Trabajo Práctico 2 - Estructuras de Datos y Algoritmos 2
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b) Especificaciones de los costos para las funciones pedidas en la implementación de secuencias con listas:
Haremos un abuso de notación para expresar que s-$\{s_i\}$ es la secuencia s sin el i-ésimo elemento.
c_2 es el costo de : (constructor de listas en haskell).

$filter_S$:

```
filterS f [] = [] filterS f (x:xs) = let (x',xs') = (f x) ||| (filterS f xs)
                                     in if x' then (x:xs')
                                        else xs'
```

$Wfilter_S(f,[]) = Sfilter_S(f,[]) = c_1$
$Wfilter_S(f,s) = Wfs_0 + Wfilter_S(f, s-\{s_0\}) + c_2 =$
$Wfilter_S(f, s-\{s_0,s_1\}) + Wfs_1 + Wfs_0 + 2c_2 =$ ...en $|s|-2$ pasos...=
$Wfilter_S(f,[]) + \sum_{i=0}^{|s-1|} Wfs_i + |s|c_2 = \sum_{i=0}^{|s-1|} Wfs_i + |s|c_2 + c_1$
$\therefore W_{filter_S}\ \epsilon\ O(\sum_{i=0}^{|s|-1} Wfs_i + |s|)$

$Sfilter_S(f,s) = max(Sfs_0, Sfilter_S(f, s-\{s_0\})) + c_2 =$
$= max(\ Sf_{s_0} + c_2, Sfilter_S(f, s-\{s_0\}) + c_2\ ) =$
$= max(\ Sf_{s_0} + c_2, max(\ Sf_{s_1}, Sfilter_S(f, s-\{s_0,s_1\})) + 2c_2) =$
$= max(\ Sf_{s_0} + c_2, max(\ Sf_{s_1} + 2c_2, Sfilter_S(f, s-\{s_0,s_1\})) + 2c_2) =$
...en $|s|-3$ pasos...
$= max(\ max_{i=0}^{|s|-1}(\ Sf_{s_i} + ic_2\ ), Sfilter_S([]) + |s|c2)=$
$= max(\ max_{i=0}^{|s|-1}(\ Sf_{s_i} + ic_2\ ), c_1 + |s|c2) \leq$
$max(\ max_{i=0}^{|s|-1}(\ Sf_{s_i} + |s|c_2\ ), c_1 + |s|c2) =$
$= max(\ max_{i=0}^{|s|-1}\ Sf_{s_i}\ , c_1) + |s|c2$
Suponiendo que $max_{i=0}^{|s|-1} S_{f_{s_i}} \geq c_1$:
$\therefore$ S$_{filter_S}$ ϵ O($max_{i=0}^{|s|-1} S_{f_{s_i}} + |s|$)

$showt_S$:

```
showtS s = case (lengthS s) of
                    0 -> EMPTY
                    1 -> ELT (nthS s 0)
                    l -> let m = div l 2
                         in NODE (takeS s m) (dropS s m)
```

Primero daremos la especificación de los costos para takeS,dropS y lengthS:
$take_S$:
$Wtake_S(0) = b_1$
$Wtake_S(|s|) = c_2 + Wtake_S(|s|) = ...$ en $|s|-1...$ $= |s|c_2 + Wtake_S(0) =$
$|s|c_2 + b_1$
$\therefore Wtake_S\ \epsilon$ O($|s|$) $drop_S$:

$Wdrop_S(0) = b_1$
$Wdrop_S(|s|) = Wtake_S(|s|-1) + b_2 =$ ... en $|s|-1$... $= |s|b_2 + Wdrop_S(0) = |s|b_2 + b_1$
$\therefore Wdrop_S \; \epsilon \; \mathrm{O}(|s|)$

$legnth_S$:
$Wlength_S(0) = b_1$
$Wlength_S(|s|) = b_2 + Wlength_S(|s|-1) =$ ...en $|s|-1$ pasos... $= |s|b_2 + b_1$
Dado que takeS,dropS y lengthS no paralelizan operaciones concluimos:

$\therefore Slength_S, Stake_S, Sdrop_S \; \epsilon \; \mathrm{O}(|s|)$

$Wshowt_S(0) = c_1$
$Wshowt_S(1) = c_2$
$Wshowt_S(|s|) = Wlength_S(|s|) + Wtake_S(\lfloor |s|/2 \rfloor) + Wdrop_S(\lceil |s|/2 \rceil) + c_3 \leq k|s| + k'(\lfloor |s|/2 \rfloor) + k''(\lceil |s|/2 \rceil) + c_3 \leq k|s| + k'|s|/2 + k''(|s|+1)/2 + c_3 = |s|(k + (k'/2) + (k''/2)) + k''/2 + c_3$
$\therefore W_{showt_S} \; \epsilon \; \mathrm{O}(|s|)$

$Sshowt_S(|s|) = Slength_S(|s|) + max(\; Stake_S(\lfloor |s|/2 \rfloor), Sdrop_S(\lceil |s|/2 \rceil)) + c_3$
$\leq k|s| + k'|s| + c = |s|(k + k') + c_3$
$\therefore S_{showt_S} \; \epsilon \; \mathrm{O}(|s|)$

Antes de dar la especificación del costo de $reduce_S$, daremos la especificación del costo de $contract_S$:

$contract_S$:
contractS f [] = []
contractS f (x:[]) = [x]
contractS f (x:y:xs) = let (x',xs') = (f x y) ||| (contractS f xs)
in x':xs'

$Wcontract_S(f, []) = Scontract_S(f, []) = b_1$
$Wcontract_S(f, [x]) = Scontract_S(f, [x]) = b_2$
$Wcontract_S(f, s) = Wf_{s_0,s_1} + Wcontract_S(f, s - \{s_0, s_1\}) + c_3 =$ ... en $\lfloor |s|/2 \rfloor$ pasos... $= \sum_{i=0}^{\lfloor |s|/2 \rfloor -1} Wf_{s_{2i},s_{2i+1}} + \lfloor |s|/2 \rfloor c_3 + b_3$
$\leq \sum_{i=0}^{\lfloor |s|/2 \rfloor -1} Wf_{s_{2i},s_{2i+1}} + (|s|/2)c_3 + b_3$ con $b_3 = b_1$ o $b_3 = b_2$

$\therefore Wcontract \; \epsilon \; \mathrm{O}(\sum_{i=0}^{\lfloor |s|/2 \rfloor -1} Wf_{s_{2i},s_{2i+1}} + |s|)$

$Scontract_S(f, s) =$
$max(Sf_{s_0,s_1}, Scontract_S(f, s - \{s_0, s_1\}) + c_3 =$
$max(\; Sf_{s_0,s_1} + c_3 \; , \; Scontract_S(f, s - \{s_0, s_1\}) + c_3) =$
$max(\; Sf_{s_0,s_1} + c_3 \; , \; max(Sf_{s_2,s_3}, Scontract_S(f, s - \{s_0, s_1, s_2, s_3\})) + 2c_3) =$
... en $\lfloor |s|/2 \rfloor$ pasos...
$max(\; max_{i=0}^{\lfloor |s|/2 \rfloor -1} \; (Sf_{s_{2i},s_{2i+1}} + ic_3), \; \lfloor |s|/2 \rfloor c_3 + b_3 \leq$
$max(\; max_{i=0}^{\lfloor |s|/2 \rfloor -1} \; Sf_{s_{2i},s_{2i+1}} + (|s|/2)c_3, \; |s|/2c_3 + b_3) =$
$max(\; max_{i=0}^{\lfloor |s|/2 \rfloor -1} \; Sf_{s_{2i},s_{2i+1}} \; , b_3) + (|s|/2)c_3 =$

$max_{i=0}^{\lfloor |s|/2 \rfloor -1}\ Sf_{s_{2i},s_{2i+1}}\ +\ (|s|/2)c_3$

$\therefore\ Scontract_S\ \epsilon$ O$(max_{i=0}^{\lfloor |s|/2 \rfloor -1}\ Sf_{s_{2i},s_{2i+1}} + |s|)$

Ahora si, $reduce_S$:
reduceS f b [] = b
reduceS f b [x] = f b x
reduceS f b xs = reduce f b (contractS xs)

$Wreduce_S(f,b,[]) = b_1$
$Wreduce_S(f,b,s) = Wcontract_{f,s}\ +\ Wf_{b,contract(f,s)}\ +\ b_2\ =$
$\sum_{i=0}^{\lfloor |s|/2 \rfloor -1} Wf_{s_{2i},s_{2i+1}}\ +\ k_1\lceil |s|/2\rceil\ +\ Wf_{b,contract(f,s)}\ +\ b_2$

Observemos que cada aplicación de $contract_S$ nos da un nivel superior del árbol de reducción de f sobre los elementos de la secuencia, es decir que podemos afirmar que se hacen $\lceil lg(|s|)\rceil$ aplicaciones de $contract_S$. Sea $O_r(f,b,s)$ el conjunto de aplicaciones de f en el árbol de reducción, continuamos el análisis:

... en $\lceil lg(|s|)\rceil$ pasos ... $=\ \sum_{i=1}^{\lceil lg(|s|)\rceil} k_i\lceil |s|/2^i\rceil\ +\ \sum_{f_{x,y}\epsilon O_r(f,b,s)} Wf_{x,y}\ +$
$\lceil lg(|s|)\rceil b_2$

Además:
$\sum_{i=1}^{\lceil lg(|s|)\rceil} k_i\lceil |s|/2^i\rceil\ \leq\ \sum_{i=1}^{\lceil lg(|s|)\rceil} k_i(|s|+1)/2^i\ \leq\ k(|s|+1)\sum_{i=1}^{\lceil lg(|s|)\rceil} 1/2^i\ \leq$
$k(|s|+1)$

Por lo tanto:

$Wreduce_S(f,b,s) \leq \sum_{f_{x,y}\epsilon O_r(f,b,s)} Wf_{x,y}\ +\ k(|s|+1)\ +\ (\lceil lg(|s|)\rceil)b_2$

$\therefore\ Wreduce_S\ \epsilon$ O$(|s|\ +\ \sum_{f_{x,y}\epsilon O_r(f,b,s)} Wf_{x,y})$

$Sreduce_S(f,b,s) = Scontract_{f,s}\ +\ Sf_{b,contract(f,s)}\ +\ b_2$

Por el análisis hecho para el trabajo de reduce, podemos afirmar que:

... en $\lceil lg(|s|)\rceil$ pasos ...

$Sreduce_S(f,b,s) \leq \sum_{i=0}^{\lceil lg(|s|)\rceil -1} k_i(|s|)/2^i\ +\ \lceil lg(|s|)\rceil \max_{f_{x,y}\epsilon O_r(f,b,s)} Sf_{x,y}\ +$
$(\lceil lg(|s|)\rceil)b_2\ \leq\ k(|s|+1)\ +\ \lceil lg(|s|)\rceil \max_{f_{x,y}\epsilon O_r(f,b,s)} Sf_{x,y}\ +\ (\lceil lg(|s|)\rceil)b_2\ \therefore$

$Sreduce_S\ \epsilon$ O$(|s|\ +\ \lceil lg(|s|)\rceil \max_{f_{x,y}\epsilon O_r(f,b,s)} Sf_{x,y})$

Antes de dar la especificación para el trabajo y la profundidad de scan, daremos la especificación de expandS .
$expand_S$:
expandS f [] [] = []
expandS f (x:[]) (z:[]) = [z]
expandS f (x:y:xs) (z:zs) = let (x',xs') = (f x z) ||| (expandS f xs zs)
                                           in z:x':xs'

$Wexpand_S(f, [], []) = b_1$
$Wexpand_S(f, [s_1], [s'_1]) = b_2$
$Wexpand_S(f, s, s') = Wf_{s'_0, s_0} + Wexpand_S(f, s - \{s_0, s_1\}, s' - \{s_0\}) + b_3 =$
... en $\lceil |s|/2 \rceil$ pasos ... $= \sum_{i=0}^{\lfloor |s|/2 \rfloor - 1} Wf_{s'_{\lfloor i/2 \rfloor}, s_{i-1}} + \lceil |s|/2 \rceil c_2 + b_3 \leq$
$\sum_{i=0}^{\lfloor |s|/2 \rfloor - 1} Wf_{s'_{\lfloor i/2 \rfloor}, s_{i-1}} + (|s|/2) c_2 + b_3$

$\therefore W_{expand_S} \; \epsilon \; O(\sum_{i=0}^{\lfloor |s|/2 \rfloor - 1} Wf_{s'_{\lfloor i/2 \rfloor}, s_{i-1}} + |s|)$

$Sexpand_S(f, s, s') = \max(\; Sf_{s'_0, s_0}, \; Sexpand_S(f, s - \{s_0\}, s' - \{s'_0\})) + c_2$
Ya hemos visto que recurrencias de esta forma tienen la siguiente resolución:
$max(\max_{i=0}^{\lfloor |s|/2 \rfloor - 1} (Sf_{s'_{\lfloor i/2 \rfloor}, s_{i-1}} + ic_2, \; c_3 + \lceil |s|/2 \rceil c_2)$
Por análisis anteriores támbien podemos concluir:
$Sexpand_S(f, s, s') \leq \max_{i=0}^{\lfloor |s|/2 \rfloor - 1} Sf_{s'_{\lfloor i/2 \rfloor}, s_{i-1}} + |s| c_2$
$\therefore Sexpand_S \; \epsilon \; O(\max_{i=0}^{\lfloor |s|/2 \rfloor - 1} Sf_{s'_{\lfloor i/2 \rfloor}, s_{i-1}} + |s|)$

$scan_S$:
```
scanS f b [] = ([],b)
scanS f b (x:[]) = ([b],f b x)
scanS f b xs = let xs' = contractS f xs
                   (s,r) = scanS f b xs'
               in (expandS f xs s, r)
```

$Wscan_S(f, b, []) = a_1$
$Wscan_S(f, b, [x]) = a_2$
$Wscan_S(f, b, s) = Wcontract_S(f, s) +$
$Wexpand_S(f, b, first(scan_S(f, b, contract_S(f, s)))) +$
$Wscan_S(f, b, contract_S(f, s)) + a_3$

Como en $reduce_S$, la cantidad de llamadas recursivas que tenemos a $scan_S$ es igual a la cantidad de contracciones que hacemos sobre la entrada o igual a la altura del árbol de reduccion de f sobre la secuencia. Luego tenemos $\lceil lg(|s|) \rceil$ llamadas recursivas a $scan_S$. Definimos $O_s(f, b, s)$ como el conjunto de aplicaciones de f sobre los elementos del árbol de reducción que realiza $contract_S$ más todas las aplicaciones de f que $expand_S$ para armar la nueva secuencia. Continuamos con el análisis:

... en $\lceil lg(|s|) \rceil$ pasos ...
$Wscan_S(f, b, s) \leq \sum_{f_{x,y} \epsilon O_s(f,b,s)} Wf_{x,y} + \sum_{i=1}^{\lceil lg(|s|) \rceil} k'_i \lceil |s|/2^i \rceil +$
$\sum_{i=1}^{\lceil lg(|s|) \rceil} k''_i \lceil |s|/2^i \rceil + \lceil lg(|s|) \rceil a_3$

Además:
$\sum_{i=1}^{\lceil lg(|s|) \rceil} k'_i \lceil |s|/2^i \rceil + \sum_{i=1}^{\lceil lg(|s|) \rceil} k''_i \lceil |s|/2^i \rceil = \sum_{i=1}^{\lceil lg(|s|) \rceil} (k'_i + k''_2) \lceil |s|/2^i \rceil \leq$
$\sum_{i=1}^{\lceil lg(|s|) \rceil} (k'_i + k''_2)(|s| + 1/2^i) \leq k(|s| + 1) \sum_{i=1}^{\lceil lg(|s|) \rceil} 1/2^i \leq k(|s| + 1)$

Por lo tanto:
$Wscan_S(f, b, s) \leq \sum_{f_{x,y} \epsilon O_s(f,b,s)} Wf_{x,y} + k(|s| + 1) + \lceil lg(|s|) \rceil a_3$

$\therefore Wscan_S \ \epsilon \ \mathrm{O}(\sum_{f_{x,y} \epsilon O_s(f,b,s)} Wf_{x,y} + |s|)$

$Sscan_S(f,b,s) = Scontract_S(f,s) +$
$Sexpand_S(f,b,first(scan_S(f,b,contract_S(f,s)))) +$
$Sscan_S(f,b,contract_S(f,s)) + a_3 \leq max_{i=0}^{\lfloor |s|/2 \rfloor -1} Sf_{s_{2i},s_{2i+1}} +$
$\max_{i=0}^{\lfloor |s|/2 \rfloor -1} Sf_{s'_{\lfloor i/2 \rfloor},s_{i-1}} + k|s|$

Por el análisis hecho para el trabajo de scan podemos afirmar:

... luego de $\lceil lg(|s|) \rceil$ pasos ...
$Sscan_S(f,b,s) \leq \lceil lg(|s|) \rceil \max_{f_{x,y} \epsilon O_s(f,b,s)} Sf_{x,y} + \sum_{i=0}^{\lceil lg(|s|) \rceil -1} k'_i(|s|/2^i) +$
$\lceil lg(|s|) \rceil a_3 \leq \lceil lg(|s|) \rceil \max_{f_{x,y} \epsilon O_s(f,b,s)} Sf_{x,y} + k(|s|+1) + \lceil lg(|s|) \rceil a_3$
$\therefore Sscan_S \ \epsilon \ \mathrm{O}(\lceil lg(|s|) \rceil \max_{f_{x,y} \epsilon O_s(f,b,s)} Sf_{x,y} + |s|)$

d)Especificaciones de los costos para las funciones pedidas en la implementación de secuencias con arreglos.

$filter_S$:

```
filterS f s = let f' x = if (f x) then (singletonS x) else emptyS
                  s' = mapS f' s
              in joinS s'
```

Primero veremos que la espeficación de los costos para las funciones llamadas en $filter_S$ coinciden con lo esperado.

$empty_S$:
$Wempty = c_1$

$singleton_S$:
$Wsingleton_S(s_i) = c_2$

$Wf'_{s_i} = Wf_{s_i} + c_3$ con $c_3 = c_1$ o $c_3 = c_2$

$\therefore Wf' \ \epsilon \ \mathrm{O}(Wf)$, $Sf' \ \epsilon \ \mathrm{O}(Sf)$ $Wsingleton_S(s_i) = WfromList(s_1) = c_2$

map_S:

```
mapS f s = let f'= ı-> f (nthS s i)
               n = lengthS s
           in tabulateS f' n
```

$WmapS(f,s) = Wlength(s) + Wtabulate(f,s) \leq k + \sum_{i=0}^{|s|-1} Wf_{s_i}$
$\therefore Wmap_S \ \epsilon \ \mathrm{O}(\sum_{i=0}^{|s|-1} Wf_{s_i})$

$SmapS(|s|) = Slength(|s|) + Stabulate(|s|) \leq k + k'$
$\therefore Smap_S \ \epsilon \ \mathrm{O}(1)$

$join_S$:
joinS s = A.flatten s

$\therefore Wjoin_S \ \epsilon$ O($|s|$ + $\sum_{i=0}^{|s|-1} O(|s!i|)$), $Sjoin_S \ \epsilon$ O($lg(|s|)$)

$Wfilter_S(f,s) = Wmap_S(f,s) + Wjoin_S(s) + d_1 \leq \sum_{i=0}^{|s|-1} Wf_{s_i} + k|s| + \sum_{i=0}^{|s|-1} k'_i + d_1 \leq \sum_{i=0}^{|s|-1} Wf_{s_i} + k|s| + k'|s| + d_1 = \sum_{i=0}^{|s|-1} Wf_{s_i} + |s|(k+k')$ $+ \ d_1$

$\therefore Wfilter_S \ \epsilon$ O($\sum_{i=0}^{|s|-1} Wf_{s_i} + |s|$)

$Sfilter_S(f,s) = Smap_S(f,s) + Sjoin_S(s) + d_3 \leq \max_{i=0}^{|s|-1} S_{s_i} + klg(|s|) + d_3$

$\therefore Sfilter_S \ \epsilon$ O($\max_{i=0}^{|s|-1} S_{s_i} + lg(|s|)$)

$showt_S$:

```
showtS s = case (lengthS s) of
                  0 -> EMPTY
                  1 -> ELT (nthS s 0)
                  l -> let m = div l 2
                            in NODE (takeS s m) (dropS s m)
```

Veamos que las especificaciones de los costos de $take_S$ y $drop_S$ cumplen lo esperado:

$take_S$:
takeS s n = A.subArray 0 n s

$Wtake_S(s) = Stake_S(s) = k$

$\therefore Wtake_S$, $Stake_S \ \epsilon$ O(1)

$drop_S$:
dropS s n = A.subArray n ((lengthS s)-n) s
$Wdrop_S(s) = Wlength(s) + WsubArrary(s) \leq k' + k''$
$Sdrop_S(s) = Slength(s) + SsubArrary(s) \leq k' + k''$
$\therefore Wdrop_S$, $Sdrop_S \ \epsilon$ O(1)

$Wshowt_S(<>) = Wempty_S = c_1$
$Wshowt_S(< s_i >) = Wnth_S(< s_i >, 0) = c_2$
$Wshowt_S(s) = Wdrop_S(s) + Wtake_S(s) + c_3 \leq k + k' + k'' + c_3$

$\therefore Wshowt_S \ \epsilon$ O(1)

$Sshowt_S(s) = Sdrop_S(s) + Stake_S(s) + \leq k + k' + k'' + c_3$

$\therefore Sshowt_S \ \epsilon$ O(1)

$reduce_S$:

```
reduceS f b s = case (lengthS s) of
                  0 -> b
                  1 -> f b (s A.! 0)
                  l -> reduceS f b (contractS f s)
```

Primero daremos la especificación de costo para $contract_S$:
$contract_S$:

```
contractS f s = case (A.length s) of
          1 -> s
          l -> let m = ceiling ((fromIntegral l)/2)
                   f' x | x /= (m-1) = f (s A.! (2*x)) (s A.! (2*x+1))
                        | otherwise = if even l then f (s A.! (l-2)) (s A.! (l-1))
                                      else (s A.! (l-1))
               in A.tabulate f' m
```

$Wf'(i) = Wnth_S(s, 2*i) + Wnth_S(s, 2i+1) + Wf_{nth_S(s,2i), nth_S(s,(2i)+1)} \leq c_1 + Wf_{nth_S(s,2i), nth_S(s,2i+1)}$ si i es par
$Wf'(i) = c_2$ sino

$Wcontract_S(f, <>) = d_1$
$Wcontract_S(f, < s_i >) = d_2$
$Wcontract_S(f,) = Wlength(s) + c_2 + Wtabulate(f', s) + d_3 \leq k + \sum_{i=0}^{\lfloor|s|/2\rfloor-1} Wf_i' + d_3 + d_4 = k + \sum_{i=0}^{\lfloor|s|/2\rfloor-1} (Wf_{s_{2i}, s_{2i+1}} + c_1) + d_3 + d_4 = k + \sum_{i=0}^{\lfloor|s|/2\rfloor-1} Wf_{s_{2i}, s_{2i+1}} + \sum_{i=0}^{\lfloor|s|/2\rfloor-1} c_1 + d_3 + d_4 = k + \sum_{i=0}^{\lfloor|s|/2\rfloor-1} Wf_{s_{2i}, s_{2i+1}} + \lfloor|s|/2\rfloor c_1 + d_3 + d_4$ con $d_4 = c_2$ o $d_4 = 0$

$\therefore Wcontract \ \epsilon$ O($\sum_{i=0}^{\lfloor|s|/2\rfloor-1} Wf_{s_{2i}, s_{2i+1}} + |s|$)

$Scontract(f, s) = Slength + Stabulate(f', s) + d_3 \leq k + max_{i=0}^{\lfloor|s|/2\rfloor-1} Sf_i' + d_3 + d_4 = k + max_{i=0}^{\lfloor|s|/2\rfloor-1} Sf_{s_{2i}, s_{2i+1}} + c_1 + d_3 + d_4$

$\therefore Scontract \ \epsilon$ O($max_{i=0}^{\lfloor|s|/2\rfloor-1} Sf_{s_{2i}, s_{2i+1}}$)

$reduce_S$:

```
reduceS f b s = case (lengthS s) of
                0 -> b
                1 -> f b (s A.! 0)
                l -> reduceS f b (contractS f s)
```

$Wreduce_S(f, b, s) = Wlength(s) + Wcontract(f, s) + Wreduce_S(f, b, contract(f, s)) + b_3 \leq k + k' \sum_{i=0}^{\lfloor|s|/2\rfloor-1} Wf_{s_{2i}, s_{2i+1}} + k'|s| + Wreduce_S(f, b, contract(f, s)) + b_3 =$ ... en $\lceil lg(|s|)\rceil$ pasos ... $= \lceil lg(|s|)\rceil k + \sum_{i=0}^{\lceil lg(|s|)\rceil} k_i' \lceil |s|/2^i \rceil + \sum_{f_{x,y} \epsilon O_r(f,b,s)} \leq \lceil lg(|s|)\rceil k + k'|s| + \sum_{f_{x,y} \epsilon O_r(f,b,s)} + \lceil lg(|s|)\rceil b_3$

$\therefore Wreduce_S \ \epsilon$ O($|s| + \sum_{f_{x,y} \epsilon O_r(f,b,s)}$)

$Sreduce_S(f, b, s) = Slength(s) + Scontract(f, s) + Sreduce_S(f, b, contract_S(f, s)) + b_3 \leq k + \max_{i=0}^{\lfloor|s|/2\rfloor-1} Sfs_{2i}, s_{2i+1} + Sreduce_S(f, b, contract_S(f, s)) + b_3$

... en $\lceil lg(|s|)\rceil$ pasos ...

$Sreduce_S(f, b, s) \leq \lceil lg(|s|)\rceil k + \lceil lg(|s|)\rceil \max_{f_{x,y} \epsilon O_r(f,b,s)} + \lceil lg(|s|)\rceil b_3$

$\therefore$ $Sreduce_S$ ϵ O($\lceil lg(|s|)\rceil \max_{f_{x,y}\epsilon O_r(f,b,s)}$)

$scan_S$:

```
scanS f b s = case (lengthS s) of
                  0 -> (emptyS,b)
                  1 -> (singletonS b,f b (s A.! 0))
                  l -> let s' = contractS f s
                           (s'',r) = scanS f b s'
                        in (expandS f s s'',r)
```

$expand_S$:

```
expandS f s s' = let f' x = let m = div x 2 in
                        if even x then s' A.! m
                        else f (s' A.! m) (s A.! (x-1))
                 in A.tabulate f' (A.length s)
```

$Wf'(i) = Wnth_S(s', \lceil i/2\rceil) = c_2$ si i es par
$Wf'(i) = Wnth_S(s', \lceil i/2\rceil) + Wnth_S(s, i-1) +$
$Wf(nth_S(s', \lceil i/2\rceil), nth_S(s, i-1))$ sino

$Wexpand_S(f, s, s') = b_3 + Wtabulate(f', |s|) + d_3 \leq b_3 + \sum_{i=0}^{|s|-1} Wf'_i + d_3$

$\therefore$ $Wexpand_S$ ϵ O($\sum_{i=0}^{|s|-1} Wf'_i$)

$Sexpand(f, s, s') = Slength(s) + Stabulate(f', |s|) + d_3 \leq b_3 + \max_{i=0}^{|s|-1} Sf'_i$
$+ d_3$

$\therefore$ $Sexpand_S$ ϵ O($\max_{i=0}^{|s|-1} Sf'_i$)

$Wscan_S(f, b, s) = c_3 + Wcontract(f, s) +$
$Wexpand(f, s, first(scan_S(f, b, contract(f, s)))) + Wscan(f, b, contract(f, s))$
$\leq c_3 + \sum_{i=0}^{\lfloor |s|/2\rfloor -1} Wf_{2i,2i+1} + \sum_{i=0}^{|s|-1} Wf'_i$

Ahora:

$\sum_{i=0}^{|s|-1} Wf'_i = \sum_{i=0}^{\lfloor |s|/2\rfloor -1} Wf'_{2i} + \sum_{i=0}^{\lfloor |s|/2\rfloor -1} Wf'_{2i+1} = \lfloor |s|/2\rfloor c_2 +$
$\sum_{i=0}^{\lfloor |s|/2\rfloor -1} Wf'_{2i+1}$

Luego:

$Wscan_S(f, b, s) \leq c_3 + \sum_{i=0}^{\lfloor |s|/2\rfloor -1} Wf_{2i,2i+1} + \sum_{i=0}^{\lfloor |s|/2\rfloor -1} Wf'_{2i+1} + \lfloor |s|/2\rfloor c_2$
$Wscan(f, b, contract(f, s)) = \ldots$ en $\lceil lg(|s|)\rceil$ pasos $\ldots = \lceil lg(|s|)\rceil c_3 +$
$\sum_{f_{x,y}\epsilon O_s(f,b,s)} + \sum_{i=1}^{\lceil lg(|s|)\rceil} \lfloor |s|/2^i\rfloor c_2$

Además:

$\sum_{i=1}^{\lceil lg(|s|)\rceil} \lfloor |s|/2^i\rfloor c_2 \leq c_2 \sum_{i=1}^{\lceil lg(|s|)\rceil} (|s|+1)/2^i \leq$
$c_2(|s|+1) \sum_{i=1}^{\lceil lg(|s|)\rceil} 1/2^i \leq c_2(|s|+1)$

Concluimos:

$Wscan_S(f, b, s) \leq \lceil lg(|s|) \rceil c_3 + \sum_{f_{x,y} \epsilon O_s(f,b,s)} + (|s| + 1)c_2$

$\therefore Wscan_S \; \epsilon \; \mathrm{O}(\sum_{f_{x,y} \epsilon O_s(f,b,s)} + |s|)$

$Sscan_S(f, b, s) = c_3 + Scontract(f, s) +$
$Sexpand(f, s, first(scan_S(f, b, contract(f, s)))) + Sscan(f, b, contract(f, s)) \leq$
$c_3 + \max_{i=0}^{\lfloor |s|/2 \rfloor} Sf_{s_{2i}, s_{2i+1}} + \max_{i=0}^{|s|-1} Sf'_i \; Sscan(f, b, contract(f, s))$

... en $\lceil lg(|s|) \rceil$ pasos ...
$Sscan_S(f, b, s) \leq \lceil lg(|s|) \rceil c_3 + \lceil lg(|s|) \rceil \max_{f_{x,y} \epsilon O_s(f,b,s)} Sf_{x,y}$

$\therefore Sscan_S \; \epsilon \; \mathrm{O}(\lceil lg(|s|) \rceil \max_{f_{x,y} \epsilon O_s(f,b,s)} Sf_{x,y})$