QUEUEING MODEL

WAITING LINE MODEL λ/hour

PARAMETER

Arrival → follows Poisson distribution

Service → Exponential distribution ↳ μ/hour

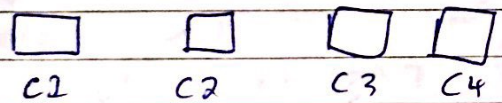
TYPES

(i) single service queueing model
→ single counter

(ii) multi-service queueing model
→ multiple counter

Both can be finite or infinite

Mechanic



if C5 comes, it will find this full. So it will leave.
This is FINITE as 4 is fixed

INFINITE POPULATION

~~Finite queue~~
~~Infinite queue~~

for mechanic, we have 4 machine. But number of people coming is infinite. So FINITE QUEUE and INFINITE POPULATION

(1) $\frac{\lambda}{\mu} < 1$

↳ This is because
(i) arrival and departure time intervals are not synchronized
(ii) Different distributions are follow

* finite ~~infinite~~ queue

(2) $\frac{\lambda}{\mu} > 1$

* infinite queue

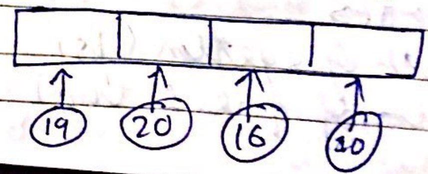
→ BALKING

(1) Mechanic sees 4 slots in which he gives service. There is no queue visible to mechanic
→ FINITE QUEUE (FORCEFUL)

(2) There is a queue where he can wait for service
→ INFINITE QUEUE (UN-FORCEFUL)

→ RANGING (leaves queue in b/w)

→ ~~JOCKENG~~ JOCKEYING



Waiting People

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person goes to queue 4 as it is smallest. But service time of this queue is slow, i.e. propagation time of queue was large

So jockeying states that selection of queue depends on
(i) queue size
(ii) Service time.

* M | M | 1 : ∞ | ∞ → CANDALS NOTATION

- M → Arrival λ/hr
- M → Service μ/hr
- 1 → No of servers
- ∞ → Queue length (infinite)
- ∞ → population (infinite)

M = memory less system

One event take place in small interval h

At steady state, what is prob of system to have 0 people, 1 people, ... n people in system

PARAMETES
(i) Length of system (L_s)
(ii) Length of queue (L_q)

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(iii) Waiting time in system (W_s)
(iv) Waiting time in queue (W_q)

L_s = total no. of people in system including the person who is being served.

L_q = no. of people waiting in queue.

W_s = include service time and time in queue.

W_q = time in queue.

IMP

(I) What is the probability that n people in system at time t+h

ANS. $P_n(t+h) = P_{n-1}(t)$ * prob of one arrival and no service

$+ P_{n+1}(t)$ * one service & no arrival

$+ P_n(t)$ * no service & no arrival

(case of 1 service + arrival is not possible because both are not possible together)

Prob of one arrival = λh
Prob of one service = μh

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$$P_n(t+h) = P_{n-1}(t) \times \lambda h (1-\mu h) + P_{n+1}(t)\, \mu h (1-\lambda h) + P_n(t)(1-\lambda h)(1-\mu h)$$

Neglecting higher order terms

$$P_n(t+h) = P_{n-1}(t)\, \lambda h + P_{n+1}(t)\, \mu h + P_n(t)\,[1 - \lambda h - \mu h]$$

$$\frac{P_n(t+h) - P_n(t)}{h} = P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)(\lambda+\mu)$$

(i) Steady state is when state is const i.e $\frac{P_n(t+h) - P_n(t)}{h} = 0$

at $h \to \infty$

$$\lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu) P_n \quad \to (1)$$

as steady state, it is independent of time

(ii) $P_0(t+h) = P_1(t) \times$ one service & no arrival $+ P_0(t) \times$ no arrival & no service

$$P_0(t+h) = P_1(t)(1-\lambda h)\mu h + P_0(t)(1-\lambda h)(1)$$

$\hookrightarrow$ as $n=0$, P(NO service) = 1

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On solving:

$$\frac{P_0(t+h) - P_0(t)}{h} = P_1(t)\mu - P_0(t)\lambda$$

$$0 = P_1(t)\mu - P_0(t)\lambda$$

$$P_1\mu - P_0\lambda = 0$$

$$P_1 = \frac{\lambda}{\mu} P_0 \quad \to (2)$$

using (1) $n = 1$

$$\lambda P_0 + \mu P_2 = (\lambda+\mu) P_1$$

$\lambda P_0 = \mu P_1$ Using (2)

$$\cancel{\mu P_1} + \mu P_2 = \lambda P_1 + \cancel{\mu P_1}$$

$$P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Let $\frac{\lambda}{\mu} = \rho$

$$P_1 = \rho P_0$$

$$P_2 = \rho^2 P_0$$

$$P_n = \rho^n P_0$$

TILL NOW, we are studying

μ λ $\to$ infinite population

M	M	1 : ∞	∞

$\to$ queue length

$\hookrightarrow$ Memoryless

Now, we will have

$\boxed{M / M / 1 : N / \infty}$

$P_0 + P_1 + \ldots + P_n = 1$

$P_0 + \delta P_0 + \delta^2 P_0 + \ldots + \delta^n P_n = 1$

$P_0 (1 + \delta + \delta^2 \ldots + \delta^n) = 1$

$P_0 \left[\frac{1-\delta^{n+1}}{1-\delta}\right] = 1$

$P_0 = \frac{1-\delta}{1-\delta^{n+1}}$

$M / M / 1 : \infty / \infty$

(μ, λ; M → memory less; ∞ → infinite population; ∞ → queue length)

$P_0 + P_1 + \ldots \infty = 1$

$P_0 + \delta P_0 + \delta^2 P_0 + \ldots \infty = 1$

$P_0 [1 + \delta + \delta^2 + \ldots \infty] = 1$

$\frac{P_0}{1-\delta} = 1$

$P_0 = 1 - \delta$

$P_n = \delta^n P_0$

$P_n = \delta^n (1-\delta)$

→ W_s = Total waiting time

W_q = Waiting time in queue

L_s = Total no. of people

L_q = Total no. of people waiting in queue

$L_s = \sum_{j=0}^{\infty} j P_j$

$= \sum_{j=0}^{\infty} j \delta^j P_0$

$= \delta P_0 \sum_{j=0}^{\infty} j \delta^{j-1}$

$= \delta P_0 \sum_{j=0}^{\infty} \frac{d \delta^j}{d\delta}$

$= \delta P_0 \frac{d}{d\delta} \sum_{j=0}^{\infty} \delta^j$

$= \delta P_0 \frac{d}{d\delta} [1 + \delta + \delta^2 + \ldots]$

$= \delta P_0 \frac{d}{d\delta} \left[\frac{1}{1-\delta}\right]$

$= \frac{\delta P_0}{(1-\delta)^2} = \frac{\delta(1-\delta)}{(1-\delta)^2}$

$= \frac{\delta}{1-\delta}$

$\therefore L_s = \frac{\delta}{1-\delta}$

LITTLE'S EQ^n

(i) $L_s = L_q + \frac{\lambda}{\mu}$ ($\frac{\lambda}{\mu} \to \delta$)

(ii) $L_s = \lambda W_s$

(iii) $L_q = \lambda W_q$

QUES. $\lambda = 8$ per hour

$\mu = 9$ per hour

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Find (i) δ
(ii) Probability of no queue
(iii) " " 10 people in system
(iv) " " atleast 2 people in the queue

ANS (i) $\delta = \frac{\lambda}{\mu} = \frac{8}{9}$

(ii) $P_0 = 1 - \delta = \frac{1}{9}$

(iii) $P_{10} = \delta^{10} P_0 = \left(\frac{8}{9}\right)^{10} \frac{1}{9}$

(ii) No queue means $P_0 + P_1$

$P_0 = \frac{1}{9} \qquad P_1 = \frac{8}{81}$

$P_0 + P_1 = \frac{17}{81}$

(iv) atleast 2 people =

$P_2 + P_3 + \text{---} + P_\infty$

$P_0(\delta^2 + \delta^3 + \text{---} + \infty)$

$\cancel{P_0}\left(\frac{\delta^2}{\cancel{1-\delta}}\right) = \delta^2 = \left(\frac{8}{9}\right)^2 = \frac{64}{81}$

Now, we will have

M	M	1:N	∞

$P_0 + P_1 + \text{--} + P_n = 1$

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$P_0 + \delta P_1 + \delta^2 P_2 + \text{--} + \delta^n P_0 = 1$

$P_0 (1 + \delta + \delta^2 + \text{--} + \delta^n) = 1$

$P_0 \left(\frac{1-\delta^{n+1}}{1-\delta}\right) = 1$

$P_0 = \frac{1-\delta}{1-\delta^{n+1}}$

$L_s = \sum_{j=0}^{n} j P_j$

$= \sum_{j=0}^{n} j \delta^j P_0$

$= \delta P_0 \sum_{j=0}^{n} j \delta^{j-1}$

$= \delta P_0 \frac{d}{d\delta}\left[\frac{1-\delta^{n+1}}{1-\delta}\right]$

$= \delta P_0 \left[\frac{(1-\delta)(-\delta^n)(n+1) + 1 - \delta^{n+1}}{(1-\delta)^2}\right]$

$= \delta P_0 \left[\frac{\delta^{n+1}(\cancel{n+1-1}) - \delta^n(n+1) + 1}{(1-\delta)^2}\right]$

$= \delta P_0 \left[\frac{\delta^n n(\delta - 1) - \delta^n + 1}{(1-\delta)^2}\right]$

$= \delta \frac{(1-\delta)}{(1-\delta^{n+1})}\left[\frac{1 + n\delta^{n+1} - (n+1)\delta^n}{(1-\delta)^2}\right]$

$= \frac{\delta(1 + n\delta^{n+1} - (n+1)\delta^n)}{(1-\delta)(1-\delta^{n+1})}$

(i) $L_s = L_q + \frac{\lambda}{\mu}$

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WHEN BALKING IS THERE

$\lambda_{eff} = \lambda(1-P_n)$

Multi-server

M | M | c : ∞ | ∞ → population.

λ/hr μ/hr No. of server queue

occupies server 1

occupies server 2

waits for server to get empty.

Occupies server that is empty

$\frac{\rho}{c} = \frac{\lambda}{c\mu} < 1$

→ no. of server (more servers = more serving rate)

$\lambda_n = \lambda$ (Arrival rate is independent)

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$\mu_n = n\mu, \quad n < c$
$= c\mu, \quad n \geq c$

$P_n = \rho^n P_0$

$P_n = \frac{\lambda^n}{\mu^n} P_0$

IF $n < c$ $\quad P_n = \frac{\lambda^n}{(\mu)(2\mu)\dots(n\mu)} P_0$

$= \left[\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0\right]$

IF $n \geq c$

$P_n = \frac{\lambda^n}{(\mu)(2\mu)(3\mu)(4\mu)(4\mu)(4\mu)\dots} P_0$ If $c = 4$

→ after this $c = 4$ as max servers $= 4$

$= \frac{\lambda^n P_0}{c!\,\mu^n\,c^{n-c}} = \left(\frac{\lambda}{\mu}\right)^n \frac{P_0}{c!\,c^{n-c}}$

$\therefore P_0 + P_1 + \dots + P_\infty = 1$

~~$P_0(n\geq c) + P_0(n<c) + P_1(n\geq c) + \dots$~~ $= 1$

$\sum_{n=0}^{c-1} \frac{\rho^n P_0}{n!} + \sum_{n=c}^{\infty} \frac{\rho^n P_0}{c!\,c^{n-c}} = 1$

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$$P_0 = \frac{1}{\left(\sum_{0}^{c-1} \frac{\delta^n}{n!}\right) + \frac{\delta^c}{c!} \frac{1}{(1-\delta/c)}}$$

L_q = Length of queue.
queue will exist only if $n > c$.

$$= \sum_{n=c}^{\infty} (n-c) P_n \qquad n > c$$

Let $n - c = j$
$n = c + j$

$$= \sum_{j=0}^{\infty} j P_{c+j}$$

$$= \sum_{j=0}^{\infty} j \frac{\delta^{c+j}}{c!\, c^j} P_0$$

$$= \frac{\delta^{c+1} P_0}{c^2 (c-1)!} \sum_{j=0}^{\infty} j \left(\frac{\delta}{c}\right)^{j-1}$$

$\hookrightarrow \frac{d x^n}{dx} = n x^{n-1}$

$$= \frac{\delta^{c+1} P_0}{c^2 (c-1)!} \frac{d}{dx} \sum_{j=0}^{\infty} \left(\frac{\delta}{c}\right)^j$$

$\frac{1}{1-x}$

$$= \frac{\delta^{c+1} P_0}{c^2 (c-1)!} \frac{d}{d\delta}\left(\frac{1}{1 - \frac{\delta}{c}}\right)$$

$\frac{+1}{(1-x)^2}$

$$= \frac{\delta^{c+1} P_0}{c^2 (c-1)!} \frac{1}{\left(1 - \frac{\delta}{c}\right)^2}$$

$$= \frac{\delta^{c+1} P_0}{(c-1)!\, (c-\delta)^2}$$

(1) $c = 2$, $\lambda = 10$, $\mu = 6$, Find
(i) P_0
(ii) prob that a person come in the queue will get the service
(iii) Find prob that there is no queue.

ANS $P_0 = \dfrac{1}{\frac{\delta^0}{0!} + \frac{\delta^2}{2!\left(1 - \frac{\delta}{c}\right)} + \frac{\delta^1}{1!} \cancel{+ \frac{\delta^2}{2!(1-\frac{\delta}{c})}}}$

$$\delta = \frac{\lambda}{\mu} = \frac{10}{6} = \frac{5}{3}$$

$$= \frac{1}{1 + \frac{5}{3} + \frac{25}{9 \times 2\left(1 - \frac{5}{6}\right)}} = \frac{1}{1 + \frac{5}{3} + \frac{25}{2}}$$

$$\cancel{= \frac{6}{6+10+75}} \qquad \frac{25}{3 \cdot 18 \times 1} \qquad \frac{1}{1 + \frac{30}{3}}$$

$$\cancel{= \frac{6}{91}}$$

$$= \frac{1}{11} = 0.09$$

(i) 0.09
(iii) P (no queue) = $P(n \leq c)$
$= P_0 + P_1 + P_2$

$$\left(\frac{8^0}{0!} + \frac{8^1}{1!} + \frac{8^2}{2!}\right) P_0$$

$$= \left(1 + \frac{5}{3} + \frac{25}{9 \times 2}\right) \times \frac{1}{11}$$

$$= \left(1 + \frac{5}{3} + \frac{25}{18}\right) \times \frac{1}{11}$$

$$= \frac{18+30+25}{18 \times 11} = \frac{73}{18 \times 11} = \frac{73}{18 \times 11} = 0.3686$$

(ii) means person is in the queue
i.e P_0 is not includes
Gets service means $\leq c$

$$P_1 + P_2 = \left(\frac{5}{3} + \frac{25}{18}\right) \frac{1}{11}$$

$$= \frac{\cancel{55}^{5}}{18 \times \cancel{11}} = \frac{5}{18}$$

$$= 0.27778$$

INVENTORY MODEL

Two types
- ↳ Single period → one period is alloted to decide
- ↳ Multi period → ~~multiple periods are~~ available to take decision

Organisation — decides how much to order? and when to order

⇅

Inventory

→ Depending on nature of demand, there are 3 types of inventory models:
(i) Deterministic
(ii) risk
(iii) uncertainity

→ DETERMINISTIC
* actual demand is known and known with certain

→ RISK
* Demand is known and it follows a certain distribution

→ UNCERTAINITY
* Distribution demand is not known and demand is not distriministic but certain parameters are known such as variance, mean,

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and standard deviation

(i) How much to order?
= order quantity → Q

(ii) When to order
= reorder level (r)

→ Cost of item (C) → Actual cost of item, Q
→ Money spend on buying annual demand = CD

→ D = Demand
(Demand is no. of end product quantity is raw materials required for the demand)

→ Order cost or ordering
= C_0 = eg: transportation, labour cost

→ Inventory holding and carrying cost (C_c)
↳ eg cost of space, security

→ lead time
↳ time required by inventory to prepare the order

→ INTEREST (i)
$C_c = i\,C$
↳ Interest of borrowing money for order. this is larger than all other costs. so neglecting other costs

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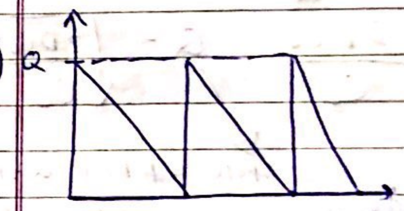
UNITS COMPULSORY

MODEL 1
SINGLE ITEM CONTINUOUS DEMAND INSTANCES

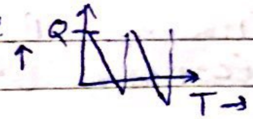
ASSUMPTIONS
→ no lead time i.e lead time = 0
→ no shortage

(→ BACK ORDER COST
If we want to deliver order on D1 and we dont have material. We take another date D2 to deliver the order. This cost is BACKORDER COST) → Not in Model I



↓ a material is available we consume it, and as no shortage, we reach the demand and order again

* If we do not assume no shortage graph goes -ve



* no. of order / year = D/Q
order/year ← → cost/~~year~~ order

* Total ordering cost = $\frac{D}{Q}$ C_0 } unit = ~~ord~~ cost/year

* Avg Inventory = $\frac{Q}{2}$ (Graph)

$\frac{10000}{Q}$

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- total holding cost $= \frac{Q}{2} C_c$
- Cost of item $= D C$
- Total cost $= TC = \frac{D}{Q} C_0 + \frac{Q}{2} C_c + DC$

TARGET: MINIMISE THIS
OPTIMISE THIS

$$TC' = \frac{-D}{Q^2} C_0 + \frac{C_c}{2} + 0 = 0$$

$$\frac{D C_0}{Q^2} = \frac{C_c}{2}$$

$$Q^2 = \frac{2 D C_0}{C_c} \qquad Q = \sqrt{\frac{2DC_0}{C_c}}$$

ECONOMIC ORDER QUANTITY

If we order more than this it will be waste.

THIS IS CALLED EOQ MODEL (Economic Order Quantity)

QUES. $D = 10{,}000/\text{year}$, $C_0 = 300$ per order
$C = 20/\text{unit}$ $i = 20\%$. Find
(i) holding cost
(ii) Q
(iii) total cost

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$C_c = iC = 0.2 \times 20 = 4$

ANS. $Q = \sqrt{\frac{2 \times 10^4 \times 300}{4}}$

$= \sqrt{\frac{6}{4}} \times 10^3 = \sqrt{\frac{3}{2}} \times 10^3$

$= \frac{1.71}{1.41} \times 10^3$

total holding cost $= \frac{1.71}{1.41} \times 10^3 \times \frac{4}{2}$

total cost ?

$= \frac{\cancel{10000} \times 300}{\frac{\cancel{Q} 1.71 \times 10^5}{1.41}} + \frac{\frac{1.71}{1.41} \times 10^3 \times 4}{2} + 10^4 \times 20$

$= \frac{10 \times 300}{\frac{1.71}{1.41}} + 2 \times \frac{1.71}{1.41} \times 10^3 + 20 \times 10$

$300 \times 8.246 + 2425 + 2 \times 10^5$

~~$= 202433$~~

$= 204898.8$

$TC = \frac{D C_0 \sqrt{C_c}}{\sqrt{2DC_0}} + \frac{1}{2}\sqrt{\frac{2DC_0}{C_c}}\, C_c + DC$

$= \sqrt{\frac{DC_0C_c}{2}} + \sqrt{\frac{2DC_0C_c}{2}} + DC$

$\cancel{= \left(\sqrt{2} + \frac{1}{\sqrt{2}}\right)\sqrt{DC_0C_c} + DC}$

$\sqrt{2DC_0C_c} + DC$

* D C-gupta

* $TC = \sqrt{2DC_oC_c} + DC$