ECE 484 Digital Control Applications
Lab 1: Modeling of the Motor System

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Introduction

This lab is the first in a series of three labs with the objective to control the position of a ball on a beam. The purpose of this lab is to model the motor system. The motor plant receives an input voltage, V, and outputs a gear angle, θ as shown in Figure 1, taken from the lab manual [1].

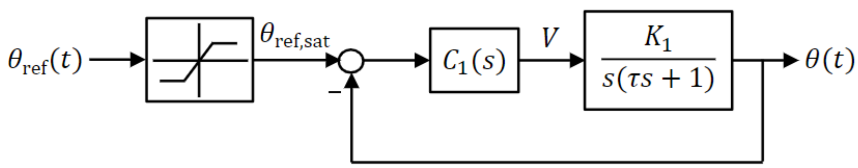

Figure 1 Inner control loop of the ball and beam system with a saturator

In this lab, motor stiction is identified and corrected, motor plant parameters K_1 and τ are determined and validated, and an input angle saturator is implemented and tested.

b) Motor Stiction

To compensate for stiction in the motor, the values of α and β needed to be determined. As shown in Figure 1-1, α is the voltage required to overcome stiction in the negative voltage direction, and β is the voltage required to overcome stiction in the positive direction [1].

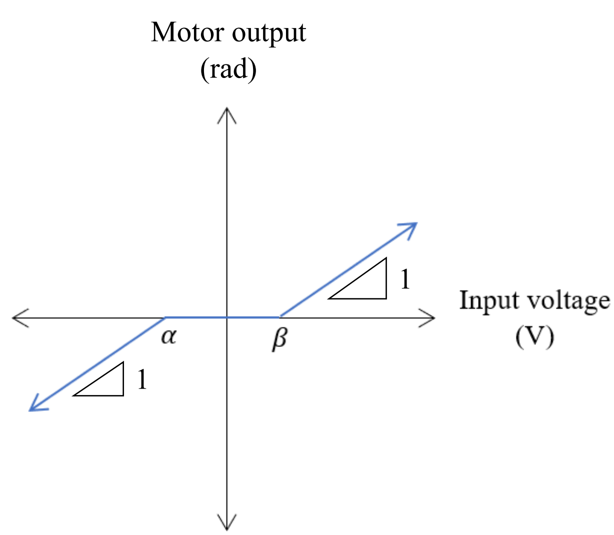

Figure 2 Motor stiction

The following method was used to identify α and β using the Simulink model shown in Figure 2. The values must be determined individually because they are different for each motor direction.

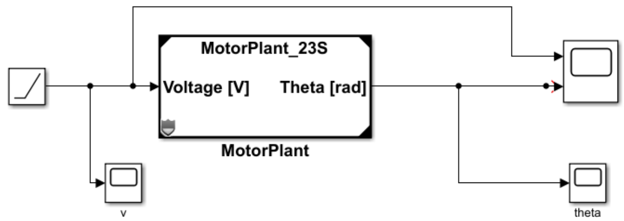

Figure 3 Simulink model for determining stiction

1. To determine α, a ramp input voltage with an initial value of 0 and slope of -1 was applied to the motor plant. The output was analyzed to find the input voltage that resulted in the first non-zero output, which was $\alpha = -0.4810$ V as shown in Figure 4a.

2. To determine β, a ramp input voltage with an initial value of 0 and slope of +1 was applied to the motor plant. The output was analyzed to find the input voltage, β, that resulted in the first non-zero output, which was $\beta = 0.5010$ V as shown in Figure 4b.

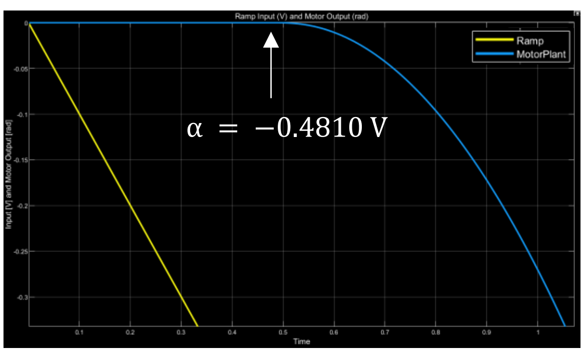

Figure 4a Simulink scope plot identifying α

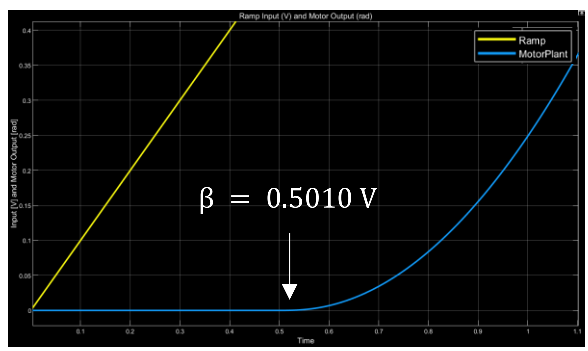

Figure 4b Simulink scope plot identifying β

To compensate for stiction the following Stiction_correction MATLAB function was added to the Simulink model, as shown in Figure 5. The function code is included below.

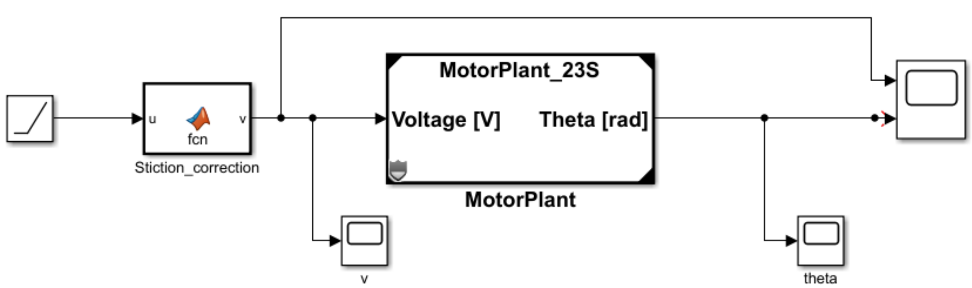

Figure 5 Simulink model with Stiction_correction MATLAB function

```
function v = fcn(u)

      a = -0.4810;
      b = 0.5010;

      if u > 0
          v = u + b;
      elseif u < 0
          v = u + a;
      else
          v = u;

end
```

The Simulink model with the Stiction_correction block was tested by applying a ramp input with an initial value of -2V and a slope of 1. The result in Figure 6 shows that the relationship between the input voltage (yellow) and motor output position (blue) is smooth and linear.

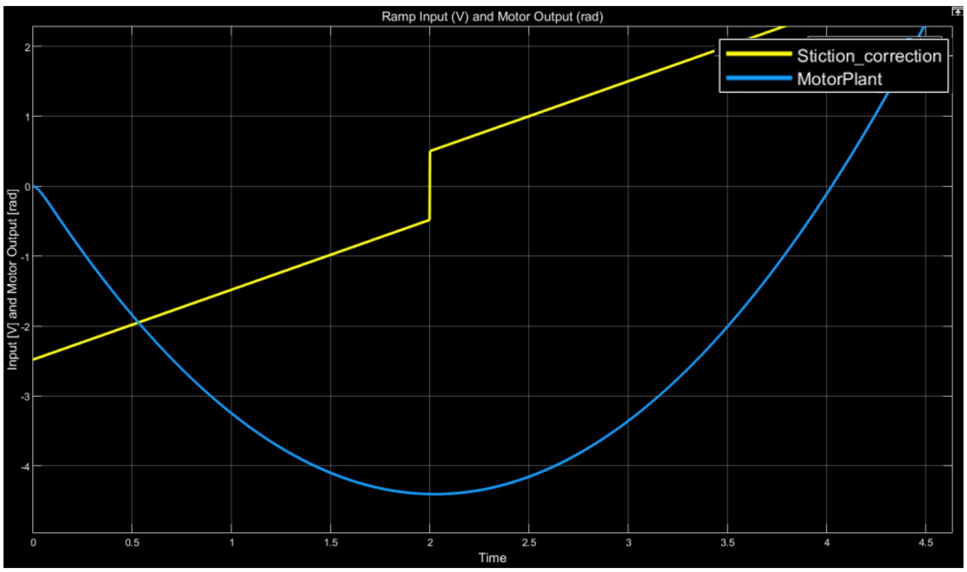

Figure 6 Test results for the Stiction_correction function

c) Motor Modeling

To determine the values of K_1 and τ for the plant, a $C_1(s)$ must first be chosen to stabilize the plant. A simple 0th order controller was selected, with a value that meets the input voltage limit of [-6.0 V, + 6.0V] and operating limit of -45 degrees < θ_{ref} < 45 degrees.

The resulting transfer function is below, based on Figure 5(a) in the lab manual [2]. The values of K_1 and τ are assumed to be positive; therefore, C must be positive to ensure that the terms in the denominator are all positive indicating closed-loop stability.

$$H(s) = \frac{\mathrm{P(s)C(s)}}{1 + \mathrm{P(s)C(s)}} = \frac{K_1 C}{\tau s^2 + s + K_1 C}$$

Via trial and error, a value of $C = 12/\pi$ was found to be the maximum gain that did not exceed the limit of [-6.0 V, + 6.0V] for a maximum θ_{ref} of 45 degrees, as shown in Figure 7. All other tested values of θ_{ref} in Table 1 has an input voltage to the motor plant within the limit.

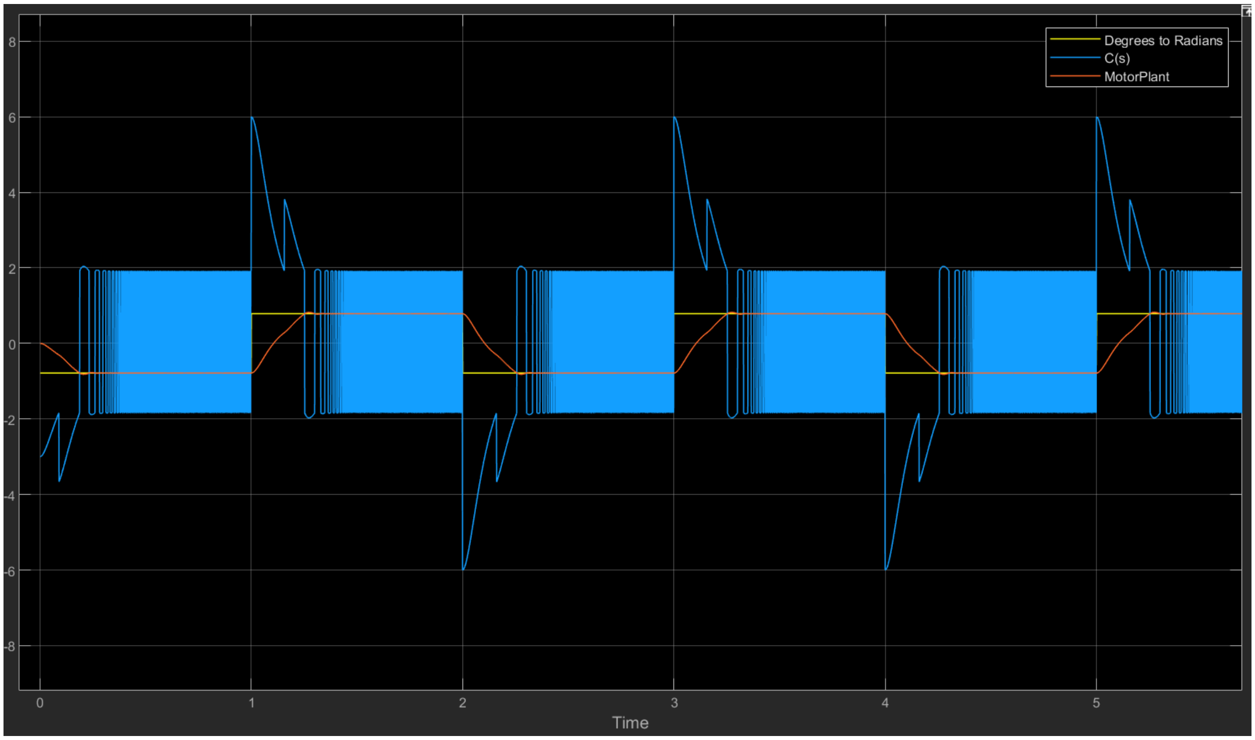

Figure 7 Maximum input voltage ≤ 6 V for maximum gain = 12/π V and $\theta_{\mathrm{ref}} = 45$ *degrees*

The system identification technique used to determine values of K_1 and τ was step response overshoot and time to first peak. Figure 8 shows the Simulink model setup used during the system identification process. A square wave generator was used as in the input source.

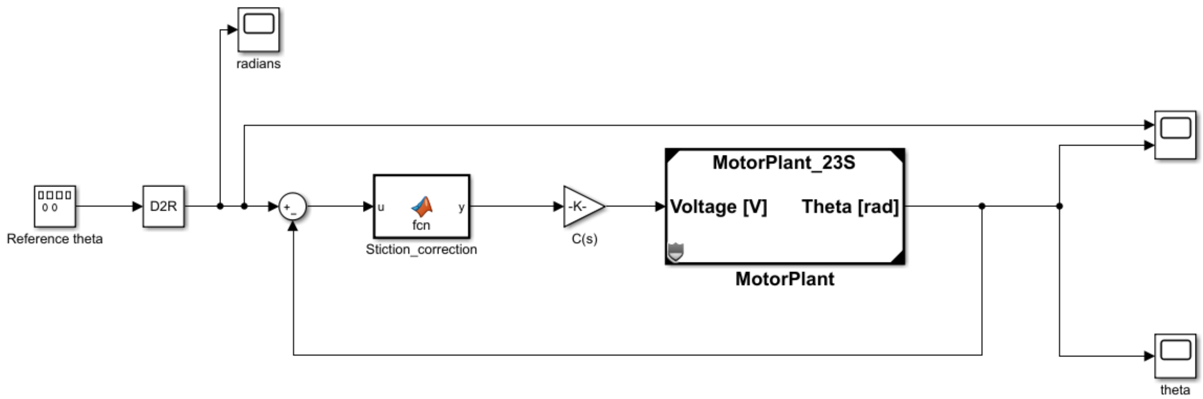

Figure 8 Simulink model for system identification

Table 1 summarizes the test data used to calculate K_1 and τ. A sample calculation is shown below for the 20 degrees input data with a square wave frequency of 0.5 Hz. Relevant points are labelled on a sample scope image in Figure 9.

Table 1 System identification data

Input (deg)	**Final value (rad)**	C	**Max value (rad)**	**Time at max value**	**% OS**	ζ	T_p	ω_n	τ	K_1
10	0.1745	12/pi	0.2073	3.112	18.797	0.4697	0.112	31.773	0.0335	8.8546
15	0.2618	12/pi	0.297	3.146	13.445	0.6202	0.146	27.430	0.0294	5.7895
20	0.3491	12/pi	0.3847	3.189	10.198	0.5879	0.189	20.548	0.0414	4.5753
25	0.4363	12/pi	0.4727	3.215	8.343	0.6202	0.215	18.627	0.0433	3.9315
30	0.5236	12/pi	0.5585	3.233	6.665	0.6529	0.233	17.802	0.0430	3.5689
35	0.6109	12/pi	0.6468	3.249	5.877	0.6698	0.249	16.992	0.0439	3.3206
40	0.6981	12/pi	0.7243	3.262	3.753	0.7225	0.262	17.342	0.0399	3.1422
45	0.7854	12/pi	0.8208	3.273	4.507	0.7023	0.273	16.166	0.0440	3.0130
								AVG	**0.0398**	**4.5245**

Sample Calculation

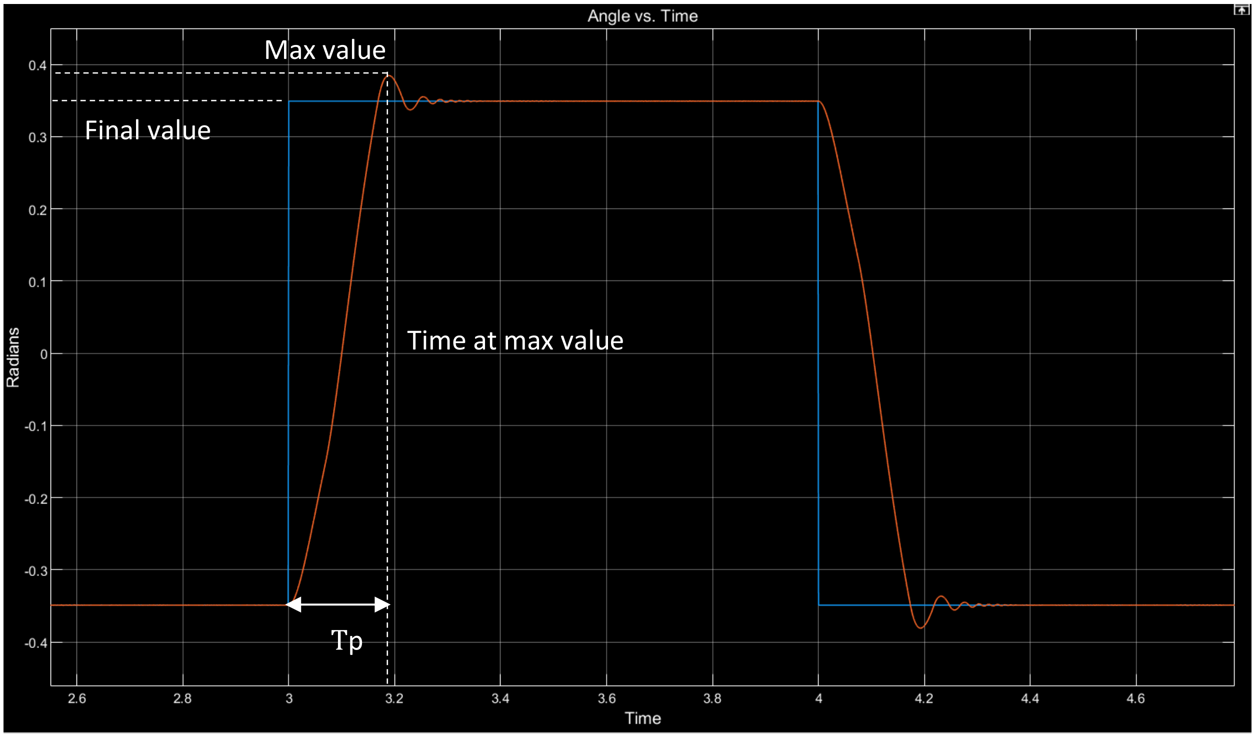

Figure 9 Sample scope data with input (blue) and output (orange) angle

Find the step response overshoot (%OS) to calculate ζ:

$$\%\mathrm{OS} = 100 \times \frac{\text{max value} - \text{final value}}{\text{final value}} = 100\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$100 \times \frac{0.3847 - 0.3491}{0.3491} = 100\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$10.1976511 = 100exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 - \ln^2\left(\frac{\%OS}{100}\right)}} = \frac{-\ln\left(\frac{10.1976511}{100}\right)}{\sqrt{\pi^2 - \ln^2\left(\frac{10.1976511}{100}\right)}} = \mathbf{0.5879}$$

Find time to first peak (T_p) to solve for ω_n:

$$T_p = \text{Time at max value} - 3 = 3.189 - 3 = 0.189 = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{0.189\sqrt{1-(0.5879)^2}} = \mathbf{20.548}\ \boldsymbol{rad/s}$$

Simplify the closed-loop system:

$$\frac{\theta}{\theta_{ref}} = \frac{P(s)C(s)}{1+P(s)C(s)} \frac{\left(\frac{K_1}{s(\tau_v s+1)}\right)C}{1+\left(\frac{K_1}{s(\tau_v s+1)}\right)C}$$

$$\frac{\theta}{\theta_{ref}} = \frac{1}{1+\frac{CK_1}{\tau_v s^2+s}} \cdot \frac{K_1 C}{\tau_v s^2+s}$$

$$\frac{\theta}{\theta_{ref}} = \frac{K_1 C}{\tau_v s^2 + s + K_1 C}$$

Equate simplified close-loop system and the standard model to derive expressions for τ, K_1:

$$\frac{K_1 C}{\tau s^2 + s + K_1 C} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{\frac{K_1 C}{\tau}}{s^2 + \frac{s}{\tau} + \frac{K_1 C}{\tau}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\zeta = \frac{1}{2\tau\omega_n} \quad \rightarrow \quad \tau = \frac{1}{2\zeta\omega_n} = \frac{1}{2(0.5879)(20.548)} = \mathbf{0.0414\ s}$$

$$\omega_n = \sqrt{\frac{K_1 C}{\tau}} \quad \rightarrow \quad K_1 = \frac{{\omega_n}^2\tau}{C} = \frac{(20.548)^2(0.0414)}{12/\pi} = \mathbf{4.5753}$$

The data for the input values of 10, 15, 25, 30, 35, 40, and 45 degrees was averaged to determine the following system parameters:

$$\boldsymbol{K_1} = \mathbf{4.52}$$

$$\boldsymbol{\tau} = \mathbf{0.0398\ s}$$

d) Motor Model Validation

The second method used to verify the parameter values of τ and K_1 is Bode plot fitting. The Simulink model used in Figure 8 was once again used; however, the input was changed to a sine wave generator. A sinusoid input with an amplitude of 25 degrees was applied to the system at frequencies of 0.5, 1, 2.5, 10, 25, and 50 Hz. The response of the system was measured and plotted. Sample data points were identified on each plot to calculate the gain and phase shift of the output relative to in the input. A sample plot is shown in Figure 10 and sample calculations below. Table 2 summarizes the data and resulting gain and phase values for all frequencies.

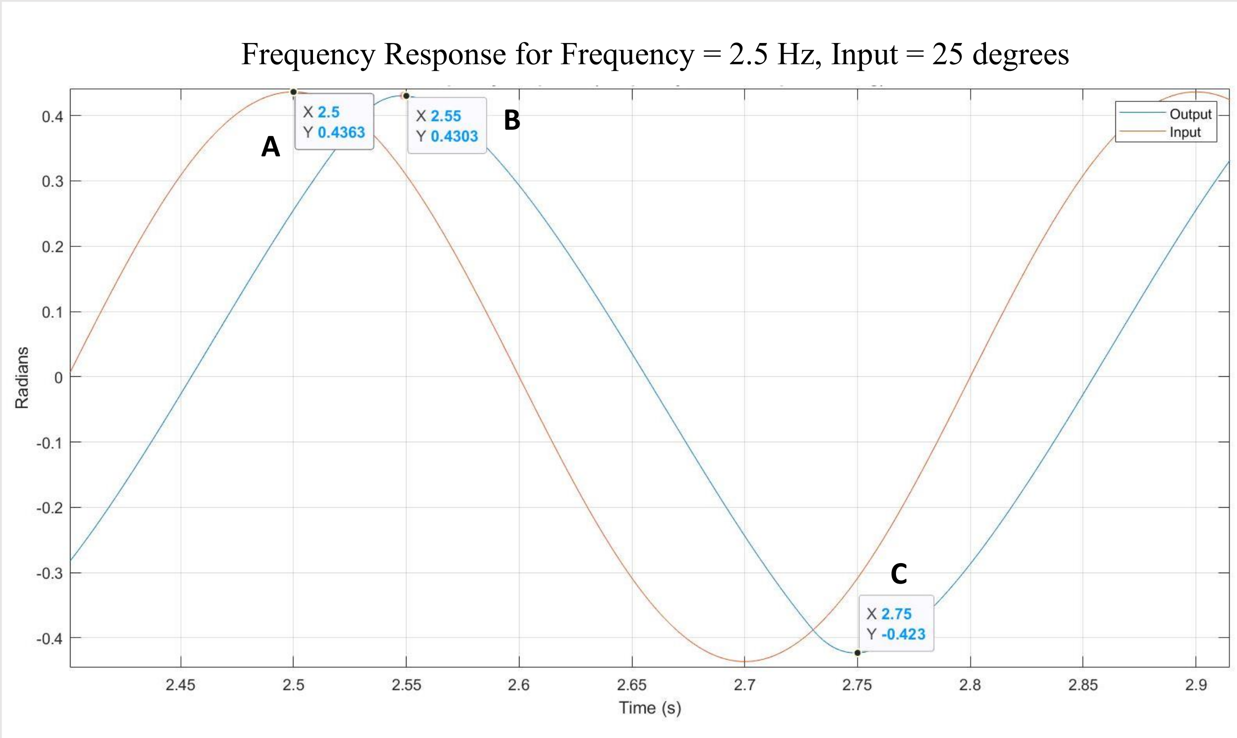

Figure 10 Example data points for bode plot calculations

$$\text{Gain } = \frac{\frac{\mathrm{B_y} - \mathrm{C_y}}{2}}{\mathrm{A_y}} = \frac{\frac{0.4363 + 0.423}{2}}{0.4363} = 0.977810475$$

$$\text{Time lag } = \mathrm{A_x} - \mathrm{B_x} = 2.50 - 2.55 = -0.05 \text{ s}$$

$$Phase\ shift\ = 360 \times Time\ lag\ \times frequency = 360(-0.05)(2.5) = -45\ degrees$$

Table 2 Bode Plot Data

Freq (Hz)	Freq (rad/s)	Input (deg)	Output (rad)	Time lag	Gain	Phase (deg)
0.5	3.1415927	0.436332	0.436332	0	1	0
1	6.2831853	0.436332	0.436332	-0.003	1	-1.08
2.5	15.707963	0.436332	0.42665	-0.05	0.977810475	-45
5	31.415927	0.436332	0.175	-0.054	0.401070744	-97.2
10	62.831853	0.436332	0.0579	-0.037	0.132697121	-133.2
25	157.07963	0.436332	0.01178	-0.018	0.026997791	-162
50	314.15927	0.436332	0.00311	-0.0095	0.0071276	-171

The frequency response of the system summarized in Table 2 is illustrated in the Bode plot in Figure 11. The experimental gain and phase for the sample frequencies is overlayed on the theoretical Bode plot for the values of τ and K_1 determined in part b).

The Bode plot was produced via the MATLAB script bode_plot.m.

$$H(s) = \frac{K_1 C}{\tau s^2 + s + K_1 C} = \frac{(4.52)(12/\pi)}{0.0398 s^2 + s + (4.52)(12/\pi)}$$

$$= \frac{17.265}{0.0398 s^2 + s + 17.265}$$

$$\approx \frac{434}{s^2 + 25.1 s + 434}$$

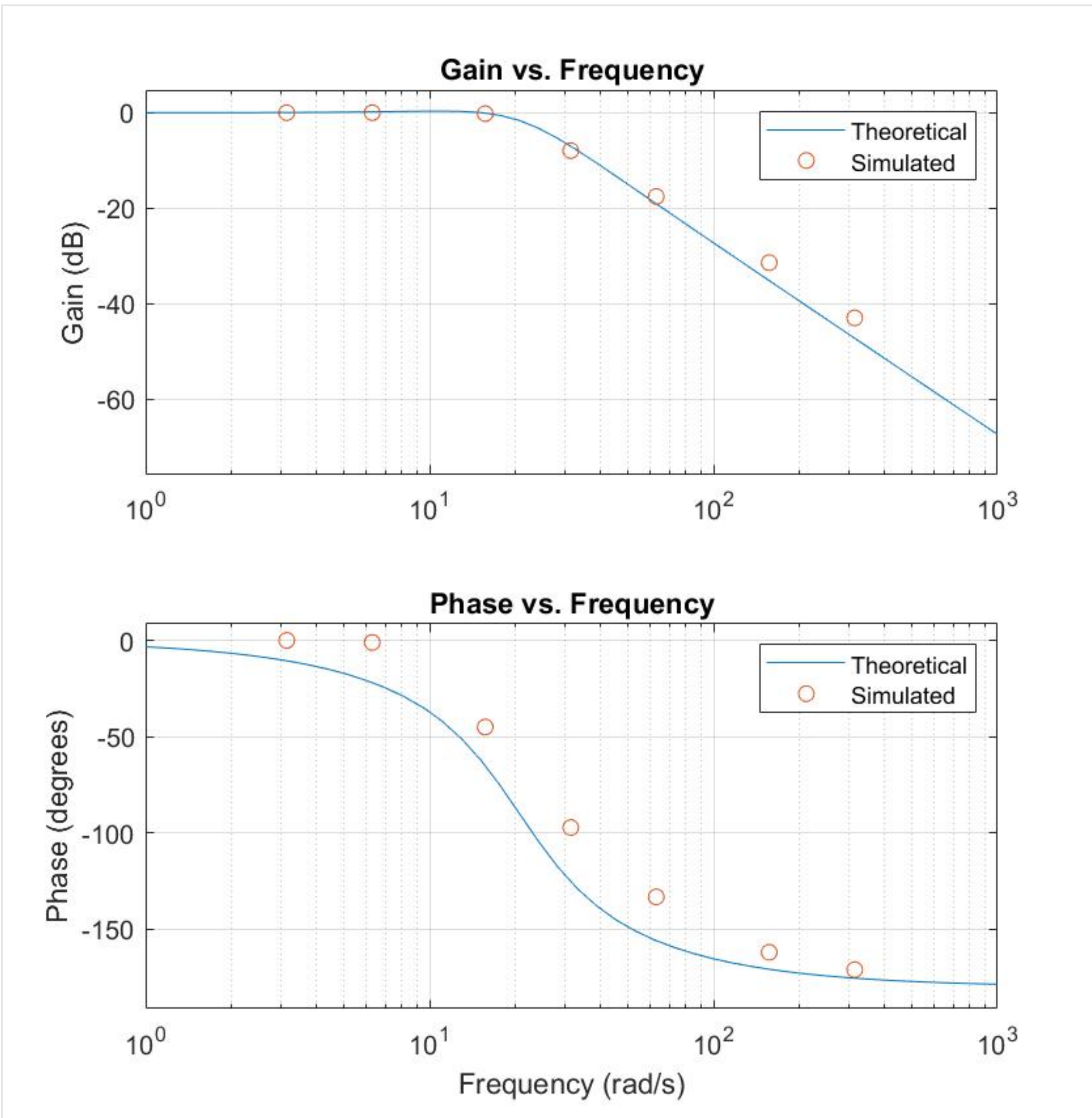

Figure 11 Comparison of theoretical and simulated Bode plots

Based on visual inspection, the theoretical and experimental results are comparable. The shape of the experimental gain data points follows the shape of the theoretical curve. In dB scale, both plots begin with a gain of zero and have a corner frequency at approximately 20 rad/s. After the corner frequency, the slope of the curve is -80 dB/decade, which is characteristic of a second order transfer function in the standard form with two poles.

The theoretical and experimental phase results are also similar. The experimental data points are shifted slightly to the right compared to the theoretical curve. Both begin at a phase of 0 degrees and end up at a phase of -180 degrees, which is also characteristic of a second order transfer function with no zeros.

The phase plot is analyzed in further detail to determine values of τ and K_1 in Figure 12. First, lines are drawn approximating the trend lines for the flat and decreasing segments of the line plot. The frequency at which they intersect is $\omega_\mathrm{n} \approx 21\ \mathrm{rad/s}$, which is the corner frequency.

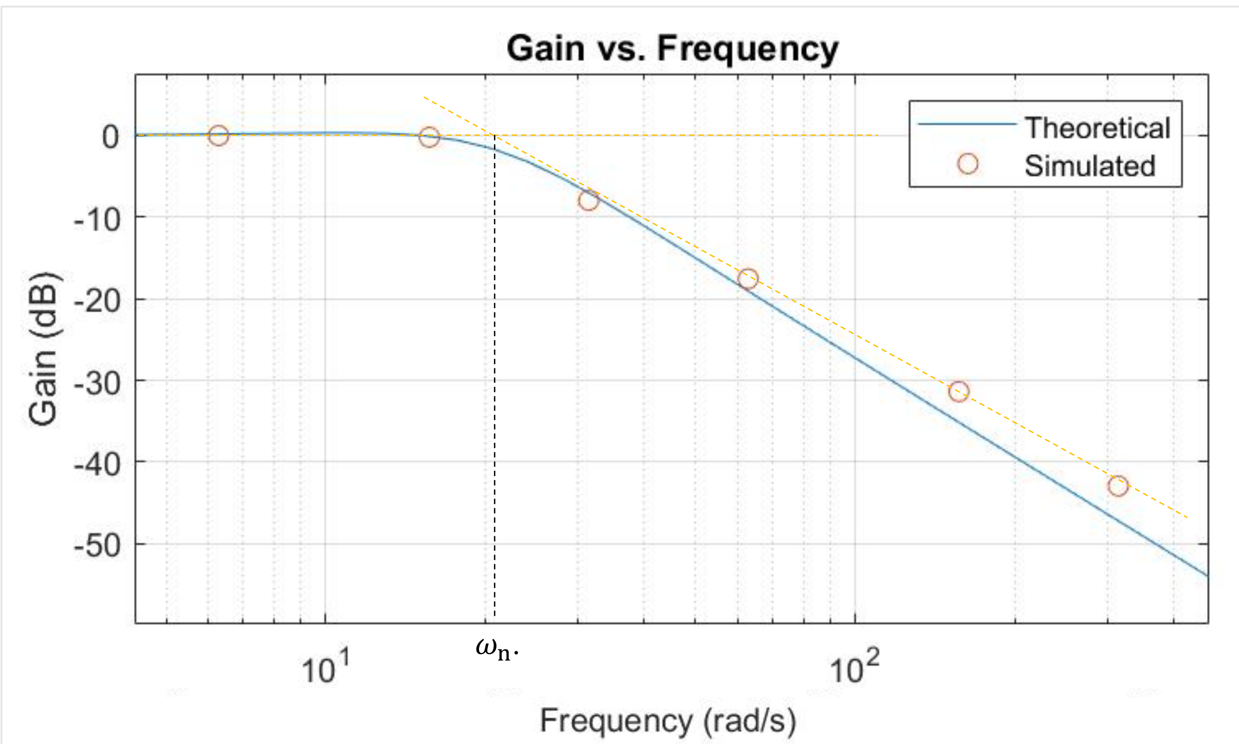

Figure 12 Gain plot analysis

The shape of the plot at the corner frequency is dependent on the value of the damping factor, ζ, as can be seen in Figure 13 [3]. Comparing Figure 12 and Figure 12, the damping factor is estimated to be in between 0.5 and 0.707. A value of $\zeta = 0.6$ is selected for use in calculations.

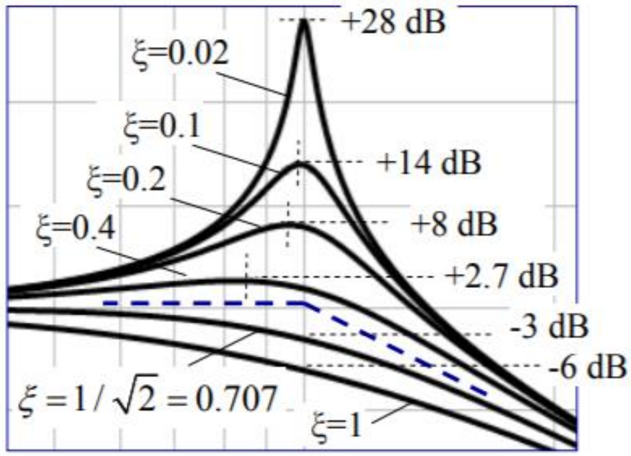

Figure 13 Behaviour near the corner frequency based on damping factor, ζ [3]

With estimate values of ω_n and ζ, the values of τ and K_1 can be calculated as follows based on equations derived in part c):

$$\tau = \frac{1}{2\zeta\omega_n} = \frac{1}{2(0.6)(21)} = 0.0397\ \mathrm{s}$$

$$K_1 = \frac{{\omega_n}^2\tau}{C} = \frac{(21)^2(0.0397)}{12/\pi} = 4.584$$

These values are similar to the values identified previously therefore, the Bode plot fitting method validates that the values of τ and K_1 determined in part c) are approximately correct.

e) Implementation and Testing of a Saturator

Figure 14 shows the addition of the reference angle saturator in between the degrees to radians (D2R) block and the summing point.

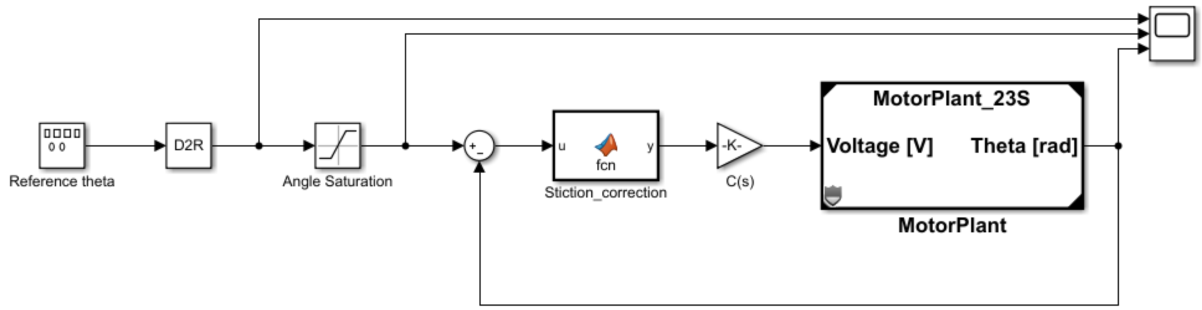

Figure 14 Simulink model with angle saturator

The saturator was verified by applying a reference angle of 75 degrees (1.309 radians). Figure 15 shows the reference angle (yellow), output of the saturator (blue), and the motor plant output, theta (orange). The blue square saturates at the maximum value of 0.7 rad and minimum value of -0.7 rad as expected.

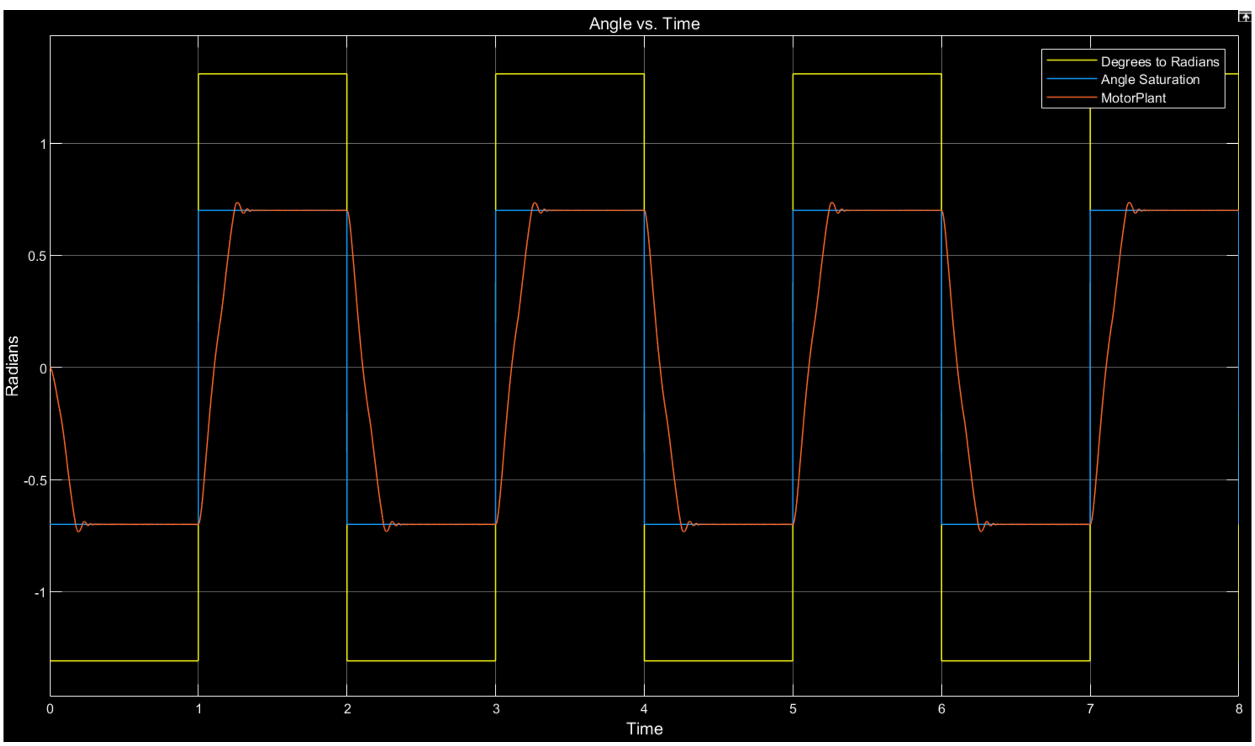

Figure 15 Angle saturation test with input of 75 degrees

Conclusion

The purpose of this lab was the model the inner loop of the motor system, shown in Figure 1.

First, motor stiction was tested by applying a ramp input to determine the offset values of $\alpha = -0.4810$ V and $\beta = 0.5010$ V. A MATLAB function block was then added to the Simulink model to correct for stiction.

Next, the system identification technique of step response overshoot and time to first peak was used to identify the values of $K_1 = 4.52$ and $\tau = 0.398$. These values were validated to be approximately correct via Bode plot fitting.

Finally, a reference angle saturator block was added to the Simulink model. It was validated to limit the output of the saturator block to the range of [-0.7 rad, 0.7 rad].

References

[1] ECE 484 Digial Control Systems: Ball and Beam Lab Manuel, Waterloo: University of Waterloo, 2020.

[2] D. Miller, "2.5.1 Inverting Simple Non-Linearities, Example 2: Stiction," in *Course Noted Chapter 2: Continous Time Performance Specifications and Some Modelling Issues*, Waterloo, University of Waterloo, 2020, p. 18.

[3] B. York, "ECE 232 Handout: Frequency Response and Bode Plots," New Jersey Institute of Technology, 2009. [Online]. Available: https://web.njit.edu/~levkov/classes_files/ECE232/Handouts/Frequency%20Response.pdf. [Accessed 5 October 2020].

Statement of Originality

We acknowledge and promise that:

(a) We are the sole authors of this lab report and associated simulation files/code.

(b) This work represents our original work.

(c) We have not shared detailed analysis or detailed design results, computer code, or Simulink diagrams with any other student.

(d) We have not obtained or looked at lab reports from any other current or former student of ECE 484/481, and we have not let any other student access any part of our lab work.

(e) We have completely and unambiguously acknowledged and referenced all persons and aids used to help us with our work.

Signed L Chambers ______________ and ______________