Mean and Variance of Distributions

		μ	σ^2
Binomial Distribution	$b(x; n, p)$	np	npq
Negative Binomial Distribution	$b^*(x; k, p)$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2} = \frac{kq}{p^2}$
Geometric Distribution	$g(x; p)$	$\frac{1}{p}$	$\frac{1-p}{p^2} = \frac{q}{p^2}$
Poisson Distribution	$p(x; \lambda t)$	λt	λt
Gamma Distribution	$f(x; \alpha, \beta)$	$\alpha\beta$	$\alpha\beta^2$
Exponential Distribution	$f(x; \beta)$	β	β^2
Normal Distribution	$n(x; \mu, \sigma)$	μ	σ^2
Standard Normal Distribution	$n(z; \mu, \sigma)$	0	1

Mean and Variance of Binomial Distribution

$\mu = np$, $\sigma^2 = npq$

Proof :- $\mu = E(X) = \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$

$= \sum_{x=1}^{n} x \, {}^nC_x \, p^x q^{n-x}$

$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$

$= \sum_{x=1}^{n} \frac{x \, n(n-1)!}{x(x-1)!(n-x)!} p^x q^{n-x}$

$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$

$= np \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)} \, p^{x-1} q^{n-x}$

$= np \left[{}^{(n-1)}C_0 \, p^0 q^{n-1} + {}^{(n-1)}C_1 \, p q^{n-2} + {}^{(n-1)}C_2 \, p^2 q^{n-3} + \cdots + {}^{n-1}C_{n-1} \, p^{n-1} q^0 \right]$

$\left[(x+y)^n = {}^nC_0 \, x^0 y^n + {}^nC_1 \, x^1 y^{n-1} + {}^nC_2 \, x^2 y^{n-2} + \cdots + {}^nC_n \, x^n y^0 \right]$

$= np \, (p+q)^{n-1}$

$\boxed{\mu = np}$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x} - n^2p^2$$

$$= \sum_{x=1}^{n} x^2 \, {}^nC_x \, p^x q^{n-x} - n^2p^2$$

$$= \sum_{x=1}^{n} (x^2 - x + x) \, {}^nC_x \, p^x q^{n-x} - n^2p^2$$

$$= \sum_{x=1}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!} p^x q^{n-x} + \sum_{x=1}^{n} x \, {}^nC_x \, p^x q^{n-x} - n^2p^2$$

$$= \sum_{x=2}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!} p^x q^{n-x} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^{x-2} q^{n-x} + np - n^2p^2$$

$$= n(n-1)p^2 [p+q]^{n-2} + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p) = npq$$

$$\boxed{\sigma^2 = npq}$$

Mean and Variance of Geometric Distribution

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{q}{p^2}$$

Proof:

$$\mu = E(X) = \sum_{x=1}^{\infty} x p q^{x-1}$$

$$= p \sum_{x=1}^{\infty} x q^{x-1}$$

$$= p\left[1 + 2q + 3q^2 + 4q^3 + \cdots + \cdots\right]$$

$$= p[1-q]^{-2} = \frac{p}{p^2} = \frac{1}{p} \Rightarrow \boxed{\mu = \frac{1}{p}}$$

$$\left[(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\right]$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 p q^{x-1} = \sum_{x=1}^{\infty} (x^2 - x + x) p q^{x-1}$$

$$= \sum_{x=2}^{\infty} x(x-1) p q^{x-1} + \sum_{x=1}^{\infty} x p q^{x-1}$$

$$= 2pq \sum_{x=2}^{\infty} \frac{x(x-1) q^{x-2}}{2} + \frac{1}{p}$$

$$= 2pq\left[1 + 3q + 6q^2 + 10q^3 + \cdots\right] + \frac{1}{p}$$

$$= \frac{2pq}{(1-q)^3} + \frac{1}{p} = \frac{2pq}{p^3} + \frac{1}{p} = \frac{2q}{p^2} + \frac{1}{p}$$

$$\sigma^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} = \frac{q+q+p-1}{p^2} = \frac{q}{p^2}$$

$$\boxed{\sigma^2 = \frac{q}{p^2}}$$

Mean and Variance of Negative Binomial Distribution

$$\mu = \frac{K}{p}, \quad \sigma^2 = \frac{Kq}{p^2}$$

Proof: $\mu = E(X) = \sum_{x=K}^{\infty} x \binom{x-1}{K-1} p^K q^{x-K}$

$$\left[x\binom{x-1}{K-1} = \frac{x(x-1)!}{(K-1)!(x-K)!} = \frac{K\,x!}{K!(x-K)!} \right]$$

$$\mu = Kp^K \sum_{x=K}^{\infty} \frac{x!}{K!(x-K)!} q^{x-K}$$

$$= Kp^K \sum_{x=K}^{\infty} \binom{x}{K} q^{x-K}$$

$$= Kp^K \sum_{x=K}^{\infty} \binom{x+1-1}{K+1-1} q^{x+1-(K+1)}$$

Let $x+1 = l$
$\Rightarrow x-K+1 = l-K$
$\Rightarrow 0+1 = l-K$
$\Rightarrow l = K+1$

$$= Kp^K \sum_{l=K+1}^{\infty} \binom{l-1}{K+1-1} q^{l-(K+1)}$$

$$= Kp^K \sum_{l=K+1}^{\infty} \binom{l-1}{K} q^{l-(K+1)}$$

$$= Kp^K \sum_{l=K+1}^{\infty} \frac{(l-1)!}{K!(l-1-K)!} q^{l-(K+1)}$$

$$= Kp^K \left[1+(K+1)q + \frac{(K+1)(K+2)}{2!} q^2 + \frac{(K+1)(K+2)(K+3)}{3!} q^3 + \cdots \right]$$

$$\left[(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \cdots \right]$$

$$\mu = Kp^K \cdot \frac{1}{(1-q)^{K+1}} = \frac{Kp^K}{p^{K+1}} = \frac{K}{p} \Rightarrow \boxed{\mu = \frac{K}{p}}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$E(X^2) = \sum_{x=K}^{\infty} x^2 \frac{(x-1)!}{(K-1)!(x-K)!} \cdot p^K q^{x-K}$$

$$= \frac{p^K}{(K-1)!} \sum_{x=K}^{\infty} \frac{x \cdot x(x-1)!}{(x-K)!} q^{x-K}$$

$$= \frac{p^K}{q^K (K-1)!} \sum_{x=K}^{\infty} \frac{x \; x!}{(x-K)!} q^x$$

$$= \left(\frac{p}{q}\right)^k \frac{1}{(k-1)!} \sum_{x=k}^{\infty} x! \, \frac{x q^{x-1} \cdot q}{(x-k)!}$$

$$= \left(\frac{p}{q}\right)^k \frac{q}{(k-1)!} \sum_{x=k}^{\infty} \frac{x!}{(x-k)!} \frac{d}{dq}\left(q^x\right)$$

Put $x-k=t$
$x=t+k$

$$= \left(\frac{p}{q}\right)^k \frac{q}{(k-1)!} \frac{d}{dq}\left(\sum_{x=k}^{\infty} \frac{x!}{(x-k)!} q^x\right)$$

$$= \left(\frac{p}{q}\right)^k \frac{q}{(k-1)!} \frac{d}{dq}\left(\sum_{t=0}^{\infty} \frac{(t+k)!}{t!} q^{t+k}\right)$$

$$= \frac{p^k}{q^{k-1}(k-1)!} \frac{d}{dq}\left(q^k \sum_{t=0}^{\infty} \frac{(t+k)!}{t!} q^t\right)$$

$$= \frac{p^k}{q^{k-1}(k-1)!} \frac{d}{dq}\left[q^k\left(k! + \frac{(k+1)!}{1!} q + \frac{(k+2)!}{2!} q^2 + \frac{(k+3)!}{3!} q^3 + \text{---}\right)\right]$$

$$= \frac{p^k}{q^{k-1}(k-1)!} \frac{d}{dq}\left[q^k k!\left(1 + (k+1)q + \frac{(k+1)(k+2)}{2!} q^2 + \frac{(k+1)(k+2)(k+3)}{3!} q^3 + \text{---}\right)\right]$$

$$= \frac{p^k}{q^{k-1}(k-1)!} k! \frac{d}{dq}\left[q^k (1-q)^{-(k+1)}\right]$$

$$= \frac{p^k}{q^{k-1}(k-1)!} k! \frac{d}{dq}\left(\frac{q^k}{(1-q)^{k+1}}\right)$$

$$= \frac{k p^k}{q^{k-1}} \left[\frac{(1-q)^{k+1} k q^{k-1} + q^k (k+1)(1-q)^k}{(1-q)^{2(k+1)}}\right]$$

$$= \frac{k p^k}{q^{k-1}} \left[q^{k-1}(1-q)^k \left(\frac{k(1-q) + q(k+1)}{(1-q)^{2k+2}}\right)\right]$$

$$= \frac{k p^k}{q^{k-1}} \left[q^{k-1}(1-q)^k \frac{k - kq + qk + q}{(1-q)^{2k}(1-q)^2}\right]$$

$$= \frac{k p^k}{\cancel{q^{k-1}}} \left[\frac{\cancel{q^{k-1}}}{(1-q)^{k+2}} (k+q)\right]$$

$$= \frac{k p^k}{p^{k+2}} (k+q) = \frac{k}{p^2}(k+q) = \frac{k^2}{p^2} + \frac{kq}{p^2}$$

$$\sigma^2 = \frac{k^2}{p^2} + \frac{kq}{p^2} - \frac{k^2}{p^2} = \frac{kq}{p^2}$$

$$\boxed{\sigma^2 = \frac{kq}{p^2}}$$

Mean and Variance of Poisson Distribution

$\mu = \lambda t, \ \sigma^2 = \lambda t$

Proof: $\mu = E(X) = \sum_{x=0}^{\infty} x \dfrac{e^{-\lambda t} (\lambda t)^x}{x!} = \sum_{x=1}^{\infty} \dfrac{x e^{-\lambda t} (\lambda t)^x}{x!}$

$$= e^{-\lambda t} (\lambda t) \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{(x-1)!}$$

$$= e^{-\lambda t} (\lambda t) \left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots \right]$$

$$= e^{-\lambda t} (\lambda t) e^{\lambda t}$$

$$\boxed{\mu = \lambda t}$$

$\left[e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \right]$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda t} (\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=1}^{\infty} (x^2 - x + x) \frac{e^{-\lambda t} (\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=1}^{\infty} (x^2 - x) \frac{e^{-\lambda t} (\lambda t)^x}{x!} + \sum_{x=1}^{\infty} \frac{x e^{-\lambda t} (\lambda t)^x}{x!} - \lambda^2 t^2$$

$$= \sum_{x=2}^{\infty} (x^2 - x) \frac{e^{-\lambda t} (\lambda t)^x}{x(x-1)(x-2)!} + \lambda t - \lambda^2 t^2$$

$$= (e^{-\lambda t}) (\lambda t)^2 \sum_{x=2}^{\infty} \frac{(\lambda t)^{x-2}}{(x-2)!} + \lambda t - \lambda^2 t^2$$

$= e^{-\lambda t} (\lambda t)^2 e^{\lambda t} + \lambda t - \lambda^2 t^2$

$= \lambda^2 t^2 + \lambda t - \lambda^2 t^2$

$\boxed{\sigma^2 = \lambda t}$

Mean and Variance of Gamma Distribution

$\mu = \alpha\beta, \quad \sigma^2 = \alpha\beta^2$

Proof :-

$\Gamma\alpha = \int_0^\infty x^{\alpha-1} e^{-x} dx$

$\Gamma(\alpha+1) = \alpha\Gamma\alpha.$

$\mu = E(X)$

$= \int_0^\infty x \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} dx$

$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty x^\alpha e^{-x/\beta} dx$

Let $\frac{x}{\beta} = t$

$x = \beta t$

$\Rightarrow dx = \beta dt$

$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty \beta^\alpha t^\alpha e^{-t} (\beta dt)$

$= \frac{\beta}{\Gamma\alpha} \int_0^\infty t^\alpha e^{-t} dt$

$= \frac{\beta}{\Gamma\alpha} \Gamma(\alpha+1) = \frac{\beta}{\Gamma\alpha} \alpha\Gamma\alpha = \alpha\beta$

$\boxed{\mu = \alpha\beta}$

$\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \int_0^\infty x^2 \frac{1}{\beta^\alpha \Gamma\alpha} x^{\alpha-1} e^{-x/\beta} dx$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty x^{\alpha+1} e^{-x/\beta} dx$$

$$= \frac{1}{\beta^\alpha \Gamma\alpha} \int_0^\infty \beta^{\alpha+1} t^{\alpha+1} e^{-t} (\beta dt)$$

$$= \frac{\beta^2}{\Gamma\alpha} \int_0^\infty t^{\alpha+1} e^{-t} dt$$

$$= \frac{\beta^2}{\Gamma\alpha} \Gamma\alpha+2 = \frac{\beta^2}{\Gamma\alpha} [\alpha(\alpha+1)\Gamma\alpha]$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

$$\sigma^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$\boxed{\sigma^2 = \alpha\beta^2}$$

Mean and Variance of Exponential Distribution

$\mu = \beta,\ \sigma^2 = \beta^2$

Proof:

$$\mu = \int_0^\infty x \frac{1}{\beta} e^{-x/\beta} dx$$

$$= \frac{1}{\beta} \int_0^\infty (\beta t) e^{-t} (\beta dt)$$

$$= \beta \int_0^\infty t e^{-t} dt = \beta \Gamma 2 = \beta\, 1! = \beta.$$

$$\boxed{\mu = \beta}$$

$$\sigma^2 = \int_0^\infty x^2 \frac{1}{\beta} e^{-x/\beta}\, dx - \beta^2$$

$$= \cancel{\frac{1}{\beta}\Gamma 3}\ \frac{1}{\beta}\int_0^\infty \beta^2 t^2 e^{-t} (\beta\, dt) - \beta^2$$

$$= \beta^2 \int_0^\infty t^2 e^{-t}\, dt - \beta^2$$

$$= \beta^2 \Gamma 3 - \beta^2$$

$$= 2\beta^2 - \beta^2 = \beta^2$$

$$\boxed{\sigma^2 = \beta^2}$$

Mean and Variance of Normal Distribution

$$Z = \frac{X-\mu}{\sigma}$$

$\Rightarrow X = \mu + \sigma Z \Rightarrow E(X) = \mu + \sigma E(Z)$ — (1)

and $Var(X) = Var(\mu) + Var(\sigma Z)$

$= 0 + \sigma^2 Var(Z)$

$\Rightarrow Var(X) = \sigma^2 Var(Z)$ — (2)

Now $E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz = 0$

Because $ze^{-z^2/2}$ is an odd function.

By (1), $\boxed{E(X) = \mu}$

Now, $Var(Z) = E(Z^2) - [E(Z)]^2 = E(Z^2)$

$$E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\left(ze^{-z^2/2}\right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[-ze^{-z^2/2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-z^2/2} dz \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{\infty} f(z)\,dz = 1 \Rightarrow E(z^2) = 1$$

$Var(z) = 1$

By (2), we have

$$\boxed{Var(X) = \sigma^2}$$