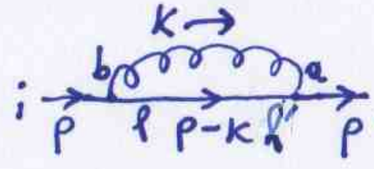
Renormalization in QCD

quark self-energy diagram 

$= \Sigma_{ij}(p)$

$$i\Sigma_{ij}(p) = \int \frac{d^n k}{(2\pi)^n} (-ig_s)\gamma^\mu \frac{\lambda^a_{j\ell'}}{2} \frac{i(\not p - \not k + m)_{\ell'\ell}}{(p-k)^2 - m^2} (-ig_s)\gamma^\nu \frac{\lambda^b_{\ell i}}{2} \frac{(-ig_{\mu\nu})}{k^2}\delta_{ab}$$

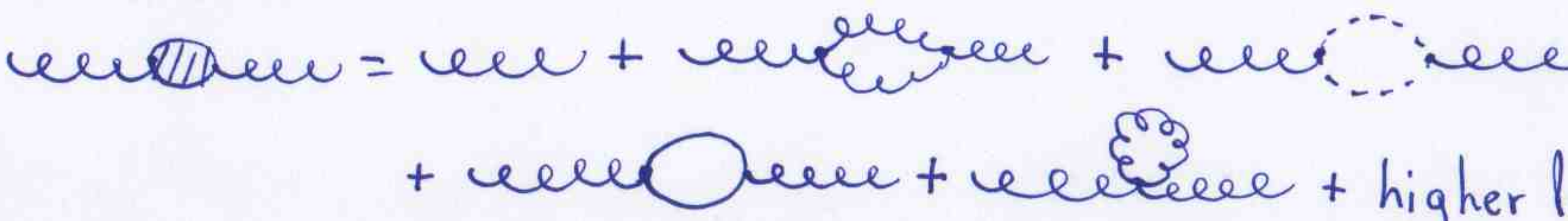
$$= \frac{1}{4}(\lambda^a\lambda^a)_{ji} (-g_s^2)\int \frac{d^n k}{(2\pi)^n} \frac{\gamma^\mu(\not p - \not k + m)\gamma_\mu}{[(p-k)^2 - m^2]k^2}$$

But $\frac{1}{4}(\lambda^a\lambda^a)_{ji} = C_F\delta_{ij}$ and the rest is the same as in QED (with $e^2 \to g_s^2$)

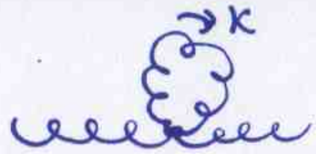
Then $\Sigma_{ij}(p) = C_F\,\delta_{ij}\,\frac{g_s^2}{8\pi^2\varepsilon}(\not p - 4m) + O(\varepsilon^0) = \frac{g_s^2}{6\pi^2\varepsilon}(\not p - 4m)\delta_{ij} + O(\varepsilon^0)$

Then $\psi_b = \sqrt{Z_\psi}\,\psi$ with $Z_\psi = 1 - \frac{g_s^2}{6\pi^2\varepsilon} = 1 - \frac{g_s^2}{8\pi^2\varepsilon}C_F$

vacuum polarization



The four-gluon-vertex diagram vanishes, and the quark-loop diagram is similar to QED, but we have additional gluon-loop and ghost-loop diagrams.
(with three-gluon-vertices) (gluon-ghost vertices)



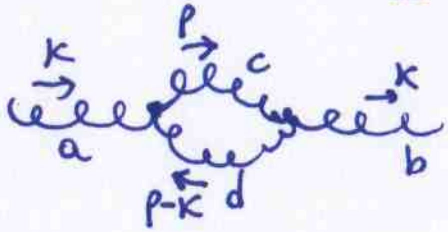
proportional to $\int \frac{d^n k}{k^2} = 0$

n_f flavors

This is $n_f \, \mathrm{tr}(T^a T^b)$ times the QED result (with $e^2 \to g_s^2$)

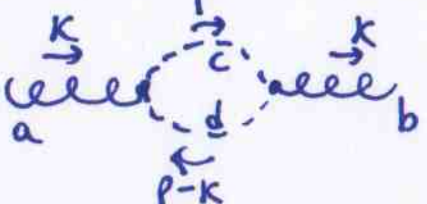
But $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$

Then $\Pi^{\mu\nu}_{\substack{\text{quark-loop}\\ ab}}(k) = \frac{n_f}{2}\delta_{ab}\frac{g_s^2}{6\pi^2\varepsilon}\left(k^\mu k^\nu - g^{\mu\nu}k^2\right) + O(\varepsilon^0)$



$\Pi^{\mu\nu}_{\substack{\text{gluon-loop}\\ ab}}(k) = -\frac{g_s^2}{16\pi^2\varepsilon} f^{acd} f^{bcd}\left(\frac{11}{3}k^\mu k^\nu - \frac{19}{6}g^{\mu\nu}k^2\right) + O(\varepsilon^0)$

since $f^{acd} f^{bcd} = C_A \delta_{ab}$ this is $\Pi^{\mu\nu}_{\substack{\text{gluon-loop}\\ ab}}(k) = -\frac{g_s^2}{16\pi^2\varepsilon} C_A \delta_{ab}\left(\frac{11}{3}k^\mu k^\nu - \frac{19}{6}g^{\mu\nu}k^2\right) + O(\varepsilon^0)$



$$\Pi^{\mu\nu}_{\substack{\text{ghost-loop}\\ ab}}(k) = \frac{g_s^2}{16\pi^2\varepsilon} f^{acd} f^{bcd}\left(\frac{1}{3}k^\mu k^\nu + \frac{1}{6}g^{\mu\nu}k^2\right) + O(\varepsilon^0)$$

$$= \frac{g_s^2}{16\pi^2\varepsilon} C_A \delta_{ab}\left(\frac{1}{3}k^\mu k^\nu + \frac{1}{6}g^{\mu\nu}k^2\right) + O(\varepsilon^0)$$

Summing over quark, gluon, and ghost loops, we get

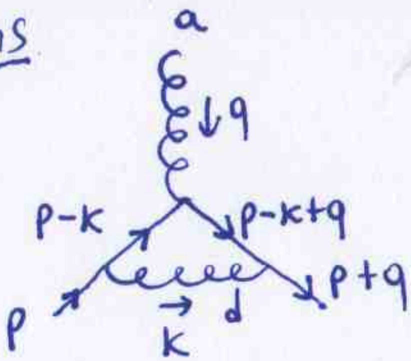
$$\Pi^{\mu\nu}_{\substack{\text{loops}\\ ab}}(k) = \frac{g_s^2}{24\pi^2\varepsilon}\delta_{ab}\left(5C_A - 2n_f\right)\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right) + O(\varepsilon^0)$$

gluon & ghost loop (opposite sign to quark) — quark loop

Then $G^a_{0\mu} = \sqrt{Z_G}\, G^a_\mu$

with $Z_G = 1 + \frac{g_s^2}{24\pi^2\varepsilon}\left(5C_A - 2n_f\right)$

Vertex diagrams

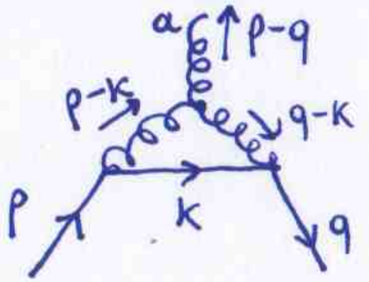


$= \Lambda^a_{1\mu}(p,q)$

This is $T^d T^a T^d$ times the QED result (with $e \to g_s$)

But $T^d T^a T^d = \left(C_F - \frac{C_A}{2}\right) T^a$ Thus $\Lambda^a_{1\mu}(p,q) = -\frac{g_s^3}{8\pi^2\varepsilon}\left(C_F - \frac{C_A}{2}\right)\gamma_\mu T^a$

Also, three-gluon vertex diagram



$$\Lambda^a_{2\mu}(p,q) = -\frac{g_s^3}{8\pi^2\varepsilon}\,\frac{3}{2}C_A\gamma_\mu T^a = -\frac{3g_s^3}{16\pi^2\varepsilon}C_A\gamma_\mu T^a$$

Then, the total is $\Lambda^a_\mu(p,q) = \Lambda^a_{1\mu}(p,q) + \Lambda^a_{2\mu}(p,q) = -\frac{g_s^3}{8\pi^2\varepsilon}(C_F + C_A)\gamma_\mu T^a$

Then $\mathcal{L}_{b\,int} = -Z_L g_s \bar{\psi}\gamma^\mu T^a \psi G^a_\mu$ with $Z_L = 1 - \frac{g_s^2}{8\pi^2\varepsilon}(C_F + C_A)$

$$\Rightarrow \mathcal{L}_{b\,int} = -Z_L\, g_s \frac{\bar{\psi}_b}{\sqrt{Z_\psi}}\gamma^\mu T^a \frac{\psi_b}{\sqrt{Z_\psi}}\frac{G^a_{b\mu}}{\sqrt{Z_G}} = -\frac{Z_L}{Z_\psi\sqrt{Z_G}}\, g_s \bar{\psi}_b \gamma^\mu T^a \psi_b G^a_{b\mu}$$

Then the bare coupling is $g_{sb} = \frac{Z_L}{Z_\psi\sqrt{Z_G}}\, g_s\, \mu^{\varepsilon/2}$ with g_s dimensionless and μ the renormaliz scale

Bare coupling $g_{sb} = Z_L Z_\psi^{-1} Z_G^{-1/2} g_s \mu^{\varepsilon/2}$

$$\Rightarrow g_{sb} = \left[1 - \frac{g_s^2}{8\pi^2\varepsilon}(C_F + C_A)\right]\left(1 - \frac{g_s^2}{8\pi^2\varepsilon}C_F\right)^{-1}\left[1 + \frac{g_s^2}{24\pi^2\varepsilon}(5C_A - 2n_f)\right]^{-1/2} g_s \mu^{\varepsilon/2}$$

$$\Rightarrow g_{sb} = \left[1 - \frac{g_s^2}{8\pi^2\varepsilon}(C_F + C_A)\right]\left(1 + \frac{g_s^2}{8\pi^2\varepsilon}C_F + \dots\right)\left[1 - \frac{g_s^2}{48\pi^2\varepsilon}(5C_A - 2n_f) + \dots\right] g_s \mu^{\varepsilon/2}$$

$$\Rightarrow g_{sb} = \left[1 - \frac{g_s^2}{48\pi^2\varepsilon}(11C_A - 2n_f)\right] g_s \mu^{\varepsilon/2}$$

The bare strong coupling is independent of μ.

Thus $\frac{\partial g_{sb}}{\partial \mu} = 0 \underset{\text{as } \varepsilon \to 0}{\Rightarrow} \beta(g_s) = \mu \frac{\partial g_s}{\partial \mu} = -\frac{g_s^3}{48\pi^2}(11C_A - 2n_f) = -\frac{g_s^3}{16\pi^2} b_0$

with $b_0 = \frac{11}{3}C_A - \frac{2n_f}{3}$

We note that $\beta(g_s) < 0$ since $C_A = 3$ and $n_f = 6$ (in fact for n_f up to 16)

so g_s decreases with increasing energy $\rightarrow$ asymptotic freedom

This is the opposite behavior of QED where $\beta(e) = \frac{e^3}{12\pi^2} > 0$

The reason is the antiscreening from gluons.

Thus perturbative QCD gets better at higher energies in its precision.

An alternative definition of the QCD beta function in terms of $a_s = \frac{g_s^2}{4\pi}$ is $\beta(a_s) = \frac{d \ln a_s}{d \ln \mu^2} = -\sum_{n=0}^{\infty} b_n \left(\frac{a_s}{4\pi}\right)^{n+1}$ (to all loops)

Note that the calculation of b_n requires $n+1$ loops

At one loop $\frac{d \ln a_s}{d \ln \mu^2} = -b_0 \frac{a_s}{4\pi} \Rightarrow \frac{da_s}{a_s^2} = -\frac{b_0}{4\pi} d \ln \mu^2 \Rightarrow \left.\frac{1}{a_s}\right|_{a_s(\mu_0)}^{a_s(\mu)} = \frac{b_0}{4\pi} \ln \mu^2 \Big|_{\mu_0}^{\mu}$

$$\Rightarrow \frac{1}{a_s(\mu)} - \frac{1}{a_s(\mu_0)} = \frac{b_0}{4\pi} \ln \frac{\mu^2}{\mu_0^2} \Rightarrow a_s(\mu) = \frac{a_s(\mu_0)}{1 + a_s(\mu_0) \frac{b_0}{4\pi} \ln \frac{\mu^2}{\mu_0^2}}$$

so we can calculate a_s at scale μ if we know it at scale μ_0

We can choose a QCD scale Λ as

$$\ln \Lambda^2 = \ln \mu_0^2 - \frac{4\pi}{b_0 a_s(\mu_0)}$$

Then $a_s(\mu) = \dfrac{4\pi}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$

Λ depends on the number of flavors $\quad \Lambda \sim 200$ MeV