

Image Processing

I-T D12

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Image: An image is a 2-D. function, $f(x, y)$ where x and y are the ~~spatial~~ plane coordinates, and the amplitude of f at any pair of coordinates (x, y) is called the intensity of the image at the level.

Digital Image: It is composed of finite number of elements, called pixel each of which has a particular location & value.

Image types:

(i) Monochrome / Bitmap Images:

1 bit ~~exists~~ to represent for pixel $(0, 1)$

$0 \rightarrow$ black

$1 \rightarrow$ white

(ii) Gray Level Image:

8 bits ~~exists~~ $(0 - 7)$ for a pixel

* 256 variations of gray levels

(iii) Color Image:

* 24 bits

* $(0 - 7)$ R, $(0 - 7)$ G, $(0 - 7)$ B

* 3 planes with regards to color

Gray level, $X = 0.30R + 0.59G + 0.11B$

(iv) Half toned image:

Gray scale image using only black & white colors

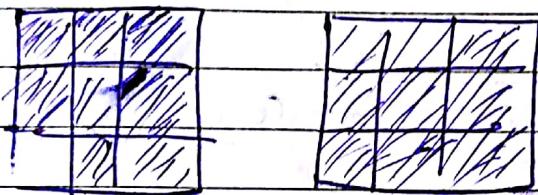
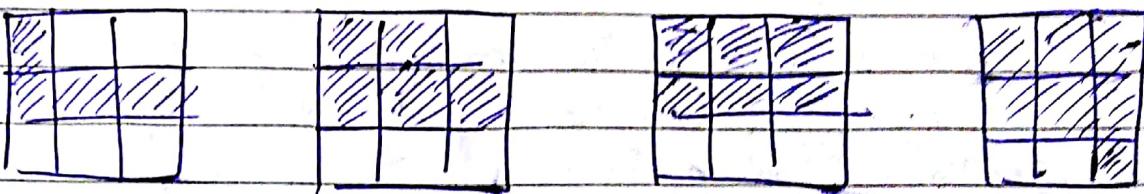
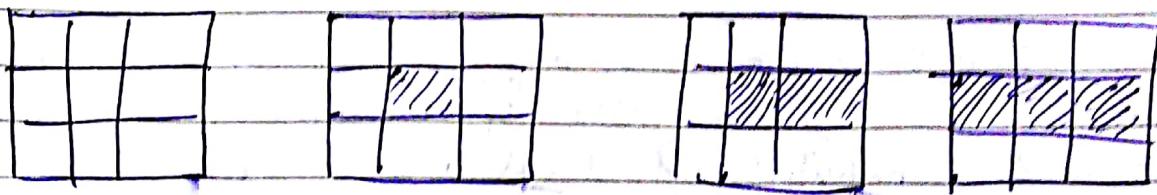


Image Processing:

It is a method to convert an image into a digital form and perform some operations on it in order to get an enhanced image or to extract some useful information of it.

Purpose of image processing:

- (i) Visualisation: Observe the objects that are not visible. Eg sharpen blurred image, x-ray image
- (ii) Image sharpening & restoration: To create a better image.
- (iii) Image retrieval: Seek for the image of interest
- (iv) Measurement of pattern: Calculate length Measure

various objects in a image

(v) Image recognition: Distinguish the objects in an image

Applications:

(i) Intelligent transport system:

(ii) Remote Sensing: Processing of satellite images to predict floods, etc.

(iii) Moving object tracking: Measure motion parameters

(iv) Defense surveillance:

(v) Biomedical imaging techniques: X-ray, MRI, CT scan

(vi) Automatic visual inspection system:

Low level Processing:

* Both input and output are images.

* Image processing operations such as noise reduction, contrast enhancement, image sharpening are performed.

Mid level Processing:

* Input are images but output are characteristics expected from those images such as ages, contours (boundary extraction) and identity of individual

Objects

- * Processing images to send for further computer processing.
- * Segmentation for object recognition and classification

High level Processing

- * Tracking or identifying objects in an image.
- * Performing cognitive functions (involved in knowing, learning and understanding typically associated with human vision)

Ques Given an 3×3 image. Plot its bit planes

1	2	0
4	3	2
7	5	2

Grayscale image

Max value = 7

\therefore 3 bit required

\therefore 3 bit planes

0	0	0
1	0	0
1	1	0

bit plane 2

MSB

0	1	0
0	1	1
1	0	1

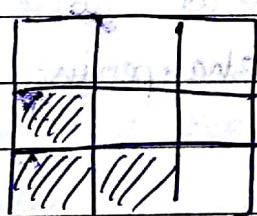
bit plane 1

Mid SP

1	0	0
0	1	0
1	1	0

bit plane 0

LSB



Steganography

(Embedding & extraction)

Similar

To hide a text in an image

$$150 \rightarrow 10010110$$

$$50 \rightarrow 00110010$$

$$51 \rightarrow 00110011$$

very
less
difference
for a gray scale
image

$$50$$

To hide image in a

$$20 \rightarrow 000010100$$

$$\begin{array}{|c|c|c|} \hline 20 & 10 & 3 \\ \hline 4 & 5 & 9 \\ \hline 13 & 12 & 11 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 20 & 10 & 3 \\ \hline 4 & 5 & 9 \\ \hline 13 & 12 & 11 \\ \hline \end{array}$$

Image formats:

(i) JPEG (.jpg):

- * Lossy but loss in quality is not very noticeable
- * Joint Photographic Experts Group
- * Not good for sharp edges or text
- * Quantization matrix, DCT

(ii) PNCh (.png):

- * Portable Network Graphics
- * Lossless
- * Compressed without loss in quality
- * Great for sharp edges, texts

(iii) GIF

- * Graphics Interchange Format
- * Used for web application / graphics
- * Limited to only 256 colours

(iv) TIFF:

- * Tagged Image File Format
- * Lossless
- * Huffman Coding

- * Produce images of very high quality
- * Large file size

(v) Bitmap (.bmp):

- * developed by microsoft
- * no compression
- * very high quality and very large size

(vi) RAW:

- * unprocessed data

(vii) EPS:

- * Encapsulated Post Script
- * Scalable to any size
- * Can be opened by graphic design software
- * Does not use colors
- * Recommended for print

Difference between gif, jpeg, and png

Neighborhood of pixel

Note:: A pixel (x, y) has 4 horizontal & 4 vertical neighbours. This set of 8 pixels is called 8 neighbours of P .

each of these neighbours is at a unit distance.

$$N_8(P) = \{(x, y-1), (x, y+1), (y, x-1), (y, x+1)\}$$

for (x, y)

$$f_{\text{diag}}(x, y) \quad N_D(P) = \{(x-1, y-1), (x+1, y+1), (x+1, y-1), (x-1, y+1)\}$$

(Diagonal pixel)

$$N_E(P) = \{\text{all neighbours of } (x, y)\}$$

$$= N_{\text{4D}}(P) + N_D(P)$$

Connectivity of Pixels

Let V be the set of gray levels used to define connectivity for 2 pts, $P, Q \in V$, $P \neq Q$

3 types of connectivity are defined:-

(i) 4-connectivity

$P, Q \in V$ are 4-connected if $Q \in N_4(P)$

(ii) 8-connectivity

$P, Q \in V$ are 8-connected if $Q \in N_8(P)$

(iii) M-connectivity (Mixed Connectivity):

$P, Q \in V$ are M-connected if

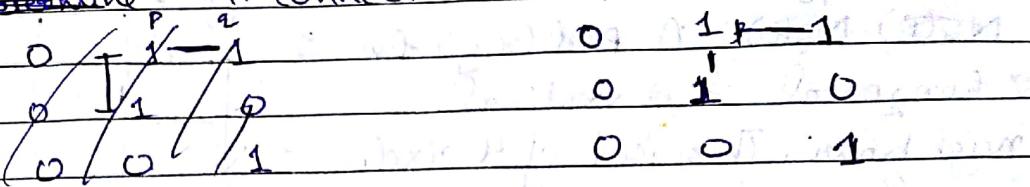
a) $Q \in N_D(P)$
OR

b) $Q \in N_D(P)$ and $N_4(P) \cap N_4(Q) = \emptyset$ (empty)

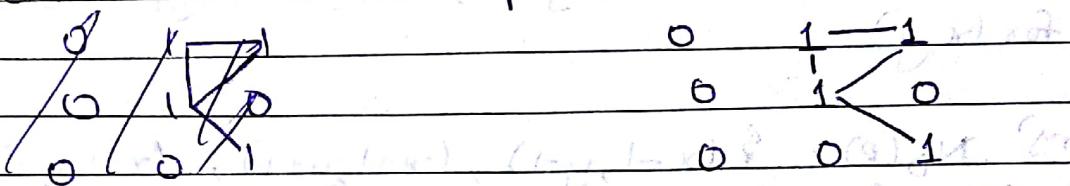
Ques

$V = \{1, 2, 3\}$

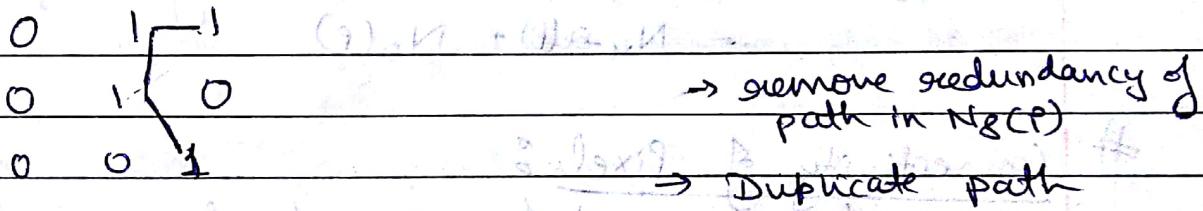
Determine 4-connected pixel.



Determine 8-connected pixels

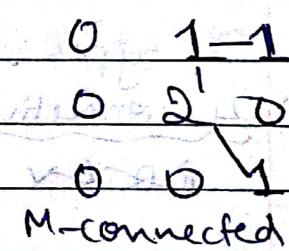
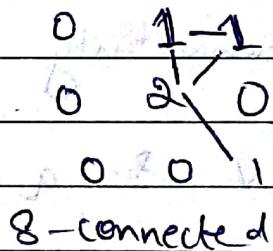
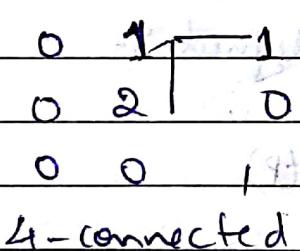


Determine M-connected Pixel



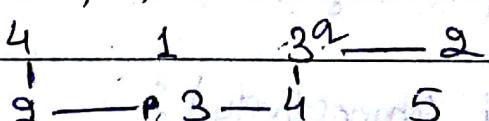
Ques

$V = \{1, 2, 3\}$



Ques

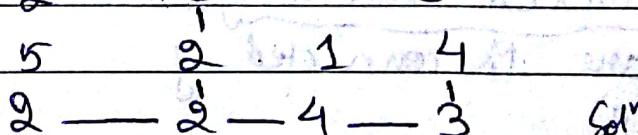
$V = \{1, 2, 3, 4\}$



Ques

\Rightarrow p, q M-connected or

not



Solⁿ Not M-connected

Ques

$$V = \{1, 3\}$$

$$\begin{matrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$$

4-connected

$$\begin{matrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$$

8-connected

$$\begin{matrix} 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$$

M-connected

#

PATH / Length of Path:

A path from $P(x,y)$ to $Q(s,t)$ is defined as a sequence of distinct pixels.

$$(x_0, y_0) \rightarrow (x_1, y_1) \rightarrow (x_2, y_2) \rightarrow \dots \rightarrow (x_n, y_n)$$

where

$$(x_0, y_0) = (x, y) \quad \text{and} \quad (x_n, y_n) = (s, t)$$

(x_i, y_i) is adjacent to (x_{i+1}, y_{i+1}) for $1 \leq i \leq n$

$n \rightarrow$ length of path.

Adjacency:

Let V be a set of gray levels used to define adjacency.

a) 4-adjacency: 2 pixels P and Q with values from V are 4-adjacent if Q is in the set $N_4(P)$

b) 8-adjacency: 2 Pixels P and Q with values from V are 8-adjacent if Q is in the set $N_8(P)$

c) M-adjacency: $P, Q \in V$

(i) $Q \in N_M(P)$

(ii) $Q \in N_M(P)$ and $N_M(P) \cap N_4(Q) = \emptyset$ (empty)

Ques

3	1	2	1
2	2	0	2
1	2	1	1
1	0	1	2

~~V₁~~, V = {2, 3}

(P) Compute the length of shortest 4, 8 and M-paths between P and Q, if particular path does not exist b/w these 2 pts, explain why.

~~P — Q~~

3	1	2	1
2	2	0	2
1	2	1	1
P	1	0	1

4- Path does not exist

4-adjacency does not exist

4-connected

3	1	2	1
2	2	0	2
1	2	1	1
P	1	0	1

8-connected

3	1	2	1
2	2	0	2
1	2	1	1
P	1	0	1

length of 8-path = 4

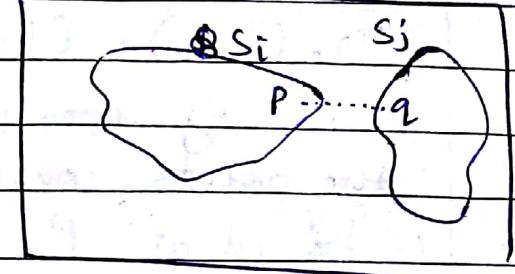
M-connected

length of M-path = 5

* Length of shortest path is 4 (8-Path)

2 Image Subsets, S_i & S_j are adjacent if there exist $P \in S_i$ and there exist $Q \in S_j$ such that P and Q are adjacent.

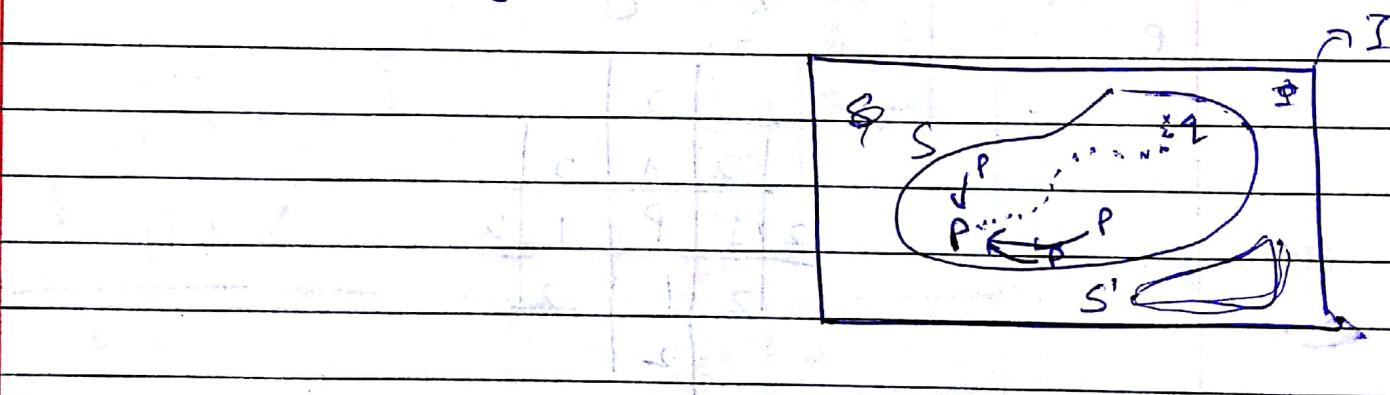
Application: ~~2 objects~~ an image can be considered 1 object because they are connected.



Connected Component:

If $S \subset I$ and $P, Q \in S$, then P is connected to Q in S , if there is a path from $P \rightarrow Q$, consisting entirely of pixel \oplus in S . For any $P \in S$, the set of pixels in S that are connected to P , is called connected component of S .

Any 2 pixels of a connected component are connected to each other. Distinct connected components are disjoint.



Distance Measure:

For pixels (P, q, z) with coordinates (x, y) ; (x', y') ; and (u, v) respectively, we call D to be a valid distance function or metric if it fulfills:

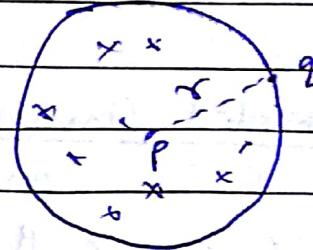
- $D(P, q) \geq 0$ [$D(P, q) = 0$ if $P = q$]
- $D(P, q) = D(q, P)$
- $D(P, z) \leq D(P, q) + D(q, z)$

Euclidean Distance b/w $P \& Q$:

$$P(x, y) \& Q(s, t)$$

$$D_e(P, Q) = [(x-s)^2 + (y-t)^2]^{1/2}$$

Set of points, $S = \{q | D(P, q) \leq r\}$ are the points contained in a disk of radius, r centred at P

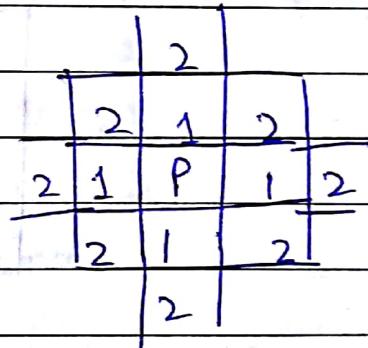


Manhattan (City - Block) Distance (Manhattan)

D_u distance or City-Block Distance,

$$D_u(P, Q) = |x-s| + |y-t|$$

Points having city block distance from P less than or equal to r form a diamond centered at P

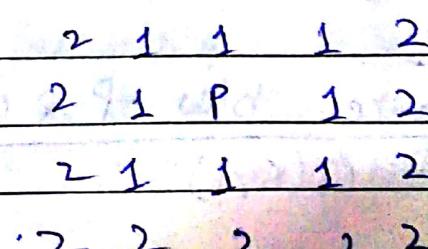


Chess-Board Distance

D_g distance

$$D_g(P, Q) = \max(|x-s|, |y-t|)$$

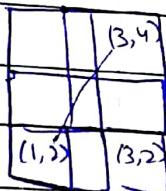
$S = \{q | D_g(P, q) \leq r\}$ forms a square centered at P



Eg

$$P(1, 2), Q(3, 4)$$

$$DE(P, Q) = \sqrt{(1-3)^2 + (2-4)^2} \\ = \sqrt{4+4} = 2\sqrt{2}$$



$$D_H(P, Q) = |1-3| + |2-4| = 4$$

$$D_S(P, Q) = \max(|1-3|, |2-4|) \\ = 2.$$

Component labelling

Component labelling

1	2	3	4
1	2	3	4
1	1	1	1
1	1	1	1
6	1	1	1
1	1	1	1

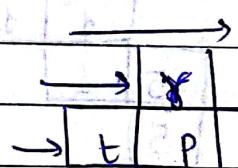
level 1

level 2

Algorithm :

Scan the image from left to right

1st Pass :



$l(P)$ = pixel value at position P

$L(P)$ = label assigned to pixel ~~value~~ location P

If $l(P) = 0$, move to the next scanning position

If $l(P)$ is in V & $l(t) = l(s) = 0$, assign new label to $l(P)$.

If $l(P)$ is in V & $L(s) \neq L(t)$, then assign one of the label to $l(P)$.

else $l(P) = l(s)$

Eg

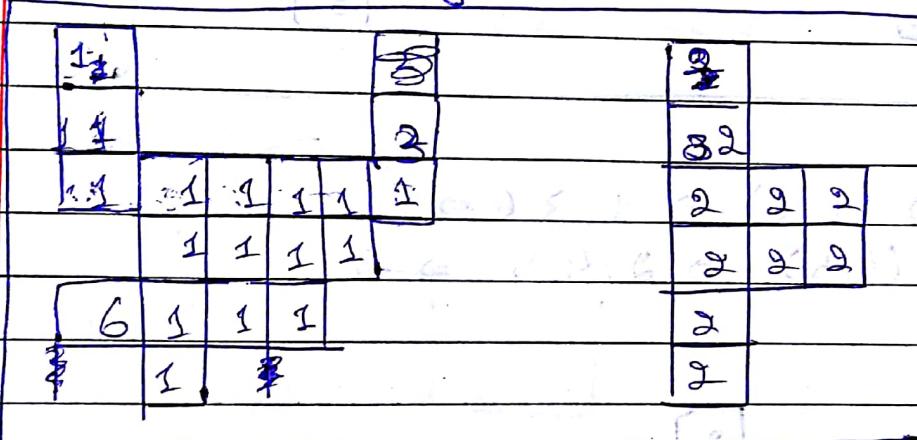
$$P(1, 2), Q(3, 4)$$

$$D_E(P, Q) = \sqrt{(1-3)^2 + (2-4)^2} \\ = \sqrt{4+4} = 2\sqrt{2}$$



$$D_H(P, Q) = |1-3| + |2-4| = 4$$

$$D_\infty(P, Q) = \max(|1-3|, |2-4|) = 2$$

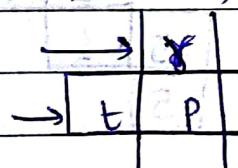
Component LabellingComponent Labelling

Level 1

Level 2

Algorithm :

Scan the image from left to right

1st Pass : $l(P)$ = pixel value at position P $L(P)$ = label assigned to pixel ~~value~~ location PIf $l(P) = 0$, move to the next scanning positionIf $l(P)$ is in V & $l(t) = l(s) = 0$, assign new label to $l(P)$.If $l(P)$ is in V & $L(s) \neq L(t)$, then assign one ofthe label to $l(P)$.else $l(P) = l(s)$ 

2nd Pass: Process equivalent pairs to form equivalence classes.

Assign new label

2
↑
4

Eg:

1		3		9
1				4
1	1	1	1	4
1	1	1	1	5
6	1	1		5
	1			5

$$(1, 3) (1, 6) \Rightarrow 1, 3, 6 \Rightarrow 1$$

$$(2, 4) (4, 5) \Rightarrow 2, 4, 5 \Rightarrow 2$$

Class

1		2		3	3
1				4	4
1	1	1	1	4	4
1	1	1		4	4
5	1			4	4
5				4	4

$$(1, 2) (1, 5) \Rightarrow 1, 2, 5 \Rightarrow 1$$

$$(3, 4) \Rightarrow 3, 4 \Rightarrow 2$$

Image Algebra :

Addition:

Application: Image Morphing

Subtraction:

Application: Movement Detection

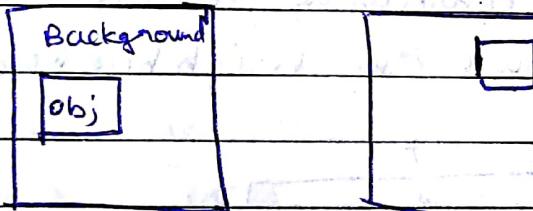
Multiplication:

Application: Brightness increases (Gives lighter image)
Gives a more saturated image

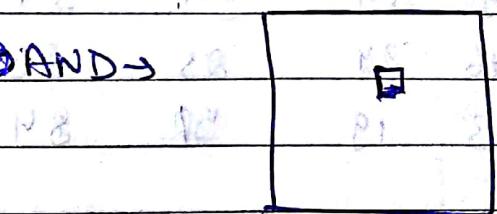
Division: (by a factor)

Application: Gives a ~~darker~~ ^{darken} image

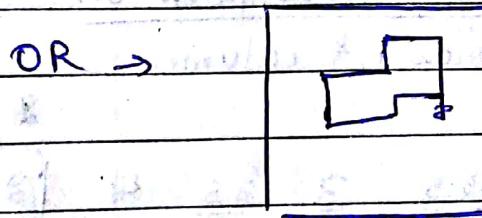
Logical Operations:



NOT \rightarrow AND \rightarrow



OR \rightarrow



XOR \rightarrow

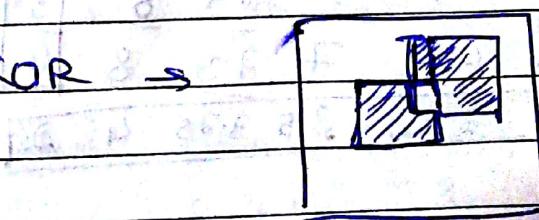


Image Zooming:

1. ~~Nearest column & row~~ ^{b/w}

24	14	54
68	24	26
18	19	84

⇒ Neighborhood zooming method

24 24 14 14 54 54

68 68 24 24 26 26 → stair effect
 18 18 19 19 84 84

Linear Interpolation

(Avg. of two consecutive columns or 2 consecutive rows in neighborhood zooming method)

Bilinear interpolation

Both row & column.

1	1.5	2	2.5	3	3.5	4	2
3	3.5	4	4.5	5	5.5	6	3
5	5.5	6	6.5	7	7.5	8	4
2.5	2.75	3	3.25	3.5	3.75	4	2

2+4
↓ After zooming
4+8

Image Shrinking:

To Retain visual quality

Remove or rows and columns

Image Resolution:

↳ Bits per pixel

Unit → b ppi (bits per inch)

For eg for gray scale image

$$2^8 = 256$$

8 bits per pixel.

Gray levels → 256

no. of gray levels

Ques Determine f, with a gray level resolution of

$$2^k \text{ for}$$

(i) $k=5$ → no. of gray levels

(ii) $k=3$

$$f =$$

123	162	200	147	93
137	157	165	232	189
151	155	152	141	130
205	101	100	193	115
250	50	75	88	100

Sol (i) No. of gray levels, $2^5 = 32$

Multiply

Divide each pixel by $\frac{32}{256}$ or $\frac{1}{8}$

i.e. Divide each pixel by 8. $[2^{5-3} = 2^2]$

~~111 111 111~~

Size of image = $M \times N \times K$
 Where m and n is no. of rows
 & columns &
 K is bits per pixel

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$f =$	15	20	25	18	11
	17	19	20	29	23
	18	19	19	17	16
	25	12	12	24	14
	31	6	9	11	12

(ii) $k = 3$

No. of gray level, $L = 2^3 = 8$
 Divide each pixel by 32 [$2^{8-3} = 2^5$]

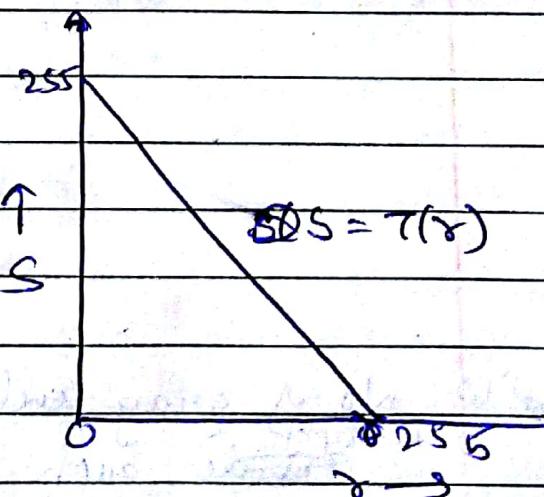
$f =$	3	5	6	4	2
	4	4	5	7	5
	4	4	4	4	4
	6	3	3	6	3
	7	1	2	2	3

Image Enhancement : Modify an image pixel values so that the output image suits for some specific application

1) Digital Negative Image :

$$S = (L-1) - \gamma$$

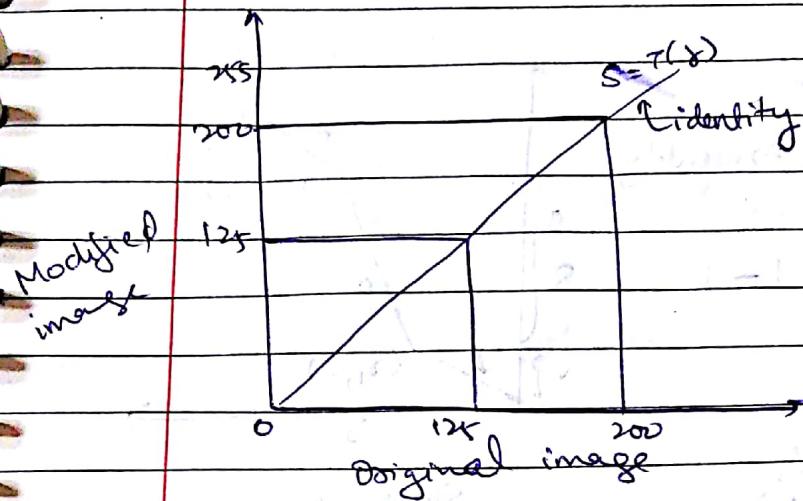
Applications: In medical x-rays



* Point Processing :

Higher bits carry significant information
lower bits stores intensity of image

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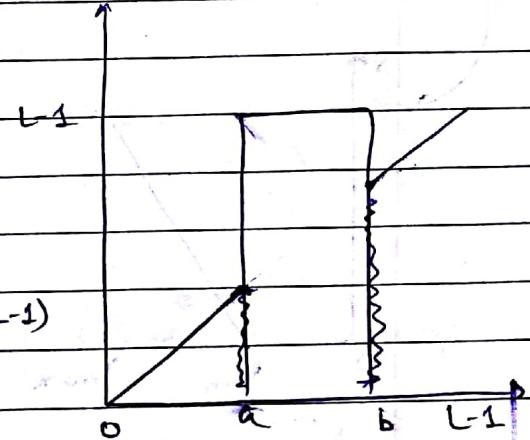
g) Intensity level Slicing : $\rightarrow S = \begin{cases} L-1, & \text{if } a \leq s \leq b \\ 0, & \text{otherwise} \end{cases}$

To retain color information

$$S = \begin{cases} L-1, & \text{if } a \leq s \leq b \\ 0, & \text{otherwise} \end{cases}$$

Application :- Satellite image.

To River intensity is known
(range of gray scale value to L-1)



3) Bit plane Slicing:

Application :- Compression

Gray scale value calculation for a compressed 4-bit image

Ex

$$\begin{aligned} \text{Gray Scale} &= \frac{1}{8} \times 2^7 + \frac{5}{8} \times 2^4 \\ &= 8 \times 128 + 5 \times 16 \\ &= 1024 + 80 = 1104 \end{aligned}$$

$$\begin{aligned} \text{Gray Scale} &= \sum_{n=0}^{n-1} \text{bit value} \times 2^{n-1} \\ (1001) &= 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 \\ &= 128 + 16 = 144 \end{aligned}$$

$$\begin{aligned} \text{Gray Scale} &= 128 \\ (1000) & \end{aligned}$$

All other
image enhancement
techniques

8 y--

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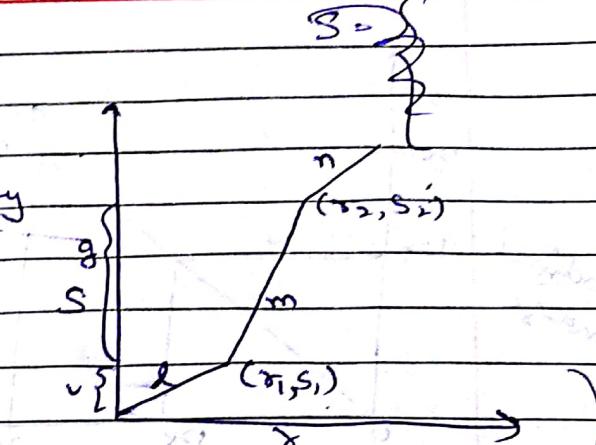
4) Contrast Stretching:

$$\tau_1 = S_1 \text{ & } \tau_2 = S_2 \Rightarrow \text{identity line}$$

$$\tau_1 = \tau_2, S_1 = 0 \text{ & } S_2 = L-1$$

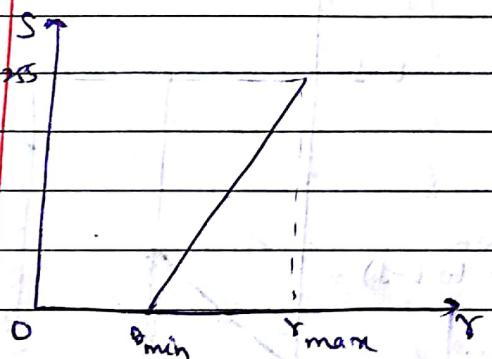
$$(\tau_1, S_1) = (\tau_{\min}, 0)$$

$$(\tau_2, S_2) = (\tau_{\max}, L-1)$$



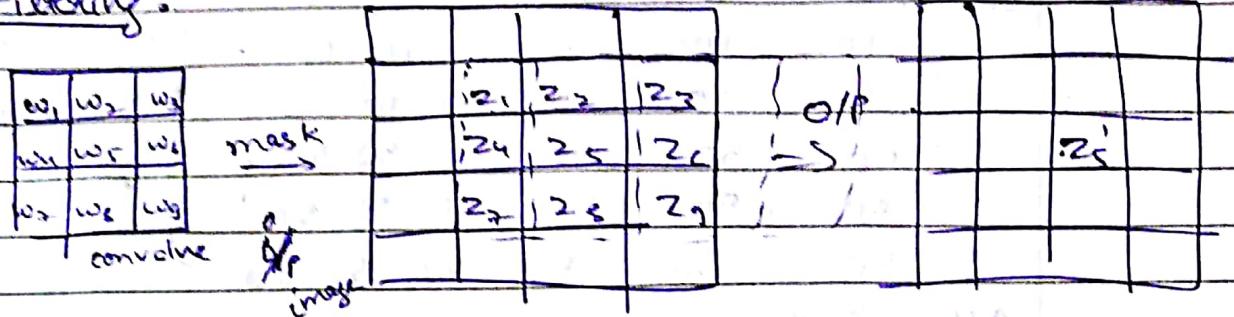
Slopes = l, m, n

Application: To utilize dynamic range of camera.



$$S = \begin{cases} l \cdot r, & 0 \leq r \leq \frac{a}{b} \\ m(r-a)+v, & a < r < b \\ n(r-b)+v+g, & b \leq r \leq L-1 \end{cases}$$

~~#~~ Filtering :



$$z_5' = \underbrace{w_1 z_1 + w_2 z_2 + w_3 z_3 + \dots + w_9 z_9}_{\text{weighted sum}}$$

Linear filter (or Weighted sum)

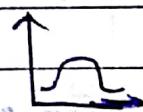
~~non linear~~ # Non-linear filter ($z_k, \forall k=1 \text{ to } 9$) [min, max, median]

~~#~~ Noise :

Impulse noise : White intensity values in the image

Salt & Pepper noise : Black & white intensity values

Gaussian Noise :



~~#~~ Low Pass filters :

Low Pass regions? are the smooth regions in an image.
 (no variation in pixel value)



High Pass region \rightarrow where intensity (color) change in an image

Averaging filter \rightarrow



<u>Eg.</u>	10 10 10 10 10 10	smooth region
	10 10 10 10 10 10	
	10 10 10 10 10 10	
	50 50 50 50 50 50	High pass region
	50 50 50 50 50 50	
	50 50 50 50 50 50	smooth region

After filtering

10	10 10 10	10	smooth region
	10 10 10	10	
	10 10 10	10	
	23.3 23.3 23.3	23.3	Blur
	36.6 36.6 36.6	36.6	
	50 50 50	50	
	10 10 10	10	smooth region
	10 10 10	10	
	10 10 10	10	

- * To maintain low frequency region, divide by 9.
- # To remove impulse noise, use ~~thresholding~~ non-linear filters (median)

padding # To maintain size of image, duplicate boundary rows & columns or pad zero padding.

For a filter of size $(f \times f)$ and image of $(n \times n)$, padding size:

$$(n + 2p - f + 1) = n + 2p - f$$

$$p = \frac{f-1}{2}$$

Stride = 1

High Pass filter:

* Mask contains the 2 - ve weights

* Sum of the mask elements is 0.

$\frac{1}{9}$	-1	-1	-1
	-1	8	-1
↑	-1	-1	-1

	0	-1	0
④	-1	4	-1
0	-1	0	0

scaling fraction

These 1000 ab. individuals were sampled.

Multiply

~~Eff~~ High Boost Filter :

* To retain background of image

Multiply positive value
with positive constant

~~so~~ $c > 1$

$c > 1$.

Eg $c = 1.1$

1	-1	-1	-1
9	-1	8*c	-1
-1	-1	-1	-1

$$\begin{matrix} 8/9 & 8/9 & 8/9 & 8/9 & 8/9 & 8/9 & 8/9 \\ 8/9 & 8/9 & 8/9 & 8/9 & 8/9 & 8/9 & 8/9 \\ -26/27 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 \\ 25/27 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 \\ 80/81 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 \\ 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 \\ 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 & 0/9 \end{matrix}$$

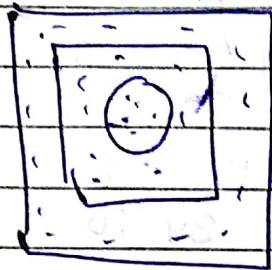
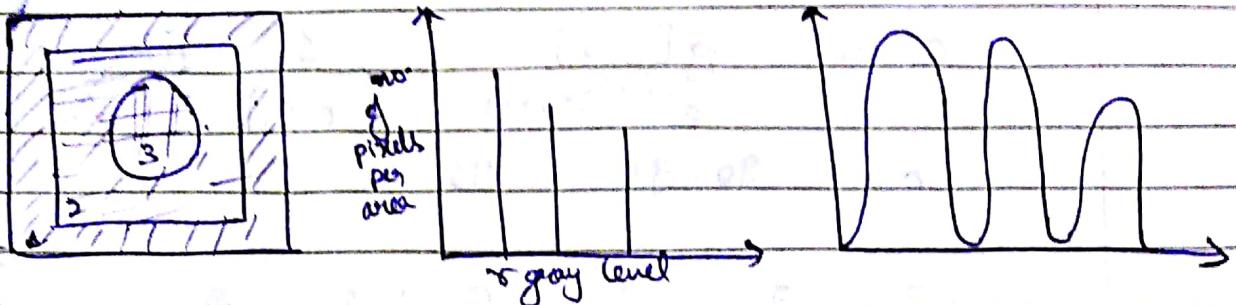
Increase constant value ~~so that~~ to get exact value of background pixel.

Eg

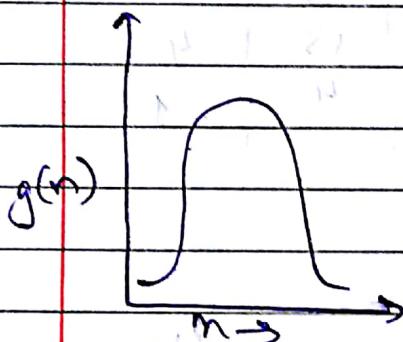
~~c = 2.125~~ $c = 2.125$

$$\begin{matrix} 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ -20 & -20 & -20 & -20 & -20 & -20 & -20 \\ 130 & 130 & 130 & 130 & 130 & 130 & 130 \\ 100 & 100 & 100 & 100 & 100 & 100 & 100 \\ 100 & 100 & 100 & 100 & 100 & 100 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Gaussian Noise



Gaussian Filter / Smoothing Filter



$$g(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{n^2}{2\sigma^2}}$$

PDF of gaussian random variable

$$g[x, y] = C e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

Discrete gaussian distribution normalizing constant

For $\sigma^2 = 9$, $n = -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$

$$\begin{array}{ccccccccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ -3 & 0.011 & 0.038 & 0.082 & 0.105 & 0.082 & 0.038 & 0.011 \end{array}$$

$$\begin{array}{ccccccccccccc} -2 & 0.038 & 0.135 & 0.286 & 0.368 & 0.286 & 0.135 & 0.038 \end{array}$$

$$\begin{array}{ccccccccccccc} -1 & 0.082 & 0.286 & 0.606 & 0.779 & 0.606 & 0.286 & 0.082 \end{array}$$

$$\begin{array}{ccccccccccccc} 0 & 0.105 & 0.368 & 0.779 & 1 & 0.779 & 0.368 & 0.105 \end{array}$$

$$\begin{array}{ccccccccccccc} 1 & 0.082 & 0.286 & 0.606 & 0.779 & 0.606 & 0.286 & 0.082 \end{array}$$

$$\begin{array}{ccccccccccccc} 2 & 0.038 & 0.135 & 0.286 & 0.368 & 0.286 & 0.135 & 0.038 \end{array}$$

$$\begin{array}{ccccccccccccc} 3 & 0.011 & 0.038 & 0.082 & 0.105 & 0.082 & 0.038 & 0.011 \end{array}$$

Max Kernel

OR

Gaussian Kernel

$$g\left[\frac{3}{c}, 3\right] = e^{-\frac{(x^2+y^2)}{20^2}}$$

$$C = \frac{g[3,3]}{e^{-(x^2+y^2)/202}} \Rightarrow \frac{1}{0.01} \quad \begin{matrix} \text{[corner value} \\ \text{should be 1]}\end{matrix}$$

$$c = \underline{90 \cdot 91} = 91$$

	-3	-2	-1	0	1	1	2	1	3	
-3	1	3	84	8	10	8	4	1	1	
-2	4	1	13	27	34	27	13	4		
-1	8	1	13	56	71	56	13	8		
0	10	1	34	71	91	71	34	10		
1	8	1	13	56	71	56	13	8		
2	4	1	13	27	34	27	13	4		
3	1	1	4	8	10	8	4	1		

Gaussian Fitter

1

scaling
value

This filter smoothes image.

function

* Scaling function = sum of gaussian filters

(1115)

* To obtain uniform intensity.

• Averaging filter \rightarrow blurry image

Canny filter \rightarrow sharp image

Ches

For $n=5$, $\sigma^2 = 1$. The PFFO distribution is:

1990-1991 First meeting 1991

~~5 - 2~~ ~~0~~ ~~5~~ ~~3~~ ~~2~~ ~~0~~ ~~1~~ ~~8~~ ~~0~~ ~~0~~ ~~0~~ ~~5~~ ~~0~~ ~~7~~ ~~2~~ ~~1~~ ~~8~~

~~Hair-1~~ 5% 21 34 0 21 85

0 8 34 55 34 8

1 5 21 34 21 . 5

2 1 S 8 5 ~~2~~ 1