Chapter 1

Fundamentals of Stacky Curves

In this chapter we will introduce and develop the basic theory of stacky curves. In the first section we cover the local and global structure results for stacky curves, relating stacky curves to classical curves. In the second section we start our analysis of coherent sheaves on stacky curves and prove analogues of many of the classical results, like the existence of a torsion filtration and a description of invertible sheaves. In the third section we will develop an analogue of the theory of projective curves and give analogues of Serre-duality, the Riemann-Roch theorem and the Riemann-Hurwitz theorem. We also discuss Hilbert polynomials and stability. In the final section we will relate vector bundles on stacky curves to parabolic vector bundles and compare the notions of stability on both sides.

1.1 Structure results for stacky curves

In this section we will describe the basic geometry of stacky curves. The main results are two structure results for stacky curves: a local structure result describing stacky curves as finite quotients of classical curves and a global structure result describing stacky curves as a classical curve together with finite data. The results in this chapter are certainly well-known; however, they are often stated in such high generality that it might obfuscate the simplicity of the case of curves. Consequently, we will restate these results in terms of curves and use the fact that we are on a curve to give simplified proofs. What is new is that we work over an arbitrary (potentially imperfect) base field. Because of this we will have to work with regular curves rather then smooth curves.

> **Definition 1.1.1** A **stacky curve** is a regular separated finite type geometrically connected Deligne-Mumford stack $\mathcal{C}$ of dimension 1 over a field k, such that there exists a scheme X and an open immersion $X \to \mathcal{C}$.

The condition that $\mathcal{C}$ contains an open subscheme excludes things like gerbes over curves and ensures that $\mathcal{C}$ has only finitely many stacky points. We will only consider regular stacky curves, which is why we include it in the definition. Note that by definition a curve is just a stacky curve that happens to be a scheme. When we want to emphasize that a curve is scheme we will call it a **classical** curve.

> **Definition 1.1.2** Let $\mathcal{C}$ be a stacky curve and p be a closed point of $\mathcal{C}$. We say that p is a **stacky point** if it has a non-trivial stabilizer group $G_p := \underline{\mathtt{Isom}}(p, p)$. If the order of G_p is invertible in k we say that p is a **tame** point. We say that $\mathcal{C}$ is tame if all of its points are tame. We define the residual gerbe of p to be the unique reduced closed substack supported on p and denote it by $\mathcal{G}_p$.

Note that in our situation this definition is equivalent to the more general definition of [15, Definition 06MU] via [15, Lemma 0H27].

The motivating example of a stacky curve is the following.

> **Example 1.1.3** Let C be a curve over a field k and G be a finite subgroup of $Aut(C)$, then the stack quotient $[C/G]$ is a stacky curve. The stacky points of $[C/G]$ correspond to the orbits of G with non-trivial inertia. Let p be a fixed point of G and denote by $G_s(p)$ and $G_i(p)$ the stabilizer group and inertia group respectively. Then the residual gerbe $\mathcal{G}_{Gp}$ is isomorphic to $[\mathsf{Spec}\big(\kappa(p)^{G_s/G_i}\big)/G_i]$.

In the next example we glue together two quotient curves to get a stacky curve that is not itself a quotient of a curve (see Theorem 1.3.6 for a proof).

> **Definition 1.1.4** The **football space** $\mathcal{F}(p, q)$, with weights $p, q \in \mathbb{N}_{\geq 1}$, is given by gluing the two stacky curves $U_0 = [\mathbb{A}^1_k/\mu_p]$ and $U_1 = [\mathbb{A}^1_k/\mu_q]$, were μ_p and μ_q act by multiplication and the gluing map $\mathsf{Spec}\big(k[x, x^{-1}]\big) \simeq [\mathbb{A}^1_k - \{0\}/\mu_p] \to [\mathbb{A}^1_k - \{0\}/\mu_q] \simeq \mathsf{Spec}\big(k[y, y^{-1}]\big)$ is defined by $y \to x^{-1}$.

The football space $\mathcal{F}(1, 1)$ is simply $\mathbb{P}^1_k$ and topologically $\mathcal{F}(p, q)$ is just $\mathbb{P}^1_k$ where the points 0 and ∞ are stacky with residual gerbes $B\mu_p$ and $B\mu_q$ respectively. Over the complex numbers we can think of this as a sphere with two pointy sides, i.e. an American football. When p and q are coprime, $\mathcal{F}(p, q)$ is isomorphic to the weighted projective stack $\mathbb{P}(p, q) := [\mathbb{A}^2_k - \{(0, 0)\}/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts as $\lambda \cdot (x, y) = (\lambda^p x, \lambda^q y)$. When $\mathsf{gcd}(p, q) = e > 1$, there is a map $\mathbb{P}(p, q) \to \mathcal{F}(p, q)$, making $\mathbb{P}(p, q)$ into a μ_e-gerbe over $\mathcal{F}(p, q)$.

> **Definition 1.1.5** Let $\mathcal{C}$ be a stacky curve. A **coarse space morphism** for $\mathcal{C}$ is a morphism $\pi : \mathcal{C} \to C$ to an algebraic space satisfying the following properties.
>
> - Any morphism $f : \mathcal{C} \to X$ to an algebraic space factors uniquely through π.

> - The induced map $|\mathcal{C}(\Omega)| \to |C(\Omega)|$ is a bijection for algebraically closed fields Ω.
>
> The algebraic space C is called the **coarse space**.

By the factorisation property, the coarse space morphism is unique if it exists. To mirror the idea that the coarse space is a rough (coarse) approximation of the stacky curve we will write stacky curves with calligraphic letters and their coarse spaces with the same non-calligraphic letter. In the literature coarse spaces are sometimes called coarse **moduli** spaces, in analogy with the concept of fine/coarse moduli spaces. Since stacky curves are not (always) moduli spaces, we omit the word “moduli”.

To show the existence of coarse spaces we can apply the much more general Keel-Mori theorem; See for example [5] for a proof.

> **Theorem 1.1.6** (Keel-Mori) Let $\mathcal{X}$ be an Artin stack that is locally of finite presentation over a field k, with finite inertia stack $I(\mathcal{X})$. Then there exists a coarse space morphism $\pi : \mathcal{X} \to X$ to an algebraic space X with the following additional properties.
>
> (1) If $\mathcal{X}$ is separated, then so is X.
>
> (2) The coarse space X is locally of finite type over k.
>
> (3) The map π is proper and quasi-finite.
>
> (4) For $X' \to X$ a flat map of algebraic spaces the pullback $\pi' : \mathcal{X} \times_X X' \to X'$ is also a coarse space morphism.

Clearly stacky curves satisfy the conditions of the Keel-Mori Theorem, so they always have a coarse space morphism. Using this fact we can give the local structure result for stacky curves we alluded to before.

> **Theorem 1.1.7** (Local form of stacky curves) Let $\mathcal{C}$ be a stacky curve with coarse space map $\pi : \mathcal{C} \to C$ and p a closed point of C with stabilizer group G_p. Then there exists an étale morphism $V \to C$ from a curve with p in its image and a (possibly disconnected) curve U with an action of G_p such that $\mathcal{C} \times_C V \simeq [U/G_p]$.

Proof. The existence of the schemes V, U and the action by G_p follows from the proof of [2, Lemma 2.2.3]. The quotient $U \to [U/G_p]$ is finite and smooth, so U is finite and smooth over $\mathcal{C}$. It follows that U is regular separated and 1-dimensional over k, so it is a curve. ⬡

We will use the following technical lemma to conclude that the coarse space of a stacky curve is a curve.

Lemma 1.1.8 Let $\mathcal{C}$ be a stacky curve and $\pi : \mathcal{C} \to C$ the coarse space morphism, then

1. C is separated,
2. C is irreducible,
3. C is 1-dimensional over k,
4. C is regular over k (and a fortiori normal).

Proof. We prove the statements one by one.

1. This follows from Theorem 1.1.6 (1).
2. Since π is a homeomorphism this follows from the irreducibility of $\mathcal{C}$.
3. By definition we have an open substack $X \to \mathcal{C}$ that is a 1-dimensional scheme. Now the coarse space of X, which is X, is an open subspace of C. Since C contains an open 1-dimensional scheme it is itself 1-dimensional.
4. By Theorem 1.1.7 we know there exists a surjective étale cover by a (disconnected) curve $f : V \to C$. It follows that C is regular.

□

Theorem 1.1.9 Let $\mathcal{C}$ be a stacky curve with coarse space C, then C is a classical curve.

Proof. By [9, Theorem V.4.4], a normal, separated, irreducible algebraic space over a field is a scheme in codimension 1. It follows from the lemma above that C is a scheme and hence a curve. □

1.1.1 Ramification theory and root stacks

We will now develop some basic ramification theory for stacky curves. This is based on [7], which gives a treatment for more general (smooth) DM-stacks. The goal is to understand the ramification of the coarse space map and see how it characterises the curves.

Definition 1.1.10 Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of stacky curves. Let $p \in \mathcal{C}$ be a closed point with image $f(p) = q \in \mathcal{D}$. Take an étale cover by a scheme $U \to \mathcal{D}$ and then another étale cover by a scheme $V \to U \times_{\mathcal{D}} \mathcal{C}$. Then take a point $v \in V$ that maps to p and let u be its image in U. Then we define the **ramification index** $e_{p/q}$ to be the ramification index $e_{v/u}$ of v over u.

Proposition 1.1.11 The definition above is independent of the chosen coverings.

Proof. Fix U be chose two different V and V'. Then $V \times_{\mathcal{C}\times_{\mathcal{D}}\times U} V'$ is also étale over $\mathcal{C} \times_{\mathcal{D}} \times U$, so we may assume there is an étale morphism $V' \to V$ commuting with the map to $\mathcal{C} \times_{\mathcal{D}} U$. Let u, v, v' be such that $v' \mapsto v \mapsto u$, then $e_{v'/u} = e_{v'/v} e_{v/u} = e_{v/u}$. Now pick two pairs of étale covers U, V and U', V'. Since $U \times_{\mathcal{D}} U'$ is étale over $\mathcal{D}$, so we may assume that there is an étale morphism $U' \to U$. By the first point we may replace V' by $V \times_{\mathcal{D}} U'$ so that we have a commutative diagram,

$$\begin{array}{ccc} V' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \end{array}$$

where the vertical arrows are étale. Now pick u, v, u', v' appropriately, then we have $e_{v/u} = e_{v'/v} e_{v/u} = e_{v'/u'} e_{u'/u} = e_{v'/u'}$. ◇

Definition 1.1.12 Let f be as above, the **ramification locus** R_f is the set of closed points $p \in \mathcal{C}$ such that $e_p > 1$. The branch locus is the image of R_f inside $\mathcal{D}$. We denote by e_f the set of multiplicities e_p for $p \in R_f$. A map f is called unramified if R_f is empty. We say that f is **tamely** ramified at p if the characteristic of k does not divide $e_{p/f(p)}$. The map f is tamely ramified if it is tamely ramified at every point.

Example 1.1.13 Let G be a finite group acting faithfully on a curve C. Consider the coarse space morphism $\pi : [C/G] \to C/G$ from the stack quotient to the schematic quotient. Assume that the orders of the inertia groups $G_i(x)$ are not divisible by the characteristic of k for any closed point $x \in U$. Then for any closed point $y \in [C/G]$, with $z := \pi(y)$, we have that the ramification index $e_{y/z}$ is equal to the order of the inertia group $G_i(x)$ for a point x lying above y.

Proof. Since C/G is already a scheme we can take the identity map as its étale cover. The map $C \to [C/G]$ is étale, so we may pick a point x in C that maps to y and compute $e_{x/z}$ of the map $C \to C/G$. Now the result is classic. ◇

Theorem 1.1.14 Let $f : \mathcal{C} \to \mathcal{D}$ be an unramified map of tame stacky curves, then f is representable.

Proof. Let $U \to \mathcal{D}$ be an étale cover for $\mathcal{D}$ by a scheme, then $\mathcal{C} \times_{\mathcal{D}} U \to U$ is also unramified, so we may assume that $\mathcal{D}$ is a scheme. Let $[V/G] \to \mathcal{C}$ be as in the local form of Theorem 1.1.7. The map $[V/G] \to \mathcal{C} \to \mathcal{D}$ is unramified and factors through the coarse space V/G. Since ramification indices are multiplicative in compositions the map

$[V/G] \to V/G$ is unramified. This means that G acts freely on V by Example 1.1.13, hence $[V/G] = V/G$. This means that the coarse space map $\mathcal{X} \to X$ is étale locally an isomorphism, so $\mathcal{X}$ is a scheme. ◇

Theorem 1.1.15 Let $f : \mathcal{C} \to \mathcal{D}$ be an unramified map of tame stacky curves that induces an isomorphism of coarse spaces $C \simeq D$, then f is an isomorphism.

Proof. Since being an isomorphism is étale local we can assume $\mathcal{D} = [V/G]$ and $D = V/G$ for a curve V and finite group G. Since f is unramified it is representable, so $V' := \mathcal{C} \times_{\mathcal{D}} V$ is a scheme. Because $V' \to \mathcal{C}$ is étale V' is also a curve. We have open subschemes $U \subset \mathcal{C}$ and $U' \subset \mathcal{D}$ and we can take their intersection $U \cap U' \subset C \simeq D$ in the coarse spaces. Now $U \cap U'$ is an open subscheme of both $\mathcal{C}$ and $\mathcal{D}$ and f restricts to an isomorphism on this open subscheme. It follows that the morphism $V' \to V$ between regular curves is birational and a bijection on points, hence an isomorphism. Consequently f is an isomorphism. ◇

Definition 1.1.16 Let $\mathcal{C}$ be a stacky curve. A **Weil divisor** D on $\mathcal{C}$ is a finite formal sum $\sum_Z n_Z Z$ of reduced closed substacks Z of codimension 1 of $\mathcal{C}$. If all the $n_Z \geq 0$ we call D **effective**.

The reduced closed substacks of codimension 1 of $\mathcal{C}$ are in one to one correspondence with the reduced closed subschemes of the coarse space C, hence they are in one to one correspondence with the closed points of both $\mathcal{C}$ and C. When p is stacky point the associated closed substack is precisely the residual gerbe $\mathcal{G}_p$ of p. This is the motivation for the following definition.

Definition 1.1.17 Let p be a stacky point of order e_p on a stacky curve. We define $\frac{1}{e_p} p$ to be the Weil divisor $\mathcal{G}_p$. This lets us write a Weil divisor as a formal sum of closed points with coefficients in $\mathbb{Q}$, namely we define $\sum_p \frac{n_p}{e_p} p := \sum_{\mathcal{G}_p} n_p \mathcal{G}_p$.

Definition 1.1.18 Let $\mathcal{C}$ be a stacky curve. An effective Cartier divisor D on $\mathcal{C}$ is a non-zero map $D : \mathcal{C} \to [\mathbb{A}^1/\mathbb{G}_m]$ i.e. a line bundle $\mathcal{L}$ on $\mathcal{C}$ together with a non-zero section s of $\mathcal{L}$.

Note that one can similarly define a (non-effective) Cartier divisor to be a map to $[\mathbb{P}^1/\mathbb{G}_m]$. This definition is more familiar then it might look on first glance, namely the isomorphism classes of maps into $[\mathbb{A}^1/\mathbb{G}_m]$ are nothing more then elements of $H^0(\mathcal{O}/\mathcal{O}^\times)$. Similarly maps into $[\mathbb{P}^1/\mathbb{G}_m]$ are parametrized by $H^0(\mathcal{M}^\times/\mathcal{O}^\times)$.

Definition 1.1.19 Let $Z \subset \mathcal{C}$ be a closed substack, we define the ideal sheaf $\mathcal{O}_{\mathcal{C}}(-Z) \subset \mathcal{O}_{\mathcal{C}}$ on étale covers of $\mathcal{C}$ as follows. Let $f : U \to \mathcal{C}$ be étale, then $\mathcal{O}_{\mathcal{C}}(-Z)|_U = \mathcal{O}_U(-Z \times_{\mathcal{C}} U) \subset \mathcal{O}_U$.

To an effective Weil divisor D we can associated the ideal sheaf

$$\mathcal{O}(-D) := \bigotimes_p \mathcal{O}(-\frac{1}{e_p}p)^{\otimes n_p} \subset \mathcal{O}_{\mathcal{C}}$$

and the effective Cartier divisor $(\mathcal{O}_{\mathcal{C}}(D), s_D)$ where $\mathcal{O}_{\mathcal{C}}(D) = \mathcal{H}om(\mathcal{O}_{\mathcal{C}}(-D), \mathcal{O}_{\mathcal{C}})$ and s_D is corresponds to the dual of the inclusion map $\mathcal{O}_{\mathcal{C}}(-D) \to \mathcal{O}_{\mathcal{C}}$. This process can be inverted by sending $(\mathcal{L}, s)$ to $\sum_p \frac{v_p(s)}{e_p} p$. Here $v_p(s)$ is defined by considering the inclusion $i : \mathcal{G}_p \to \mathcal{C}$ and setting $v_p(s)$ to be one less then length of the extension of local rings $i^{-1}s : i^{-1}\mathcal{O}_{\mathcal{C}} \to i^{-1}\mathcal{L}$. To see that these two operations are inverse to each other we can pass to an étale cover, where it follows from the case of regular curves.

Definition 1.1.20 Let $f : \mathcal{C} \to \mathcal{D}$ be a non-constant map of stacky curves and D an effective Cartier divisor on $\mathcal{D}$. We define the **pullback** f^*D of D to be the composition $\mathcal{C} \to \mathcal{D} \to [\mathbb{A}^1/\mathbb{G}_m]$.

The following proposition expresses the pullback of a divisor in terms of Weil divisors and ramification data.

Proposition 1.1.21 Let $f : \mathcal{C} \mapsto \mathcal{D}$ be a tamely ramified map of stacky curves and $q \in \mathcal{D}$ with pre-images $\{p_i\} = f^{-1}(q)$. We have $f^*\mathcal{G}_q = \sum_{p_i} e_{p_i/q}\mathcal{G}_{p_i}$.

Proof. We first show the case where $\mathcal{C} = C$ is a scheme and f is étale. We then have $f^*\mathcal{G}_q := (\mathcal{O}(G_q \times_{\mathcal{D}} C), s_{G_q \times_{\mathcal{D}} C}) = \sum_{p_i} p_i$.

For the general case we let $u : U \to \mathcal{D}$ be an étale neighbourhood of q such that q has a unique preimage $\tilde{q}$ and let $V \to U \times_{\mathcal{D}}$ be an étale cover, so we have the following diagram.

$$\begin{tikzcd} V \arrow[d, "v"] & \\ U \times_{\mathcal{D}} \mathcal{C} \arrow[r, "g"] \arrow[d, "w"] & U \arrow[d, "u"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

We can now verify the equality by passing to the cover V, i.e. we have to show $v^*w^*f^*\mathcal{G}_q = v^*w^*\sum_{p_i} e_{p_i/q}\mathcal{G}_{p_i}$.

Note that $v^*w^*f^*\mathcal{G}_q = v^*g^*u^*\mathcal{G}_q = (v \circ g)^*\tilde{q}$. Let r_{ij} be the preimages of the p_i under $(v \circ w)$, then by the first case $v^*w^*\sum_{p_i} e_{p_i/q}\mathcal{G}_{p_i} = \sum_{r_{ij}} e_{p_i/q} r_{ij}$. Note that the r_{ij} are exactly the preimages of $\tilde{q}$ under $(v \circ g)$ and $e_{p_i/q} = e_{r_{ij}/\tilde{q}}$. So we have reduced to the case of classical curves, which is [11, Chapter 7, Exercise 2.3(b)] ◇

We now go over the construction of root stacks, which should be viewed as “degree 1 covers” with specified ramification data. We will prove that all stacky curves are actually root stacks over their coarse space in Theorem 1.1.32. For a more general treatment on root stacks see [4]

Definition 1.1.22 Let $\mathcal{C}$ be a stacky curve, p a closed point and $e > 1$ a natural number not divisible by the characteristic of k. Consider the Cartier divisor $(\mathcal{O}(\mathcal{G}_p), s_p)$ associated to p. The root stack $\sqrt[e]{p/\mathcal{C}}$ is defined as the fibre product of the diagram

$$\begin{tikzcd} \sqrt[e]{p/\mathcal{C}} \arrow[r] \arrow[d, "\rho"] & {[\mathbb{A}^1_k/\mathbb{G}_m]} \arrow[d, "\wedge e"] \\ \mathcal{C} \arrow[r, "(\mathcal{O}(\mathcal{G}_p), s_p)"] & {[\mathbb{A}^1_k/\mathbb{G}_m]} \end{tikzcd},$$

where the right arrow is induced by the e-th power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$ and the bottom arrow is induced by p. The top map $\sqrt[e]{p/\mathcal{C}} \to [\mathbb{A}^1/\mathbb{G}_m]$ defines an effective Cartier divisor (T_p, s_p), which is called the tautological divisor, we refer to T_p as the tautological line bundle. The left arrow $\rho : \sqrt[e]{p/\mathcal{C}} \to \mathcal{C}$ is called the root morphism.

For a finite set of points $\underline{p} = (p_1, \ldots p_n)$ and multiplicities $\underline{e} = (e_1, \ldots e_n)$ we define the **iterated root stack**

$$\sqrt[\underline{e}]{\underline{p}/\mathcal{C}} := \sqrt[e_1]{p_1/\mathcal{C}} \times_{\mathcal{C}} \sqrt[e_2]{p_2/\mathcal{C}} \times_{\mathcal{C}} \cdots \times_{\mathcal{C}} \sqrt[e_n]{p_n/\mathcal{C}},$$

which comes with tautological Cartier divisors $(\mathrm{T}_{p_i}, s_{p_i})$ for each i and an iterated root morphism $\sqrt[\underline{e}]{\underline{p}/\mathcal{C}} \to \mathcal{C}$.

Technically the root construction also allows us to root in non-reduced divisors, however rooting in $n \cdot D$ with degree e is the same as rooting in D with degree $e/\gcd(n, e)$.

Since root stacks commute with pullback by construction, the following example explains the local structure of root stacks.

Example 1.1.23 Let $C = \mathrm{Spec}(A)$ be an affine curve. Let $x \in A$ be a section corresponding to a point $p = (x)$, we have

$$\sqrt[e]{p/C} \simeq [\mathrm{Spec}(A[t]/(t^e - x))/\mu_e],$$

where μ_e acts by multiplication on the variable t.

Proof. Since $\mathcal{O}_C(p) \simeq \mathcal{O}_C$ the morphism $C \xrightarrow{p} [\mathbb{A}^1/\mathbb{G}_m]$ factors as $C \xrightarrow{x} \mathbb{A}^1 \to$

$[\mathbb{A}^1/\mathbb{G}_m]$. We first claim that $X := \mathbb{A}^1 \times_{[\mathbb{A}^1/\mathbb{G}_m]} [\mathbb{A}^1/\mathbb{G}_m] \simeq [\mathbb{A}^1/\mu_e]$. To see this consider the diagram of Cartesian squares.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\wedge e"] \\ \mathbb{A}^1 \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\ \mathsf{Spec}(k) \arrow[r] & B\mathbb{G}_m \end{tikzcd}$$

It follows that $X = \mathsf{Spec}(k) \times_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m] \simeq [\mathbb{A}^1/(\mathsf{ker} \wedge e : \mathbb{G}_m \to \mathbb{G}_m)] = [\mathbb{A}^1/\mu_e]$. Now consider another commutative diagram of Cartesian squares.

$$\begin{tikzcd} \mathsf{Spec}(A[t]/(t^e - x)) \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] & \\ \sqrt[e]{p/C} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mu_e]} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\ C \arrow[r] & \mathbb{A}^1 \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

The action of μ_e on $\mathbb{A}^1$ pulls back to an action on $\mathsf{Spec}(A[t]/(t^e - x))$ and $\sqrt[e]{p/C} \simeq [\mathsf{Spec}(A[t]/(t^e - x))/\mu_e]$. ⬡

Remark 1.1.24 In the case that we are rooting in a non-stacky point the example shows that the Weil divisor associated to (T_p, s_p) is supported on the single closed point lying above p and has stabilizer μ_e. We abuse notation and the point lying above p will also be called p, so that the corresponding divisor is denoted by $\frac{1}{e}p$ By construction we have $\pi^*(\mathcal{O}(p)) = \mathcal{O}(\frac{1}{e}p)^e$, which motivates the "root" terminology.

Lemma 1.1.25 The root morphism $\rho : \sqrt[e]{p/\mathcal{C}} \to \mathcal{C}$ is an isomorphism away from the rooted point.

Proof. Away from the rooted point the section s_p does not vanish, so the restriction $\mathcal{C} - \{p\} \to [\mathbb{A}^1_k/\mathbb{G}_m]$ factors through the open substack $\mathsf{Spec}(k) = [\mathbb{G}_m/\mathbb{G}_m] \subset [\mathbb{A}^1_k/\mathbb{G}_m]$ and the restricted map $e : [\mathbb{G}_m/\mathbb{G}_m] \to [\mathbb{G}_m/\mathbb{G}_m]$ is the identity. ⬡

For completeness we will prove two lemmas on the regularity/smoothness properties of branched coverings.

Lemma 1.1.26 Let A be a regular local ring with maximal ideal $\mathfrak{m}$ and $k = A/\mathfrak{m}$. Let $s \in A - 0$ such that $A/(s)$ is regular and e a positive integer invertible in A. Then

$B := A[t]/(t^e - s)$ is regular.

Proof. We split up the proof into two cases. First assume $s \notin \mathfrak{m}$, then we claim that $A \to B$ is étale. Indeed $\Omega_{A/B} = \langle dt | et^{e-1}dt = 0\rangle$ and $et^{e-1} \in B^\times$ by assumption. Hence $\Omega_{A/B} = 0$.

Now assume that $s \in \mathfrak{m}$. We see that $\mathfrak{m} + (t)$ is the unique maximal ideal of B and we compute

$$\begin{aligned}\mathtt{dim}_k \frac{\mathfrak{m} + (t)}{(\mathfrak{m} + (t))^2} &= \mathtt{dim}_k \frac{\mathfrak{m} \oplus tA \oplus \cdots \oplus t^{e-1}A}{(\mathfrak{m}^2 + (s)) \oplus t\mathfrak{m} \oplus t^2A \oplus \cdots \oplus t^{e-1}A} \\ &= \mathtt{dim}_k \frac{\mathfrak{m}}{\mathfrak{m}^2 + (s)} + \mathtt{dim}_k A/\mathfrak{m} < \mathtt{dim}_k \frac{\mathfrak{m}}{\mathfrak{m}^2} + 1.\end{aligned}$$

The final inequality follows as $s \in \mathfrak{m}$, but $s \notin \mathfrak{m}^2$, because $A/(s)$ was assumed to be regular. It follows that we must have $\mathtt{dim}_k \frac{\mathfrak{m}+(t)}{(\mathfrak{m}+(t))^2} = \mathtt{dim}_k \frac{\mathfrak{m}}{\mathfrak{m}^2}$, so B is regular. ⬠

Lemma 1.1.27 Let A be a smooth k-algebra, s an irreducible section and $e \geq 2$ an integer invertible in k. Let $B = A[t]/(t^e - s)$. Then B is smooth if and only if $A/(s)$ is smooth.

Proof. First notice that B_t is smooth, since it is étale over A_s. Any prime of B containing s also contains t so they are in bijection with the primes of $B/(t, s) = A/(s)$. Let $\mathfrak{p} \subset B$ be such a prime and $\mathfrak{q}$ the corresponding prime in $A/(s)$.

We may assume that A has a standard smooth presentation $k[x_1, \ldots x_n]/(f_1, \ldots, f_c)$, and write $B = k[x_1, \ldots x_n, t]/(f_1, \ldots, f_c, h)$, where $h = t^e - s$. If $A/(s)$ is smooth, then by [15, Lemma 00TE], for any $\mathfrak{q}$, we can rename variables so that

$$\mathtt{det} \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial s}{\partial x_j} \end{bmatrix}_{1 \leq i \leq c, 1 \leq j \leq c+1}$$

does not map to $\mathfrak{q}$. It then follows that

$$\mathtt{det} \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial t^e - s}{\partial x_j} \end{bmatrix}_{1 \leq i \leq c, 1 \leq j \leq c+1}$$

does not map to $\mathfrak{p}$, so B is smooth at $\mathfrak{p}$ for all $\mathfrak{p}$. (Note that $\frac{\partial t^e - s}{\partial x_j} = \frac{\partial s}{\partial x_j}$, so the determinant does not have any t-terms.)

On the other hand assume that $A/(s)$ is singular. Then, again by [15, Lemma 00TE], there is a prime $\mathfrak{q}$ such that for every relabelling of the x_i the determinant

$$\mathtt{det} \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial s}{\partial x_j} \end{bmatrix}_{1 \leq i \leq c, 1 \leq j \leq c+1}$$

maps to $\mathfrak{q}$. It follows that if we want a relabelling on the level of B we need to include t. Now consider

$$\mathtt{det}\begin{bmatrix}\frac{\partial f_i}{\partial x_j} & \frac{\partial t^e - s}{\partial x_j}\\ \frac{\partial f_i}{\partial t} & \frac{\partial t^e - s}{\partial t}\end{bmatrix}_{1\leq i,j\leq c} = et^{e-1}\,\mathtt{det}\left[\frac{\partial f_i}{\partial x_j}\right]_{1\leq i,j\leq c},$$

where we use $\frac{\partial f_i}{\partial t} = 0$ and $\frac{\partial t^e - s}{\partial t} = et^{e-1}$. So we see also for relabellings containing t the determinant lands in $\mathfrak{q}$. It follows that B is not smooth at $\mathfrak{q}$. ◯

> **Proposition 1.1.28** Let $\mathcal{C}$ be a stacky curve p a closed point and $e > 1$ a natural number not divisible by the characteristic of k. The root stack $\sqrt[e]{p/\mathcal{C}}$ is a stacky curve. Moreover, $\sqrt[e]{p/\mathcal{C}}$ is smooth over k if and only if $\mathcal{C}$ and $\mathcal{G}_p$ are smooth over k.

Proof. The only non-trivial facts are that $\sqrt[e]{p/\mathcal{C}}$ is DM and that $\sqrt[e]{p/\mathcal{C}}$ is regular. By Theorem 1.1.7 and Example 1.1.23 we can cover $\mathcal{C}$ by affine curves $\mathsf{Spec}(A) \to \mathcal{C}$ such that $\mathsf{Spec}(A) \times_{\mathcal{C}} \sqrt[e]{p/\mathcal{C}} \simeq [\mathsf{Spec}(B)/\mu_e]$, where $B = A[t]/(t^e - s)$ and $s \in A$ is a section corresponding to a reduced point. Since s is assumed to be reduced B is regular by Lemma 1.1.26 and it follows that $\sqrt[e]{p/\mathcal{C}}$ is a regular DM stack. The smoothness statement is immediate from Lemma 1.1.27. ◯

The proposition shows that root stacks naturally give rise to regular but non-smooth stacky curves, since over an imperfect base we can have closed points of a smooth curve that are not smooth themselves.

> **Example 1.1.29** Let $k = \mathbb{F}_p(t)$ and consider the curve $\mathbb{A}^1_k = \mathsf{Spec}(k[x])$, with the point $(-x^p - t)$. Then $\sqrt[e]{p/\mathbb{A}^1}$ is the curve $[\mathsf{Spec}(k[x,y]/(x^p + y^e + t))/\mu_e]$, so it is singular at the point $y = 0, x = t^{1/p}$ by [18, Example 3].

> **Proposition 1.1.30** Let C be a curve and let $\underline{p}$ be a set of closed points together with a set of multiplicities $\underline{e}$ and consider the root stack $\mathcal{X} := \sqrt[\underline{e}]{\underline{p}/C}$. The root morphism $\mathcal{X} \to C$ is the coarse space morphism.

Proof. Let $\pi : \mathcal{X} \to X$ be the coarse space morphism. By the universal property of the coarse space $\mathcal{X} \to C$ factors through a map $X \to C$. We can check that this is an isomorphism zariski-locally. Take an affine open $\mathsf{Spec}(A) = U \subset C$ containing a single of the $p \in \underline{p}$. By Example 1.1.23 we have $\mathcal{X} \times_C U = [\mathsf{Spec}(A[t]/(t^e - s))/\mu_e]$ and $X \times_C U = \mathsf{Spec}(A[t]/(t^e - s)^{\mu_e}) = \mathsf{Spec}(A) = U$. Since C can be covered by affine opens of this type $X \to C$ is an isomorphism. ◯

Proposition 1.1.31 Let $\mathcal{C}$ be a stacky curve and p a closed point on $\mathcal{C}$. The root morphism $\mathcal{X} = \sqrt[e]{p/\mathcal{C}} \to \mathcal{C}$ is ramified above p with degree e and it is universal (terminal) with respect to this property.

Proof. The ramification at p can computed using Example 1.1.23 and Example 1.1.13. Let $f : \mathcal{X} \to \mathcal{C}$ be a map of stacky curves and q a point of $\mathcal{X}$ ramified with degree e above $p \in \mathcal{C}$, then $(\mathcal{O}(\frac{1}{e_q}q), s_q)$ defines a map to $[\mathbb{A}^1_k/\mathbb{G}_m]$ and $f^*(\mathcal{O}_{\mathcal{C}}(\frac{1}{e_p}p), s_p) = (\mathcal{O}_{\mathcal{X}}(\frac{1}{e_q}q)^{\otimes e}, s_q^e)$ by Proposition 1.1.21. Hence f factors through $\mathcal{X}$ by the universal property of the fibre product. ◯

Theorem 1.1.32 Let $\mathcal{C}$ be a tame stacky curve with coarse space $\pi : \mathcal{C} \to C$ and let R_π be the ramification locus. Identifying the ramification locus with the branch locus we have that $\mathcal{C}$ is canonically isomorphic to $\sqrt[e_\pi]{R_\pi/C}$.

Proof. By the universal property of root stacks it follows that π factors via a map $\mathcal{C} \to \sqrt[e_\pi]{R_\pi/C}$. This map is unramified and induces an isomorphism of coarse spaces. By Theorem 1.1.15 it is an isomorphism. ◯

One immediate consequence of this important structure result is the following corollary.

Corollary 1.1.33 A stronger form of Theorem 1.1.7 holds for tame stacky curves, where we replace the étale morphism $V \to C$ by a Zariski neighbourhood of p. Moreover, the groups appearing are cyclic groups μ_e.

Corollary 1.1.34 Let G be a finite group of order not divisible by the characteristic of k acting on a smooth curve C. Then is for any fixed point x the residue field $\kappa(x)$ is separable over k.

Remark 1.1.35 In [16] they define a **separably rooted** smooth stacky curve to be a smooth stacky curve such that the residue fields of the stacky points are separable field extensions of the base. By Theorem 1.1.32 and Proposition 1.1.28 it follows that all smooth stacky curves are separably rooted.

The root stack description also defines a canonical isomorphism from the residual gerbe of a stacky point to $B\mu_e$.

Theorem 1.1.36 Consider the following commutative diagram.

$$\begin{tikzcd}
\mathcal{G}_p \arrow[r] \arrow[d] & \sqrt[e]{p/C} \arrow[r] \arrow[d] & \left[\mathbb{A}^1/\mathbb{G}_m\right] \arrow[r] \arrow[d] & B\mathbb{G}_m \arrow[d] \\
p \arrow[r] & C \arrow[r] & \left[\mathbb{A}^1/\mathbb{G}_m\right] \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

The outer square is a 2-Cartesian diagram. As a consequence the residual gerbe $\mathcal{G}_p$ is naturally isomorphic to $B\mu_e$, where $B\mu_e$ is considered as the kernel of the map $e : B\mathbb{G}_m \to B\mathbb{G}_m$.

Proof. By the universal property of the 2-fibre product we get a morphism $\mathcal{G}_p \to B\mu_e = B\mathbb{G}_m \times_{B\mathbb{G}_m} p$. On the other hand the morphism $B\mu_e \to B\mathbb{G}_m$ factors through $\left[\mathbb{A}^1/\mathbb{G}_m\right]$, so again by the universal property of 2-fibre products the morphism in fact factors via a morphism $B\mu_e \to \sqrt[e]{p/C}$. The image of this morphism is precisely p and since $B\mu_e$ is reduced it follows that it factors through $\mathcal{G}_p$. Summarizing we get a factorisation $\mathcal{G}_p \to B\mu_e \to \mathcal{G}_p \to \sqrt[e]{p/C} \to B\mathbb{G}_m$, showing that the natural morphism $\mathcal{G}_p \to B\mu_e$ is an isomorphism. ◯

We end this section with a technical definition that will be used when we want to reduce to the case of a stacky curve with a single stacky point.

Definition 1.1.37 Let $\mathcal{C}$ be a tame stacky curve. A **coarsening** $f : \mathcal{C} \to \mathcal{C}'$ is a map to a tame stacky curve $\mathcal{C}'$ inducing an isomorphism on coarse spaces.

Theorem 1.1.38 Let $\pi : \mathcal{C} \to \mathcal{C}'$ be a coarsening of tame stacky curves. Then $\mathcal{C}$ is canonically isomorphic to $\sqrt[e_\pi]{R_\pi/\mathcal{C}'}$.

Proof. This follows immediately from applying Theorem 1.1.32 to $\mathcal{C}$ and $\mathcal{C}'$. ◯

Example 1.1.39 let $\mathcal{C}$ be a stacky curve with coarse space $\pi : \mathcal{C} \to C$ and ramification divisor $R_\pi = \sum_{i=1}^n e_{p_i} p_i$. Set $\mathcal{C}_0 = C$ and $\mathcal{C}_i = \sqrt[e_i]{p_i/\mathcal{C}_{i-1}}$ then $\mathcal{C}_n = \mathcal{C}$ and the maps $r_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ are all coarsenings, such that $\pi = r_1 \circ \cdots \circ r_{n-1} \circ r_n$.

1.2 Sheaves on stacky curves

In this section we will develop the basic theory of coherent sheaves on stacky curves. We start by giving technical results relating sheaves on a stacky curve to sheaves on the coarse space. We then describe the discrete data of coherent sheaves and give several computational tools that use them. We classify the invertible bundles relative to the invertible bundles on the coarse space and we describe torsion sheaves in terms of cyclic quiver representations. We then compute the Grothendieck group of a stacky curve by showing that a coherent sheaf has a torsion filtration and that a locally free sheaf has a filtration by invertible sheaves. We end with a computation of the canonical sheaf of a stacky curve.

The functors π_* and π^*

We begin by giving an equivalent characterization of the tameness condition in terms of coherent sheaves.

Theorem 1.2.1 Let $\mathcal{C}$ be a stacky curve with coarse space map $\pi : \mathcal{C} \to C$, then $\mathcal{C}$ is tame if and only if the pushforward on the categories of quasi-coherent sheaves $\pi_* : \mathfrak{QCoh}(\mathcal{C}) \to \mathfrak{QCoh}(C)$ is exact.

Proof. The forward implication is [2, Lemma 2.3.4]. The full equivalence is proven in [1, Theorem 3.2]. ◯

Proposition 1.2.2 Let $\mathcal{C}$ be a tame stacky curve with coarse space morphism $\pi : \mathcal{C} \to C$. The functor π_* restricts to a functor of coherent sheaves $\mathfrak{Coh}(\mathcal{C}) \to \mathfrak{Coh}(C)$ and to a functor of vector bundles $\mathfrak{Vect}(\mathcal{C}) \to \mathfrak{Vect}(C)$.

Proof. This is [2, Lemma 2.3.4]. ◯

Proposition 1.2.3 Let $\mathcal{C}$ be a tame stacky curve with coarse space morphism $\pi : \mathcal{C} \to C$. The functor $\pi^* : \mathfrak{Coh}(C) \to \mathfrak{Coh}(\mathcal{C})$ is exact.

Proof. By Theorem 1.1.32 the map π is a root morphism and since the map $e : [\mathbb{A}^1_k/\mathbb{G}_m] \to [\mathbb{A}^1_k/\mathbb{G}_m]$ is faithfully flat, so is π. ◯

These formal properties of π_* are essential for our applications to coherent sheaves, so from this point onwards all our stacky curves will be assumed to be tame unless stated otherwise.

Theorem 1.2.4 Let $\pi^* : \mathcal{C} \to C$ be a stacky curve and $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{C}$.

1. The natural map $\mathcal{O}_C \to \pi_*\mathcal{O}_{\mathcal{C}}$ is an isomorphism.

2. The natural map $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \pi^*\pi_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \mathcal{F})$ is an isomorphism.
3. There is a natural isomorphism $\mathrm{Hom}_C(\mathcal{O}_C, \pi_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \mathcal{F})$ and as a consequence $H^i(\mathcal{C}, \mathcal{F}) = H^i(C, \pi^*\mathcal{F})$.

Proof.

1. Let $U \to C$ be étale then $U \times_C \mathcal{C} \to U$ is a coarse space morphism by Theorem 1.1.6, so any morphism $U \times_C \mathcal{C} \to \mathbb{A}^1$ factors uniquely through a morphism $U \to \mathbb{A}^1$.
2. There is an inverse given by sending a section $s : \mathcal{O}_{\mathcal{C}} \to \mathcal{F}$ to the composition

$$\mathcal{O}_{\mathcal{C}} \to \pi^*\mathcal{O}_C \to \pi^*\pi_*\mathcal{O}_{\mathcal{C}} \to \pi^*\pi_*\mathcal{F}.$$

3. We can compose a series of natural isomorphisms.

$$\mathrm{Hom}_C(\mathcal{O}_C, \pi_*\mathcal{F}) \to \mathrm{Hom}_C(\pi_*\mathcal{O}_{\mathcal{C}}, \pi_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \pi^*\pi_*\mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \mathcal{F}).$$

□

The optimistic interpretation of this theorem is that it is easy to compute sheaf cohomology on stacky curves, in fact it is just as easy as computing sheaf cohomology on classical curves. The pessimistic interpretation is that sheaf cohomology does not help us understand anything about the stacky structure of either the curve or the sheaves. However the above theorem is very specific to the structure sheaf $\mathcal{O}_{\mathcal{C}}$, so there is no analogue for ext groups. In other words ext groups do see stacky structure. Because of this we will phrase our results in terms of ext groups whenever possible.

Using the local form we can make the functors π_* and π^* very concrete.

Theorem 1.2.5 Let V be a curve together with the action of a finite group G, such that $[V/G]$ is a stacky curve. View a coherent sheaf on $[V/G]$ as a G-equivariant sheaf $\mathcal{F}$ on V. Then $\pi_*\mathcal{F} = \mathcal{F}^G$ is the G-invariant part of $\mathcal{F}$. If F is a coherent sheaf on V/G then π^*F is the pullback to V together with the trivial action.

Proof. This follows from the definitions. □

Corollary 1.2.6 Let $\pi : \mathcal{C} \to C$ be a stacky curve and let F be a coherent sheaf on C. Then the canonical morphism $F \to \pi_*\pi^*F$ is an isomorphism.

Discrete Data

Classically coherent sheaves on curves contain two pieces of discrete data, the rank and the degree. For stacky curves there is an additional class of discrete data at every stacky point of the curves.

Definition 1.2.7 Let $\mathcal{C}$ be a tame stacky curve and $\mathcal{E}$ a coherent sheaf on $\mathcal{C}$. Let p be a stacky point with multiplicity e_p and $i_p : \mathcal{G}_p \simeq B\mu_{e_p} \to \mathcal{C}$ be the inclusion of the residual gerbe at p, where the isomorphism is the canonical one from Theorem 1.1.36. Then the coherent sheaf $i^*\mathcal{E}$ defines a μ_{e_p} representation, which splits as $i^*\mathcal{E} \simeq \bigoplus_{i\in\mathbb{Z}/e_p\mathbb{Z}} V_i^{m_{p,i}}$, where V_i is the 1-dimensional μ_{e_p} representation corresponding to $i \in \mathbb{Z}/e_p\mathbb{Z}$. The numbers $m_{p,i}$ are called the **multiplicities** of $\mathcal{E}$ at p. We take the convention that $0 \le i \le e_p - 1$ and define the **multiplicity vector** $m_p(\mathcal{E}) = m_p := (m_{p,0}, \cdots, m_{p,e_p-1})$. Finally the collection of all the multiplicity vectors m_p for every stacky point p is called the multiplicities of $\mathcal{E}$ denoted by $\underline{m}(\mathcal{E}) = \underline{m}$.

Example 1.2.8 Let $\mathcal{C} := \sqrt[e]{p/C}$. The tautological sheaf $\mathcal{T}_p = \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p)$ has multiplicity vector $m_p = (0, 1, 0, \ldots, 0)$.

Proof. The pullback of the tautological sheaf corresponds to the composition $B\mu_e \to \sqrt[e]{p/C} \to [\mathbb{A}^1/\mathbb{G}_m] \to B\mathbb{G}_m$, which is the inclusion map by Theorem 1.1.36. ⬡

Since pullback commutes with taking tensor products and $V_i \otimes V_1 = V_{i+1}$ we can see that tensoring with the tautological sheaf acts as a shift operator on the multiplicities.

Example 1.2.9 Let $\mathcal{C} := \sqrt[e]{p/C}$ and let F be a coherent sheaf on C, then π^*F has multiplicity vector $m_p = (n, 0, \ldots, 0)$.

Proof. We have a commutative diagram.

$$\begin{array}{ccc} \mathcal{G}_p & \xrightarrow{i} & \mathcal{C} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(k) & \xrightarrow{\bar{i}} & C \end{array}$$

So we have $i_*\pi^*F = \phi^*\bar{i}^*F$, so $i_*\pi^*F$ is a trivial representation. ⬡

The above example actually classifies the coherent sheaves with "trivial" multiplicities.

Theorem 1.2.10 Let $\mathcal{C}$ be a stacky curve and $\mathcal{F}$ a coherent sheaf on $\mathcal{C}$, such that $m_p = (n, 0, \ldots, 0)$ for every stacky point p, then the canonical morphism $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Proof. Consider the local from Theorem 1.1.7.

$$\begin{array}{ccc} [V/\mu_e] & \xrightarrow{f} & \mathcal{C} \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ V/\mu_e & \xrightarrow{g} & C \end{array}$$

Where we now assume that μ_e is the stabilizer of a single point $p \in V$. By [13, Proposition 1.5] we have $f^*\pi^*\pi_*\mathcal{F} = \pi'^*g^*\pi_*\mathcal{F} = \pi'^*\pi'_*f^*\mathcal{F}$, so we can check that the canonical isomorphism is an isomorphism locally. View $\mathcal{F}$ as a μ_e-equivariant sheaf on V, so that $\mathcal{F} \simeq \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathcal{F}_i$ decomposes into eigensheaves. Then $\pi^*\pi_*\mathcal{F} = \mathcal{F}_0$, so we have to show that $\mathcal{F}_i = 0$ for $i \neq 0$. We have a Cartesian square.

$$\begin{array}{ccc} \mathsf{Spec}(k) & \xrightarrow{p} & V \\ \big\downarrow & & \big\downarrow \\ B\mu_e & \xrightarrow{i} & [V/G] \end{array}$$

Showing that $i^*\mathcal{F}$ is the same as the fibre of $\mathcal{F}$ at p together with the μ_e action on this fiber. Since $i^*\mathcal{F}$ is a trivial representation it follows that $\mathcal{F}_i = 0$ for $i \neq 0$. ◯

Corollary 1.2.11 Let $\mathcal{C}$ be a stacky curve and let $\mathcal{L}$ be a line bundle on $\mathcal{C}$. For each stacky point p of order e_p, let a_p be the unique number such that $m_{p,a_p} \neq 0$. We have $\mathcal{L} \simeq \pi^*L \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes a_p}$ for a unique (up to isomorphism) line bundle L on C.

Proof. We can apply the above theorem to $\mathcal{L} \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}$ to see

$$\pi^*\pi_*\left(\mathcal{L} \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}\right) = \mathcal{L} \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}.$$

Now set $L = \pi_*\left(\mathcal{L} \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}\right)$ to get

$$\pi^*L \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p} = \mathcal{L}.$$

Now let $\pi^*L \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p} \simeq \pi^*L' \otimes \bigotimes_p \mathcal{O}(\frac{1}{e_p}p)^{\otimes -a_p}$, then $\pi^*L \simeq \pi^*L'$, so $L = \pi_*\pi^*L \simeq \pi_*\pi^*L' = L'$. ◯

Corollary 1.2.12 Let $\pi : \mathcal{C} \to C$ be a stacky curve with stacky points p_i of order e_i for $1 \leq i \leq n$. Denote by $\mathtt{Pic}_{\mathcal{C}}$ the (set theoretic) Picard group of $\mathcal{C}$. We have an isomorphism of groups

$$\mathtt{Pic}_{\mathcal{C}} \simeq \mathtt{Pic}_C[\mathcal{O}_C(p_1)/e_1, \ldots, \mathcal{O}_C(p_n)/e_n],$$

given by $\mathcal{O}_C(p_i)/e_i \mapsto \mathcal{O}_{\mathcal{C}}(\frac{1}{e_i}p_i)$.

For completeness we also rephrase Corollary 1.2.11 in terms of Weil divisors.

Corollary 1.2.13 Let $\pi : \mathcal{C} \to C$ be a stacky curve and $p \in \mathcal{C}$ be a stacky point of order e. For $m \in \mathbb{Z}$ we have $\pi^*(mp) = \frac{em}{e}p$ and $\pi_*(\frac{m}{e}p) = \lfloor \frac{m}{e} \rfloor p$. Where $\lfloor x \rfloor$ is the floor of x, i.e. the largest integer n such that $n \leq x$.

Locally Free Sheaves

Having classified the line bundles on stacky curves, we now show that every torsion-free sheaf is a vector bundle (locally free) and that vector bundles are iterated extensions of line bundles, as in the case of classical curves.

Definition 1.2.14 Let $\mathcal{C}$ be a stacky curve and $\mathcal{E}$ be a coherent sheaf on $\mathcal{C}$. We define the **torsion subsheaf** $\mathcal{T} \subset \mathcal{E}$ to be the maximal subsheaf of $\mathcal{E}$ that is torsion. We say that $\mathcal{E}$ is **torsion-free** if $\mathcal{T} = 0$.

Theorem 1.2.15 Let $\mathcal{C}$ be a stacky curve and $\mathcal{E}$ be a torsion free sheaf, then $\mathcal{E}$ is locally free.

Proof. By Theorem 1.1.7 there is an étale cover $f : U \to \mathcal{C}$ of $\mathcal{C}$ by classical curves. Then $f^*\mathcal{E}$ is a torsion free sheaf on a (disconnected) classical (regular) curve U, hence locally free. ◯

Note that locally free should be interpreted in the étale topology. For a stacky point p there is no Zariski neighbourhood U of p such that $\mathcal{O}_{\mathcal{C}}|_U \simeq \mathcal{O}_{\mathcal{C}}(\frac{1}{e}p)|_U$, since they are not isomorphic after pulling back to $\mathcal{G}_p$.

Corollary 1.2.16 Let $\mathcal{C}$ be a stacky curve and $\mathcal{E}$ a coherent sheaf on $\mathcal{C}$. We have a short exact sequence

$$0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{T}$ is the torsion subsheaf of $\mathcal{E}$ and $\mathcal{F}$ is locally free.

Proof. Let $q : \mathcal{E} \to \mathcal{E}/\mathcal{T} =: \mathcal{F}$ be the quotient map and let $\mathcal{T}'$ be the torsion subsheaf of $\mathcal{F}$, then $q^{-1}(\mathcal{T}') + \mathcal{T}$ is torsion, so by maximality of $\mathcal{T}$ we have that $q^{-1}(\mathcal{T}') \subset \mathcal{T}$, so $\mathcal{T}' = 0$. ◇

Proposition 1.2.17 Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of locally free sheaves on a stacky curve then $\underline{m}(\mathcal{E}) + \underline{m}(\mathcal{G}) = \underline{m}(\mathcal{F})$.

Proof. This is immediate as the pullback functor to the residual gerbe $\mathcal{G}_p$ is exact on locally free sheaves. ◇

The above proposition is false for general coherent sheaves. Consider for example a short exact sequence of the form $0 \to \mathcal{O}(-\mathcal{G}_p) \to \mathcal{O}_{\mathcal{C}} \to \mathcal{T} \to 0$. Then pulling back to $\mathcal{G}_p$ we get the short exact sequence $V_{e-1} \to V_0 \to i^*\mathcal{T} \to 0$. The first arrow must be the zero map, so we get $m_p(\mathcal{T}) = m_p(\mathcal{O}_{\mathcal{C}})$.

Lemma 1.2.18 Let $\mathcal{F}$ be a locally free sheaf of rank r on a stacky curve $\mathcal{C}$. There exists a sequence of surjective maps

$$\mathcal{F} = \mathcal{E}_0 \twoheadrightarrow \mathcal{E}_2 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}_r,$$

such that $\mathcal{E}_i$ is locally free and $\mathcal{L}_i := \ker\left(\mathcal{E}_i \to \mathcal{E}_{i+1}\right)$ is an invertible sheaf. Moreover $\underline{m}(\mathcal{F}) = \sum_{i=1}^{r} \underline{m}(\mathcal{L}_i)$.

Proof. Let $D >> 0$ be a positive divisor of large degree on the coarse space C, then $\pi_*\mathcal{F}(D)$ admits a non-zero section, so by Theorem 1.2.4 we get a non-zero section $\mathcal{O}_{\mathcal{C}} \to \mathcal{F} \otimes \pi^*\mathcal{O}_C(D)$. This gives rise to a subsheaf $\pi^*\mathcal{O}_C(-D) \to \mathcal{F}$. Let $\mathcal{T}$ be the torsion sheaf of $\mathcal{F}/\pi^*\mathcal{O}_C(-D)$ and take the saturation

$$\mathcal{L}_0 = \overline{\pi^*\mathcal{O}_{\mathcal{C}}(D)} := \ker \mathcal{F} \to (\mathcal{F}/\pi^*\mathcal{O}_C(-D))/\mathcal{T}$$

and set $\mathcal{E}_1 := \mathcal{F}/\pi^*\mathcal{O}_C(-D))/\mathcal{T}$. The saturation of an invertible sheaf is again an invertible sheaf and $\mathcal{E}_1$ is locally free by construction. The vector bundle $\mathcal{E}_1$ has rank $r-1$, so iteratively applying this construction finishes the proof. ◇

Torsion sheaves

Now that we have a basic understanding of vector bundles we move on to torsion sheaves. We start by giving a very explicit description of torsion sheaves in terms of quiver representations.

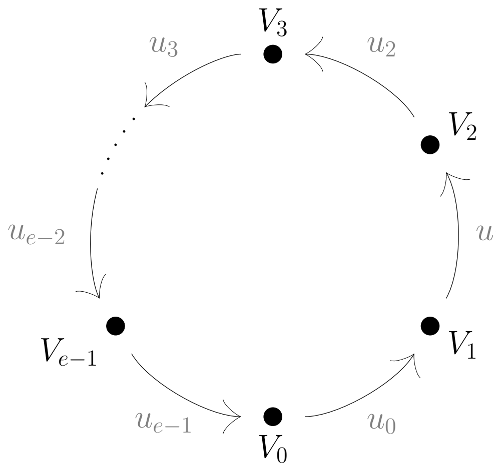

Figure 1.1: A quiver representation of the cyclic quiver

Definition 1.2.19 A **k-quiver representation of the cyclic quiver with e vertices** is a $\mathbb{Z}/e\mathbb{Z}$-graded k-vector space together with a degree 1 map. More explicitly, it is a collection of k-vector spaces V_i and linear maps $u_i : V_i \to V_{i+1}$ indexed by $i \in \mathbb{Z}/e\mathbb{Z}$. See Figure 1.1 for a pictorial interpretation. A morphism of quiver representations $(V_i, u_i) \to (W_i, w_i)$ is a collection of linear maps $\phi_i : V_i \to W_i$, such that $\phi_i \circ u_i = w_i \circ \phi_i$.

A quiver representation is said to be **nillpotent** if the map is nilpotent.

Theorem 1.2.20 Let $\mathcal{C}$ be a stacky curve and p a stacky point of order e. There is an equivalence of categories between the category of torsion sheaves supported on p and the category of nilpotent $\kappa(p)$-quiver representations of the cyclic quiver with e vertices.

Proof. Take a local form $[V/\mu_e]$ around the point p, such that the μ_e action fixes a unique point $q \in V$. Now the category of torsion sheaves on $\mathcal{C}$ supported on p is equivalent to the category of μ_e-equivariant torsion sheaves on V supported on q.

Let $R := \mathcal{O}_{V,q}$ be the local ring at q with maximal ideal $\mathfrak{m}$, then there is an induced μ_e-action on $\mathrm{Spec}(R)$, which induces a $\mathbb{Z}/e\mathbb{Z}$-grading $R = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} R_i$. Since the μ_e action fixes $\mathfrak{m}$ it is a homogeneous ideal of R for this grading. It follows that there is a homogeneous uniformizer $u \in \mathfrak{m}$, which using the conventions of Theorem 1.1.36 has degree 1.

Now the category of μ_e-equivariant torsion sheaves supported on q is naturally equivalent to the category of $\mathbb{Z}/e\mathbb{Z}$-graded torsion modules over R.

Next we notice that a torsion module over R is an R-module M such that $u^n M = 0$ for some n. This means that the category of torsion R-modules is equivalent to the category of pairs M, n, where M is an $R/\mathfrak{m}^n$-module such that $u^{n-1}M \neq 0$ (together with the pair $(0, -\infty)$.) and the morphisms are morphisms of R-modules after extending scalars. Moreover $R/\mathfrak{m}^n$ inherits the grading of R and this equivalence respects gradings. Since $R/\mathfrak{m}^n = \hat{R}/\mathfrak{m}^n$ it follows that the category of graded torsion modules over R is equivalent to the category of graded torsion modules over $\hat{R}$. (Note that $\hat{R}$ has a natural grading, since we complete in a homogeneous ideal.)

Finally, by the Cohen structure theorem we know that $\hat{R} \simeq \kappa(p)[[X]]$, where we can chose X to map to u. Then the induced grading on $\kappa(p)[[X]]$ is the one where X^i is homogeneous of degree i. A $\kappa(p)[[X]]$-module is torsion if and only if it is finite dimensional as a $\kappa(p)$ vector space. It follows that the category of graded torsion $\kappa(p)[[X]]$-modules is equivalent to the category pairs (V, u), where V is a $\mathbb{Z}/e\mathbb{Z}$-graded $\kappa(p)$-vector space and $X : V \to V$ is a degree 1 map.

◯

If we view non-stacky points as stacky points of order 1 we recover the fact that a torsion sheaf on a curve supported on single point corresponds to nilpotent representation of the Jordan quiver.

Example 1.2.21 Let $\mathcal{C}$ be a stacky curve and p a stacky point of order e. Define the torsion sheaf $\mathcal{T}_i$ via the exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}}(-\frac{i}{e}p) \to \mathcal{O}_{\mathcal{C}} \to \mathcal{T}_i \to 0,$$

for $1 \leq i \leq e$. On the level of $\kappa(p)[[u]]$-modules this exact sequence becomes

$$0 \to u^i\kappa(p)[[u]] \to \kappa(p)[[u]] \to \kappa(p)[[u]]/\langle u^i \rangle.$$

We can now see that $\mathcal{T}_i$ corresponds to the quiver representation $V_0 = V_1 = \cdots V_{i-1} = \kappa(q)$ and $V_i = \cdots = V_{e-1} = 0$ with the identity maps wherever possible. Except for $\mathcal{T}_e$, where the map $V_{e-1} \to V_0$ is the zero map.

Remark 1.2.22 Chasing through all of the definitions we can see that for a torsion sheaf supported on a stacky point p the multiplicities $m_{p,i}(\mathcal{T})$ are exactly the dimensions of the vector spaces V_i.

Theorem 1.2.23 Let $\mathcal{C}$ be a stacky curve with a stacky point p of order e. The irreducible torsion sheaves supported on p are all fit in the exact sequence

$$0 \to \mathcal{O}(-\frac{i+1}{e}p) \to \mathcal{O}(-\frac{i}{e}p) \to \mathcal{T}_i \to 0.$$

Proof. Let $\mathcal{T}$ be an irreducible torsion sheaf supported on p and consider the associated quiver representation $u : V \to V$. Since u is nilpotent it must send some nonzero vector

$v_i \in V_i \subset V$ to 0. Then the we have a subrepresentation $\mathcal{T}_i \simeq u_i : k \cdot v_i \to 0$, which by irreducibility must be an isomorphism. Such a quiver representation corresponds to the module $u^i\kappa(p)[[u]]/u^{i+1}\kappa(p)[[u]]$. ○

Corollary 1.2.24 Let $\mathcal{C}$ be a stacky curve. The Grothendieck group $K(\mathcal{C}) \simeq \mathbb{Z} \oplus \mathtt{Pic}_{\mathcal{C}}$.

Proof. This follows immediately from Lemma 1.2.18 and Theorem 1.2.23. ○

Definition 1.2.25 Consider the composition

$$K(\mathcal{C}) \to \mathtt{Pic}_{\mathcal{C}} \to \mathtt{Pic}_{C}[\mathcal{O}(p_1)/e_1, \ldots, \mathcal{O}(p_n)/e_n] \to \mathbb{Z}[d_1/e_1, \ldots, d_n/e_n] \subset \mathbb{Q},$$

where the first map is the projection and the last map is induced by the degree map on $\mathtt{Pic}_C$. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We define the **degree** deg $\mathcal{F}$ to be the image under this composition.

Note that we allow fractional degrees, but the denominators of the fractions are bounded in terms of the orders of the stacky points. This definition is chosen so that the pullback from the coarse space $\pi^* : K(C) \to K(\mathcal{C})$ is degree preserving and in fact it is uniquely defined by this property.

The rank of a vector bundle and its pushforward to the coarse space agree. The same is not true for the degree, but the difference can be expressed in terms of the multiplicities.

Theorem 1.2.26 Let $\mathcal{E}$ be a locally free sheaf with multiplicities $\underline{m}$. We have deg $\mathcal{E} =$ deg$(\pi_*\mathcal{E}) + \sum_p \frac{1}{e_p} \sum_{i=0}^{e_p-1} im_{p,i}$.

Proof. Both sides of the equation are additive in short exact sequences, so we can reduce to the case of invertible sheaves by Lemma 1.2.18. The case of line bundles follows from Corollary 1.2.11. ○

The Cotangent sheaf

We end this section with a discussion on the cotangent sheaf of stacky curves. We will start from a very abstract definition and then show that it can be very concretely described. The abstract definition is not necessary for any of our results, so it should only be viewed as motivation for the concrete description which we will actually use.

Definition 1.2.27 Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of DM-stacks, following [8] we define the **cotangent sheaf** $\Omega_{\mathcal{X}/\mathcal{Y}}$ on the étale site of $\mathcal{X}$ as follows. Let $\mathcal{I}$ be the kernel of the multiplication morphism $\mathcal{O}_{\mathcal{X}} \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$, then $\Omega_{\mathcal{X}/\mathcal{Y}} := \mathcal{I}/\mathcal{I}^2$.

We have two canonical exact sequences.

Theorem 1.2.28 Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$ be morphisms of DM-stacks. We have a short exact sequence
$$f^*\Omega_{\mathcal{Y}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0.$$
If $\mathcal{O}_{\mathcal{X}}$ is a locally free $f^{-1}\mathcal{O}_{\mathcal{Y}}$-module then we can extend the sequence to
$$0 \to f^*\Omega_{\mathcal{Y}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Z}} \to \Omega_{\mathcal{X}/\mathcal{Y}} \to 0.$$

Proof. This follows from [8, (1.1.2.12)] and [8, (1.1.2.13)] ◯

Theorem 1.2.29 Let $i : \mathcal{Y} \to \mathcal{X}$ be a closed immersion of DM-stacks with ideal sheaf $\mathcal{J}$. We have a canonical short exact sequence
$$\mathcal{J}/\mathcal{J}^2 \to i^*\Omega_{\mathcal{X}} \to \Omega_{\mathcal{Y}} \to 0.$$

Proof. This follows from [8, (1.1.6.2)]. ◯

Theorem 1.2.30 Let $\pi : \mathcal{C} \to C$ be a smooth tame stacky curve with stacky points $\underline{p}$. We have
$$\Omega_{\mathcal{C}} \simeq \pi^*\Omega_C \otimes \bigotimes_{p\in\underline{p}} \mathcal{O}(\frac{1}{e_p}p)^{\otimes e_p - 1}.$$

Proof. Let $u : U \to \mathcal{C}$ be an étale atlas for $\mathcal{C}$, then U is smooth and Ω_U is a line bundle. From Theorem 1.2.28 we get an exact sequence $0 \to u^*\Omega_{\mathcal{C}} \to \Omega_U \to \Omega_{U/\mathcal{C}} = 0$, so $\Omega_{\mathcal{C}}$ is a line bundle.

Now apply Theorem 1.2.28 to the coarse space map $\pi : \mathcal{C} \to C$ to get a short exact sequence
$$\pi^*\Omega_C \to \Omega_{\mathcal{C}} \to \Omega_{\mathcal{C}/C} \to 0.$$

The sequence extends to the left since $\pi^*\Omega_C \to \Omega_{\mathcal{C}}$ is a map of line bundles that is generically an isomorphism, hence injective. Since $\Omega_{\mathcal{C}/C}$ is supported on the stacky points it follows from Corollary 1.2.11 that $\Omega_{\mathcal{C}} = \pi^*\Omega_C \otimes \bigotimes_{p\in\underline{p}} \mathcal{O}_{\mathcal{C}}(\frac{1}{e_p}p)^{\otimes n_p}$ for non-negative integers n_p.

To compute n_p we can take a local form around p.

$$\begin{array}{ccc} U & & \\ \downarrow{\scriptstyle \phi} & & \\ [U/\mu_{e_p}] & \xrightarrow{g} & \mathcal{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ U/\mu_{e_p} & \xrightarrow{f} & C \end{array}$$

Denote the preimage of p under g also by p and let q be the unique point in U sitting above p. Then pulling back along g we get

$$\Omega_{[U/\mu_{e_p}]} = g^*\Omega_{\mathcal{C}} = g^*\pi^*\Omega_C \otimes \mathcal{O}_{[U/\mu_{e_p}]}(\frac{1}{e_p}p)^{\otimes n_p} = \pi'^*\Omega_{U/\mu_{e_p}} \otimes \mathcal{O}_{[U/\mu_{e_p}]}(\frac{1}{e_p}p)^{\otimes n_p}.$$

Pulling back once more along ϕ we see

$$\Omega_U = \phi^*\Omega_{[U/\mu_{e_p}]} = (\phi \circ \pi')^*\Omega_{U/\mu_{e_p}} \otimes \mathcal{O}_U(q)^{\otimes n_p}.$$

Now it follows from the ramification theory of classical curves that $n_p = e_p - 1$. ⬠

To get a similar result for non-smooth curves one should work with the canonical sheaf instead, but we will not develop the theory of canonical sheaves for DM-stacks here.

1.3 Projective stacky curves

In this section we develop a theory of projective stacky curves analogous to the theory of classical projective curves. The main difference from the classical theory is that the polarization of a stacky curve is not given by a line bundle, but by a higher rank vector bundle called a generating sheaf. It is important to note that on a classical curve many results do not depend on the choice of a polarizing line bundle, but in the stacky setting this is no longer the case.

> **Definition 1.3.1** A projective stacky curve $\mathcal{C}$ is a smooth tame stacky curve with a coarse space C that is projective.

Warning: The definition of a projective stack is more subtle, but for stacky curves this naive definition is good enough. See [10] for the higher dimensional.

> **Proposition 1.3.2** Let $\mathcal{C}$ be a tame stacky curve with coarse space C. If $\mathcal{C}$ is proper, then $\mathcal{C}$ is projective.

Proof. By Theorem 1.1.6, C is proper if and only if $\mathcal{C}$ is. Since a proper classical curve is projective the result follows. ⬠

> **Definition 1.3.3** Let $\mathcal{C}$ be a projective stacky curve. We define the euler characteristic $\chi_{\mathcal{C}} := -\deg \omega_{\mathcal{C}}$. We then define the genus $g_{\mathcal{C}}$ via $2 - 2g_{\mathcal{C}} = \chi_{\mathcal{C}}$.

Since the canonical bundle can have rational degree, the Euler characteristic and genus are not integers in general. This means for example that there is no cohomological description like $h^1(\mathcal{O}_C) = g_C$. One big motivation for this definition is that it satisfies an analogue of the Riemann-Hurwitz theorem. We first state the Riemann-Hurwitz theorem applied to the coarse space map.

> **Theorem 1.3.4** Let $\pi : \mathcal{C} \to C$ be a projective stacky curve with stacky points $\underline{p}$. We have
> $$\chi_{\mathcal{C}} = \chi_C - \sum_{p \in \underline{p}} \frac{e_p - 1}{e_p} [\kappa(p) : k]$$
> and
> $$g_{\mathcal{C}} = g_C + \frac{1}{2} \sum_{p \in \underline{p}} \frac{e_p - 1}{e_p} [\kappa(p) : k]$$

Proof. This follows immediately from Theorem 1.2.30. ⬠

> **Theorem 1.3.5** (Riemann-Hurwitz) Let $f : \mathcal{C} \to \mathcal{D}$ be a map of stacky curves tamely ramified at the points p_i with degree e_i. We have
> $$f^* \omega_{\mathcal{D}} = \omega_{\mathcal{C}} \bigotimes_i \mathcal{O}(\mathcal{G}_{p_i})^{e_i - 1}.$$
> And as a consequence
> $$\chi_{\mathcal{C}} = (\deg f) \cdot \chi_{\mathcal{D}} - \sum_i (e_i - 1) \deg(\mathcal{G}_{p_i}).$$

Proof. Let $\pi_C : \mathcal{C} \to C$ and $\pi_D : \mathcal{D} \to D$ be the coarse space morphisms and let $g : C \to D$ be the map induced by $\pi_D \circ f$. By Theorem 1.2.30, we know the theorem holds for $\pi_{\mathcal{C}}$ and $\pi_{\mathcal{D}}$, by the classical Riemann-Hurwitz theorem the theorem holds for g as well. An easy computation then shows that the theorem holds for f. ⬠

We give a short proof of the following well known result to highlight the effectiveness of the genus.

> **Theorem 1.3.6** Let $m \neq n$ by natural numbers not divisible by the characteristic of k, then the football space $\mathcal{F}(m, n)$ is not the quotient of a classical curve by a finite group.

Proof. Assume there is a classical curve C with an action of a finite group G such that $[C/G] \simeq \mathcal{F}(m,n)$. Then $C/G \simeq \mathbb{P}^1_k$, so C is projective. The map $C \to \mathcal{F}(m,n)$ is unramified, so we can apply Riemann-Hurwitz to see

$$\chi_C = |G|\chi_{\mathcal{F}(m,n)} = |G|(2 - (\frac{m-1}{m} + \frac{n-1}{n}) = |G|\frac{m+n}{mn}.$$

Since the right hand side is positive it follows that $\chi_C = 2$. Now write d for the greatest common divisor of m and n so that $m = da$ and $n = db$ for positive integers a and b. Since G contains subgroups of order m and n (the stabilizers of $0, \infty \in \mathcal{F}(m,n)$) we must have that dab divides $|G|$. Write $|G| = xdab$ so the equation $2 = |G|\frac{m+n}{mn}$ becomes $2 = x(a+b)$, which implies that $a = b = 1$, but then $m = n$ is a contradiction. ◯

We move on to proving Serre duality.

Theorem 1.3.7 (Serre Duality) Let $\mathcal{E}$ be a coherent sheaf on a projective stacky curve $\mathcal{C}$, we have a natural isomorphism

$$\mathtt{Ext}^i(\mathcal{E}, \omega_{\mathcal{C}}) \simeq \mathtt{Ext}^{1-i}(\mathcal{O}_{\mathcal{C}}, \mathcal{E})^\vee,$$

for $i = 0, 1$.

Proof. We can reduce to the case that $\mathcal{E}$ is a line bundle $\mathcal{L} \simeq \pi^* L \otimes \bigotimes_p \mathcal{O}(\frac{i_p}{e_p}p)$. Now we apply Serre duality on C to get

$$\mathtt{Ext}^i(\mathcal{L}, \omega_{\mathcal{C}}) \simeq \mathtt{Ext}^i\big(\mathcal{O}_C, L^\vee \otimes \omega_C\big) \simeq \mathtt{Ext}^{1-i}(\mathcal{O}_C, L)^\vee \simeq \mathtt{Ext}^{1-i}(\mathcal{O}_{\mathcal{C}}, \mathcal{L})^\vee.$$

The first isomorphism follows as

$$\pi_*(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}) = L^\vee \otimes \omega_C \otimes \pi_* \bigotimes_p \mathcal{O}(\frac{e_p - 1 - i_p}{e_p}p) = L^\vee \otimes \omega_C.$$

◯

Remark 1.3.8 Even though in general we have $\pi_*(\mathcal{F}^\vee) \neq (\pi_*\mathcal{F})^\vee$ the above proof shows that the Serre duals $S_{\mathcal{C}} : \mathcal{F} \to \mathcal{H}om_{\mathcal{C}}(\mathcal{F}, \omega_{\mathcal{C}})$ and $S_C : F \to \mathcal{H}om_C(F, \omega_C)$ do commute with π_*, i.e. $\pi_* \circ S_{\mathcal{C}} = S_C \circ \pi_*$.

We now state the naive Riemann-Roch theorem for stacky curve. The reason we call this the naive Riemann Roch theorem is that it does not involve any stacky structure of the line bundles nor the curve itself.

Theorem 1.3.9 (Naive Riemann-Roch) Let $\mathcal{C}$ be a projective stacky curve, with coarse space $\pi : \mathcal{C} \to C$. Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$. Then

$$h^0(\mathcal{L}) - h^0(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}) = \deg \pi_*\mathcal{L} + 1 - g_C$$

Proof. By the remark above we have $h^0(\mathcal{L}) - h^0(\mathcal{L}^\vee \otimes \omega_{\mathcal{C}}) = h^0(\pi_*\mathcal{L}) - h^0((\pi_*\mathcal{L})^\vee \otimes \omega_C) = \deg \pi_*\mathcal{L} + 1 - g_C$. ⬠

Generating sheaves

We will now spend some time defining generating sheaves, which will serve as a polarization of a projective curve. Generating sheaves where first introduced in [14] in order to embed quot schemes on tame DM stacks into quot schemes over their coarse space.

Definition 1.3.10 Let $\pi : \mathcal{C} \to C$ be a stacky curve $\mathcal{E}$ a locally free sheaf on $\mathcal{C}$. Following [14] we define the functor $F_{\mathcal{E}} : \mathsf{Coh}(\mathcal{C}) \to \mathsf{Coh}(C)$ as

$$F_{\mathcal{E}}(\mathcal{F}) := \pi_*\mathcal{H}om(\mathcal{E}, \mathcal{F}) = \pi_*(\mathcal{F} \otimes \mathcal{E}^\vee).$$

And in the other direction $G_{\mathcal{E}} : \mathsf{Coh}(C) \to \mathsf{Coh}(\mathcal{C})$

$$G_{\mathcal{E}}(F) := \pi^*(F) \otimes \mathcal{E}.$$

Definition 1.3.11 The identity map $\pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \to \pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F}))$ has a left adjoint

$$\pi^*\pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \to \mathcal{H}om(\mathcal{E}, \mathcal{F}),$$

which has a left adjoint

$$\pi^*\pi_*(\mathcal{H}om(\mathcal{E}, \mathcal{F})) \otimes \mathcal{E} \to \mathcal{F}.$$

We denote this left adjoint of the left adjoint by $\theta_{\mathcal{E}} : G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}) \to \mathcal{F}$.

Definition 1.3.12 Let $\mathcal{E}$ be a locally free sheaf on a stacky curve $\mathcal{C}$. If $\theta_{\mathcal{E}}(\mathcal{F})$ is surjective, then $\mathcal{E}$ is called a **generator** for $\mathcal{F}$. If $\mathcal{E}$ is a generator for all coherent sheaves $\mathcal{F}$ on $\mathcal{C}$, then $\mathcal{E}$ is a **generating sheaf** for $\mathcal{C}$.

It is not so obvious how to verify if a sheaf is generating directly, but the following condition is easy to check in practice.

Theorem 1.3.13 (Local condition of generation) Let $\mathcal{C}$ be a stacky curve with stacky points $\underline{p}$ and $\mathcal{E}$ a locally free sheaf. Then $\mathcal{E}$ is a generating sheaf if and only if $m_{p,j} > 0$ for every $p \in \underline{p}$ and $0 \leq j \leq e_p - 1$. In other words the representations $\iota^*\mathcal{E}$ for

$\iota : \mathcal{G}_p \hookrightarrow \mathcal{C}$ contain all irreducible representations of μ_{e_p}.

Proof. First off all the surjectivity of $\theta_{\mathcal{E}}(\mathcal{F})$ can be checked locally, so we will assume that $\mathcal{C}$ has a single stacky point p of order e.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent sheaves. We get a commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0 \\
0 \arrow[r] & G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}_1) \arrow[r] \arrow[u, "\theta_{\mathcal{F}_1}"] & G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}_2) \arrow[r] \arrow[u, "\theta_{\mathcal{F}_2}"] & G_{\mathcal{E}} \circ F_{\mathcal{E}}(\mathcal{F}_3) \arrow[r] \arrow[u, "\theta_{\mathcal{F}_3}"] & 0
\end{tikzcd}$$

Because of this we know that if two of the $\theta_{\mathcal{E}}(\mathcal{F}_i)$ are surjective, so is the third. Since $K_0(\mathcal{C})$ is generated by line bundles it follows that we only have to show surjectivity for line bundles. To verify if $\mathcal{E}$ generates the line bundles $\mathcal{L} \simeq \pi^*L \otimes \bigotimes_i \mathcal{O}(\frac{j}{e_i}p_i)$. We can rewrite $\theta_{\mathcal{L}}$ as

$$\pi^*\pi_*(\mathcal{H}om\left(\mathcal{E} \otimes \mathcal{O}(\frac{-j}{e}p), \mathcal{O}_{\mathcal{C}}\right)) \otimes \pi^*L \otimes \mathcal{E} \to \pi^*L \otimes \mathcal{O}(\frac{j}{e}p).$$

Tensoring both sides by $\mathcal{L}^\vee$ and denoting $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(\frac{-j}{e}p)$ we get the morphism

$$\theta_{\mathcal{L}} \otimes \mathcal{L}^\vee : \pi^*\pi_*(\mathcal{H}om(\mathcal{E}', \mathcal{O}_{\mathcal{C}})) \otimes \mathcal{E}' \to \mathcal{O}_{\mathcal{C}},$$

which is precisely $\theta_{\mathcal{E}'}(\mathcal{O}_{\mathcal{C}})$. Since $\mathcal{E}'$ also satisfies the local condition of generation we have reduced to the case $\mathcal{L} = \mathcal{O}_{\mathcal{C}}$.

Now we apply Lemma 1.2.18 to $\mathcal{E}'$ and get a chain of surjective maps $\mathcal{E}' = \mathcal{E}_0 \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_r$. From the local condition it follows that there exists a line bundle $\mathcal{L} = \ker(\mathcal{E}_i \to \mathcal{E}_{i+1})$ with $m_p(\mathcal{L}) = (1, 0, \ldots, 0)$, i.e. $\mathcal{L} \simeq \pi^*L$ for some L on C. Now we have a commutative diagram.

$$\begin{tikzcd}
\pi^*\pi_*\mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{O}_{\mathcal{C}} \arrow[ddddr, bend left] & \\
\pi^*\pi_*\mathcal{E}_i^\vee \otimes \mathcal{L} \arrow[u, two heads] \arrow[d, hook] \arrow[dddr, bend left] & \\
\pi^*\pi_*\mathcal{E}_i^\vee \otimes \mathcal{E}_i \arrow[ddr, bend left] & \\
\pi^*\pi_*\mathcal{E}_i^\vee \otimes \mathcal{E}_0 \arrow[u, two heads] \arrow[d, hook] \arrow[dr] & \\
\pi^*\pi_*\mathcal{E}_0^\vee \otimes \mathcal{E}_0 \arrow[r] & \mathcal{O}_{\mathcal{C}}
\end{tikzcd}$$

The top arrow is an isomorphism and it follows that all of the other horizontal arrows are surjective. ○

The general case of the above theorem can be found in [14]. However there it is claimed that for a stacky point $\zeta : \mathrm{Spec}(k) \to C$ with stabilizer G_ζ we have $\mathrm{Spec}(k) \times_C \mathcal{C} = BG_\zeta$. This is of course not true, since π is ramified above ζ. We do have $(\mathrm{Spec}(k) \times_C \mathcal{C})_{\mathrm{red}} = BG_\zeta$, which is enough to make their proofs work.

Example 1.3.14 Let $\mathcal{C}$ be a stacky curve with stacky points p_i of order e_i, then $\mathcal{E} := \bigotimes_{p_i} \bigoplus_{j=0}^{e-1} \mathcal{O}(\frac{j}{e_i} p_i) \oplus \bigotimes_{p_i} \bigoplus_{j=0}^{e-1} \mathcal{O}(\frac{-j}{e_i} p_i)$ is a generating sheaf, which we will call the **standard** generating sheaf for $\mathcal{C}$.

From the local condition it is immediate that the standard generating sheaf is indeed a generating sheaf. The standard generating sheaf is definitely not very canonical, however it plays a very special role from a computational perspective. Often formulas massively simplify whenever we apply them to the standard generating sheaf.

We now give a notion of degree that is relative to a generating sheaf, it is this degree that will show up in the stacky Riemann-Roch theorem.

Definition 1.3.15 Let $\mathcal{C}$ be a projective curve, $\mathcal{E}$ be a locally free sheaf $\mathcal{F}$ a coherent sheaf. We define the $\mathcal{E}$-degree

$$d_{\mathcal{E}}(\mathcal{F}) = \mathrm{deg}(\pi_* \mathcal{H}om(\mathcal{E}, \mathcal{F}))) - \mathrm{rank}\, \mathcal{F} \deg \pi_* \mathcal{H}om(\mathcal{E}, \mathcal{O}).$$

Note that the $\mathcal{E}$-degree is additive in short exact sequences in both entries. Moreover $d_{\mathcal{E}}(-) = d_{\mathcal{E} \otimes \pi^* L}$ for any line bundle L on the coarse space. It follows from Lemma 1.2.18 that the $\mathcal{E}$-degree only depends on the multiplicities of $\mathcal{E}$. We now give a notion of "weights", which is simply a repackaging of the multiplicities, that is useful for computations with $\mathcal{E}$-degrees.

Definition 1.3.16 Let $\mathcal{E}$ be a locally free sheaf with multiplicities $m_{p,j}$. We define the weights of $\mathcal{E}$ to be $w_{p,j} = w_{p,j}(\mathcal{E}) := \frac{\sum_{l=1}^{j} m_{p,l}(\mathcal{E})}{\mathrm{rank}\, \mathcal{E}}$, where j runs from 0 to $e_p - 1$.

Note that by construction $0 = w_{p,0} \leq w_{p,1} \leq \cdots \leq w_{p,e_p-1} \leq 1$. The inequalities are strict if and only if $\mathcal{E}$ is a generating sheaf.

Example 1.3.17 Let $\mathcal{E}$ be the standard generating sheaf, then $w_{p,i} = \frac{i}{e_p}$.

In fact we can find a locally free sheaf with arbitrary rational weights.

Example 1.3.18 Let $\mathcal{C}$ be a stacky curve and for each stacky point p let $w_{p,i} = \frac{a_{p,i}}{d_p}$ be rational numbers with a common denominator d_p, such that the numerators satisfy

$$0 = a_{p,0} \leq a_{p,1} \leq \cdots \leq a_{p,e_p-1} \leq d_i.$$

Set $b_{p,i} = a_{p,i} - a_{p,i-1}$ for $0 < i \leq e_p - 1$ and $b_{p,0} = d_p - a_{p,e_p-1}$. The locally free sheaf $\mathcal{E} := \bigotimes_p \bigoplus_{i=0}^{e_p-1} \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p}p)^{\oplus b_{p,i}}$ has weights $w_{p,i}$.

The weights allow us to give a formula for the $\mathcal{E}$-degree in terms of invariants defined on the coarse space and multiplicities.

Theorem 1.3.19 Let $\mathcal{E}$ be a locally free sheaf with weights $w_{p,i}$ and let $\mathcal{F}$ be a locally free sheaf with multiplicities $m_{p,i}$. We have

$$d_{\mathcal{E}}(\mathcal{F}) = \operatorname{rank}\mathcal{E}\deg(\pi_*\mathcal{F}) + \operatorname{rank}\mathcal{E}\sum_p \sum_{i=0}^{e-1} m_{p,i}w_{p,i}.$$

In particular, when $\mathcal{E}$ is the standard generating sheaf $\frac{d_{\mathcal{E}}(\mathcal{F})}{\operatorname{rank}\mathcal{E}} = \deg\mathcal{F}$, where $\mathcal{F}$ only needs to be a coherent sheaf.

Proof. Note that all the terms of the formula are additive in short exact sequences of vector bundles, for both $\mathcal{F}$ and $\mathcal{E}$, so we may assume $\mathcal{F}$ and $\mathcal{E}$ are line bundles. The case of line bundles is immediate from the description in Corollary 1.2.11. For the case of the standard generating sheaf the result follows from Theorem 1.2.26 and the fact that the formula $d_{\mathcal{E}}(\mathcal{F}) = \operatorname{rank}\mathcal{E}\deg\mathcal{F}$ is additive in all short exact sequences for $\mathcal{F}$. ◯

Now we state a more refined version of the Riemann-Roch theorem.

Theorem 1.3.20 (Stacky Riemann-Roch) Let $\mathcal{C}$ be a projective curve $\mathcal{E}$ a locally free sheaf $\mathcal{F}$ a coherent sheaf. We have

$$\mathtt{ext}^0(\mathcal{E},\mathcal{F}) - \mathtt{ext}^1(\mathcal{E},\mathcal{F}) = d_{\mathcal{E}}(\mathcal{F}) + \operatorname{rank}(\mathcal{F})\left(\mathtt{ext}^0(\mathcal{E},\mathcal{O}_{\mathcal{C}}) - \mathtt{ext}^1(\mathcal{E},\mathcal{O}_{\mathcal{C}})\right).$$

In particular when $\mathcal{E}$ is the standard generating sheaf we have

$$\frac{\mathtt{ext}^0(\mathcal{E},\mathcal{F}) - \mathtt{ext}^1(\mathcal{E},\mathcal{F})}{\operatorname{rank}\mathcal{E}} = \deg\mathcal{F} + \operatorname{rank}(\mathcal{F})(1 - g_{\mathcal{C}}).$$

Proof. The proof is analogous to the classical case. Everything is additive in short exact sequences, so we may assume $\mathcal{F}$ is a line bundle. Assume $\mathcal{F} = \mathcal{O}_{\mathcal{C}}$, then $d_{\mathcal{E}}(\mathcal{O}_{\mathcal{C}}) = 0$, so the formula holds. Assume the formula holds for a line bundle $\mathcal{L}$ and we have a non-zero map $\mathcal{L} \to \mathcal{L}'$ then let $\mathcal{T}$ be the cokernel of this map, which is a torsion sheaf. From the additivity of $\mathcal{E}$-degrees we get $d_{\mathcal{E}}(\mathcal{L}') - d_{\mathcal{E}}(\mathcal{L}) = d_{\mathcal{E}}(\mathcal{T})$. We also get the long exact sequence

$$\begin{aligned}\to \mathtt{Ext}^0(\mathcal{E},\mathcal{L}) \to \mathtt{Ext}^0\big(\mathcal{E},\mathcal{L}'\big) \to \mathtt{Ext}^0(\mathcal{E},\mathcal{T}) \to \\ \to \mathtt{Ext}^1(\mathcal{E},\mathcal{L}) \to \mathtt{Ext}^1\big(\mathcal{E},\mathcal{L}'\big) \to \mathtt{Ext}^1(\mathcal{E},\mathcal{T}) = 0.\end{aligned}$$

The last ext group is 0 because $\mathtt{Ext}^1(\mathcal{E},\mathcal{T}) = H^1(\pi_*(\mathcal{T}\otimes\mathcal{E}^\vee)) = 0$. Also $\mathtt{ext}^0(\mathcal{E},\mathcal{T}) = h^0(\pi_*(\mathcal{T}\otimes\mathcal{E}^\vee)) = d_{\mathcal{E}}(\mathcal{T})$. Now taking the Euler characteristic of the long exact sequence we see that the formula also holds for $\mathcal{L}'$. A completely analogous argument works when we have a non-zero map $\mathcal{L}' \to \mathcal{L}$.

Now any line bundle $\mathcal{L} \simeq \mathcal{O}_{\mathcal{C}}(D)$ for some Weil-divisor D. Let D_+ be the positive part of D. We have a non-zero map $\mathcal{O}_{\mathcal{C}} \to \mathcal{O}(D_+)$ and a non-zero map $\mathcal{O}_{\mathcal{C}}(D) \to \mathcal{O}_{\mathcal{C}}(D_+)$ showing that the formula holds for $\mathcal{L}$.

Finally when $\mathcal{E}$ is the standard generating sheaf we already saw $d_{\mathcal{E}}(\mathcal{F}) = \mathsf{rank}\,\mathcal{E}\,\mathsf{deg}\,\mathcal{F}$ and by the naive Riemann-Roch theorem

$$\mathtt{ext}^0(\mathcal{E},\mathcal{O}_{\mathcal{C}}) - \mathtt{ext}^1(\mathcal{E},\mathcal{O}_{\mathcal{C}}) = \mathsf{deg}(\pi_*\mathcal{E}^\vee) + \mathsf{rank}\,\mathcal{E}(1-g_C) =$$
$$-\,\mathsf{rank}\,\mathcal{E}\frac{1}{2}\left(\sum_{p_i}\frac{e_i-1}{e_i}\right) + \mathsf{rank}\,\mathcal{E}(1-g_C) = \mathsf{rank}\,\mathcal{E}(1-g_{\mathcal{C}}).$$

⬠

Hilbert polynomials and stability conditions

We will now explain how to define Hilbert polynomials for sheaves on stacky curves.

> **Definition 1.3.21** Let $\mathcal{C} \to C$ be a projective stacky curve. We define a **polarization** of $\mathcal{C}$ to be a pair $(\mathcal{E}, \mathcal{O}_C(1))$, where $\mathcal{E}$ is a generating sheaf for $\mathcal{C}$ and $\mathcal{O}_C(1)$ is a polarizing line bundle for C. For a coherent sheaf $\mathcal{F}$ we write $\mathcal{F}(m) := \mathcal{F}\otimes\pi^*\mathcal{O}_C(1)$.

In [6] it is explained how a generating sheaf together with a polarization of the coarse space induces an embedding of the stacky curve into a twisted Grassmanian stack. The twisted Grassmanians are simultaneous generalizations of weighted projective spaces and Grassmanians. This justifies the calling the pair $(\mathcal{E}, \mathcal{O}_C(1))$ a polarization.

> **Definition 1.3.22** Let $\pi : \mathcal{C} \to C$ be a projective stacky curve with polarization $(\mathcal{E}, \mathcal{O}_C(1))$. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We define the **$\mathcal{E}$-Hilbert polynomial** of $\mathcal{F}$ to be
>
> $$P_{\mathcal{E}}(\mathcal{F})(m) := \chi(\mathcal{H}om(\mathcal{E}, \mathcal{F}\otimes\pi^*\mathcal{O}_C(m))) = \mathtt{ext}^0(\mathcal{E},\mathcal{F}(m)) - \mathtt{ext}^1(\mathcal{E},\mathcal{F}(m)).$$
>
> We define the reduced $\mathcal{E}$-Hilbert polynomial $p_{\mathcal{E}}(\mathcal{F})$ to be $P_{\mathcal{E}}(\mathcal{F})$ divided by its leading coefficient.

From the stacky Riemann-Roch theorem it follows that

$$P_{\mathcal{E}}(\mathcal{F})(m) = \mathsf{rank}\,\mathcal{F}\,\mathsf{rank}\,\mathcal{E}\,\mathsf{deg}\,\mathcal{O}_C(1)\cdot m + d_{\mathcal{E}}(\mathcal{F}) + \mathsf{rank}\,\mathcal{F}\cdot C_{\mathcal{E}},$$

where $C_{\mathcal{E}}$ is a constant that does not depend on $\mathcal{F}$. It follows that we can completely reconstruct the Hilbert polynomial if we know the rank, degree and multiplicities of $\mathcal{F}$. We will see later that the connected components of the moduli space of coherent sheaves are parametrized by Hilbert polynomials, so the rank, degree and multiplicities really are the only discrete invariants.

Definition 1.3.23 Let $\mathcal{C}$ be a stacky curve with generating sheaf $\mathcal{E}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{C}$. We say that $\mathcal{F}$ is (semi)stable if for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $p_{\mathcal{E}}(\mathcal{F}') \underset{(-)}{\leq} p_{\mathcal{E}}$. Define the **slope** of $\mathcal{F}$ to be $\mu_{\mathcal{E}}(\mathcal{F}) := \frac{d_{\mathcal{E}}\mathcal{F}}{\operatorname{rank}\mathcal{F}}$. We say that $\mathcal{F}$ is $\mathcal{E}$-slope-(semi)stable if for every proper subsheaf we have $\mu_{\mathcal{E}}(\mathcal{F}') \underset{(-)}{\leq} \mu_{\mathcal{E}}(\mathcal{F})$.

It follows immediately from the Stacky Riemann-Roch theorem that slope-(semi)stability and (semi)stability are equivalent.

1.4 Parabolic vector bundles

One important reason to study vector bundles on stacky curves is their close relation to parabolic bundles. Parabolic bundles where originally considered by Seshadri to give a generalization of the Narasimhan-Seshadri correspondence to the case of punctured curves. In this section we start by recalling the basic concepts surrounding parabolic bundles. The goal is then to give a dictionary between the parabolic language and the stacky curve language.

Definition 1.4.1 ([12, Definition 1.5]) Let C be a classical curve and $\underline{p}$ a set of points of C. A **quasi-parabolic vector bundle** $\mathbb{F}$ on $(C, \underline{p})$ is a vector bundle F on C together with filtrations $F = F_0^p \supset F_1^p \supset \ldots \supset F_{e_p}^p = F \otimes \mathcal{O}_C(-p)$ for each $p \in \underline{p}$. The integer e_p is called the length of the parabolic structure at p. The collection of quasi-parabolic vector bundles of fixed length forms a category $\mathfrak{qpar}(C, \underline{p}, \underline{e})$, where the morphisms are given by morphism of the underlying vector bundles respecting the filtration. Explicitly the morphisms are morphisms $\phi : F \to G$ such that $\phi(F_j^p) \subset \phi(G_j^p)$ for all p, j.

Remark 1.4.2 Instead of a filtration at each point, it is equivalent to give at each point p a flag of quotients of the fibre $F|_p = V_0^p \twoheadrightarrow V_1^p \twoheadrightarrow \cdots \twoheadrightarrow V_{e_p-1}^p \twoheadrightarrow V_e^p = 0$. To see this send a filtration $F_\bullet$ to $V_i^p = \operatorname{coker}\left(F_{e_p-i}^p \to F_0^p\right)|_p$. To obtain a flag of injections $F_p = W_0 \supset W_1 \supset \cdots W_{e_p} = 0$ instead simply consider $W_i = \ker(V_0 \twoheadrightarrow V_{e_p-i})$.

Note that, contrary to the classical definition, we do not require the inclusions of the filtrations to be strict. One reason is that this gives much better categorical properties. For example a parabolic subbundle is simply a subobject in the category $\mathfrak{qpar}(C, \underline{p}, \underline{e})$, whereas classically subbundles might have shorter length filtrations, as the length would be bounded by the rank.

We now describe how to obtain a quasi-parabolic vector bundle from a vector bundle on a stacky curve.

Definition 1.4.3 Let $\pi : \mathcal{C} \to C$ be a stacky curve with stacky points $\underline{p}$ of degree $\underline{e}$. We define a functor $\mathtt{par} : \mathfrak{Vect}(\mathcal{C}) \to \mathfrak{qpar}(C, \underline{p}, \underline{e})$ as follows. Let $\mathcal{F}$ be a vector bundle on $\mathcal{C}$. Then $\mathtt{par}(\mathcal{F})$ is the vector bundle $\pi_*\mathcal{F}$ together with the filtrations

$$\pi_*\mathcal{F} \supset \pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)) \supset \cdots \supset \pi_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}(-\frac{e_p}{e_p}p)),$$

for each $p \in \underline{p}$. A morphism $f : \mathcal{F} \to \mathcal{G}$ gets sent to $\mathtt{par}(f) := \pi_* f : \pi_*\mathcal{F} \to \pi_*\mathcal{G}$.

There is also an inverse functor, but it is much harder to define, so we will omit it here.

Theorem 1.4.4 ([3, Théorème 4]) The functor $\mathtt{par}$ defines an equivalence of categories.

We will now look at how the functor $\mathtt{par}$ interacts with multiplicities.

Definition 1.4.5 Let $\mathbb{F}$ be a quasi-parabolic bundle. We define the multiplicities $m_{p,i}(\mathbb{F}) := \mathtt{dim}\,\mathtt{coker}(F_{i+1} \to F_i)|_p$, where $0 \leq i < e_p$.

In the surjective flag picture we have $m_{p,i} = \mathtt{dim}\, V^p_{e_p-i-1} - \mathtt{dim}\, V^p_{e_p-i}$ or in the injective picture $m_{p,i} = \mathtt{dim}\, W^p_i - \mathtt{dim}\, W^p_{i+1}$.

Proposition 1.4.6 Let $\mathcal{F}$ be a vector bundle on a stacky curve $\mathcal{C} := \sqrt[\underline{e}]{\underline{p}/C}$, with multiplicities $m_{p,i}$, then $\mathtt{par}(\mathcal{F})$ has multiplicities $m_{p,i}$.

Proof. We see that the $m_{p,i}(\mathtt{par}\,\mathcal{F})$ is additive in short exact sequences, so it suffices to show this for line bundles. Then for a line bundle $\mathcal{L} = \pi^* L \otimes \bigotimes_{p \in \underline{p}} \mathcal{O}_{\mathcal{C}}(\frac{n_p}{e_p}p)$ we see that the filtrations of $\mathtt{par}(\mathcal{L})$ are given by $L^p_i = L$ for $0 \leq i \leq n_p$ and $L^p_i = L(-p)$ for $n_p < i \leq e_p$. This shows that $m_{p,n_p}(\mathtt{par}(\mathcal{L})) = 1$ and the other multiplicities are 0 as required. ◇

Now we will discuss the notion of weights and (semi)stability for quasi-parabolic bundles.

Definition 1.4.7 Let C be a classical curve and $\underline{p}$ a set of points of C. A **parabolic bundle** on C is a quasi-parabolic bundle together with a set $\underline{\alpha}$ of **parabolic weights** $\alpha_{p,j} \in \mathbb{R}$ for $p \in \underline{p}$ and $0 \leq j < e_p$, satisfying

$$0 \leq \alpha_{p,0} < \cdots < \alpha_{p,e_p-1} < 1.$$

Let $(\mathbb{F}, \underline{\alpha})$ be a parabolic bundle, we define the **parabolic degree** $\mathtt{pardeg}(\mathbb{F}, \underline{\alpha}) = \deg F + \sum_p \sum_{i=1}^{e_p-1} m_{p,i}(\mathbb{F})\alpha_{p,i}$ and the **parabolic slope** $\mu(\mathbb{F}, \underline{\alpha}) = \frac{\mathtt{pardeg}(\mathbb{F},\underline{\alpha})}{rk(F)}$. We say that a parabolic bundle $(\mathbb{F}, \underline{\alpha})$ is **(semi)stable** if for every proper quasi-parabolic subbundle $\mathbb{F}' \subset \mathbb{F}$ we have $\mu(\mathbb{F}', \underline{\alpha}) \underset{(-)}{\leq} \mu(\mathbb{F}, \underline{\alpha})$.

The functor $\mathtt{par}$ respects stability.

Theorem 1.4.8 Let $\mathcal{F}$ be a vector bundle on a stacky curve $\mathcal{C} := \sqrt[\underline{e}]{\underline{p}/C}$ stacky curve C. Let $\mathcal{E}$ be a generating sheaf with weights $w_{p,i}$, then $\mathcal{F}$ is $\mathcal{E}$-(semi)stable if and only if $\mathtt{par}(\mathcal{F})$ together with the parabolic weights $\alpha_{p,i} = w_{p,i}$ is a (semi)stable parabolic bundle.

Proof. This follows immediately from the fact that $\deg_{\mathcal{E}}(\mathcal{F}) = \mathtt{pardeg}(\mathtt{par}(\mathcal{F}), \underline{w})$, which follows from combining Proposition 1.4.6 with Theorem 1.3.19. ◯

Theorem 1.4.9 Let $\mathfrak{qpar}(C, \underline{p}, \underline{e})^{\underline{\alpha}-(s)s} \subset \mathfrak{qpar}(C, \underline{p}, \underline{e})$ be the full subcategory of bundles that are (semi)stable when endowed with the parabolic weights $\underline{\alpha}$. Then there exists a generating sheaf $\mathcal{E}$ on $\mathcal{C} = \sqrt[\underline{e}]{\underline{p}/C}$, such that the category of (semi)stable vector bundles $\mathfrak{Vect}(\mathcal{C})^{\mathcal{E}-(s)s}$ is equivalent to $\mathfrak{qpar}(C, \underline{p}, \underline{e})^{\underline{\alpha}-(s)s}$.

Proof. By [12, Corollary 2.9] we can always perturb the weights $\underline{\alpha}$ to be rational without changing the notion of stability. Secondly we can shift the parabolic weights by a constant without changing the notion of (semi)stability by [12, Remark 2.10], so we might as well assume that $\alpha_{p,0} = 0$. This means we can pick $\mathcal{E}$ as in Example 1.3.18. ◯

We end this section with some comments on “strongly” parabolic homomorphisms and Higgs fields.

Definition 1.4.10 Let $\mathbb{F}, \mathbb{G} \in \mathfrak{qpar}(C, \underline{p}, \underline{e})$ be quasi-parabolic bundles. We define a **strongly parabolic morphism** to be a morphism $f : F \to G$, such that $f(F_i^p) \subset G_{i+1}^p$ for every p, i. The set of strongly parabolic morphisms is denoted by $\mathtt{sHom}(\mathbb{F}, \mathbb{G})$.

Let $D := \sum_{p \in \underline{p}} p$ be the **parabolic divisor**. A **Higgs field** on $\mathbb{F}$ is a strongly parabolic parabolic morphism $\phi : \mathbb{F} \to \mathbb{F} \otimes \omega_C(D)$. (Here the tensor product should be done term-wise on every term of the filtrations of $\mathbb{F}$.)

The notion of a strongly parabolic morphisms might seem quite ad-hoc. In fact the only reason that it shows up is that the “logarithmic” canonical sheaf $\omega_C(D)$ has the wrong parabolic structure. On the level of stacky curves this will be apparent.

Theorem 1.4.11 Let $\mathcal{C} = \sqrt[e]{\underline{p}/C}$ and let $\mathcal{F}, \mathcal{G} \in \mathfrak{Vect}(\mathcal{C})$ be two vector bundles. We have a natural isomorphism

$$\phi : \mathtt{Hom}\left(\mathcal{F}, \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)\right) \to \mathtt{sHom}(\mathtt{par}(\mathcal{F}), \mathtt{par}(\mathcal{G})).$$

In particular we have a correspondence of Higgs fields $\mathtt{sHom}(\mathtt{par}(\mathcal{F}), \mathtt{par}(\mathcal{F}) \otimes \omega_C(D)) = \mathtt{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_{\mathcal{C}})$.

Proof. Denote by ι the inclusion $\iota : \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p) \hookrightarrow \mathcal{G}$. We define ϕ by sending a morphism $f : \mathcal{F} \to \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)$ to $\phi(f) := \mathtt{par}(\iota \circ f)$. By definition this defines a strongly parabolic morphism and clearly ϕ is injective. To see that it is surjective take any strongly parabolic morphism $h : \mathtt{par}(\mathcal{F}) \to \mathtt{par}(\mathcal{G})$, by Theorem 1.4.4 it lifts to a unique morphism $\tilde{h} : \mathcal{F} \to \mathcal{G}$. We need to show that $\tilde{h}$ factors through ι. To see this consider the generating sheaf $\mathcal{E} = \bigoplus_{p \in \underline{p}} \bigoplus_{i=0}^{e_p - 1} \mathcal{O}_{\mathcal{C}}(\frac{i}{e_p}p)$. The fact that h is strongly parabolic ensures that $F_{\mathcal{E}}(\tilde{h})$ factors through $F_{\mathcal{E}}(\mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p))$. Now consider the following commutative diagram.

$$\begin{tikzcd}
G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{F}) \arrow[r] \arrow[d, two heads] \arrow[rr, bend left, "G_{\mathcal{E}}F_{\mathcal{E}}(\tilde{h})"] & G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)) \arrow[r, hook] \arrow[d, two heads] & G_{\mathcal{E}}F_{\mathcal{E}}(\mathcal{G}) \arrow[d, two heads] \\
\mathcal{F} \arrow[r, dotted] \arrow[rr, bend right, "\tilde{h}"'] & \mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p) \arrow[r, hook] & \mathcal{G}
\end{tikzcd}$$

This shows that the image of $\tilde{h}$ lies inside $\mathcal{G} \otimes \bigotimes_p \mathcal{O}_{\mathcal{C}}(-\frac{1}{e_p}p)$. ⬡

The theorem above also explains why Serre duality [17, Proposition 3.7] for parabolic bundles is perhaps not what we would expect naively. Namely we have

$$\begin{aligned}
\mathtt{Ext}^1(\mathtt{par}(\mathcal{F}), \mathtt{par}(\mathcal{G})) = \mathtt{Ext}^1(\mathcal{F}, \mathcal{G}) &= \mathtt{Hom}(\mathcal{G}, \mathcal{F} \otimes \omega_{\mathcal{C}})^\vee \\
&= \mathtt{sHom}(\mathtt{par}(\mathcal{G}), \mathtt{par}(\mathcal{F}) \otimes \omega_C(D))^\vee.
\end{aligned}$$

All the equivalences in this section are on the level of categories, but we will see in the next chapter that they also hold on the level of moduli stacks.

Bibliography

[1] Dan Abramovich, Martin Olsson, and Angelo Vistoli. Tame stacks in positive characteristic. **Ann. Inst. Fourier (Grenoble)**, 58(4):1057–1091, 2008.

[2] Dan Abramovich and Angelo Vistoli. Compactifying the space of stable maps. **J. Amer. Math. Soc.**, 15(1):27–75, 2002.

[3] Niels Borne. Fibrés paraboliques et champ des racines. **Int. Math. Res. Not. IMRN**, (16):Art. ID rnm049, 38, 2007.

[4] Charles Cadman. Using stacks to impose tangency conditions on curves. **Amer. J. Math.**, 129(2):405–427, 2007.

[5] Brian Conrad. Keel–mori theorem via stacks. 2005.

[6] Mitchell Faulk and Chiu-Chu Melissa Liu. Embedding deligne-mumford stacks into git quotient stacks of linear representations. **arXiv preprint arXiv:2111.01068**, 2021.

[7] Anton Geraschenko and Matthew Satriano. A "bottom up" characterization of smooth Deligne-Mumford stacks. **Int. Math. Res. Not. IMRN**, (21):6469–6483, 2017.

[8] Luc Illusie. **Complexe cotangent et déformations. I**. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[9] Donald Knutson. **Algebraic spaces**. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971.

[10] Andrew Kresch. On the geometry of Deligne-Mumford stacks. In **Algebraic geometry—Seattle 2005. Part 1**, volume 80, Part 1 of **Proc. Sympos. Pure Math.**, pages 259–271. Amer. Math. Soc., Providence, RI, 2009.

[11] Qing Liu. **Algebraic geometry and arithmetic curves**, volume 6 of **Oxford Graduate Texts in Mathematics**. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[12] V. B. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. **Math. Ann.**, 248(3):205–239, 1980.

[13] Fabio Nironi. Moduli spaces of semistable sheaves on projective deligne-mumford stacks, 2009.

[14] Martin Olsson and Jason Starr. Quot functors for Deligne-Mumford stacks. volume 31, pages 4069–4096. 2003. Special issue in honor of Steven L. Kleiman.

[15] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2024.

[16] John Voight and David Zureick-Brown. The canonical ring of a stacky curve. **Mem. Amer. Math. Soc.**, 277(1362):v+144, 2022.

[17] Kôji Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. **Internat. J. Math.**, 6(1):125–148, 1995.

[18] Oscar Zariski. The concept of a simple point of an abstract algebraic variety. **Trans. Amer. Math. Soc.**, 62:1–52, 1947.