Example:

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

$$f(x,y) = P(X=x, Y=y)$$

Find the covariance of 'X' and 'Y'.

$$Cov.(X,Y) = E(XY) - E(X)E(Y).$$

$f(x,y)$		$x=0$	1	2	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$P(X=x) = g(x)$		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xy\, f(x,y)$$

$$f(x)$$

$$E(X) = \mu_X = \sum_{x=0}^{2} x P(X=x)$$

$$P(X=1) = g(1) = \quad \sum_{x=0}^{2} x\, g(x) \qquad P(X=2)$$

From Example 4.6, we see that $E(XY) = 3/14$. Now

$P(X=0) = g(0)$

$$E(X) = \mu_X = \sum_{x=0}^{2} x g(x) = (0)\left(\frac{5}{14}\right) + (1)\left(\frac{15}{28}\right) + (2)\left(\frac{3}{28}\right) = \frac{3}{4},$$

$$= (0)g(0) + (1)g(1) + (2)g(2)$$

$P(X=2) = g(2)$. $\quad P(X=0) \quad P(X=1)$

and

$$E(Y) = \mu_Y = \sum_{y=0}^{2} y h(y) = (0)\left(\frac{15}{28}\right) + (1)\left(\frac{3}{7}\right) + (2)\left(\frac{1}{28}\right) = \frac{1}{2}.$$

$$g(x) = \sum_{y=0}^{2} f(x,y)$$

$$g(x) = f(x,0) + f(x,1) + f(x,2)$$

$$g(0) = f(0,0) + f(0,1) + f(0,2)$$

Therefore,

$$Cov.(X,Y) = \sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

$$h(y) = \sum_{x=0}^{2} f(x,y)$$

Example:

The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$\cancel{P(X=x) = g(x)}$$

$$f(x, y) = \begin{cases} 8xy, & 0 \leq y \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.

$$Cov.(X,Y) = E(XY) - E(X)E(Y),$$

We first compute the marginal density functions. They are

$$g(x) = \begin{cases} 4x^3, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

$$E(X) = \int_{0}^{1} x\, g(x)\, dx = \int_0^x x \times 4x^3\, dx.$$

and

$$h(y) = \begin{cases} 4y(1 - y^2), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$g(x) = \int_0^x f(x,y)\, dy = \int_0^x 8xy\, dy$$

$$\sum_{y=0}^{2} f(x,y)$$

$$= 8x \left.\frac{y^2}{2}\right|_0^x$$

From these marginal density functions, we compute

$$\mu_X = E(X) = \int_0^1 4x^4\, dx = \frac{4}{5} \text{ and } \mu_Y = \int_0^1 4y^2(1 - y^2)\, dy = \frac{8}{15}.$$

$$= 4x(x^2 - 0) = 4x^3$$

From the joint density function given above, we have

$$h(y) = \int_y^1 f(x,y)\, dx$$

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2\, dx\, dy = \frac{4}{9}.$$

$$\int_0^1 \int_0^x 8x^2y^2\, dy\, dx$$

Then

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

$$E(XY) = \iint xy\, f(x,y)\, dx\, dy = \iint xy \times 8xy\, dx\, dy = \iint 8x^2y^2\, dx\, dy$$

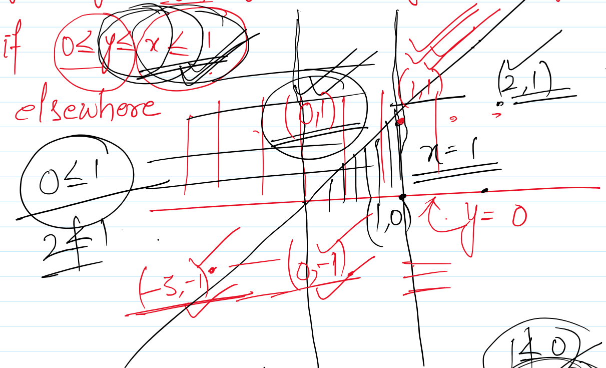
$9 \quad (5)(15) \quad 225$

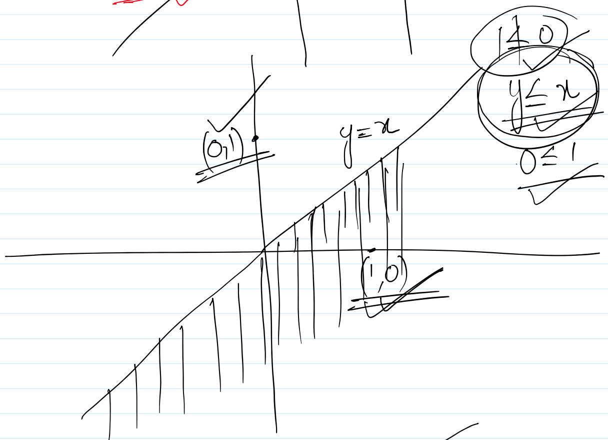
$$E(XY) = \iint xy\, f(x,y)\,dx\,dy = \iint xy \times 8xy\,dx\,dy = \iint 8x^2y^2\,dx\,dy$$

$$f(x,y) = \begin{cases} 8xy, & \text{if } 0 \le y \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$y \geq 0$ $\quad 1 \geq 0$

$y = 0$ $\quad -1 \ngeq 0$

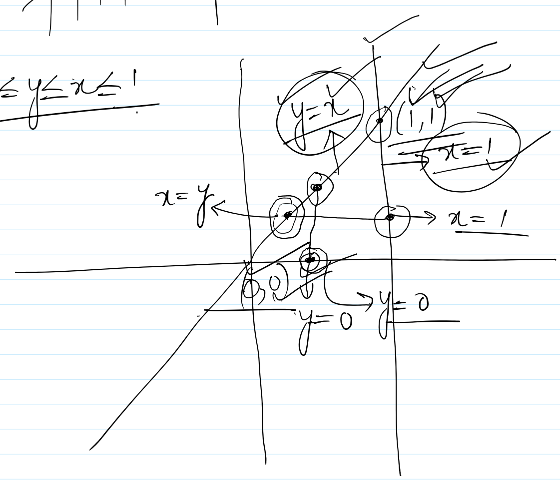




$0 \le y \le x \le 1$

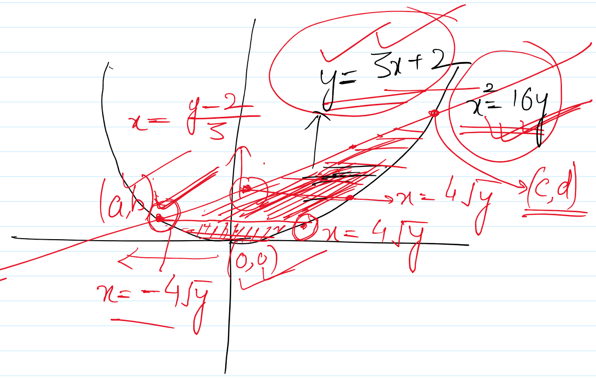
$$\int_0^1 \int_0^x 8x^2y^2\,dy\,dx$$

$$\int_0^1 \int_y^1 8x^2y^2\,dx\,dy$$



$$\iint f(x,y)\,dx\,dy$$

$$= \int_b^d \int_{\frac{y-2}{3}}^{4\sqrt{y}} f(x,y)\,dx\,dy + \int_0^b \int_{-4\sqrt{y}}^{4\sqrt{y}} f(x,y)\,dx\,dy$$



(Chebyshev's Theorem) The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - 1/k^2$. That is,

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

(Chebyshev's Theorem) The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - 1/k^2$. That is,

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$-4 = \mu - k\sigma = 8 - k(3)$

$-4 = 8 - 3k$

$k = 4$

EXAMPLE:

A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find

(a) $P(-4 < X < 20)$,

(b) $P(|X - 8| \geq 6)$.

$1 - \frac{1}{4^2}$

$20 = 8 + 4(3)$

(a) $P(-4 < X < 20) = P[8 - (4)(3) < X < 8 + (4)(3)] \geq \frac{15}{16}$.

(b) $P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6) = 1 - P(-6 < X - 8 < 6)$ $= 1 - P(2 < X < 14)$

$ = 1 - P[8 - (2)(3) < X < 8 + (2)(3)] \leq \frac{1}{4}$.

$+8 \quad +8 \quad +8$

$X - 8 = Z.\quad P[|Z| \geq 6] + P[|Z| < 6] = 1$

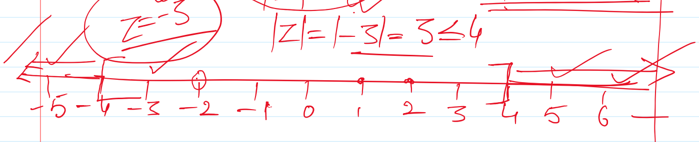
If $|Z| \leq k$ then $-k \leq Z \leq k$.

$|Z| \leq 4$ " $-4 \leq Z \leq 4$

$Z = -3$

$|Z| = |-3| = 3 \leq 4$

If $|Z| \geq k$ $(= 4)$ then either $Z \geq k$ or $Z \leq -k$



$P(2 < X < 14)$

$= P[8 - 2(3) < X < 8 + 2(3)]$

$2 = 8 - k(3)$

$k = 2.$

$\geq 1 - \frac{1}{2^2} = \frac{3}{4}$

$1 - P(2 < X \leq 14) \leq 1 - \frac{3}{4}$

$1 - P[6 < X - 8 < 6] \leq \frac{1}{4}$

$1 - P[|X - 8| < 6] \leq \frac{1}{4}$