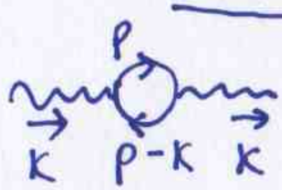
Photon self-energy at one loop and renormalization



$$= \Pi^{\mu\nu}(k) = \frac{e^2}{6\pi^2\varepsilon}\left(k^\mu k^\nu - g^{\mu\nu}k^2\right) + O(\varepsilon^0)$$

photon propagator  $D_{0\mu\nu}(k) = \dfrac{-ig_{\mu\nu}}{k^2+i\epsilon}$

dressed photon propagator 

$$D_{\mu\nu}(k) = D_{0\mu\nu}(k) + D_{0\mu\rho}(k)\, i\Pi^{\rho\sigma}(k)\, D_{0\sigma\nu}(k) + \dots$$

$$\Rightarrow D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2} - i\frac{g_{\mu\rho}}{k^2}\frac{ie^2}{6\pi^2\varepsilon}\left(k^\rho k^\sigma - g^{\rho\sigma}k^2\right)(-i)\frac{g_{\sigma\nu}}{k^2} + \dots = -i\frac{g_{\mu\nu}}{k^2} - \frac{ie^2}{6\pi^2\varepsilon}\frac{k_\mu k_\nu}{k^4} + \frac{ie^2}{6\pi^2\varepsilon}\frac{g_{\mu\nu}}{k^2} + \dots$$

or $$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2}\left(1 - \frac{e^2}{6\pi^2\varepsilon}\right) - \frac{ie^2}{6\pi^2\varepsilon}\frac{k_\mu k_\nu}{k^4}$$

We add to $\mathcal{L}_{gauge} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$ the counterterms $-\frac{C}{4}F^{\mu\nu}F_{\mu\nu} - \frac{N}{2}(\partial_\mu A^\mu)^2$

with $C = \dfrac{-e^2}{6\pi^2\varepsilon} = N$

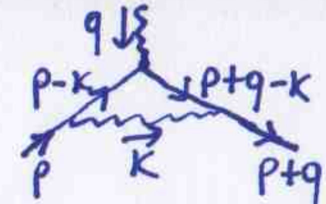
Then bare gauge Lagrangian is $\mathcal{L}_{b\,gauge} = -\dfrac{(1+C)}{4}F^{\mu\nu}F_{\mu\nu} - \dfrac{(1+N)}{2}(\partial_\mu A^\mu)^2$

$\Rightarrow \mathcal{L}_{b\,gauge} = -\frac{1}{4}Z_A F^{\mu\nu}F_{\mu\nu} +$ gauge-fixing with $Z_A = 1 + C = 1 - \dfrac{e^2}{6\pi^2\varepsilon}$

bare photon field $A_b^\mu = \sqrt{Z_A}\, A^\mu$ Thus $\mathcal{L}_{b\,gauge} = -\frac{1}{4}(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu)(\partial_\mu A_{b\nu} - \partial_\nu A_{b\mu}) +$ gauge-fixing

We have renormalized the field of the photon

Vertex diagram at one loop and renormalization



$= \Lambda^\mu(p,q)$

$$i\Lambda^\mu(p,q)=\int\frac{d^nk}{(2\pi)^n}(-ie\gamma^\nu)\frac{i(\not p+\not q-\not k+m)}{(p+q-k)^2-m^2}(-ie\gamma^\mu)\frac{i(\not p-\not k+m)}{(p-k)^2-m^2}(-ie\gamma^\rho)\frac{(-ig_{\nu\rho})}{k^2}$$

$$\Rightarrow \Lambda^\mu(p,q)=\frac{ie^3}{(2\pi)^n}\int d^nk\ \frac{\gamma^\nu(\not p+\not q-\not k+m)\gamma^\mu(\not p-\not k+m)\gamma_\nu}{[(p-k)^2-m^2][(p+q-k)^2-m^2]\,k^2}$$

$$=\frac{2ie^3}{(2\pi)^n}\int_0^1dx\int_0^{1-x}dy\int d^nk\ \frac{\gamma^\nu(\not p+\not q-\not k+m)\gamma^\mu(\not p-\not k+m)\gamma_\nu}{[x(p-k)^2-xm^2+y(p+q-k)^2-ym^2+(1-x-y)k^2]^3}$$

$$=\frac{2ie^3}{(2\pi)^n}\int_0^1dx\int_0^{1-x}dy\int d^nk\ \frac{\gamma^\nu(\not p+\not q-\not k+m)\gamma^\mu(\not p-\not k+m)\gamma_\nu}{[k^2+2k\cdot(-y(p+q)-xp)+xp^2+y(p+q)^2-(x+y)m^2]^3}$$

After several steps we find $\Lambda^\mu(p,q)=\frac{-e^3}{8\pi^2\varepsilon}\gamma^\mu+O(\varepsilon^0)$

We add to $\mathcal{L}_{int}=-e\bar\psi\gamma^\mu\psi A_\mu$ the counterterm $-Le\bar\psi\gamma^\mu\psi A_\mu$ with $L=-\frac{e^2}{8\pi^2\varepsilon}$

The bare interaction Lagrangian is $\mathcal{L}_{b\,int}=-(1+L)e\bar\psi\gamma^\mu\psi A_\mu=-Z_Le\bar\psi\gamma^\mu\psi A_\mu$ with $Z_L=1+L=1-\frac{e^2}{8\pi^2\varepsilon}\underset{\text{(Ward identity)}}{=}Z_\psi$ Thus $\mathcal{L}_{b\,int}=-e_b\bar\psi_b\gamma^\mu\psi_bA_{b\mu}$ with $e_b=e/\sqrt{Z_A}$ the bare charge. We have renormalized the electric charge of the electron.

The complete bare QED Lagrangian is then

$$\mathcal{L}_{b\,QED}=i\bar\psi_b\gamma^\mu\partial_\mu\psi_b-m_b\bar\psi_b\psi_b-e_b\bar\psi_b\gamma^\mu\psi_bA_{b\mu}-\frac{1}{4}(\partial^\mu A_b^\nu-\partial^\nu A_b^\mu)(\partial_\mu A_{b\nu}-\partial_\nu A_{b\mu})+\text{gauge-fixing}$$

Dimensional analysis The action $S = \int d^n x \, \mathcal{L}$ is dimensionless

Hence $\mathcal{L}$ has dimensions of $(\text{length})^{-n}$ in n dimensions

In natural units $\hbar = c = 1$ $\quad \underset{\text{energy}}{[E]} = \underset{\text{length}}{[L]}^{-1}$ as can be seen from $E = \frac{hc}{\lambda}$

Also $\underset{\text{energy}}{[E]} = \underset{\text{momentum}}{[p]} = \underset{\text{mass}}{[m]}$ as can be seen from $E^2 = \vec{p}^{\,2} c^2 + m^2 c^4$ with $c = 1$

From the term $-m\bar{\psi}\psi$ in the QED Lagrangian - with mass dimension n - we have $[m\bar{\psi}\psi] = n \Rightarrow [\bar{\psi}\psi] = n-1 \Rightarrow [\psi] = \frac{n-1}{2}$ which is $\frac{3}{2}$ in 4 dimensions

From the term $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ in the QED Lagrangian, we have

$$[F^{\mu\nu} F_{\mu\nu}] = n \Rightarrow [F^{\mu\nu}] = \frac{n}{2} \Rightarrow [\partial^\mu A^\nu - \partial^\nu A^\mu] = \frac{n}{2} \Rightarrow [\partial^\mu A^\nu] = \frac{n}{2}$$

$$\Rightarrow \left[\frac{\partial A^\nu}{\partial x_\mu} \right] = \frac{n}{2} \Rightarrow [A^\nu] = \frac{n}{2} - 1$$ which is 1 in 4 dimensions

From the term $-e\bar{\psi}\gamma^\mu \psi A_\mu$ in the QED Lagrangian, we have

$$[e\bar{\psi}\psi A_\mu] = n \Rightarrow [e] + 2[\psi] + [A_\mu] = n \Rightarrow [e] + n - 1 + \frac{n}{2} - 1 = n$$

$$\Rightarrow [e] = 2 - \frac{n}{2}$$ which is 0 in 4 dimensions

If we want e to be a dimensionless quantity, then $-e\bar{\psi}\gamma^\mu\psi A_\mu \to -e\mu^{2-\frac{n}{2}} \bar{\psi}\gamma^\mu \psi A_\mu$ or $-e\mu^{\varepsilon/2} \bar{\psi}\gamma^\mu\psi A_\mu$ where μ is a renormalization scale

Then the bare charge is $e_b = \frac{e\mu^{\varepsilon/2}}{\sqrt{Z_A}} = \frac{e\mu^{\varepsilon/2}}{\sqrt{1-\frac{e^2}{6\pi^2\varepsilon}}} = e\mu^{\varepsilon/2}\left(1+\frac{e^2}{12\pi^2\varepsilon}+\dots\right)$

The bare charge is independent of μ.

Thus $\frac{\partial e_b}{\partial \mu} = 0 \Rightarrow \frac{\partial}{\partial\mu}\left[\mu^{\varepsilon/2}\left(e+\frac{e^3}{12\pi^2\varepsilon}\right)\right] = 0 \Rightarrow \frac{\varepsilon}{2}\mu^{\frac{\varepsilon}{2}-1}\left(e+\frac{e^3}{12\pi^2\varepsilon}\right) + \mu^{\varepsilon/2}\left(\frac{\partial e}{\partial\mu}+\frac{3e^2}{12\pi^2\varepsilon}\frac{\partial e}{\partial\mu}\right) = 0$

$\Rightarrow \frac{\varepsilon}{2\mu}\left(e+\frac{e^3}{12\pi^2\varepsilon}\right) + \frac{\partial e}{\partial\mu}\left(1+\frac{e^2}{4\pi^2\varepsilon}\right) = 0 \Rightarrow \mu\frac{\partial e}{\partial\mu}\left(1+\frac{e^2}{4\pi^2\varepsilon}\right) = -\frac{\varepsilon}{2}\left(e+\frac{e^3}{12\pi^2\varepsilon}\right)$

$\Rightarrow \mu\frac{\partial e}{\partial\mu} = -\frac{\varepsilon}{2}\left(e+\frac{e^3}{12\pi^2\varepsilon}\right)\left(1-\frac{e^2}{4\pi^2\varepsilon}+\dots\right) \Rightarrow \mu\frac{\partial e}{\partial\mu} = -\frac{\varepsilon}{2}e + \frac{e^3}{12\pi^2}+\dots$

The β (beta) function is defined as $\beta(e) = \mu\frac{\partial e}{\partial\mu} \Rightarrow \beta(e) = \frac{e^3}{12\pi^2} > 0$

Since β is positive, we find that e <u>increases</u> with scale.

$\mu\frac{\partial e}{\partial\mu} = \frac{e^3}{12\pi^2} \Rightarrow \int_{\mu_0}^{\mu}\frac{\partial\mu}{\mu} = 12\pi^2\int_{e(\mu_0)}^{e(\mu)}\frac{\partial e}{e^3} \Rightarrow \ln\frac{\mu}{\mu_0} = -6\pi^2\left(\frac{1}{e^2(\mu)}-\frac{1}{e^2(\mu_0)}\right)$

$\Rightarrow e^2(\mu) = \frac{e^2(\mu_0)}{1-\frac{e^2(\mu_0)}{6\pi^2}\ln\frac{\mu}{\mu_0}}$ running coupling – diverges at very high energy

fine structure "constant" $\alpha(\mu) = \frac{e^2(\mu)}{4\pi}$ Note that $\alpha \simeq \frac{1}{137}$ at low energies
(actually <u>not</u> constant but running)