MODELLING AND SIMULATION

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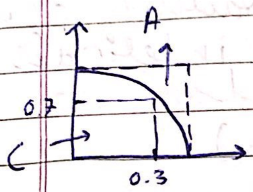
MONTE CARLO SIMULATION

A method of estimating the value of an unknown quantity using the principles of INFERENTIAL STATISTICES

→ Variance should be minimum in parent sample for this to be applicable

eg →



Let Area $A = 1$
Area $C = \pi r^2/4$
and $r = 1$

$C = \pi/4$

A = parent set

assume that $C = 75\%$ of A

$\frac{\pi}{4} = \frac{75}{100} \times 1 \qquad \pi = 3.0$

* If total no. of points in A = 10,000, and P = no. of points in area C

$C = \frac{P}{10000} \times A = \frac{P}{10000} =$

$\Rightarrow \frac{\pi}{4} = \frac{P}{10000}$

$\pi = \frac{4P}{10000}$

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STEPS IN MONTE CARLO

(i) Establishing probability distribution (given)
(ii) Cumulative probability distribution
(iii) Setting random no. intervals
(iv) Generating Random no. (generally given)

(1) Strong is a dentist who schedule all of his patients for 30 min appointment. Some of the patients take less or more 30 min depending on the type of dental work to be done. The following summary shows various categories of work, time needed to complete the work, and their responsibilities.
Stimulate the dentist clinic for 4 hrs & find out avg waiting time for patient as well as idleness for the doctor.
Assume that all patients show up at the clinic at exactly on their scheduled arrival time, starting at 8 am. Use following random no's for handling above problem.
40, 82, 11, 34, 25, 66, 17, 79.

CATEGORY	TIME REQ.	NO. OF PATIENTS
Filling	45 min	40
Crown	60	15
Cleaning	15	15
Extracting	45	10
Checkup	15	20
		100

ANS.

1. → Cummulative Prob

Category	time req	prob	CP	R. No Interval
Filling	45	0.40	0.40	0–39
Crown	60	0.15	0.55	40–54
Cleaning	15	0.15	0.70	55–69
Extracting	45	0.10	0.80	70–79
Checkup	15	0.20	1.00	80–99

2.

R. No.	Patient	Category	Time Req.
40	1	Crown	60
82	2	Checkup	15
11	3	Filling	45
34	4	Filling	45
25	5	Filling	45
66	6	Cleaning	15
17	7	Filling	45
79	8	Extracting	45

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3.

Patient	Scheduled Arrival	Service Start	Service end	Waiting (min)	Idleness
1	8:00	8:00	9:00	0	0
2	8:30	9:00	9:15	30	0
3	9:00	9:15	10:00	15	0
4	9:30	10:00	10:45	30	0
5	10:00	10:45	11:30	45	0
6	10:30	11:30	11:45	60	0
7	11:00	11:45	12:30	45	0
8	11:30	12:30	1:15	60	0
				285	0

Avg waiting = $\frac{285}{8}$ = 35.62 min

idle time = 0 min

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(1)

DAILY DEMAND	PROB
0	0.01
15	0.15
25	0.20
35	0.50
45	0.12
50	0.02

48, 78, 09, 51, 56, 77, 15, 14, 68, 9

Check Simulation for next 10 days

RANDOM NO.

cakes mode = 35/day

Avg Daily Demand = ?

RN - ~~40, 82, 11, 34, 66, 17, 79~~

cummulative Prob

ANS.

Daily Demand	Prob	CP	Regular No. interval
0	0.01	0.01	00
15	0.15	0.16	1 - 15
25	0.20	0.36	16 - 35
35	0.50	0.86	36 - 85
45	0.12	0.98	86 - 97
50	0.02	1.00	98 - 99

GIVEN

Day	RNO	Demand	35/day	Stock
1	48	35	35	0
2	78	35	35	0
3	09	15	35	20
4	51	35	35	0
5	56	35	35	0
6	77	35	35	0
7	15	15	35	20
8	14	15	35	20
9	68	35	35	0

RNo belongs to which regular no. interval based on Daily Demand of that interval

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absolute diff

10	9	15	35	20

Avg demand = $\frac{\sum \text{Demand}}{10} = \frac{270}{10} = \boxed{27}$

(2) The automobile company manufactures around 150 scooters. Daily production varies from 146–150. The finished scooter are transported in container accumulated 150 scooters. Use following random no to find answer.
RN: 80, 81, 76, 75, 64, 43, 18, 26, 10, 12, 65, 58, 69, 61, 57

Simulate (i) avg no. of scooters waiting in a factory.
(ii) Avg no. of empty space in container.

A

Production/day	Prob
146	0.04
147	0.09
148	0.12
149	0.14
150	0.11
151	0.10
152	0.20
153	0.12
154	0.08

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ANS.

Production/day	Prob	CP	Regular Interval
146	0.04	0.04	00–03
147	0.09	0.13	04–12
148	0.12	0.25	13–24
149	0.14	0.39	25–38
150	0.11	0.50	39–49
151	0.10	0.60	50–59
152	0.20	0.80	60–79
153	0.12	0.92	80–91
154	0.08	1.00	92–99

Day	R.N.	Demand	150/day	weighting	empty
1	80	153	150	3	
2	81	153	150	3	
3	76	152	150	2	
4	75	152	150	2	
5	64	152	150	2	
6	43	150	150	0	
7	18	148	150	−2	
8	26	149	150	−1	
9	10	147	150	−3	
10	12	147	150	−3	
11	65	152	150	2	
12	58	151	150	1	

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13	69	152	150	2
14	61	152	150	2
15	57	151	150	1

avg waiting scooters = sum of positive no's = $\frac{3+3+2+2+2+2+1+2+2+1}{15}$

$= \frac{20}{15} = \frac{4}{3}$

avg empty space = sum of negative no's = $\frac{2+1+3+3}{15} = \frac{9}{15} = \frac{3}{5}$

→ EXPECTED OR MEAN VALUE

$E(x) = \sum_{all\ i} x_i\, p(x_i)$ for discrete 'x'

$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$ for continuous
↳ PDF

* Variance → always gives +ve value
* Standard deviation → gives a range i.e -ve to +ve.
* Standard deviation which we find is always corresponding to a particular point in that range thats why +ve

→ RANDOM VARIABLE
↳ DISCRETE
↳ CONTINUOUS

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→ R_x (Range of x) of a RANDOM VARIABLE is finite numbers or countably infinite for discrete random variable

PROPERTIES :
(i) $P(x_i)$ → probability that X takes x_i

(ii) $P(x_i) \geq 0$; $\sum_{all\ i} P(x_i) = 1$

(iii) $(x_i, p(x_i))$ is called prob distribution func.

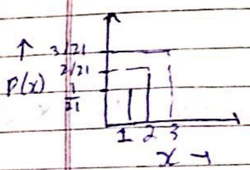
(iv) $p(x_i)$ is called probability mass func.

CONTINUOUS FUNCTION
eg → Dice tossing experiment
x = no. of spots on up face
$R_x = \{1, 2, 3, 4, 5, 6\}$

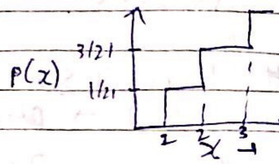
Assum prob of a given face getting landed is proportional to no. of spots shown.

ANS.

x_i	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$



PROBABILITY MASS FUNC



PROBABILITY DISTRIBUTION FUNC

CONTINUOUS RANDOM VARIABLE

If R_x of a random variable is an interval or collection of intervals, X is continuous Random variable.

→ Collection: collection of heart
collection of spade
collection of face cards.

→ Prob that X lie in interval (a,b) is

$$P(a \le x \le b) = \int_a^b f(x)\,dx.$$

RANDOM VARIABLE

(i) $f(x) \ge 0 \quad \forall\ x$ in R_x

(ii) $\int_{R_x} f(x)\,dx = 1$

(iii) $f(x) = 0$ → when x is not in R_x

Eg→ $f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & \forall\ x \ge 0 \\ 0 & \text{else} \end{cases}$

If, prediction to be made from life 2-3 years

ANS. $\int_2^3 \frac{1}{2} e^{-x/2}\,dx = \frac{1}{2} \frac{e^{-x/2}}{\frac{1}{2}} =$

$= e^{-3/2} - e^{-2/2} = 0.145$

COMMULATIVE DISTRIBUTION FUNC

(i) If X is discrete:

$$CDF = \sum_{x_i < X} P(x_i)$$

(ii) If X is continuous:

$$F(x) = CDF = \int_{-\infty}^{x} f(t)\,dt$$

* If F is a non-decay function then $\lim_{x \to \infty} F(x) = 1$

* If F is a decay function then $\lim_{x \to -\infty} F(x) = 0$

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VARIANCE

$V = E[x - E[x]]^2$
$V = E[x^2] - (E(x))^2$

BERNOULLI DISTRIBUTION

* useful for independent cases
* For experiments consisting of n trials
* each can be a success or a failure
$x_j = 1$ If jth experiment is success and
$x_j = 0$ If It is a failure.
* If n Bernoulli trials are independent It is called Bernoulli Process distribution

$$P_j(x_j) = p(x_j) = \begin{cases} p, & x_j = 1, \ j = 1,2,3..n \\ 1-p = q, & x_j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean $= E[x_j] = 0 \times q + 1 \times p = p$
Variance $= [0^2 q + 1^2 \times p] - p^2 = \cancel{pq}$
$= p(1-p) = pq$

RANDOM NO:

* eg OTP, captha
* every random no. generator is stochastic process

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* Disadvantage of R.No generator like coin-toss
(i) limited outcome
(ii) limited no's
(iii) storage required

FREQUENCY TEST → * each no. should occuring uniformly distributed

INDEPENDENT TEST → * occurence of random no. should ~~to~~ be independent of other random no.

* Every stochastic process contains random no.
Eg, process - dice ~~through~~ throw
random no. = result (1,2,_ 6)

* CONSIDERATION:
→ Generation should be fast
→ It should be portable
→ Should have a sufficiently long cycle: it should generate random no. for long cycles, i.e it should generate many numbers
→ Should be replicable
we may generate same no again
→ Independent
→ Uniform

* PSEUDO-RANDOM NO:
It should try to achieve properties of random no.

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CHALLENGES IN GENERATING PSEUDO R. NO

- → Generation should be uniformly distributed
- → mean of generated no. should not be too high or low
- → variance of generated number should not be too high or too low
- ↳ Autocorrelation b/w no's (should not follow any pattern)

→ MID-SQUARE METHOD is arithmetic method, i.e which follow a basic calculation but not some formula

→ TECHNIQUES TO GENERATE PSEUDO RANDOM NO.

(i) Congruential generator (CG)
(a) Linear CG
(b) Mixed CG
(c) Multiplicative CG
(d) Combined linear CG

(a) Linear CG

$Z_i = (Z_{i-1} + c) \bmod M$

Random no. → $\frac{Z_0}{M}$ (From 0 → 1)

Z_0 = seed value

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1. Parameters $M=63$, $a=22$, $c=4$, $Z_0=19$
$Z_i = [22 Z_{i-1} + 4] \bmod 63$

ANS.

i	$22Z_{i-1}+4$	Z_i	R. NO
0		19	—
1	22(19)+4=422	44	0.6984
⋮			
63	334	19	0.3016
64	422	44	0.2354

↳ repetation after M iterations

→ CHI-SQUARE TEST → For large data

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

O_i = observed no.
E_i = Expected no.

From uniform distribution
no in each class $\left[\frac{N}{n} = E_i\right]$ → SUMMATION

* $\chi_\alpha^2 > \chi^2$ → UNIFORM
↳ GIVEN

* $\chi_\alpha^2 < \chi^2$ → NON-UNIFORM

→ CALCULATE EXPECTED FREQ 2x2 TABLE

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A/B	B_1	B_2	Total
A_1	F_{11}	F_{12}	N_1
A_2	F_{21}	F_{22}	N_2
Total	N_3	N_4	N

$m \times n$ table

ν = degree of freedom
$= (m-1)(n-1)$

$f_{e_{11}} = \frac{N_1 \times N_3}{N}$ $f_{e_{12}} = \frac{N_1 \times N_4}{N}$

$f_{e_{21}} = \frac{N_2 \times N_3}{N}$ $f_{e_{22}} = \frac{N_2 \times N_4}{N}$

(1) In a sample survey of public opinion answer to the question
(1) DO you drink
(2) Are you in favour of local opinion sale liquor are tabulated below.

		QUES-1		
		Yes	No	Total
QUES2	Yes	56	31	87
	NO	18	6	24
	Total	74	37	111

can you infer or not the local opinion on sale of liquor is dependent on individual drink.
Given that $\chi_\alpha^2 = 3.841$ corresponding to 5% = α

~~$f_{e_{11}} = \frac{87 \times 24}{111}$ $f_{e_{12}} =$~~

ANS. H_0 = hypothesis that sale of liquor is dependent on individual drink.

$f_{e_{11}} = \frac{87 \times 74}{111} = \cancel{18.511}\ 58$

$f_{o_{11}} = 56$

$f_{e_{12}} = \frac{87 \times 37}{111} = 29$

$f_{o_{12}} = 31$

$f_{e_{21}} = \frac{24 \times 74}{111} = 16$

$f_{o_{21}} = 18$

$f_{e_{22}} = \frac{24 \times 37}{111} = 8$

$f_{o_{22}} = 6$

$\chi^2 = \frac{\cancel{1383}\ 4}{58\ \cancel{18.511}} + \frac{4}{\cancel{31}\ 29} + \frac{4}{16} + \frac{4}{8}$

$$\chi^2 = \frac{4}{58} + \frac{4}{29} + \frac{4}{16} + \frac{4}{8}$$

$= \cancel{0.068} <$

$0.9568 < 3.841$

$\therefore$ IT IS UNIFORM

AUTO-CORRELATION TEST

(i) Concerned with dependency b/w no's in sequence

(ii) Computation of auto-correlation b/w every m numbers (m is known as "lag") starting with ith no.

(iii) δ_{im} b/w numbers $R_i, R_{i+m}, R_{i+2m}, \ldots R_{i+(M+1)m}$ is to be found

m → lag

i → starting point

M → highest integer

M = largest integer s.t.

$$i + (M+1)m \le N$$

(iv) A non-zero auto-correlation implies lack of dependence, so following details test is appropriate

$H_0 : \delta_{im} = 0 \rightarrow$ Always dependent

$H_1 : \delta_{im} \neq 0 \rightarrow$ may or may not be independent

$\rightarrow \delta_{im} \neq 0$

$$z_0 = \frac{\delta_{im}}{\sigma_{\delta im}} \qquad Z_{\alpha/2} = \text{given in graph}$$

~~given in graph~~ $-Z_{\alpha/2} \le z_0 \le Z_{\alpha/2}$

$$\delta = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

$$\sigma_{\delta im} = \frac{\sqrt{13M+7}}{12(M+1)}$$

Q. Starting from second, check auto correlation for 7th and 12th given 40 Random no.'s

ANS. $N = 40 \qquad m = 12 - 7 = 5 \qquad i = 2$

$i + (M+1)m \le N$

$2 + (M+1)5 \le 40$

$2 + 5M + 5 \le 40$

$5M \le 33$

$M \le 6 \qquad M = 6$

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$$\rho = \frac{1}{7}\left[R_2R_7 + R_7R_{12} + R_{12}R_{17} + R_{17}R_{22} + R_{22}R_{27} + R_{27}R_{32} + R_{32}R_{37}\right] - 0.25$$

$$\sigma = \frac{\sqrt{13 \times 6 + 7}}{12(7)}$$

Now compare $Z_{\alpha/2}$ and $\frac{\rho}{\sigma}$

→ If $\rho > 0$ → +ve auto-correlation
If $\rho < 0$ → −ve auto-correlation

* +ve auto-correlation means high random no's are followed by high and low by low
* −ve auto-correlation means low random no's are followed by high and high by low

(1) Test 3rd, 8th, 13th and so on in sequence are autocorrelated using $\alpha = 0.05$

0.12, 0.01, 0.23, 0.28, 0.89, 0.31,
0.64, 0.28, 0.83, 0.93, 0.99, 0.15
0.33, 0.35, 0.91, 0.41, 0.60, 0.27,
0.75, 0.88, 0.68, 0.49, 0.05, 0.43,
0.95, 0.58, 0.19, 0.36, 0.69, 0.87

→