Find three numbers z such that $z^3 = 8i$.

Task 65 (D)**.** Fix $n \geq 3$. Express the modulus and argument of z^n in terms of the modulus and argument of z. What is the effect of the function $z \mapsto z^n$ on $\mathbb{C}$?

Task 66 (CD)**.** The solutions to $z^n = 1$ are called *nth roots of unity*. Plot on separate Argand diagrams the nth roots of unity for $n = 2, 3, 4, 5, 6$.

Task 67 (D)**.** Find a subset U of $\mathbb{C}$ whose image by the function $z \mapsto z^n$ covers all of $\mathbb{C}$ except 0 and the negative real axis.

Task 68 (DE)**.** Define a *principal nth root* function $z \mapsto \sqrt[n]{z}$ on $\mathbb{C}$ for $n \geq 3$. Explain why all the nth roots of z can be found by multiplying $\sqrt[n]{z}$ by the nth roots of unity. When $z \neq 0$, what geometric figure is formed by the nth roots of z?

Task 69 (C)**.** Solve the following equations in $\mathbb{C}$.

- $z^4 = -4$
- $z^4 = 4 - 4i$
- $z^6 = -1$
- $z^6 = i$

Because any nonzero complex number has n distinct roots, the meaning of $z^{1/n}$ is ambiguous; several solutions to the equation $w^n = z$ are possible. Some sources treat $z \mapsto z^{1/n}$ as a *multivalued function*. When we need an nth root function, however, we will usually restrict ourselves to the principal root $\sqrt[n]{z}$, or possibly a multiple of this function by some nth root of unity.

2.8 Exponential function

Definition 2.8.1. The *complex exponential* is the function $\exp z = \exp(x + yi) = e^x(\cos y + i \sin y)$.

Definition 2.8.1 may seem strange: why introduce trigonometric functions into a formula and call the result an exponential function? We shall see later* that this is the "right" way to extend the exponential function from $\mathbb{R}$ to $\mathbb{C}$,** but for now let's work with this definition and see what the function does.

Task 70 (E)**.** Show that $\exp(z + w) = (\exp z)(\exp w)$ for all $z, w \in \mathbb{C}$.

Task 71 (D)**.**

- Express the modulus and argument of $\exp z$ in terms of the real and imaginary parts of z.†
- What is the effect of $z \mapsto \exp z$ on vertical lines?
- What is the effect of $z \mapsto \exp z$ on horizontal lines?
- Optional: What is the effect of $z \mapsto \exp z$ on a line that is neither vertical nor horizontal?
- Give as complete a description as you can of the function $z \mapsto \exp z$ as a transformation.

Task 72 (E)**.**

- Explain why $\exp z$ is never 0.
- Show that, if $w \neq 0$, then $w = \exp z$ for infinitely many values of z.
- Find a subset of $\mathbb{C}$ whose image by $z \mapsto \exp z$ covers all of $\mathbb{C}$ except 0 and the negative real axis.

2.9 Logarithms

As you have just seen in Task 72, the complex exponential function is not one-to-one, even though e^x is one-to-one as a function $\mathbb{R} \to \mathbb{R}$. Thus $\exp z$ does not have

*When we study power series, in Chapter 5. At that point we will give a separate definition for the expression e^z and show that it equals $\exp z$, but for now we will just use $\exp z$.

**What does $\exp z$ equal when $\operatorname{Im} z = 0$?

†Read this part of the task carefully before answering.

a single-valued inverse on $\mathbb{C} \setminus \{0\}$, and we must make a choice, just as we did for the square root and the nth root functions, in order to define a logarithm function (not a multivalued function).

Definition 2.9.1. For any $z \neq 0$, a *logarithm* of z is $\log z = \ln|z| + i \arg z$, where $\ln :]0, \infty[\to \mathbb{R}$ is the natural logarithm and $\arg z$ is any argument of z. The *principal logarithm* of z, written $\operatorname{Log} z$, is $\ln|z| + i \operatorname{Arg} z$.*

Task 73 (C). What are the possible values of $\log 1$? of $\log(-1)$? of $\log 4i$? of $\log(1 + i)$?

Task 74 (C). Compute and plot $\operatorname{Log}\big((1+i)^n\big)$ and $\operatorname{Log}\big((1 - i\sqrt{3})^n\big)$ for $n = 1, 2, 3, 4$. What do you notice?

Task 75 (E). Show that, if $z \neq 0$ and $\log z$ is any logarithm of z, then $\exp(\log z) = z$.

Task 76 (D). Explain why the domain of $\operatorname{Log} z$ is $\mathbb{C} \setminus \{0\}$. What is the effect of $\operatorname{Log} z$ on its domain?

*In these notes, log will never mean a base-10 logarithm. Instead it will be an inverse value of the complex exponential, as defined here. This notation is standard in advanced mathematics. Notice that if x is real and positive, then $\operatorname{Log} x = \ln x$. (Why?)

Chapter 3

Topology and limits

Now we turn to properties of $\mathbb{C}$ and its subsets that can be described in terms of "nearness." This notion is formalized in the vocabulary of *topology*. The Greek word τόπος ("topos") means "place," while λόγος ("logos") means "study." Thus, etymologically "topology" means "the study of location."* Topological properties are those that have only to do with the idea of points being "sufficiently close," not with particular distances, which is why topology employs terms like "neighborhood," as seen below.

3.1 Neighborhoods and open sets

Definition 3.1.1. Given $z \in \mathbb{C}$ and $\varepsilon > 0$,** the *ε-neighborhood of z*, written $N_\varepsilon(z)$, is the set of all $w \in \mathbb{C}$ whose distance from z is less than ε. In symbols,

$$N_\varepsilon(z) = \{w \in \mathbb{C} : |z - w| < \varepsilon\}.$$

Task 77 (D)**.** Draw a picture of $N_{1/2}(2+3i)$, the $\frac{1}{2}$-neighborhood of $2+3i$. What can you say in general about the geometric appearance of an ε-neighborhood?

Task 78 (E)**.** Suppose $z \in N_1(0)$, and set $\varepsilon = 1 - |z|$. Explain why $\varepsilon > 0$, and show that $N_\varepsilon(z) \subseteq N_1(0)$.

Definition 3.1.2. A subset $U \subseteq \mathbb{C}$ is *open* if, for every $z \in U$, there exists $\varepsilon > 0$ such that $N_\varepsilon(z)$ is entirely contained in U. If U is an open subset of $\mathbb{C}$, then we write $U \mathring{\subseteq} \mathbb{C}$, or possibly $U \mathring{\subset} \mathbb{C}$ if $U \neq \mathbb{C}$.***

Roughly speaking, an open set has "wiggle room" around each of its points. If one point is in an open set U, then all sufficiently nearby points are also in U. With Definition 3.1.2 in hand, the content of Task 78 can be expressed as "$N_1(0)$ is open." We also write $N_1(0)$ as $\mathbb{D}$ and call it the *open unit disk*. Task 79 includes other examples of open sets.

*Topology was originally called, in Latin, *analysis situs*, with the same meaning.

**The Greek letter ε ("epsilon") is often used to denote an arbitrary positive number, usually when we want to think of it as being very small. The 20th century mathematician Paul Erdős was known to call children "epsilons" due to their smallness.

***The use of the symbols $\mathring{\subseteq}$ and $\mathring{\subset}$ to mean "is an open subset of" is entirely nonstandard, but I find it a convenient shorthand.

Task 79 (E)**.** Show that each of the following is an open subset of $\mathbb{C}$.[†]

- the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$
- the punctured plane $\mathbb{C} \setminus \{0\}$
- an annulus such as $A = \{z : 1 < |z| < 5\}$
- the empty set $\varnothing$

Task 80 (E)**.** Explain why each of the following subsets of $\mathbb{C}$ is not open.[††]

- the real line $\mathbb{R} = \{z : \operatorname{Im} z = 0\}$
- $\{z : -\pi < \operatorname{Im} z \leq \pi\}$
- $\{1/n : n \in \mathbb{N}, n \neq 0\}$
- $\mathbb{C} \setminus \{1/n : n \in \mathbb{N}, n \neq 0\}$

Task 81 (DE)**.** Give an example of another subset of $\mathbb{C}$ that you think is open, and show that it is.

Definition 3.1.3. A *neighborhood* of $z \in \mathbb{C}$ is any open set that contains z.

3.2 Boundaries and closed sets

Definition 3.2.1. Let $A \subseteq \mathbb{C}$. A *boundary point* of A is a point $z \in \mathbb{C}$ such that for all $\varepsilon > 0$, $N_\varepsilon(z)$ contains elements of both A and $\mathbb{C} \setminus A$. The set of all boundary points of A is written ∂A and called the *boundary* of A. In symbols,

$$z \in \partial A \qquad \Longleftrightarrow \qquad \forall \varepsilon > 0, \quad N_\varepsilon(z) \cap A \neq \varnothing \quad \text{and} \quad N_\varepsilon(z) \cap (\mathbb{C} \setminus A) \neq \varnothing.$$

Note that a boundary point of A may or may not itself belong to A.

Task 82 (D)**.** Determine the boundary of the open unit disk $\mathbb{D}$ and of each set in Tasks 79 and 80.

Definition 3.2.2. A subset $B \subseteq \mathbb{C}$ is *closed* if it contains all its boundary points.

Task 83 (E)**.** Show that each of the following is a closed subset of $\mathbb{C}$.

- $\mathbb{R} = \{z : \operatorname{Im} z = 0\}$
- $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$
- $\mathbb{C}$
- the empty set $\varnothing$

Task 84 (DE)**.** Give an example of a subset of $\mathbb{C}$ that is neither open nor closed. Explain why neither condition holds.

Definition 3.2.3. The *closure* of a set $A \subseteq \mathbb{C}$, written $\overline{A}$, is the union of A and its boundary: $\overline{A} = A \cup \partial A$.

Task 85 (D)**.** Find the closure of each of the sets in Tasks 79, 80, and 84.

3.3 Sequences

In this and future sections, it will occasionally be useful to separate 0 from the rest of the natural numbers. Hereafter, 0 will often be used as an index to indicate a

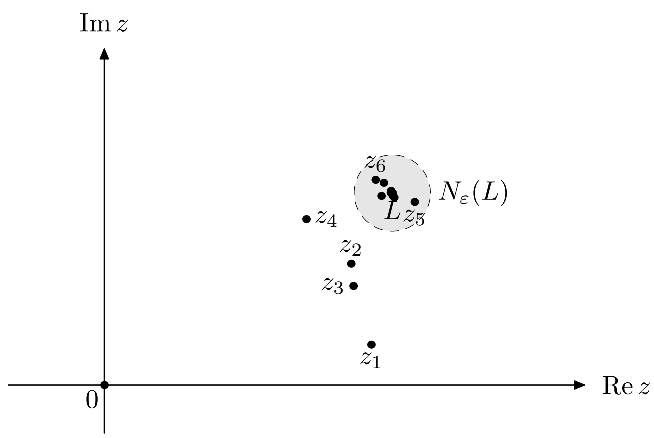
[†]You must show that every point of the set has an ε-neighborhood that is also contained in the set. For these examples, you should be able to find an explicit value of ε (which may depend on the point).

[††]If we negate the definition of "U is open," we get the statement: "There exists $z \in U$ such that every ε-neighborhood of z intersects the complement of U." Find such a z in each set.

particular point of interest, while the indices 1, 2, 3, ... will be used to label terms in a sequence.*

We will use $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ to denote the set of positive integers.

Definition 3.3.1. A *sequence* in $\mathbb{C}$ is a function $\mathbb{N}_+ \to \mathbb{C}$. In other words, it is an ordered list of numbers $z_1, z_2, z_3, \ldots$, called *terms* of the sequence. A sequence *converges* to the *limit* L if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $z_n \in N_\varepsilon(L)$ for all $n > n_0$. If such a number L exists, we write $\lim\limits_{n\to\infty} z_n = L$.



A sequence converges to L if, no matter how small ε is chosen, the terms of the sequence eventually remain inside $N_\varepsilon(L)$.

Task 86 (C)**.** Determine whether each of the following sequences converges, and if so, to what limit.

- $z_n = i^n$
- $z_n = i^n/n$
- $z_n = \left(\frac{1+i}{2}\right)^n$
- $z_n = (1 - i\sqrt{3})^n$
- $z_n = 1 - \left(\frac{i}{\sqrt{2}}\right)^n$
- $z_n = \left(\frac{3}{5} + \frac{4}{5}i\right)^n$

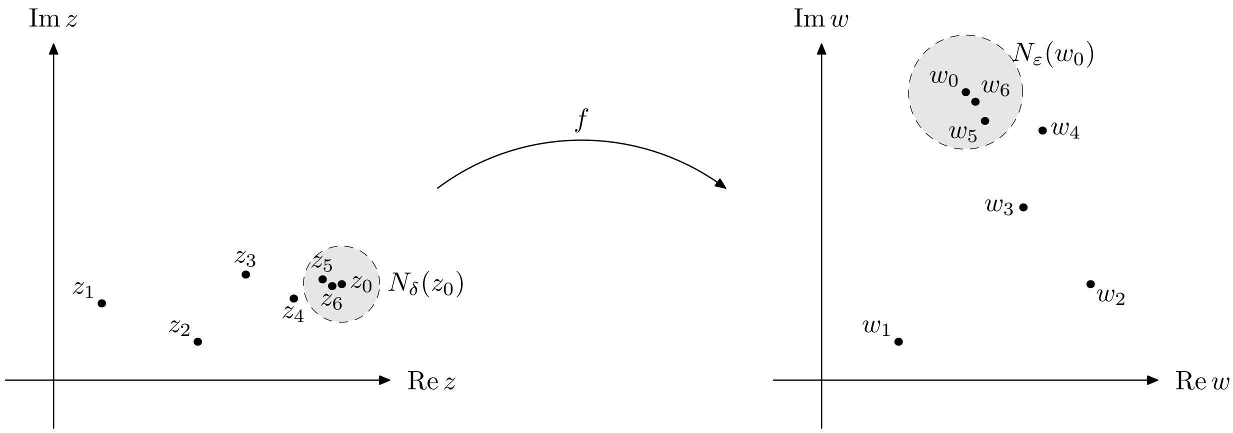
We can study the convergence of sequences of complex numbers by considering their real and imaginary parts separately. The following theorem comes from real analysis; you may find it intuitively clear, but its proof reaches farther into analysis than we intend to delve, so we will accept it without proof.**

Theorem 1. *Let $z_n = x_n + y_n i$ be a sequence of complex numbers, with $x_n, y_n \in \mathbb{R}$. Then z_n converges to $L \in \mathbb{C}$ if and only if x_n and y_n converge to the real and imaginary parts of L, respectively. In symbols,*

$$\lim_{n\to\infty} z_n = L \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \operatorname{Re} z_n = \operatorname{Re} L \quad \textit{and} \quad \lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} L.$$

*This convention will not be universally followed. For instance, sometimes it will be convenient to start sequences with the index 0, such as when we work with power series. We will even want to allow negative indices at times.

**Accepting this kind of result is somewhat different from accepting a new axiom, as we did when defining $\mathbb{R}$ at the start of the course. This is not a statement whose truth is independent of earlier assumptions, a claim that can be accepted or rejected at will. It is a consequence of axioms and definitions that were previously established. We are accepting Theorem 1 "on authority," trusting that someone, somewhere, has proved it. This kind of acceptance is a questionable practice in mathematics generally, but it is a reasonable choice pedagogically, if the proof of the result strays too far from the primary material.



This way of thinking about limits in $\mathbb{C}$ means that we can have reasonable expectations about when a sequence in $\mathbb{C}$ converges by relying on the Dedekind completeness of $\mathbb{R}$ (see Definition 0.2.3).*

Task 87 (E)**.** Suppose $\lim_{n\to\infty} z_n = L_1$ and $\lim_{n\to\infty} w_n = L_2$. Use Theorem 1 and convergence properties of sequences of real numbers† to explain why

$$\lim_{n\to\infty}(z_n + w_n) = L_1 + L_2 \qquad \text{and} \qquad \lim_{n\to\infty} z_n w_n = L_1 L_2.$$

Task 88 (D)**.**

- Find sequences z_n and w_n such that $\lim_{n\to\infty}(z_n + w_n)$ exists but $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$ do not.
- Find sequences z_n and w_n such that $\lim_{n\to\infty} z_n w_n$ exists but $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$ do not.

Task 89 (E)**.** Suppose $B \subseteq \mathbb{C}$ is closed. Show that if $z_1, z_2, z_3, \ldots$ is a sequence of points in B that converges to a limit L, then $L \in B$.††

3.4 Limits of functions

Definition 3.4.1. Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$ be a function. Suppose z_0 is in the closure of A. We say that the *limit* of f at z_0 equals w_0 if, for every sequence z_n of points in $A \setminus \{z_0\}$ that converges to z_0, the sequence $w_n = f(z_n)$ converges to w_0. If such an w_0 exists, we write $\lim_{z\to z_0} f(z) = w_0$.

Definition 3.4.1 is illustrated in the figure above. The function f maps points of any sequence z_n in A to points of a corresponding sequence w_n. As long as the sequence z_n converges to z_0 and does not contain z_0 as a term, the sequence

*Because $\mathbb{C}$ is not an ordered field, it does not make sense to discuss whether it is Dedekind complete or not. There are other notions of completeness that do apply to $\mathbb{C}$, but for us it is enough to use the fact that $\mathbb{C}$ is constructed from $\mathbb{R}^2$.

†Try writing $z_n = x_n + y_n i$ and $w_n = u_n + v_n i$.

††One approach is to show that if L is not in B, then it is a boundary point of B. What would this imply?

w_n must converge to w_0. (We require that z_0 not appear among the terms of the sequence $z_1, z_2, z_3, \ldots$ for technical reasons. In many cases of interest, the function is not defined at z_0, but only at nearby points.) To illustrate convergence, the figure shows a small neighborhood $N_\delta(z_0)$ of z_0 and a small neighborhood $N_\varepsilon(w_0)$ of w_0. These are chosen so that when z_n is in $N_\delta(z_0)$, it follows that w_n is in $N_\varepsilon(w_0)$. If $\lim\limits_{z\to z_0} f(z) = w_0$, then the definition essentially says that $\varepsilon \to 0$ as $\delta \to 0$.*

Task 90 (E). Use Definition 3.4.1 and the results of Task 87 to show the following.

- If $a, b \in \mathbb{C}$ are constants, then $\lim\limits_{z\to z_0} (az + b) = az_0 + b$.
- $\lim\limits_{z\to z_0} z^n = z_0^n$ for all $n \geq 1$.†

In general, in order to determine whether a function f has a limit at $z_0 \in \mathbb{C}$, it is not sufficient to consider only sequences that approach z_0 "from the left" and "from the right", nor even just sequences that approach z_0 along straight lines. For the next task, however, you should use your intuition to guess whether each limit exists, and if it seems like one of them does not, then considering a few well-chosen sequences should suffice to rule out its existence.

Task 91 (C). Determine whether each of the following limits exists, and if so what number it equals.

- $\lim\limits_{z\to 1+i} iz^2 - 2\overline{z} + 1$
- $\lim\limits_{z\to i} \dfrac{z^2+1}{z-i}$
- $\lim\limits_{z\to 1+2i} |z|$
- $\lim\limits_{z\to 0} \dfrac{\overline{z}}{z}$
- $\lim\limits_{z\to 1} \dfrac{|z|-1}{z-1}$ ††
- $\lim\limits_{z\to 0} \left(\dfrac{\overline{z}}{z}\right)^2$

3.5 Continuity

Definition 3.5.1. Suppose $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, and $z_0 \in A$. We say that f is *continuous at* z_0 if $\lim\limits_{z\to z_0} f(z) = f(z_0)$. If f is continuous at every point of A, then we simply call it *continuous*.

Task 92 (D).

- Explain why $z \mapsto \overline{z}$ is continuous.
- Explain why $z \mapsto z^2$ is continuous.
- Explain why $z \mapsto \sqrt{z}$ is not continuous.†

It seems unfortunate that the square root function, an old friend, should become discontinuous when extended to the complex realm. The next theorem,

*There are several other definitions of limits that are equivalent to the sequential definition that we have adopted. In analysis courses one often encounters a definition that dispenses with sequences and just uses δ- and ε-neighborhoods. The definition we are using fits well with the perspective on functions as transformations, which is part of why we have chosen it.

†One approach is to use Task 87 and induction.

††Consider sequences that approach 1 along the real axis and along the unit circle.

†Remember that $\sqrt{z}$ denotes the <u>principal</u> square root of z.

which we will again accept on the basis of authority*, suggests that it is possible to obtain a continuous function from $\sqrt{z}$ by restricting its domain only mildly.

Theorem 2. *Suppose $f : U \to \mathbb{C}$ is continuous and one-to-one, with $U \mathring{\subseteq} \mathbb{C}$, and set $V = f(U)$. Then $V \mathring{\subseteq} \mathbb{C}$, and the inverse function $f^{-1} : V \to U$ is also continuous.*

Task 93 (D)**.** Find an open subset U of $\mathbb{C}$, as large as possible, on which $z \mapsto z^2$ is one-to-one. What is the image of U? Call this image V. How can you tell that $z \mapsto \sqrt{z}$ is continuous on V?

Theorem 2 provides another way to view the "multivalued functions" that we encountered in Chapter 2. Each point $z_0 \in \mathbb{C}$ has a neighborhood U on which the function $z \mapsto z^n$ or $z \mapsto \exp z$ is one-to-one (in the case of z^n we must assume $z_0 \neq 0$), and thus *locally invertible* (having an inverse when restricted to U).** The different values of $z^{1/n}$ or $\log z$ come from these "local inverses", each of which may be treated as a continuous function, even though z^n and $\exp z$ do not have "global inverses", defined on all of $\mathbb{C}$.

*Theorem 2 is a special case of the Invariance of Domain Theorem, which was published by L. E. J. Brouwer in 1912. Unlike the case of Theorem 1 and most other theorems whose truth we will assume, the proofs of Theorem 2 are quite sophisticated, going far beyond a standard undergraduate curriculum. Fortunately, we will not need to use it much beyond this point.

**The association between the notions of "neighborhood" and "local" is deliberate.

Chapter 4

Derivatives

4.1 Limit definition of a derivative

The definition of a derivative should look familiar; we just introduce complex arithmetic and complex limits into the formula(s) you learned in calculus. Many of the same derivative properties hold for complex-valued functions as in the real-valued case. However, we shall see that in the complex realm derivatives also have new and surprising properties beyond those of real derivatives.

Definition 4.1.1. Suppose f is a complex-valued function defined on a neighborhood of $z_0 \in \mathbb{C}$. We say that f is *complex differentiable at* z_0 with *complex derivative*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

provided either limit exists (in which case both do, and they are equal).

Wherever the function $z \mapsto f'(z)$ is defined, we also call this function f' the *complex derivative* of f.

Task 94 (E). Using Definition 4.1.1, show the following.

- The complex derivative of a constant function is zero at every point.
- The complex derivative of an affine function is a constant function.
- The complex derivative of $z \mapsto z^2$ at z_0 is $2z_0$.

Task 95 (E). Show that the following functions are not complex differentiable:[†]

- $z \mapsto \operatorname{Re} z$
- $z \mapsto \overline{z}$
- $z \mapsto |z|^2$

Task 96 (E). Show that if f is differentiable at z_0, then it is continuous at z_0.[††]

The following (unsurprising) theorem will be useful in the future.

[†] The first two are not complex differentiable at any point. The third is complex differentiable at 0, but nowhere else.

[††] Try writing $f(z) - f(z_0)$ as $\frac{f(z)-f(z_0)}{z-z_0} \cdot (z - z_0)$ and taking the limit of this expression as $z \to z_0$.

Theorem 3. *If $f : \mathbb{C} \to \mathbb{C}$ has a complex derivative everywhere equal to zero, then f is constant.*[*]

In parallel with the notation used for real derivatives, we also write $\frac{d}{dz} f(z)$ for the complex derivative of f and $f^{(n)}$ or $\frac{d^n}{dz^n} f(z)$ for the nth complex derivative of f (assuming it exists).

4.2 Derivative rules

From now on, we will usually just say "derivative" to mean "complex derivative" and "differentiable" to mean "complex differentiable." If another meaning of differentiable is intended (for example, "real differentiable" as in §4.3 below), we will specify that.

Task 97 (E)**.** Use the limit definition of the derivative to show that if f and g are differentiable and $c \in \mathbb{C}$, then the following rules apply:

- Sum rule $\frac{d}{dz}\big(f(z) + g(z)\big) = f'(z) + g'(z)$
- Coefficient rule $\frac{d}{dz}\big(cf(z)\big) = cf'(z)$
- Product rule $\frac{d}{dz}\big(f(z)g(z)\big) = f(z)g'(z) + f'(z)g(z)$

Task 98 (E)**.** Show that, for any $n \in \mathbb{N}_+$, the derivative of $f(z) = z^n$ is $f'(z) = nz^{n-1}$.[†]

Task 99 (E)**.** Show that any polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is complex differentiable.

We will also accept these additional rules without proof:

- Quotient rule $\frac{d}{dz}\big(f(z)/g(z)\big) = \big(g(z)f'(z) - f(z)g'(z)\big)/\big(g(z)\big)^2$
- Chain rule $\frac{d}{dz}\big(f(g(z))\big) = f'(g(z))g'(z)$

Task 100 (E)**.** Show that, for any $n \in \mathbb{N}_+$, the derivative of $f(z) = z^{-n} = 1/z^n$ is $f'(z) = -nz^{-n-1}$.[††]

Task 101 (E)**.** Show that, for any $n \in \mathbb{N}_+$, the derivative of $f(z) = z^{1/n}$ is $f'(z) = \frac{1}{n} z^{(1/n)-1}$.[†††]

Recall from §3.5 that $z^{1/n}$ can be thought of as a local inverse of z^n. There is a bit of a trick to applying the result of Task 101: in order for it to be true, we have to use the <u>same</u> inverse of z^n to define both $z^{1/n}$ and $z^{(1/n)-1} = (z^{1/n})^{1-n}$.

[*] Theorem 3 can be proved from the version for real functions by considering the real and imaginary parts of $f(z)$ separately and observing that $\operatorname{Re} f$ and $\operatorname{Im} f$ must each be constant on horizontal and vertical lines.

[†] There are several approaches to this task. You could try factoring $z^n - z_0^n$ in the first limit from Definition 4.1.1, or expand $(z_0 + h)^n$ in the second limit from Definition 4.1.1, or use induction along with the product rule.

[††] Try using the quotient rule.

[†††] Try using the chain rule.

4.3 Cauchy–Riemann equations

The condition of complex differentiability on a complex-valued function f imposes noteworthy restrictions on the partial derivatives of the real and imaginary parts of f. The most fundamental of these restrictions are the *Cauchy–Riemann equations*,* which you will derive in the next task.

Task 102 (E)**.** Suppose $f(z) = u(x,y) + iv(x,y)$, with $z = x + iy$, and assume that f is complex differentiable at $z_0 = x_0 + iy_0$.

- Consider values Δx that approach 0 along the real axis. Interpret

$$\lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}$$

in terms of (real) partial derivatives, and show that it equals

$$\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

- Consider values $i\Delta y$ that approach 0 along the imaginary axis. Show that

$$\lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$

- Use the calculations from the previous two parts of this exercise, along with the fact that both results must equal $f'(z_0)$, to find an equation that relates $(\partial u/\partial x)(x_0, y_0)$ and $(\partial v/\partial y)(x_0, y_0)$. Likewise find an equation that relates $(\partial u/\partial y)(x_0, y_0)$ and $(\partial v/\partial x)(x_0, y_0)$. (These are the Cauchy–Riemann equations.)

Task 103 (C)**.** Write each of the following functions in the form $x + iy \mapsto u(x,y) + iv(x,y)$, then verify the Cauchy–Riemann equations using derivative rules you know from calculus with real variables:

- $z \mapsto z^2$
- $z \mapsto 1/z$
- $z \mapsto \exp z$

Task 104 (C)**.** Investigate whether the Cauchy–Riemann equations are true for the functions in Task 95.

As a partial converse to the result of Task 102, we have the following result, which we will accept without proof.

Theorem 4. *If $u(x,y)$ and $v(x,y)$ have continuous (real) partial derivatives and they satisfy the Cauchy–Riemann equations at (x_0, y_0), then $f(x+iy) = u(x,y) + iv(x,y)$ is complex differentiable at $x_0 + iy_0$.*

Task 105 (C)**.** Where is the function $f(x + iy) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ complex differentiable?

*You have probably heard of Bernhard Riemann (pronounced "ree-mahn"), since his method of defining integrals is usually taught in calculus classes. If you have studied real analysis, you have likely also heard of Augustin-Louis Cauchy (pronounced "koh-shee"), through the type of sequence that bears his name. Both of them made immense contributions to the theory of complex functions; we shall encounter their names again.

Task 106 (C). Show that the complex derivative of $\exp z$ is $\exp z$.

Task 107 (C). Show that the complex derivative of $\log z$ is $1/z$.

Recall from §3.5 that $\log z$ can be thought of as representing many different local inverses of $\exp z$. Task 107 shows, however, that regardless of which local inverse is chosen, the derivative is always the same. As we shall see, the fact that $\log z$ is "multivalued" is related to the fact that $1/z$ is not defined at 0.

4.4 Interpretation in terms of affine transformations

To get at the geometric meaning of the derivative, let's first revisit the notion of real derivatives in light of the transformational view of functions $\mathbb{R} \to \mathbb{R}$. Recall that the *first-order Taylor polynomial* (or *linearization*) of $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ is the affine function

$$T_1(x) = T_{f,x_0,1}(x) = f(x_0) + f'(x_0)(x - x_0).$$

This definition is chosen so that, near x_0, $f(x)$ "acts like" the function $T_1(x)$. To be more explicit, both f and T_1 send x_0 to $f(x_0)$, and while T_1 stretches (or contracts) all distances by exactly a factor of $f'(x_0)$, f multiplies distances near x_0 by approximately $f'(x_0)$.*

Task 108 (CD). Compute the first-order Taylor polynomial of each function at the given value of x_0, and compare the effect of the original function on $\mathbb{R}$ near x_0 with the effect of the linearized version. (Go back to Tasks 2 and 3 for descriptions of the original functions.)

- $x \mapsto x^2$, $x_0 = 1$
- $x \mapsto x^2$, $x_0 = -1$
- $x \mapsto x^2$, $x_0 = 0$
- $x \mapsto e^x$, $x_0 = 0$

In the same way, if f is complex-differentiable at $z_0 \in \mathbb{C}$, then it has a first-order Taylor polynomial,

$$T_1(z) = T_{f,z_0,1}(z) = f(z_0) + f'(z_0)(z - z_0).$$

As you saw in §2, a non-constant affine function $\mathbb{C} \to \mathbb{C}$ carries out some combination of scaling, rotating, and translating. It's most useful to think of $T_1(z)$ as first translating z_0 to 0, then scaling and rotating around 0 via multiplication by $f'(z_0)$, then translating 0 to $f(z_0)$. At a small scale, this is what we expect a complex-differentiable function to do.

Task 109 (CD). Compare the effect of each function on $\mathbb{C}$ near the given value of z_0 with the effect of the first-order Taylor polynomial at that point.

- $z \mapsto z^2$, $z_0 = 1$
- $z \mapsto z^2$, $z_0 = i$
- $z \mapsto z^2$, $z_0 = 0$
- $z \mapsto \exp z$, $z_0 = 0$
- $z \mapsto \exp z$, $z_0 = \pi i$
- $z \mapsto \exp z$, $z_0 = 2\pi i$

*In case you are familiar with little-o notation, the formal statement is $f(x) - T_1(x) = o(|x - x_0|)$ as $x \to x_0$.

4.5 Holomorphic functions

Definition 4.5.1. Let $A \subseteq \mathbb{C}$, and let $f : A \to \mathbb{C}$. We say that f is *holomorphic** *at* z_0 if it is complex differentiable on a neighborhood of z_0. f is called *holomorphic on* A, or simply *holomorphic*, if it is holomorphic at every point of A (in other words, if A is open and f is complex differentiable at every point of A).

Differentiability is a pointwise condition: to be differentiable at z_0, only the limit that defines $f'(z_0)$ needs to exist. The property of being holomorphic at z_0, however, is a local condition: it requires complex differentiability on a neighborhood of z_0 (i.e., for all points "sufficiently close to z_0").

Task 110 (CD)**.** Determine the domain of each function below, and explain why it is holomorphic on its domain, using the derivative rules developed in this section.

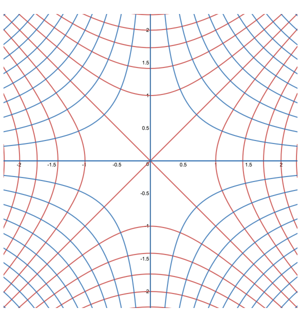
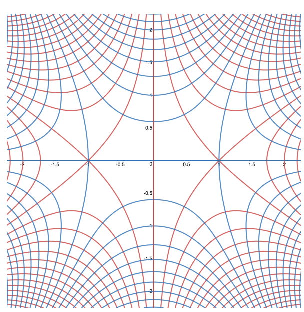
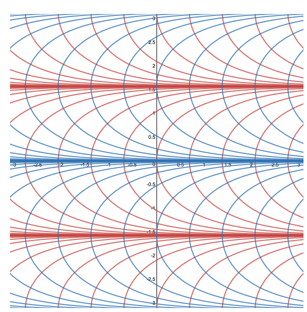
- $\dfrac{2z+1}{z(z^2+4)}$
- $\exp \dfrac{1}{z}$
- $\dfrac{\exp z}{z^2-3z+2}$
- $\dfrac{z^3-i}{z^2+2iz-1}$

Task 111 (C)**.** Where is the function f of Task 105 holomorphic?

Holomorphic functions are the main object of study in complex variables. As we shall see, they have many fantastic properties, which is what makes this study both engaging and useful. For starters, here is one nice geometric property relating the real and imaginary parts of a holomorphic function.

Task 112 (E)**.** Suppose $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ is holomorphic. Set $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$. Show that the gradients $\nabla u(x,y)$ and $\nabla v(x,y)$ are orthogonal at every point of U, and that $\nabla u(x,y) = 0$ if and only if $\nabla v(x,y) = 0$.†

In multivariable calculus, you learned that the gradient of a real-valued function is orthogonal to the level curves** of the function. The result of the previous task thus implies that the level curves of the real and imaginary parts of a holomorphic function are at every point orthogonal to each other, unless their gradients vanish. The images on the next page illustrate this property.

*From the Greek words ὅλος (*holos*), meaning "whole" or "complete", and μορφή (*morphe*), meaning "shape".

†Recall that the gradient of a real-valued function is the vector whose components are the first partial derivatives of the function. Two vectors are orthogonal if their dot product (a.k.a. inner product) is zero.

**Also known as contour lines. We will later use both "curve" and "contour" with different meanings, however.

Level curves of $\operatorname{Re} f$ (in red) and $\operatorname{Im} f$ (in blue) for z^2, $z^3 - 3z$, and $\exp z$.

The level curves of z^2 take values 1 unit apart. The level curves of $z^3 - 3z$ take values 2 units apart. The level curves of $\exp z$ are spaced so that the values on adjacent curves differ by a factor of 2.

Task 113 (CD)**.** Plot some level curves of $\operatorname{Re} z^{-1}$ and $\operatorname{Im} z^{-1}$ on the same diagram. What do you notice?

This property of the orthogonality of level curves is a special case of a more general geometric property, which we can obtain by considering the first-order Taylor polynomials of a holomorphic function (introduced in §4.4). Because the operations of scaling, rotating, and translating do not change the measurement of angles, holomorphic functions preserve angles wherever their derivatives are nonzero.

Definition 4.5.2. A function $f : A \to \mathbb{C}$ is *angle-preserving*, or *conformal*, at $z_0 \in A$ if f is differentiable at z_0 and $f'(z_0) \neq 0$.

Task 114 (D)**.** A power function $z \mapsto z^n$, where $n \geq 2$, is conformal at every point of $\mathbb{C}$ except 0. Why? What happens to the angles between lines that pass through 0?

Chapter 5

Series, power series, and analytic functions

5.1 Series

A series is a particular kind of sequence, written in the form of a sum with infinitely many terms. To make sense of such an expression, we must add the terms sequentially and examine the limiting behavior.

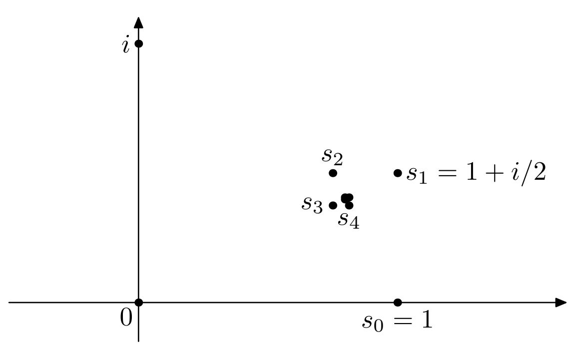
Definition 5.1.1. An *infinite series* (usually just called a *series*) is an expression of the form

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + z_3 + \cdots.$$

The number z_n is the *nth term* of the series. The *partial sums* of an infinite series are the finite sums

$$s_k = \sum_{n=0}^{k} z_n = z_0 + z_1 + \cdots + z_k.$$

An infinite series is said to *converge* if its sequence of partial sums converges, in which case the limit of the partial sums is called the *sum* of the series. A series that does not converge is said to *diverge*.



Partial sums s_k of the series $\sum_{n=0}^{\infty}(i/2)^n$.

Task 115 (CD). Find the first few partial sums of the series $\sum_{n=0}^{\infty}(i/2)^n$. Can you predict whether this series converges?

Task 116 (D)**.**

a. Explain why if the series $\sum_{n=0}^{\infty} z_n$ converges, then its sequence of terms z_n must converge to zero.
b. The contrapositive of the statement in part a. is called the *nth term test*. State this test.

5.2 Power series

Definition 5.2.1. A *power series* with *center* $z_0 \in \mathbb{C}$ (or *centered at* z_0) is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

(We follow the convention that $w^0 = 1$ for all $w \in \mathbb{C}$, even $w = 0$.)

A power series contains a variable z. It always converges when $z = z_0$, with the sum a_0, because $(z - z_0)^0 = 1$ and all other terms are zero. It may converge for some or all other values of z. If a power series converges for every point z in a set $A \subseteq \mathbb{C}$, then we say the series converges *on* A.

The simplest example of a power series (apart from polynomials, which may be considered as power series having only finitely many nonzero coefficients) is the case in which all coefficients are 1. As it turns out, this is also one of the most important examples; we will return to it often.

Definition 5.2.2. The *geometric series* is defined for $z \in \mathbb{C}$ by

$$\sum_{n=0}^{\infty} z^n.$$

Task 117 (E)**.**

- Show that the partial sums of the geometric series, when $z \neq 1$, are

$$s_k = \frac{1 - z^{k+1}}{1 - z}.$$

  Conclude that the geometric series converges to $1/(1-z)$ on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$.
- Show that the geometric series does not converge if $|z| \geq 1$.†

Task 118 (C)**.** Find the sums of the following four series. Express your answers in the form $x + iy$.*

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n, \qquad \sum_{n=0}^{\infty} \left(\frac{1 - i\sqrt{2}}{3}\right)^n, \qquad \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n,$$
$$2 + (1+i) + \frac{(1+i)^2}{2} + \frac{(1+i)^3}{2^2} + \cdots$$

†The nth term test is sufficient.
*Notice that the first of these is the series from Task 115.

Part of the utility of power series comes from the ease with which they can be manipulated "formally," without regard to questions of convergence. Definitions 5.2.3 and 5.2.4 contain examples of two such operations. They mimic the corresponding operations (differentiation and multiplication) for polynomials.

Definition 5.2.3. The *formal derivative* of a power series centered at z_0 is defined by

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$

The d/dz notation in Definition 5.2.3 should not be interpreted as an ordinary derivative. When calculating the formal derivative of a series, we do not make any assumptions about the values of z for which the series converges, in particular whether it defines a function in a neighborhood of z_0 or not.

Definition 5.2.4. The *formal product* of two power series† centered at z_0 is defined by

$$\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-z_0)^n\right) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Again, the power series that appear in Definition 5.2.4 may or may not define functions. Each coefficient c_n in the formal product only depends on finitely many values of a_k and b_k, so it does not matter whether or not the series converge for purposes of this definition.

Task 119 (C)**.** Show that the power series obtained from the geometric series in Definition 5.2.2 by

1. formal differentiation
2. formal squaring

is the same in either case.

The next two theorems can be proved by optional tasks included at the end of this chapter. For now, you may assume that they are true. They state that, when a power series converges to a function, the formal operations defined above correspond to the ordinary operations of differentiation and multiplication.

Theorem 5. *If a power series converges to $f(z)$ on an open set, then the formal derivative of the power series converges to $f'(z)$ on the same open set.*

In the statement of Theorem 5, we did not need to worry about the center of the power series. However, Theorem 6 involves two different series, and so in its statement they must have the same center.

Theorem 6. *If two power series have the same center z_0, and they converge to $f(z)$ and $g(z)$ on a neighborhood of z_0, then the formal product of these power series converges to the product $f(z)g(z)$ on the same neighborhood.*

Task 120 (CD)**.** Show that $\frac{d}{dz}\left(\frac{1}{1-z}\right) = \left(\frac{1}{1-z}\right)^2$. How does this equation, together with the result of Task 119, illustrate the claims of Theorems 5 and 6?

†While formal differentiation requires the presence of a variable, formal multiplication does not. For a pair of ordinary series, we have the product formula: $\left(\sum_{n=0}^{\infty} z_n\right)\left(\sum_{n=0}^{\infty} w_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} z_k w_{n-k}\right)$.

Now we turn to another important example of power series.*

Definition 5.2.5. The *exponential series* is defined for $z \in \mathbb{C}$ by

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

As we shall see in a little while (in Task 143), this series converges for all values of z. (For the moment, assume this is true.) We define e^z to be the sum of the exponential series for any $z \in \mathbb{C}$.**

The next two tasks show that e^z possesses certain properties that should be familiar from the real case.

Task 121 (C)**.** Show that the formal derivative of the exponential series is again the same power series.

Task 122 (E)**.** Use Definition 5.2.5 to show that $e^{z+w} = e^z \cdot e^w$ for any $z, w \in \mathbb{C}$.†

In the next task you will show that e^z equals the function $\exp z$ that was defined in §2.8.

Task 123 (E)**.**

a. Show that the function $e^z / \exp z$ is constant.††
b. Show that $e^0 / \exp 0 = 1$, and conclude from this equation, together with the result of part a., that $e^z = \exp z$ for all $z \in \mathbb{C}$.

Task 124 (C)**.** Evaluate $e^{\pi i} + 1$.***

Task 125 (C)**.** Find the sums of the following series, and locate them on an Argand diagram.

$$\sum_{n=0}^{\infty} \frac{(-1+i\pi)^n}{n!}, \qquad \sum_{n=0}^{\infty} \frac{(\ln 4 - i\pi/3)^n}{n!}, \qquad \sum_{n=0}^{\infty} \frac{i^n}{n!}$$

By completing Task 122, you demonstrated a fundamental property of the exponential function. (You also showed the same property in Task 70, using the definition of $\exp z$ instead of e^z.) We pause briefly in our study of power series to consider some of its consequences.

Task 126 (E)**.** Show that $e^{-z} = (e^z)^{-1}$.

Task 127 (E)**.** Show that, for any $n \in \mathbb{N}$, $z^n = \exp(n \log z)$, regardless of which value of $\log z$ is chosen.

Task 128 (E)**.** Show that, for any $n \in \mathbb{N}$, $z^{-n} = \exp(-n \log z)$, regardless of which value of $\log z$ is chosen.

*The geometric series and exponential series pop up all over the place, seriously. Any sustained study of mathematics should lead to familiarity with them.

**This formula provides a convenient definition of the number $e = e^1$ as the sum of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.

†You may use the binomial theorem: $(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

††Try using Theorem 3.

***The resulting equation is one of the most famous formulas in mathematics, often called *Euler's identity*.

Task 129 (E). Show that, if $n \in \mathbb{N}_+$, then $\exp\left(\frac{1}{n}\log z\right)$ can be made to equal any nth root of z by choosing an appropriate value of $\log z$.

Inspired by these last few results, we adopt the following definition of z^w for general values of w.

Definition 5.2.6. If $z, w \in \mathbb{C}$ and $z \neq 0$, then $z^w = \exp(w \log z)$. The *principal value* of z^w is $\exp(w \operatorname{Log} z)$.

Task 130 (C). Find all values of i^i. What is its principal value?

Task 131 (C). Find the principal value of each of these expressions:

$$(1+i)^i, \qquad 2^{1+i\pi/\ln 4}, \qquad (-1)^{1/\pi}.$$

5.3 Taylor series

Definition 5.3.1. Suppose $f^{(n)}(z_0)$ exists for all $n \in \mathbb{N}$. Then the *Taylor series** of f centered at z_0 is

$$T(z) = T_{f,z_0}(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

Taylor series extend the first-order Taylor polynomials described in §4.4. Conceptually, the Taylor series of a function contains the data of all the derivatives of that function at a single point in a way most likely to recover the function itself.

Task 132 (C). For each of the following functions, calculate the Taylor series centered at the given point z_0.

- $z \mapsto z^3,\ z_0 = i$
- $z \mapsto e^{iz},\ z_0 = 0$
- $z \mapsto 1/z,\ z_0 = -2i$

Taylor series are, in particular, power series, to which we can apply the usual formal operations. The next two tasks show that these operations are compatible with Definition 5.3.1.

Task 133 (E). Show that the Taylor series of f' at z_0 is the formal derivative of the Taylor series of f at z_0.

*Named after Brook Taylor, who wrote about such series in *Methodus Incrementorum Directa & Inversa*, published in 1715.

Task 134 (E)**.**

- Show that $\frac{d^n}{dz^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$. (The 0th derivative of a function is the function itself.)
- Show that the Taylor series of a product fg centered at z_0 is the formal product of the Taylor series of f and g each centered at z_0.

Task 135 (C)**.**

- Find the Taylor series of $1/(1+z)$ centered at 0.†
- Show that $\frac{d}{dz}\operatorname{Log}(1+z) = 1/(1+z)$, and use this fact to find the Taylor series of $\operatorname{Log}(1+z)$ centered at 0.*

Task 136 (C)**.** Find the rational function $K(z)$ ** that has the following Taylor series centered at 0.†

$$\sum_{n=1}^{\infty} nz^n.$$

5.4 Trigonometric and hyperbolic functions

Task 137 (E)**.** Let $\theta \in \mathbb{R}$.††

- Show that $\overline{e^{i\theta}} = e^{-i\theta}$.
- Show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

If we replace θ with a complex variable z in the expressions above, we obtain definitions of the cosine and sine functions on all of $\mathbb{C}$:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{for all } z \in \mathbb{C}.$$

Task 138 (CE)**.** Using the definitions of $\cos z$ and $\sin z$ in terms of e^{iz} and e^{-iz}, do the following:

- Show that $\cos^2 z + \sin^2 z = 1$.
- Show that $\sin' z = \cos z$ and $\cos' z = -\sin z$.
- Find the Taylor series of $\cos z$ and $\sin z$ centered at 0.

Task 139 (E, Optional)**.** Show that the angle sum formulas (stated immediately prior to Task 40) are true even when α and β are complex numbers.

Closely related to the trigonometric functions are the *hyperbolic functions*. The hyperbolic cosine, written cosh, and the hyperbolic sine, written sinh, are defined by

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

†Use results you already know about the geometric series.

*The resulting series is called the *Mercator series*, after Nicholas Mercator (not the same as Gerardus Mercator, after whom the map projection is named). It was published by Mercator in his 1668 text *Logarithmotechnia* ("The making of logarithms").

**$K(z)$ is the *Koebe function*, named after Paul Koebe, who showed that this function is important in studying a certain class of holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$.

††For this task, you might find some results from Task 23 helpful.

Task 140 (CE, Optional)**.** Using the above definitions of $\cosh z$ and $\sinh z$, do the following:

- Show that $\cosh^2 z - \sinh^2 z = 1$.[*]
- Show that $\sinh' z = \cosh z$ and $\cosh' z = \sinh z$.
- Find the Taylor series of $\cosh z$ and $\sinh z$ centered at 0.

Although the trigonometric and hyperbolic functions arise from very different physical contexts,[**] when expressed as functions of a complex variable they demonstrate a tight relationship.

Task 141 (E, Optional)**.** Show that $\cosh iz = \cos z$ and $\sinh iz = i \sin z$.

5.5 Review of convergence tests

We can tell whether certain series converge or not by direct calculation. The most notable example is the geometric series, for which we can calculate the partial sums in an explicit form. For most series, however, we cannot directly prove convergence or divergence; instead we must compare them to other series whose convergence properties are known. This is easiest when the terms of the series are real and non-negative. The following theorem, familiar from calculus, can be proved using the Dedekind completeness property (DC) from chapter 0.

Theorem 7 (Comparison test)**.** *Suppose $0 \leq x_n \leq y_n$ for all n and $\sum_{n=0}^{\infty} y_n$ converges. Then $\sum_{n=0}^{\infty} x_n$ converges.*

Most tests of whether a series of complex numbers converges depend on the following notion.

Definition 5.5.1. We say that a series $\sum_{n=0}^{\infty} z_n$ *converges absolutely* if the (related, but different) series $\sum_{n=0}^{\infty} |z_n|$ converges.

The goal of the next task is to show that if a series converges absolutely, then it converges in the ordinary sense.[***]

[*] When $\cosh z$ and $\sinh z$ are real-valued, this equation means that $(\cosh z, \sinh z)$ lies on the hyperbola $x^2 - y^2 = 1$, which is one explanation for why these functions are named "hyperbolic". In parallel, sometimes the usual trigonometric functions are called "circular functions".

[**] The ordinary sine and cosine of course come from the study of triangles and circles. The hyperbolic cosine is perhaps most famous as the solution to the "catenary problem"—that is, its graph over $\mathbb{R}$ models the shape of a hanging chain (in Latin, "catenaria"), suspended from two ends and subject only to the forces of gravity and its own tension. The hyperbolic tangent, defined by $\tanh z = \sinh z / \cosh z$, appears in the formula for the velocity of a falling object that is subject to drag forces.

[***] This is an example of a kind of statement, not uncommon in mathematics, that is much more profound than it sounds. After all, you might ask, if we say that a series converges absolutely, then haven't we already said that it converges, and the adverb "absolutely" just means that it converges in some stronger sense as well? The terminology is indeed meant to have that connotation, but it is not as immediate a fact as it seems. The definition of "absolute convergence" does not say anything about whether $\sum z_n$ converges, but whether the completely different series $\sum |z_n|$ converges. Hence the need for Task 142.

Task 142 (E)**.** Suppose that $\sum_{n=0}^{\infty} |z_n|$ converges. Show that each of the following series converges.†

$$\sum_{n=0}^{\infty} \big(|z_n| + \operatorname{Re} z_n\big), \qquad \sum_{n=0}^{\infty} \big(|z_n| - \operatorname{Re} z_n\big),$$
$$\sum_{n=0}^{\infty} \big(|z_n| + \operatorname{Im} z_n\big), \qquad \sum_{n=0}^{\infty} \big(|z_n| - \operatorname{Im} z_n\big), \qquad \sum_{n=0}^{\infty} z_n$$

Because we know that a geometric series converges if the ratio of successive terms is less than 1 in absolute value, the convergence of many other series is tested by comparison with geometric series.

Theorem 8 (Ratio test)**.** *Suppose that there exist $0 < \rho < 1$ and $N \in \mathbb{N}$ such that the terms of the series $\sum_{n=0}^{\infty} z_n$ satisfy $|z_{n+1}/z_n| \le \rho$ for all $n \ge N$. Then the series $\sum_{n=0}^{\infty} z_n$ converges.*

Task 143 (CE)**.** Show that, for any $z \in \mathbb{C}$, the ratio of successive terms in the exponential series is eventually less than $1/2$ in absolute value. Conclude that the exponential series converges for all $z \in \mathbb{C}$.

Task 144 (C)**.** Show that the power series for the function $K(z)$ in Task 136 converges for any $|z| < 1$.††

The ratio test is not sufficient to determine whether every series converges, however. Given $p \in \mathbb{R}$, the *p-series* is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

For any value of p, the ratio of successive terms tends to 1 as $n \to \infty$. (Check this!) Thus the ratio test cannot tell us whether or not the series converges. The following result comes from calculus.

Theorem 9 (p-series test)**.** *The p-series $\sum_{n=1}^{\infty} n^{-p}$ converges if $p > 1$ and diverges if $p \le 1$.*

The p-series with $p = 1$ is called the *harmonic series*. Because the terms $1/n$ in the harmonic series tend to 0 as $n \to \infty$, this series shows the nth term test is <u>not</u> sufficient to determine if a series diverges.

†For the first four series, try using the comparison test. Keep in mind $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.
††Try using the ratio test with a ratio slightly larger than $|z|$.

Task 145 (CE, Optional)**.**

- Show that if $n \in \mathbb{N}_+$ and $s \in \mathbb{C}$, then $|n^s| = n^{\mathrm{Re}\, s}$. (Use principal values on both sides of the equality.)
- Show that if $\mathrm{Re}\, s > 1$, then the series $\sum_{n=1}^{\infty} n^{-s}$ converges absolutely.*

We recall one more test, which is useful for certain series that do not converge absolutely.

Theorem 10 (Alternating series test)**.** *Suppose x_n is a sequence of positive numbers such that $x_n > x_{n+1}$ for all n and $\lim_{n\to\infty} x_n = 0$. Then $\sum_{n=0}^{\infty}(-1)^n x_n$ converges.*

In certain cases, none of the above tests are sufficient to determine whether a series converges or not, and additional methods are required. We will not encounter such situations in this course, however.

5.6 Radius of convergence

Even though the shape of the region on which a function is defined may be very complicated, the shape of the region on which a power series converges is almost as simple as possible. Here we investigate the possibilities.

Task 146 (E)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series centered at z_0, and suppose that it converges at some $z = z_1 \neq z_0$. Let z_2 be a point in $N_{|z_1-z_0|}(z_0)$.†

- Explain why the terms of the series $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ must be bounded; that is, there exists $M > 0$ such that $|a_n(z_1-z_0)^n| \le M$ for all $n \in \mathbb{N}$.
- Show that $|a_n(z_2-z_0)^n| \le M\left|\frac{z_2-z_0}{z_1-z_0}\right|^n$ for all $n \in \mathbb{N}$, where M is chosen as in the previous part.
- Show that $\sum_{n=0}^{\infty} a_n(z_2-z_0)^n$ converges absolutely.

The <u>technique</u> of the previous task will be useful again: if we know that a power series converges at a particular point, we can get information about its convergence (or the convergence of a related power series) at a point nearer to the center. The <u>result</u> of the previous task is what permits us to give a (nearly) complete description of the shape of a set on which a power series converges.

Given a power series $S = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, consider the following two sets:

$$A_S = \{\rho \in [0,\infty[: S \text{ converges for some } z = z_1 \text{ with } |z_1 - z_0| = \rho\}$$
$$B_S = \{\rho \in [0,\infty[: S \text{ diverges for all } z = z_1 \text{ such that } |z_1 - z_0| = \rho\}$$

Task 147 (E)**.** Explain why the two sets A_S and B_S just defined form a partition of the non-negative real numbers: that is, every $\rho \ge 0$ is in either A_S or B_S, but not in both. Moreover, show that if $a \in A_S$ and $b \in B_S$, then $a < b$.

*The function $\zeta(s)$ defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ whenever $\mathrm{Re}\, s > 1$ is called the *Riemann zeta function*. It is conventional to use s as the input variable for this function. The Riemann zeta function is at the heart of one of the most famous open problems in mathematics, called the *Riemann hypothesis*, which I will not try to explain here.

†That is, $|z_2 - z_0| < |z_1 - z_0|$. Try drawing a picture of what this means, with representative points z_0, z_1, and z_2.

If both sets A_S and B_S are nonempty, then Task 147 and the Dedekind completeness property (Definition 0.2.3) together imply that there is a real number R_{conv} such that $a \le R_{\text{conv}}$ for all $a \in A_S$ and $R_{\text{conv}} \le b$ for all $b \in B_S$. Because $0 \in A_S$ (why?), A_S is never empty, and so there are three possibilities:

- $R_{\text{conv}} = 0$. This means that S converges only when $z = z_0$.
- $R_{\text{conv}} > 0$. Then S converges whenever $|z - z_0| < R_{\text{conv}}$ and diverges whenever $|z - z_0| > R_{\text{conv}}$.
- B_S is empty. Then S converges for all z. We represent this case by setting $R_{\text{conv}} = \infty$.

Definition 5.6.1. The *radius of convergence* of a power series S is the value R_{conv} defined above.

For example, you showed in Task 117 that the radius of converges of the geometric series is 1, and in Task 143 that the radius of convergence of the exponential series is ∞.

Task 148 (C)**.** Show that each of the following power series has radius of convergence equal to 1.†

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad \sum_{n=1}^{\infty} 2^n z^{n!}$$

Task 149 (C)**.** Show that the radius of convergence of $\sum_{n=0}^{\infty} n! z^n$ is 0.

Task 150 (C)**.** Show that the radius of convergence of $\sum_{n=0}^{\infty} (cz)^n$, with $c \in \mathbb{C} \setminus \{0\}$ a constant, is $1/|c|$.

If a power series S centered at z_0 has radius of convergence $R = R_{\text{conv}} > 0$, then Task 146 shows that S is guaranteed to converge (absolutely) at every point of the open disk $N_R(z_0)$. However, S may or may not converge at points on the boundary of $N_R(z_0)$. We have seen that the geometric series converges on $N_1(0)$ and diverges at every point of $\partial N_1(0)$, which is the unit circle. The next task illustrates two other possibilities: a power series may converge at all points of $\partial N_R(z_0)$, or at some points but not others.

Task 151 (C)**.** Consider the three series from Task 148.

- Show that the first series converges for all z on the unit circle.
- Show that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.
- (Optional) Show that the third series diverges for infinitely many points on the unit circle.††

The theory of complex power series can clear up some otherwise inscrutable facts about power series of a real variable.

Task 152 (CD)**.** Use the power series for $1/(1 - z)$ to find the power series for $1/(1 + z^2)$. What is the radius of convergence R for this power series? What happens to the function on the boundary of $N_R(0)$, particularly where it intersects the imaginary axis? Note that $1/(1 + x^2)$ is infinitely differentiable everywhere on the real line, but its Taylor series at 0 does not converge for all real numbers. How does the complex picture help explain this phenomenon?

†That is, show that each series converges when $|z| < 1$ and diverges when $|z| > 1$.

††Consider roots of unity.

5.7 Analyticity

Definition 5.7.1. A function $f : U \to \mathbb{C}$ is *analytic at* $z_0 \in U$ if there is a neighborhood of z_0 on which

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some coefficients $a_n \in \mathbb{C}$. If f is analytic at every point of U, we say that it is *analytic on* U, or if the domain U is clear, simply *analytic*.

Task 153 (D)**.** Explain why each of the following functions is analytic†:

- a power function z^n on $\mathbb{C}$, where $n \in \mathbb{N}$
- the complex exponential $z \mapsto e^z$ on $\mathbb{C}$
- the function $z \mapsto 1/z$ on $\mathbb{C} \setminus \{0\}$

Task 154 (D)**.** Using Theorem 5, explain why every analytic function is holomorphic.

The converse of the result from Task 154—i.e., the statement that every holomorphic function is analytic—is astonishingly also true, and one of the major results of this subject. Just as astonishing, perhaps, is that in order to reach this result about complex differentiability, we need to first pass through complex integration, which will be introduced in the next chapter.

For real-valued functions, it is still true that analyticity implies differentiability, but even if a real-valued function is differentiable everywhere it is by no means guaranteed to be analytic.

Task 155 (E)**.** Show that an analytic function has a unique power series at each point of its domain. That is, if $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ converge to the same values on a neighborhood of z_0, then $a_n = b_n$ for all n.

5.8 Proofs of Theorems 5 and 6 (Optional)

Here you will prove that the formal operations of multiplying and differentiating power series behave as expected when the series converge on an open set. First, some general comments.

You have seen that if a power series converges at more than a single point (its center), then it converges on an open disk (or all of $\mathbb{C}$), and possibly at some or all points on the boundary of that disk, but not outside the disk. Therefore, if a power series converges on an open set, that set must be contained in an open disk on which the power series converges as well, so there is no loss of generality in assuming that the open set is a disk. Also, within its radius of convergence from z_0, a power series does not merely converge, it converges absolutely, as shown in Task 146. This feature is essential in the next two tasks.

Task 156 (E, Optional)**.** Suppose the series $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=0}^{\infty} \beta_n$ converge absolutely. For each $n \in \mathbb{N}$, set $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k}$.

†Given $z_0 \in \mathbb{C}$, try writing $z = z_0 + z - z_0$, then use properties of each function to express it as a power series centered at z_0.

- Show that $\sum_{n=0}^{\infty} \gamma_n$ converges absolutely by comparing $\sum_{n=0}^{N} |\gamma_n|$ to $\left(\sum_{n=0}^{N} |\alpha_n|\right)\left(\sum_{n=0}^{N} |\beta_n|\right)$.†
- Show that $\lim_{N\to\infty} \left(\sum_{n=0}^{N} \gamma_n - \left(\sum_{n=0}^{N} \alpha_n\right)\left(\sum_{n=0}^{N} \beta_n\right)\right) = 0$.
- Obtain Theorem 6 by substituting $\alpha_n = a_n(z - z_0)^n$ and $\beta_n = b_n(z - z_0)^n$.

Task 157 (E, Optional)**.** Let S be a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ with radius of convergence $R > 0$.

- Show that if S converges at some $z_1 \neq z_0$, then the formal derivative of S converges absolutely at any z_2 such that $|z_2 - z_0| < |z_1 - z_0|$.†† Conclude that the formal derivative of S also has radius of convergence R.
- Suppose $|z_1 - z_0| < R$. For each n, show that

  $$\lim_{z\to z_1} \frac{a_n(z - z_0)^n - a_n(z_1 - z_0)^n}{z - z_1}$$

  equals $na_n(z_1 - z_0)^{n-1}$. Then use the definition of the derivative at z_1 to prove Theorem 5. (You will have to exchange a limit and a summation, which is non-trivial, but possible in this case, again thanks to absolute convergence.)

†Use the triangle inequality. Observe that the sequence $\left(\sum_{n=0}^{N} |\alpha_n|\right)\left(\sum_{n=0}^{N} |\beta_n|\right)$ converges to $\left(\sum_{n=0}^{\infty} |\alpha_n|\right)\left(\sum_{n=0}^{\infty} |\beta_n|\right)$.

††Adapt the method of Task 146. It may help to consider the result of Task 144, as well.

Chapter 6

Contours and contour integrals

6.1 Curves and contours in $\mathbb{C}$

When integrating functions of a complex variable, we are primarily interested in integrals over (real) one-dimensional subsets of $\mathbb{C}$ called "contours". The notion of "dimension" is somewhat slippery; here we'll take "one-dimensional" to mean that the set is parametrized by a certain kind of function (which we'll call a "curve") whose domain is an interval in $\mathbb{R}$. The next definition provides the necessary details.

Definition 6.1.1. A *curve* in $\mathbb{C}$ is a continuous function $\gamma : I \to \mathbb{C}$, where $I \subseteq \mathbb{R}$ is an interval. If $\gamma(t) = x(t) + iy(t)$ and the real-valued functions $x(t)$ and $y(t)$ are differentiable at t_0, then the *derivative* of γ at t_0 is $\gamma'(t_0) = x'(t_0) + iy'(t_0)$. A curve is *piecewise smooth* if it has a unit tangent vector $\gamma'(t)/|\gamma'(t)|$ at all but finitely many points, at which it has one-sided unit tangent vectors.* A *contour* in $\mathbb{C}$ is the set C of points parametrized by a piecewise smooth curve γ whose domain is a closed and bounded interval $[a,b]$.** We say in this case that C is a contour *from* $\gamma(a)$ *to* $\gamma(b)$, or that C *joins* $\gamma(a)$ to $\gamma(b)$.***

Task 158 (CD)**.** Sketch the contour parametrized by each of the following curves, either by hand or using a graphing program. Label the point(s) where each curve begins and ends. Using one or more arrows, indicate the direction in which the contour is traveled as the parameter increases.

- $\gamma(t) = e^{it}$, $t \in [0, \pi]$
- $\gamma(t) = 2e^{-it}$, $t \in [0, \pi]$
- $\gamma(t) = \begin{cases} 1 - e^{it} & \text{if } t \in [0, 2\pi] \\ e^{it} - 1 & \text{if } t \in \,]2\pi, 4\pi] \end{cases}$

*The one-sided unit tangent vectors at $\gamma(t_0)$ are $\lim\limits_{t \to t_0^-} \frac{\gamma'(t)}{|\gamma'(t)|}$ and $\lim\limits_{t \to t_0^+} \frac{\gamma'(t)}{|\gamma'(t)|}$.

**It is common in mathematics to use "curve" to refer to both the function $\gamma : I \to \mathbb{C}$ and its image $C = \gamma(I)$. We will be slightly more careful to distinguish them. Fortunately, the terminology of "contour" is standard in the theory of complex variables. As an analogy, imagine tracing with a pen: a contour is what you draw, while a curve is how you draw it.

***While the reason for using the letter C in this context is likely to be obvious, the reason for using γ ("gamma") may seem less so. In the Greek alphabet, γ is the third letter, so it corresponds in order to the letter c in the Roman alphabet. (Greek does not have a letter c, although in older scripts the capital sigma Σ may appear written as C.)

- $\gamma(t) = \begin{cases} 1 - e^{it} & \text{if } t \in [0, 2\pi] \\ e^{-it} - 1 & \text{if } t \in]2\pi, 4\pi] \end{cases}$
- $\gamma(t) = \begin{cases} e^{it} + 1 & \text{if } t \in [0, 2\pi] \\ 2e^{it} & \text{if } t \in]2\pi, 4\pi] \end{cases}$

Task 159 (CD)**.** Same instructions as the previous task.

- $\gamma(t) = t^2 + it$, $t \in [-1, 1]$
- $\gamma(t) = t^2 + it^3$, $t \in [-1, 1]$
- $\gamma(t) = t^2 + i(t^3 - t)$, $t \in [-1, 1]$
- $\gamma(t) = t^2 + i(t^3 - t)$, $t \in [-2, 2]$

Task 160 (CD)**.** Find curves that parametrize each of these contours.

- the line segment from 0 to $1 + i$
- the line segment from 1 to i
- the circle with center $3 + 4i$ and radius 5, traveled counterclockwise
- the triangle with vertices 0, 3, and $3 + 4i$, traveled counterclockwise

Given a contour, we are usually concerned only with the order in which its points are covered by a parametrization, not the particular rate. Thus we introduce the following notion, which states when we will treat two curves that parametrize the same set of points as defining "the same" contour.

Definition 6.1.2. Suppose C is a contour. Two parametrizations $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ of C are *equivalent* if there is an increasing function $\phi : [a_1, b_1] \to [a_2, b_2]$ such that $\phi(a_1) = a_2$, $\phi(b_1) = b_2$, and $\gamma_1(\phi(t)) = \gamma_2(t)$.

If two parametrizations of a contour C are equivalent, then they have the same starting and ending points, and their unit tangent vectors have the same direction at each point of C. In other words, both curves endow C with the same *orientation*, i.e., the direction in which C is traveled.

Definition 6.1.3. If C is a contour, then we write $-C$ to mean the contour with the same set of points but the opposite orientation.

Task 161 (D)**.** If $\gamma : [a, b] \to \mathbb{C}$ parametrizes the contour C, then one parametrization of $-C$ is the curve* $\eta(t) = \gamma(a + b - t)$, with the same domain $[a, b]$. Explain why.

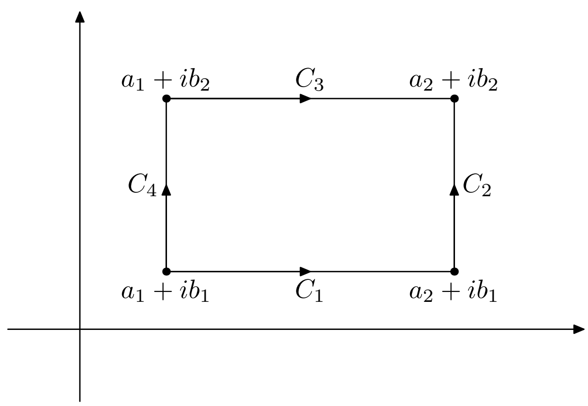
Under appropriate conditions, we can also "add together" two contours to produce a new contour. The fancy name for this process is *concatenation*, but we will simply call it a sum.

Definition 6.1.4. If C_1 and C_2 are contours, parametrized by $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ respectively, such that $\gamma_1(b_1) = \gamma_2(a_2)$, then the *sum* of C_1 and C_2 is the contour $C_1 + C_2$ that joins $\gamma_1(a_1)$ to $\gamma_2(b_2)$ by following first C_1, then C_2.

Task 162 (C)**.** Suppose C_1 and C_2 are parametrized by curves γ_1 and γ_2 that have the properties listed in Definition 6.1.4. Construct a curve $\gamma_3 : [a_3, b_3] \to \mathbb{C}$ that parametrizes $C_1 + C_2$.

*When γ is the name of one curve and a name for a second curve is needed, we often use η ("eta") because it corresponds to the Roman letter h, which follows g, which is the Roman form of γ, which we chose in place of c. Perfectly sensible.

Task 163 (C)**.** Let a_1, a_2, b_1, b_2 be real numbers such that $a_1 < a_2$ and $b_1 < b_2$. Let C be the boundary of the rectangle with vertices $a_1 + ib_1$, $a_2 + ib_1$, $a_2 + ib_2$, and $a_1 + ib_2$, oriented counterclockwise. Write C as a sum of four contours, $C = C_1 + C_2 - C_3 - C_4$, where C_1 and C_3 are horizontal segments oriented left-to-right and C_2 and C_4 are vertical segments oriented bottom-to-top. (See figure below.) Find curves that parametrize each C_k.† (We'll use these parametrizations in some later tasks.)



Definition 6.1.5. Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. We say that γ is *closed* if $\gamma(a) = \gamma(b)$, and γ is *simple* if $\gamma(t) \neq \gamma(s)$ for all $t, s \in [a, b]$, except possibly if $t, s \in \{a, b\}$. We will also use "closed" and "simple" to describe contours that are parametrized by the corresponding type of curve.*

Task 164 (D)**.** Identify which contours in Tasks 158–160 are simple, which are closed, and which are both.

Intuitively, we can think of a simple closed contour as dividing $\mathbb{C}$ into exactly two pieces, the "inside" and "outside" of the contour. The next definition and theorem formalize this intuition.

Definition 6.1.6. Suppose C is a simple closed contour and p is any complex number such that $p \notin C$. We call p an *exterior point* of C if there exists an $R > 0$ such that for all p' outside of $N_R(0)$, p can be joined to p' by a contour that does not intersect C. Otherwise, p is an *interior point* of C.

Theorem 11. *If C is a simple closed contour, then any two interior points of C can be joined by a contour that does not intersect C.*

Theorem 11 is known as the *Jordan curve theorem*.* It is infamous as a statement that is seemingly obvious, but surprisingly tricky to prove. The next task illustrates part of the challenge: it is not always immediately evident whether a given point is an interior or exterior point of the contour.

†Use x or y as a parameter for each curve, with the domain $[a_1, a_2]$ or $[b_1, b_2]$.

*Loosely speaking, a simple curve or contour "does not cross itself." We allow for the endpoints of a simple curve to coincide in order to define a "simple closed curve," which will be important in our study.

*Named after Camille Jordan, who proved the general theorem (that is, even for curves that are not piecewise smooth) in his 1887 textbook *Cours d'analyse de l'École Polytechnique*. Simple closed curves are often called "Jordan curves" in his honor.

Task 165 (C)**.** Verify that the image below shows a simple closed contour. Of the points p and q, which is an interior point of the contour and which is an exterior point? How can you tell?

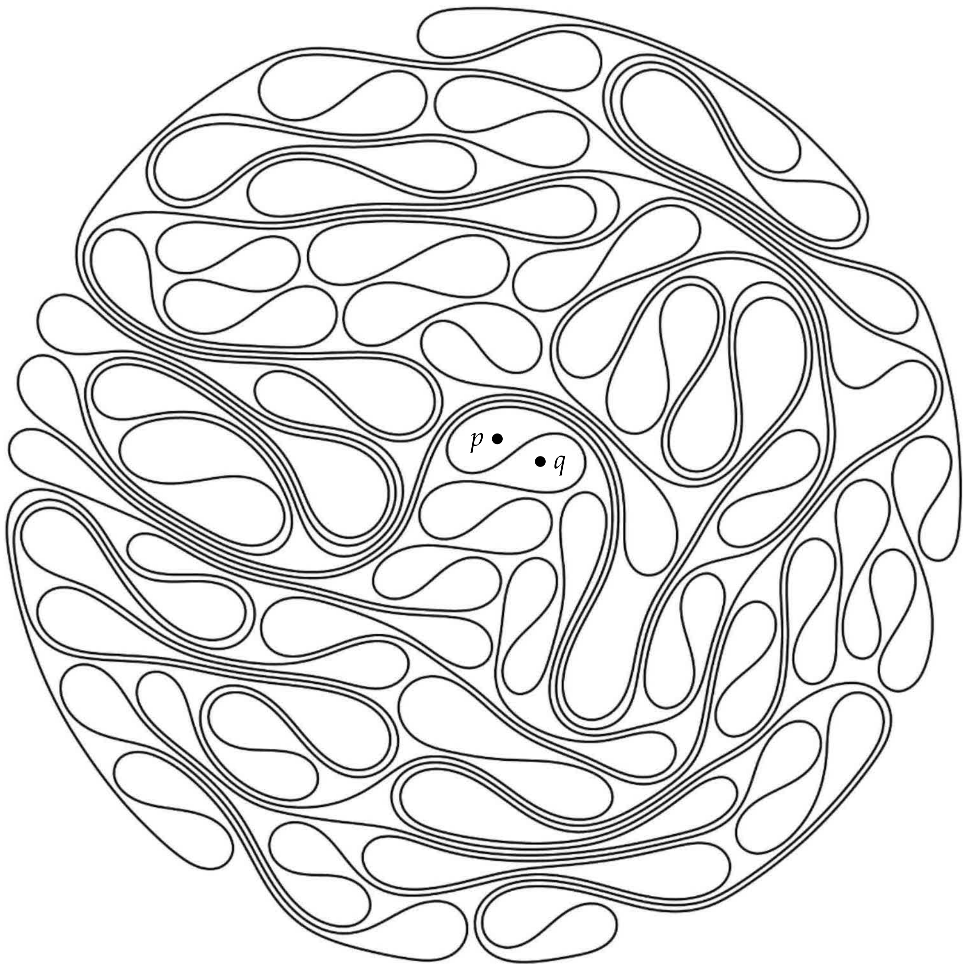
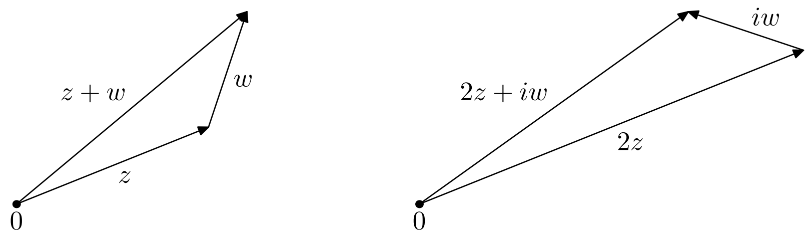


Image source: Karthik Dondeti. Used with permission. (The points p and q have been added.)

6.2 Riemann sums

Recall that a sum of complex numbers can be visualized as a vector sum, with each number represented by an arrow that starts where the previous arrow ended. (See the image below on the left.) The same is true if each number has a coefficient, which transforms the number prior to completing the addition. (See the image below on the right. The numbers z and w are the same in the two images.)



Definition 6.2.1. Let $a_1, \dots, a_n$ and $z_1, \dots, z_n$ be complex numbers. The *weighted sum* of $z_1, \dots, z_n$ with (*complex*) *weights** $a_1, \dots, a_n$ is

$$\sum_{k=1}^{n} a_k z_k = a_1 z_1 + \cdots + a_n z_n.$$

Task 166 (C)**.** Compute the following weighted sums, and draw a picture to represent each.

- $1 + e^{i2\pi/3} + e^{i4\pi/3}$
- $1 \cdot 1 + 2 \cdot e^{i2\pi/3} + 3 \cdot e^{i4\pi/3}$
- $1 + e^{i\pi/2} + e^{i\pi} + e^{i3\pi/2}$
- $1 \cdot 1 + 2 \cdot e^{i\pi/2} + 3 \cdot e^{i\pi} + 4 \cdot e^{i3\pi/2}$
- $i \cdot 1 + i^2 \cdot e^{i\pi/2} + i^3 \cdot e^{i\pi} + i^4 \cdot e^{i3\pi/2}$

The kind of integrals we wish to study are essentially continuous versions of weighted sums.

Let $f : U \to \mathbb{C}$ be a continuous complex-valued function on $U \overset{\circ}{\subseteq} \mathbb{C}$, and let $\gamma : [a,b] \to \mathbb{C}$ be a curve that parametrizes a contour C, which is contained in U. If we divide $[a,b]$ into n equal pieces, each of length $\Delta t = (b-a)/n$, then we obtain $n+1$ endpoints $t_k = a + k\Delta t$ in $[a,b]$, with $t_0 = a$ and $t_n = b$. By applying γ to each of these points in $[a,b]$ we get $n+1$ points in $\mathbb{C}$:

$$z_0 = \gamma(a), \qquad z_1 = \gamma(t_1), \qquad \dots, \qquad z_k = \gamma(t_k), \qquad \dots, \qquad z_n = \gamma(b).$$

When n is large, the segment from z_{k-1} to z_k approximates the piece of the contour C that starts at z_{k-1} and ends at z_k, and by telescoping sums we have

$$z_k = z_0 + \sum_{j=1}^{k} (z_j - z_{j-1}).$$

Each $(z_j - z_{j-1})$ may be thought of as a vector, which approximates a tangent vector to the curve γ.

Now we create weighted sums using the function $f(z)$ to provide the weights. As in the case of real integrals, we could use any point along the kth piece of C to determine the weight on that piece. For simplicity, however, we will only use the analogues of "left-hand" and "right-hand" sums.

Definition 6.2.2. The *nth Riemann sums* of $f(z)$ over a contour C with respect to a parametrization $\gamma : [a,b] \to \mathbb{C}$ are defined by

$$L_n(f,\gamma) = \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1}) \qquad \text{and} \qquad R_n(f,\gamma) = \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}),$$

*A weighted sum is also called a *linear combination*. The weights are then called *coefficients*.

where $z_k = \gamma(t_k)$ with $t_k = a + k\Delta t = a + k(b-a)/n$. $L_n(f,\gamma)$ is the *nth left Riemann sum*, and $R_n(f,\gamma)$ is the *nth right Riemann sum*.

Task 167 (C). Let $f(z) = \overline{z}$ and let C be the first contour from Task 158. Find $L_n(f,\gamma)$ and $R_n(f,\gamma)$ for $n = 2,3,4$. Find general expressions that work for any n.

The definition we have given for Riemann sums depends on the curve γ, not just the contour C. You should anticipate, however, that we will take limits of $L_n(f,\gamma)$ and $R_n(f,\gamma)$ as $n \to \infty$. As we do so, the points z_k will become distributed more and more densely along C, and the particular choice of γ will not matter. We will see this property illustrated in several examples; the general principle is established in analysis courses.

6.3 Contour integrals

Here we will see how contour integrals are continuous analogues of weighted sums.

Definition 6.3.1. If $C = \gamma([a,b])$ is a contour contained in the open set $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ is continuous, then the *contour integral of f over C* is*

$$\oint_C f(z)\,dz = \lim_{n\to\infty} L_n(f,\gamma) = \lim_{n\to\infty} R_n(f,\gamma).$$

Although we do not prove it here, the fact that the sequences of left-hand sums $L_n(f,\gamma)$ and right-hand sums $R_n(f,\gamma)$ converge and their limits are equal is a consequence of the continuity of f and the piecewise smoothness of γ. Moreover, the chain rule guarantees that equivalent parametrizations of the contour C yield the same value for the integral. (This is clearer from the formula in Theorem 12 below.)

Task 168 (C). Use Definition 6.3.1 to find

$$\oint_C dz \qquad \text{and} \qquad \oint_C c\,dz,$$

where $C = \gamma([a,b])$ is any contour and $c \in \mathbb{C}$ is any constant.

Task 169 (C). Find expressions for the nth Riemann sums of $f(z) = z$ over any contour $C = \gamma([a,b])$. Find the average of $L_n(z,\gamma)$ and $R_n(z,\gamma)$, and explain why

$$\oint_C z\,dz = \lim_{n\to\infty} \frac{1}{2}\big(L_n(z,\gamma) + R_n(z,\gamma)\big).$$

Then use this equality to compute the integral.

Task 170 (C). Interpret the limit of the expressions you found in Task 167 as an integral, and evaluate it.†

*The distinction in meaning between the integral signs $\oint$ and $\int$ is not entirely standardized. These notes use $\oint$ to indicate that the integral is defined over a parametrized set in $\mathbb{C}$, and $\int$ when the integral is over an interval in $\mathbb{R}$.

†Try using the power series for e^z.

Task 171 (E)**.** Suppose that $U \overset{\circ}{\subseteq} \mathbb{C}$, $C = \gamma([a,b])$ is a contour contained in U, f and g are continuous functions $U \to \mathbb{C}$, and $c \in \mathbb{C}$ is a constant. Using Definition 6.3.1, show that the following equalities are true:

$$\oint_C cf(z)\,dz = c\oint_C f(z)\,dz, \qquad \oint_C \big(f(z)+g(z)\big)\,dz = \oint_C f(z)\,dz + \oint_C g(z)\,dz.$$

The next task justifies the use of $+$ and $-$ to represent operations on contours.

Task 172 (D)**.** Suppose that $U \overset{\circ}{\subseteq} \mathbb{C}$, $C_1 + C_2$ is a sum of contours C_1 and C_2 that are contained in U, and f is a continuous function $U \to \mathbb{C}$. Give plausible explanations (not necessarily formal proofs) for why

$$\oint_{C_1+C_2} f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz \qquad \text{and} \qquad \oint_{-C_1} f(z)\,dz = -\oint_{C_1} f(z)\,dz.$$

We can rewrite the definition of Riemann sums in a way that permits us to convert contour integrals into ordinary real integrals. Using the notation $\Delta t = (b-a)/n$, we have

$$L_n(f,\gamma) = \sum_{k=1}^{n} f(z_{k-1})\frac{z_k - z_{k-1}}{\Delta t}\Delta t \qquad \text{and} \qquad R_n(f,\gamma) = \sum_{k=1}^{n} f(z_k)\frac{z_k - z_{k-1}}{\Delta t}\Delta t.$$

Notice that the expression $(z_k - z_{k-1})/\Delta t$ approaches a tangent vector $\gamma'(t)$ as $\Delta t \to 0$. The above manner of rewriting the nth Riemann sums of f over C suggests the following theorem.*

Theorem 12. *Suppose $U \overset{\circ}{\subseteq} \mathbb{C}$, $f : U \to \mathbb{C}$ is continuous, and $C = \gamma([a,b])$ is contained in U. Then*

$$\oint_C f(z)\,dz = \int_a^b f\big(\gamma(t)\big)\cdot\gamma'(t)\,dt.$$

The resulting integral on the right has a complex-valued integrand, although the variable t is real. Notice that the integrand looks like a tangent vector $\gamma'(t)$ times a weight $f(\gamma(t))$, which is why this integral resembles a weighted sum. To evaluate such an integral, we compute its real and imaginary parts separately, using standard tools from calculus.

Task 173 (C)**.** Given a positive number $R > 0$, let C be the circle parametrized by $\gamma(t) = Re^{it}$, $0 \le t \le 2\pi$. Use Theorem 12 to compute the following integrals:

$$\oint_C \bar{z}\,dz, \qquad \oint_C z^2\,dz, \qquad \oint_C \frac{1}{z}\,dz.$$

Task 174 (C)**.** Suppose $C = C_1 + C_2 - C_3 - C_4$ is the boundary of a rectangle with vertices $a_1 + ib_1$, $a_2 + ib_1$, $a_2 + ib_2$, and $a_1 + ib_2$, as in Task 163, and $f = u + iv$ is a continuous function defined on an open set U that contains C. Write $\oint_C f(z)\,dz$ in terms of real integrals of $u(x,y)$ and $v(x,y)$ over the intervals $[a_1,a_2]$ and $[b_1,b_2]$. (Hang on to the formulas you obtain. We'll need them in the next section.)

*Actually proving this theorem is finicky, and we will not attempt it. Some texts avoid this difficulty by using the equality stated in Theorem 12 as the definition of a contour integral.

6.4 Bounding the modulus of a contour integral

It is often useful, for practical or theoretical purposes, to estimate how large (in absolute value) a contour integral might possibly be, without determining its exact value. In Task 176 you will derive such an estimate, but first we need to define the length of a contour; the definition is the same as in multivariable calculus.

Definition 6.4.1. If C is a contour parametrized by $\gamma : [a,b] \to \mathbb{C}$, then the *length* of C is*

$$\oint_C |dz| = \lim_{n\to\infty} \sum_{k=1}^{n} |z_k - z_{k-1}| = \int_a^b |\gamma'(t)|\, dt.$$

Here, as in Definition 6.2.2, we use the notation $z_k = \gamma(t_k)$, with $t_k = a + k(b-a)/n$.

Task 175 (C)**.** Find $\oint_C |dz|$, where C is each of the contours in Tasks 158 and 160 .

Although the length of a contour is an interesting geometric quantity in its own right**, for our purposes its primary utility is in the next formula.

Task 176 (E)**.** Suppose that C is a contour, f is a continuous function, and $M > 0$ is chosen such that $|f(z)| \le M$ for all $z \in C$. Let L be the length of C. Using Definitions 6.3.1 and 6.4.1, show that†

$$\left| \oint_C f(z)\, dz \right| \le M \cdot L.$$

Task 177 (C)**.** Let C be the second contour from Task 158. Show that

$$\left| \oint_C \frac{1}{z^2+1} dz \right| \le \frac{2\pi}{3}\ .^{\dagger\dagger}$$

Task 178 (D)**.** Use the results of Task 173 to illustrate that sometimes the inequality in Task 176 is strict (i.e., the left side is definitely smaller than the right side), and sometimes it is an equality.

*In multivariable calculus classes, the "arc length element" $|dz|$ is often written ds instead.

**We could even extend the definition of the length integral to include a function $f(z)$ and have $\oint_C f(z)\,|dz|$ represent the *integral of f with respect to arc length*; in multivariable calculus this is just the usual (unoriented) line integral. However, we will not need such line integrals in this course.

†One appraoch is to apply the triangle inequality from Task 30 to the Riemann sums.

††Show that on C the modulus of $z^2 + 1$ is always at least 3.

Chapter 7

Fundamental Theorems of Calculus and Cauchy's Theorem

7.1 First fundamental theorem

Recall from calculus that if f is a differentiable real-valued function on an interval containing $[a,b]$, and its derivative f' is continuous, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This is one of the Fundamental Theorems of Calculus.* You have likely already used it several times in this course. To achieve its analogue in complex variables, we introduce the following definition.

Definition 7.1.1. Let $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$. We will say that f is C^1*-holomorphic*** if the complex derivative f' exists and is continuous on U.

Task 179 (E)**.** Suppose f is C^1-holomorphic on U, and C is a contour parametrized by $\gamma : [a,b] \to U$. Explain why†

$$\oint_C f'(z)\,dz = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Definition 7.1.2. Let $U \overset{\circ}{\subseteq} \mathbb{C}$. If $f : U \to \mathbb{C}$ is continuous, and $F : U \to \mathbb{C}$ is a holomorphic function such that $F' = f$, then we call F an *antiderivative* of f.

As in the real case, the result of Task 179 means that if we can find an antiderivative for the integrand of a contour integral, then we can compute the integral simply by evaluating the antiderivative at the endpoints of the contour and calculating the difference of those values.

*Sources differ in their conventions of whether this is the "first" or "second" fundamental theorem of calculus. It is almost certainly the one that is more familiar to students.

**Read as "cee-one-holomorphic." We are borrowing from the language of analysis, in which a C^k function (where $k \in \mathbb{N}$) is assumed to have k continuous derivatives. The terminology of "C^1-holomorphic" is not standard, however, for reasons that will be explained in §7.3. It will be for us a useful notion until then.

†Try using the chain rule on $f(\gamma(t))$.

Task 180 (C)**.** Use the result of Task 179 to compute the following integrals. Write your answers in the form $x + yi$.

- $\oint_C z^2\,dz$, where C is either the first or second contour from Task 159
- $\oint_C e^{iz}\,dz$, where C is any contour from 0 to π

Task 181 (E)**.** Suppose $f : U \to \mathbb{C}$ is C^1-holomorphic and C is a closed contour in U. Explain why

$$\oint_C f'(z)\,dz = 0.$$

7.2 Second fundamental theorem

The second Fundamental Theorem of Calculus states that, if f is a continuous real-valued function on an open interval I and $a \in I$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable on I, with $F' = f$. In other words, every continuous function on an open interval has an antiderivative on that interval.

An analogous result for functions of a complex variable requires a bit more care. First, we will need to assume that our function f is not just continuous, but C^1-holomorphic.* The next task illustrates that, for complex-valued functions, continuity is not sufficient to guarantee the existence of an antiderivative.

Task 182 (E)**.** Explain how one result from Task 173 implies that $z \mapsto \bar{z}$ does <u>not</u> have an antiderivative.

Second, open subsets of $\mathbb{C}$ can have much more complicated shapes than open intervals in $\mathbb{R}$ can. The next definition provides the necessary restrictions.

Definition 7.2.1. Let $U \overset{\circ}{\subseteq} \mathbb{C}$. We say that U is *connected* if any two points of U can be joined by a contour in U. We say that U is *simply connected* if it is connected and, in addition, every simple closed contour in U has only points of U as interior points.

If U is connected, then the following is equivalent to saying that U is simply connected: for every simple closed contour C in U, if $z \notin U$ then z can be joined to a point sufficiently far from 0 by a contour that does not intersect C.

Task 183 (D)**.** Explain, using Definition 7.2.1, why the following sets are simply connected†: $\mathbb{C}$, $\mathbb{D}$, $\mathbb{H}$

*Recall from Task 96 that a holomorphic function is itself automatically continuous; we are adding an additional condition on the <u>derivative</u> of the function.

†To review the definitions of $\mathbb{D}$ and $\mathbb{H}$, see §3.1.

Task 184 (D)**.** Determine whether each of the following sets is simply connected or not.

- $\mathbb{C} \setminus \{0\}$
- $\mathbb{C} \setminus]-\infty, 0]$ (the complement of the nonpositive real numbers)
- $\mathbb{C} \setminus \mathbb{R}$

Tasks 185–187 will complete the second part of the "holomorphic fundamental theorem of calculus." (Task 179 is the first part.) That is, after completing these tasks, you will have proved that

If f is a C^1-holomorphic function on a simply connected open set, then f has an antiderivative.

We begin with a result that may be called "Cauchy's Theorem for rectangles".

Task 185 (CE)**.** Let a_1, a_2, b_1, b_2 be real numbers with $a_1 < a_2$ and $b_1 < b_2$. Let C be the boundary of the rectangle with vertices $a_1 + ib_1$, $a_2 + ib_1$, $a_2 + ib_2$, and $a_1 + ib_2$, oriented counterclockwise. Suppose $f = u + iv$ is a C^1-holomorphic function defined on an open set that contains C and all its interior points.

a. Explain why, for any fixed value of y,

$$u(a_2, y) - u(a_1, y) = \int_{a_1}^{a_2} \frac{\partial u}{\partial x}(x,y)\,dx \quad \text{and} \quad v(a_2, y) - v(a_1, y) = \int_{a_1}^{a_2} \frac{\partial v}{\partial x}(x,y)\,dx.$$

b. Explain why, for any fixed value of x,

$$u(x, b_2) - u(x, b_1) = \int_{b_1}^{b_2} \frac{\partial u}{\partial y}(x,y)\,dy \quad \text{and} \quad v(x, b_2) - v(x, b_1) = \int_{b_1}^{b_2} \frac{\partial v}{\partial y}(x,y)\,dy.$$

c. Use the results of Task 174 along with parts a. and b. to write the real and imaginary parts of $\oint_C f(z)\,dz$ as double integrals over the rectangle $[a_1, a_2] \times [b_1, b_2]$.[†]

d. Show that the double integrals you obtained in part c. are both equal to zero, so that

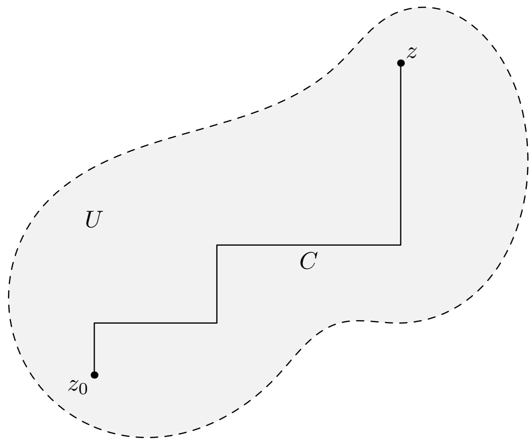
$$\oint_C f(z)\,dz = 0.$$

The result of Task 185 will find fuller expression in Task 189. At the moment, it provides an essential element for defining an antiderivative of any C^1-holomorphic function on a simply connected open set. The goal of Tasks 186–187 is to demonstrate the existence of such an antiderivative.[*]

For this paragraph and the next two tasks, U is a simply connected open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a C^1-holomorphic function. We construct a new function $F : U \to \mathbb{C}$ in the following way. Pick a point $z_0 \in U$. If z is any other point of U, then z_0 can be connected to z by a contour consisting of a finite number of horizontal and vertical segments. (See the figure below for an example.)

[†]You will probably need Fubini's Theorem: if $\psi(x,y)$ is continuous, then $\int_{a_1}^{a_2}\int_{b_1}^{b_2} \psi(x,y)\,dy\,dx = \int_{b_1}^{b_2}\int_{a_1}^{a_2} \psi(x,y)\,dx\,dy$.

[*]These two tasks can be conceptually challenging, but the individual steps should guide you in the right direction. Make sure you read them carefully, and apply previous results when appropriate.

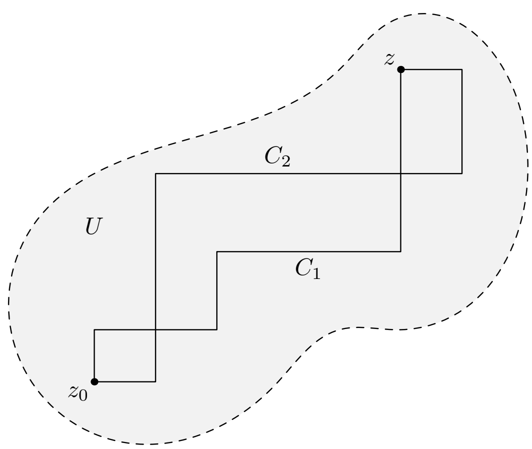


Define $F(z)$ by

($\star$) $F(z) = \oint_C f(w)\,dw$, where C is any contour that joins z_0 to z via horizontal and vertical segments.

We now need to verify two things: first (Task 186), that this function F is well-defined—meaning that the value of $F(z)$ does not depend on which contour C we choose—and second (Task 187), that $F' = f$.

Task 186 (E)**.** In this task you will show that Equation ($\star$) produces a well-defined function F. Suppose that C_1 and C_2 are two contours in U from z_0 to z, each composed of horizontal and vertical segments. (See the next figure for an illustration.)



a. Justify (informally is fine) why any region between C_1 and C_2 can be decomposed into a finite number of rectangles with horizontal and vertical sides.
b. Use part a. along with the result of Task 185 to explain why, in between any two points where C_1 and C_2 intersect, the integrals over the respective pieces of C_1 and C_2 are equal.
c. Conclude that
$$\oint_{C_1} f(w)\,dw = \oint_{C_2} f(w)\,dw,$$
and so $F(z)$ is well-defined.

Task 187 (E). In order to show that $F' = f$, we return to the definition of the derivative. Let $z \in U$.

a. Explain why, if $z+h$ is also in U, then $F(z+h) - F(z)$ is the integral of f over a contour C_h that joins z to $z+h$.[†]
b. Explain why, if $h \in \mathbb{C}$ is small enough, then a contour C_h that joins z to $z+h$ can be chosen to have at most one horizontal and one vertical segment.
c. For h and C_h as in part b., explain why $\oint_{C_h} dw = h$, and use this equality to show that

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \oint_{C_h} \big(f(w) - f(z)\big)\, dw \right| \le \frac{M}{|h|}\big(|\mathrm{Re}\, h| + |\mathrm{Im}\, h|\big) \le 2M,$$

where M is the maximum of $|f(w) - f(z)|$ for $w \in C_h$.[*] (Justify each equality or inequality.)
d. Explain why, as $h \to 0$, the value of M in part c. also approaches 0.[††]
e. Conclude that $F'(z) = f(z)$.

Task 188 (D). Where in Tasks 186 and 187 are each of the following assumptions used?

- U is open
- U is connected
- U is simply connected
- f is holomorphic

7.3 Cauchy–Goursat Theorem

Using results from sections 7.1 and 7.2, you should now be able to prove *Cauchy's Theorem*, which is stated in the next task. You should recognize this as a generalization of Task 185.

Task 189 (E). Show that if f is C^1-holomorphic on a simply connected open set U, and C is any closed contour in U, then[†††]

$$\oint_C f(z)\, dz = 0.$$

For some time after Cauchy proved his eponymous theorem,[*] it was an open question whether the requirement that f be C^1-holomorphic could be dropped and replaced with the assumption that f simply be holomorphic. In other words, is it possible to assume only that f' exists at every point—but not assume that f' is continuous—and still reach the same conclusion? Eventually, Édouard Goursat showed that, yes, this weakening of assumptions still produces a true statement.[**]

Cauchy's Theorem (which is also called the Cauchy–Goursat Theorem, in honor of Goursat's contribution) may seem unimpressive on the surface; after all, it just

[†]Notice that if C is any contour joining z_0 to z, then $C + C_h$ joins z_0 to $z+h$.

[*]The existence of such a maximum follows from the Extreme Value Theorem, which is often stated in calculus and proved in analysis classes.

[††]Use the facts that f is continuous and $f(w) \in N_{|h|}(z)$ for all $w \in C_h$.

[†††]Combine Task 181 with the result of Tasks 186–187.

[*]First in 1825, then again in 1846 using essentially the tools outlined in these notes.

[**]Goursat seems to have first shared his proof in an 1883 letter to Charles Hermite. It was published in 1884 as an article in *Acta Mathematica*, under the title "Démonstration du théorème de Cauchy (Extrait d'une lettre adressée à M. Hermite)".

says that a certain class of integrals are all equal to zero. But it could just as well be called the *Fundamental Theorem of Complex Variables*, because much of the rest of the theory is based on it. For example, we will see that this theorem implies the derivative of a holomorphic function is automatically continuous—in fact, a holomorphic function has derivatives of all orders, and therefore all its derivatives are continuous. (In analysis courses, one encounters examples of real-differentiable functions whose derivatives are not continuous.)

Henceforth we will assume Goursat's strengthening of Cauchy's Theorem to be true, and we will drop the terminology of "C^1-holomorphic" because it is not standard (as the previous paragraph explains, the Cauchy–Goursat Theorem implies that it means the same thing as "holomorphic").

Task 190 (C)**.** Find $\oint_C \frac{\sin z}{z+i}\,dz$ and $\oint_C \mathrm{Log}(1+z)\,dz$ when C is either of the last two contours from Task 160.

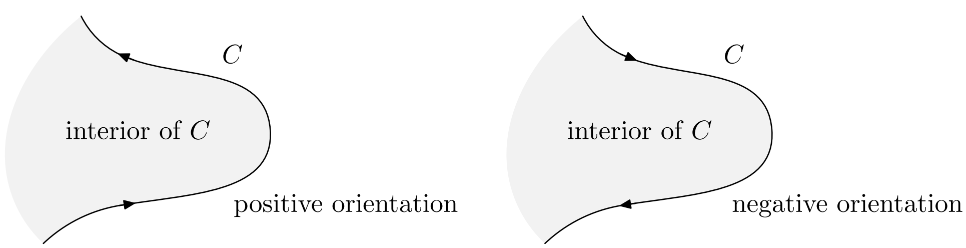
Task 191 (CD)**.** Show that, even if the conditions of Cauchy's Theorem are not satisfied, the conclusion of the theorem may still be true, by calculating

$$\oint_C (\overline{z})^2\,dz \qquad \text{and} \qquad \oint_C \frac{1}{z^2}\,dz$$

where C is the unit circle, oriented counterclockwise. Make sure you explain why Cauchy's Theorem does not apply in each case, and then make sure not to use Cauchy's Theorem to compute the integrals![†]

7.4 A corollary of Cauchy's Theorem

Every contour C has an orientation—which we have described as the "direction" of its unit tangent vectors—that may be found from any of its equivalent parametrizations γ. When C is a simple closed contour, we can classify its orientation as "positive" or "negative".[***] The distinction depends on how the orientation relates to the interior points of C. (See figure below and the following definition.)



Definition 7.4.1. Let C be a simple closed contour. The orientation of C is *positive* if, while following C, interior points of C are "to the left". The orientation of C is *negative* if, while following C, interior points of C are "to the right".

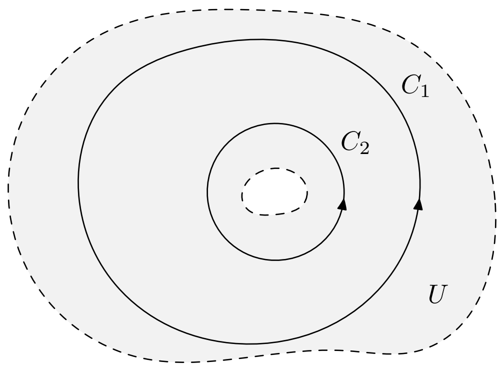
Task 192 (C)**.** Add a few arrows to the contour in Task 165 to indicate the positive orientation. Give special attention to the parts of the contour near p and q.

[†]Theorem 12 may be helpful again.

[***]These orientations are also known as "counterclockwise" and "clockwise", respectively, which coincides with the usual meaning of these terms in the case that C is a circle. The names "positive" and "negative" will be justified in the next section.

The next task contains an important consequence of Cauchy's Theorem that will be used repeatedly in the future. Loosely stated, if a function is holomorphic "in between" two simple closed contours, and both contours are oriented positively, then the integrals of the function over the two contours are equal.

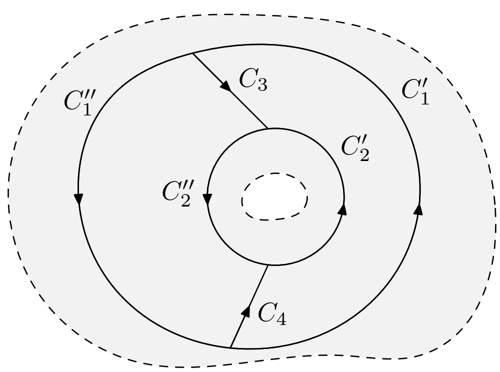
Task 193 (E)**.** Let $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ be holomorphic. Suppose C_1 and C_2 are positively-oriented simple closed contours such that C_2 is contained in the interior of C_1, and all points that are both interior to C_1 and exterior to C_2 are contained in U. (This situation is illustrated in the figure below.)



The goal of this task is to show that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

We begin by introducing disjoint contours C_3 and C_4, each joining a point of C_1 to a point of C_2, all of whose points (except the endpoints) are interior to C_1 and exterior to C_2. (See the next figure.)



The endpoints of C_3 and C_4 split C_1 and C_2 into two separate contours; let C_1' and C_2' be the portions that go from C_4 to C_3, and let C_1'' and C_2'' be the portions that go from C_3 to C_4.

a. Explain why $C_1' + C_3 - C_2' - C_4$ and $C_1'' + C_4 - C_2'' - C_3$ are closed contours.
b. Show that the integral of f over each of the contours in part a. is 0.
c. Add together the integrals from part b. and use Task 172 to get the desired equality.

Task 194 (D)**.** Use one result from Task 173 along with the result of Task 193 to explain why, if C is any positively-oriented simple closed contour that has 0 as an interior point, then

$$\oint_C \frac{1}{z}\,dz = 2\pi i.$$

Task 195 (D)**.** Formulate an extension of Task 193 to the case where C and $C_1, \ldots, C_n$ are all positively-oriented simple closed curves, where each C_k is contained in the interior of C and no C_j is contained in the interior of any C_k, and f is holomorphic on an open set that contains all these contours and all the points between C and the C_ks. Sketch a proof of your statement.

Chapter 8

Integral formulas and their applications

In this chapter we will consider several formulas that are consequences of Cauchy's Theorem, as well as applications of these formulas, starting with a simple topological property and concluding with a proof of the Fundamental Theorem of Algebra. Along the way we will encounter other remarkable features of holomorphic functions, including the astonishing fact that every holomorphic function is analytic.

8.1 Winding number

Task 196 (C)**.** Show that, if C is any positively-oriented simple closed contour, then

$$\oint_C \frac{1}{z - z_0}\, dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ is an interior point of } C, \\ 0 & \text{if } z_0 \text{ is an exterior point of } C. \end{cases}$$

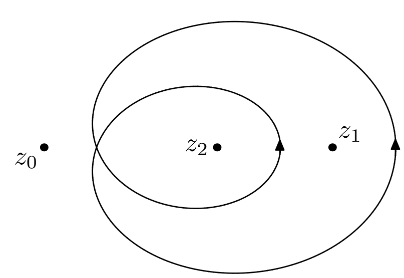
Definition 8.1.1. Let $z_0 \in \mathbb{C}$, and let C be a (not necessarily simple) closed contour that does not pass through z_0. Then the *winding number of C around z_0* is

$$\operatorname{wind}(C, z_0) = \frac{1}{2\pi i} \oint_C \frac{dz}{z - z_0}.$$

The expression $\operatorname{wind}(C, z_0)$ is not defined if z_0 is on C.

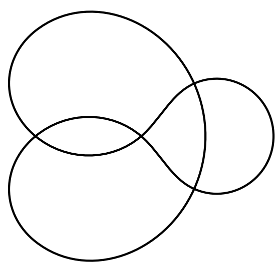
Thus, the orientation of a simple closed curve is positive if its winding number around any of its interior points is positive. A circle with <u>clockwise</u> orientation has winding number -1 around its center.

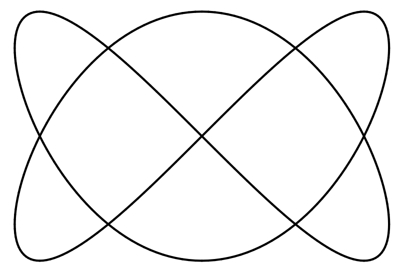
The general idea is that the winding number measures how often a closed contour "goes around" a chosen point, keeping track of positive and negative orientations. For example, given the contour C and the points z_0, z_1, z_2 shown to the right, the winding number of C around z_0 is 0, around z_1 is 1, and around z_2 is 2.



It is often useful to split a closed contour into a sum of simple closed contours, find the winding number of each summand around a point, then add together the results. The contour on the previous page is a sum of two simple closed contours, one inside of the other, both positively oriented. Thus the winding number of the full contour around a point counts whether that point is interior to none, one, or both of these simple closed contours.

Task 197 (C)**.** For each contour C shown below, choose an orientation. Then label each region in the complement of C with the winding number of C around a point of that region.





Often, the winding number of a contour around a point can be found visually, as in Task 197. In such cases, we can use Definition 8.1.1 to evaluate certain integrals. This is a special case of a powerful computational method for contour integrals, which will be developed in the ensuing sections.

Task 198 (C)**.** Show that

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

Use this equation together with winding numbers to find the integral of $1/(z^2-1)$ over each of the three closed contours in Task 158.

8.2 Cauchy integral formula

The theme of several upcoming results is that holomorphic functions are more "rigid" than might be expected; that is, they can be determined by relatively small amounts of data. The result of Task 200 will provide the basis for many of these rigidity properties. Task 199 sets the stage.

Task 199 (E)**.** Let $U \mathring{\subseteq} \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Given $z_0 \in U$, define $g : U \to \mathbb{C}$ by

$$g(z) = \begin{cases} \big(f(z) - f(z_0)\big)/(z - z_0) & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$$

- Explain why $g(z)$ is holomorphic on $U \setminus \{z_0\}$.
- Explain why $g(z)$ is continuous at z_0.
- Suppose C is a simple closed contour in U whose interior is also in U. Show that if z_0 is an interior point of C, then

$$\oint_C g(z)\,dz = 0$$

by justifying each of the following statements: We can replace C with a circle C_ρ centered at z_0 having arbitrarily small radius $\rho > 0$, without changing the value of the integral. Because g is continuous at z_0, there exists $M > 0$ such that $|g(z)| \le M$ for all z in a sufficiently small ε-neighborhood $N_\varepsilon(z_0)$, and thus on any contour contained in $N_\varepsilon(z_0)$. Therefore the modulus of $\oint_{C_\rho} g(z)\,dz$ can be bounded by a quantity that tends to 0 as $\rho \to 0$.

Task 200 (E)**.** Let $U \mathring{\subseteq} \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that C is a positively-oriented simple closed contour in U whose interior is also in U, and z_0 is an interior point of C. Using the result of Task 199, show that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz.$$

The formula in Task 200 is called the *Cauchy integral formula*. It has numerous theoretical and practical uses. One consequence that relates to the notion of “rigidity” is as follows:

*A holomorphic function is determined at every point inside a simple closed contour by its values on the contour.**

But the integral formula can also be used for the purpose of calculation, as in the following tasks.†

Task 201 (C)**.** Evaluate the following integrals.

- $\displaystyle\oint_C \frac{z^2 - 1}{z - i}\,dz$, where C is the circle $|z| = 2$, oriented positively
- $\displaystyle\oint_C \frac{z^3 + 2z}{z + i}\,dz$, where C is the circle $|z| = 2$, oriented positively

Task 202 (C)**.** Evaluate the following integrals.

- $\displaystyle\oint_C \frac{\cos z}{z - \pi}\,dz$, where C is the circle $|z| = 2$, oriented positively
- $\displaystyle\oint_C \frac{\cos z}{z - \pi}\,dz$, where C is the circle $|z| = 4$, oriented positively

Task 203 (C)**.** Evaluate the following integrals.

- $\displaystyle\oint_C \frac{2z + 1}{z(z^2 + 4)}\,dz$, where C is the unit circle, oriented positively
- $\displaystyle\oint_C \frac{e^z}{z^2 + 1}\,dz$, where C is the circle $|z - 3 - 4i| = 5$, oriented positively

8.3 Mean value property

Holomorphic functions satisfy a very strong averaging property. For comparison, the integral mean value theorem from calculus states the following: if a function

*Think of examples that illustrate how this differs from the case of real-valued functions.

†Tasks 201–203 highlight various details that must be heeded when applying the Cauchy integral formula to evaluate integrals. Pay attention to each of the conditions stated in Task 200, and also to the form of the integral in that task.

$f : [a,b] \to \mathbb{R}$ is continuous, then at some point of $[a,b]$ it equals its average (i.e., mean) value on that interval: that is, there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The integral mean value theorem does not, in general, give any information about where c is located.

In the case of a holomorphic function, however, when we average over a circle, we can tell exactly where the mean value occurs. In fact:

The value of a holomorphic function at the center of a disk equals its average value on the boundary of the disk.

You will prove this result in the next task.

Task 204 (E)**.** Let $U \overset{\circ}{\subseteq} \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that $z_0 \in U$, and $R > 0$ is chosen so that the closed disk $\overline{N_R(z_0)}$ is contained in U. Use the Cauchy integral formula to show that[†]

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt.$$

In what way does the integral in this equation represent an average value?

Task 205 (C)**.**

- Evaluate $\displaystyle\int_0^{2\pi} e^{e^{it}}\,dt$ and $\displaystyle\int_0^{2\pi} e^{e^{e^{it}}}\,dt$.[††]
- Using the fact that $e^{e^{it}} = e^{\cos t}(\cos(\sin t) + i\sin(\sin t))$, evaluate the (real) integrals

$$\int_0^{2\pi} e^{\cos t}\cos(\sin t)\,dt \qquad \text{and} \qquad \int_0^{2\pi} e^{\cos t}\sin(\sin t)\,dt.$$

8.4 Holomorphicity implies analyticity

We have never assumed anything stronger of a holomorphic function f than that its derivative f' exist on an open set, or perhaps that this derivative be continuous.[*] We certainly never insisted that a holomorphic function should have more than one derivative. However, the next task shows that *a holomorphic function is analytic at every point of its domain*, and thus it is infinitely differentiable. This is an incredible "upgrade" from differentiable to analytic, which has no parallel in the world of real-differentiable functions.

Task 206 (E)**.** Suppose that $U \overset{\circ}{\subseteq} \mathbb{C}$ and $f : U \to \mathbb{C}$ is holomorphic on U, and let z_0 be a point in U.

[†]Parametrize $\partial\overline{N_R(z_0)}$, giving it the positive orientation.

[††]The integrands can also be written, respectively, as $\exp(e^{it})$ and $\exp(\exp(e^{it}))$, in case that helps you decipher them.

[*]See the discussion in section 7.3.

a. Justify each step in this chain of equalities, assuming that $|z - z_0| < |w - z_0|$ and $N \in \mathbb{N}$.

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n \\ &= \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \cdots + \frac{(z-z_0)^N}{(w-z_0)^N} + \left(\frac{z-z_0}{w-z_0}\right)^{N+1} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}}\right) \\ &= \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \cdots + \frac{(z-z_0)^N}{(w-z_0)^{N+1}} + \frac{1}{w-z}\left(\frac{z-z_0}{w-z_0}\right)^{N+1}\end{aligned}$$

b. Let $R > 0$ be chosen so that the closed disk $\overline{N_R(z_0)}$ is contained in U, and set $C = \partial\overline{N_R(z_0)}$, with the positive orientation. Use the result of part a. together with the Cauchy integral formula to show that, if $z \in N_R(z_0)$, then

$$f(z) = \sum_{n=0}^{N} \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw + \frac{1}{2\pi i} \oint_C \frac{f(w)(z-z_0)^{N+1}}{(w-z)(w-z_0)^{N+1}}\,dw.$$

c. Now we need to show that the "remainder term" in part b. tends to zero as N goes to infinity: that is, if C and z are chosen as in part b., then

$$\lim_{N\to\infty} \oint_C \frac{f(w)(z-z_0)^{N+1}}{(w-z)(w-z_0)^{N+1}}\,dw = 0.$$

Find a bound for the modulus of the integral on the left in terms of the modulus of the integrand and the length of C, and show that this bound goes to 0 as $N \to \infty$.[†]

The result of the previous task implies that, if f is holomorphic at z_0 and C is the boundary of a disk that contains z_0 and is in the domain of f, then for all z within some ε-neighborhood of z_0, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw,$$

and so f is also analytic at z_0. As the next task shows, we thereby get an extension of the Cauchy integral formula that expresses the derivatives of f in terms of contour integrals.

Task 207 (E)**.** Let f be holomorphic on a simply connected neighborhood U of z_0. Using the power series obtained above along with the form of the Taylor coefficients for f, show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where C is <u>any</u> positively-oriented simple closed curve in U having z_0 as an interior point.

[†]Keep in mind that z and z_0 are both fixed, and $|z - z_0| < |w - z_0| = R$ for all $w \in C$.

8.5 Using derivatives to compute integrals

Task 207 allows us to extend the class of contour integrals that we can compute, and surprisingly we can use derivatives to do so. (This is something of a reversal from our usual habit, which is to compute integrals by finding <u>anti</u>derivatives.) For example, we now have, under the appropriate conditions,

$$\oint_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0), \qquad \oint_C \frac{f(z)}{(z-z_0)^3}\,dz = \pi i f''(z_0),$$
$$\oint_C \frac{f(z)}{(z-z_0)^4}\,dz = \frac{\pi i}{3} f'''(z_0),$$

and so on. (Be careful of the shift by one from the degree of the denominator inside the integral on the left to the number of derivatives on the right.)

Task 208 (C)**.** Evaluate the following integrals. In all cases, C is the circle $|z| = 4$, oriented positively.

$$\oint_C \frac{z}{(z-1)^2}\,dz \qquad \oint_C \frac{z}{(z-1)^3}\,dz \qquad \oint_C \frac{\cos z}{(z-\pi)^4}\,dz \qquad \oint_C \frac{\cos z}{(z-\pi)^5}\,dz$$

Task 209 (C)**.** Evaluate $\displaystyle\oint_C \frac{e^z}{(z-1)^n}\,dz$ for all $n \in \mathbb{N}_+$, where C is the circle $|z| = 2$, oriented positively.

8.6 Maximum modulus principle (Optional)

Definition 8.6.1. Let $A \subseteq \mathbb{C}$ and $g : A \to \mathbb{R}$. We say that g has a *maximum value on A at z* if $g(z) \geq g(w)$ for all $w \in A$. If g has a maximum value at z on some neighborhood of $z \in A$, then we say it has a *local maximum value at z*.

Another surprising property of holomorphic functions is the following:

If f is a non-constant holomorphic function on a connected open set U, then $|f|$ cannot have a maximum value on U.

Through the next two tasks, you will prove a local version of this statement, or rather its contrapositive: if $|f|$ has a local maximum at z_0, then f must be constant on a neighborhood of z_0.

Task 210 (E)**.** Let f be holomorphic at z_0. Suppose there exists $R > 0$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in N_R(z_0)$. Let C be a circle with center z_0 and radius less than R, and let M be the maximum of f on C.

- Use the results of Tasks 176 and 204 to show that $|f(z_0)| \leq M$. Conclude that $|f(z_0)| = M$.
- Suppose that at some point $z \in C$ the strict inequality $|f(z)| < M$ is satisfied. Explain, again using Tasks 176 and 204, why this would lead to the claim that $|f(z_0)| < M$. Conclude that $|f(z)| = M$ for all $z \in C$.
- By letting the radius of C vary from 0 to R, show that $|f|$ is constant on $N_R(z_0)$.

Task 211 (E)**.** Show that, if f is holomorphic on U and $|f|^2$ is constant on U, then f is constant on U.[†] Use Task 210 to conclude that if $|f|$ has a local maximum at $z_0 \in U$, then f must be constant on a neighborhood of z_0.

The local version of the maximum modulus principle from Task 211, combined with the analyticity of holomorphic functions, has the following global consequence.

Theorem 13. *Let $U \subseteq \mathbb{C}$ be a connected open set, and $f : U \to \mathbb{C}$ be holomorphic. If $|f|$ has a maximum value on U, then f is constant.*

One proof follows a line of reasoning that is typical for analytic functions, by which information near one point can be transferred to other points by moving along a contour. Suppose $|f|$ has a maximum value at z_0. Then the local version of the maximum modulus principle implies that f must be constant near z_0. Consequently, the Taylor series of f at any point near z_0 must have only a constant term; all other derivatives vanish. By joining any other point $z \in U$ to z_0 with a contour, we can conclude that the power series of f at every point of the contour must be constant, and so $f(z) = f(z_0)$.

Task 212 (E)**.** Suppose that U is an open set in which any two points can be joined by a contour, and $f : U \to \mathbb{C}$ is holomorphic. Explain why, if A is a closed set in U, then the maximum value of $|f|$ on A must occur at a point of ∂A, if at all.[††]

Task 213 (C)**.** Find the maximum value of $|z^3 - z|$ on the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$.

In the next section, you will see that if f is holomorphic on all of $\mathbb{C}$, then even if $|f|$ is bounded (without assuming it has a maximum value) f must be constant!

8.7 Liouville's theorem and the Fundamental Theorem of Algebra (Optional)

Definition 8.7.1. An *entire function* is a function that is holomorphic on all of $\mathbb{C}$.

Familiar examples of entire functions include polynomials, $\exp z$, $\cos z$, and $\sin z$. We already knew that polynomials and the exponential function take on arbitrary large values on the real line (at least when their coefficients are real), but it was perhaps surprising to find that $\cos z$ and $\sin z$, which are bounded along the real axis, also become arbitrarily large (in modulus) along the imaginary axis. This is no accident; among other things, it is a consequence of *Liouville's theorem*[*], which you will prove in the next task.

Task 214 (E)**.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, and $M > 0$ is a constant such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Given $z_0 \in \mathbb{C}$ and $R > 0$, let C_R be the circle of radius

[†]Consider the partial derivatives of $|f|^2$, and show that f' is zero on U.

[††]If $z_0 \in A$ and $z_0 \notin \partial A$, then z_0 has an ε-neighborhood that is contained in A. What do you know if $|f|$ has a maximum value at such a point of A?

[*]Liouville's theorem may actually be due to Cauchy. Joseph Liouville, however, included a version of it in his 1847 lectures entitled *Leçons sur les fonctions doublement périodiques*, and this is perhaps how the theorem got its name.

R centered at z_0, oriented positively. Use the formula obtained in Task 207 to get an upper bound for $f'(z_0)$ in terms of M and R, and show that this upper bound tends to 0 as $R \to \infty$. Conclude that $f'(z_0) = 0$ for all z_0, and so f is constant.†

In short, Liouville's theorem says that

If an entire function is bounded, then it is constant.

This implies, for instance, that an entire function can have at most one direction in which it is periodic, as in the case of $\cos z$, $\sin z$, and $\exp z$. An entire function that is periodic in two ($\mathbb{R}$-)independent directions in $\mathbb{C}$ must be constant.**

A corollary of Liouville's theorem is the *Fundamental Theorem of Algebra*, which states the following:

*If $p(z)$ is any non-constant polynomial, then the equation $p(z) = 0$ has at least one solution in $\mathbb{C}$.****

This statement shows that $\mathbb{C}$ differs from $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ in another remarkable way. The equation $2z - 1 = 0$ involves only coefficients from $\mathbb{Z}$, but its solution requires introducing the rational number $1/2$. Likewise, the equation $z^2 - 2 = 0$ uses coefficients from $\mathbb{Q}$, but its solutions $\pm\sqrt{2}$ lie in $\mathbb{R} \setminus \mathbb{Q}$. Finally, the equation $z^2 + 1 = 0$ with coefficients from $\mathbb{R}$ can only be solved by introducing the imaginary unit i. After all this, it is reasonable to wonder whether some equation with coefficients from $\mathbb{C}$ would require introducing some larger algebraic object in order to solve it. The Fundamental Theorem of Algebra states that this is not the case: as long as the coefficients of a polynomial equation come from $\mathbb{C}$, the equation has a solution in $\mathbb{C}$ or no solutions at all. You will prove it in the next task.

Task 215 (E)**.** Let $p(z) = a_n z^n + \cdots + a_1 z + z_0$ be a polynomial. Suppose that $p(z) = 0$ has no solutions. The goal is to show that $p(z)$ is constant.

- Explain why, under the above assumptions, $1/p(z)$ is entire.
- Show that, if $p(z)$ is not constant, then $|1/p(z)| \to 0$ as $|z| \to \infty$. Conclude that $1/p(z)$ is bounded. How does this contradict Liouville's Theorem?

†Use Theorem 3.

**An example of a real-valued function that has two independent periods is $\cos x \cos y$, but this function is nowhere holomorphic. The Weierstrass p-function, defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\frac{1}{(m\omega_1 + n\omega_2 + z)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right)$$

is periodic in two directions, with periods ω_1 and ω_2 (assuming $\arg\omega_1 \neq \arg\omega_2$), and it is holomorphic except at points of the form $m\omega_1 + n\omega_2$.

***Another formulation of the Fundamental Theorem of Algebra is that any complex polynomial can be factored into a product of affine functions; that is, if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, then there exist numbers $z_1, \ldots, z_n$, not necessarily distinct, such that $p(z) = a_n(z - z_1) \cdots (z - z_n)$. This alternate formulation appears on the face of it to be a stronger statement, but getting the factorization is not the hard part of the theorem. Any solution z_0 to the equation $p(z) = 0$ makes it possible to factor $p(z)$ as $p(z) = (z - z_0)q(z)$, where the degree of $q(z)$ is lower than that of $p(z)$. An induction proof therefore shows that once we can find a single solution to any polynomial equation, we can factor any polynomial into a product of the form $a_n(z - z_1) \cdots (z - z_n)$.

Chapter 9

Singularities

9.1 Residue theorem

We now turn to holomorphic functions on sets that are not simply connected. We will mostly be interested in the behavior of a function near isolated points at which it is not holomorphic. The following definitions will be useful.

Definition 9.1.1. Let $z \in \mathbb{C}$, $\varepsilon > 0$. The *punctured ε-neighborhood of z* is the open set

$$N'_\varepsilon(z) = \{w \in \mathbb{C} : 0 < |w - z| < \varepsilon\} = N_\varepsilon(z) \setminus \{z\}.$$

Definition 9.1.2. Suppose $f : U \to \mathbb{C}$ is holomorphic and $z_0 \notin U$. We call z_0 an *isolated singularity* of f if there exists some $\varepsilon > 0$ such that $N'_\varepsilon(z_0) \subseteq U$.

Definition 9.1.3. Let $z_0 \in \mathbb{C}$. Suppose f is holomorphic on $N'_\varepsilon(z_0)$ for some $\varepsilon > 0$. The *residue* of f at z_0 is

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \oint_{C_\rho} f(z)\, dz,$$

where C_ρ is any positively-oriented circle centered at z_0 having radius $\rho < \varepsilon$.

Task 216 (C)**.** Let $n \in \mathbb{Z}$. Show that

$$\operatorname{Res}(z^n, 0) = \begin{cases} 1 & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

By applying the definition of residues and winding numbers to the result of Task 195, we have the following result, known as the *residue theorem*.

Theorem 14. *Suppose $U \overset{\circ}{\subseteq} \mathbb{C}$ is simply connected and f is holomorphic on U except at finitely many points $\{z_1, \ldots, z_n\}$. If C is any closed contour in $U \setminus \{z_1, \ldots, z_n\}$, then*

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{wind}(C, z_k) \cdot \operatorname{Res}(f, z_k).$$

Task 217 (E, Optional)**.** Justify Theorem 14.

Many of the calculations in the previous section can be viewed as applications of the residue theorem. For instance, in Task 198, you essentially computed integrals using the fact that $\operatorname{Res}\big((z^2-1)^{-1}, 1\big) = 1/2$ and $\operatorname{Res}\big((z^2-1)^{-1}, -1\big) = -1/2$.

Take some time to look through your calculations using integral formulas to see how the residue theorem applies in each case. Here are a couple more examples, too.

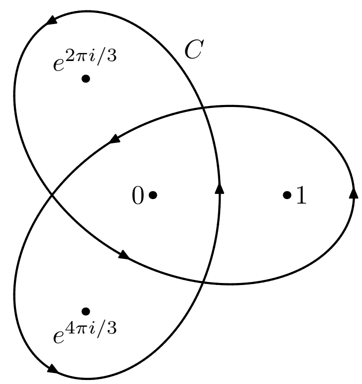
Task 218 (C)**.** Find $\oint_C \frac{e^z}{z^2 - 3z + 2}\,dz$, where C is the circle $|z| = 3$, positively oriented.

Task 219 (C)**.** Let $f(z) = \dfrac{1}{z(z^3 - 1)}$. Then $f(z)$ is holomorphic except at 0, 1, $e^{2\pi i/3}$, and $e^{4\pi i/3}$. (Why?)

If C is the contour shown in the figure to the right, find

$$\oint_C \frac{1}{z(z^3 - 1)}\,dz.$$

Hint. $z^3 - 1 = (z-1)(z - e^{2\pi i/3})(z - e^{4\pi i/3})$.



9.2 Laurent series

You showed in Task 207 that, when f is holomorphic at z_0 and C is a sufficiently small circle centered at z_0, the integrals

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

for $n \geq 0$ provide the coefficients of the Taylor series of f (after multiplying by the appropriate constants, which are independent of f). According to Definition 9.1.3, for $n = -1$ the above integral equals $2\pi i$ times the residue of f at z_0 when z_0 is an isolated singularity. It is reasonable to ask whether these integrals have any meaning for $n < -1$; that is, what do the integrals

$$\oint_C (z - z_0)^N f(z)\,dz$$

tell us about f when $N \geq 0$? Of course, if f is holomorphic at z_0 then these integrals are all zero. If f has a singularity at z_0, however, then they can be used to describe $f(z)$ by a generalization of power series.

Definition 9.2.1. Suppose $z_0 \in \mathbb{C}$ and f is holomorphic on $N'_\varepsilon(z_0)$ for some $\varepsilon > 0$.

Then the *Laurent series** *of* f *at* z_0 is the doubly infinite series

$$L(z) = L_{f,z_0}(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad \text{where}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad \text{for all } n \in \mathbb{Z},$$

with C chosen to be a circle centered at z_0, contained in $N'_\varepsilon(z_0)$.

By combining the techniques of Tasks 193 and 206, one can obtain the following theorem.

Theorem 15. *Let z_0 be an isolated singularity of a holomorphic function f. Then the Laurent series of f at z_0 converges to $f(z)$ on a punctured neighborhood of z_0.*

Task 220 (E, Optional)**.** Justify Theorem 15.

Thus the Laurent series of a function f at a point z_0 is expressed in terms of both positive and negative powers of $z - z_0$.** Just as we consider polynomials to be power series with only finitely many nonzero terms, it is not necessary that all, or even infinitely many, of the terms of a Laurent series be nonzero. For example, the Laurent series of $1/z$ at $z_0 = 0$ is simply $1/z$.

Rarely do we use Definition 9.2.1 directly to find Laurent series. In many cases, we can instead calculate Laurent series from power series that we already know.

Task 221 (C)**.** Find the Laurent series of $\dfrac{1}{z(1-z)}$ at $z_0 = 0$ and at $z_0 = 1$.†

Task 222 (C)**.** Find the Laurent series of $\exp \dfrac{1}{z}$, $z \exp \dfrac{1}{z}$, and $\exp \dfrac{1}{z^2}$ at $z_0 = 0$.

From the definition of the coefficients a_n for the Laurent series of f at z_0, we see that $a_{-1} = \operatorname{Res}(f, z_0)$. Loosely speaking, the residue of f at z_0 is the "minus-one coefficient" of the Laurent series. (Compare this observation with the result of Task 216.) Sometimes it is easier to find the entire Laurent series of f at z_0 than it is to calculate the residue directly from its definition as an integral.

Task 223 (C)**.** Using the results of Task 222, find $\operatorname{Res}\left(\exp \dfrac{1}{z}, 0\right)$, $\operatorname{Res}\left(z \exp \dfrac{1}{z}, 0\right)$, and $\operatorname{Res}\left(\exp \dfrac{1}{z^2}, 0\right)$.

*Named after Pierre Alphonse Laurent. Unfortunately, none of his work on this topic appeared in print during his lifetime, despite recommendations from Cauchy that two of his papers be published by the Académie des Sciences.

**In fact, given a function $f(z)$ that is holomorphic on an annulus $0 \le R_1 < |z - z_0| < R_2 \le \infty$, a Laurent series for f in this annulus can be obtained from the same formulas as in Definition 9.2.1 by integrating over a circle $|z - z_0| = \rho$, with $R_1 < \rho < R_2$, and the Laurent series will converge to $f(z)$ on this annulus. For example, the series $1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$ converges only when $|z| > 1$. What function does it converge to?

†At 0, use the geometric series to expand $1/(1-z)$. At 1, write $z = 1 - (1 - z)$ and use the geometric series to expand $1/z$.

9.3 Classification of singularities

We can classify the behavior of a holomorphic function f near an isolated singularity z_0 by the number of terms of negative degree its Laurent series has. This presents three possibilities.

Definition 9.3.1. Let f be holomorphic, and suppose z_0 is an isolated singularity of f. Let a_n, $n \in \mathbb{Z}$, be the coefficients of the Laurent series of f at z_0, as in Definition 9.2.1. Then z_0 is called:

- a *removable singularity* if $a_n = 0$ for all $n < 0$;
- a *pole of order* N if $N > 0$, $a_{-N} \neq 0$, and $a_n = 0$ for all $n < -N$;
- an *essential singularity* if $a_n \neq 0$ for infinitely many values of $n < 0$.

When f has a removable singularity at z_0, the Laurent series of f at z_0 becomes an ordinary power series, and so $\lim_{z \to z_0} f(z)$ exists and equals the constant term of the Laurent series; this is where the terminology of "removable" comes from. In this case, $|f|$ must be bounded in a neighborhood of z_0. The converse is also true, as the next task shows.

Task 224 (E)**.** Let z_0 be an isolated singularity of a holomorphic function f. Suppose there exist $\varepsilon > 0$ and $M > 0$ such that $|f| \le M$ on $N'_\varepsilon(z_0)$.

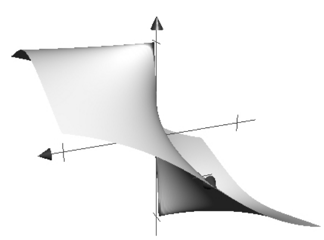
a. For each $N > 0$, show that

$$\lim_{\rho \to 0} \left| \oint_{C_\rho} (z - z_0)^N f(z)\, dz \right| = 0.$$

b. Conclude that z_0 is a removable singularity of f.†

In other words, if $|f|$ is bounded on a punctured neighborhood of z_0, then z_0 must be a removable singularity. Thus isolated singularities of holomorphic functions cannot have "jump" discontinuities in their modulus, such as we encounter among real-valued functions.

For example, the image to the right shows the graph in $\mathbb{R}^3$ of the real-valued function $(\operatorname{Re} z)/|z| = \cos(\arg(z))$ near 0. Because this function is bounded but also has different limits when 0 is approached from different directions, it cannot be written as $|f|$, or even $|f| - 1$, where f is holomorphic on $N'_\varepsilon(0)$. (What other reasons can you find to rule out this function as having the form $|f|$, where f is holomorphic on $N'_\varepsilon(0)$?)



Task 225 (C)**.** Show that $\dfrac{\sin z}{z}$ has a removable singularity at 0.†† What is $\lim_{z \to 0} \dfrac{\sin z}{z}$?

Task 226 (E)**.** Show that $(z - z_0)^N f(z)$ has a removable singularity at z_0 if and only if f has a pole of at most order N at z_0 (this includes the possibility that f itself has a removable singularity at z_0).

†What does part a. imply about the coefficients of the Laurent series of f at z_0?
††Use the power series for $\sin z$.

Task 227 (C)**.** Define the cotangent function as usual by $\cot z = \cos z / \sin z$. After writing

$$\cot z = \frac{1}{z} \cdot \frac{z}{\sin z} \cdot \cos z,$$

do the following.

a. Use the Cauchy integral formula and the result of Task 225 to find $\operatorname{Res}(\cot z, 0)$.
b. Use the result of Task 226 to show that $\cot z$ has a pole of order 1 at 0.
c. Conclude that $\cot z$ has a pole of order 1 at every $k\pi$, $k \in \mathbb{Z}$, and determine $\operatorname{Res}(\cot z, k\pi)$.

We have so far examined the first two ways a holomorphic function f can behave near an isolated singularity: it may have a limit (in the case of a removable singularity), or it may "approach infinity" (in the case of a pole—we'll examine this behavior more carefully in the next chapter). On the other hand, near an essential singularity z_0, the behavior of f is extremely wild: on every punctured neighborhood of z_0, f takes values arbitrarily close to every point of $\mathbb{C}$!

Task 228 (E)**.** Let z_0 be an isolated singularity of a holomorphic function f.

a. Suppose that there exist $\varepsilon_1, \varepsilon_2 > 0$ and $w_0 \in \mathbb{C}$ such that $|f(z) - w_0| \geq \varepsilon_2$ for all $z \in N'_{\varepsilon_1}(z_0)$. Explain why in this case $1/(f(z) - w_0)$ has a removable singularity at z_0.† Conclude that $f(z) - w_0$ has either a removable singularity at z_0 (if $\lim_{z \to z_0} 1/(f(z) - w_0) \neq 0$) or a pole of some order $N > 0$ (if the first N coefficients of the Laurent series of $1/(f(z) - w_0)$ are zero). Hence $f(z)$ also has either a removable singularity or a pole at z_0.
b. Now suppose that z_0 is an essential singularity of f. Based on part a., what can you conclude about the values of f near z_0?††

Task 229 (CD)**.** Show that 0 is an essential singularity of $\exp \frac{1}{z}$. Explain why, for all $\varepsilon > 0$ and for every $w \neq 0$, there exists $z \in N'_\varepsilon(0)$ such that $\exp \frac{1}{z} = w$.*

†Rewrite the given inequality as $|1/(f(z) - w_0)| \leq 1/\varepsilon_2$, and use the result of Task 224.
††Use the contrapositive.
*Notice that this is a much stronger property than the one given by Task 228. It will help to use the periodicity of $\exp z$.

Chapter 10

The Riemann sphere

10.1 Stereographic projection

We temporarily move away from the real plane $\mathbb{R}^2$ (which we have been treating as $\mathbb{C}$) to consider another surface: the *unit sphere* in $\mathbb{R}^3$, defined by*

$$\mathbb{S}^2 = \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}.$$

The equation that defines $\mathbb{S}^2$ means that it includes all points in $\mathbb{R}^3$ that are 1 unit away from the origin $(0,0,0)$. We call $P_N = (0,0,1)$ the *north pole* and $P_S = (0,0,-1)$ the *south pole*.

Now think of the complex plane $\mathbb{C}$ as the (X,Y)-plane in $\mathbb{R}^3$ by identifying $x + yi$ with $(x,y,0)$. Imagine standing at the north pole of $\mathbb{S}^2$ and drawing a line ℓ from P_N through another point $(X,Y,Z) \in \mathbb{S}^2$. Because this point must have Z-coordinate less than 1, the line ℓ will intersect the (X,Y)-plane, which is to say, the complex plane $\mathbb{C}$, at a point $\sigma(X,Y,Z)$. (See the figure on the next page for an illustration.) A bit of geometry with similar triangles shows that the formula for $\sigma(X,Y,Z)$ is

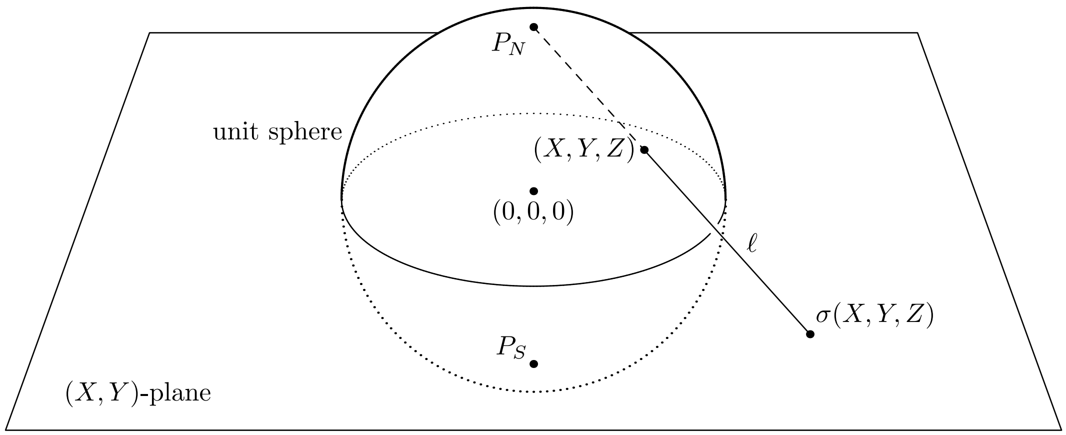
$$\sigma(X,Y,Z) = \frac{X}{1-Z} + \frac{Y}{1-Z}\, i.$$

The function $\sigma : \mathbb{S}^2 \setminus \{P_N\} \to \mathbb{C}$ is called *stereographic projection*. It is undefined at the north pole. (Why?)

Task 230 (C)**.** Find where in $\mathbb{C}$ each of the following points of $\mathbb{S}^2$ is sent by stereographic projection.

- the south pole P_S
- $(2/3, 2/3, 1/3)$
- $\left(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$
- $(3/5, -4/5, 0)$
- $(3/5, 0, -4/5)$

*In order to avoid confusion with the complex variable z and its real and imaginary parts, we will use the capital letters X, Y, and Z for coordinates in $\mathbb{R}^3$.



The point (X,Y,Z) on the unit sphere is projected stereographically to $\sigma(X,Y,Z)$.

We can also invert σ to get a function τ from $\mathbb{C}$ to $\mathbb{S}^2 \setminus \{P_N\}$, defined by

$$\tau(z) = \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Task 231 (E)**.** Show that $\tau(z) \in \mathbb{S}^2$ for all $z \in \mathbb{C}$.

Task 232 (E)**.** Show that $\sigma(\tau(z)) = z$ for all $z \in \mathbb{C}$.

Definition 10.1.1. We use the symbol ∞ to represent a new point, called the *point at infinity*. The *extended complex plane* is*

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

An element of $\widehat{\mathbb{C}}$ is called an *extended complex number*. That is, every extended complex number $\hat{z} \in \widehat{\mathbb{C}}$ is either an ordinary complex number or the point at infinity.

The preceding discussion shows that there is a one-to-one correspondence between the extended complex numbers and the points of $\mathbb{S}^2$, by which ∞ is associated to P_N. Another name for $\widehat{\mathbb{C}}$ is the *Riemann sphere*.**

On the next page, you can see a world map obtained via stereographic projection. In the map projection, the roles of the north and south poles are switched from our convention in these notes, so that the north pole corresponds to the origin in the plane, and the south pole corresponds to the point at infinity.

*The symbol $\widehat{\mathbb{C}}$ is read "cee-hat."

**The Riemann sphere is just one example of what are known as *Riemann surfaces*, which are studied in advanced courses on complex analysis and still used in many active areas of research. Riemann introduced abstract Riemann surfaces in his 1851 doctoral dissertation, entitled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. It seems to be Carl Neumann who first systematically developed the theory of the Riemann sphere itself, in his 1865 book *Vorlesungen über Riemann's Theorie der Abel'schen Integrale*.

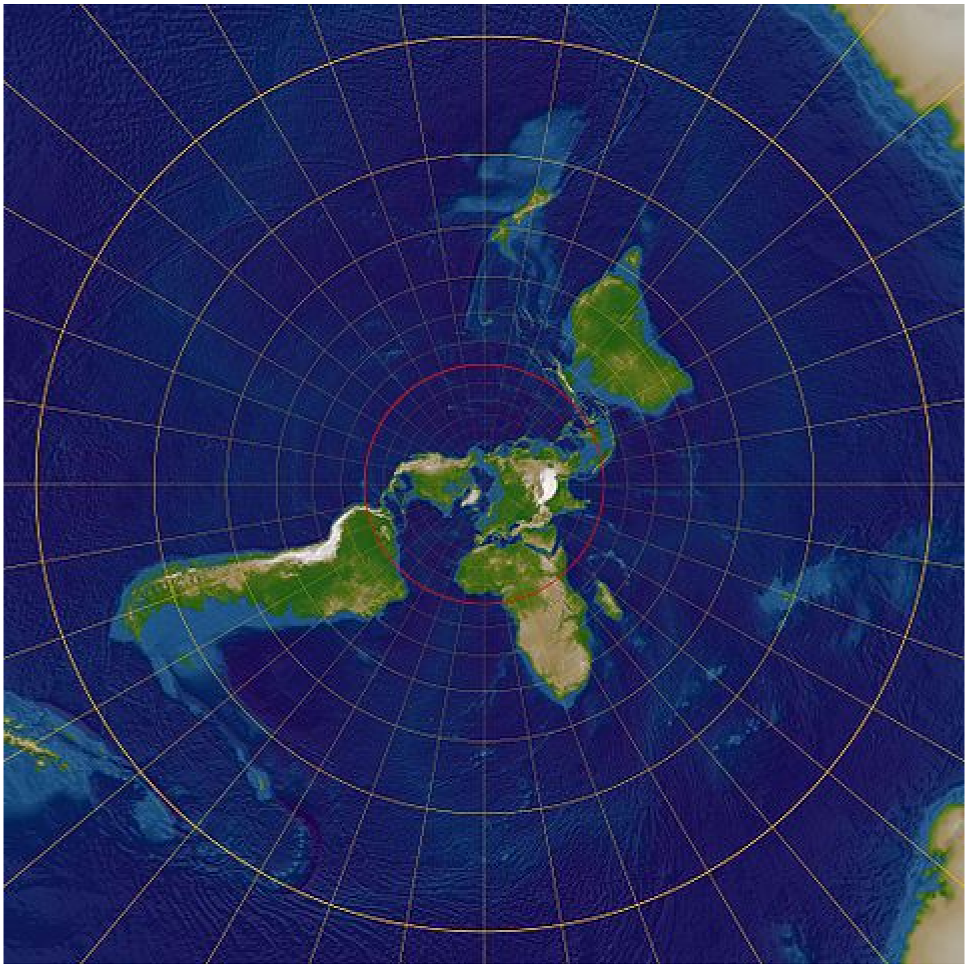


Image source: Lars H. Rohwedder (User:RokerHRO), "Stereographic Projection Polar Extreme." Obtained from Wikimedia Commons. Used under the CC BY-SA 3.0 licence.

The radial lines in this image are longitude lines, extending from the north pole (in the center) to the south pole (at infinity). The circles are latitude lines. The red circle indicates the equator, which separates the northern and southern hemispheres. Notice that some edges of Antarctica, which surrounds the south pole, are visible in the corners of the image.

Task 233 (C)**.** Let $\tau : \mathbb{C} \to \mathbb{S}^2 \setminus \{P_N\}$ be the inverse of stereographic projection. Show that, if $z \neq 0$ and $\tau(z) = (X, Y, Z)$, then $\tau(1/z) = (X, -Y, -Z)$. That is, the function $1/z$ may be visualized as rotating the sphere $\mathbb{S}^2$ by π around the X-axis in $\mathbb{R}^3$. (Compare this with the description you gave in Task 51.)

The rules of arithmetic for complex numbers can be partially extended to $\widehat{\mathbb{C}}$ by defining

$$\begin{aligned} \hat{z} + \infty &= \infty && \text{for any extended complex number } \hat{z}\text{, and} \\ \hat{z} \cdot \infty &= \infty && \text{for any nonzero extended complex number } \hat{z}. \end{aligned}$$

Task 233 suggests we may also reasonably define $1/0 = \infty$ and $1/\infty = 0$. (Notice that rotation by π around the X-axis in $\mathbb{R}^3$ exchanges the north and south poles of $\mathbb{S}^2$.) However, we cannot generally make sense of the expressions $\infty - \infty$, $0 \cdot \infty$, or ∞/∞.* When one of these expressions makes an appearance in a function, we must turn to limits in order to uncover whether a reasonable value exists.

10.2 Limits involving infinity

Next we introduce neighborhoods of the new point ∞, in order to extend topological notions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. Because no complex number z is at a finite distance from ∞, we define "close to ∞" in terms of being "far from 0."

Definition 10.2.1. Given $\varepsilon > 0$, the *ε-neighborhood of infinity* is the set of points at least $1/\varepsilon$ away from 0, together with ∞ itself:

$$N_\varepsilon(\infty) = \{z \in \mathbb{C} : |z| > 1/\varepsilon\} \cup \{\infty\}$$

Task 234 (D)**.** What does $N_{1/2}(\infty) \cap \mathbb{C}$ look like? Draw a picture. What about $N_\varepsilon(\infty) \cap \mathbb{C}$ generally?

Task 235 (E)**.** Let $\tau : \mathbb{C} \to \mathbb{S}^2$ be the inverse of stereographic projection. Show that if $z \in N_\varepsilon(\infty)$, then the Z-coordinate of $\tau(z)$ is greater than $(1-\varepsilon^2)/(1+\varepsilon^2)$.

For comparison, when working on the real line, we often introduce two points at infinity, $+\infty$ and $-\infty$. Sometimes $\mathbb{R} \cup \{+\infty, -\infty\}$ is called the set of *extended real numbers* and written $[-\infty, +\infty]$ (with closed brackets on both sides). In the complex plane, however, "all roads lead to Rome," so to speak: a sequence that escapes every bounded set of $\mathbb{C}$ always goes to the same point in $\widehat{\mathbb{C}}$, which we simply call ∞.

Definition 10.2.2. A sequence $\hat{z}_1, \hat{z}_2, \hat{z}_3, \ldots$ of extended complex numbers *converges to* ∞ if, given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\hat{z}_n \in N_\varepsilon(\infty)$ for all $n > n_0$.

Compare this definition with Definition 3.3.1, and notice the following two things:

- We can now consider sequences that include ∞ among their terms.
- A sequence of complex numbers that converges in $\widehat{\mathbb{C}}$ to ∞ is divergent in $\mathbb{C}$. It is essential, therefore, when discussing whether a sequence of points converges or diverges, to keep in mind the context of the ambient set.

Task 236 (C)**.** Determine which of the following sequences, if any, converge to ∞.

- $z_n = n/2$
- $z_n = -n$
- $z_n = \left(\frac{1+i}{2}\right)^n$
- $z_n = 1 - (i/3)^n$
- $z_n = (1-i)^n$
- $z_n = \left(\frac{3}{5} + \frac{4}{5}i\right)^n$

We can directly adapt Definition 3.4.1 to define what it means for a function to converge to ∞.

*In calculus, the expressions $\infty - \infty$, $0 \cdot \infty$, and ∞/∞, along with $0/0$, often go by the name of "indeterminate forms".

Definition 10.2.3. Let $A \subseteq \mathbb{C}$, and let $f : A \to \mathbb{C}$ be a function. Suppose z_0 is in the closure of A. Then we say f *converges to* ∞ *as* z *approaches* z_0, and we write

$$\lim_{z \to z_0} f(z) = \infty,$$

if for every sequence $z_1, z_2, z_3, \ldots$ of points in A that converges to z_0, the sequence $f(z_n)$ converges to ∞.

Task 237 (C)**.** Find the following limits.

- $\lim\limits_{z \to 0} \dfrac{2z+1}{z(z^2+4)}$
- $\lim\limits_{z \to 1+i} \dfrac{z+3i}{z^4+4}$
- $\lim\limits_{z \to -i} \dfrac{z^2+1}{z^2+2iz-1}$

Task 238 (D)**.** Formulate definitions for the expressions $\lim\limits_{z \to \infty} f(z) = L$, where $L \in \mathbb{C}$, and $\lim\limits_{z \to \infty} f(z) = \infty$. Are your definitions consistent with previous ones?

Task 239 (C)**.** Find the following limits.

- $\lim\limits_{z \to \infty} \dfrac{2z+1}{z(z^2+4)}$
- $\lim\limits_{z \to \infty} \dfrac{iz+1}{z-\pi}$
- $\lim\limits_{z \to \infty} \dfrac{z^3-i}{z+3i}$

We can now precisely describe the behavior of a holomorphic function near a pole.

Task 240 (E)**.** Suppose f has a pole of order $N > 0$ at z_0. Using Task 226, explain why $\lim\limits_{z \to z_0} f(z) = \infty$.

If f is holomorphic on a punctured neighborhood of infinity $N'_\varepsilon(\infty) = \{z \in \mathbb{C} : |z| > 1/\varepsilon\}$, then we can classify the behavior of f at ∞ just as we do for any other isolated singularity: we use the fact that in this case $f(1/z)$ is holomorphic on a punctured neighborhood of 0.

Definition 10.2.4. Suppose f is holomorphic on $\{z \in \mathbb{C} : |z| > R\}$ for some $R > 0$. Then f has a removable singularity, an essential singularity, or a pole at ∞ according to whether $f(1/z)$ has, respectively, a removable singularity, an essential singularity, or a pole at 0.

Task 241 (C)**.** Classify ∞ as an isolated singularity of each of these functions.

- $z \mapsto z^2$
- $z \mapsto \exp z$
- $z \mapsto z/(1+z^2)$

10.3 Circles in $\widehat{\mathbb{C}}$ (Optional)

In the next two tasks, you will show that stereographic projection has a somewhat surprising geometric property. Recall that in $\mathbb{R}^3$, a plane is defined by an equation of the form $AX + BY + CZ = D$, where A, B, and C are not all zero.

Task 242 (CE)**.** Show that the set of points $(X, Y, Z) \in \mathbb{S}^2$ such that $\sigma(X, Y, Z)$ lies on a circle $|z - c| = r$ for some $c \in \mathbb{C}$, $r > 0$, is contained in a plane.

Task 243 (CE)**.** Show that the set of points $(X, Y, Z) \in \mathbb{S}^2$ such that $\sigma(X, Y, Z)$ lies on a real line $ax + by = c$ for some $a, b, c \in \mathbb{R}$ is contained in a plane that passes through P_N.

Because the intersection of $\mathbb{S}^2$ with a plane in $\mathbb{R}^3$ is a circle (unless it is a point, or empty), the previous two tasks show that circles and lines in $\mathbb{C}$ both correspond to circles in $\mathbb{S}^2$, which we have identified with $\widehat{\mathbb{C}}$. Thus we can think of circles and lines as the same kind of object in $\widehat{\mathbb{C}}$: lines are simply circles that happen to pass through ∞.

10.4 Möbius transformations (Optional)

In this section, we will consider a special class of rational functions that have particularly nice properties on the Riemann sphere $\widehat{\mathbb{C}}$.

Definition 10.4.1. A *Möbius transformation** is a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form**

$$f(z) = \frac{az + b}{cz + d} \qquad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

This definition has a small subtlety: the given formula doesn't work for every point of $\widehat{\mathbb{C}}$. The next task clarifies how to treat a Möbius transformation as a function on all of $\widehat{\mathbb{C}}$.

Task 244 (CD)**.** Suppose $f(z) = \dfrac{az + b}{cz + d}$ is a Möbius transformation.

- Find $\lim\limits_{z \to \infty} f(z)$.
- Find a point z_0 such that $\lim\limits_{z \to z_0} f(z) = \infty$.

Following Task 244, when f is a Möbius transformation, we will simply write $f(\infty)$ instead of $\lim\limits_{z \to \infty} f(z)$ and $f(z_0) = \infty$ instead of $\lim\limits_{z \to z_0} f(z) = \infty$.

Task 245 (D)**.** Show that a Möbius transformation f is an affine function if and only if $f(\infty) = \infty$.

Task 246 (CD)**.** Show that, if $ad - bc = 0$, then $f(z) = \dfrac{az + b}{cz + d}$ is a constant function.

Task 247 (CD)**.** Show that every Möbius transformation has an inverse as a function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

Task 248 (CD)**.** Show that a composition of Möbius transformations is a Möbius transformation.

*Named after August Ferdinand Möbius

**These functions are also called *fractional linear transformations*, as a descriptive way of referring to this form.

Task 249 (E). Show that every Möbius transformation except $z \mapsto z$ has either 1 or 2 fixed points in $\widehat{\mathbb{C}}$.†

Task 250 (E). Show that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is affine and $C \subset \widehat{\mathbb{C}}$ is a circle, then the image of C by f is a circle.

Task 251 (E). Show that if $C \subset \widehat{\mathbb{C}}$ is a circle, then the image of C by the map $f : z \mapsto 1/z$ is a circle.

Task 252 (E). Show that every Möbius transformation can be written as a composition of affine functions and the function $z \mapsto 1/z$.

Task 253 (E). Show that a Möbius transformation sends any circle in $\widehat{\mathbb{C}}$ to a circle in $\widehat{\mathbb{C}}$.

Task 254 (C). Let f be the Möbius transformation defined by $f(z) = \dfrac{iz+1}{z+i}$.

- Show that if $z \in \mathbb{R}$, then $|f(z)| = 1$.
- Find $f(\infty)$ and $f^{-1}(\infty)$.
- Find a formula for f^{-1}.

The function of the previous task can be used to show that, from the perspective of complex variables, the upper half plane $\mathbb{H}$ and the unit disk $\mathbb{D}$ are essentially the same.

Task 255 (E). Let $f(z) = \dfrac{iz+1}{z+i}$. Show that the image of $\mathbb{H}$ by f is $\mathbb{D}$. (That is, if $\operatorname{Im} z > 0$, then $|f(z)| < 1$.)

10.5 Rational functions (Optional)

In this final section, we consider properties of general rational functions on $\widehat{\mathbb{C}}$.

Definition 10.5.1. A function f is called a *rational function* if $f(z) = p(z)/q(z)$ for some polynomials p, q.

We have already considered some examples of limits of rational functions in Tasks 91, 237, and 239. The next two tasks cover some general properties that are likely familiar from calculus.

Task 256 (E). Let $p(z)$ and $q(z)$ be polynomials. Explain why, if $q(z_0) = 0$, then z_0 is either a pole or a removable singularity of $p(z)/q(z)$.†

Task 257 (E). Let $p(z)$ and $q(z)$ be polynomials with degrees $\deg p(z)$ and $\deg q(z)$.

- Show that if $\deg p(z) > \deg q(z)$, then $p(z)/q(z)$ has a pole at ∞.
- Show that if $\deg p(z) \le \deg q(z)$, then $p(z)/q(z)$ has a removable singularity at ∞.

†Handle the case of affine functions separately. Use Task 245.

†The Fundamental Theorem of Algebra implies that $q(z) = (z - z_0)^d q_0(z)$ for some $d \ge 1$, where q_0 is a polynomial such that $q_0(z_0) \neq 0$. Use Task 226.

Tasks 256 and 257 may interpreted as saying that any rational function $f : \mathbb{C} \to \mathbb{C}$ can be extended to a continuous function $\hat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. We conclude this section with an observation about the set of all rational functions.

Task 258 (E)**.** Show that the rational functions on $\widehat{\mathbb{C}}$ form a field.

Appendices

A review of sets and logic

The text of the notes assumes that you are familiar with certain facts about sets and logic. Here is a brief review of the terms and expressions you should know.

- A (*logical*) *statement* is a claim that is either definitely true or definitely false (though we may not know which at any particular time). We often use letters like p and q to represent statements.
- Suppose p and q are statements and whenever p is true, q is also guaranteed to be true. Then we say "if p then q" or "p implies q" or, symbolically, $p \implies q$. In the implication $p \implies q$, p is called the *hypothesis* (or *assumption*) and q is called the *conclusion*. It may happen that the conclusion is true in some cases in which the hypothesis is false, but that does not affect the implication. Sometimes we say "when" in place of "if"; the logical content is the same.
- Two statements p and q are *logically equivalent* if one is always true whenever the other one is true (in both directions). We then say "p if and only if q" or, symbolically, $p \iff q$. This breaks down into two statements: "p if q" (which is the same as "q implies p") and "p only if q" (which is the same as "p implies q", because q cannot be false when p is true). Another way of saying "if and only if" is "exactly when".
- The conjunctions "and" and "or" have very particular meanings in logical constructions. Suppose p and q are statements. The statement "p and q" is true exactly when p and q are both true. The statement "p or q" is true when either p or q is true, possibly both.
- A statement may depend on a variable, and the truth of the statement may then change depending on the value of the variable. For example, the statement "x is greater than 2" is true if $x = 3$ but false if $x = 1$. We often use $P(x)$ to represent a statement that depends on x.
- A *set* is a well-defined collection of objects, each of which is called an *element* of the set. The expression $x \in S$ is read "x is an element of S," and $x \notin S$ is read "x is not an element of S." The elements of a set do not have to be numbers; they can be functions, sequences of numbers, or even other sets.
- A set is often specified in mathematical writing by some property that its elements are required to possess. The expressions $S = \{x : P(x)\}$ and $S = \{x \mid P(x)\}$ both mean that an object x is an element of S exactly when $P(x)$ is true. (In these notes, we usually use the colon : instead of the vertical bar $|$.) If we wish to consider only elements drawn from a particular set T, we may write $\{x \in T : P(x)\}$ as shorthand for $\{x : x \in T \text{ and } P(x)\}$.
- The set with no elements is called the *empty set* and written $\varnothing$.
- If A and B are sets, and every element of A is also an element of B, then we say that A is a *subset* of B, and we write $A \subseteq B$. $\varnothing$ is a subset of every set.
- Suppose A and B are sets. The *union* of A and B is the set $A \cup B$ such that $x \in A \cup B$ exactly when x is in either A or B (or possibly both). The *intersection* of A and B is the set $A \cap B$ such that $x \in A \cap B$ exactly when x is in both A and B. In particular, $A \cap B \subseteq A \cup B$. If $A \cap B = \varnothing$, meaning that A and B have no elements in common, then A and B are said to be *disjoint*. The *set difference* of A and B is the set $A \setminus B$ such that $x \in A \setminus B$ exactly when $x \in A$ and $x \notin B$. It is not necessary for B to be a subset of A in order for $A \setminus B$ to make sense. Sometimes $A \setminus B$ is written $A - B$, but be aware that the set difference operation does not behave like ordinary subtraction.

- Sometimes we want to make a statement involving multiple possible values of a variable. In such cases, we use *quantifiers*. There are two kinds of quantifiers.
  - The *universal quantifier*, usually expressed by "for all". When we say "For all x, $P(x)$" we mean that the statement $P(x)$ is true regardless of what value of x is chosen. If we want to consider only elements of a particular set S, we can say "For all $x \in S$, $P(x)$." In these notes, some other equivalent expressions are used, like "Whenever $x \in S$, $P(x)$" or "Suppose $x \in S$; then $P(x)$."
  - The *existential quantifier*, usually expressed by "there exists". When we say "There exists x such that $P(x)$" we mean that the statement $P(x)$ is true for at least one value of x. If we want to consider only elements of a particular set S, we can say "There exists $x \in S$ such that $P(x)$."

  The order in which quantifiers apply to a statement is important! The statement "For all x, there exists y such that $P(x,y)$" means that for each value of x we can select a different value of y that will make the statement $P(x,y)$ true. However, "There exists y such that for all x, $P(x,y)$" means that we must have a single value of y that makes the statement $P(x,y)$ true regardless of the value of x.