A direct solution of Ampère's law

by Mathias Hoppe August 11, 2021

In DREAM, the initial value of the poloidal flux ψ is calculated from an inverted Ampére's law. In other words, the Ampére's law which is usually solved in DREAM is rewritten on an integral form and evaluated numerically. While this is analytically consistent, the numerical solutions to the differential and integral forms of the equation are generally different. This in turn gives rise to a mismatch in the electric field and poloidal flux when the latter is eventually being evolved according to the differential form of Ampére's law in the main part of the simulation.

The implications of this for disruption simulations are minor since $E \approx 0$ at the start of the simulation (at least compared to the electric field in the current quench). However, in the startup simulations conducted with STREAM, the exact value of the initial electric field could potentially play an important role. In STREAM it might therefore be crucial to solve Ampére's law consistently for both the initial value of ψ as well as its time evolution.

In this document we derive the solution to the differential form of Ampére's law in the special case $N_r = 1$ (one radial grid point), which could be straightforwardly implemented in DREAM and improve the solution accuracy.

Derivation

Ampére's law, as stated in DREAM and STREAM, is

$$\frac{1}{V'}\frac{\partial}{\partial r}\left[V'\left\langle\frac{|\nabla r|^2}{R^2}\right\rangle\frac{\partial\psi}{\partial r}\right] = 2\pi\mu_0\left\langle\boldsymbol{B}\cdot\nabla\phi\right\rangle\frac{j_{\mathrm{tot}}}{B}, \tag{1}$$

where V' is the spatial jacobian, r the minor radius coordinate, R the major radius coordinate, $\boldsymbol{B}$ the magnetic field vector, ϕ the toroidal angle coordinate, and j_{tot} the plasma current density.

We proceed by discretizing equation (1) in the special case of one radial grid point. According to equation (56) of the DREAM paper, equation (1) discretizes as

$$\frac{1}{V_1'\Delta r_1}\left[V_{3/2}'\left\langle\frac{|\nabla r|^2}{R^2}\right\rangle\bigg|_{3/2}\frac{\partial\psi}{\partial r}\bigg|_{3/2} - V_{1/2}'\left\langle\frac{|\nabla r|^2}{R^2}\right\rangle\bigg|_{1/2}\frac{\partial\psi}{\partial r}\bigg|_{1/2}\right] = 2\pi\mu_0\left.\left\langle\boldsymbol{B}\cdot\nabla\phi\right\rangle\right|_1\frac{j_1}{B(r_1)}. \tag{2}$$

The second term on the LHS vanishes identically, either by noting that $V_{1/2}' = 0$, or by noting that because of symmetry $\partial\psi/\partial r = 0$ in the center of the plasma, $r = r_{1/2}$. The remaining r-derivative can be evaluated with the help of equation (59) in the DREAM paper, but with r_2 (which is so far undefined in this single-radial-grid-point system) taken as the plasma edge:

$$\frac{\partial\psi}{\partial r}\bigg|_{3/2} = \frac{\psi_2-\psi_1}{\Delta r_{3/2}}. \tag{3}$$

Taking $r_2 = a$, with a the plasma minor radius, we obtain

$$\begin{aligned}\psi_2 &\equiv \psi_{\mathrm{edge}},\\ \Delta r_{3/2} &\equiv a - r_1.\end{aligned} \tag{4}$$

Plugging equations (3) and (4) into equation (1) we obtain the following algebraic expression for ψ_1:

$$V'_{3/2} \left\langle \frac{|\nabla r|^2}{R^2} \right\rangle \bigg|_{3/2} \frac{\psi_{\mathrm{edge}} - \psi_1}{(a - r_1) V'_1 \Delta r_1} = 2\pi\mu_0 \left\langle \boldsymbol{B} \cdot \nabla\phi \right\rangle |_1 \frac{j_1}{B(r_1)}. \tag{5}$$

Finally, solving for ψ_1 we arrive at

$$\psi_1 = \psi_{\mathrm{edge}} - \frac{2\pi\mu_0 \left\langle \boldsymbol{B} \cdot \nabla\phi \right\rangle |_1 (a - r_1) V'_1 \Delta r_1}{V'_{3/2} \left\langle \frac{|\nabla r|^2}{R^2} \right\rangle \Big|_{3/2} B(r_1)} j_1. \tag{6}$$