ECE 484 Digital Control Applications

Lab 2: Design of the Inner-Loop Controller and Modeling of the Ball & Beam

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Introduction

This lab is the second in a series of three labs with the objective to control the position of a ball on a beam. The purpose of this lab is to design a dynamic controller, $C_1(s)$, to control the motor angle and discretize it to give $D_1[z]$. In addition, the motion of the ball and beam are modelled to determine the transfer functions $\frac{\emptyset(s)}{\theta(s)}$ and $\frac{Y(s)}{\emptyset(s)}$.

Lab 1 Values

Table Lab 1 vs. Lab 2 Values

Parameter	Lab 1	Lab 2
Stiction offset, α	-0.4810 V	-0.48 V
Stiction offset, β	$0.5010\ V$	$0.5\ V$
K_1	4.52	2.2
τ	$0.0398\ s$	$0.025\ s$

In Lab 1, I placed the stiction correction block in the wrong location before $C_1(s)$ instead of after. This resulted in incorrect values of K_1 and τ.

Part 1: Inner Loop Controller Design

a) Dynamic Controller Design - $C_1(s)$

The following process was used to design a controller to meet the specifications listed below for the model show in Figure 1.

- The (linearized) feedback loop is stable
- When the reference signal steps from $\theta_{ref} = -0.7\ to\ \theta_{ref} = 0.7\ rad$:
  - The steady-state step response tracking error is zero
  - The step response 2% settling time (T_s) is no more than 0.23 seconds
  - The step response overshoot is no more than 5%
  - The motor voltage does not saturate

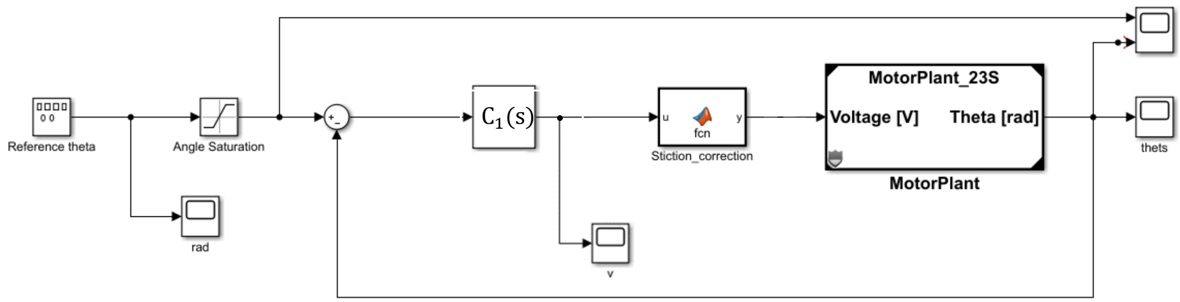

Figure 1 Simulink model showing controller $C_1(s)$ to be designed

The plant includes in integrator, $\frac{1}{s}$, therefore, a steady-state step response tracking error of zero can be achieved with a first order controller of the form:

$$C_1(s) = \frac{as + b}{s + c}$$

Iteration A: Pole Placement Method

The first iteration of the controller design was done using the pole placement method [1]. First, the good region of the s-plane (Figure 2) was determined using the transient specs:

$$T_s \leq 0.23 \quad \rightarrow \quad Re(s) \leq -\frac{4}{T_s^{max}} = -\frac{4}{0.23} = -17.4$$

$$\%OS \leq 5\% \quad \rightarrow \quad \theta_{\min} = \tan^{-1}\left\{-\frac{1}{\pi}\ln\left(\frac{5}{100}\right)\right\} = 43.64°$$

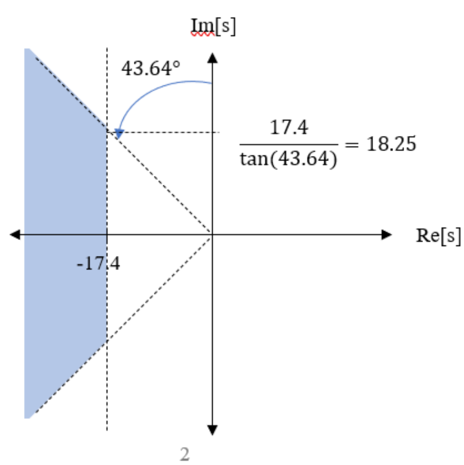

Figure 2 Good region of the s-plane based on %OS and settling time

Three desirable closed loop poles were selected – one real pole from the plant and two complex conjugate poles just inside the edge of the good region of the s-plane. The complex conjugate poles are put near rather than on the edge to provide room for error during discretization.

$$poles \ = \ \{-40, -18 \pm \mathrm{j}18\}$$

A zero is chosen for the controller to cancel out the good pole in the plant, giving the following $P(s)$ and $C(s)$:

$$P(s) = \frac{88}{s(s+40)} \qquad\qquad C(s) = \mathrm{K}\frac{(s+40)}{g_1 s + g_0}$$

Based on the chosen closed loop poles, the desired characteristic polynomial is:

$$\Delta_{des}(s) = (\mathrm{s}+40)(\mathrm{s}+18+\mathrm{j}18)(\mathrm{s}+18-\mathrm{j}18)$$
$$= (\mathrm{s}+40)(s^2+36s+648)$$

Based on $P(s)$ and $C(s)$, the actual characteristic polynomial is:

$$\Delta_{act}(s) = s(s+40)(g_1 s + g_0) + 88K(s+40)$$
$$= (s+40)(g_1 s^2 + g_0 s + 88K)$$

Comparing $\Delta_{des}(s)$ and $\Delta_{act}(s)$, the following values were determined by inspection:

$$g_1 = 1, \ \ g_0 = 36, \ \ \ K = 7.364$$

The resulting controller was:

$$C_{\mathrm{a}}(s) = 7.364\left(\frac{s+40}{s+36}\right)$$

This controller was then tested in the Simulink model shown in Figure 1 with a step input from -0.7 to 0.7 rad/s. As can be seen in Figures 3 and 4, the resulting system behaviour exceeded the overshoot specification of 5% and the maximum allowable motor voltage of 6V, suggesting the controller gain was too high.

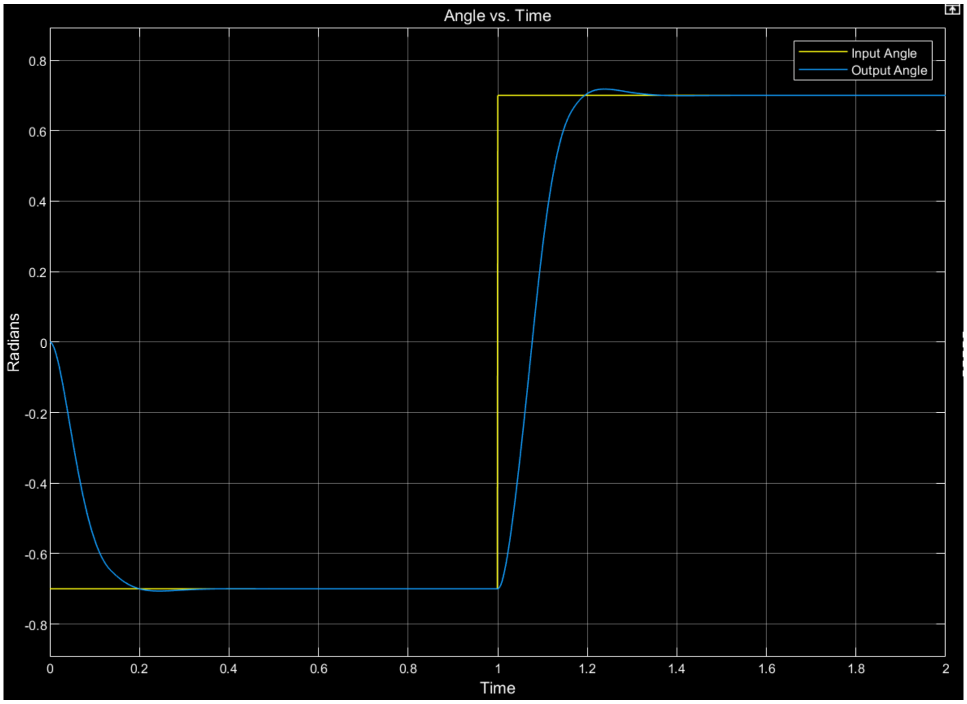

Figure 3 Simulink model step response for $C_a(s)$

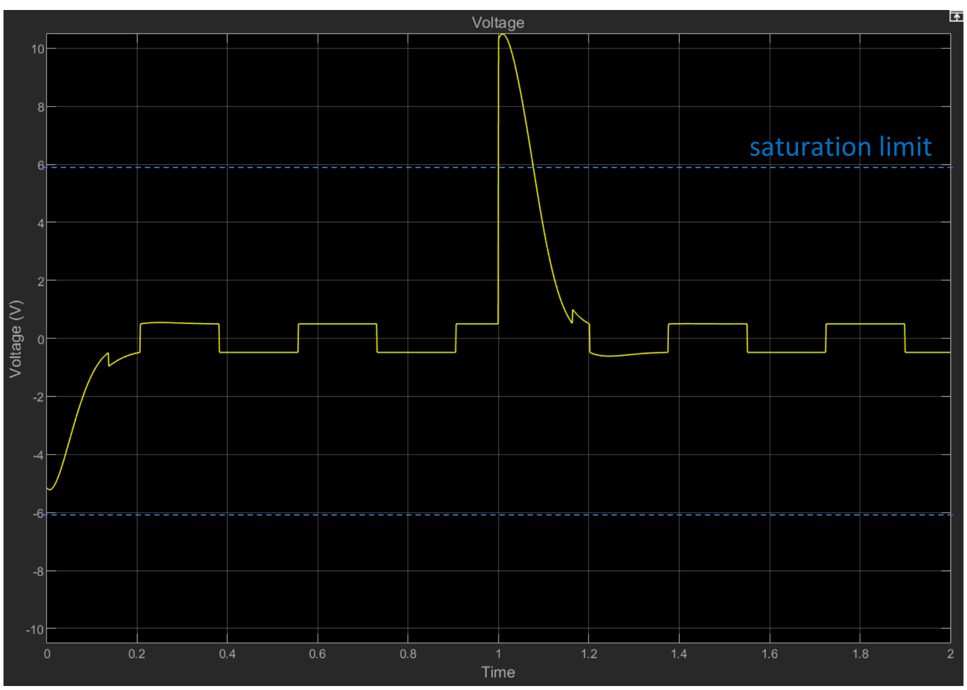

Figure 4 Simulink model motor voltage during step response for $C_a(s)$

Further iteration was needed to refine the controller design to meet specifications.

Iteration B: Minimizing Gain

Sisotool was then used to refine the controller design with the goal of minimizing the controller gain to avoid motor saturation. The system architecture is shown in Figure 5. Block F was set to a value of 1.4 to make the step input equivalent to an input of -0.7 to 0.7 rad. The first iteration of the controller, $C_a(s)$, was used as a starting point for block C.

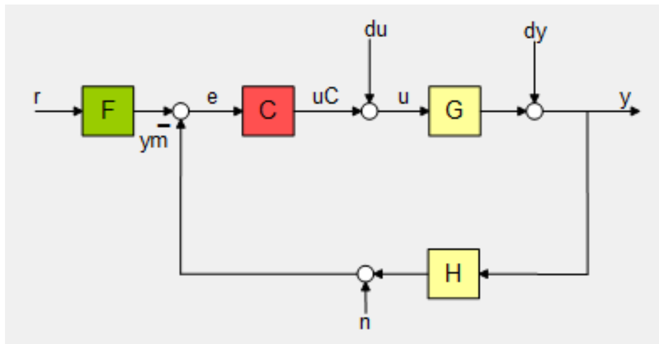

Figure 5 Sisotool system architecture

The transient design requirements were added to the step response plot to provide visual guidelines as the controller was adjusted. The imaginary parts of the complex closed loop poles were then moved in the Root Locus Editor until they reached the lowest value at which the setting time and peak amplitude specifications were met. Figure 6 shows the resulting step response for the following controller:

$$C_b(s) = 4.6156\left(\frac{s+40}{s+36}\right)$$

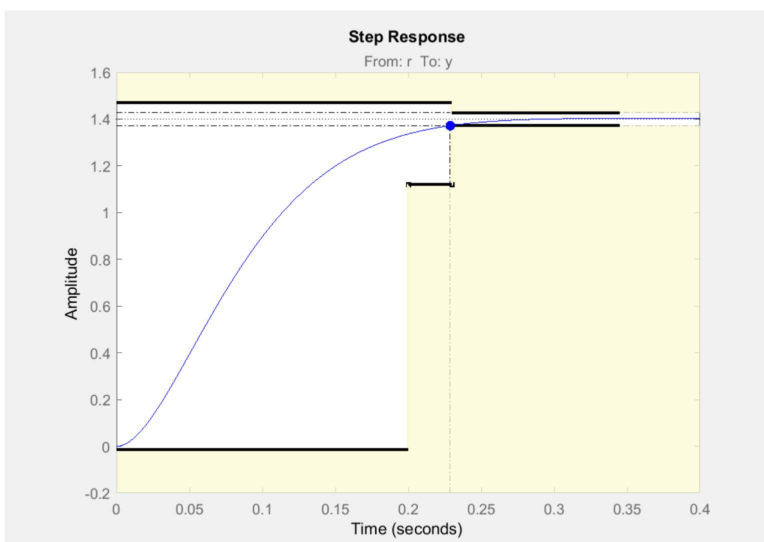

Figure 6 Step response for $C_b(s)$ with design requirement boundaries

$C_b(s)$ was then tested in the Simulink model. As can be seen in Figures 7 and 8, the step response was too slow, and the motor voltage still exceeded the 6V maximum.

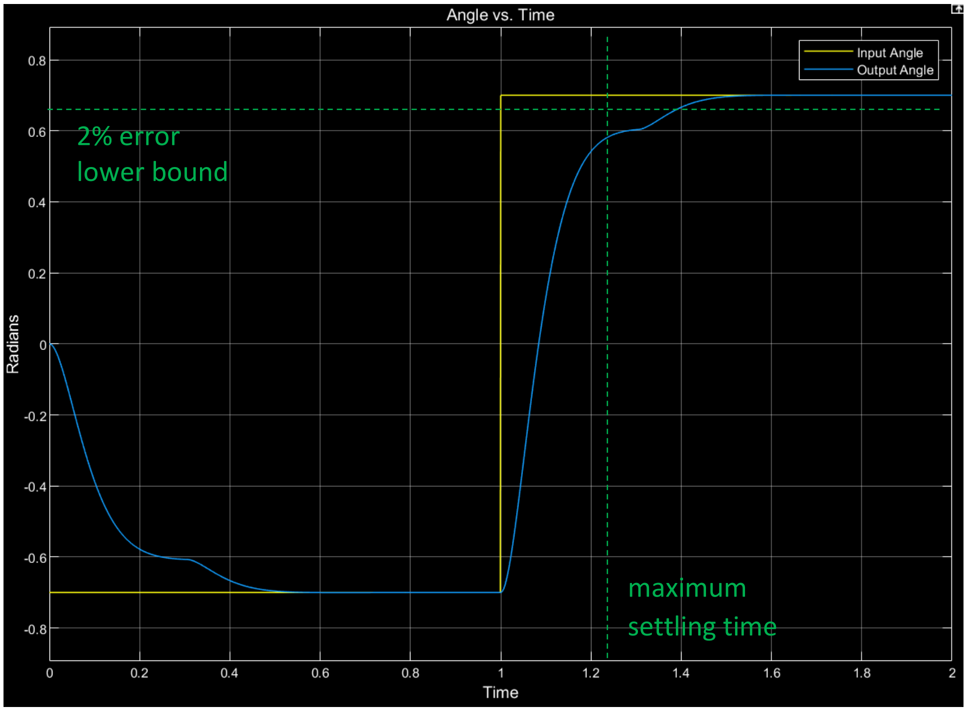

Figure 7 Simulink model step response for $C_b(s)$

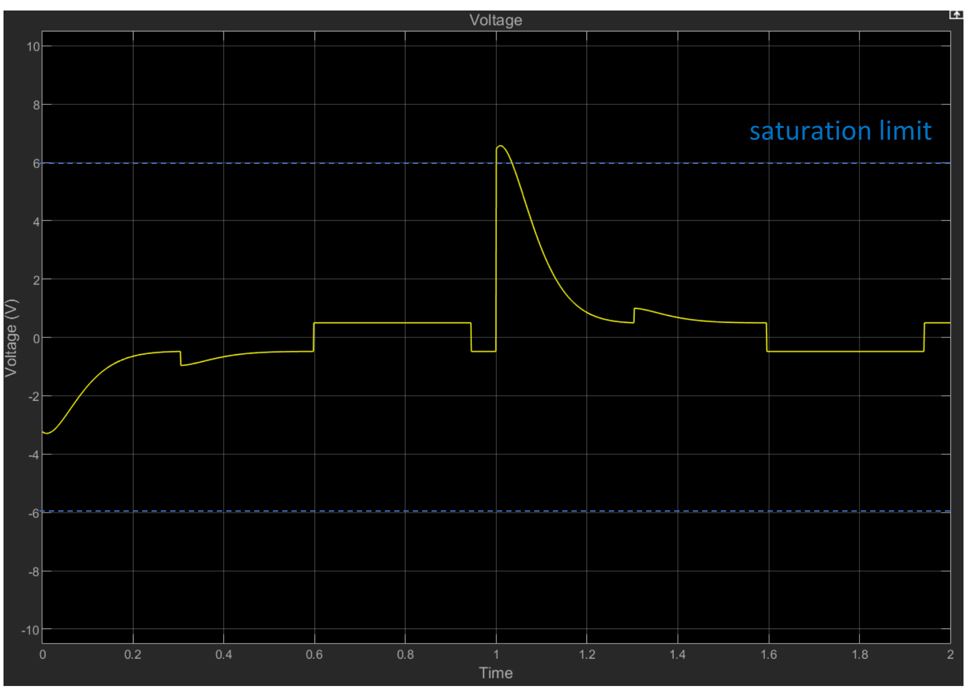

Figure 8 Simulink model motor voltage during step response for $C_b(s)$

Iteration C: Zero and Gain Modification

It was clear that reducing the value K was not enough to avoid motor saturation. In fact, the minimum possible value of K with real poles $\{-40, -18, -18\}$ still caused saturation. Controller gain needed to be reduced further while the response of the system needed to be sped up. This meant that moving the pole further into the LHP to reduce settling time was not a viable option.

Another way to speed up the response of the system is by changing the location of the zero. A zero in the left-hand plane decreases the time to peak while also increasing overshoot [1]. In the controller designs thus far, the zero in the controller cancelled with the pole in the plant. Sisotool was once again used to iterate on the zero and gain values with the goal of reducing gain and setting time. Closed loop zero and complex pole locations were adjusted in the Root Locus Editor and the effect was observed in the step response plot.

Figure 9 shows the resulting step response for the following controller:

$$C_c(s) = 1.647\left(\frac{s + 97.35}{s + 36}\right)$$

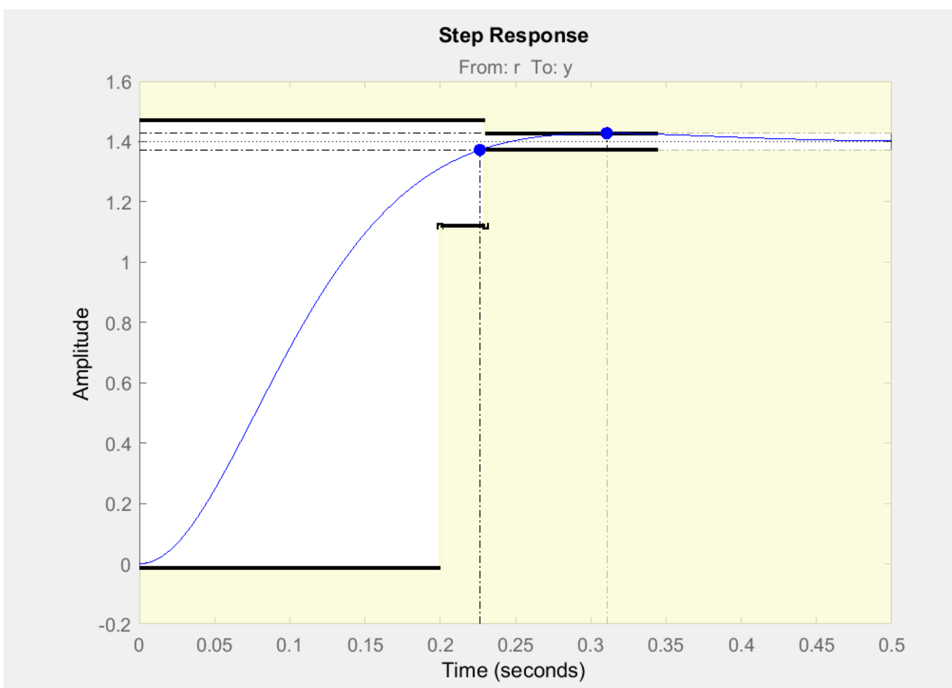

Figure 9 Step response for $C_c(s)$ with design requirement boundaries

$C_c(s)$ was then tested in the Simulink model. As can be seen in Figures 10 and 11, the step response was too slow, but the motor voltage was within an acceptable range.

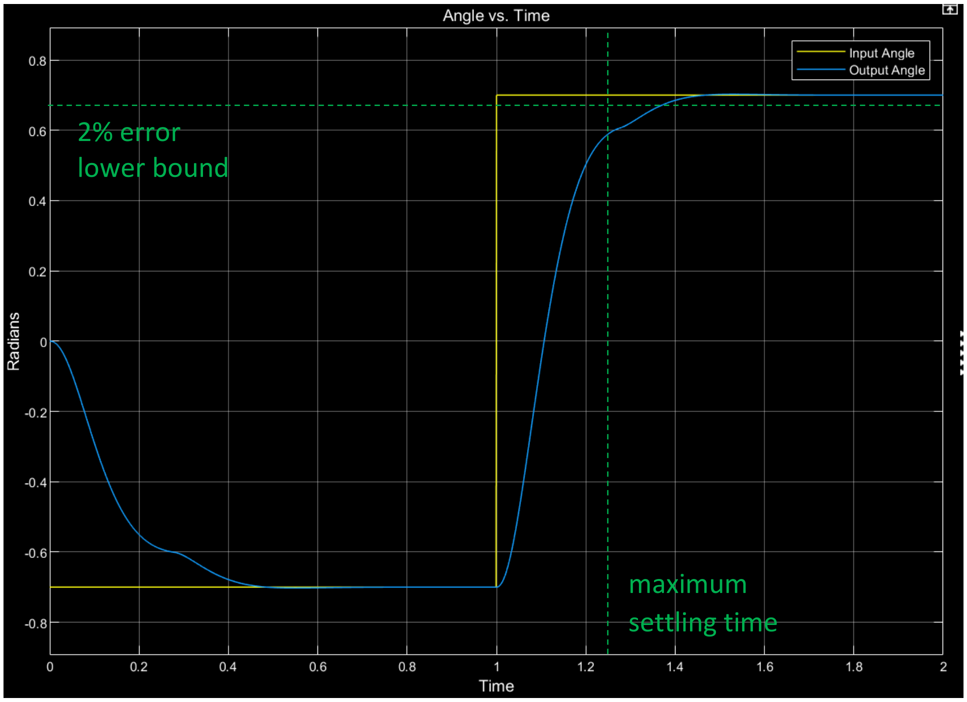

Figure 10 Simulink model step response for $C_c(s)$

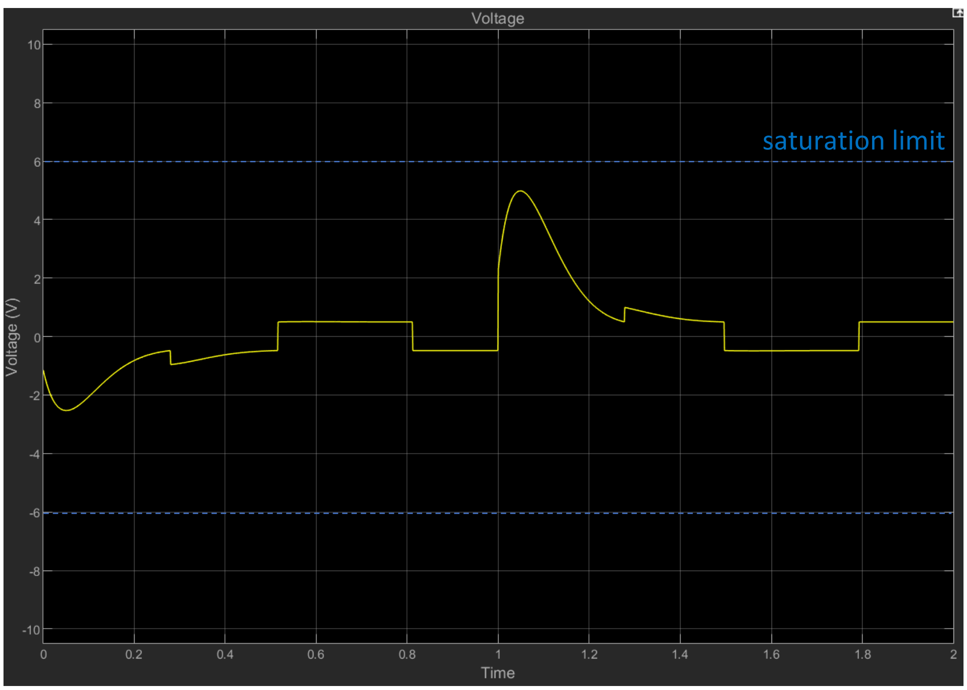

Figure 11 Simulink model motor voltage during step response for $C_c(s)$

Given the motor voltage did not saturate, there was room to increase the controller gain slightly to speed up the response. The zero location and value of K were tweaked to produce the final controller design:

$$C_1(s) = 2.2\left(\frac{s+90}{s+36}\right)$$

Figures 12 and 13 show the step response for the final controller $C_1(s)$. The design requirements are met as follows:

- ✓ Steady-state error = 0
- ✓ $T_s \approx 0.228\ s < 0.23\ s$
- ✓ $\%OS = \frac{peak\ value - final\ value}{final\ value} \times 100 = \frac{0.7139-0.7}{0.7} \times 100 \approx 2\% < 5\%$
- ✓ $|\text{Peak motor voltage}| = 5.987\ V < 6V$

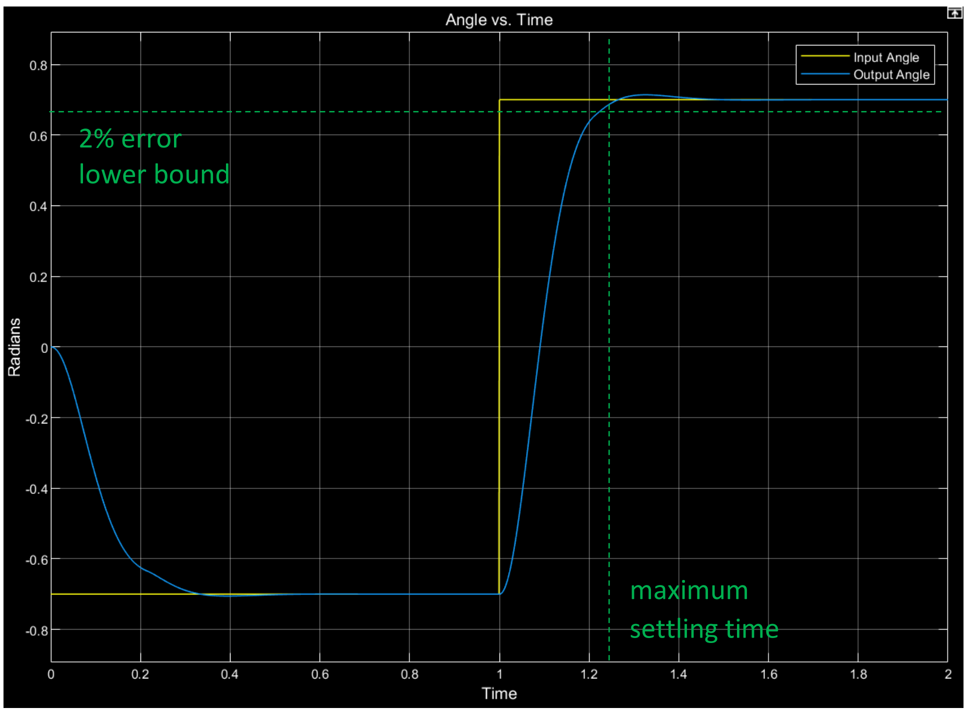

Figure 12 Simulink model step response for final controller, $C_1(s)$

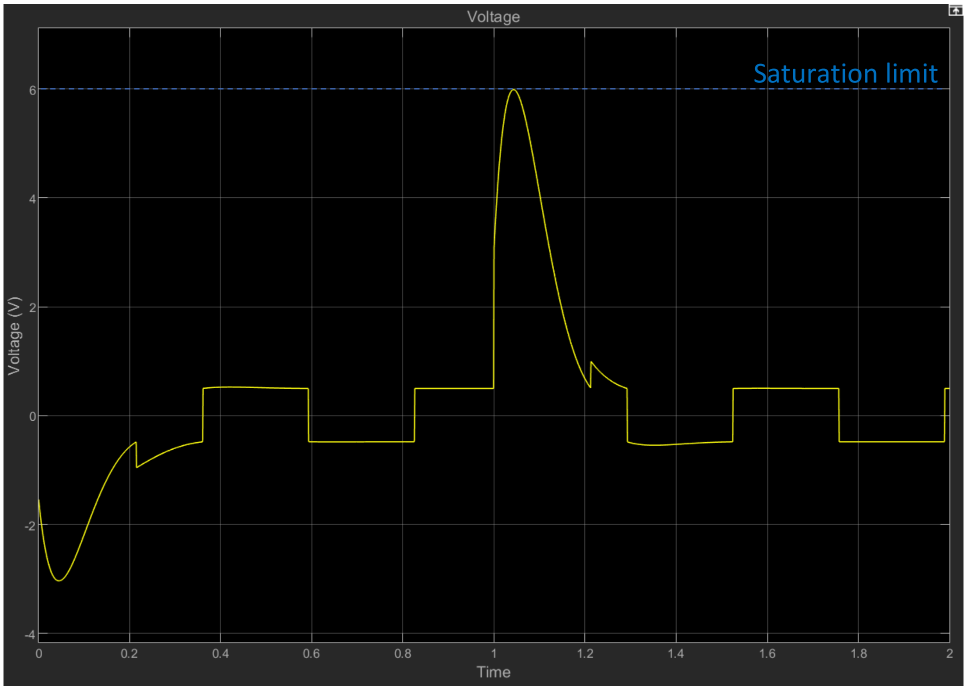

Figure 13 Simulink model motor voltage during step response for final controller, $C_1(s)$

b) Discrete-Time Controller - $D_1[z]$

The emulation controller design approach works best for small sampling periods. The sampling period was determined to meet the following rule of thumb [3]:

$$\omega_{sampling} \leq 10\omega_{bandwidth}$$

The system bandwidth was determined from the system Bode plot by locating the frequency at which the gain is equal to -3 dB. As shown in Figure 14, $\omega_{bandwidth} \approx 20\ rad/s$.

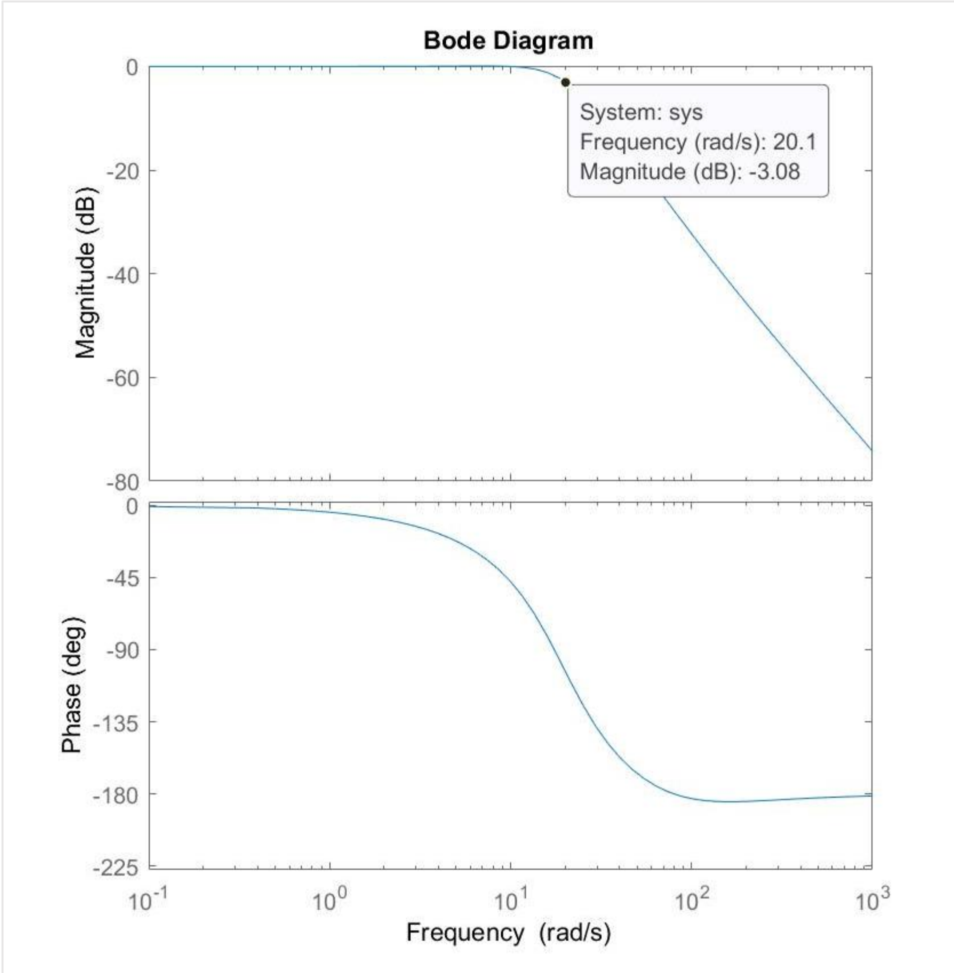

Figure 14 Bode plot identifying system bandwidth

This value was subbed into the equality to solve for the sampling period, T:

$$\frac{2\pi}{T} \leq 90(20\ rad/s) \quad \rightarrow \quad T \leq \frac{2\pi}{200} \approx 0.03\ s$$

The Trapezoid Rule is used to discretize the controller, $C_1(s)$:

$$\mathrm{D}_1[\mathrm{z}] = C_1(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}} = 2.2\left(\frac{\frac{2}{\mathrm{T}}\frac{\mathrm{z}-1}{\mathrm{z}+1}+90}{\frac{2}{\mathrm{T}}\frac{\mathrm{z}-1}{\mathrm{z}+1}+36}\right) = 2.2\left[\frac{2(\mathrm{z}-1)+90T(\mathrm{z}+1)}{2(\mathrm{z}-1)+36T(\mathrm{z}+1)}\right]$$

$$= 2.2\left(\frac{2z-2+90Tz+90T}{2z\ -1+36Tz+36T}\right)$$

$$= 2.2\frac{(90T-2)+(90T+2)z}{(36T-1)+(36T+2)z}$$

Subbing in the sampling period $T = 0.03$ s:

$$\mathrm{D}_1[\mathrm{z}] = 2.2\frac{(90 \cdot 0.03 - 2) + (90 \cdot 0.03 + 2)z}{(36 \cdot 0.03 - 1) + (36 \cdot 0.03 + 2)z} = 2.2\frac{0.7 + 4.7z}{0.08 + 3.08z}$$

The discrete time controller, $\mathrm{D}_1[\mathrm{z}]$, was then added to the Simulink model in place of $C_1(s)$ using a discrete transfer function block and tested. Figure 15 shows the simulation results (red) overlaid on the that of $C_1(s)$ (blue). The discrete controller system behaviour had a slower response with increased settling time and decreased overshoot.

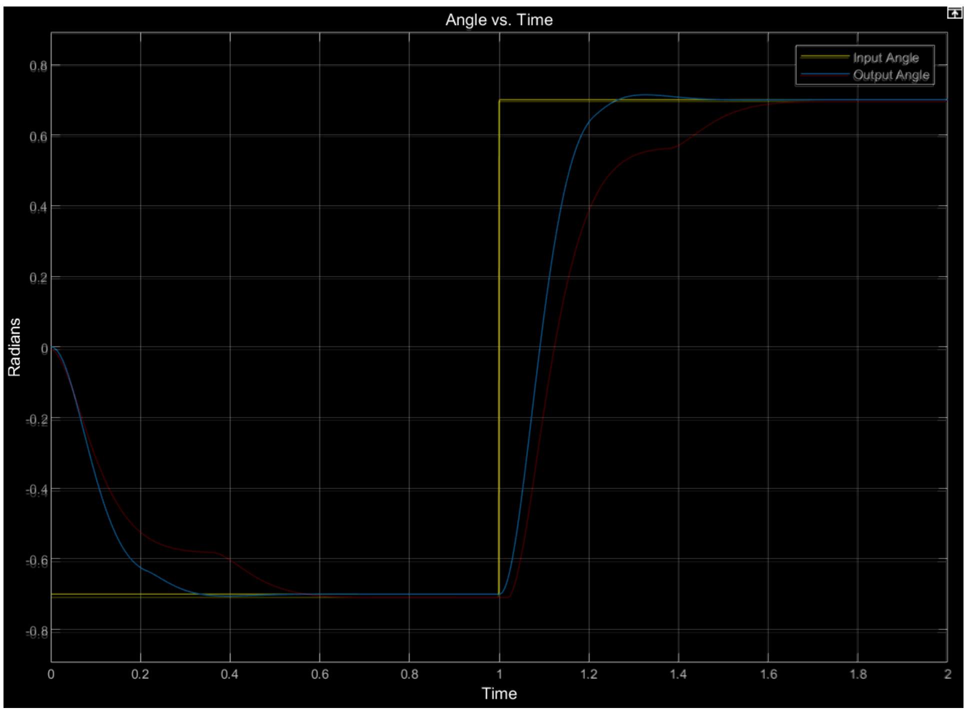

Figure 15 Comparison of$C_1(s)$ (blue) and $D_1[z]$ (red) system output

The input motor voltage can also be compared for the continuous and discrete time controllers, shown in Figure 16. The motor voltage with $\mathrm{D}_1[\mathrm{z}]$ reached a lower maximum value of 0.4889 V.

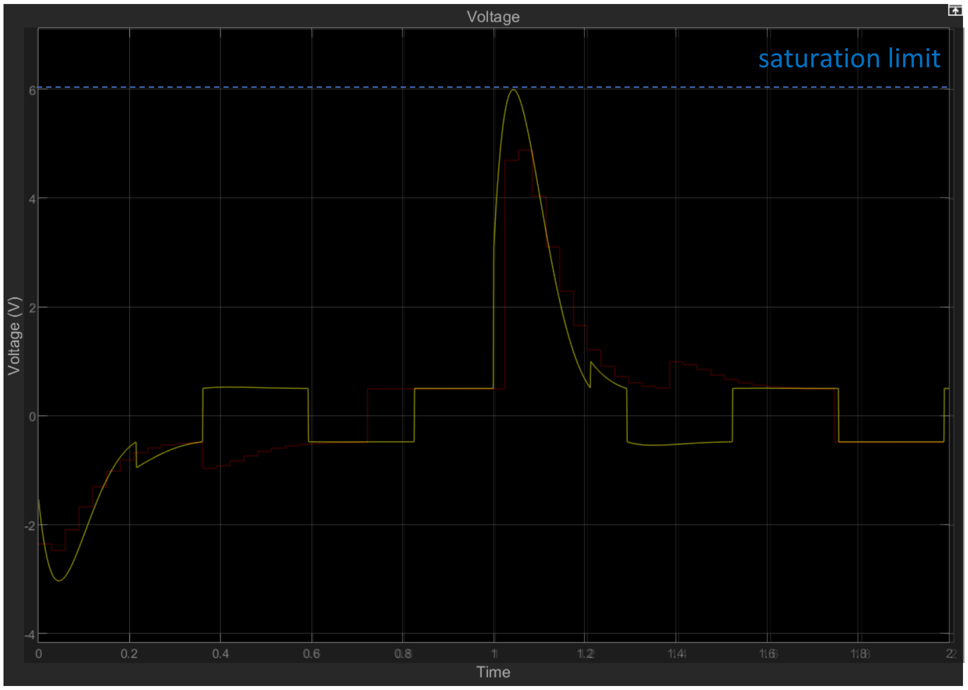

Figure 16 Comparison of motor voltage with $C(s)$ (yellow) and $D[z]$ (red)

Since the motor voltage did not reach saturation, there was room to increase the gain of $\mathrm{D_1[z]}$ to speed up the response of the system. A value of $\mathrm{K} = 2.75$ was found to improve the system performance without saturating the input motor voltage. The final discrete time controller is:

$$\mathrm{D_1[z]} = 2.75\frac{0.7 + 4.7z}{0.08 + 3.08z}$$

c) Time Domain Controller Equation

The inverse z-transform of $\mathrm{D_1[z]}$ is derived as follows:

$$\mathrm{D_1[z]} = \frac{\mathrm{V}[z]}{E[z]} = 2.75\frac{0.7 + 4.7z}{0.08 + 3.08z}$$

$$Z^{-1}\{\mathrm{V}[z](0.08 + 3.08z)\} = Z^{-1}\{E[z](1.925 + 12.925z)\}$$

$$3.08\mathrm{v[k-1]} + 0.08\mathrm{v[k]} = 12.925\mathrm{e[k-1]} + 1.925\mathrm{e[k]}$$

Part 2: Modeling of the Ball and Beam

d) Modeling and Linearization

Derivation of K_2:

The value of K_2 can be derived from the relationship between the beam angle $\emptyset$ and θ, which is purely geometric (Figure 17). It as assumed that the lever arm is perpendicular to the horizontal.

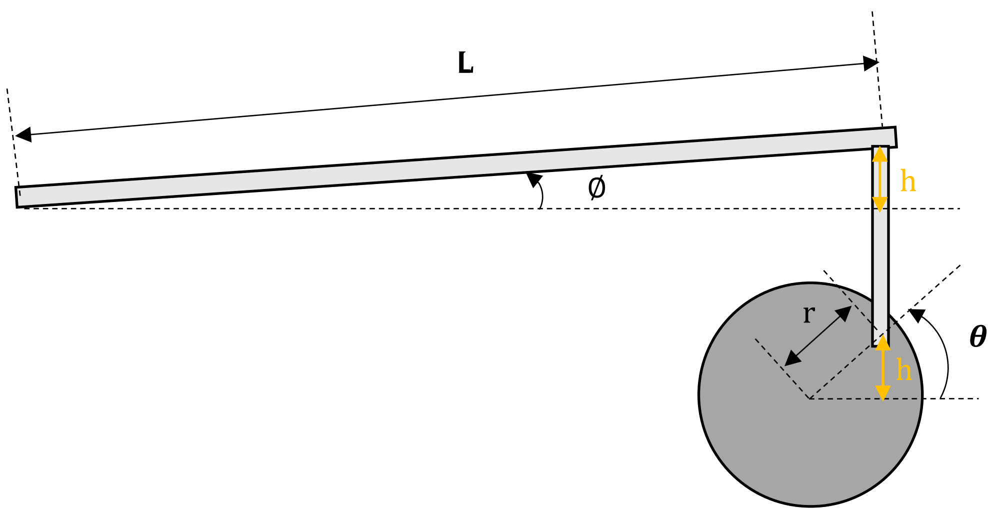

Figure 17 Beam and lever arm geometry

$$\mathrm{h} = \mathrm{L}\sin\emptyset = \mathrm{r}\sin\theta$$

The small angle approximation for the sine function can be used to simplify the relation. This approximation is acceptable to use since the maximum values of $\emptyset$ and θ are less than 0.7 radians and more frequently, the operating regions for the angles will be less than that.

$$\therefore\ \mathrm{L}\emptyset = \mathrm{r}\theta$$

Taking the Laplace transform and rearranging:

$$\mathrm{K}_2 = \frac{\emptyset(\mathrm{s})}{\theta(\mathrm{s})} = \frac{\mathrm{r}}{\mathrm{L}} = \frac{2.54\ \mathrm{cm}}{45\ \mathrm{cm}} = 0.056444$$

Derivation of K_3:

The value of K_3 is derived by formulating the equation of motion for the system based on the free body diagram shown in Figure 18.

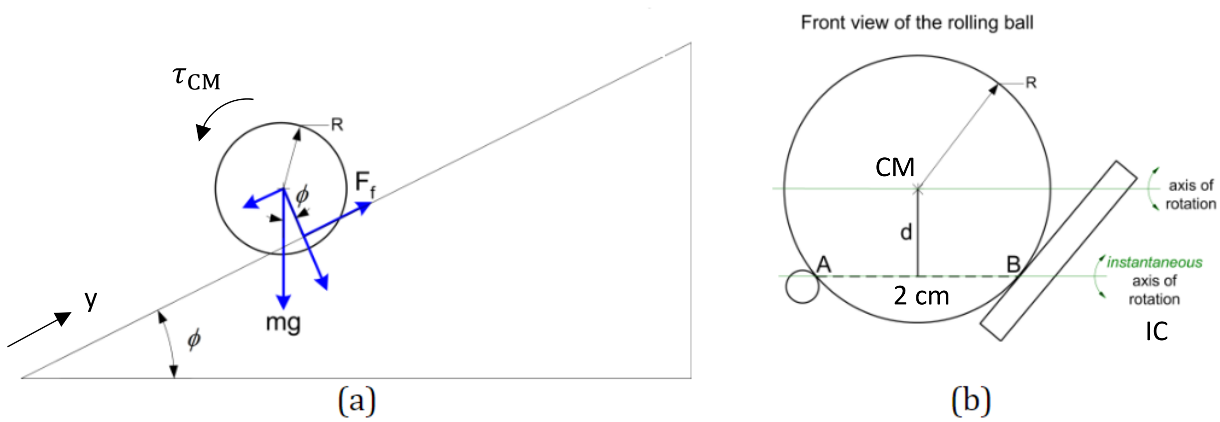

Figure 18 Diagram of ball on the beam: a) side view; b) cross-sectional front view

Determining length d in metres:

$$\mathrm{d} = \sqrt{\mathrm{R}^2 - 0.01^2} = \sqrt{\mathrm{R}^2 - 0.0001}$$

Using Newton's second law:

$$\sum F_y = F_f - mg\sin\emptyset = ma_y \tag{1}$$

Using torque about the centre of mass (CM):

$$\tau_{\mathrm{CM}} = F_f\mathrm{d} = F_f\sqrt{\mathrm{R}^2 - 0.0001} = \mathrm{I}_{\mathrm{cm}}\alpha$$

$$F_f = \frac{\mathrm{I}_{\mathrm{cm}}\alpha}{\sqrt{\mathrm{R}^2-0.0001}} = \frac{2\mathrm{mR}^2\alpha}{5\sqrt{\mathrm{R}^2-0.0001}} \tag{2}$$

Relating the ball's velocity at its CM and angular velocity based on the instantaneous centre of rotation (IC):

$$v_{CM} = r_{CM/IC}\omega = d\omega = \sqrt{\mathrm{R}^2 - 0.0001}\omega$$

Differentiating:

$$a_{CM} = a_y = \sqrt{\mathrm{R}^2 - 0.0001}\alpha$$

$$\alpha = \frac{a_y}{\sqrt{\mathrm{R}^2-0.0001}} \tag{3}$$

Subbing (3) into (2):

$$F_f = \frac{2\mathrm{mR}^2 a_y}{5\sqrt{\mathrm{R}^2-0.0001}\sqrt{\mathrm{R}^2-0.0001}} = \frac{2\mathrm{mR}^2 a_y}{5(\mathrm{R}^2-0.0001)} \tag{4}$$

Subbing (4) into (1):

$$\frac{2mR^2 a_y}{5(\mathrm{R}^2 - 0.0001)} - mg\sin\emptyset = ma_y$$

$$\frac{2R^2 a_y}{5(\mathrm{R}^2 - 0.0001)} - a_y = g \sin\emptyset$$

$$a_y \left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right) = g \sin\emptyset$$

$$a_y = \frac{g \sin\emptyset}{\left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right)}$$

For the likely operating range, the small angle approximation for the sine function can be used:

$$a_y = \frac{g}{\left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right)} \emptyset$$

Let $a_y = \ddot{y}$:

$$\ddot{y} = \frac{g}{\left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right)} \emptyset$$

Taking the Laplace transform:

$$L\{\ddot{y}\} = L \left\{ \frac{g}{\left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right)} \emptyset \right\}$$

$$Y(s)s^2 = \frac{g}{\left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right)} \emptyset(s)$$

$$\frac{Y(s)}{\emptyset(s)} = \frac{g}{\left(\frac{2R^2}{5(\mathrm{R}^2 - 0.0001)} - 1 \right) s^2} = \frac{9.81}{\left(\frac{2(0.0127)^2}{5(0.0127^2 - 0.0001)} - 1 \right) s^2} = \frac{\mathrm{K}_3}{s^2} = \frac{186.378}{s^2}$$

$$\therefore \mathrm{K}_3 = 186.378$$

e) Implementation in Simulink

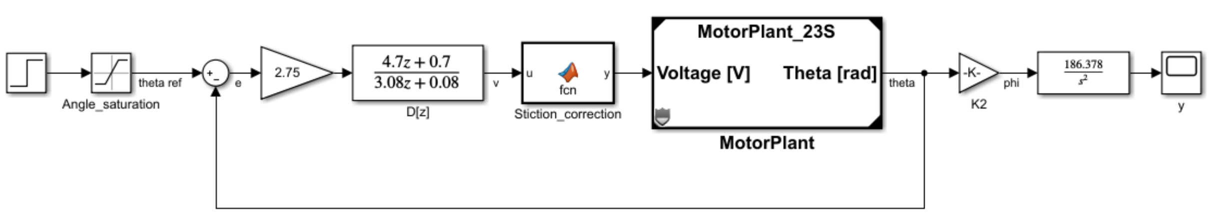

Figure 18 Ball and beam model implemented in Simulink

Conclusion

The purpose of this lab was to design the motor angle controller and model the ball and beam system to determine values for K_2 and K_3.

Multiple iterations of controller design resulted in the following continuous time motor angle controller:

$$C_1(s) = 2.2\left(\frac{s+90}{s+36}\right)$$

The trapezoid rule was used to discretize $C_1(s)$ with a sampling time of $T = 0.03$ s. Gain was then adjusted to improve performance, resulting in the following discrete time controller:

$$D_1[z] = 2.75\frac{0.7+4.7z}{0.08+3.08z}$$

The value of K_2 was determined based on the geometric relationship between θ and $\emptyset$:

$$K_2 = 0.056444$$

Finally, the value of K_3 was determined based on the equation of motion for the ball and beam:

$$K_3 = 186.378$$

References

[1] D. Miller, "Chapter 3 The Pole Placement Design Problem," in *ECE 484 Course Notes*, Waterloo, University of Waterloo, 2020.

[2] University of Illinois, Grainger College of Engineering, "ECE 486 Effects of Zeros and Poles on Step Resposne," Fall 2008. [Online]. Available: https://courses.engr.illinois.edu/ece486/fa2017/documents/lecture_notes/effects_zero_pole.pdf. [Accessed 1 November 2020].

[3] C. Caradima, "Question @51 Part B Lab 2," ECE 484 Piazza, [Online]. Available: https://piazza.com/class/kelhvtzdj5m77k?cid=51. [Accessed 1 November 2020].

[4] ECE 484 Digial Control Systems: Ball and Beam Lab Manuel, Waterloo: University of Waterloo, 2020.

Statement of Originality

We acknowledge and promise that:

(a) We are the sole authors of this lab report and associated simulation files/code.

(b) This work represents our original work.

(c) We have not shared detailed analysis or detailed design results, computer code, or Simulink diagrams with any other student.

(d) We have not obtained or looked at lab reports from any other current or former student of ECE 484/481, and we have not let any other student access any part of our lab work.

(e) We have completely and unambiguously acknowledged and referenced all persons and aids used to help us with our work.

Signed L Chambers and ______________________________