Introduction to Markov Processes in Healthcare Supply Chains

Harsha Halgamuwe Hewage
Data Lab for Social Good Research Lab, Cardiff University, UK

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Assumptions

- You should be familiar with basic probability and random variables.
- You are expected to be comfortable with R (or Python) for basic simulation and matrix operations.
- This is not a theory-heavy workshop—we will use simple examples to build intuition, not derive theorems.

What we will cover

- Key concepts in Markov chains
- Transition matrices and system evolution
- Steady-state distributions and their interpretations
- Applications of Markov models in healthcare supply chains
- Code demonstrations and simulations

What we will `not cover`

- Proofs of Markov chain convergence theorems
- Advanced topics like Hidden Markov Models or Semi-Markov processes
- Continuous-time Markov processes in full generality
- Formal classification of all chain types (e.g., reducibility, ergodicity)
- Marcov process with rewards

Materials

You can find the workshop materials here.

Note: These materials are based on my learnings at NATCOR Taught Course Centre: Stochastic Modelling Course.

Outline

- Why stochastic modeling?
- What are Markov processes?
- Brand switching as a DTMC example
- Code walk-throughs in R

Why stochastic modeling?

Why stochastic modelling in healthcare supply chains?

- `Demand is unpredictable`: Patient arrivals, seasonal outbreaks, and changing usage patterns
- `Supply is uncertain`: Delivery delays, stock losses, funding gaps, partial shipments
- `Helps quantify risks`: Probability of stockouts, unmet demand, cold chain failures
- `Enables simulation of long-run behavior`: nderstand steady-state stock levels, refill patterns, or equipment uptime
- `Useful for evaluating interventions`: E.g., What if we promote a local brand? Add a backup supplier? Use mobile delivery?

What are Markov processes?

System states

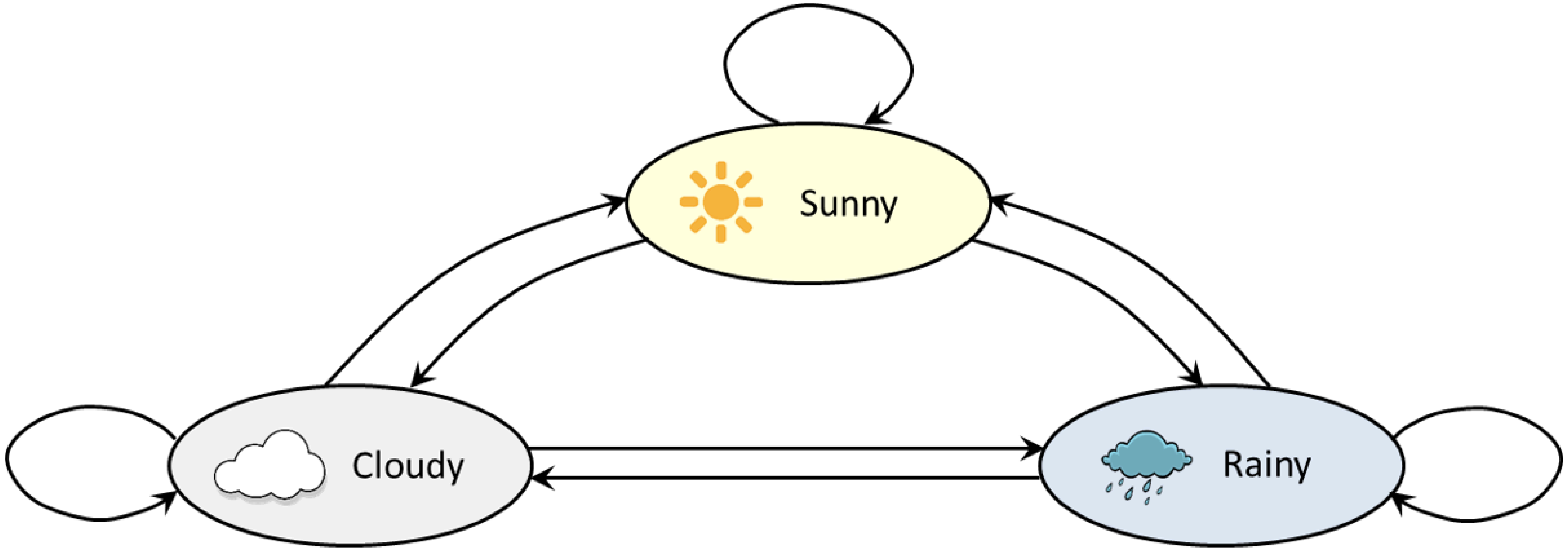
We want to model the behaviour of a `system` which can change its `state` from one period to the next.

For example:

System	States	Time unit
Weather	Sunny, cloudy, rainy, …	Day or hour
No. of people in a healtcare centre	0, 1, 2, 3, …	Minute or second
Status of job application	"In preparation", "Submitted", "Invited for interview", …	Day or hour?

System states: `weather example`

We are interested in the `transitions` between different states. Suppose there are only 3 possible states in weather:



From any of the 3 states we can get to any of the other states in a single transition (or stay in the same state). A sample trajectory of the system could be:

$$[Sunny, Cloudy, Cloudy, Rainy, Sunny, Cloudy, Rainy, Rainy, \ldots]$$

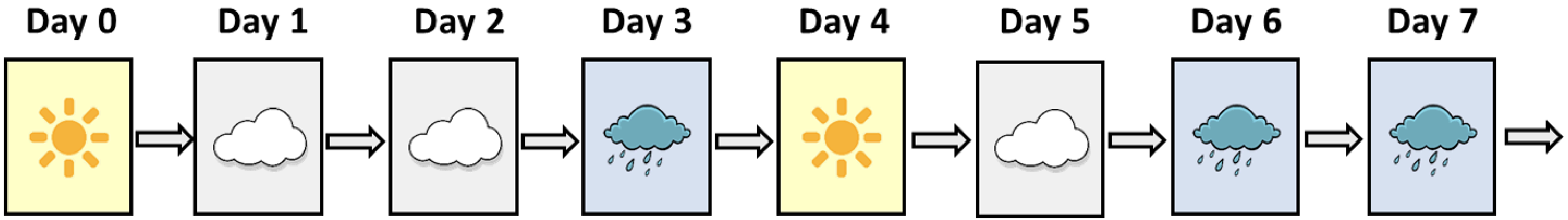
System states: weather example cont.

For convenience, let's give each of the 3 states a number: $0 - Sunny$, $1 - Cloudy$, $2 - Rainy$

Let S be the set of system states. So in our example: $S = 0, 1, 2$

Let X_n be the state of the system after n time units (e.g. days).

For example:



$X_0 = 0$ $X_1 = 1$ $X_2 = 1$ $X_3 = 2$ $X_4 = 0$ $X_5 = 1$ $X_6 = 2$ $X_7 = 2$

System states: `weather example cont.`

Suppose that the system begins as follows:

$X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 2$

Given the above, what is the probability that $X_4 = 0$, i.e. it is sunny after 4 days? We can write this as a conditional probability:

$$Pr(X_4 = 0 \,|\, \underbrace{X_0 = 0,\ X_1 = 1,\ X_2 = 1,\ X_3 = 2)}_{\text{entire history of the process}}$$

Suppose we assume that the probability that $X_4 = 0$ depends only on the value of X_3, and not on X_2, X_1 or X_0. We can then write:

$$Pr(X_4 = 0 \,|\, \underbrace{X_0 = 0,\ X_1 = 1,\ X_2 = 1,\ X_3 = 2)}_{\text{entire history of the process}} = Pr(X_4 = 0 \,|\, \underbrace{X_3 = 2)}_{\text{current state}}$$

Markov process: definition

A stochastic process with the `Markov Property`:

$$(X_1, X_2, X_3, \dots)$$

in which the probability distribution for state X_{n+1} depends only on the state X_n, and not on any of the states from X_0 up to X_{n-1} (for all $n \geq 0$).

Writing this more mathematically, we can say:

$$\Pr\big(X_{n+1} = j \;\big|\; X_0 = x_0,\ X_1 = x_1,\ \dots,\ X_n = x_n\big) \;=\; \Pr\big(X_{n+1} = j \;\big|\; X_n = x_n\big)$$

`Key idea`: *"The future depends only on the present, not on the past."*

So the probability distribution for the next state depends only on the current state, not on the history of previous states. Can be discrete or continuous in time.

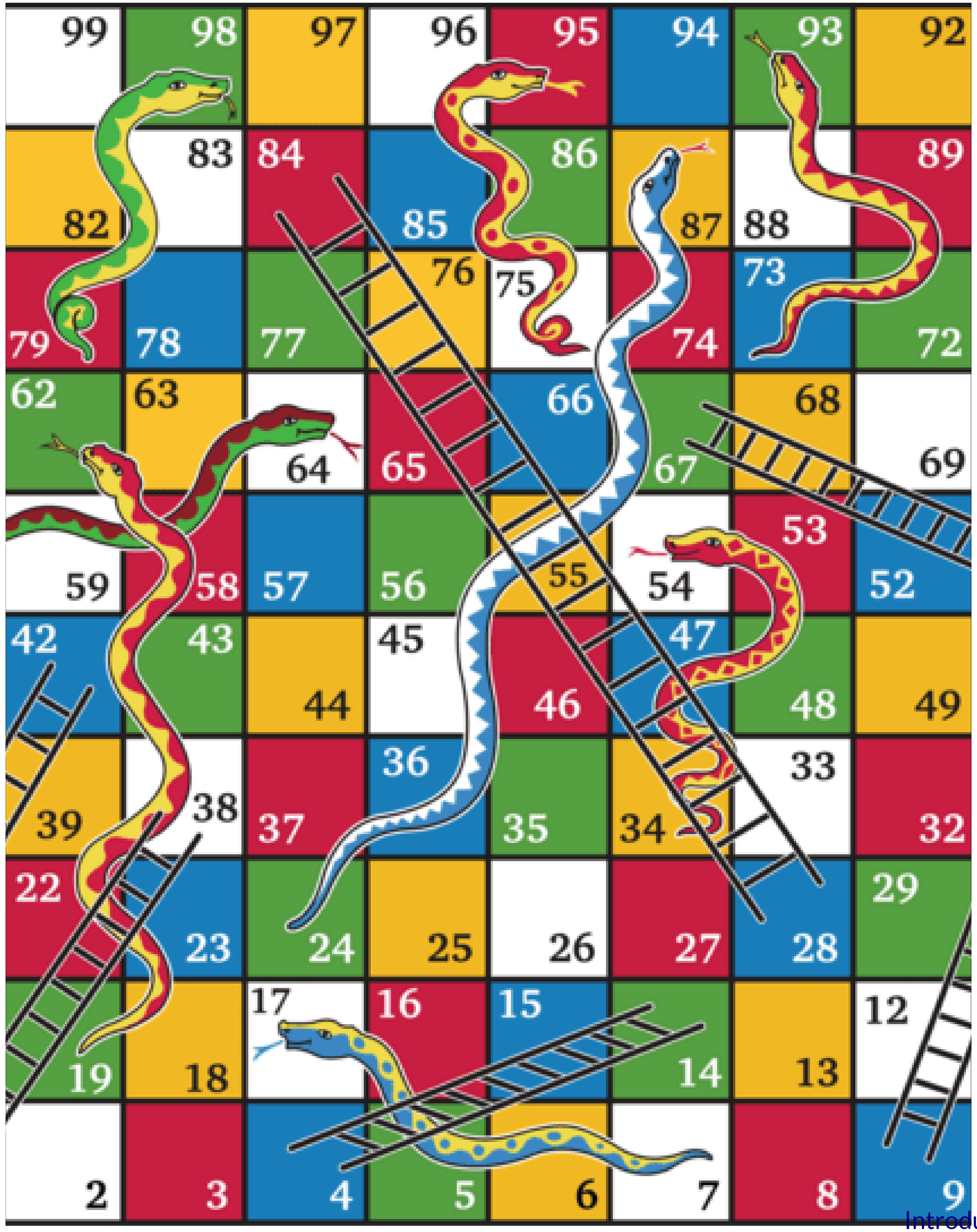
Pause for a thought... 💬

Is it a good idea to model the `weather` using a Markov chain?

00:30

What is a `good example` of a Markov chain?

If the weather is not a good example of a Markov chain, then what is?



For example, in Snakes and Ladders, if $X_n = 50$ then:

- $\Pr\left(X_{n+1} = 67 \mid X_n = 50\right) = \frac{1}{6}$
- $\Pr\left(X_{n+1} = 52 \mid X_n = 50\right) = \frac{1}{6}$
- $\Pr\left(X_{n+1} = 53 \mid X_n = 50\right) = \frac{1}{6}$
- $\Pr\left(X_{n+1} = 34 \mid X_n = 50\right) = \frac{1}{6}$
- $\Pr\left(X_{n+1} = 55 \mid X_n = 50\right) = \frac{1}{6}$
- $\Pr\left(X_{n+1} = 56 \mid X_n = 50\right) = \frac{1}{6}$

Assumptions of Markov chain models

Remember, simple Markov chain models rely upon some important assumptions:

- The Markov chain is in exactly one state on any particular time step
- The probability distribution for the next state only depends on the current state `[Markov property]`
- The transition probabilities are the same on every time step

Discrete time `Markov Chains`

Transition probability matrix

Let $S = \{s_1, s_2, \ldots, s_n\}$ be the finite state space. The one-step transition probabilities are:

$$p_{ij} = \Pr(X_{t+1} = s_j \mid X_t = s_i)$$

We collect these into the **transition matrix**:

$$P = [p_{ij}], \quad \text{where each row sums to 1}$$

The n-step transition matrix is:

$$P^{(n)} = P^n = \underbrace{P \cdot P \cdot \cdots \cdot P}_{n \text{ times}}$$

We can use the **Chapman-Kolmogorov Equation**:

$$P^{(n)} = P^{(k)} P^{(n-k)}$$

Steady-state distribution

When a Markov chain evolves over time, the probability distribution of states may converge to a fixed vector — called the `steady-state distribution`, π.

A steady-state distribution exists if the chain is:

- **Irreducible**: all states communicate
- **Aperiodic**: not cyclic
- **Positive recurrent**: expected return time is finite

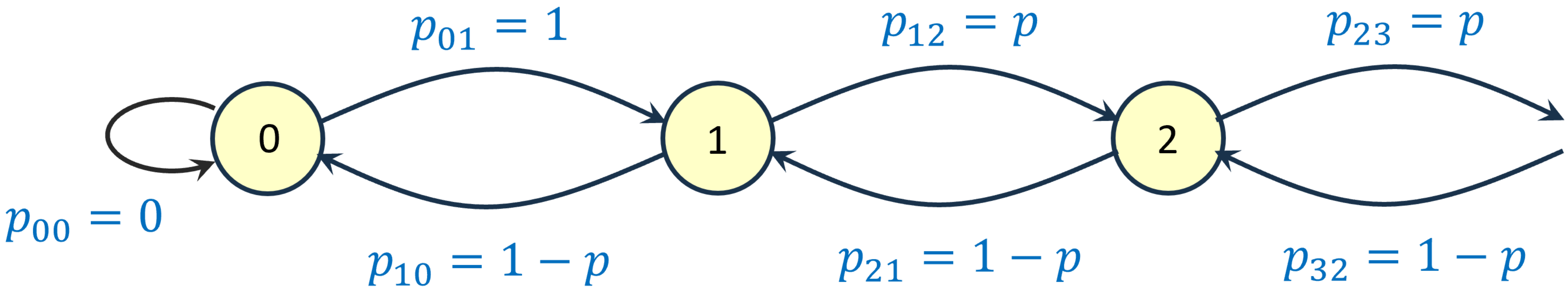
Steady-state equation: $\pi P = \pi, \quad \sum_i \pi_i = 1$

This means:

- π_i: long-run proportion of time spent in state i
- It's a left eigenvector of P corresponding to eigenvalue 1

Classifying states: infinitely many states

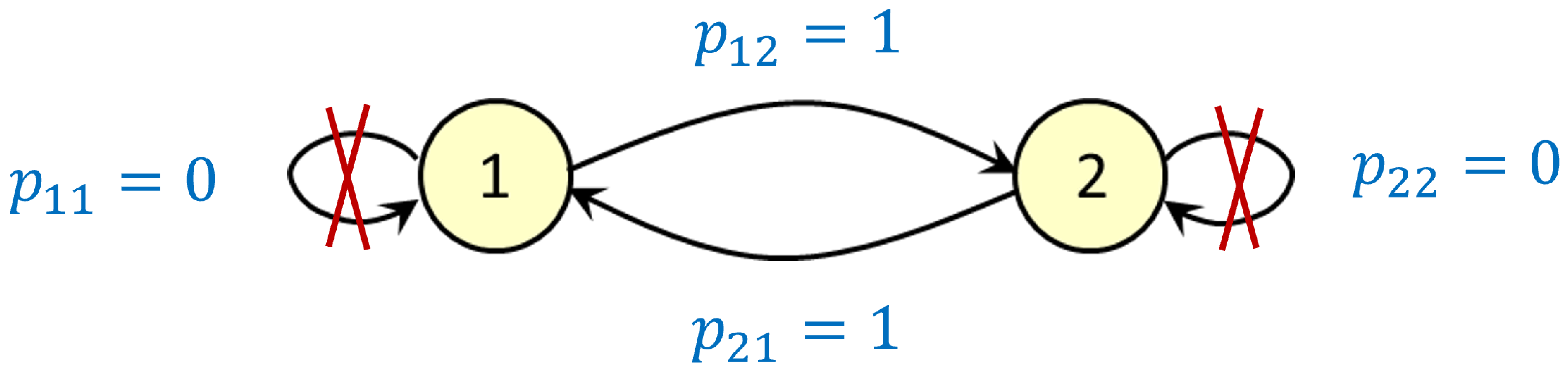
Suppose we have a Markov chain defined on the infinite state space $0, 1, 2, \ldots$.. If it is in state 0 then it moves to state 1 with probability 1. If it is in any other state then it moves up with probability p and moves down with probability $1 - p$, where $0 < p < 1$.



Key point: *if a Markov chain has infinitely many states, a steady-state distribution might not exist.*

Classifying states: periodic states

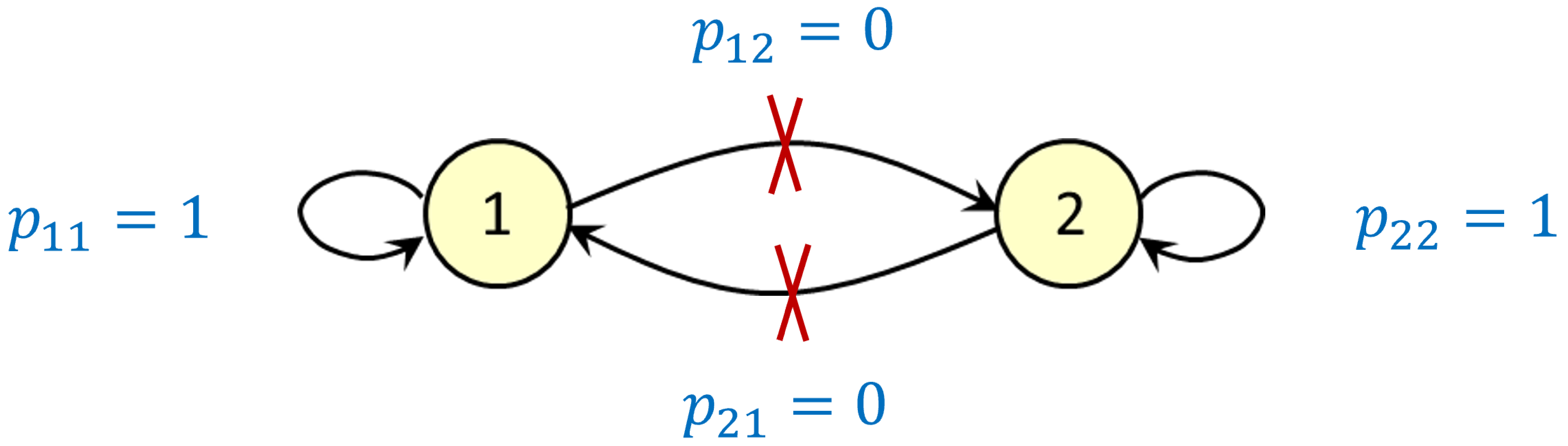
Suppose we have a Markov chain which just goes back and forth between states 1 and 2. We call this a periodic Markov chain with a period of 2, because it can only return to the same state after an even number of time steps.



Key point: *even if a steady-state distribution exists, the Markov chain might not "converge" to the steady-state distribution unless it actually starts there.*

Classifying states: absorbing states

This time, we assume that if you're in state 1 you stay there forever – and the same applies to state 2. In this case we call states 1 and 2 absorbing states.



Key point: *if states do not all "communicate" with each other (meaning that you cannot necessarily find a path from one state to another), there could be multiple steady-state distributions.*

Brand switching

Imagine there are two brands of paracetamol 100mg tablets: Brand A (locally manufactured) and Brand B (imported). As the healthcare center manager, you're planning to promote Brand A via healthcare campaigns to encourage local sourcing. You want to estimate how the market share evolves over time and determine the expected steady-state market share for each brand under this strategy.

Lets promote `Brand A`

Assume patients switch weekly between two brands according to the probabilities shown in the table below:

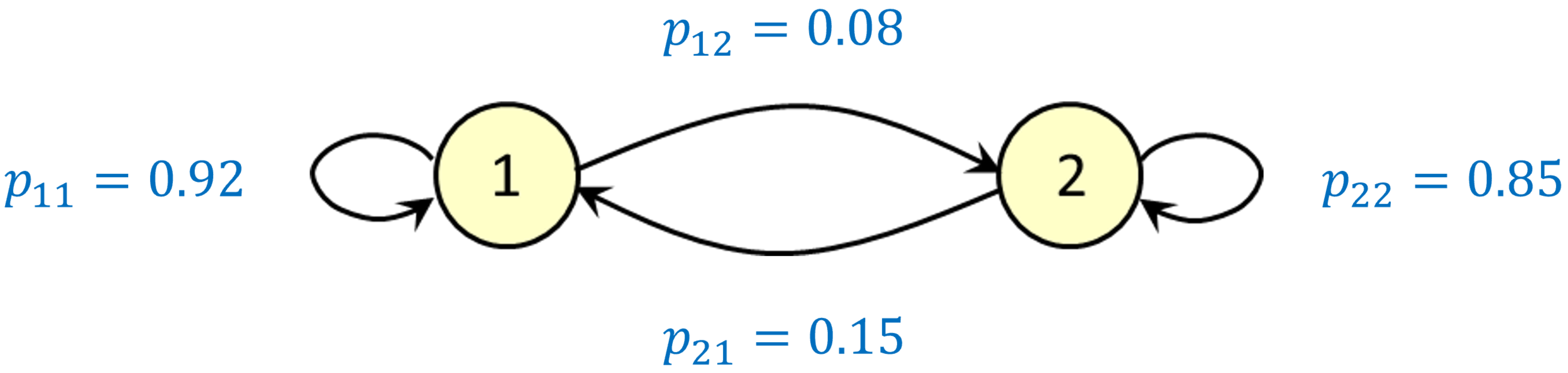
From To	Brand A	Brand B
Brand A	0.92	0.08
Brand B	0.15	0.85

Let X_n denote the preferred brand (either A or B) of a randomly-chosen customer after n weeks. From the table:

$$\Pr(X_{n+1} = 1 \mid X_n = 1) = 0.92 \, \Pr(X_{n+1} = 2 \mid X_n = 1) = 0.08$$
$$\Pr(X_{n+1} = 1 \mid X_n = 2) = 0.15 \, \Pr(X_{n+1} = 2 \mid X_n = 2) = 0.85$$

Lets promote `Brand A` cont.

We can represent this situation using a discrete-time Markov chain:



We are using some shorthand notation: $p_{ij} = Pr(X_{n+1} = j \,|\, X_n = i)$

Lets promote `Brand A` cont.

Suppose that after zero weeks, both brands have a 50% market share. This means a randomly-chosen patient has a 50% chance of preferring Brand A.

So: $Pr(X_0 = 1) = 0.5$ and $Pr(X_0 = 2) = 0.5$

Using the switching probabilities and invoking the `law of total probability`, we can calculate the preferred brand of a randomly-chosen patient after 1 week:

$$Pr(X_1 = 1) = \underbrace{Pr(X_0 = 1)Pr(X_1 = 1|X_0 = 1)}_{\text{(begin in state 1)}} + \underbrace{Pr(X_0 = 2)Pr(X_1 = 1|X_0 = 2)}_{\text{(begin in state 2)}}$$

$$= (0.5 \times 0.92) + (0.5 \times 0.15) = 0.535$$

$$Pr(X_1 = 2) = \underbrace{Pr(X_0 = 1)Pr(X_1 = 2|X_0 = 1)}_{\text{(begin in state 1)}} + \underbrace{Pr(X_0 = 2)Pr(X_1 = 2|X_0 = 2)}_{\text{(begin in state 2)}}$$

$$= (0.5 \times 0.08) + (0.5 \times 0.85) = 0.465$$

Lets promote `Brand A` cont.

We can make these calculations look neater by using matrix-vector notation. Let p_{ij} denote the probability of switching from i to j. Obviously, this implies:

$p_{ij} \geq 0, \ \forall\, i, j \in S, \sum_{j\in S} p_{ij} = 1, \ \forall\, i \in S.$

Let P denote the `transition matrix`:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.92 & 0.08 \\ 0.15 & 0.85 \end{pmatrix}$$

Also let $q^{(0)}$ be the vector of initial market shares:

$q^{(0)} = (0.5, 0.5)$

To get the expected market shares after one week, we multiply $q^{(0)}$ by P to get $q^{(1)}$

$$\mathbf{q}^{(0)}\mathbf{P} = (0.5,\ 0.5) \begin{pmatrix} 0.92 & 0.08 \\ 0.15 & 0.85 \end{pmatrix} = (0.535,\ 0.465) = \mathbf{q}^{(1)}$$

Lets promote Brand A cont.

To find the expected market shares after two weeks, we repeat the process, starting from the expected market shares after one week. This means we need to calculate

$$\underset{\mathbf{q}^{(1)}}{\text{expected market shares after one week}} \times \underset{P}{\text{transition matrix}} = \underset{\mathbf{q}^{(2)}}{\text{expected market shares after two weeks}}$$

$$\mathbf{q}^{(1)}\mathbf{P} = (0.535,\ 0.465)\begin{pmatrix} 0.92 & 0.08 \\ 0.15 & 0.85 \end{pmatrix} = (0.56195,\ 0.43805) = \mathbf{q}^{(2)}$$

Similarly, to find the expected market shares after three weeks:

$$\mathbf{q}^{(2)}\mathbf{P} = (0.56195,\ 0.43805)\begin{pmatrix} 0.92 & 0.08 \\ 0.15 & 0.85 \end{pmatrix} = (0.5827015,\ 0.4172985) = \mathbf{q}^{(3)}$$

Lets promote `Brand A` cont.

In general, to find the expected market shares after n weeks, we calculate

$$\mathbf{q}^{(n-1)}\,\mathbf{P} = \mathbf{q}^{(n)}$$

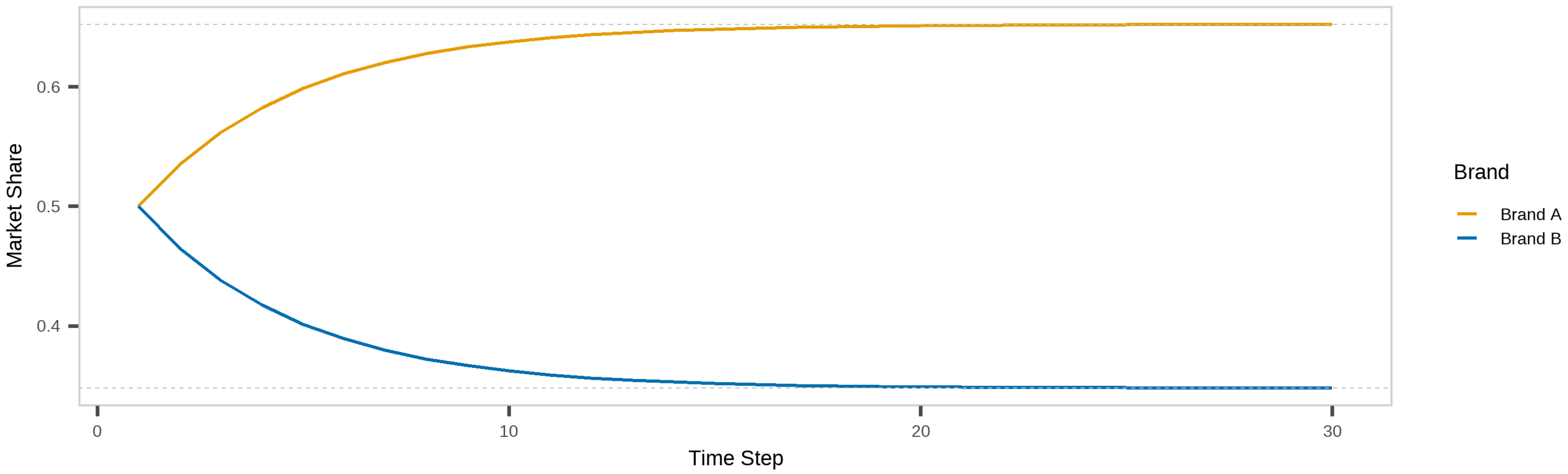
This is the same as:

$$\mathbf{q}^{(0)}\underbrace{\mathbf{P}\,\mathbf{P}\,\cdots\,\mathbf{P}}_{\text{(n times)}} \;=\; \mathbf{q}^{(0)}\,\mathbf{P}^{n}$$

i.e. the vector of initial market shares multiplied by $\mathbf{P}$ to the power n.

Lets promote `Brand A` cont.

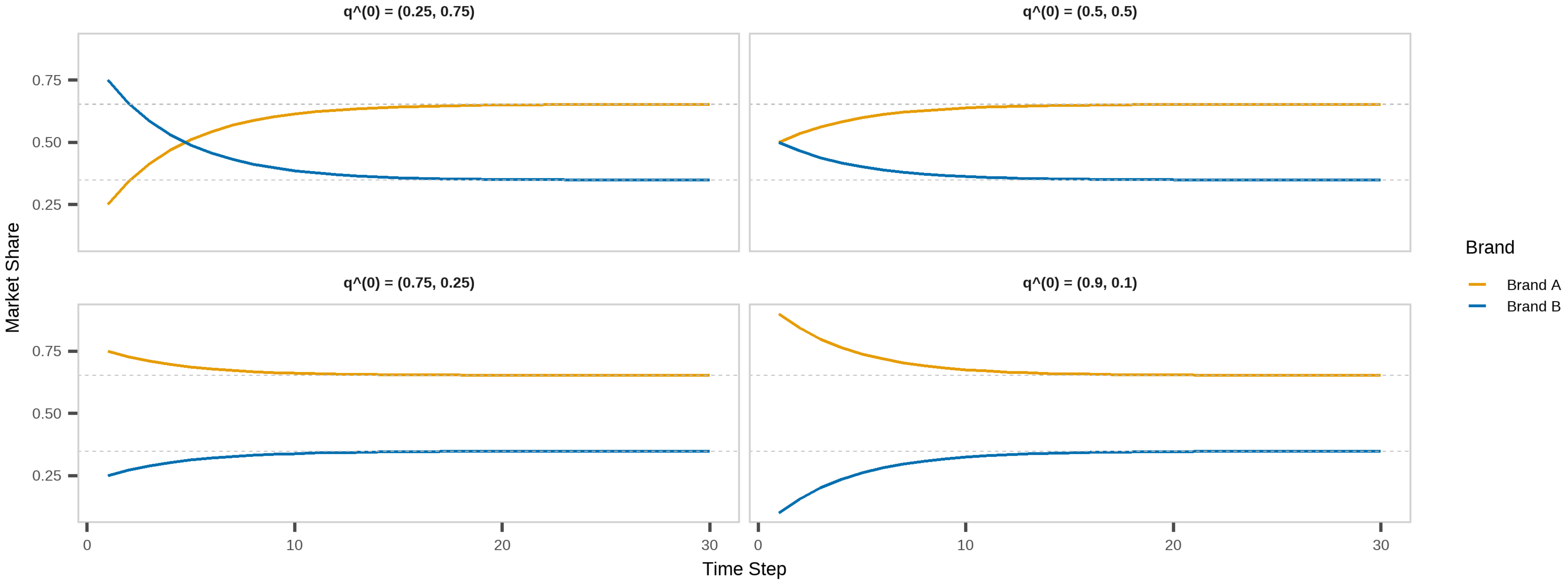
If we use a line graph to show how the expected market shares change with time, we find that both of them appear to `converge` to fixed values.



The market share for `Brand A` converges to about 65.2%, and the market share for `Brand B` converges to about 34.8%. We call (0.652, 0.348) the `steady-state distribution` in this example.

Lets promote `Brand A` cont.

Changing the initial probability vector $\mathbf{q}^{(0)}$ has no effect on the steady-state distribution.



Lets promote `Brand A` cont.

Recall that: $\mathbf{q}^{(n)}$ = vector of state probabilities after n time steps (*this is the vector of expected market shares in our task*) and $\mathbf{P}$ = transition matrix.

We have already seen that $\mathbf{q}^{(n)}$ appears to converge towards a limit as n increases. If we use $\boldsymbol{\pi}$ to denote the limiting vector, i.e. $\boldsymbol{\pi} = \lim_{n\to\infty} \mathbf{q}^{(n)}$, then we can take limits to obtain:

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

Note: $\boldsymbol{\pi} = (\pi_1, \pi_2)$, where the numbers π_1 and π_2 are "unknowns."

Lets promote `Brand A` cont.

By solving the equations $\boldsymbol{\pi} = (\pi_1, \pi_2)$ (where $\boldsymbol{\pi}$ is a 'vector of unknowns') we can calculate the steady-state distribution of the Markov chain.

Recall the market shares example from earlier. Let π_1 and π_2 denote the (unknown) steady-state expected market shares for brands A and B respectively. We have:

$$\boldsymbol{\pi} P = \boldsymbol{\pi} \quad \Longleftrightarrow \quad (\pi_1, \pi_2) \begin{pmatrix} 0.92 & 0.08 \\ 0.15 & 0.85 \end{pmatrix} = (\pi_1, \pi_2).$$

This gives us a couple of linear equations in π_1 and π_2:

$$\Longleftrightarrow \begin{cases} \pi_1 = 0.92\,\pi_1 + 0.15\,\pi_2 \\ \pi_2 = 0.08\,\pi_1 + 0.85\,\pi_2 \end{cases} \Longleftrightarrow \begin{cases} 0.08\,\pi_1 - 0.15\,\pi_2 = 0 \\ 0.08\,\pi_1 - 0.15\,\pi_2 = 0 \end{cases} \quad (\text{These equations are the same$$

To solve these equations we will also have to use the fact that $\pi_1 + \pi_2 = 1$.

Lets promote `Brand A` cont.

Let's replace one of the two identical equations with $\pi_1 + \pi_2 = 1$. Then we have:

$$\begin{cases} 0.08\,\pi_1 - 0.15\,\pi_2 = 0 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

If we substitute $\pi_2 = 1 - \pi_1$ into the first equation, this gives:

$$0.08\,\pi_1 \;-\; 0.15\,(1 - \pi_1) \;=\; 0 \iff 0.23\,\pi_1 \;=\; 0.15$$

Therefore;

$$\pi_1 \;=\; \frac{0.15}{0.23} \approx 0.652, \qquad \pi_2 \;=\; 1 - \pi_1 \approx 0.348$$

So the steady-state expected market shares are roughly: 65.2% for `Brand A` and 34.8% for `Brand B`.

Now it is `your turn`

15:00

Any questions or thoughts?

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