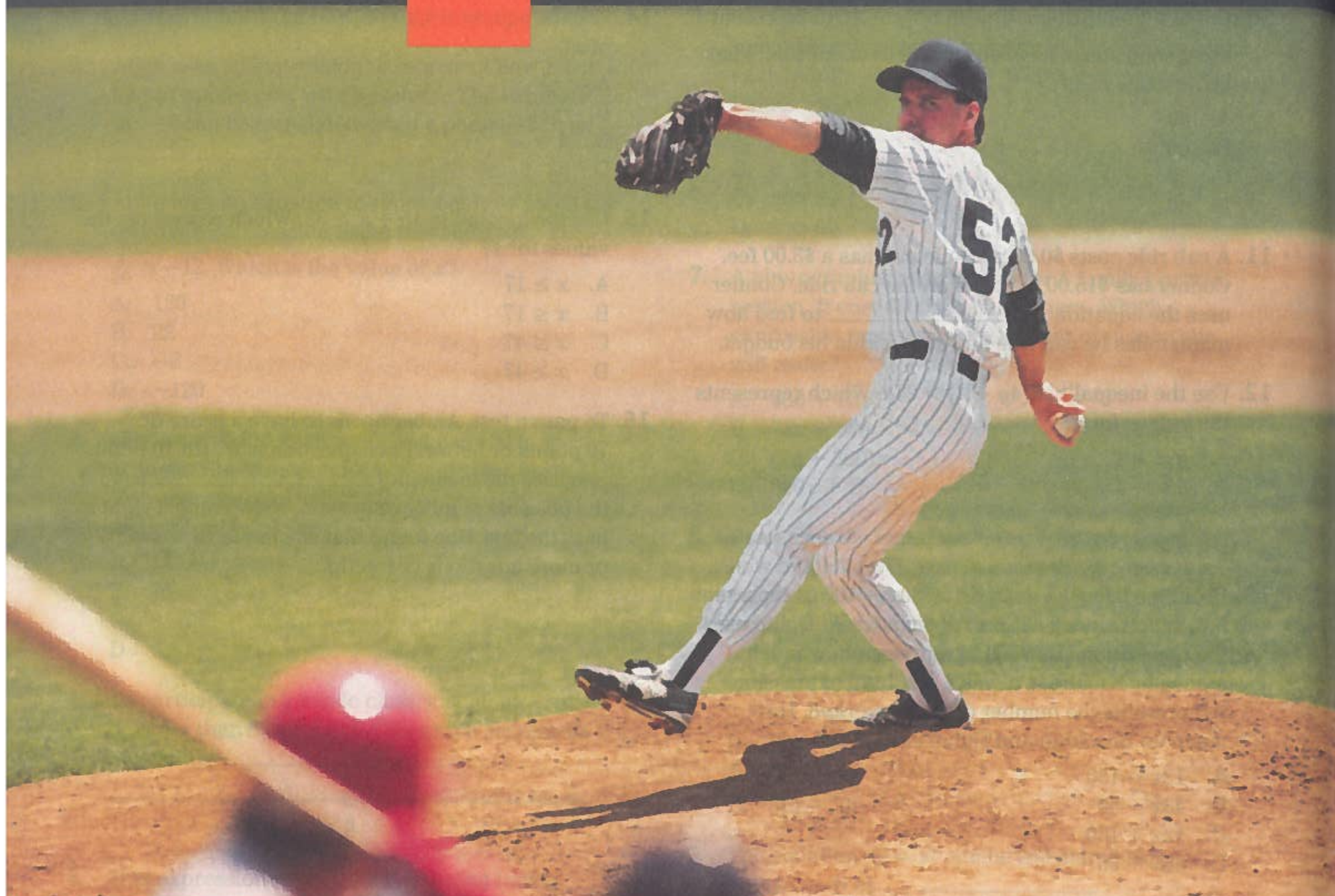
Chapter 4



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Polynomials and Rational Expressions

In baseball, pitchers need to know exactly where they are throwing the ball to strike out the opposing team. This skill develops over time with practice and an understanding of their own particular pitch. Mathematically, you can model the movement of dropped or thrown objects using polynomials. While a linear expression can be used to model many simple situations, a polynomial is critical for modeling more complex situations. You are already familiar with variables and linear expressions, but when they are combined by multiplication and division you create polynomials and quadratic and rational expressions.

Lesson 4.1
Evaluate Polynomials

How do you calculate and work with numbers raised to exponents? How can you apply what you observed with real numbers to variables raised to exponents? Learn how to identify and classify different types of polynomials.

Lesson 4.2
Factor Polynomials

How do you approach a difficult problem at work? Often you might break the problem down into smaller chunks that are simpler to solve. In math, understanding how to factor polynomials will help you solve more complex problems. Learn tricks and methods for factoring polynomials.

Lesson 4.3
Solve Quadratic Equations

Linear equations are solved by applying inverse operations to get the variable alone. In a quadratic equation, new solving methods are needed because the variable is squared. Learn how to solve quadratic equations by using factoring and formulas.

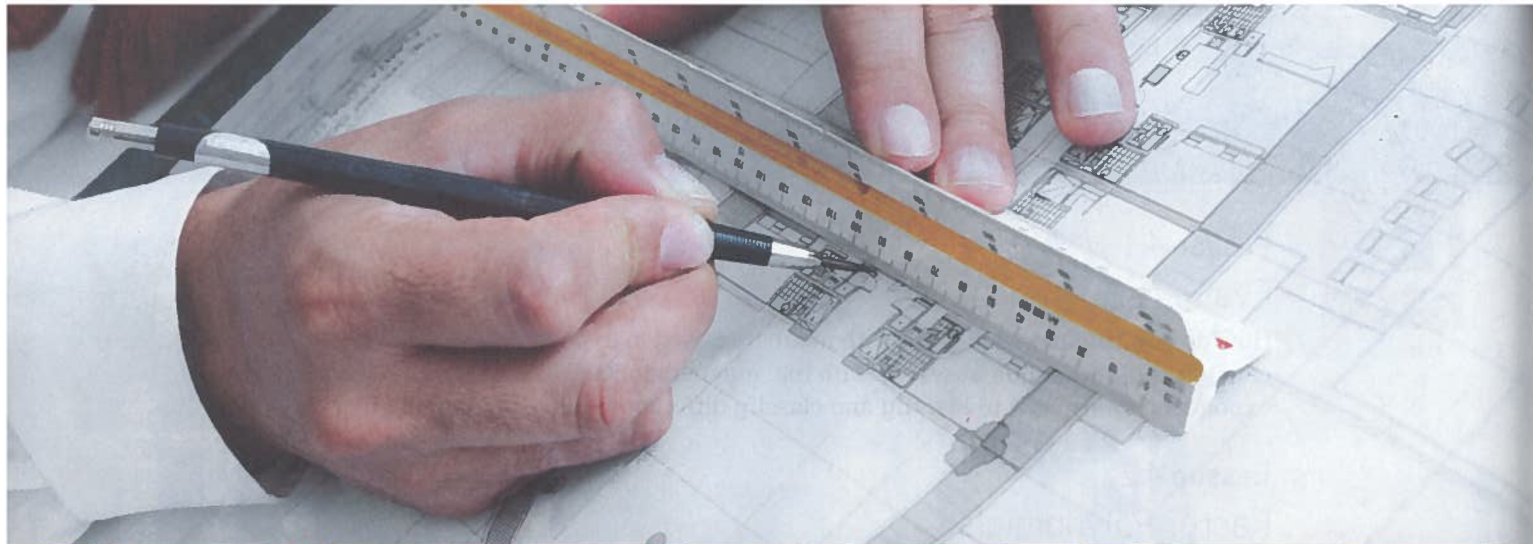
Lesson 4.4
Evaluate Rational Expressions

When you learned about fractions you were told never to have 0 in the denominator. What happens when a variable expression is in the denominator? How can you avoid having that expression be 0? Learn how to simplify and restrict the values for rational expressions.

Goal Setting

Think about being presented with math problems to solve. When did you have the easiest time? What made you feel confident about solving the problem? Did you know a shortcut or process to solve the problem? What are some examples of math tools, tips, or shortcuts that you have learned in the past? How did you use them to solve problems?

How might breaking down a quadratic equation help you solve it? What previous knowledge might you need to solve quadratic equations?



LESSON 4.1 Evaluate Polynomials

LESSON OBJECTIVES

- Identify different polynomials
- Evaluate polynomials
- Add, subtract, multiply, and divide polynomials

CORE SKILLS & PRACTICES

- Use Math Tools Appropriately
- Evaluate Expressions

Key Terms

polynomial
an algebraic expression consisting of one or more terms in which each term is a number or a product of numbers and variables with whole-number exponents

degree
the value of the greatest exponent in a polynomial

standard form
the form of a polynomial that shows the terms listed from left to right with the powers of the variables from greatest to least

Vocabulary

opposite polynomial
the polynomial with all of its signs changed to their opposites

substitute
to replace a variable in an expression with a numerical value

Key Concept

Polynomials are special types of variable expressions with one or more terms. Each term has a variable raised to a whole number exponent or is a constant.

Identifying Polynomials

Small business owners can use linear expressions to model some simple costs and everyday situations. However, in order to model their profit and other more complex situations, they need to use squares, cubes, and other higher-order variables. These situations can be modeled with polynomials.

Types of Polynomials

Polynomials are algebraic expressions with one or more terms. Each term is a number or a product of numbers and variables with whole-number exponents. The **degree** of a polynomial is the value of the greatest exponent in the polynomial. A linear expression is a polynomial of degree 1. A term that contains only a number is a constant term and has a degree of 0.

$4x^2$	$2z - 6$	$3c^4 + 2c^2 - 1$	$2a^3 + 7a + a$
degree 2	degree 1	degree 4	degree 3

Three types of polynomials are named by the number of terms in the polynomial expression. The names of these polynomials are chosen because of their prefixes. The prefix *poly-* means many.

A *monomial* has one term.	$3x$
A *binomial* has two terms.	$4x + 2$
A *trinomial* has three terms	$2x^2 + x - 5$

A polynomial is written in **standard form** when the terms are listed from left to right with the powers of the variables from greatest to least.

- The polynomial $4x + 5 + 3x^2$ is not written in standard form.
- The polynomial $3x^2 + 4x + 5$ is written in standard form.

Simplifying Polynomials

In addition to writing polynomials in standard form, simplifying polynomials can also make them easier to work with. Polynomials are simplified when all like terms have been combined. Like terms have the same variable raised to the same power. They can be combined by combining their coefficients.

Example 1: Combining Like Terms

Simplify and write in standard form: $11c^2 - 3c + 4c^2 - 2c^3 + c$

Step 1 Use the commutative and associative properties to group like terms.

$$11c^2 - 3c + 4c^2 - 2c^3 + c = 11c^2 + (-3c) + 4c^2 + (-2c^3) + c$$
$$= (11c^2 + 4c^2) + (-3c + c) + (-2c^3)$$

Step 2 Combine like terms by adding the coefficients.

$$(11c^2 + 4c^2) + (-3c + c) + (-2c^3) = (11 + 4)c^2 + (-3 + 1)c + (-2c^3)$$
$$= 15c^2 + (-2c) + (-2c^3)$$

Step 3 Write the polynomial in standard form.

$$15c^2 + (-2c) + (-2c^3) = -2c^3 + 15c^2 - 2c$$

Think about Math

Directions: Choose the best answer to each question.

1. Which of the following binomials is not written in standard form?
   A. $8m^3 + 4$
   B. $7 - n$
   C. $x^2 + x$
   D. $3y^4 - 2y$

2. What is the degree of the polynomial $4x - 2x^2 + 1 + 5x^3$?
   A. 0
   B. 1
   C. 2
   D. 3

CORE PRACTICE

Use Math Tools Appropriately

"Standard forms" exist for a reason. It is usually easiest to work with numbers and other mathematical objects when they are expressed in standard forms. Polynomials are no exception; they should be simplified and written in standard form when you need to add, subtract, or multiply them. Remember: *Simplify* means to combine like terms, or terms that share the same variable raised to the same power. *Standard form* refers to the terms written with decreasing exponents from left to right.

Simplify the polynomial by combining like terms and express the result in standard form.

$5 + 3m - 2m^3 + 6m - 5m^2 - 3m^3 + 7m$

CORE SKILL

Evaluate Expression

Being able to evaluate expressions is a useful skill in solving real-world problems. For example, suppose a baseball player hits a ball with an initial speed of 100 feet per second. The height of the ball t seconds after the ball is hit can be represented by the polynomial $-32t^2 + 100t + 6$. To find the height of the ball 2 seconds after it was hit, you can substitute $t = 2$ into the expression and evaluate.

$$
\begin{aligned}
&-32t^2 + 100t + 6\\
&= -32(2)^2 + 100(2) + 6\\
&= -32(4) + 100(2) + 6\\
&= -128 + 200 + 6\\
&= 78
\end{aligned}
$$

What is the height of the ball 3 seconds after it has been hit?

Evaluating Polynomials

Polynomials can be used to represent different real-world phenomena, such as the height of a falling object at a specific time. To find the height of an object at a specific time, you simply need to evaluate the polynomial for the specific value of the variable.

Substitution for Variables

To evaluate a polynomial for a given value of the variable, you **substitute** the value into each place the variable appears in the polynomial. Then, evaluate the expression according to the order of operations.

Example 2: Evaluate a Polynomial

Find the value of $x^2 - 3x + 1$ when $x = -2$.

Step 1 Substitute the value -2 in the polynomial for each x.

$$x^2 - 3x + 1 = (-2)^2 - 3(-2) + 1$$

Step 2 Simplify using the order of operations.

$$
\begin{aligned}
(-2)^2 - 3(-2) + 1 &= 4 - 3(-2) + 1 && \text{Exponents}\\
&= 4 + 6 + 1 && \text{Multiplication}\\
&= 11 && \text{Addition}
\end{aligned}
$$

Operations with Polynomials

You can calculate the distance an object has traveled by multiplying its speed by the time it has traveled. If the speed of an object and the time it travels are each given by polynomials, then we can determine its distance by multiplying the two polynomials.

Adding Polynomials

Adding polynomials is similar to simplifying polynomials. When adding polynomials, use properties to group and simplify like terms.

Example 3: Add Polynomials

Find the sum of the polynomials: $(4x^2 - 3x + 1) + (5 + 2x^2)$

Step 1 Use the commutative and associative properties to group like terms.

$$
\begin{aligned}
(4x^2 - 3x + 1) + (5 + 2x^2) &= 4x^2 - 3x + 1 + 2x^2 + 5\\
&= 4x^2 + (-3x) + 1 + 2x^2 + 5\\
&= (4x^2 + 2x^2) + (-3x) + (1 + 5)
\end{aligned}
$$

Step 2 Combine like terms and write the sum in standard form.

$$(4x^2 + 2x^2) + (-3x) + (1 + 5) = (4 + 2)x^2 + (-3x) + (1 + 5)$$

$$= 6x^2 - 3x + 6$$

You could also show the addition vertically, using coefficients of 0 for any missing terms in order to keep the like terms aligned.

$$
\begin{array}{r}
4x^2 - 3x + 1\\
+\ 2x^2 + 0x + 5\\
\hline
6x^2 - 3x + 6
\end{array}
$$

The Opposite of a Polynomial

To subtract one polynomial from another, you need to add the **opposite polynomial**. The opposite polynomial simply reverses the sign of each term of the polynomial.

- **Polynomial:** $6t^3 + 3t^2 - 4t - 2$
- **Opposite Polynomial:** $-6t^3 - 3t^2 + 4t + 2$

Example 4: Subtract Polynomials

Find the difference of the polynomials: $(3x^2 + 6x - 5) - (2x^2 - 4x + 2)$

Step 1 Rewrite the subtraction as addition of the opposite polynomial.

$$(3x^2 + 6x - 5) - (2x^2 - 4x + 2) = (3x^2 + 6x - 5) + (-2x^2 + 4x - 2)$$

Step 2 Use the commutative and associative properties to group like terms.

$$
\begin{aligned}
(3x^2 + 6x - 5) - (2x^2 - 4x + 2) &= 3x^2 + 6x + (-5) + (-2x^2) + 4x + (-2)\\
&= (3x^2 + (-2x^2)) + (6x + 4x) + (-5 + (-2))
\end{aligned}
$$

Step 3 Combine like terms and write in standard form.

$$
\begin{aligned}
&(3x^2 + (-2x^2)) + (6x + 4x) + (-5 + (-2))\\
&= (3 + (-2))x^2 + (6 + 4)x + (-5 + (-2))\\
&= 1x^2 + 10x + (-7)\\
&= x^2 + 10x - 7
\end{aligned}
$$

Multiplying Polynomials

Multiplying two polynomials is similar to multiplying numerical expressions. To multiply polynomials, use the Distributive Property to multiply each pair of terms from the polynomials.

Example 5: Multiply Polynomials

Find the product of the polynomials: $(5x - 7)(2x^2 + 6x - 3)$

Step 1 Use the Distributive Property to multiply each term of the first polynomial by the second polynomial.

$$(5x - 7)(2x^2 + 6x - 3) = 5x(2x^2 + 6x - 3) - 7(2x^2 + 6x - 3)$$

Step 2 Use the Distributive Property again to multiply each monomial by the second polynomial. Use caution when distributing any negative terms.

$$
\begin{aligned}
&5x(2x^2 + 6x - 3) - 7(2x^2 + 6x - 3)\\
&= 5x(2x^2) + 5x(6x) + 5x(-3) - 7(2x^2) - 7(6x) - 7(-3)\\
&= 10x^3 + 30x^2 + (-15x) + (-14x^2) + (-42x) + 21
\end{aligned}
$$

Step 3 Use the Commutative and Associative properties to combine like terms.

$$
\begin{aligned}
&10x^3 + 30x^2 + (-15x) + (-14x^2) + (-42x) + 21\\
&= 10x^3 + 30x^2 + (-14x^2) + (-15x) + (-42x) + 21
\end{aligned}
$$

Step 4 Combine like terms and write in standard form.

$$
\begin{aligned}
&10x^3 + 30x^2 + (-14x^2) + (-15x) + (-42x) + 21\\
&= 10x^3 + (30 - 14)x^2 + (-15 - 42)x + 21\\
&= 10x^3 + 16x^2 + (-57x) + 21\\
&= 10x^3 + 16x^2 - 57x + 21
\end{aligned}
$$

21ST CENTURY SKILL

Economic Literacy

Profit is a key concept when running a business. Profit is the difference when the costs are subtracted from the revenue, or income. If the revenue and costs for a company are given by polynomials, then you can find the company's profit by subtracting polynomials.

The polynomials shown represent the cost and revenue for a soup company, where x is the number of cans of soup sold.

Revenue: $1.5x - 125$
Cost: $0.0002x^2 + 50$

Write a polynomial in standard form that represents the company's profit.

Vocabulary Review

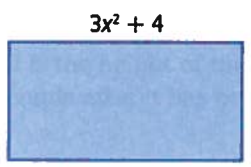
Directions: Match each term to its definition.

1. ____	degree	a.	to replace a variable in an expression with a numerical value
2. ____	opposite polynomial	b.	the value of the greatest exponent in a polynomial
3. ____	polynomial	c.	the form of a polynomial that shows the terms listed from left to right with the powers of the variables from greatest to least
4. ____	standard form	d.	the polynomial with all of its signs changed to the opposite sign
5. ____	substitute	e.	an algebraic expression consisting of one or more terms in which each term is a number or a product of numbers and variables raised to whole-number exponents

Skill Review

Directions: Read each problem and complete the task.

1. Which gives the area of the rectangle as a polynomial in standard form?



A. $12x^2 - 6x - 8$
B. $12x^2 - 16x + 8$
C. $12x^3 + 6x^2 + 16x^2 - 8$
D. $12x^3 - 6x^2 + 16x - 8$

2. Explain how to determine the degree of a polynomial that is written in standard form. Give an example in your explanation.

3. Determine whether the statement below is true or false. Explain your reasoning.
The product of 2 monomials is a binomial.

4. What is the value of the polynomial expression below when $y = 4$?
$-3y^3 + 2y^2 - 5y + 7$

A. -173
B. -133
C. 147
D. 187

5. What is the difference of these two polynomials?
$(6x^5 + 7x^3 - 4x^2 + 9) - (2x^4 + 7x^5 + x^2)$

A. $13x^5 - 2x^4 + 7x^3 - 3x^2 - 9$
B. $-x^5 - 2x^4 + 7x^3 - 5x^2 + 9$
C. $8x^5 + 3x^2 - 9$
D. $4x^5 - 5x^2 + 9$

Skill Practice

Directions: Read each problem and complete the task.

1. Which of the following expressions has the greatest value when $x = -1$?

A. $x^3 - 4x^2 + 5$
B. $-x^3 + 3x + 3$
C. $2x^3 - 3x^2 + x$
D. $-2x^3 + x^2 - 4x + 1$

2. What is the degree of the product of a polynomial of degree 2 and a polynomial of degree 3?

A. 2
B. 3
C. 5
D. 6

3. Is it possible to subtract two polynomials of degree 4 and get a polynomial of degree 2? Give an example to support your explanation.

4. What is the degree of the product of two linear polynomials?

5. The table shows a company's costs for labor and the costs of materials to produce x items. What polynomial expression represents the company's total cost to produce x items?

Costs for x Items	
Labor	$3x^2 + 300x + 10$
Materials	$x^2 + 80x + 100$

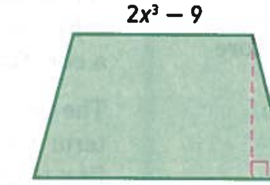
A. $2x^2 + 220x - 90$
B. $3x^2 + 400x + 90$
C. $4x^2 + 380x + 110$
D. $4x^2 + 220x - 110$

6. What is the degree of the sum of these two polynomials?
$(5x^3 - 2x^2 + 1) + (2x^2 - 6x^4 - 5)$

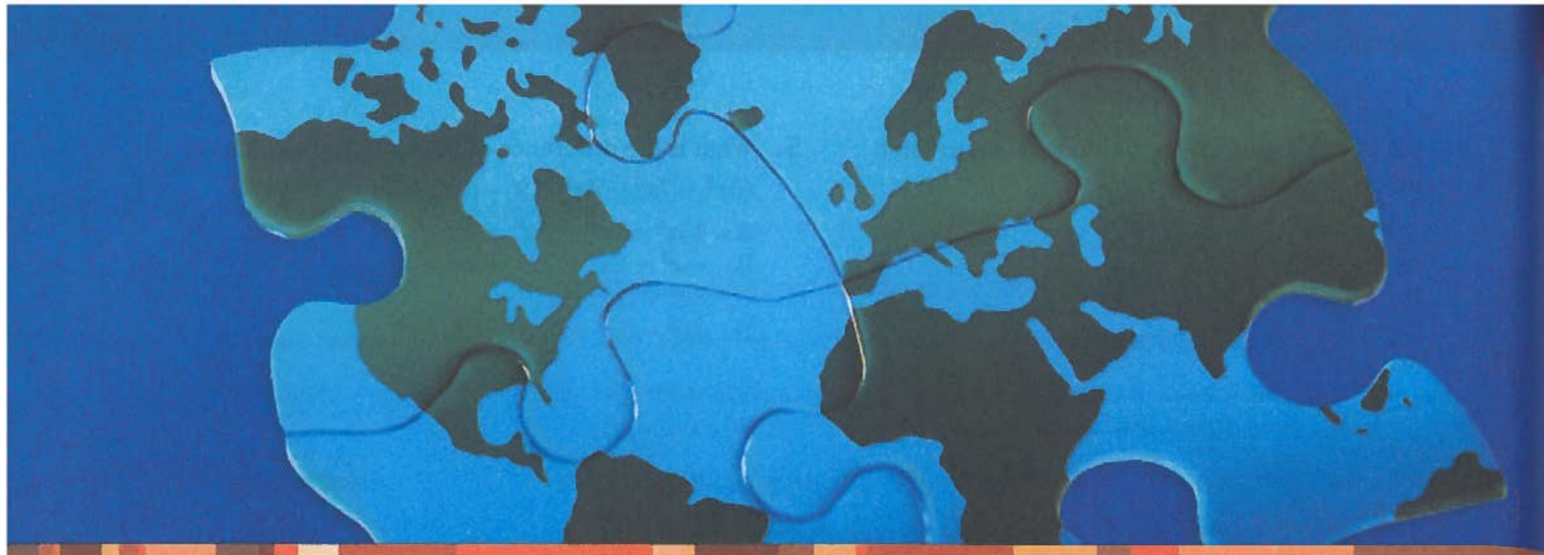
A. 2
B. 3
C. 4
D. 5

7. Is the following statement sometimes, always, or never true? Give examples to support your answer.
The sum of two monomials is a binomial.

8. Which gives the area of the trapezoid as a polynomial in standard form? (Remember, the area of a trapezoid with bases b_1 and b_2 and height h is given by the expression $\frac{1}{2}(b_1 + b_2)h$.)



A. $8x^4 + 20x^3$
B. $4x^4 + 10x^2$
C. $4x^3 + 10x^2$
D. $8x^2 + 20x$



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LESSON 4.2 Factor Polynomials

LESSON OBJECTIVES

- Read, write, and evaluate expressions with variables
- Identify the parts of an expression
- Factor polynomials
- Factor quadratic expressions

CORE SKILLS & PRACTICES

- Build Lines of Reasoning
- Make Use of Structure

Key Terms

coefficient
the number that appears before a variable that multiplies the variable in an algebraic expression

degree
the term in a polynomial with the greatest exponential power

polynomial
an expression with one or more monomial(s), or term(s)

Vocabulary

factor
to divide a monomial by another monomial with no remainder

leading coefficient
the coefficient accompanying the first term in a polynomial that has been written in standard form.

monomial
An expression with one term, such as 10, $2x$, and $3xy$

Key Concept

People practicing a variety of professions and hobbies write, simplify, and evaluate polynomial expressions. Polynomial expressions can be classified by their number of terms or by the greatest exponential power.

Factoring Out Monomials

Just as a puzzle is made up of individual pieces that make the puzzle whole, a polynomial is also made up of pieces—its factors. **Monomials**, or polynomials with one term, are certain types of polynomials that can be factored out of larger polynomials to help make them easier to work with.

Polynomial Language

A **polynomial** is a set of terms (or expressions) that include one or more variables raised to a whole-numbered power. Any term may also include a number called a **coefficient**. In the term $5x^2$, for example, the coefficient is the number 5.

The terms in a polynomial written in standard form are ordered from the term with the greatest power to the term with the least power. Because the first term in a polynomial written in standard form has the greatest power, its coefficient is called the **leading coefficient**.

leading coefficient constant

$$3x^3 + 2x^2 + x - 9$$

Classifying Polynomials

One way to classify polynomials is by the number of terms in the expression.

Classification	Number of Terms	Example
monomial	1	$5x^3$
binomial	2	$5x^3 + 8x^2$
trinomial	3	$5x^3 + 8x^2 + 12$

You can also classify polynomials by degree. The **degree** of a polynomial is the greatest power of the variable. When polynomials are written in standard form, the degree appears in the first term.

Classification	Degree	Example
constant	0	$3x^0 = 3$ ($x^0 = 1$ for $x \neq 0$)
linear	1	$2x + 9$
quadratic	2	$5x^2 - 2x + 7$

Factoring Using the Greatest Common Factors

The greatest common factor, or GCF of two or more polynomials is the polynomial with the greatest degree that divides evenly into both polynomials. When factoring a polynomial, treat each term as a separate polynomial and find the GCF of the terms.

Example 1: Find the GCF of a Polynomial

Find the GCF of the polynomial $4x^2y^3 - 2xy^2$.

Step 1 Find the GCF of the coefficients and the GCF of each variable. The GCF is the greatest power shared by every variable that appears in every term.

- The GCF of the coefficients 4 and 2 is 2.
- The greatest power that x^2 and x share is 1, so the GCF of the power of x is x.
- The greatest power that y^3 and y^2 share is 2, so the GCF of the power of y is y^2.

Step 2 Multiply these GCFs to determine the GCF of the polynomial.

The GCF is $2 \times x \times y^2 = 2xy^2$.

Once you have found the GCF of a polynomial, you can use it to **factor** the polynomial, rewriting it as a product of its smaller parts.

Example 2: Factor a Polynomial

Factor the polynomial $4x^2y^3 - 2xy^2$.

Step 1 Find the GCF of the two terms.

The GCF is $2 \times x \times y^2 = 2xy^2$.

Step 2 Divide both terms by the GCF.

$$4x^2y^3 - 2xy^2 = \frac{4x^2y^3}{2xy^2} - \frac{2xy^2}{2xy^2}$$

Step 3 Rewrite the problem by expressing the exponents in the denominator as negative exponents and combining all exponents. (Remember, a number raised to the 0 power is equal to 1.)

$$\frac{4x^2y^3}{2xy^2} - \frac{2xy^2}{2xy^2} = \frac{4}{2}x^{2-1}y^{3-2} - \frac{2}{2}x^{1-1}y^{2-2} = 2xy - 1$$

Step 4 Multiply the GCF of the polynomial, or $2xy^2$, by $(2xy - 1)$ to rewrite the polynomial.

$$4x^2y^3 - 2xy^2 = 2xy^2(2xy - 1)$$

CORE SKILL

Build Lines of Reasoning

When you divide exponents that share the same base, you apply the following rule:

$$\frac{x^p}{x^q} = x^{p-q} \text{ for } x \neq 0$$

To understand this rule, you can show the process of division in nonstandard form, showing all multiples of the variables. For example, you could simplify the expression from Example 2 $\frac{4x^2y^3}{2xy^2} - \frac{2xy^2}{2xy^2}$ as follows.

$$\frac{4x^2y^3}{2xy^2} - \frac{2xy^2}{2xy^2} = \frac{4xxyyy}{2xyy} - \frac{2xyy}{2xyy}$$

$$= \frac{\overset{2}{\cancel{4}}\,\cancel{x}\,x\,\cancel{y}\,\cancel{y}\,y}{\cancel{2}\,\cancel{x}\,\cancel{y}\,\cancel{y}} - \frac{\overset{1}{\cancel{2}}\,\cancel{x}\,\cancel{y}\,\cancel{y}}{\cancel{2}\,\cancel{x}\,\cancel{y}\,\cancel{y}}$$

$$= 2xy - 1$$

For the expression $\frac{8x^3 4y^2}{4xy} + \frac{4xy}{4xy}$, verify that the result is the same using the rule for dividing two powers as well as expanding each power and dividing out common factors.

CORE PRACTICE

Make Use of Structure

Recognizing that the coefficients of $x^2 + cx + d$ are either a sum or product of parts of the factors helps to check your work easily. To factor the quadratic expression $x^2 - 3x + 2$, you must find two numbers whose product is 2 and whose sum is -3. First, find two numbers whose product is 2 (1 and 2, or -1 and -2) Then from that list, determine which numbers sum to -3 (-1 and -2).

So, you can factor the quadratic expression $x^2 - 3x + 2$ as $(x + (-1))(x + (-2))$, or $(x - 1)(x - 2)$.

Factor the quadratic expression $x^2 + x - 12$.

Think about Math

Directions: Rewrite each expression by factoring the GCF of the terms.

1. $14x^3 + 4x^9$ 2. $2x^7y - 3x^2y^3$ 3. $4x^3y^2 + 2x^4y^4 - 6x^2y^3$

Factoring Quadratic Expressions

Quadratic expressions help model most scenarios that we know, from the equations of the orbits of planets to the paths of ballistic objects.

Quadratic Expressions

The names "second degree polynomials," "quadratic trinomials," and "quadratic expressions" all mean the same thing. They all have a variable raised to the second power, as in $x^2 + cx + d$, where c and d are real numbers. The following are examples of quadratic expressions.

$x^2 + 8x - 4$ $2x^2 + 3x + 5$ $3x^2 + 5x - 2$ $10x^2 - 12x - 8$

When you factor the quadratic expression $x^2 + cx + d$ for integers c and d, you rewrite the expression as the product of two binomials $(x + a)(x + b)$. If an expression can be factored, then you will be able to find two numbers a and b whose sum equals c, and whose product equals d.

Example 3: Factor a Quadratic Expression

Factor the quadratic $x^2 + 6x + 8$.

1	Is the constant 8 positive or negative?	positive
2	What numbers can you multiply to get a positive product?	two positive or two negative numbers
3	Is the coefficient 6 positive or negative?	positive
4	Review the answer to Question 2. You're looking for two positive or two negative numbers. Can you add two negative numbers to get a positive number?	no
5	So, what two positive numbers can you add to make the coefficient 6?	1 + 5; 2 + 4; 3 + 3
6	Of the numbers that add to make 6, which two can you multiply to make the product 8?	2 × 4
7	What numbers replace the question marks in the terms $(x + ?)(x + ?)$?	2 and 4

So, you can factor $x^2 + 6x + 8$ as $(x + 2)(x + 4)$.

Check your work by expanding your factored expression.

$$\begin{aligned}(x + 2)(x + 4) &= x(x + 4) + 2(x + 4)\\ &= x^2 + 4x + 2x + 8\\ &= x^2 + 6x + 8\end{aligned}$$

Leading Coefficients Not Equal to 1

Sometimes, it is possible to factor across a quadratic expression with a leading coefficient not equal to 1. In that case, apply the GCF first. Then factor the expression.

Example 4: Factor the GCF First

Factor the quadratic $2x^2 + 4x - 6$.

Step 1 Factor the GCF of the terms, 2.

$2x^2 + 4x - 6 = 2(x^2 + 2x - 3)$

Step 2 The quadratic inside the parentheses has a leading coefficient of 1. Look for the factors of -3.

Factors of -3: 1 and -3, -1 and 3

Step 3 Identify the factors of -3 that sum to 2.

From the previous list, only -1 and 3 have a sum of 2.

Step 4 Since the leading coefficient is now 1, we can use our basic binomial factoring to complete the factoring.

$2(x^2 + 2x - 3) = 2(x + 3)(x - 1)$

Sometimes the leading coefficient in a quadratic expression is not equal to 1. You can write such expressions as $ax^2 + bx + c$. Because the leading coefficient is not equal to 1, you must deal with 3 coefficients, a, b, and c.

Example 5: Factor $ax^2 + bx + c$

Factor the quadratic $4x^2 + 14x - 8$ as the product of two binomials.

Step 1 Unlike simpler quadratic expressions, the product of the two numbers in this example must equal ac. In this case, ac = –32.

Factors of –32: 1 and -32, -1 and 32, 2 and -16, -2 and 16, 4 and -8, -4 and 8

Step 2 The sum of the two factors you select must equal b, or in this case, 14.

From the previous list, only -2 and 16 have a sum of 14.

Step 3 Now you are ready to factor by grouping. Rewrite the quadratic by writing the middle term as the sum of the factors. Group the terms in pairs, and factor the GCF out of each pair.

$$\begin{aligned}4x^2 + 14x - 8 &= 4x^2 - 2x + 16x - 8\\ &= (4x^2 - 2x) + (16x - 8)\\ &= 2x(2x - 1) + 8(2x - 1)\end{aligned}$$

Step 4 Now, each term has the binomial $(2x - 1)$ in it as a factor. So factor out $(2x - 1)$ as the GCF.

$2x(2x - 1) + 8(2x - 1) = (2x - 1)(2x + 8)$

CORE PRACTICE

Understand the Question

When you are given a question, make sure you read it carefully so you do not miss any important information. You might also be able to eliminate any unnecessary information by reading the question carefully.

Ginger has factored a quadratic to find that her binomial factors are $(x + 5)$ and $(x - 2)$. She also remembers that her quadratic had a leading coefficient of 4. What was her quadratic expression?

Vocabulary Review

Directions: Match each term to its definition.

1. ______ coefficient
2. ______ degree
3. ______ factor
4. ______ leading coefficient
5. ______ monomial
6. ______ polynomial

a. The exponent in the term of a polynomial with the greatest exponential power
b. A polynomial with one term
c. The number that appears before a variable that multiplies the variable in an algebraic expression
d. An expression with one or more monomials
e. The coefficient accompanying the first term in a polynomial that has been written in standard form
f. To divide a monomial by another monomial exactly, meaning with no remainder

Skill Review

Directions: Read each problem and complete the task.

1. What is the leading coefficient of the polynomial shown below?
$4x^2 - 2x^3 - 5$
A. -5
B. -2
C. 3
D. 4

2. If $x^2 + bx + c = (x + n)(x + m)$, what is the value of $n + m$?
A. b
B. c
C. bc
D. $b + c$

3. Write the following polynomial expression as the product of linear terms.
Show your work.
$3x^2 - 8x + 4$

4. Which factor pair can you use to find r and s in the equation $(x + r)(x + s) = x^2 - 8x + 15$?
A. -2 and 4
B. -4 and 2
C. 3 and 5
D. -3 and -5

5. Marquise wrote the quadratic $4x^2 + 10x - 6$ in the factored form $(ax + b)(x + c)$, where a, b, and c are nonzero integers. Which of the following correctly gives the values of a, b, and c in order from least to greatest?
A. a, b, c
B. b, a, c
C. b, c, a
D. c, a, b

Skill Practice

Directions: Read each problem and complete the task.

1. If $x^2 + bx + c = (x + n)(x + m)$, what is the value of nm?
A. b
B. c
C. bc
D. $b + c$

2. Rebecca said that she was able to factor the expression $x^2 + x + 1$ as the product of 3 linear factors. Do you agree or disagree with her statement? Explain your reasoning.

3. Which of the following shows the correct factorization of $4x^2 + 8x - 60$?
A. $2(x - 3)(x - 5)$
B. $2(x + 3)(x - 5)$
C. $4(x - 3)(x + 5)$
D. $4(x + 3)(x + 5)$

4. Write the following polynomial expression as the product of linear terms. Show your work.
$4x^3 + 2x^2y - 2xy^2$

5. Given that $-5x - 21 + 6x^2 = (ax + b)(cx + d)$, what is the value of bd?
A. -21
B. -5
C. 6
D. 11

6. Which of the following is not a factor of the polynomial $6x^2 + 15x + 6$?
A. 3
B. $(x - 1)$
C. $(x + 2)$
D. $(2x + 1)$

7. Determine whether the statement below is true or false. Give an example to demonstrate your reasoning.
Factoring a quadratic expression always results in the product of two linear binomials.

8. For the quadratic expression $ax^2 + bx + c$ where a, b, and c are all nonzero integers, the product of the leading coefficient and the constant term is 36. Which of the following could not be a value of b?
A. -13
B. 15
C. 20
D. 35

LESSON 4.3 Solve Quadratic Equations

Thinkstock

LESSON OBJECTIVES

- Solve a quadratic equation by inspection, by factoring, by completing the square, and by using the quadratic formula

CORE SKILLS & PRACTICES

- Reason Abstractly
- Solve Real-World Problems

Key Terms

quadratic formula
a formula that can be used to solve any quadratic equation by substituting the coefficients of the equation

discriminant
the part of the quadratic formula that is under the square root

Vocabulary

completing the square
a technique of manipulating quadratic equations so that they can be solved by taking the square root of both sides

solving by inspection
determining the solution(s) of an equation simply by looking at the equation

perfect square trinomial
a quadratic expression that can be written as a perfect square of a linear expression

Key Concept

Quadratic equations can be solved in several ways. Simple quadratic equations can be solved by inspection. More complex ones can be solved by factoring, completing the square, or using the quadratic formula.

Solving a Quadratic Equation by Factoring

Quadratic equations can be used to describe the motion of objects. Solving a quadratic equation can tell you when an object will land on the ground after being thrown, or even how high it will be off the ground during its travel. Factoring is one way to solve these types of quadratic equations.

Solving a Factored Quadratic Equation

You can find the roots of a quadratic equation by setting the equation equal to zero and applying the zero-product principle: If $a \times b = 0$, then $a = 0$ or $b = 0$ or both a and $b = 0$. In other words, if you know that the product of two factors is 0, then one or both of the factors must be 0.

Example 1: Use the Zero-Product Principle

Solve the equation $(x - 1)(x + 5) = 0$.

Step 1 Set the factors equal to 0. $\quad x - 1 = 0 \quad x + 5 = 0$

Step 2 Solve for x. $\quad x = 1 \quad x = -5$

The solutions are 1 and −5.

Factoring to Solve a Quadratic Equation

If you are given a quadratic equation that contains a trinomial that is equal to 0, you first need to find the factors. Then you can set the factors equal to 0 and find the solutions to the equation.

Example 2: Solve by Factoring

Solve the quadratic equation $x^2 + 14x + 48 = 0$.

Step 1 Factor the left side of the equation. $\quad (x + 6)(x + 8) = 0$

Step 2 Set each factor equal to 0 and solve for x. $\quad x + 6 = 0 \quad x + 8 = 0$; $\quad x = -6 \quad x = -8$

The solutions are −6 and −8.

Think about Math

Directions: Solve each quadratic equation.

1. $(3n + 6)(n - 2) = 0$
2. $x^2 + 4x - 21 = 0$

Completing the Square

When you are cooking, you might follow recipes and measure ingredients carefully, or you may know the recipe so well that you can add ingredients by feel. Similarly, some quadratic equations require special tools and methods to solve, and others you can see the solution quickly without any additional help.

Solving by Inspection

Not all quadratic equations can be factored. Some quadratic equations can be solved by taking the square root of both sides. For simple equations, you may be able to do this mentally. This is called **solving by inspection**.

Example 3: Solve by Inspection

Solve the equation $x^2 = 49$.

In this example, 49 is a perfect square. Taking the square root of both sides is easy to do mentally. Remember that a positive number has two square roots, one positive and one negative. The solutions are 7 and −7.

$$x^2 = 49$$
$$\sqrt{x^2} = \sqrt{49}$$
$$x = \pm 7$$

Example 4: Take the Square Root of Both Sides

Solve the equation $x^2 = 77$.

You can take the square root of both sides even though the right side is not a perfect square. In this example, the solutions, $\sqrt{77}$ and $-\sqrt{77}$, are not integers.

$$x^2 = 77$$
$$\sqrt{x^2} = \sqrt{77}$$
$$x = \pm\sqrt{77}$$

CALCULATOR SKILL

Use the $\sqrt{x}$ function on a calculator to find the square root of a number that is not a perfect square. To find the square root of 77, press $\sqrt{x}$ 77 ENTER. Be aware that the answer given by the calculator (8.774964387) is only an approximation of $\sqrt{77}$. The actual value of $\sqrt{77}$ is an irrational number; it has infinitely many numbers after the decimal point, and these numbers do not repeat in any kind of pattern. Notice that the calculator returns only the positive square root, so you will have to remember that the negative square root is also a solution.

TEST-TAKING SKILL

Gather Information

When taking an exam, it is always important to understand not only what is being asked but also how to convert verbal information to an algebraic equation.

The area of a rectangular patio is 144 ft^2 and the length of the patio is 10 feet longer than the width. You can write and solve an equation to find the length and width of the patio. Let x represent the width of the patio. Then the length of the patio is $x + 10$. Remember that the formula for the area of a rectangle is $A = \ell w$.

$$x(x + 10) = 144$$
$$x^2 + 10x = 144$$
$$x^2 + 10x - 144 = 0$$
$$(x + 18)(x - 8) = 0$$
$$x + 18 = 0 \qquad x - 8 = 0$$
$$x = -18 \qquad x = 8$$

There are two solutions to this quadratic equation, one positive and one negative. Because x represents the width of a rectangle, only positive values make sense. Therefore, the width is 8 feet and the length is $8 + 10 = 18$ feet.

Suppose you're designing a kitchen island. You want the width to be 2 feet shorter than the length, with a total area of 15 ft^2. What should the length and width be?

A trinomial that can be factored as a square is a **perfect square trinomial.** If the quadratic expression in a quadratic equation is a perfect square trinomial, you can take the square root of both sides.

Example 5: Perfect Square Trinomial

Solve the equation $x^2 + 4x + 4 = 9$.

Step 1 Factor the left side. $(x + 2)(x + 2) = 9$

Step 2 Write the left side as a square. $(x + 2)^2 = 9$

Step 3 Take the square root of both sides. $\sqrt{(x + 2)^2} = \sqrt{9}$

$x + 2 = \pm 3$

Step 4 Solve for x. $x + 2 = 3$ or $x + 2 = -3$

$x = 1$ or $x = -5$

The solutions are 1 and -5.

Solving by Completing the Square

How do you take the square root of a quadratic trinomial that cannot be written as a square? You can use a technique called **completing the square** to make the quadratic trinomial into a perfect square trinomial.

For example, consider the equation $x^2 + 12x + 11 = 0$. Notice that the left side of the equation cannot be written as a square, so we cannot solve by taking the square root of both sides. However, we can manipulate the left side so that it can be written as a square.

The goal is to write the left side in the form $x^2 + 2bx + b^2$, so that we can write it as a square, $(x + b)^2$. We must find the value of b.

Example 6: Completing the Square

Solve the equation $x^2 + 12x + 11 = 0$.

Step 1 Isolate the terms with variables on one side. In this case, that means subtracting 11 from both sides. $x^2 + 12x + 11 = 0$; $x^2 + 12x = -11$

Step 2 Find the values of b and b^2. The x-term is $12x$, so $2bx = 12x$. $2bx = 12x$; $b = 6$; $b^2 = 36$

Step 3 Use the value of b^2 to write the left side of the equation as $x^2 + 12x + 36$. To keep the equation balanced, add 36 to the right side of the equation. $x^2 + 12x + 36 = -11 + 36$; $x^2 + 12x + 36 = 25$

Step 4 Write the left side of the equation as a square. $(x + 6)^2 = 25$

Step 5 Take the square root of both sides. $\sqrt{(x + 6)^2} = \sqrt{25}$; $x + 6 = \pm 5$

Step 6 Solve for x. $x + 6 = 5$ or $x + 6 = -5$; $x = -1$ or $x = -11$

The solutions are -1 and -11.

Think about Math

Directions: Solve each quadratic equation.

1. $(x + 5)^2 = 49$
2. $x^2 + 6x = 0$
3. $x^2 + 4x = 12$

CORE SKILL

Reason Abstractly

Consider the equation $x^2 = -16$. What happens when we take the square root of both sides?

$$x^2 = -16$$
$$x = \pm\sqrt{-16}$$
$$x = ???$$

There is no real number that can be squared to produce a negative number, so a negative number does not have a real square root. If at any point in solving an equation you must take the square root of a negative number, the equation has no real solutions.

Which equation or equations below have no real solutions?

Equation 1: $x^2 + 100 = 0$

Equation 2: $x^2 + 4x + 10 = 4$

Equation 3: $-x^2 = -81$

The Quadratic Formula

Solving a Quadratic Equation with the Quadratic Formula

The **quadratic formula** is a formula that allows you to solve any quadratic equation. All you need to do is identify each of the coefficients in the equation and substitute them in the formula to find the solution or solutions.

The Quadratic Formula

For the quadratic equation $ax^2 + bx + c = 0$, the solution is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Note that a quadratic equation must be in the form $ax^2 + bx + c = 0$ before you can use the quadratic formula.

Example 7: The Quadratic Formula

Solve the equation $x^2 - 5x - 14 = 0$.

Step 1 Identify the values of a, b, and c. $a = 1, b = -5, c = -14$

Step 2 Substitute the values of a, b, and c into the quadratic formula and simplify.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-14)}}{2(1)}$$
$$= \frac{5 \pm \sqrt{25 - (-56)}}{2}$$
$$= \frac{5 \pm \sqrt{81}}{2}$$
$$= \frac{5 \pm 9}{2}$$
$$x = 7 \text{ or } x = -2$$

The solutions are 7 and -2.

Knowing When a Quadratic Equation has No Real Solutions

You can tell whether a quadratic equation has no real solutions without solving it. You just need the part of the quadratic formula that is under the square root. This expression, $b^2 - 4ac$, is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If the discriminant is positive, then the equation has two real solutions.
- If the discriminant is 0, then the equation has one real solution.
- If the discriminant is negative, then the equation has no real solutions.

Example 8: Use the Discriminant

Without solving, tell how many real solutions each equation has.

a. $x^2 + 10x + 25 = 0$ b. $2x^2 - 10x + 9 = 0$ c. $x^2 - 3x + 7 = 0$

Step 1 Identify the values of a, b, and c.

a. $a = 1, b = 10, c = 25$ b. $a = 2, b = -10, c = 9$ c. $a = 1, b = -3, c = 7$

Step 2 Find the discriminant, $b^2 - 4ac$.

a. $b^2 - 4ac = (10)^2 - 4(1)(25)$
$= 100 - 100$
$= 0$

b. $b^2 - 4ac = (-10)^2 - 4(2)(9)$
$= 100 - 72$
$= 28$

c. $b^2 - 4ac = (-3)^2 - 4(1)(7)$
$= 9 - 28$
$= -19$

Step 3 Use the discriminant to determine the number of real solutions.

a. The discriminant is 0, so the equation has one real solution.

b. The discriminant is positive, so the equation has two real solutions.

c. The discriminant is negative, so the equation has no real solutions.

Think about Math

Directions: Use the quadratic equation $x^2 + 2x - 8 = 0$ to answer the following questions.

1. What are the values of a, b, and c?
2. What is the discriminant?
3. How many real solutions does the equation have?
4. Give the real solutions, if they exist.

CORE SKILL

Solve Real-World Problems

Sometimes when you are solving a real-world problem, one or more of the solutions may not make sense in the real-world situation. Always check that your solutions are reasonable in the context of the problem.

A rock is thrown upward at a speed of 38 ft/sec from a height of 5 ft. Its height in h feet after t seconds is given by the equation $-16t^2 + 38t + 5$. When will the rock hit the ground?

When the rock hits the ground, its height will be 0. Solve the equation $0 = -16t^2 + 38t + 5$.

$a = -16, b = 38, c = 5$

$$t = \frac{-(38) \pm \sqrt{(38)^2 - 4(-16)(5)}}{2(-16)}$$
$$= \frac{-38 \pm \sqrt{1764}}{-32}$$
$$= \frac{-38 \pm 42}{-32}$$
$$t = \frac{5}{2} = 2.5 \text{ or } t = -\frac{1}{8} = -0.125$$

There are two possible solutions. Remember that t represents time. Time cannot be negative, so only the positive solution makes sense. The rock will hit the ground after 2.5 seconds.

Now you try. An object is shot into the air. Its height is given by the equation $h = -5t^2 + 30t$, where h represents height and t represents elapsed time in seconds. How long will it take the object to reach the ground?

Vocabulary Review

Directions: Write the missing term in the blank.

completing the square	**discriminant**	**perfect-square trinomial**
quadratic formula	**solving by inspection**	

1. ________________ is a way to manipulate a quadratic equation so that one side is a ________________, which is an expression that can be written as a square.

2. Solving simple equations mentally is called ________________.

3. The ________________ is a formula that can be used to find the solutions of any quadratic equation. The part of this formula that is under the square root is called the ________________ and it can be used to determine the number of real solutions of the equation.

Skill Review

Directions: Read each problem and complete the task.

1. Solve the equation $(x + 7)(x - 3) = 0$.

2. Solve the equation $x^2 - 15x + 36 = 0$ by factoring.

3. Solve the equation $x^2 + x - 72 = 0$ by factoring.

4. Solve the equation $(x + 4)^2 = 100$ by taking the square root of both sides.

5. A company is installing a swimming pool in a customer's backyard. In order for the pool to fit in the yard, the area must be 195 square feet and the width of the pool must be 2 feet shorter than the length. What should the width of the pool be?
   A. 11 feet
   B. 13 feet
   C. 15 feet
   D. 17 feet

6. Which of the following equations has no real solutions?
   A. $x^2 = 49$
   B. $(x - 7)^2 = 256$
   C. $(x + 11)^2 = -121$
   D. $-4x^2 = -100$

7. What must be added to the expression below to make it a perfect square trinomial?
   $x^2 + 24x +$ ____
   A. 12
   B. 48
   C. 144
   D. 576

8. How many real solutions does the equation $x^2 + 6x - 12 = 0$ have?

9. Solve the equation $-3x^2 - 5x + 2 = 4$ using the quadratic formula.

Skill Practice

Directions: Read each problem and complete the task.

1. Solve the equation $2x^2 + 10x - 3 = x^2 + 15x - 9$.

2. The first several steps to solve the equation $3(x^2 - 4) - 6 = -3(3x + 2)$ are shown below. Complete the solution process to solve the equation.

$$3(x^2 - 4) - 6 = -3(3x + 2)$$
$$3x^2 - 12 - 6 = -9x - 6$$
$$3x^2 - 18 = -9x - 6$$
$$3x^2 + 9x - 18 = -6$$

3. Solve the equation $(x + 3)^2 + 5 = -2x - 2$.

4. A rocket is launched into the air at a velocity of 256 feet per second from ground level. The height h of the rocket in feet after t seconds is given by the equation $h = -16t^2 + 256t$. How long will it take the rocket to reach the ground?
   A. 0 seconds
   B. 4 seconds
   C. 16 seconds
   D. 40 seconds

5. You can use the methods in this lesson to solve formulas for a variable. The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle.
   a. Solve this formula for r.
   b. Ana said that there are two solutions for r, one positive and one negative. Is Ana correct? If so, explain why. If not, describe Ana's error.
   c. Find the radius of a circular tabletop whose area is 12 ft². Round your answer to the nearest whole number.

6. The distance d in feet that a dropped object falls in t seconds is given by the equation $d = 16t^2$. If an object is dropped from a height of 900 feet, how long will it take to reach the ground?
   A. 7.5 seconds
   B. 8.7 seconds
   C. 56 seconds
   D. 144 seconds

7. Raul has 100 feet of fencing that he wants to use to make a rectangular pen with an area of 525 ft². He needs to determine the necessary length and width of the rectangle. The equation $w(50 - w) = 525$ models this situation. What do the two solutions to this equation represent? What are the length and width of the rectangle that Raul should make?

8. Yuri operates a food truck that sells sandwiches. After reviewing her financial information, she has determined that her daily profit p in dollars is modeled by the function $p = 70s - s^2 - 1225$, where s is the number of sandwiches sold. How many sandwiches does Yuri have to sell each day to break even? (Hint: Yuri will break even when her profit is $0.)

9. For what value of c does the equation $x^2 + 6x + c = 0$ have one real solution? Explain how you found your answer.

10. For the quadratic equation $3x^2 - 2x + 1 = 4x - 3$, Zach said that $a = 3$, $b = -2$, and $c = 1$. What error did Zach make? Explain how to find the correct values of a, b, and c, and identify these values. Then solve the equation.

Garry Moore/moodboard/Glow Images

LESSON 4.4 Evaluate Rational Expressions

LESSON OBJECTIVES

- Evaluate rational expressions
- Simplify rational expressions
- Add, subtract, multiply, and divide rational expressions

CORE SKILLS & PRACTICES

- Evaluate Expressions
- Perform Operations

Key Terms

rational expression
a ratio of two polynomials

restricted value (of a rational expression)
a value of the variable for which the denominator of the rational expression is equal to 0

Vocabulary

polynomial
an expression made up of numbers and variables with whole-numbered exponents and any sum, difference, or product of them

prime number
a whole number > 1 whose only two factors are 1 and itself

reciprocals
two numbers or expressions whose product is 1

least common denominator (LCD)
the least common multiple of two or more denominators

Key Concept

A rational expression is a ratio of two polynomials. Rational expressions are similar to fractions and can be simplified, multiplied, divided, added, and subtracted using methods similar to those for fractions.

Simplifying Rational Expressions

A **polynomial** is an expression containing one or more terms made up of numbers and/or variables with whole-number exponents. A **rational expression** is a ratio of two polynomials. The word "rational" stems from the word "ratio," indicating a comparison of a numerator and a denominator. Rational expressions appear often in math and science. For example, in physics, rational expressions can be used to describe the motion of objects along a curved or circular path.

Rational Expressions

Below are examples and non-examples of rational expressions. Note that the numerator and the denominator in the rational expressions are both polynomials. Remember that terms in a polynomial cannot have negative exponents, division by a variable, variable exponents, or variables under a radical.

Examples of Rational Expressions		
$\frac{5}{x}$	$\frac{n+1}{n-1}$	$\frac{r^2+5r}{r^2+7r+10}$

Non-Examples of Rational Expressions			
$\frac{5}{2^x}$	$\frac{3-\sqrt{n}}{2n}$	$\frac{5}{1+\frac{1}{y}}$	$\frac{2x^{-2}+3}{x}$

A rational expression is undefined when the denominator is equal to 0. A **restricted value** of a rational expression is any value of the variable for which the denominator is equal to 0.

Example 1: Find Restricted Values

Find the restricted value(s) for each rational expression.

a. $\frac{5}{x}$ b. $\frac{r^2+5r}{r^2+7r+10}$

Step 1 Set the denominator equal to 0.

a. $x = 0$ b. $r^2 + 7r + 10 = 0$

Step 2 Solve the equation. The solution or solutions are the restricted values.

a. $x = 0$

The restricted value is 0.

b. $(r+2)(r+5) = 0$
$r + 2 = 0$ or $r + 5 = 0$
$r = -2$ or $r = -5$

The restricted values are -2 and -5.

Simplifying Rational Expressions

A rational expression is simplified when its numerator and denominator have no common factors other than 1. To simplify a rational expression, you may need to factor the numerator, denominator, or both to recognize common factors that can be divided out. The original expression and the simplified expression, under the same restrictions, are equivalent.

Example 2: Simplify a Rational Expression

Simplify the rational expression $\frac{x^2+x}{x^2+3x+2}$

Step 1 Factor the numerator and the denominator. $\frac{x^2+x}{x^2+3x+2} = \frac{x(x+1)}{(x+1)(x+2)}$

Step 2 Identify restricted values. Restricted values: $x = -1$ and $x = -2$

Step 3 Cancel out common factors in the numerator 1 and denominator. Because a quantity divided by itself equals 1, replace factors that are divided out with 1. $\frac{x\cancel{(x+1)}^{1}}{\cancel{(x+1)}_{1}(x+2)}$

Step 4 Write the simplified expression and indicate the restricted values. $\frac{x}{x+2}; x \neq -1, x \neq -2$

Think about Math

Directions: Simplify the rational expressions and state the restricted values.

1. $\frac{x^3-3x^2}{x^2+2x-15}$
2. $\frac{5x}{5x+10}$
3. $\frac{x^2-3x-18}{x^2-x-12}$

Multiplying and Dividing Rational Expressions

The formula $d = rt$ describes the distance traveled at a constant rate in a time period. Rational expressions are often used to represent rates, because rates are ratios of one quantity to another. To use the formula $d = rt$ when r is a rational expression, multiply and divide with rational expressions.

CORE SKILL

Evaluate Expressions

You can evaluate rational expressions the same way you evaluate other algebraic expressions. Just replace the variable for whichever value you want to substitute in. Then simplify both the numerator and denominator using the appropriate order of operations. Finally, reduce the fraction by dividing out any common factors in the numerator and denominator.

Now you try. Evaluate the rational expression $\frac{-3g^2-1}{-4g+2}$ when $g = -3$.

CALCULATOR SKILL

You can use a calculator to evaluate rational expressions, but remember to use parentheses around the numerator and the denominator. For example, to evaluate $\frac{x-3}{2x+1}$ when $x = 2$, you must enter $(2-3) \div (2 \times 2 + 1)$. Try entering the expression with and without parentheses to see that the calculator displays different answers. Always include the parentheses to be sure your answer is correct.

CORE SKILL

Perform Operations

To multiply fractions, you multiply the numerators, multiply the denominators, and then simplify your answer. However, it may be easier to divide out common factors before you multiply. Begin by rewriting each fraction so that its numerator and denominator are products of prime numbers. A **prime number** is a whole number greater than 1 whose only two factors are 1 and itself.

Rewrite the fractions below so that their numerators and denominators are products of prime numbers to find the product. $\frac{22}{27} \times \frac{63}{66}$.

Multiplying Fractions

Multiplying rational expressions is similar to multiplying fractions.

Example 3: Multiply Fractions

Find the product $\frac{2}{3} \times \frac{3}{4}$.

Step 1 Multiply numerators and denominators. $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$

Step 2 Simplify the answer by dividing out common factors. $\frac{6}{12} = \frac{1 \times \cancel{6}}{2 \times \cancel{6}} = \frac{1}{2}$

Multiplying Rational Expressions

When multiplying rational expressions, factor each numerator and denominator and divide out common factors before multiplying.

Example 4: Multiply Rational Expressions

Find the product $\frac{2x}{x+4} \times \frac{x^2+5x+4}{4x}$.

Step 1 Factor the numerators and the denominators.

$$\frac{2x}{x+4} = \frac{2 \times x}{(x+4)} \qquad \frac{x^2+5x+4}{4x} = \frac{(x+1)(x+4)}{2 \times 2 \times x}$$

Step 2 Divide out common factors. As with fractions, you can divide a factor in either numerator by a factor in either denominator.

$$\frac{\overset{1}{\cancel{2 \times x}}}{\underset{1}{\cancel{(x+4)}}} \times \frac{(x+1)\overset{1}{\cancel{(x+4)}}}{2 \times \underset{1}{\cancel{2 \times x}}}$$

Step 3 Multiply the remaining factors.

$$\frac{1 \times (x+1) \times 1}{1 \times 2 \times 1} = \frac{x+1}{2}$$

Dividing Fractions

Two numbers or expressions are reciprocals if their product is 1. Remember that dividing is the same as multiplying by the reciprocal. Dividing rational expressions is similar to dividing fractions.

Example 5: Divide Fractions

Find the quotient $\frac{8}{11} \div \frac{4}{33}$.

Step 1 Rewrite division as multiplication by the reciprocal. (The reciprocal of $\frac{4}{33}$ is $\frac{33}{4}$). $\frac{8}{11} \div \frac{4}{33} = \frac{8}{11} \times \frac{33}{4}$

Step 2 Rewrite each fraction so that its numerator and denominator are products of prime numbers.

$$\frac{8}{11} = \frac{2 \times 2 \times 2}{11} \qquad \frac{33}{4} = \frac{3 \times 11}{2 \times 2}$$

Step 3 Divide out common factors.

$$\frac{\overset{1}{\cancel{2 \times 2}} \times 2}{\underset{1}{\cancel{11}}} \times \frac{3 \times \overset{1}{\cancel{11}}}{\underset{1}{\cancel{2 \times 2}}}$$

Step 4 Multiply the remaining factors and simplify the answer.

$$\frac{1 \times 2 \times 3 \times 1}{1 \times 1} = \frac{6}{1} = 6$$

Dividing Rational Expressions

To divide rational expressions, first rewrite the division as multiplication by the reciprocal. Then follow the steps for multiplying rational expressions.

Example 6: Divide Rational Expressions

Find the quotient $\frac{3x-9}{3x} \div \frac{x^2-9}{9x+3}$.

Step 1 Rewrite division as multiplication by the reciprocal.

$$\frac{3x-9}{3x} \times \frac{9x+3}{x^2-9}$$

To write the reciprocal, interchange the numerator and the denominator.

Step 2 Factor the numerators and the denominators.

$$\frac{3x-9}{3x} = \frac{3(x-3)}{3 \times x} \qquad \frac{9x+3}{x^2-9} = \frac{3(3x+1)}{(x-3)(x+3)}$$

Step 3 Divide out common factors.

$$\frac{\overset{1}{3\cancel{(x-3)}}}{\underset{1}{\cancel{3}} \times x} \times \frac{\overset{1}{\cancel{3}}(3x+1)}{\underset{1}{\cancel{(x-3)}}(x+3)}$$

Step 4 Multiply the remaining factors.

$$\frac{3 \times 1 \times 1 \times (3x+1)}{1 \times x \times 1 \times (x+3)} = \frac{3(3x+1)}{x(x+3)}$$

Think about Math

Directions: Perform the operations.

1. $\frac{6x}{x^2-3x-18} \times \frac{x+3}{2x^2+8x}$

2. $\frac{x^2-16}{21x} \div \frac{4x+16}{7x^2}$

Adding and Subtracting Rational Expressions

Rational expressions are often used in work problems. If you know the amount of time it takes for several individual people to complete a task when working alone, you can add rational expressions to determine how long it will take them to complete the task working together.

Adding with Like Denominators

Adding rational expressions with like denominators is similar to adding fractions with like denominators. To add fractions with like denominators, add the numerators and keep the like denominator. Simplify the answer if necessary.

$$\frac{11}{17} + \frac{2}{17} = \frac{11+2}{17} = \frac{13}{17} \qquad \frac{3}{8} + \frac{5}{8} = \frac{3+5}{8} = \frac{8}{8} = 1$$

Example 7: Add Rational Expressions with Like Denominators

Find the sum $\frac{3x-1}{x+4} + \frac{2x+4}{x+4}$.

Step 1 Add the numerators and keep the like denominator.

$$\frac{(3x-1)+(2x+4)}{x+4}$$

Step 2 Combine like terms in the numerator. Simplify the answer, if necessary.

$$\frac{3x-1+2x+4}{x+4} = \frac{3x+2x-1+4}{x+4} = \frac{5x+3}{x+4}$$

WORKPLACE SKILL

Plan and Organize

A production manager is considering hiring a new employee. The new employee would help another employee complete production jobs. The present employee can complete a production job working alone in x hours. The manager estimates that the new employee will be able to complete a production job working alone in $x + 5$ hours. To help decide whether she should hire the new employee, the manager must determine how long it will take the two employees working together to complete a production job. Add the rational expressions below to determine the fraction of a production job that the two employees will complete per hour when working together.

The present employee completes $\frac{1}{x}$ job per hour.

The new employee will complete $\frac{1}{x+5}$ job per hour.

Adding with Unlike Denominators

One way to add fractions with unlike denominators is to use the **least common denominator (LCD)**. The LCD is the least common multiple of two or more denominators.

Example 8: Add Fractions with Unlike Denominators

Find the sum $\frac{5}{12} + \frac{7}{18}$.

Step 1 Factor each denominator into a product of prime numbers. Write the products using exponents.

$12 = 2 \times 2 \times 3 = 2^2 \times 3 \qquad 18 = 2 \times 3 \times 3 = 2 \times 3^2$

Step 2 For every prime-number factor in the denominators, identify its greatest power. Multiply these powers to find the LCD.

$\text{LCD} = 2^2 \times 3^2$
$= 4 \times 9 = 36$

The prime-number factors in the denominators are 2 and 3. The greatest power of 2 that appears in either denominator is 2^2 and the greatest power of 3 that appears in either denominator is 3^2.

Step 3 Rewrite each fraction as an equivalent fraction whose denominator is the LCD.

$$\frac{5}{12} \xrightarrow{\times 3} \frac{15}{36} \qquad \frac{7}{18} \xrightarrow{\times 2} \frac{14}{36}$$

Step 4 Add the fractions. Simplify the answer, if necessary.

$$\frac{15}{36} + \frac{14}{36} = \frac{15 + 14}{36} = \frac{29}{36}$$

To add rational expressions with unlike denominators, first identify the LCD. Then use the LCD to write rational expressions with like denominators.

Example 9: Add Rational Expressions with Unlike Denominators

Find the sum $\frac{2}{x+3} + \frac{2x}{x-1}$.

Step 1 Factor all denominators, if possible. In this case the denominators cannot be factored. Proceed to Step 2.

Step 2 Identify the LCD. Because the denominators cannot be factored, the LCD is the product of the denominators.

$$\text{LCD} = (x + 3)(x - 1)$$

Step 3 Rewrite each rational expression as an equivalent expression whose denominator is the LCD.

$$\frac{2}{x+3} \xrightarrow{\times (x-1)} \frac{2(x-1)}{(x+3)(x-1)} \qquad \frac{2x}{x-1} \xrightarrow{\times (x+3)} \frac{2x(x+3)}{(x+3)(x-1)}$$

Step 4 Add the rational expressions. Simplify the answer, if necessary.

$$\frac{2(x-1)}{(x+3)(x-1)} + \frac{2x(x+3)}{(x+3)(x-1)} = \frac{2(x-1) + 2x(x+3)}{(x+3)(x-1)}$$

Distributive Property $\rightarrow = \frac{2x - 2 + 2x^2 + 6x}{(x+3)(x-1)}$

Combine like terms. $\rightarrow = \frac{2x^2 + 8x - 2}{(x+3)(x-1)}$

Subtracting with Like Denominators

Subtracting rational expressions is similar to adding rational expressions.

Example 10: Subtract with Like Denominators

Find the difference $\frac{2x-3}{x+2} - \frac{x+5}{x+2}$.

Step 1 Subtract the numerators. Keep the like denominator.

$$\frac{(2x-3) - (x+5)}{x+2}$$

Step 2 Combine like terms in the numerator. Simplify the answer, if necessary.

$$\frac{(2x-3) - (x+5)}{x+2} = \frac{2x - 3 - x - 5}{x+2} = \frac{x-8}{x+2}$$

Subtracting with Unlike Denominators

To subtract rational expressions with unlike denominators, first find a common denominator.

Example 11: Subtract with Unlike Denominators

Find the difference $\frac{3x}{x^2-1} - \frac{x+2}{x-1}$.

Step 1 Factor all denominators.

$x^2 - 1 = (x - 1)(x + 1) \qquad x - 1$

The denominator $x - 1$ cannot be factored.

Step 2 Identify the LCD. To find the LCD, multiply every factor in the denominators.

$\text{LCD} = (x - 1)(x + 1)$

The factors in the denominators are $x - 1$ and $x + 1$.

Step 3 Rewrite each rational expression as an equivalent expression whose denominator is the LCD.

$$\frac{3x}{x^2-1} = \frac{3x}{(x-1)(x+1)} \qquad \frac{x+2}{x-1} \xrightarrow{\times (x+1)} \frac{(x+2)(x+1)}{(x-1)(x+1)}$$

Step 4 Subtract the rational expressions with like denominators by subtracting the numerators and keeping the like denominator. Simplify the answer, if necessary.

$$\frac{3x}{(x-1)(x+1)} - \frac{(x+2)(x+1)}{(x-1)(x+1)} = \frac{3x - (x+2)(x+1)}{(x-1)(x+1)}$$

Multiply $(x + 2)(x + 1)$ in the numerator $\rightarrow = \frac{3x - (x^2 + 3x + 2)}{(x-1)(x+1)}$

$= \frac{3x - x^2 - 3x - 2}{(x-1)(x+1)}$

Combine like terms. $\rightarrow = \frac{(x^2 - 2)}{(x-1)(x+1)}$

Vocabulary Review

Directions: Write the missing term in the blank.

LCD	**prime number**	**restricted values**
polynomial	**rational expression**	**reciprocals**

1. An algebraic expression with one or more terms in which each variable is raised to a whole-number exponent is called a ________________.
2. If two numbers or expressions are multiplied and yield a product of 1, the numbers or expressions are ________________.
3. A ratio of two polynomials is called a ________________.
4. The least common multiple of two or more denominators is the ________________.
5. ________________ make the denominator of a rational expression equal to 0.
6. A whole number greater than 1 whose only two factors are 1 and itself is a ________________.

Skill Review

Directions: Read each problem and complete the task.

1. Determine which of the following are rational expressions:

$$\frac{r+1}{r-4}, \frac{4}{3x}, \frac{8^{-2}+3x}{4x}, \frac{n^2-81}{n-1}, \frac{9-\sqrt{n}}{5}$$

2. Find the restricted value(s) for each rational expression.

a. $\frac{x+1}{x-1}$

b. $\frac{7}{x}$

c. $\frac{x^2-64}{x^2-x-12}$

3. Which shows the rational expression $\frac{x+2}{x^2+5x+6}$ correctly simplified with its restricted values?

A. $\frac{1}{x+2}; x \neq -2$

B. $\frac{1}{x+2}; x \neq -2; x \neq -3$

C. $\frac{1}{x+3}; x \neq -3$

D. $\frac{1}{x+3}; x \neq -2; x \neq -3$

Directions: Perform each operation.

4. $\frac{2x}{5x} \times \frac{x+1}{x}$

5. $\frac{7r}{8r+1} + \frac{7}{8r+1}$

6. $\frac{4n-3}{n-3} - \frac{1}{n-3}$

7. A rectangular plot of land has area $x^3 + 8x^2 + 15x$ and length $x + 3$. Which expression represents the width of the plot of land?

A. $x + 5$
B. $x - 5$
C. $x(x + 5)$
D. $5(x - 5)$

Skill Practice

Directions: Read each problem and complete the task.

1. When the rational expressions below are added and the sum is simplified, which term appears in the numerator?

$14 + 2n^2 - 1 + n - 3n - 4$

A. -16
B. $13n^2$
C. $21n$
D. $-20n$

2. A student subtracted two rational expressions and arrived at the incorrect answer below. Explain the student's error. What is the correct answer?

$$\frac{7+x}{x} - \frac{x+1}{x+2}$$

Step 1: $\frac{(7+x)(x+2) - x(x+1)}{x(x+2)}$

Step 2: $\frac{7x + 14 + x^2 + 2x - x^2 - 1x}{x^2 + 2x}$

Step 3: $\frac{2x^2 + 8x + 14}{x^2 + 2x}$

3. Write two rational expressions whose sum is $\frac{x-1}{x+4}$.

4. Identify the missing numerator.

$$\frac{x+2}{x-5} \div \frac{?}{x^2-25} = x + 5$$

5. Simplify the rational expression $\frac{a^2-b^2}{2a-2b}$.

6. It takes Rider 3 hours longer than Morgan to complete a repair job.

a. Let m = the length of time it takes Morgan to complete the job. Write an expression that represents the amount of time it takes Rider to complete the job.

b. The fraction of the job that Morgan completes in one hour is $\frac{1}{m}$. Write an expression that represents the fraction of the job that Rider completes in one hour.

c. Add your answers to parts b and c. This sum represents the fraction of the job that Morgan and Rider complete in one hour when they are working together.

d. Suppose that Morgan completes the job in 3 hours. What fraction of the job will Morgan and Rider complete in one hour working together? How long will it take Morgan and Rider to complete the job working together?

Directions: Choose the best answer to each question.

1. Nathan lays carpet with a length of $x + 3$ and a width of $2x^2 + 5$. So, the area of his room is ________________.

2. Ellen uses the ________________ to rewrite $(x - 5)(2x + 6) = 0$ as $x - 5 = 0$ and $2x + 6 = 0$.

3. Which polynomial is not in standard form?
   A. $5x(7 + 3x)$
   B. $5x^2 + 12x - 3$
   C. $3 - 14x + 4x^2$
   D. $3x^2 + 6x + 4 + 2x$

4. Find the quotient of $\frac{x+2}{3x^2} \div \frac{x^2+x-2}{15x}$.
   A. $\frac{x^2 - x}{5}$
   B. $\frac{x^3 + 3x^2 - 4}{45x^3}$
   C. $\frac{5}{x(x-1)}$
   D. $\frac{5}{x^2 + 10}$

5. Factor the expression $2x^2 + 4x - 30$.
   A. $(x + 10)(x - 6)$
   B. $2(x + 5)(x - 3)$
   C. $(2x + 5)(x - 3)$
   D. $(2x + 10)(2x - 6)$

6. Leah is solving the equation $\frac{3m + 24}{m^2 + 6m - 16}$. She knows that there are restrictions to the possible values of m. The restricted values are ________________.

7. The GCF (greatest common factor) of the expression $12x^3y - 8x^2y^3$ is ________________.

8. Factor the expression $16xy^2 + 24x^2y^3$.
   A. $40x^3y^5$
   B. $8(xy^2 + 3x^2y^3)$
   C. $8xy(2 + 3xy)$
   D. $8xy^2(2 + 3xy)$

9. Theresa's company uses the expression $2x^2 + 16x + 50$ to represent their earnings based on the number of items ordered, x. What is the value when 25 items are ordered?
   A. 70
   B. 500
   C. 1,075
   D. 1,700

10. Find the solution(s) for the equation: $x^2 - 4x + 4 = 16$.
    A. $x = 6$
    B. $x = 6; x = -2$
    C. $x = 18$
    D. $x = -6; x = 2$

11. A rectangular plot of land has an area of $x^2 + x - 12$ and a width of $x^2 + 4x$. So the length of the plot is ________________.

12. Anthony has 64 square tiles and would like to arrange them in a square. He uses the equation $x^2 = 64$ to model the situation. How many tiles wide will his arrangement be?
    A. –8
    B. 0
    C. 8
    D. 32

Directions: Choose the best answer to each question.

13. Factor the expression $x^2 + 2x - 48$.
    A. $(x - 8)(x + 6)$
    B. $(x - 12)(x + 4)$
    C. $(x + 8)(x - 6)$
    D. $(x + 12)(x - 4)$

14. Amber wrote the equation $5x^3 + 4x^2 - 3x + 10$ and said that the ________________ was 5.

15. Mila studies plants that grow both above and below sea level. She expresses the depths and heights that a certain plant can grow with the equation $x^2 + 16x + 20 = 0$ where x represents the number of feet above or below sea level. What are the depths that the plant can grow?
    A. –2 ft; –14 ft
    B. –2 ft
    C. –8 ft
    D. 8 ft, 2 ft

16. The sum of $\frac{x}{x-3}$ and $\frac{6}{x+3}$ is equal to ________________.

17. Jerrod uses the equation $x^2 - x - 30 = 0$ to represent the incomes and outflows in his budget. What are the values of x?
    A. $x = 10, x = -3$
    B. $x = -10, x = 3$
    C. $x = 6, x = -5$
    D. $x = -6, x = 5$

Check Your Understanding

On the following chart, circle the items you missed. The last column shows pages you can review to study the content covered in the question. Review those lessons in which you missed half or more of the questions.

Lesson	Item Number(s)			Review Page(s)
	Procedural	Conceptual	Problem Solving	
4.1 Evaluate Polynomials	1	3, 7, 14	9	118–123
4.2 Factor Polynomials	5, 8, 13	2		124–129
4.3 Solve Quadratic Equations	10		12, 15, 17	130–137
4.4 Evaluate Rational Expressions	4, 16	6	11	138–145