We will calculate the n-point Green's function

$$G(x_1, x_2, \dots, x_n) = \langle 0 | T\{\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\}|0\rangle$$

$$= \langle 0|U^\dagger(t_1,t_0)\varphi_I(x_1)U(t_1,t_0)\,U^\dagger(t_2,t_0)\varphi_I(x_2)U(t_2,t_0)\dots U^\dagger(t_n,t_0)\varphi_I(x_n)U(t_n,t_0)|0\rangle$$

This can be rewritten as

$$G(x_1,x_2,\dots,x_n) = \langle 0|U^\dagger(t,t_0)\,T\{\varphi_I(x_1)\varphi_I(x_2)\dots\varphi_I(x_n)\,U(t,t_1)U(t_1,t_2)\dots U(t_n,-t)\}U(-t,t_0)|0\rangle$$

$$= \langle 0|U^\dagger(t,t_0)\,T\{\varphi_I(x_1)\varphi_I(x_2)\dots\varphi_I(x_n)\exp\left[-i\int_{-t}^{t}dt' H_I(t')\right]\}U(-t,t_0)|0\rangle$$

with $t \gg t_1 > t_2 > \dots > t_n \gg -t$

Next we choose $t_0 = -t$ and take the limit $t \to \infty$

Then $U(-t,t_0) = 1$ and $U^\dagger(t,t_0) \to U^\dagger(\infty,-\infty)$

Also $U(\infty,-\infty)|0\rangle = e^{i\theta}|0\rangle$ with θ a phase

$$\Rightarrow \langle 0|U(\infty,-\infty)|0\rangle = \langle 0|e^{i\theta}|0\rangle = e^{i\theta} \Rightarrow \langle 0|T\{\exp[-i\int_{-\infty}^{+\infty}dt' H_I(t')]\}|0\rangle = e^{i\theta}$$

Then $\langle 0|U^\dagger(\infty,-\infty) = \langle 0|e^{-i\theta} = \langle 0| \big/ \langle 0|T\{\exp[-i\int_{-\infty}^{+\infty}dt' H_I(t')]\}|0\rangle$

Then

$$\langle 0|T\{\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\}|0\rangle = \frac{\langle 0|T\{\varphi_I(x_1)\varphi_I(x_2)\dots\varphi_I(x_n)\exp[-i\int_{-\infty}^{+\infty}dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-\infty}^{+\infty}dt\, H_I(t)]\}|0\rangle}$$

Feynman propagator $\langle 0|T\{\varphi_I(x)\varphi_I(y)\}|0\rangle$ or simply $\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle$

Write $\varphi(x)=\varphi^+(x)+\varphi^-(x)$

with $\varphi^+(x)=\int\frac{d^3p}{(2\pi)^3(2p^0)^{1/2}}a(p)e^{-ip\cdot x}$ and $\varphi^-(x)=\int\frac{d^3p}{(2\pi)^3(2p^0)^{1/2}}a^\dagger(p)e^{ip\cdot x}$

If $x^0>y^0$ then $T\{\varphi(x)\varphi(y)\}=\varphi(x)\varphi(y)=(\varphi^+(x)+\varphi^-(x))(\varphi^+(y)+\varphi^-(y))$

$=\varphi^+(x)\varphi^+(y)+\varphi^+(x)\varphi^-(y)+\varphi^-(x)\varphi^+(y)+\varphi^-(x)\varphi^-(y)$

$=\varphi^+(x)\varphi^+(y)+\varphi^-(y)\varphi^+(x)+[\varphi^+(x),\varphi^-(y)]+\varphi^-(x)\varphi^+(y)+\varphi^-(x)\varphi^-(y)$

$=:\varphi(x)\varphi(y):+[\varphi^+(x),\varphi^-(y)]$

If $x^0<y^0$ then $T\{\varphi(x)\varphi(y)\}=:\varphi(x)\varphi(y):+[\varphi^+(y),\varphi^-(x)]$

Thus $T\{\varphi(x)\varphi(y)\}=:\varphi(x)\varphi(y):+D(x-y)$

where $D(x-y)=\theta(x^0-y^0)[\varphi^+(x),\varphi^-(y)]+\theta(y^0-x^0)[\varphi^+(y),\varphi^-(x)]$

Then $\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle=\langle 0|:\varphi(x)\varphi(y):|0\rangle+\langle 0|D(x-y)|0\rangle$

$= \quad 0 \quad + D(x-y)\langle 0|0\rangle$

So $D(x-y)=\langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle$ is the Feynman propagator

It is the amplitude for the propagation of a particle from spacetime point x^μ to spacetime point y^μ

$$D(x-y) = \theta(x^0-y^0)\left(\varphi^+(x)\varphi^-(y) - \varphi^-(y)\varphi^+(x)\right) + \theta(y^0-x^0)\left(\varphi^+(y)\varphi^-(x) - \varphi^-(x)\varphi^+(y)\right)$$

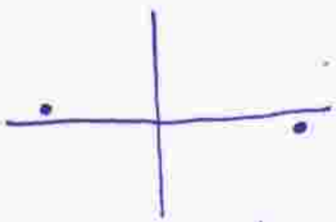
$$= \int \frac{d^3p\, d^3q}{(2\pi)^6 2\sqrt{p^0 q^0}} \left\{ \theta(x^0-y^0)[a(p), a^\dagger(q)] e^{-ip\cdot x} e^{iq\cdot y} + \theta(y^0-x^0)[a(p), a^\dagger(q)] e^{-ip\cdot y} e^{iq\cdot x} \right\}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6 2\sqrt{p^0 q^0}} (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \left\{ \theta(x^0-y^0) e^{-ip\cdot x} e^{iq\cdot y} + \theta(y^0-x^0) e^{-ip\cdot y} e^{iq\cdot x} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} \left\{ \theta(x^0-y^0) e^{-ip\cdot(x-y)} + \theta(y^0-x^0) e^{ip\cdot(x-y)} \right\}$$

This is equal to $D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$ with $\varepsilon \to 0^+$

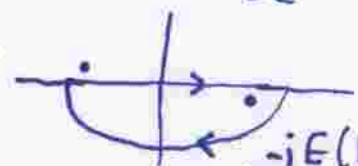
Proof: $D(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - \vec{p}^2 - m^2 + i\varepsilon} e^{-ip^0(x^0-y^0)}$



poles in complex p^0 plane

Poles at $p^0 = \pm\sqrt{\vec{p}^2+m^2-i\varepsilon} = \pm\sqrt{E^2-i\varepsilon}$ (with $E = \sqrt{\vec{p}^2+m^2}$)

$\approx \pm E\left(1 - \frac{i\varepsilon}{2E^2}\right)$

For $x^0 > y^0$ choose contour  Residue at $p^0 = E\left(1-\frac{i\varepsilon}{2E^2}\right)$

Then $\oint \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - E^2 + i\varepsilon} e^{-ip^0(x^0-y^0)} = -2\pi i \frac{i}{2\pi} \frac{e^{-iE\left(1-\frac{i\varepsilon}{2E^2}\right)(x^0-y^0)}}{2\cdot E\left(1-\frac{i\varepsilon}{2E^2}\right)} = \frac{1}{2E} e^{-iE(x^0-y^0)}$

For $x^0 < y^0$  $\rightarrow \frac{1}{2E} e^{iE(x^0-y^0)}$ This concludes the proof.

In momentum space $D(p) = \frac{i}{p^2-m^2+i\varepsilon}$

Note that $(\partial_{\mu x}\partial_x^\mu + m^2) D(x-y) = \int \frac{d^4p}{(2\pi)^4} i/(p^2-m^2+i\varepsilon)\, (-p^2+m^2) e^{-ip\cdot(x-y)} = -i\delta^4(x-y)$ So $D(x-y)$ is Green's function of $\partial_{\mu x}\partial_x^\mu + m^2$

<u>Wick's theorem</u> $T\{\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\} = :\varphi(x_1)\varphi(x_2)\dots\varphi(x_n): +$ all possible contractions

where a contraction of two fields is by definition the Feynman propagator

This generalizes the expression $T\{\varphi(x)\varphi(y)\} = :\varphi(x)\varphi(y): + D(x-y)$

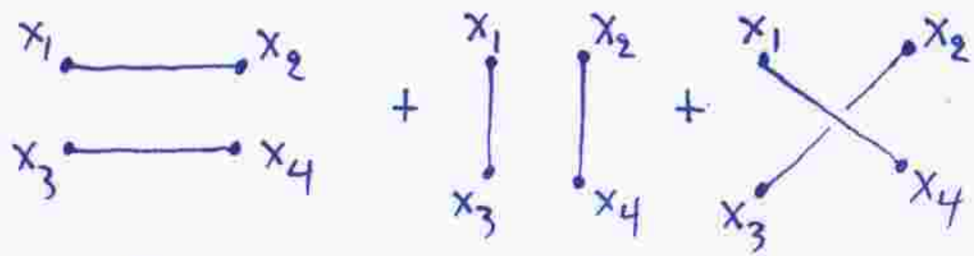
Example: (use $\varphi_i = \varphi(x_i)$) $T\{\varphi_1\varphi_2\varphi_3\varphi_4\} = :\varphi_1\varphi_2\varphi_3\varphi_4: + D_{12}:\varphi_3\varphi_4: + D_{13}:\varphi_2\varphi_4: + D_{14}:\varphi_2\varphi_3:$

$+ D_{23}:\varphi_1\varphi_4: + D_{24}:\varphi_1\varphi_3: + D_{34}:\varphi_1\varphi_2: + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$

Since $\langle 0|a^\dagger = 0$ and $a|0\rangle = 0$, $\langle 0|::|0\rangle = 0$

and thus $\langle 0|T\{\varphi_1\varphi_2\varphi_3\varphi_4\}|0\rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$

or $\langle 0|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|0\rangle = D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3)$

= 

Feynman diagrams Three ways to propagate between points

The total amplitude is the sum of these three diagrams