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MTH302

UNIT 4 The Central limit theorem and the point estimation -

The Central limit theorem - (1st Form)

If $x_1, x_2 \ldots x_n \ldots$ be a sequence of independent RV's with $E(x_i) = \mu_i$ and $var(x_i) = \sigma_i^2$, $i = 1, 2, 3 \ldots$ and if $S_n = x_1 + x_2 + x_3 \ldots + x_n$, then under certain general conditions, S_n follows a normal distributions with mean

$$\mu = \sum_{i=1}^{n} \mu_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2 \quad \text{as } n \to \infty$$

The Central Limit Theorem - (2nd Form)

If $x_1, x_2 \ldots x_n \ldots$ be a sequence of independent identically distributed RVs with $E(x_i) = \mu$, and $var(x_i) = \sigma^2$, $i = 1, 2, 3 \ldots n \ldots$ and if $S_n = x_1 + x_2 \ldots + x_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n \to \infty$

$$\bar{x} = \frac{x_1 + x_2 + x_3 \ldots + x_n}{n}$$

$$E(\bar{x}) = \frac{1}{n}\left[E(x_1) + E(x_2) \ldots + E(x_n) \right]$$

	Population		Sample
Mean	-	μ	$\bar{x}$
Variance	-	σ^2	s^2
Size	-	N	n

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$$= \frac{1}{n} \times n\mu = \mu$$

$$Var(\bar{x}) = Var\left(\frac{x_1 + x_2 \cdots + x_n}{n}\right)$$

$$= \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

$\bar{x}$ follows normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ i.e

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \text{ as } n \to \infty$$

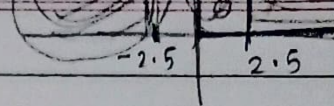
Q. An electrical firm manufactures light bulbs that have a length of life i.e normally distributed with mean equal to 800 hours and standard deviation of 40 hours. Find the probability that a ~~sa~~ random sample of 16 bulb will have an average life of 775 hours given $P(Z < -2.5) = 0.0062$ or $P(0 < Z < 2.5) = 0.4938$

Sol. $P(\bar{x} < 775) = ?$

$n = 16$

$\mu = 800$, $\sigma = 40$

$$\mu_{\bar{x}} = 800, \quad \sigma_{\bar{x}} = \frac{40}{\sqrt{16}} = \frac{40}{4} = 10$$



$$Z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{775 - 800}{10} = -2.5$$

$14.28 \to 14.2 \quad 0.08$

$$Z = -2.5$$

$$P(\bar{X} < 775) = P(Z < -2.5) = 0.0062$$

$14.287 \to 14.2 \quad 0.08$

$$P(\bar{X} < 775) = P(Z < -2.5)$$
$$= 0.5 - P(0 < Z < 2.5)$$
$$= 0.5 - 0.4938$$
$$= 0.0062$$

Q. Travelling between two campus of a university via bus takes on average 28 minutes with standard deviation of 5 minutes. In a given week a bus transported passengers 40 times. What is the probability that the average transport time was more than 30 minutes given $P(Z < 2.53) = 0.9943$ or $P(0 < Z < 2.53) = 0.4943$

Sol. $P(\bar{X} > 30) = ?$

$$\mu_{\bar{x}} = 28, \quad \sigma_{\bar{x}} = \frac{5}{\sqrt{40}} = 0.7906$$

$$P(\bar{X} > 30) = P(Z > 2.53) = 1 - 0.9943$$

$$Z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{30 - 28}{0.7906} = 2.53$$

$$P(\bar{X} > 30) = 0.0057$$

or

$P(Z > 2.53) = 0.5 - 0.4943 = 0.0057$

Q. If $X_1, X_2, X_3 \dots X_n$ are poisson variates with parameter $\lambda = 2$. Find the probability of $P(120 \leq S_n \leq 160)$ with $S_n = X_1 + X_2 \dots X_n$ and $n = 75$

Sol. $$p(x; \lambda) = \frac{e^{-\lambda}(\lambda)^x}{x!}$$

$E(X_i) = 2$ $\quad$ $Var(X_i) = 2$ $\quad$ for $1 \leq i \leq n$

S_n follows a normal distribution

with mean $(2+2+2+2 \dots +2)$ (75 times)

$= 2 \times 75 = 150$

and

Standard deviation $= \sqrt{2 \times 75} = \sqrt{150}$

$$Z = \frac{S_n - Mean(S_n)}{Var(S_n)}$$

$P(120 \leq S_n \leq 160)$

$$Z_1 = \frac{120 - 150}{\sqrt{150}} = -2.45$$

$$Z_2 = \frac{160 - 150}{\sqrt{150}} = 0.82$$

$P(-2.45 < Z < 0.82) = 0.7868$

Estimators -

Parameters : numerical value which describes the population data
$\{ \mu, \sigma^2, N \}$

Statistic : numerical value which describes the sample data
$\{ \bar{x}, s^2, n \}$

Parameters are Unknown.

$$\text{Statistic} \xrightarrow{\text{Estimate}} \overset{\text{Unknown}}{\text{Parameter}}$$
$\downarrow$
Estimators

- Estimator -

Any function of the random sample $x_1, x_2, x_3 \dots x_n$ say $T_n(x_1, x_2 \dots x_n)$ function is called Statistic. If it is used to estimate an unknown parameter θ, it is called an estimator.

Characteristics of Estimators -

- It is unbiased.
- consistent
- Efficient
- sufficient

Unbiased Estimator -

An estimator T_n which is function of $x_1, x_2 \dots x_n$ is said to be unbiased estimator of $\gamma(\theta)$ if $E(T_n) = \gamma(\theta)$ for all θ

$$T(n) = T(x_1, x_2, x_3 \dots x_n)$$
$$E(T_n) = \gamma(\theta) \quad \text{— unbiased}$$

If $E(T_n) > \gamma(\theta)$ — Positively biased
If $E(T_n) < \gamma(\theta)$ — Negatively biased

Q. If $x_1, x_2, \dots x_n$ is a random sample from a normal population with mean μ and variance $= 1$. Show that $t = \frac{1}{n} \sum_1^n x_i^2$ is an unbiased estimator of $(\mu^2 + 1)$.

Sol. To prove - $E(t) = \mu^2 + 1$

$$E(t) = E\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right)$$
$$= \frac{1}{n} \sum_{i=1}^{n} E(x_i^2)$$

$$Var(x_i) = E(x_i^2) - [E(x_i)]^2$$
$$1 = E(x_i^2) - \mu^2$$
$$E(x_i^2) = \mu^2 + 1$$

$$E(t) = \frac{1}{n} \sum_{i=1}^{n} (\mu^2+1)$$

$$E(t) = \frac{\cancel{n}(\mu^2+1)}{\cancel{n}} = \mu^2+1$$

$$E(t) = \mu^2+1$$

Hence Proved

Q. If T is an unbiased estimator of θ, then show that T^2 is a biased estimator of θ^2

Sol. Given - $E(T) = \theta$

To prove - $E(T^2) \neq \theta^2$

$$Var(T) = E(T^2) - [E(T)]^2$$
$$= E(T^2) - \theta^2$$

$$E(T^2) = Var(T) + \theta^2$$

$E(T^2) \neq$ ~~var~~ θ^2 $\qquad Var(T) > 0$ — given

T^2 is a biased estimator of θ^2

Q. Show that $\frac{\Sigma x_i(\Sigma x_i - 1)}{n(n-1)}$ is an unbiased estimator of θ^2, for the sample $x_1, x_2 \dots x_n$ drawn on X which takes the values 1 or 0 with respect to probabilities θ and $1-\theta$.

Sol.

x_i	1	0
p	θ	$(1-\theta)$

$x_1, x_2 \dots x_n$ is a random sample from a Bernoulli population.

Let $T = \Sigma x_i \sim B(n, \theta)$

$E(T) = n\theta$, $\quad Var(T) = n\theta(1-\theta)$

To prove : $E\left(\frac{T(T-1)}{n(n-1)}\right) = \theta^2$

$$E\left(\frac{T(T-1)}{n(n-1)}\right) = \frac{1}{n(n-1)} E(T^2 - T)$$

$$= \frac{1}{n(n-1)} [E(T^2) - E(T)]$$

$$Var(T) = E(T^2) - E(T)^2$$

$$n\theta(1-\theta) = E(T^2) - n^2\theta^2$$

$$E(T^2) = n\theta - n\theta^2 + n^2\theta^2$$

$$= \frac{1}{n(n-1)} [n\theta - n\theta^2 + n^2\theta^2 - n\theta]$$

$$= \frac{\theta^2 n\cancel{(n-1)}}{n\cancel{(n-1)}} = \theta^2$$

Q. Let X be a distributed in the poison form with parameter θ. Show that ~~x is~~ poison variate only unbiased

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} \dots$$

estimator of $e^{-(K+1)\theta}$, $K > 0$ is $T(X) = (-K)^X$

Sol. To prove - $E(T(X)) = e^{-(K+1)\theta}$

$E(X) = \theta$, $Var(X) = \theta$

$$E(T(X)) = \sum_{x=0}^{\infty} \frac{(-K)^x e^{-\theta} \theta^x}{x!} = e^{-\theta} \sum_{x=0}^{\infty} \frac{(-K\theta)^x}{x!}$$

$$= e^{-\theta} e^{-\theta K}$$

$$= e^{-(K+1)\theta}, \quad K > 0$$

Consistent Estimator -

- An estimator $T_n = T(x_1, x_2, x_3 \ldots x_n)$ based on a random sample of size n is said to be a consistent estimator of $\gamma(\theta)$ if T_n converges to $\gamma(\theta)$ in probability i.e $T_n \xrightarrow{p} \gamma(\theta)$ as $n \to \infty$.

- If there exist an consistent estimator T_n of $\gamma(\theta)$, there exist infinitely many such estimators can be constructed

$$T_n \xrightarrow{p} \gamma(\theta) \quad \text{as} \quad n \to \infty$$

$$T_n' = \left(\frac{n-a}{n-b}\right) T_n$$

$$= \left(\frac{1 - a/n}{1 - b/n}\right) T_n \longrightarrow T_n \longrightarrow \gamma(\theta) \text{ as } n \to \infty$$

Sufficient conditions for consistency -

T_n is a consistent estimator of $\gamma(\theta)$ if

(i) $E(T_n) \to \gamma(\theta)$ as $n \to \infty$

(ii) $Var(T_n) \to 0$ as $n \to \infty$

Q. Show that in sampling from normal population with mean μ and standard deviation σ, the sample mean is a consistent estimator of μ.

Sol. $\bar{x}$ is a consistent estimator of μ

To prove -

(i) $E(\bar{x}) \to \mu$ as $n \to \infty$

(ii) $Var(\bar{x}) \to 0$ as $n \to \infty$

Let $x_1, x_2, x_3 \dots x_n$ be a random sample from $N(\mu, \sigma)$

$$E(x_i) = \mu, \quad Var(x_i) = \sigma^2 \quad \forall \; i = 1, 2, \dots n$$

$$\bar{x} = \frac{x_1 + x_2 \dots + x_n}{n}$$

$$E(\bar{x}) = \frac{1}{n} \sum_{i=1}^{n} E(x_i)$$

$$= \frac{1}{n}(n\mu) = \mu$$

$$E(\bar{x}) \to \mu \quad \text{as} \quad n \to \infty$$

$$Var(\bar{x}) = \frac{1}{n^2} \sum_{i=1}^{n} Var(x_i)$$

$$= \frac{1}{n^2}(n\sigma^2)$$

$$= \frac{\sigma^2}{n} \to 0 \quad \text{as} \quad n \to \infty$$

Since $E(\bar{x}) \to \mu$ as $n \to \infty$

and $Var(\bar{x}) \to 0$ as $n \to \infty$

Therefore, $\bar{x}$ is a consistent estimator of μ.

Q. If T_1 and T_2 be a consistent estimator of γ_1 and γ_2 respectively. Prove that $aT_1 + bT_2$ is a consistent estimator of $a\gamma_1 + b\gamma_2$ where a and b are constant and independent of population

Sol. Since T_1 and T_2 are consistent estimators of γ_1 and γ_2 respectively this implies

$$E(T_1) \rightarrow \gamma_1, \quad E(T_2) \rightarrow \gamma_2 \quad \text{as} \quad n \rightarrow \infty$$

and

$$Var(T_1) \rightarrow 0, \quad Var(T_2) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

To prove -

$$E(aT_1 + bT_2) \rightarrow a\gamma_1 + b\gamma_2 \quad \text{as} \quad n \rightarrow \infty$$

$$Var(aT_1 + bT_2) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

$$E(aT_1 + bT_2) = aE(T_1) + bE(T_2)$$

$$\rightarrow a\gamma_1 + b\gamma_2 \quad \text{as} \quad n \rightarrow \infty$$

$$Var(aT_1 + bT_2) = a^2 Var(T_1) + b^2 Var(T_2)$$

$$\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

This implies
$\Rightarrow$ $aT_1 + bT_2$ is consistent estimator of $a\gamma_1 + b\gamma_2$

Q. If $x_1, x_2 \ldots x_n$ are random observation on bernoulli variate X taking the value 1 with probability p and value 0 with probability $1-p$. Show that $\frac{\Sigma x_i}{n}\left(1 - \frac{\Sigma x_i}{n}\right)$ is a consistent estimator of $p(1-p)$.

Sol. $X_1, X_2 \dots X_n$

X	1	0
	p	$1-p$

$$\frac{\Sigma x_i}{n}\left(1-\frac{\Sigma x_i}{n}\right)$$

$$\downarrow$$

$$\bar{x}(1-\bar{x})$$

Given that $X_1, X_2 \dots X_n$ are Bernoulli variables with parameter p.

$\therefore \quad T = \Sigma x_i \sim B(n, p)$

$E(T) = np \quad , \quad var(T) = np(1-p)$

Now, $\bar{x} = \frac{\Sigma x_i}{n} = \frac{T}{n}$

$E(\bar{x}) = E\left(\frac{T}{n}\right) = \frac{1}{n} E(T) = \frac{1}{n} \times np = p$

$E(\bar{x}) \rightarrow p$ as $n \rightarrow \infty$

$var(\bar{x}) = var\left(\frac{T}{n}\right) = \frac{1}{n^2} var(T)$

$= \frac{1}{n^2} np(1-p) = \frac{p(1-p)}{n}$

$var(\bar{x}) \rightarrow 0$ as $n \rightarrow \infty$

$\bar{x}$ is a consistent estimator of p.
By invariance property of consistency,
$\bar{x}(1-\bar{x}) = \frac{\Sigma x_i}{n}\left(1-\frac{\Sigma x_i}{n}\right)$ is a C.E of $p(1-p)$.

Invariance property of consistency -

If T_n is a consistent estimator of $\gamma(\theta)$ and f is a continous function, then $f(T_n)$ is a consistent estimator of $f(\gamma(\theta))$

Example -

$\bar{x}$ be a consistent estimator of p

$F(t) = t(t-1)$ is a continous function.

$F(\bar{x}) = \bar{x}(\bar{x}-1)$ is a C.E of $F(p) = p(p-1)$

$X_1, X_2, \dots X_n$ — random variable.

Q. Show that s^2 is a biased estimator of σ^2 where $x_1, x_2 \dots x_n$ is random sample with $E(x_i) = \mu$, $Var(x_i) = \sigma^2$, for $i = 1, 2, \dots n$.

Sol. To prove : $E(s^2) \neq \sigma^2$

$$s^2 = E(x_i^2) - [E(x_i)]^2$$

$$= E(x_i^2) - \bar{x}^2$$

$$= \frac{\sum_{i=1} x_i^2}{n} - (\bar{x})^2$$

$$E(s^2) = \frac{1}{n} \sum_{i=1}^{n} E(x_i^2) - E(\bar{x})^2$$

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

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$$\text{var}(x_i) = E(x_i^2) - [E(x_i)]^2$$

$$\sigma^2 = E(x_i^2) - \mu^2$$

$$E(x_i^2) = \sigma^2 + \mu^2$$

$$\text{var}(\bar{x}) = E(\bar{x}^2) - [E(\bar{x})]^2$$

$$\frac{\sigma^2}{n} = E(\bar{x}^2) - \mu^2$$

$$E(\bar{x}^2) = \frac{\sigma^2}{n} + \mu^2$$

$$E(s^2) = \frac{1}{n}\sum_{i=1}^{n}(\sigma^2 + \mu^2) - \frac{\sigma^2}{n} - \mu^2$$

$$= \frac{1}{\cancel{n}} \times \cancel{n}(\sigma^2 + \mu^2) - \frac{\sigma^2}{n} - \mu^2$$

$$= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2$$

$$E(s^2) = \sigma^2\left(1 - \frac{1}{n}\right)$$

$$E(s^2) \neq \sigma^2$$

$\therefore$ s^2 is a biased estimator of σ^2.

Maximum Likelihood Estimation (MLE).

Most probable estimate of the parameter i.e, provides estimate whose probability is maximum.

$x_1, x_2, \ldots x_n$ is a random sample from a population.

$$L = L(\theta) = f(x_1, \theta)\, f(x_2, \theta) \dots f(x_n, \theta)$$
$$= \prod_{i=1}^{n} f(x_i / \theta)$$

$$\frac{dL}{d\theta} = 0$$

$$\frac{d^2L}{d\theta^2} < 0$$

Let $x_1, x_2 \dots x_n$ be independent observations drawn from a probability distribution that depends on some parameter θ.

The MLE is used to maximize the likelihood function

$$L = L(\theta) = \frac{f(x_1, x_2 \dots x_n)}{\theta} = \prod_{i=1}^{n} f(x_i / \theta)$$

- for maximization, $\frac{dL}{d\theta} = 0$ and $\frac{dL^2}{d\theta^2} < 0$
- for maximizing L, it is equivalent to maximize log L.

Q. Consider a poison distribution with probability mass function $f(x/\mu)$ $= \frac{e^{-\mu} \mu^x}{x!}$, $x = 0, 1, 2 \dots$

Suppose that a random sample $x_1, x_2 \dots x_n$ is taken from a ~~population~~ distribution. What is the maximum livelihood of μ

Sol. The likelihood function is

$$L = L(\mu) = f(x_1, x_2 \dots x_n / n)$$

$$= \prod_{i=1}^{n} f(x_i / \mu)$$

$$= \prod_{i=1}^{n} \frac{e^{-\mu} \mu^{x_i}}{x_i!} = \frac{e^{-n\mu} \mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\log L = \log\left(e^{-n\mu} \mu^{\sum_{i=1}^{n} x_i}\right) - \log\left(\prod_{i=1}^{n} x_i!\right)$$

$$= \log e^{-n\mu} + \log \mu^{\sum_{i=1}^{n} x_i} - \sum_{i=1}^{n} \log(x_i!)$$

$$= -n\mu + \sum_{i=1}^{n} x_i \log \mu - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{d(\log L)}{d\mu} = -n + \sum_{i=1}^{n} x_i \frac{1}{\mu}$$

$$\frac{d(\log L)}{d\mu} = 0 \Rightarrow -n + \sum_{i=1}^{n} x_i \frac{1}{\mu} = 0$$

$$\sum_{i=1}^{n} x_i \frac{1}{\mu} = n$$

$$\mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

$$\frac{d^2(\log L)}{d\mu^2} = \sum_{i=1}^{n} x_i \left(\frac{-1}{\mu^2}\right) < 0 \qquad \therefore \hat{\mu} = \bar{x}$$

The MLE of μ is a sample mean $\bar{x}$.

Q. In random sampling from normal population with mean μ and variance σ^2. Find the maximum likelihood estimation for
(i) μ when σ^2 is known
(ii) σ^2 when μ is known.
(iii) The simultaneous estimation of μ and σ^2.

Sol. The likelihood function is –

$$L(x_1, x_2, \dots x_n/\theta) = \prod_{i=1}^{n} f(x_i/\theta)$$

Probability density function $= \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$

$$L(x_1, x_2 \dots x_n/\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\log L = \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n + \log\left(e^{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}\right)$$

$$\log L = -n\log\left(\sqrt{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= \frac{-n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\log L = \frac{-n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

(i) $\frac{d(\log L)}{d\mu} = \frac{-1}{2\sigma^{2}} \sum_{i=1}^{n} 2(x_i-\mu)(-1)$

$\frac{d(\log L)}{d\mu} = 0$

cannot be zero $\boxed{\frac{1}{\sigma^2}} \sum_{i=1}^{n}(x_i-\mu) = 0$

$\sum_{i=1}^{n}(x_i-\mu) = 0$

$\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu = 0$

$\sum_{i=1}^{n} x_i - n\mu = 0$

$\mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

$\frac{d^2(\log L)}{d\mu^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n}(-1) < 0$

$\hat{\mu} = \bar{x}$

MLE of μ is $\bar{x}$

(ii) $\log L = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$

$\frac{d(\log L)}{d\sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2$

$\frac{d(\log L)}{d\sigma^2} = 0$

$\frac{n}{\cancel{2}}\frac{1}{\cancel{\sigma^2}} = \frac{1}{\cancel{2}(\sigma^2)^{\cancel{2}}}\sum_{i=1}^{n}(x_i-\mu)^2$

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

Q. Question solution continues ....

$$\frac{d(\log L)}{d(\sigma^2)^2} = \frac{n}{2}\frac{1}{(\sigma^2)^2} - \frac{1}{2}\left(\frac{-2}{(\sigma^2)^3}\right)\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= \frac{n}{2}\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^{\cancel{3}2}}(n\cancel{\sigma^2})$$

$$\frac{\partial^2(\log L)}{(\partial\sigma^2)^2} < 0$$

The MLE of $\hat{\sigma}^2$ is $\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

(iii) $\frac{\partial(\log L)}{\partial\mu} = 0 \Rightarrow \hat{\mu} = \bar{x}$

$\frac{\partial(\log L)}{\partial\sigma^2} = 0 \Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$\therefore \hat{\sigma}^2 = s^2$$

Q. Prove that the MLE of parameter α of a population having the density function $\frac{2}{\alpha^2}(\alpha-x)$, $0 < x < \alpha$ for a sample of a unit size is $2x$, x being the sample value show that estimate is baised.

Sol.

$$L(x/\alpha) = \frac{2}{\alpha^2}(\alpha-x)$$

$$\log L = \log\left(\frac{2}{\alpha^2}\right) + \log(\alpha-x)$$

$= \log 2 - 2\log\alpha + \log(\alpha - x)$

$\frac{d(\log L)}{\partial \alpha} = \frac{-2}{\alpha} + \frac{1}{\alpha - x}$

$\frac{d(\log \alpha)}{\partial \alpha} = 0$

$\frac{2}{\alpha} = \frac{1}{\alpha - x}$

$2\alpha - 2x = \alpha$

$\alpha = 2x$

$\frac{\partial^2 (\log L)}{\partial \alpha^2} = \frac{2}{\alpha^2} - \frac{1}{(\alpha - x)^2}$

$= \frac{2}{4x^2} - \frac{1}{x^2} = \frac{1}{2x^2} - \frac{1}{x^2}$

$= \frac{-1}{2x^2} < 0$

$\hat{\alpha} = 2x \Rightarrow$ MLE of α is $2x$.

To prove biased -

$E(\alpha) = E(2x) \neq \alpha$

$E(\alpha) = \frac{4}{\alpha^2}\left[\frac{\alpha^3}{2} - \frac{\alpha^3}{3}\right]$

$= \frac{4}{\alpha^2} \times \frac{\alpha^3}{6}$

$= \frac{2\alpha}{3} \neq \alpha$

$E(\alpha) \neq \alpha$

$\alpha = 2x$ is a biased estimator of α.