HW2
PHYS4500: Quantum Field Theory

Casey Hampson

August 26, 2024

Question 1. (16.2)

Let's just plug in:

$$\begin{aligned}\{\gamma^0,\ \gamma^1\} &= \gamma^0\gamma^1 + \gamma^1\gamma^0,\\ &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\sigma^1 & 0\\ 0 & -\sigma^1\end{pmatrix} + \begin{pmatrix}\sigma^1 & 0\\ 0 & -\sigma^1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\ &= \begin{pmatrix}0 & \sigma^1\\ \sigma^1 & 0\end{pmatrix} + \begin{pmatrix}0 & -\sigma^1\\ -\sigma^1 & 0\end{pmatrix},\\ &= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 2g^{01} = 0,\end{aligned}$$

as expected.

Question 2. (16.7)

We are now looking at the *standard representation* for the gamma matrices:

$$\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \text{and}\ \gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}. \tag{1}$$

We want to show that these satisfy the representation-independent commutation relation:

$$\{\gamma^\mu,\ \gamma^\nu\} = 2g^{\mu\nu}. \tag{2}$$

We have three cases to work out:

i. When $\mu = \nu = 0$:

$$\begin{gathered}\{\gamma^0, \gamma^0\} = 2\left(\gamma^0\right)^2 = 2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 2,\\ \rightarrow 2 = 2g^{00} = 2.\ \checkmark\end{gathered}$$

ii. When $\mu = 0,\ \nu = i$:

$$\begin{aligned}\{\gamma^0,\ \gamma^i\} &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\ &= \begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix} + \begin{pmatrix}0 & -\sigma^i\\ -\sigma^i & 0\end{pmatrix} = 0,\end{aligned}$$

$$\rightarrow 0 = 2g^{0i} = 0.\ \checkmark$$

iii. When $\mu = i,\ \nu = j$:

$$\begin{aligned}\{\gamma^i,\ \gamma^j\} &= \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix},\\ &= \begin{pmatrix}0 & -\sigma^i\sigma^j\\ -\sigma^i\sigma^j & 0\end{pmatrix}\begin{pmatrix}0 & -\sigma^j\sigma^i\\ -\sigma^j\sigma^i & 0\end{pmatrix} = -\begin{pmatrix}\{\sigma^i,\ \sigma^j\} & 0\\ 0 & \{\sigma^i,\ \sigma^j\}\end{pmatrix}.\end{aligned}$$

We know that the Pauli spin matrices satisfy the following anti-commutation relation:

$$\{\sigma_i,\ \sigma_j\} = 2\delta_{ij}, \tag{3}$$

so

$$\{\gamma^i,\ \gamma^j\} = -2\delta^{ij} = 2g^{ij}.$$

When $i = j$, we have $-2 = -2$, which is what we expect, and when $i \neq j$, we have $0 = 0$. So, this holds. ✓

Question 3. (17.1 a,b)

a) The definition of rapidity is

$$\theta = \tanh \beta. \tag{4}$$

Using the definition of $\tanh^{-1} x$, we can expand this into

$$\theta = \frac{1}{2} \tanh \left(\frac{1+\beta}{1-\beta} \right).$$

When $\beta \to -1^+$, we have

$$\lim_{\beta \to -1^+} \theta = \frac{1}{2} \ln \left(\frac{1-1^+}{1+1^+} \right).$$

The quantity inside the natural log approaches zero, so we have that

$$\boxed{\lim_{\beta \to -1^+} \theta = -\infty.}$$

When $\beta \to 1^-$, we have

$$\lim_{\beta \to 1^-} \theta = \frac{1}{2} \ln \left(\frac{1+1^-}{1-1^-} \right).$$

The denominator will continue growing while remaining positive, and the numerator will also remain positive. So,

$$\boxed{\lim_{\beta \to 1^-} = \infty.}$$

b) Starting with $\sinh \theta$, we just plug in the $\tanh \theta$ expansion and simplify:

$$\begin{aligned}
\sinh \theta &= \frac{1}{2} \left[\exp \left(\frac{1}{2} \ln \frac{1+\beta}{1-\beta} \right) - \exp \left(-\frac{1}{2} \ln \frac{1+\beta}{1-\beta} \right) \right], \\
&= \frac{1}{2} \left[\exp \left(\ln \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \right) - \exp \left(\ln \left(\frac{1+\beta}{1-\beta} \right)^{-1/2} \right) \right], \\
&= \frac{1}{2} \left[\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}} \right], \\
&= \frac{1}{2} \left[\sqrt{\frac{(1+\beta)^2}{1-\beta^2}} - \sqrt{\frac{(1-\beta^2)}{1-\beta^2}} \right], \\
&= \frac{1}{2} \left[\gamma(1+\beta) - \gamma(1-\beta) \right], \\
\end{aligned}$$

$$\boxed{\sinh \theta = \gamma \beta.}$$

For $\cosh \theta$, we have an identical expression in the final line above, except we flip the sign of the third and fourth terms in the brackets due to the definition of cosh compared to that of sinh. Thus,

$$\cosh \theta = \frac{1}{2} \left[\gamma(1+\beta) + \gamma(1-\beta) \right],$$

so

$$\boxed{\cosh \theta = \gamma.}$$

Question 4.

Here, we are to prove that the spinor

$$u^{(2)}(p) = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{(E+mc^2)} \\ -\frac{cp_z}{(E+mc^2)} \end{pmatrix} \tag{5}$$

satisfies the normalization condition

$$u^{(2)\dagger}(p)u^{(2)}(p) = \frac{2E}{c}. \tag{6}$$

All we really need to do here is just plug in:

$$\begin{aligned} u^{(2)\dagger}u^{(2)} &= \left(\frac{E+mc^2}{c}\right)\begin{pmatrix} 0 & 1 & \frac{c(p_x+ip_y)}{(E+mc^2)} & -\frac{cp_z}{(E+mc^2)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{(E+mc^2)} \\ -\frac{cp_z}{(E+mc^2)} \end{pmatrix}, \\ &= \left(\frac{E+mc^2}{c}\right)\left[1 + \frac{c^2(p_x+ip_y)(p_x-ip_y)}{(E+mc^2)^2} + \frac{c^2p_z^2}{(E+mc^2)^2}\right], \\ &= \left(\frac{E+mc^2}{c}\right)\left[1 + \frac{\boldsymbol{p}^2c^2}{(E+mc^2)^2}\right]. \end{aligned}$$

From the energy-momentum relation:

$$\boldsymbol{p}^2c^2 = E^2 - m^2c^4 = (E-mc^2)(E+mc^2),$$

so

$$u^{(2)\dagger}u^{(2)} = \left(\frac{E+mc^2}{c}\right)\left[1 + \frac{E-mc^2}{E+mc^2}\right] = \frac{E+mc^2}{c} + \frac{E-mc^2}{c} = \frac{2E}{c}. \checkmark$$