Hypothesis Testing

Hypothesis :- An assumption or claim about the population that we want to test.

Null Hypothesis (H_0) A hypothesis which assumes that there is no significant difference between the sample statistic and the corresponding population parameter or between two sample statistics.

Alternative Hypothesis (H_1 or H_a)

A hypothesis that is complementary to the null hypothesis, is called an alternative hypothesis.

Test of hypothesis :- A procedure for deciding whether to accept or to reject a null hypothesis (and hence to reject or to accept the alternative hypothesis resp) is called the test of hypothesis.

Ex :- If we want to test the null hypothesis that population has a specified mean μ_0,

i.e. $H_0: \mu = \mu_0$, then alternative hypothesis could be (i) $H_1: \mu \neq \mu_0$ (i.e. $\mu > \mu_0$ or $\mu < \mu_0$) (Two-tailed alternative)

(ii) $H_1: \mu > \mu_0$ (Right tailed alternative)

(iii) $H_1: \mu < \mu_0$ (Left tailed alternative)

Errors in sampling

		Decision	
		H_0 is true	H_0 is false
True state	Reject H_0	Type I Error Probability $= \alpha$	Correct decision
	Accept H_0	Correct decision	Type II Error Probability $= \beta$.

Type I Error : Reject H_0 when it is true.

Type II Error : Accept H_0 when it is false.

If P(Type I Error)

$= P($ Reject H_0 when it is true $)$

$= P($ Reject $H_0 / H_0) = \alpha$

P(Type II Error) $=$ P(Accept H_0 when it false)

$= P($ Accept $H_0 / H_1) = \beta$,

then α and β are called the sizes of type I Error and type II Error, respectively.

Level of Significance

Level of significance is the size of type I error, i.e., α.

Level of significance is the probability of rejecting a null hypothesis when it is true.

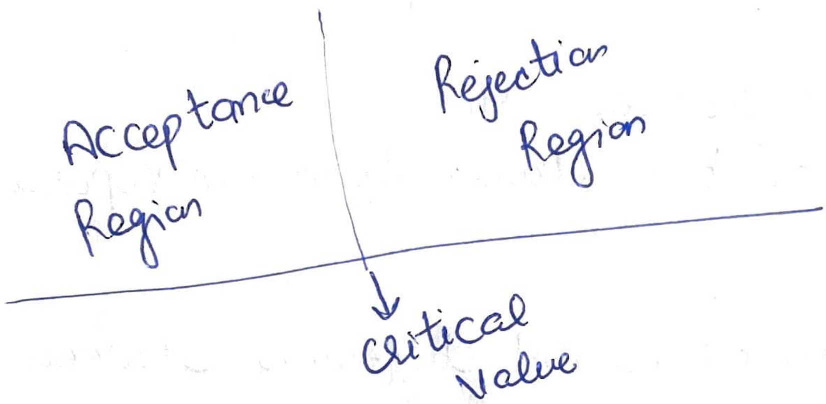
Generally it is taken at 1% or 5%.

If no level of significance is given then $\alpha = 0.05$ will be taken.

Confidence Level

Confidence level is the probability of not occuring type I error. It is denoted by $1-\alpha$. Confidence level is meant by accepting null hypothesis when it is true.

Critical Region (Rejection Region)



Confidence Interval

at α level of significa

$$\bar{x} \pm \frac{S}{\sqrt{n}} t_\alpha.$$

degrees of freedom

The maximum number of independent values, which have the freedom to vary, in the data sample.

$\nu = (n-1)$ if there are n independent observations in the sample.

Confidence Interval | Fiducial Interval

$$\bar{x} \pm \frac{S}{\sqrt{n}} t_\alpha$$

Ex: Given the frequency function:

$$f(x,\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{elsewhere} \end{cases}$$

and that you are testing the null hypothesis $H_0: \theta = 1$ against $H_1: \theta = 2$ by means of a single observed value of x. What would be the sizes of the type I and type II errors, if you choose the interval (i) $0.5 \le x$ (ii) $1 \le x \le 1.5$ as the critical regions? Also obtain the power function of the test.

Sol: We want to test $H_0: \theta = 1$ against $H_1: \theta = 2$.

(i) Here $W = \{x : 0.5 \le x\} = \{x : x \ge 0.5\}$

$\bar{W} = \{x : x \le 0.5\}$

$$\alpha = P\{x \in W \mid H_0\} = P\{x \ge 0.5 \mid \theta = 1\}$$

$$= P\{0.5 \le x \le \theta \mid \theta = 1\}$$

$$= P\{0.5 \le x \le 1 \mid \theta = 1\}$$

$$= \int_{0.5}^{1} 1\, dx = [x]_{0.5}^{1} = 1 - 0.5 = 0.5$$

$\alpha = 0.5$

$$\beta = P\{x \in \bar{W} \mid H_1\} = P\{x \le 0.5 \mid \theta = 2\} = P\{0 \le x \le 0.5 \mid \theta = 2\}$$

$$= \int_{0}^{0.5} \frac{1}{2}\, dx = \frac{1}{2}(0.5) = 0.25$$

$\beta = 0.25$ and power function of test $= 1 - \beta = 0.75$.

(ii) $W = \{1 \leq x \leq 1.5\}$

$$\alpha = P\{x \in W \mid \theta = 1\} = \int_1^{1.5} dx = 0.$$

$$\left[\because f(x,1) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases} \right]$$

$$\beta = P\{x \in \overline{W} \mid \theta = 2\} = 1 - P\{x \in W \mid \theta = 2\}$$

$$= 1 - \int_1^{1.5} \frac{1}{2}\, dx = 1 - \frac{1}{2}[1.5 - 1] = 1 - \frac{1}{2}(0.5)$$

$$= 0.75$$

$\therefore$ Power function $= 1 - \beta = 0.25.$

Ex: Let p be the probability that a coin will fall head in a single toss in order to test $H_0: p = \frac{1}{2}$ against $H_1: p = \frac{3}{4}$. The coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the probability of type I error and power of the test.

Sol: Since $X \sim B(n, p)$.

$$\therefore P(X = x) = {}^{n}C_x\, p^x (1-p)^{n-x} = {}^{5}C_x\, p^x (1-p)^{5-x}$$

Here, $W = \{x : x > 3\} \Rightarrow \overline{W} = \{x : x \leq 3\}$

$$\alpha = P[X \geq 4 \mid H_0] = P\{X \geq 4 \mid H_0 : p = \tfrac{1}{2}\}$$

$$= {}^{5}C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + {}^{5}C_5 \left(\frac{1}{2}\right)^5 = \frac{6}{32} = \frac{3}{16}$$

$$\beta = P[X \leq 3 \mid H_1]$$

$$= 1 - \left[{}^{5}C_4 \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^1 + {}^{5}C_5 \left(\frac{3}{4}\right)^5 \right]$$

$$= 1 - \frac{648}{1024} = 1 - \frac{81}{128} = \frac{47}{128}$$

Power of the test $= 1 - \dfrac{47}{128} = \dfrac{81}{128}$.

Student's t- test for single mean

Suppose we want to test:

(i) If a random sample x_i $(i=1,2,\ldots,n)$ of size n has been drawn from a normal population with a specified mean, say, μ_0, or

(ii) If the sample mean differs significantly from the hypothetical value μ_0 of the population mean.

Under the null hypothesis, H_0:

i) The sample has been drawn from the population with mean μ_0 or

(ii) there is ~~a~~ no significant difference between the sample mean $\bar{x}$ and the population mean μ_0,

the statistic

$$t = \frac{\bar{x}-\mu_0}{S/\sqrt{n}},$$

$$\text{where } \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2, \quad ns^2 = (n-1)S^2$$

follows Student's t-distribution with $(n-1)$ d.f.

* S^2 is unbiased estimator of σ^2.

We now ~~calculated the~~ compare the calculated value of t with the tabulated value at certain level of significance. If calculated $|t| >$ tabulated t, null hypothesis is rejected and if calculated $|t| <$ tabulated t, H_0 may be accepted at the level of significance adopted.

Assumption for Student's t-test

(i) The parent population from which the sample is drawn is normal.

(ii) The sample observations are independent, i.e., the sample is random.

(iii) The population standard deviation σ is unknown.

Ex.: A machinist is making engine parts with diameters of 0.700 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a standard deviation of 0.040 inch. Check whether the work is meeting the specifications.

Sol.: Given, $\mu = 0.700$ inch, $\bar{x} = 0.742$ inch, $s = 0.040$ inch and $n = 10$.

Null hypothesis: $H_0 : \mu = 0.7$, i.e., the product is conforming to specifications.

Alternative hypothesis: $H_1 : \mu \neq 0.700$

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \sim t_{(n-1)}$$

$$\therefore \quad t = \frac{0.742 - 0.7}{0.040/\sqrt{9}} = \frac{0.042}{0.01333} = 3.15$$

Now, the test statistic follows Student's t-distribution with 9 d.f.

$\left[t_{tabulated} = 2.26 \quad \text{at } 5\% \text{ level of significance.}\right]$

~~Tab~~ $t_{0.05}$ for $10-1=9$ d.f. is 2.26 for two-tailed test.

Conclusion :- Since, the calculated t is greater than tabulated $t_{0.05}$ for 9 d.f., H_0 may be rejected at 5% level of significance and we may conclude that the work ~~product~~ is not meeting the specifications.

Ex :- The mean weekly sales of soap bars in departmental stores was 146.3 ~~hours~~ bars per store. After an advertising compaign the mean weekly sales in 22 stores for a typical week increased to 153.7 and showed a standard deviation of 17.2. Was the advertising compaign successful?

Sol :- Given, $n = 22$, $\bar{x} = 153.7$, $s = 17.2$

Null Hypothesis :- $H_0: \mu = 146.3$

i.e, The advertising compaign is not successful.

Alternative :- $H_1: \mu > 146.3$ (Right tail)

Test Statistic :- Under H_0, the test statistic is :

$$t = \frac{\bar{x} - \mu}{\sqrt{S^2/n}} = \frac{\bar{x} - \mu}{\sqrt{s^2/n-1}} \sim t_{22-1} = t_{21}$$

$$\therefore \quad t = \frac{153.7 - 146.3}{\sqrt{(17.2)^2/21}} = \frac{7.4}{\sqrt{14.08}} = \frac{7.4}{3.75} = 1.97$$

Conclusion Tabulated $t_{0.05}$ for 21 d.f. = 1.72

Since, calculated value of test statistic is greater than tabulated value, it is highly significant.

Hence, we reject the null hypothesis and conclude that the advertising compaign was successful.

Ex A random sample of 10 boys had the following I.Q.'s : 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q. of 100? Find a reasonable range in which most of the mean I.Q. values of samples of 10 boys lie. $\boxed{\text{Confidence interval: } \bar{x} \pm t_{df}(\alpha/2)\frac{S}{\sqrt{n}}}$

Sol: Null Hypothesis: ~~Ho = 100~~ $H_0: \mu = 100$

i.e, The data are consistent with the assumption of a population mean I.Q. of 100 in the population.

Alternative Hypothesis; $H_1: \mu \neq 100$ (Two tailed)

Test statistic; Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \sim t_{n-1} = t_9.$$

x	$x-\bar{x}$	$(x-\bar{x})^2$
70	-27.2	739.84
120	22.8	519.84
110	12.8	163.84
101	3.8	14.44
88	-9.2	84.64
83	-14.2	201.64
95	-2.2	4.84
98	0.8	0.64
107	9.8	96.04
100	2.8	7.84
$\Sigma x = 972$		1833.6

$$\bar{x} = \frac{\Sigma x}{n} = \frac{972}{10} = 97.2$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{9}(1833.6) = 203.73$$

$$\therefore |t| = \frac{|97.2-100|}{\sqrt{203.73/10}} = \frac{2.8}{\sqrt{20.373}} = \frac{2.8}{4.514} = 0.62$$

Tabulated $t_{0.05}$ for $(10-1)$ d.f. $= 2.262$

Conclusion :- Since calculated t is less than the tabulated $t_{0.05}$ for 9 d.f., H_0 may be accepted at 5% level of significance and we may conclude that the data are consistent with the assumption of mean I.Q. of 100 in the population.

The 95 % Confidence limits within which the mean I.Q. values of samples of 10 boys will lie are given by

$$\bar{x} \pm t_{0.05} S/\sqrt{n} = 97.2 \pm (2.262) \times 4.514$$
$$= 97.2 \pm 10.21.$$
$$= 86.99 \text{ and } 107.41$$

Hence, the required 95% Confidence interval is $[86.99, 107.41]$.

Q: The heights of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches? Test at 5% significance level assuming that for 9 d.f. $P(t > 1.83) = 0.05$.

Sol: Null hypothesis: The average height is 64 inches.
i.e. $H_0: \mu = 64$

Alternative hypothesis: $H_1: \mu > 64$

x	70	67	62	68	61	68	70	64	64	66
$(x-\bar{x})$	4	1	-4	2	-5	2	4	-2	-2	0
$(x-\bar{x})^2$	16	1	16	4	25	4	16	4	4	0

$\bar{x} = \frac{660}{10} = 66.$, $\Sigma(x-\bar{x})^2 = 90$

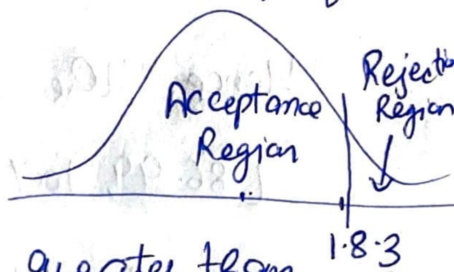
$$S^2 = \frac{1}{n-1} \Sigma(x-\bar{x})^2 = \frac{90}{9} = 10.$$

Test statistic:- Under H_0, the test statistic is

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{66 - 64}{\sqrt{10}/\sqrt{10}} = \cancel{3.16} \quad 2$$

which follows Student's t-distribution with $10-1 = 9$ d.f.

Tabulated value of t for 9 d.f. at 5% level of significance for right tail test is 1.83.



Conclusion: Since calculated value of t is greater than the tabulated value, it is significant.

Hence, H_0 is rejected at 5% level of significance and we conclude that the average height is greater than 64 inches.

Ex: A random sample of 16 values from a normal population showed a mean of 41.5 inches and the sum of squares of deviations from this mean equal to 135 sq inches. Show that the assumption of a mean of 43.5 inches for the population is not reasonable. Obtain 95 percent and 99 percent limits for the same.

~~Sol~~. You may use

$$\nu = 15, \begin{cases} P = 0.05, & t = 2.131 \\ P = 0.01, & t = 2.947. \end{cases}$$

Sol: $n = 16$, $\bar{x} = 41.5$ inches and $\sum (x-\bar{x})^2 = 135$ sq. inches

$$\therefore S^2 = \frac{1}{n-1}\sum (x-\bar{x})^2 = \frac{1}{15}(135) = 9 \Rightarrow S^2 = 9 \Rightarrow S = 3$$

Null hypothesis: $H_0: \mu = 43.5$ inches,

i.e., the mean height in the population is 43.5 inches.

Alternative hypothesis: $H_1: \mu \neq 43.5$ inches.

Test statistic: Under H_0, the test statistic is:

$$t = \frac{\bar{x}-\mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

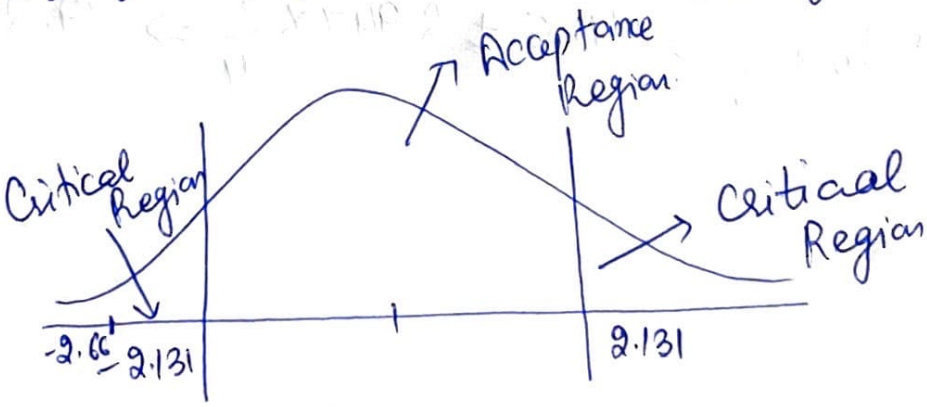
$$\therefore t = \frac{41.5 - 43.5}{3/4} = \frac{-2}{0.75} = -2.667.$$

Here number of d.f. is 15.

We are given $t_{0.05}$ for 15 d.f. $= 2.131$ and $t_{0.01}$ for 15 d.f. $= 2.947$

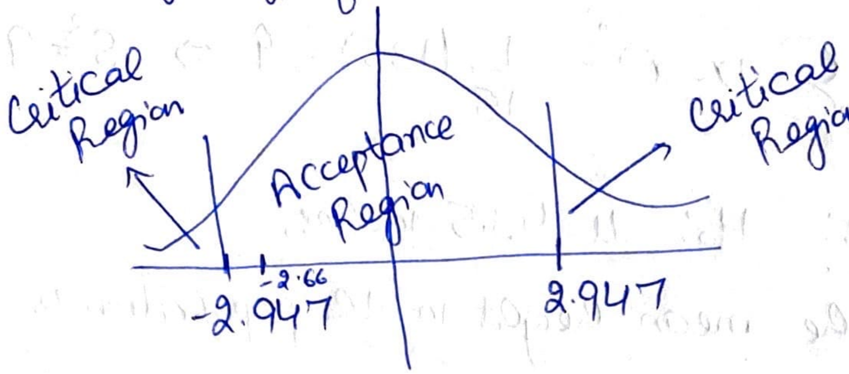
Conclusion

At 5% level of significance,



Since, the calculated value of t lies in the critical region, thus, H_0 is rejected at 5% level of significance. So, we conclude that the assumption of a mean of 43.5 inches for the population is not reasonable

At 1% level of significance,



Since, the calculated value of t lies in the acceptance region, thus, the null hypothesis may be accepted at 1% level of significance.

So, we conclude that the assumption of a mean of 43.5 inches for the population is reasonable.

* 95% Confidence limits for μ: (d.f. = 15)

$$\bar{x} \pm t_{0.05}\frac{S}{\sqrt{n}} = 41.5 \pm 2.131 \times \frac{3}{4} \Rightarrow 39.902 < \mu < 43.098$$

* 99% Confidence limits for μ: (d.f. = 15)

$$\bar{x} \pm t_{0.01}\frac{S}{\sqrt{n}} = 41.5 \pm 2.947 \times \frac{3}{4} \Rightarrow 39.29 < \mu < 43.71.$$

Student's t-test for difference of means

Suppose, we have two independent samples x_i $(i=1, 2, 3, \ldots, n_1)$ and y_j $(j=1, 2, \ldots, n_2)$ of sizes of n_1 and n_2, respectively, drawn from two normal populations with means μ_x and μ_y respectively.

Assumptions of t-test for difference of means

(i) Parent populations, from which the samples have been drawn are normally distributed.

(ii) The two samples are random and independent of each other.

(iii) The population variances are equal and unknown, i.e., $\sigma_x^2 = \sigma_y^2 = \sigma^2$ (say), where σ^2 is unknown.

t-test for difference of means

Step 1: Define Hypothesis.

$H_0: \mu_1 - \mu_2 = D$, where D is some specified tasks that we wish to test.

$H_1: \mu_1 - \mu_2 \neq D$ (Two tailed)

$H_1: \mu_1 - \mu_2 > D$ (right-tailed)

or $H_1: \mu_1 - \mu_2 < D$ (left-tailed)

Step 2:

$$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}},$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j$

and $S^2 = \frac{1}{n_1+n_2-2}\left[\sum_i (x_i-\bar{x})^2 + \sum_j (y_j-\bar{y})^2\right] = \frac{1}{n_1+n_2-2}\left[n_1 s_x^2 + n_2 s_y^2\right]$

is an unbiased estimate of the common population variance σ^2, follows Student's t- ~~test~~ distribution with (n_1+n_2-2) d.f.

Step 3: Conclusion

If $|t| > t_\alpha$, null hypothesis is rejected and if $|t| < t_\alpha$, null hypothesis is accepted at α level of significance.

Ex: Samples of two types of electric light bulbs were tested for length of life and following data were obtained:

	Type I	Type II
Sample No.	$n_1 = 8$	$n_2 = 7$
Sample Means	$\bar{x}_1 = 1234$ hrs	$\bar{x}_2 = 1036$ hrs.
Sample S.D.'s	$s_1 = 36$ hrs	$s_2 = 40$ hrs.

Is the difference in the means sufficient to warrant that type I is superior to type II regarding length of life? (The value of t at 5% level of significance for $\nu = 13$ is 1.77)

Sol: Null hypothesis, H_0: $\mu_1 = \mu_2$, i.e., the two types I and II of electric bulbs are identical.

Alternative hypothesis, H_1: $\mu_1 > \mu_2$, i.e., type I is superior to type II.

Test statistic: Under H_0, the test statistic is

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{n_1+n_2-2} = t_{13},$$

where
$$S^2 = \frac{1}{n_1+n_2-2}\left[\sum (x_i - \bar{x}_1)^2 + \sum (x_2 - \bar{x}_2)^2\right]$$

$$= \frac{1}{13}\left[n_1 s_1^2 + n_2 s_2^2\right] = \frac{1}{13}\left[8 \times (36)^2 + 7 \times (40)^2\right]$$

$$= \frac{1}{13}\left[10368 + 11200\right]$$

$$= 1659.08$$

$$t = \frac{1234-1036}{\sqrt{1659.08\left(\frac{1}{8}+\frac{1}{7}\right)}} = \frac{198}{\sqrt{1659.08 \times 0.2679}}$$

$$= \frac{198}{21.0824} = 9.39$$

Now $t_{0.05}(13) = 1.77$ for single tailed test. [$t_{0.10}(13) = 1.77$ for two tailed test]

Conclusion: Since calculated t is much greater than tabulated t, it is higly significant and H_0 is rejected.

Hence, we conclude that type I is superior to type II.

Ex: The heights of six randomly choosen sailors are (in inches): 63, 65, 68, 69, 71 and 72. Those of 10 randomly choosen soldiers are 61, 62, 65, 66, 69, 69, 70, 71, 72 and 73. Discuss, the suggestion that sailors are on the average taller than soldiers. (The value of t at 5% level of significance for $\nu = 14$ is 1.76)

Sol: If the heights of sailors and soldiers be represented by the variables X and Y respectively then the

Null hypothesis, $H_0: \mu_X = \mu_Y$, i.e., the ~~soldiers~~ sailors are not on the average taller than the soldiers.

Alternative hypothesis, $H_1: \mu_X > \mu_Y$ (right-tailed), i.e., the sailors are on the average taller than soldiers.

Under H_0, the test statistic is

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim t_{n_1+n_2-2} = t_{14}.$$

X	$X-\bar{X}$	$(X-\bar{X})^2$
63	-5	25
65	-3	9
68	0	0
69	1	1
71	3	9
72	4	16
$\Sigma X = 408$		$\Sigma(X-\bar{X})^2 = 60$

Y	$Y-\bar{Y}$	$(Y-\bar{Y})^2$
61	-6.8	46.24
62	-5.8	31.9
65	-2.8	7.84
66	-1.8	3.24
69	1.2	1.44
69	1.2	1.44
70	2.2	4.84
71	3.2	10.24
72	4.2	17.64
73	5.2	27.04
$\Sigma Y = 678$		$\Sigma(Y-\bar{Y})^2 = 151.86$

$\bar{X} = \frac{408}{6} = 68$, $\bar{Y} = \frac{678}{10} = 67.8$

$\Sigma(X-\bar{X})^2 = 60$, $\Sigma(Y-\bar{Y})^2 = 151.86$

$$S^2 = \frac{1}{n_1+n_2-2}\left[\Sigma(X-\bar{X})^2 + \Sigma(Y-\bar{Y})^2\right] = \frac{1}{14}[211.86]$$

$= 15.1328$

$$\therefore t = \frac{0.2}{\sqrt{15.1328 \times 0.2667}} = \frac{0.2}{\sqrt{4.0359}} = \frac{0.2}{2.0089} = 0.099$$

Now $t_{0.05}(14) = 1.76$ for single-tail test.

Since, calculated t is less than tabulated t,

Conclusion: it is not significant at 5% level of significance.

Hence, null hypothesis may be accepted at 5% level of significance. We conclude that the data are inconsistent with the suggestion that sailors are on average taller than soldiers.

Ex: In a certain experiment to compare two types of animal foods A and B, the following results of increase in weights were observed in animals:

Animal numbers		1	2	3	4	5	6	7	8	Total
Increase weight in lb	Food A	49	53	51	52	47	50	52	53	407
	Food B	52	55	52	53	50	54	54	53	423

Assuming that the samples are independent, can we conclude that food B is better than food A? ($t_{0.05}(14) = 1.76$ for one tail)

Sol: Null Hypothesis: $H_0: \mu_x = \mu_y$, i.e., there is no significant difference in increase in weights due to diets A and B, where the increase in weights due to foods A and B are denoted by X and Y, respectively.

Alternative Hypothesis: $H_1:$ ~~$\mu_x \neq \mu_y$~~ $\mu_x < \mu_y$ (left-tailed)

Test statistic: Under $H_0: \mu_x = \mu_y$, the test statistic is

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{n_1+n_2-2} = t_{14},$$

where $s^2 = \frac{1}{n_1+n_2-2}\left[\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2\right]$

$$= \frac{1}{14}\left[30.878 + 16.878\right]$$

$$= \frac{47.756}{14}$$

$$= 3.4111$$

X	$X-\bar{X}$	$(X-\bar{X})^2$	Y	$Y-\bar{Y}$	$(Y-\bar{Y})^2$
49	-1.875	3.516	52	-0.875	0.766
53	2.125	4.516	55	2.125	4.516
51	0.125	0.016	52	-0.875	0.766
52	1.125	1.266	53	0.125	0.016
47	-3.875	15.016	50	-2.875	8.266
50	-0.875	0.766	54	1.125	1.266
52	1.125	1.266	54	1.125	1.266
53	2.125	4.516	53	0.125	0.016
407		30.878			16.878

$$\bar{X} = \frac{407}{8} = 50.875, \quad \bar{Y} = \frac{423}{8} = 52.875$$

$$S^2 = \frac{1}{14}\left[47.756\right] = \cancel{4.111}\ 3.4111$$

$$\therefore t = \frac{-2}{\sqrt{3.4111\left(\frac{1}{8}+\frac{1}{8}\right)}} = \frac{-2}{\sqrt{0.8528}} = \frac{-2}{0.9235} = -2.166$$

Now, $t_{0.05}(14) = 1.76$ for one-tail test.

Conclusion: Since calculated $|t| > t_{0.05}(14)$.

Hence, H_0 is rejected at 5% level of significance.

Hence, we conclude that the foods A and B differ significantly and further, since $\mu_X < \mu_Y$, food B is superior to food A.

Paired t-test for difference of Means

Consider the case (i) Sample sizes are equal, i.e; $n_1 = n_2 = n$ (say)

(ii) The samples are not independent but sample observations are paired together.

Under Ho, the test statistic is

$$t = \frac{\bar{d}}{S/\sqrt{n}}, \quad \text{where } \bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i, \; d_i = x_i - y_i,$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2$$

follows t-distribution with $(n-1)$ d.f.

Ex In a certain experiment to compare two types of animal foods A and B, the following results of increase in weights were obtained in animals:

Animal number		1	2	3	4	5	6	7	8	Total
Increase in weight	Food A	49	53	51	52	47	50	52	53	407
	Food B	52	55	52	53	50	54	54	53	423

Assuming that the same set of eight animals were used in both the foods, can we conclude that food B is better than food A?

Sol: Let the increase in weights ~~be~~ due to foods A and B are denoted by X and Y.

Ho: $\mu_X = \mu_Y$, i.e., there is no significant difference in increase in weights due to foods A and B.

$H_1: \mu_Y > \mu_X$ or $\mu_X < \mu_Y \Rightarrow H_1: \mu_X - \mu_Y < 0$.

X	Y	$d_i = X - Y$	$(d_i - \bar{d})^2$
49	52	-3	1
53	55	-2	0
51	52	-1	1
52	53	-1	1
47	50	-3	1
50	54	-4	4
52	54	-2	0
53	53	0	4
		-16	12

$$\bar{d} = \frac{-16}{8} = -2, \quad s^2 = \frac{1}{7}(12) = 1.714$$

Under Ho,

$$t = \frac{-2}{\sqrt{1.714/8}} = \frac{-2}{0.46} = -4.34$$

$$\Rightarrow |t| = 4.34$$

$t_{0.05}$ for 7 d.f. = 1.895

We reject the null hypothesis and conclude that food B is superior to food A.

Z-test for single mean

When $n \geq 30$
– Apply Z-test
When $n < 30$
– Apply t-test.

Step 1 : Define H_0 and H_1.

Step 2 : Test statistic :

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$ [Population σ is known]

$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim N(0,1)$$ [Population σ is unknown].

$\because$ When sample is large $\sigma^2 = s^2$

$\left(S.E. = \frac{\sigma}{\sqrt{n}}\right)$

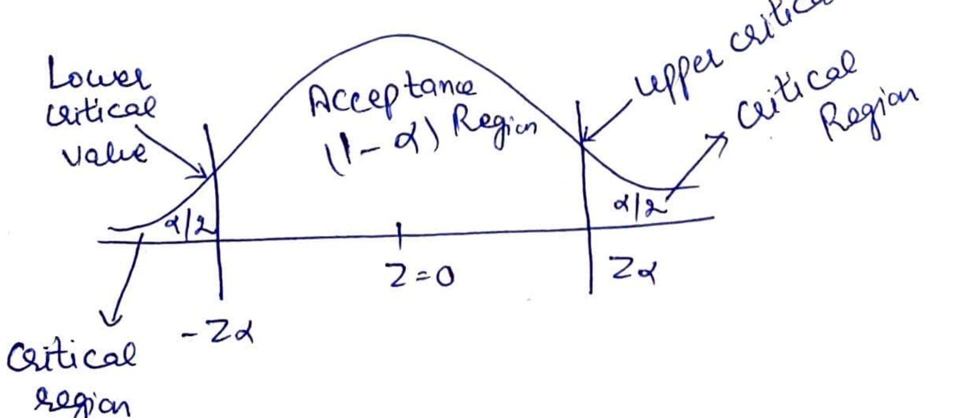
Step 3 Conclusion

If Calculated $|Z| > Z_\alpha \Rightarrow$ Reject H_0

If calculated $|Z| < Z_\alpha \Rightarrow$ Accept H_0

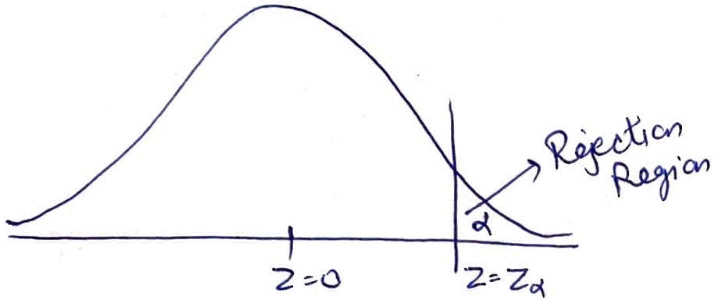
Critical values (Z_α)

Critical value (Z_α)	Level of Significance (α) 1%	5%	10%
Two tailed test	$\lvert Z_\alpha \rvert = 2.58$	$\lvert Z_\alpha \rvert = 1.96$	$\lvert Z_\alpha \rvert = 1.645$
Right tailed test	$Z_\alpha = 2.33$	$Z_\alpha = 1.645$	$Z_\alpha = 1.28$
Left tailed Test	$Z_\alpha = -2.33$	$Z_\alpha = -1.645$	$Z_\alpha = -1.28$

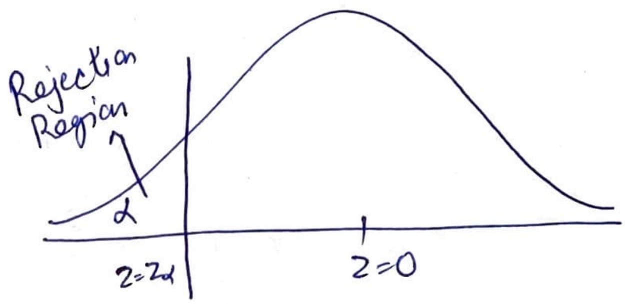


Two-tailed test

Right tailed Test



Left tailed Test



Confidence Limits | Fiducial Limits

$$\cancel{\bar{x} \leq} \quad \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Ex A random sample of 100 recorded deaths in the United states during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance?

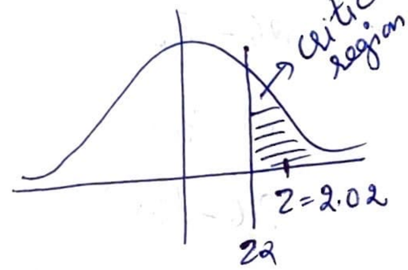
Sol: $H_0: \mu = 70$, i.e, average life span is 70 years.

$H_1: \mu > 70$

$n = 100$, $\bar{x} = 71.8$, $\sigma = 8.9$

Under Ho, The test statistic is

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{71.8 - 70}{8.9/\sqrt{100}} = \frac{1.8}{0.89} = 2.02$$



$Z_\alpha = 1.645$

Since calculated $Z > Z_\alpha$.

Conclusion: We reject Ho and Conclude that the mean life span today is greater than 70 years.

Ex: An insurance agent has claimed that the average age of policy holders who insure ~~them~~ through him is less than the average for all agents, which is 30.5 years.

A random sample of 100 policy holders who had insured through him gave the following age distribution:

Age :	16-20	21-25	26-30	31-35	36-40
No. of persons :	12	22	20	30	16

Test the claim at 5% level of significance.
[Given $Z(1.645) = 0.95$]

Sol: Null Hypothesis, $H_0: \mu = 30.5$, i.e., the sample mean and population mean do not differ significantly.

Alternative hypothesis, $H_1: \mu < 30.5$ (Left tailed)

Age	No. of persons (f)	x (Mid point)	fx	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
16-20	12	18	216	-10.8	116.64	1399.68
21-25	22	23	506	-5.8	33.64	740.08
26-30	20	28	560	-0.8	0.64	12.8
31-35	30	33	990	4.2	17.64	529.2
36-40	16	38	~~59~~ 608	9.2	84.64	1354.24
			2880			4036

$\cancel{\bar{x} = 140}$

$$\bar{x} = \frac{2880}{100} = 28.8$$

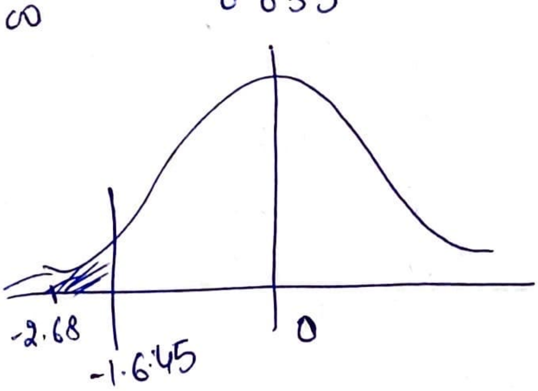
$$s^2 = \frac{\sum f(x-\bar{x})^2}{n} = \frac{4036}{100} = 40.36 \Rightarrow s = 6.35$$

Test statistic, Under H_0, test statistic is

$$Z = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{28.8-30.5}{6.35/\sqrt{100}} = \frac{-1.7}{0.635} \quad \cancel{= 2.8}$$

$$\Rightarrow Z = -2.68$$

$$Z_\alpha = 1.645$$



Conclusion, Since $|Z| > Z_\alpha$, we reject H_0 at 5% level of significance.

We conclude that insurance agent's claim is valid.

Ex: A sample of 900 members has a mean 3.4 cms and s.d. 2.61 cms. Is the sample drawn from a large population of mean 3.25 cms? If the population is normal and its mean is unknown, and s.d. 2.61cm find the 95% and 98% fiducial limits of true mean.

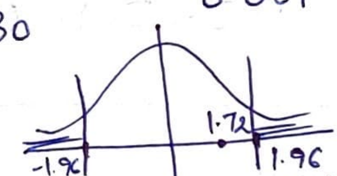
Sol: <u>Null hypothesis</u>: $H_0: \mu = 3.25$, i.e., the sample has been drawn from the population with mean 3.25 cms.

<u>Alternative hypothesis</u>: $H_1: \mu \neq 3.25$ (Two-tailed)

<u>Test-statistic</u>: Under H_0, test statistic is

$$Z = \frac{3.4 - 3.25}{2.61/\sqrt{900}} = \frac{3.4 - 3.25}{2.61/30} = \frac{0.15}{0.087}$$

$$\Rightarrow Z = 1.72$$



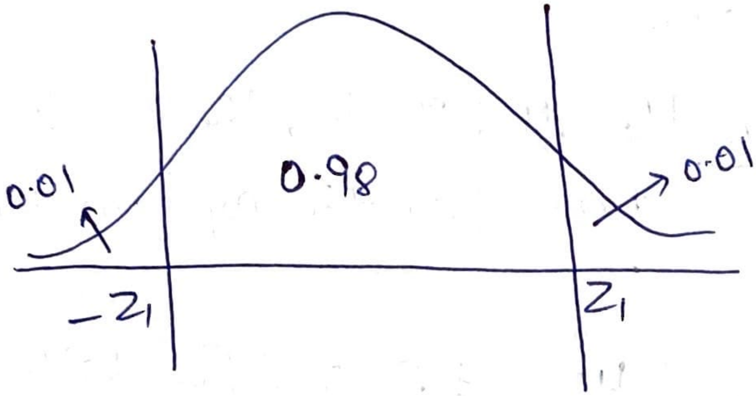
<u>Conclusion</u>: Since $|z| < 1.96$, we accept the null hypothesis at 5% level of significance. We conclude that the sample has been drawn from the population with mean 3.25 cms.

<u>95% fiducial limits</u>: $\bar{x} \pm \frac{\sigma}{\sqrt{n}} 1.96$

$$= 3.4 \pm 1.96 \frac{(2.61)}{\sqrt{900}} = 3.4 \pm 0.17052$$

95% Confidence interval is $[3.2295, 3.5705]$.

<u>98% fiducial limits</u> $\bar{x} \pm \frac{\sigma}{\sqrt{n}} (2.33)$



$P(|z| < z_1) = 0.98$

$\Rightarrow P(-z_1 < z < z_1) = 0.98$

$\Rightarrow P(z < z_1) = 0.99 \Rightarrow z_1 = 2.33$

98% Condifendence interval: $3.4 \pm \frac{2.33(2.61)}{\sqrt{900}}$

$= 3.4 \pm 0.2027$

$= 3.1973, 3.6027$

98% Confidence interval is
$[3.1973, 3.6027]$.

Ex → The mean muscular endurance score of a random sample of 60 subjects was found to be 145 with a s.d. of 40. Construct a 95% Confidence interval for the true mean. Assume the sample size to be large enough for normal ~~population~~ approximation. What size of sample is required to estimate the mean within 5 of the true mean with a 95% Confidence?

Sol: $n = 60,\ \bar{x} = 145,\ s = 40.$

95% Confidence interval for true mean is

$$\bar{x} \pm 1.96 \frac{s}{\sqrt{n}} = 145 \pm \frac{1.96(40)}{\sqrt{60}}$$

$$= 145 \pm \frac{78.4}{7.746} = 145 \pm 10.121$$

$$= 134.88, 155.12$$

Hence, 95% Confidence interval for μ is

$(134.88, 155.12)$

Now, $|\bar{x} - \mu| < 5$, $Z_{0.05} = 1.96$, $\hat{\sigma} = s = 40.$

$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}} \Rightarrow \frac{s}{\sqrt{n}} = \frac{\bar{x} - \mu}{Z}$$

$$\Rightarrow \frac{\sqrt{n}}{s} = \frac{Z}{\bar{x} - \mu}$$

$$\Rightarrow n = \frac{Z^2 s^2}{(\bar{x} - \mu)^2}$$

$$\Rightarrow n = \left(\frac{Z_\alpha s}{\bar{x} - \mu} \right)^2$$

$$= \left(\frac{1.96 \times 40}{5} \right)^2$$

$$= 245.86$$

$$\simeq 246$$

$\boxed{n = 246}$

Z-test for difference of means

(1) Define $H_0: \mu_1 - \mu_2 = d$, where d is some specified difference that we wish to test.

$H_1: \mu_1 - \mu_2 \neq d$ (Two tailed)

$\mu_1 - \mu_2 > d$ or $\mu_1 - \mu_2 < d$ (One tailed)

(2) Test statistic :

$$Z = \frac{\bar{x}_1 - \bar{x}_2 - d}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

(If σ_1, σ_2 are known)

If σ_1 and σ_2 are unknown,

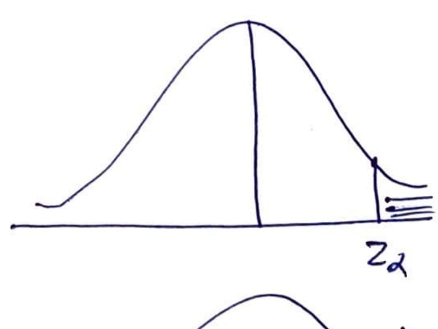
$$Z = \frac{\bar{x}_1 - \bar{x}_2 - d}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim N(0,1)$$

[$\because$ For large samples, $\hat{\sigma}_1^2 = s_1^2$, $\hat{\sigma}_2^2 = s_2^2$]

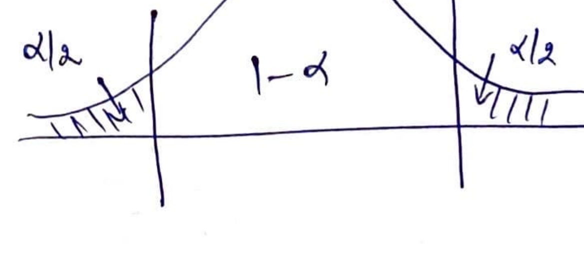
(3) Rejection Region

If $|Z| > Z_\alpha \Rightarrow$ Reject H_0

If $|Z| < Z_\alpha \Rightarrow$ Accept H_0.



(4) Draw Conclusion.



Ex The means of two single large samples of 1,000 and 2,000 members are 67.5 inches and 68.0 inches respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 inches? (Test at 5% level of significance)

Sol: Given, $n_1 = 1000$, $n_2 = 2000$, $\bar{x}_1 = 67.5$, $\bar{x}_2 = 68$, $\sigma = 2.5$

Null Hypothesis: $H_0: \mu_1 = \mu_2$ and $\sigma = 2.5$, i.e., samples are drawn from the same population of S.D. 2.5 inches.

Alternative hypothesis, $H_1: \mu_1 \neq \mu_2$ (Two tailed)

Test statistic: Under H_0, test statistic is

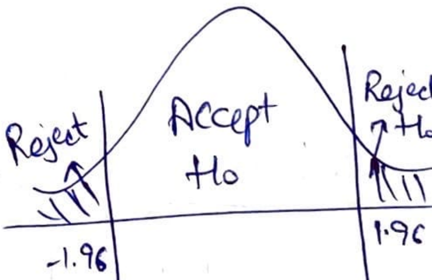
$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

$$\Rightarrow Z = \frac{67.5 - 68}{2.5\sqrt{\frac{1}{1000} + \frac{1}{2000}}} = \frac{-0.5}{2.5 \times 0.0387} = \frac{-0.5}{0.0968}$$

$$= -5.165$$

Conclusion: Since $|Z| > 1.96$,

we reject H_0. We conclude that the samples are not from the same population with S.D. 2.5 inches.



Ex: In a survey of buying habits, 400 women shoppers are chosen at random in super market 'A' located in a certain section of the city. Their average weekly food expenditure is Rs. 250 with a S.D. of Rs. 40.

For 400 women shoppers choosen at random in super market B in another section of the city, the average weekly food expenditure is Rs. 220 with S.D. of Rs. 55. Test at 1% level of significance whether the average weekly food expenditure of two populations of shoppers are equal.

Sol: $n_1 = 400$, $n_2 = 400$, $\bar{x}_1 = 250$, $\bar{x}_2 = 220$, $s_1 = 40$, $s_2 = 55$.

Null hypothesis, $H_0: \mu_1 = \mu_2$, i.e., the average weekly expenditures of two populations of shoppers are equal.

Alternative hypothesis: $H_1: \mu_1 \neq \mu_2$ (Two tailed)

Test statistic: Under H_0, the test statistic is

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Since, σ_1^2 and σ_2^2 are unknown, we can take $\hat{\sigma}_1^2 = s_1^2$, $\hat{\sigma}_2^2 = s_2^2$ for large samples.

$$Z = \frac{250 - 220}{\sqrt{\frac{(40)^2}{400} + \frac{(55)^2}{400}}} = \frac{30}{3.4}$$

$$= 8.82$$

Conclusion, Since, $|Z| > 2.58$, the null hypothesis is rejected at 1% level of significance.

We conclude that the average weekly food expenditure of two populations differ significantly.

Ex: The average hourly wage of a sample of 150 workers in a plant A was Rs. 2.56 with a S.D. of Rs. 1.08. The average hourly wage of a sample of 200 workers in plant B was Rs. 2.87 with a S.D. of Rs. 1.28. Can an applicant safely assume that the hourly wages paid by plant B are higher than those paid by plant A?

Sol: $n_1 = 150$, ~~$s_1 = 2.56$~~, $\bar{x}_1 = 2.56$, $s_1 = 1.08 = \hat{\sigma}_1$
$n_2 = 200$, $\bar{x}_2 = 2.87$, $s_2 = 1.28 = \hat{\sigma}_2$ } since samples are large

Null hypothesis, ~~$H_0: \mu_1 > \mu_2$~~ $H_0: \mu_1 = \mu_2$, i.e., there is no significant difference between hourly wages paid by plant A and plant B.

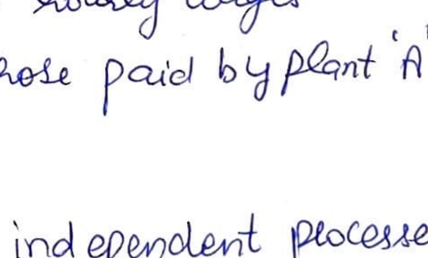
Alternative hypothesis, $H_1: \mu_2 > \mu_1$ or $\mu_1 < \mu_2$ (left tailed)

Test statistic, Under H_0, the test statistic is

$$Z = \frac{2.56 - 2.87}{\sqrt{\frac{(1.08)^2}{150} + \frac{(1.28)^2}{200}}} = \frac{-0.31}{\sqrt{0.0078 + 0.0082}} = \frac{-0.31}{0.126} = -2.46.$$

Conclusion: Since calculated

$|Z| > 1.645$,



we reject H_0 at 5% level of significance. We conclude that average hourly wages paid by ~~paid~~ plant 'B' are higher than those paid by plant 'A'.

Ex: In a certain factory there are two independent processes manufacturing the same item. The average weight in a sample of 250 items produced from one process is found to be 120 ~~ounces~~ gm with S.D. of 12 gm. While the corresponding figures in a sample of 400 items from the other process are 124 and 14. Obtain the standard error of difference between ~~to~~ the sample means. Is this difference significant? Also find the 99% confidence limits for the difference in average weights of items produced by the two processes resp.

Sol: Given, $n_1 = 250$, $\bar{x}_1 = 120$, $s_1 = 12 = \hat{\sigma}_1$, $n_2 = 400$, $\bar{x}_2 = 124$, $s_2 = 14 = \hat{\sigma}_2$ } since samples are large

$$S.E.(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{(12)^2}{250} + \frac{(14)^2}{400}}$$

$$= \sqrt{0.576 + 0.49}$$

$$= 1.03$$

Null hypothesis, $H_0: \mu_1 = \mu_2$, i.e, the sample means do not differ significantly.

Alternative hypothesis: $H_1: \mu_1 \neq \mu_2$ (two tailed test)

Test statistic, under H_0, the test statistic is

$$Z = \frac{120 - 124}{1.03} = -3.88$$

$$|Z| = 3.88$$

Conclusion: Since calculated $|z| > 2.58$, we reject H_0 at 1% level of significance and conclude that there is a significant difference between the sample means.

99% Confidence limits for $|\mu_1 - \mu_2|$ are

$$|\bar{x}_1 - \bar{x}_2| \pm 2.58\, S.E.(\bar{x}_1 - \bar{x}_2) = 4 \pm 2.58(1.03) = 4 \pm 2.66$$

$$= 1.34, 6.66$$

$$\therefore \quad 1.34 < |\mu_1 - \mu_2| < 6.66.$$

F-test for equality of two population variances

① Null hypothesis – $H_0: \sigma_X^2 = \sigma_Y^2 = \sigma^2$;
i.e., the population variances are equal or two independent estimates of the population variances are homogeneous.
Alternative hypothesis H_1: $\sigma_X^2 \neq \sigma_Y^2$ or $\sigma_X^2 > \sigma_Y^2$ or $\sigma_X^2 < \sigma_Y^2$

② Test statistic :-

Under H_0, the test statistic is given by

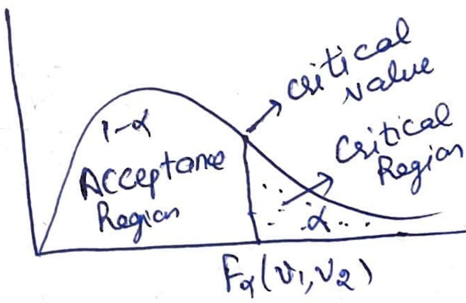
$$F = \frac{S_X^2}{S_Y^2} \qquad (S_X^2 > S_Y^2)$$

where $S_X^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n} (x_i - \bar{x})^2$, $S_Y^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n} (y_j - \bar{y})^2$

are unbiased estimates of the common population variances σ^2 obtained from two independent samples and follows F-distribution with (ν_1, ν_2) d.f., where $\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$.

③ Critical Region :-

If Calculated $F < F_\alpha(\nu_1, \nu_2)$, we accept H_0 and if calculated $F > F_\alpha(\nu_1, \nu_2)$, we reject H_0.



or

$$F_\alpha(\nu_1, \nu_2) = \frac{1}{F_{1-\alpha}(\nu_2, \nu_1)}$$

The critical values of F for left tail test are $F < F_{\nu_1,\nu_2}(1-\alpha)$ and for two tailed test are $F > F_{\nu_1,\nu_2}(\alpha/2)$ and $F < F_{\nu_1,\nu_2}(1-\alpha/2)$,
for right tailed test are $F > F_{\nu_1,\nu_2}(\alpha)$

Ex: Pumpkins were grown under two experimental conditions. Two random samples of 11 and 9 pumpkins show the sample S.D. of their weights as 0.8 and 0.5 respectively. Assuming that the weight distributions are normal, test the hypothesis that the true variances are equal, against the alternative that they are not, at the 10% level.

~~Assume that $P[F_{10,8} \geq 3.35]$ and $P[F_{8,10} \geq 3.07] = 0.05$.~~

Assume that $P(F_{10,8} \geq 3.35) = 0.05$ and $P(F_{8,10} \geq 3.07) = 0.05$.

Sol: Null hypothesis, $H_0: \sigma_X^2 = \sigma_Y^2$, i.e., two population variances are equal.

Alternative hypothesis, $H_1: \sigma_X^2 \neq \sigma_Y^2$.

Test statistic: Under H_0, the test statistic is

$$F = \frac{S_X^2}{S_Y^2} \text{ follows } F \text{ distribution with } (n_1-1, n_2-1) \text{ d.f.}$$

Now, $n_1 s_X^2 = (n_1 - 1) S_X^2 \Rightarrow (11)(0.8)^2 = 10 S_X^2$

$\Rightarrow S_X^2 = 0.704$

and $n_2 s_Y^2 = (n_2 - 1) S_Y^2 \Rightarrow 9(0.5)^2 = 8 S_Y^2$

$\Rightarrow S_Y^2 = 0.28125.$

$$\therefore F = \frac{0.704}{0.28125} = 2.5$$

The significant values of F for two tailed test at level of significance $\alpha = 0.10$ are

$$F > F_{10,8}(\alpha/2) = F_{10,8}(0.05)$$

and $F < F_{10,8}(1-\alpha/2) = F_{10,8}(0.95)$

Given, $P(F_{10,8} \geq 3.35) = 0.05 \Rightarrow F_{10,8}(0.05) = 3.35$

and $P(F_{8,10} \geq 3.07) = 0.05 \Rightarrow P\left(\frac{1}{F_{8,10}} \leq \frac{1}{3.07}\right) = 0.05$

$\Rightarrow P(F_{10,8} \leq 0.326) = 0.05$

$\Rightarrow P(F_{10,8} \geq 0.326) = 0.95$

$\Rightarrow F_{10,8}(0.95) = 0.326$

Hence, the critical values for testing $H_0: \sigma_x^2 = \sigma_y^2$ against $H_1: \sigma_x^2 \neq \sigma_y^2$ at 10% level of significance are

$F > 3.35$ and $F < 0.33$

Conclusion:
Since, the calculated value of $F = 2.5$ lies between 0.33 and 3.35, we accept H_0 at 10% level of significance.
We conclude that the two variances of populations may be equal.

Ex In one sample of 8 observations, the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in the other sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level. Given that the 5% point of F for $n_1 = 7$, $n_2 = 9$ d.f. is 3.29.

Sol: ~~$n_1 = 8$, $\bar{x}_1 = 84.4$, $n_2 = 10$, $\bar{x}_2 = 102.6$~~

$n_1 = 8$, $n_2 = 10$, $\sum(x-\bar{x})^2 = 84.4$, $\sum(y-\bar{y})^2 = 102.6$

$H_0: \sigma_x^2 = \sigma_y^2 = \sigma^2$ i.e., the estimates of σ^2 given by the samples are homogeneous.

$H_1: \sigma_X^2 \neq \sigma_Y^2$

Test statistic: The test statistic F, under H_0 is

$$F = \frac{S_X^2}{S_Y^2},$$

where $S_X^2 = \frac{1}{n_1 - 1} \Sigma (x-\bar{x})^2 = \frac{1}{7}(84.4) = 12.057$

$$S_Y^2 = \frac{1}{n_2 - 1} \Sigma (y-\bar{y})^2 = \frac{1}{9}(102.6) = 11.4$$

$$\therefore F = \frac{12.057}{11.4} = 1.058.$$

Now tabulated $F_{(7,9)}(0.05) = 3.29$.

Conclusion: Since Calculated $F <$ tabulated $F_{(7,9)}(0.05)$,

we may accept H_0 at 5% level of significance.

We conclude that the estimates of σ^2 given by the samples are homogeneous.

Ex: Two random samples gave the following results:

Sample	Size	Sample mean	Sum of square deviations from the mean
1	10	15	90
2	12	14	108

Test whether the samples come from the same normal population at 5% level of significance.

[Given, $F_{0.05}(9,11) = 2.90$, $F_{0.05}(11,9) = 3.10$, $t_{0.05}(20) = 2.086$, $t_{0.05}(22) = 2.07$]

Sol: A normal population has two parameters, mean μ and variance σ^2.

To test if two independent samples have been drawn from the same normal ~~distribution~~ population, we have to test,

(i) The equality of population means

(ii) The equality of population variances.

Null hypothesis : The two samples have been drawn from the same normal population, i.e.,

$H_0: \mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$.

Equality of means will be tested by applying t-test and equality of variances will be tested by applying F-test.

Since, t-test assumes $\sigma_1^2 = \sigma_2^2$, we shall first apply F-test and then t-test.

$n_1 = 10$, $n_2 = 12$, $\bar{x}_1 = 15$, $\bar{x}_2 = 14$, $\sum_{i=1}^{n_1}(x_i - \bar{x}_1)^2 = 90$, $\sum_{j=1}^{n_2}(x_j - \bar{x}_2)^2 = 108$.

$$S_1^2 = \frac{1}{n_1 - 1}\sum(x_i - (\bar{x}_1)^2) = \frac{90}{9} = 10$$

$$S_2^2 = \frac{1}{n_2 - 1}\sum(x_j - \bar{x}_2)^2 = \frac{108}{11} = 9.82$$

Since $S_1^2 > S_2^2$, under $H_0: \sigma_1^2 = \sigma_2^2$, the test statistic is

$$F = \frac{S_1^2}{S_2^2} = \frac{10}{9.82} = 1.018$$

Tabulated $F_{0.05}(9,11) = 2.90$.

Since, calculated $F <$ Tabulated F, we accept H_0 for equality of variances.

Since $\sigma_1^2 = \sigma_2^2$, we can now apply t-test for testing

$H_0: \mu_1 = \mu_2$.

t-test

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S^2}{n_1} + \frac{S^2}{n_2}}} \sim t_{n_1+n_2-2} = t_{20},$$

where $$S^2 = \frac{1}{n_1+n_2-2}\left[\Sigma(x_i-\bar{x}_1)^2 + \Sigma(y_j-\bar{x}_2)^2\right]$$

$$= \frac{1}{20}\left[90+108\right]$$

$$= 9.9$$

$$\therefore\ t = \frac{15-14}{\sqrt{9.9\left(\frac{1}{10}+\frac{1}{12}\right)}} = \frac{1}{\sqrt{1.815}} = \frac{1}{1.347} = 0.742$$

Tabulated $t_{0.05}$ for 20 d.f. $= 2.086$.

Since $|t| <$ tabulated $t_{0.05}$, we accept H_0 for equality of means.

Since both $H_0: \mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$ are accepted, we may conclude that the sample have been drawn from the same normal population.

χ^2- test of goodness of fit

If f_i $(i=1,2,\dots,n)$ is a set of observed (experimental) frequencies and e_i $(i=1,2,\dots,n)$ is the corresponding set of expected (theoretical) frequencies, then

$$\chi^2 = \sum_{i=1}^{n} \frac{(f_i - e_i)^2}{e_i}, \quad \left[\sum_{i=1}^{n} f_i = \sum_{i=1}^{n} e_i\right]$$

follows χ^2 distribution with $(n-1)$ d.f.

<u>Remark</u> :- 1. Chi-square is an approximate test for large values of n. Conditions for the validity of the χ^2-test of goodness of fit are

(i) The sample observations should be independent.

(ii) $\sum_{i=1}^{n} f_i = \sum_{i=1}^{n} e_i$

(iii) N, the total frequency should be greater than 50.

(iv) No e_i $(i=1,2,\dots,n)$ should be less than 5.

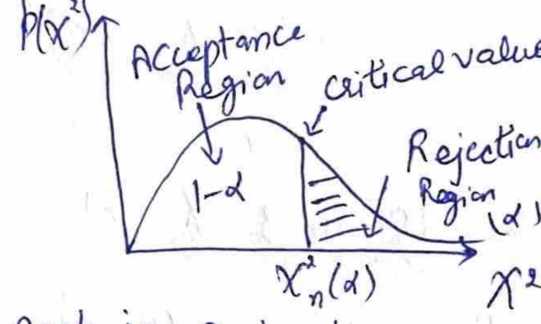
If any e_i $(i=1,2,\dots,n)$ is less than 5, then for the application of χ^2- test, it is pooled with the preceding or succeeding frequency so that the pooled frequency is more than 5 and finally adjust for the d.f. lost in pooling.

2. Since, χ^2-test depends only on the set of observed and expected frequencies and on d.f. It does not make any assumption about the parent population, thus this test is known as Non- Parametric Test or Distribution -Free test.

Decision Rule : If $\chi^2 \le \chi^2_\alpha(n-1) \Rightarrow$ Accept H_0.

If $\chi^2 > \chi^2_\alpha(n-1) \Rightarrow$ Reject H_0,

where χ^2 is the calculated value and $\chi^2_\alpha(n-1)$ is the tabulated value of χ^2 for $(n-1)$ d.f. and α level of significance.



* Expected frequencies: Np, the total frequency and p is the probability.

Ex The demand for a particular spare part in a factory was found to vary from day to day. In a sample study the following information was obtained:

Days	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
No. of parts demanded :	1124	1125	1110	1120	1126	1115

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

(Given: the values of chi-square significance at 5, 6, 7, d.f. are resp. 11.07, 12.59, 14.07 at 5% level of significance.)

Sol: H_0: Null hypothesis : The number of parts demanded does not depend on the day of the week.

Under H_0, the expected frequencies of the spare part demanded on each of the six days would be:

$$e_i = Np = \frac{1}{6}(1124 + 1125 + 1110 + 1120 + 1126 + 1115)$$

$$= \frac{6720}{6} = 1120$$

Days	Frequency Observed (f_i)	Frequency Expected (e_i)	$(f_i - e_i)^2$	$\frac{(f_i - e_i)^2}{e_i}$
Mon	1124	1120	16	0.014
Tues	1125	1120	25	0.022
Wed	1110	1120	100	0.089
Thur	1120	1120	0	0
Fri	1126	1120	36	0.032
Sat	1115	1120	25	0.022
Total	6720	6720		~~0.0179~~ 0.179

$$\chi^2 = \Sigma \frac{(f_i - e_i)^2}{e_i} = 0.179$$

d.f. $= 6 - 1 = 5$ [since we are given 6 frequencies subjected to only one constraint: $\Sigma f_i = \Sigma e_i = 6720$]

The tabulated $\chi^2_{0.05}$ for 5 d.f. $= 11.07$.

Since, the calculated value is less than tabulated value, it is not significant and we accept the null hypothesis at 5% level of significance. Hence, we conclude that the number of parts demanded are same over the 6-day period.

Ex: A sample analysis of examination results of 200 MBA's was made. It was found that 46 students had failed, 68 secured a third division, 62 secured a second division and the rest were placed in first division.

Are these figures agree with the general examination result which is in the ratio of 4:3:2:1 for various categories respectively?

Sol: Null hypothesis: The given data agree with the general examination result which is in the ratio of 4:3:2:1 for various categories.

Under H_0, the expected frequencies are

Category	Observed (f_i)	Expected (e_i)	$(f_i-e_i)^2$	$\frac{(f_i-e_i)^2}{e_i}$
Failed	46	$\frac{4}{10}\times 200 = 80$	1156	14.45
IIIrd division	68	$\frac{3}{10}\times 200 = 60$	64	1.067
2nd division	62	$\frac{2}{10}\times 200 = 40$	484	12.1
1st division	24	$\frac{1}{10}\times 200 = 20$	16	0.8
Total	200	200		28.417

$$\chi^2 = \Sigma \frac{(f_i-e_i)^2}{e_i} = 28.417$$

d.f. = 3

$\chi^2_{0.05}$ for 3 d.f. = 7.815

Since calculated χ^2 is greater than tabulated $\chi^2_{0.05}$ for 3 d.f., it is significant and the null hypothesis is rejected at 5% level of significance. Hences, we may conclude that data ~~are~~ ~~not~~ do not agree with the general examination result.

Ex: A survey of 800 families with four children each revealed the following distribution:

No. of boys	0	1	2	3	4
No. of girls	4	3	2	1	0
No. of families	32	178	290	236	64

Is this result consistent with the hypothesis that male and female births are equally probable?

Sol: Null hypothesis: The male and female births are equally probable.

Under the null hypothesis:

$p =$ Prob. of ~~fem~~ male births $= \frac{1}{2} = q$

$p(r) =$ Prob. of r male births in a family of 4 is

$$4c_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{4-r} = 4c_r \left(\frac{1}{2}\right)^4, \quad r = 0, 1, 2, 3, 4.$$

The frequency of r male births is given by

$$f(r) = N.p(r) = 800 \times 4c_r \left(\frac{1}{2}\right)^4, \quad r = 0, 1, 2, 3, 4$$

$$= 50 \times 4c_r$$

Expected frequencies are

$$f(0) = 800 \times 4c_0 \left(\frac{1}{2}\right)^4 = 50 \times 1 = 50$$

$$f(1) = 50 \times 4c_1 = 200, \quad f(2) = 50 \times 4c_2 = 300$$

$f(3) = 50 \times {}^4C_3 = 200$, $f(4) = 50 \times {}^4C_4 = 50$.

No. of male births	Frequency Observed (f_i)	Expected (e_i)	$(f_i - e_i)^2$	$\frac{(f_i - e_i)^2}{e_i}$
0	32	50	324	6.48
1	178	200	484	2.42
2	290	300	100	0.33
3	236	200	1296	6.48
4	64	50	196	3.92
Total =	800	800		19.63

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i} = 19.63$$

$\chi^2_{0.05}$ for 4 d.f. is 9.488.

Since, calculated χ^2 is greater than tabulated $\chi^2_{0.05}$ for 4 d.f., we reject null hypothesis at 5% level of significance and conclude that male and female births are not equally probable.

Ex: When the first proof of 392 pages of a book of 1200 pages were read, the distribution of printing mistakes were found to be as follows:

No. of mistakes in a page (x):	0	1	2	3	4	5	6
No. of pages (f):	275	72	30	7	5	2	1

Fit a Poisson distribution to the above data and test the goodness of fit.

Sol: Mean of given distribution $\therefore \bar{x} = \frac{\Sigma fx}{N} = \frac{189}{392} = 0.482$

x :	0	1	2	3	4	5	6
f :	275	72	30	7	5	2	1
fx :	0	72	60	21	20	10	6

In order to fit a Poisson distribution to the given data, we take the mean of the ~~given data~~ Poisson distribution equal to the mean of the given distribution.

The frequency of r mistakes per page is given by

$$f(r) = Np(r) = 392 \times \frac{e^{-0.482} (0.482)^r}{r!}, \quad r = 0, 1, 2, ---, 6.$$

$$\therefore f(0) = 392 \times \frac{e^{-0.482} \times (0.482)^0}{0!} = 392 \times 0.6176$$
$$= 242.1$$

$$f(1) = 392 \times \frac{e^{-0.482} \times (0.482)^1}{1!} = 116.69$$

$f(2) = 28.12$

$f(3) = 4.518$

$f(4) = 0.544$

$f(5) = 0.052$

$f(6) = 0.004$

Mistakes per page (X)	Frequency Observed (f_i)		Frequency Expected (e_i)		$(f_i-e_i)^2$	$\frac{(f_i-e_i)^2}{e_i}$
0	275		242.1		1082.41	4.471
1	72		116.69		1997.2	17.085
2	30		28.12		3.53	0.126
3	7	15	4.518	5.118	97.65	19.05
4	5		0.544			
5	2		0.052			
6	1		0.004			
	392		392			40.732

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i} = 40.732$$

$$d.f. = 7-3-1-1 = 2$$

$$\chi^2_{0.05} \text{ for } 2 \text{ d.f} = 5.991.$$

Since, Calculated χ^2 > tabulated χ^2, we reject null hypothesis and conclude that Poisson distribution is not a good fit to the given data.

Test of Independence of Attributes – Contingency Tables

Ex Two sample polls of votes for two candidates A and B for a public office are taken, one from among the residents of rural areas. The results are ~~set~~ given in the following table

Votes

Area	A	B	Total
Rural	620	380	1000
Urban	550	450	1000
Total	1170	830	2000

Examine whether the nature of the area is related to voting preference in this election.

Sol: Under the null hypothesis that the nature of the area is independent of the voting preference in the election, we get the expected frequencies,

$$E(620) = \frac{1170 \times 1000}{2000} = 585$$

$$E(380) = \frac{830 \times 1000}{2000} = 415 \quad \text{or} \quad 1000 - 585 = 415$$

$$E(550) = \frac{1170 \times 1000}{2000} = 585 \quad \text{or} \quad 1170 - 585 = 585$$

$$E(450) = \frac{830 \times 1000}{2000} = 415 \quad \text{or} \quad 830 - 415 = 415$$

$$\therefore \chi^2 = \sum_i \frac{(b_i - e_i)^2}{e_i} = \frac{(620-585)^2}{585} + \frac{(380-415)^2}{415} + \frac{(550-585)^2}{585} + \frac{(450-415)^2}{415}$$

$$= 2.094 + 2.9518 + 2.094 + 2.9518$$

$$= 10.09$$

$\chi^2_{0.05}$ for $(2-1)(2-1) = 1$ d.f. is 3.841.

Since calculated value is greater than the tabulated value, it is highly significant and we reject null hypothesis at 5% level of significance. Thus, we conclude that the nature of area is related to the voting preferences in the election.

Ex: Out of 8,000 graduates in a town 800 are females, out of 1600 graduate employees 120 are female. Use χ^2 to determine if any distinction is made in appointment on the basis of gender.

Sol: Null hypothesis: No distinction is made in appointment on the basis of gender.

Alternative hypothesis: Distinction is made in appointment on the basis of gender.

Observed frequencies

	Employed	Not Employed	Total
Male	1480	5720	7200
Female	120	680	800
Total	1600	6400	8000

Expected frequencies

	Employed	Not Employed	Total
Male	1440	5760	7200
Female	160	640	800
Total	1600	6400	8000

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i} = \frac{(1440-1480)^2}{1440} + \frac{(5760-5720)^2}{5760} + \frac{(160-120)^2}{160} + \frac{(640-680)^2}{640}$$

$$= 1.11 + 0.28 + 10 + 2.5$$

$$= 13.89$$

$d.f. = (2-1)(2-1) = 1$

Tabulated $\chi^2_{0.05}$ for 1 d.f. $= 3.841$

Conclusion: Since calculated value is greater than the tabulated value of χ^2, so it is significant and we reject null hypothesis. Hence, we conclude that the distinction is made on the basis of gender.

Ex: A random sample of students of XYZ University was selected and asked their opinions about 'autonomous colleges'. The results are given below. Test the hypothesis at 5% level of significance that opinions are independent of the class groupings.

Class	favouring 'autonomous colleges'	Opposed to 'autonomous colleges'	Total
B.A/B.Sc/B.Com Part I	120	80	200
B.A/B.Sc/B.Com Par II	130	70	200
B.A/B.Sc/B.Com Part III	70	30	100
M.A/M.Sc/M.Com	80	20	100
Total	400	200	600

Sol: Null hypothesis: Opinions about autonomous colleges are independent of the class groupings.

Alternative hypothesis: Opinions about autonomous colleges are dependent on the class groupings.

$E(120) = 133.33$, $E(80) = 66.67$, $E(130) = 133.33$,

$E(70) = 66.67$, $E(80) = 66.67$,

Expected frequencies

133.33	66.67
133.33	66.67
66.67	33.33
66.67	33.33

$$\chi^2 = \sum \frac{(f_i - e_i)^2}{e_i} = 1.3327 + 0.0832 + 0.1663 + 2.6652 + 2.6652 + 0.1663 + 0.3327 + 5.3312$$

$$= 12.7428$$

$d.f. = (4-1)(2-1) = 3.$

$\chi^2_{0.05}$ for 3 d.f. = 7.815.

Conclusion:- Since calculated value > $\chi^2_{0.05}$ for 3 d.f.,
we reject Ho. Hence, we conclude that opinions about autonomous colleges are dependent on the class-groupings.

t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

F Values for $\alpha = 0.10$

d_2 \ d_1	1	2	3	4	5	6	7	8	9
1	39.86	49.5	53.59	55.83	57.24	58.2	58.91	59.44	59.86
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.3	2.27
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68
inf	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63

F Value for $\alpha = 0.10$

d_2 \ d_1	10	12	15	20	24	30	40	60	120	inf
1	60.19	60.71	61.22	61.74	62	62.26	62.53	62.79	63.06	63.33
2	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
4	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
6	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
7	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
8	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
10	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
11	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
13	2.40	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
15	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76
16	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72
17	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
18	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
19	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
20	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
21	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
22	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
24	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
26	1.86	1.81	1.76	1.71	1.80	1.65	1.61	1.58	1.54	1.50
27	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
28	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
29	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
30	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
40	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29
120	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19
inf	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00

F Values for $\alpha = 0.05$

d_2 \ d_1	1	2	3	4	5	6	7	8	9
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5
2	18.51	19.00	19.16	19.25	19.3	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96
inf	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

F Values for $\alpha = 0.05$

d_2 \ d_1	10	12	15	20	24	30	40	60	120	inf
1	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	19.4	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.5
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.10	1.55	1.50	1.43	1.35	1.25
inf	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

F Values for $\alpha = 0.01$

					d_1				
d_2	1	2	3	4	5	6	7	8	9
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.2	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.14
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
inf	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

F Values for $\alpha = 0.01$

d_2 \ d_1	10	12	15	20	24	30	40	60	120	inf
1	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
inf	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Critical values of chi-square (right tail)

Degrees of freedom (*df*)	Significance level (α) .99	.975	.95	.9	.1	.05	.025	.01
1	-------	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217
13	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892
40	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691
50	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154
60	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379
70	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425
80	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329
100	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116
1000	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807