15.10. Methods of Estimation. So far we have been discussing the requisites of a good estimator. Now we shall briefly outline some of the important methods for obtaining such estimators. Commonly used methods are

- (i) Method of Maximum Likelihood Estimation.
- (ii) Method of Minimum Variance.
- (iii) Method of Moments.
- (iv) Method of Least Squares.
- (v) Method of Minimum Chi-square
- (vi) Method of Inverse Probability.

Likelihood Function. Definition. Let x_1, x_2, \dots, x_n random sample of size n from a population with density function $f(x, \theta)$. Then the likelihood function of the sample values $x_1, x_2, ..., x_n$, usually denoted by $L = L(\theta)$ is their joint density function, given by

$$L = f(x_1, \theta) f(x_2, \theta) \dots f(x_n, \theta) = \prod_{i, p=1}^n f(x_i, \theta). \quad . (15.53)$$

L gives the relative likelihood that the random variables assume a particular set of values x_1, x_2, \dots, x_n . For a given sample x_1, x_2, \dots, x_n , L becomes a function of the variable θ , the parameter.

The principle of maximum likelihood consists in finding an estimator for the unknown parameter $\theta = (\theta_1, \theta_2, ..., \theta_k)$, say, which maximizes the likelihood function $L(\theta)$ for variations in parameter *i.e.*, we wish to find $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k)$ so that

$$L(\hat{\theta}) > L(\theta) \quad \forall \theta \in \Theta$$

 $L(\hat{\theta}) = \operatorname{Sup} L(\theta) \ \forall \theta \in \Theta.$

Thus if there exists a function $\hat{\theta} = \hat{\theta}(x_1, x_2, ..., x_n)$ of the sample values which maximises L for variations in θ , then $\hat{\theta}$ is to be taken as an estimator of

 θ , $\hat{\theta}$ is usually called Maximum Likelihood Estimator (M.L.E.). Thus $\hat{\theta}$ is the solution, if any, of

Since L > 0, and $\log L$ is a non-decreasing function of L; L and $\log L$ attain their extreme values (maxima or minima) at the same value of $\hat{\theta}$. The first of the two equations in (15-54) can be rewritten as

uniform in (15.54c) and the rewritten as
$$\frac{1}{L} \cdot \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial \log L}{\partial \theta} = 0, \quad ...(15.54c)$$

a form which is much more convenient from practical point of view.

If θ is vector valued parameter, then $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_k)$, is given by the solution of simultaneous equations:

$$\frac{\partial}{\partial \theta_{i}} \log L = \frac{\partial}{\partial \theta_{i}} \log L \ (\theta_{1}, \theta_{2}, ..., \theta_{k}) = 0 \ ; \ i = 1, 2, ..., k$$

$$...(15.54b) \qquad \qquad \bigcirc \bigcirc \bigcirc \bigcirc$$

Equations (15.54a) and (15.54b) are usually referred to as the Likelihood Equations for estimating the parameters.

Example 15:31. In random sampling from normal population $N(\mu; \sigma^2)$, find the maximum likelihood estimators for

- (i) u when o is known,
- (ii) o² when μ is known, and
- (iii) the simultaneous estimation of μ and σ^2 .

Solution.
$$X \sim N \ (\mu, \sigma^2)$$
 then
$$L = \prod_{i=1}^{n} \left[\frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2\sigma^2} \left(x_i - \mu \right)^2 \right\} \right]$$

$$= \left(\frac{1}{\sigma \sqrt{2\pi}} \right)^n \exp \left\{ -\frac{\sum_{i=1}^{n} \left(x_i - \mu \right)^2 / 2\sigma^2 \right\}$$

$$\log L = -\frac{n}{2} \log (2\pi) - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(x_i - \mu \right)^2$$

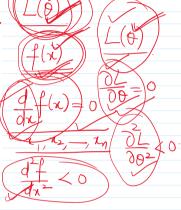
Case (i). When
$$\sigma^2$$
 is known, the likelihood equation for estimating μ is
$$\frac{\partial}{\partial \mu} \log L = 0 \implies -\frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(x_i - \mu)(-1) = 0$$

$$\sum_{i=1}^{n} (x_i - \mu) = 0 \implies \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\frac{\partial}{\partial \mu} \log L = 0 \implies \sum_{i=1}^{n} 2(x_i - \mu)(-1) = 0$$

$$\frac{\partial}{\partial \mu} \log L = 0 \implies \sum_{i=1}^{n} 2(x_i - \mu)(-1) = 0$$

Hence M.L.E. for μ is the sample mean $\bar{\lambda}$



$$\frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial^2 (\log L)}{\partial \theta} (\log L) < 0$$

$$\frac{\partial (\log L)}{\partial \theta} = 0$$

Hation N(
$$\mu$$
; σ^2).
$$L = \prod_{i=1}^{n} f(x_i, \theta) = f(x_i, \theta) \times f(x_2, \theta) \times \dots \times f(x_n, \theta)$$

$$= \frac{-1}{\sqrt{2\pi}} \left(\frac{x_1 - \mu}{\sigma} \right)^2 \qquad \frac{-1}{2} \left(\frac{x_2 - \mu}{\sigma} \right)$$

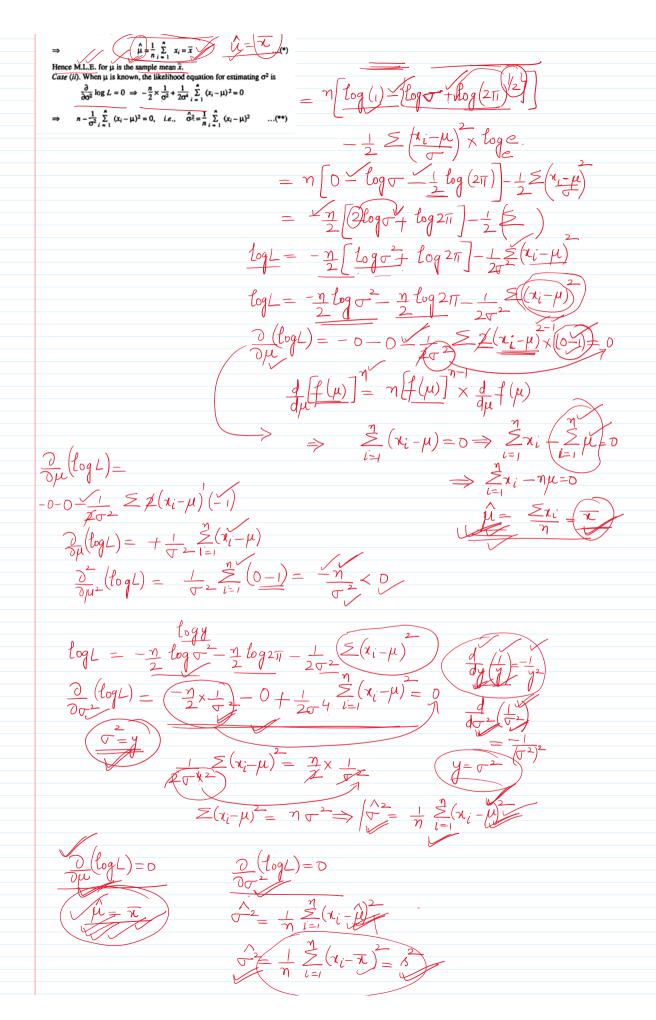
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x_1 - \mu}{\sigma} \right)^2 \qquad \frac{-1}{2} \left(\frac{x_2 - \mu}{\sigma} \right)$$

$$L = \left(\frac{1}{\sqrt{\sqrt{2\pi}}} \right)^n e^{-\frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2} \qquad \frac{m_1}{2} \sum_{i=1}^{m_1} \left(\frac{x_i - \mu}{\sigma} \right)^2 \qquad = e^{m_1 + m_2 + \dots + m_n}$$

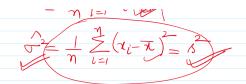
$$\log L = \log \left(\frac{1}{\sqrt{\sqrt{2\pi}}} \right)^n + \log e^{-\frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2} \qquad = e^{m_1 + m_2 + \dots + m_n}$$

$$= e^{m_1 + m_2 + \dots + m_n}$$

$$= e^{m_1 + m_2 + \dots + m_n}$$







Poll Que. (i) Which of the following is the maximum likelihood estimate of the parameter ' α ' of

<u>a</u> population having density function: $\frac{2}{\alpha^2}(\alpha - x)$, $0 < x < \alpha$ when a sample of unit size is drawn

from the population? (Here 'x' is the sample value

(a) 2x

(b) *x* (c

(d) None of these.

 $0 \rightarrow \alpha$ $0 \rightarrow (0, 0, 0)$ $\mu \qquad \tau$

Poll Que. (ii) Is the above MLE is biased

(a) Yes (b) No.

$$f(x,0) = f(x,\alpha) = \frac{2}{\alpha^{2}}(\alpha - x) \quad 0 < x < \infty$$

$$i = 1, 2, -(x) \quad (x = 1)$$

$$- \iint_{i=1}^{\infty} f(x_{i}\alpha) = f(x_{i},\alpha) = f(x_{i}\alpha) = \frac{2}{\alpha^{2}}(\alpha - x)$$

$$\log L = \frac{\log 2 + \log (\alpha - x) - 2\log \alpha}{2\alpha - 2x} = 0$$

$$\frac{2}{\alpha} = \frac{1}{\alpha - x} \Rightarrow 2\alpha - 2x = \alpha \Rightarrow \hat{\alpha} = 2x$$

$$E(\hat{\alpha}) = E(2x) = \int_{0}^{\infty} 2x \times \frac{2}{\alpha^{2}}(\alpha - x) dx = \frac{4}{\alpha^{2}} \left(\frac{x^{2}}{2} - \frac{x^{3}}{3}\right) = \frac{4}{\alpha^{2}} \left(\frac{x^{3}}{3} - \frac{2x}{3} + \frac{x^{3}}{3}\right)$$

$$= \frac{4}{\alpha^{2}} \left(\frac{x^{3}}{2} - \frac{x^{3}}{3}\right) = \frac{4}{\alpha^{2}} \left(\frac{x^{3}}{3} - \frac{2x}{3} + \frac{x^{3}}{3}\right)$$