HW4

PHYS4210: Quantum Mechanics

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October 12, 2024

Problem 1.

The eigenvalue equation for the position operator is

$$\hat{x}f(x) = x_0 f(x) \quad \rightarrow \quad xf(x) = x_0 f(x). \tag{1.1}$$

where x_0 is the eigenvalue. The only function that is the same when multiplied by *any* x and the singular x_0 is the Dirac delta:

$$f(x) = \delta(x - x_0). \tag{1.2}$$

The eigenfunctions of the position operator being the Dirac delta sort of make sense - such a function (roughly speaking) is a localization entirely at $x = x_0$, which is exactly what the position is!

With this in mind, it is pretty straightforward to see in this case that the eigenvalues $\{ x_n \}$ are all the possible positions the particle can take. Since this spectrum is continuous and since the particle can be anywhere in space, then the eigenvalues must be the set of all real numbers.

Problem 2.

Considering two observables A and B, we can define

$$|f\rangle = (\hat{A} - \langle A \rangle) |\psi\rangle, \quad \text{and} \quad |g\rangle = (\hat{B} - \langle B \rangle) |\psi\rangle, \tag{2.1}$$

just as the book did. The product of the variances is given, by the Schwartz inequality, as

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2. \tag{2.2}$$

To achieve minimum uncertainty, this means that we have an equality rather than a inequality in the above equation:

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle \langle g|g\rangle = |\langle f|g\rangle|^2. \tag{2.3}$$

However, the only way for this to be the case in the vector space of square-integrable functions (Hilbert space) is for the two functions f and g to be equal, up to a constant which can in general be complex:

$$f = ag. \tag{2.4}$$

But, Griffiths states that for a complex number z:

$$|z|^2 = [\text{Re}(z)]^2 + [\text{Im}(x)]^2 \geq [\text{Im}(z)]^2. \tag{2.5}$$

Again, in the minimum-uncertainty regime, this is now an equality, meaning

$$|\langle f|g\rangle|^2 = |a \langle f|f\rangle|^2 = [\text{Im}(z)]^2, \tag{2.6}$$

or

$$[\text{Re}(a \langle f|f\rangle)]^2 = 0. \tag{2.7}$$

We could have taken it to be equal to the

However, since f lies in Hilbert space, it is square-integrable, meaning its inner product with itself must be real. For the entire quantity to be zero, a must be purely imaginary:

$$a = i\alpha. \tag{2.8}$$

We now have that

$$f = i\alpha g \quad \rightarrow \quad (\hat{A} - \langle A \rangle)\psi = i\alpha(\hat{B} - \langle B \rangle)\psi, \tag{2.9}$$

or replacing $\hat{A} \rightarrow \hat{p} \rightarrow -i\hbar \mathrm{d}/\mathrm{d}x$ and $\hat{B} \rightarrow \hat{x} = x$, we get

$$\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x} - \langle p\rangle\right)\psi = i\alpha(x - \langle x\rangle)\psi, \tag{2.10}$$

$$-i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}x} - \langle p\rangle\,\psi = i\alpha(x - \langle x\rangle)\psi, \tag{2.11}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{i}{\hbar}[i\alpha(x - \langle x\rangle) + \langle p\rangle]\psi, \tag{2.12}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \left(-\frac{\alpha}{\hbar}(x - \langle x\rangle) + \frac{i}{\hbar}\langle p\rangle\right)\psi. \tag{2.13}$$

This will be an exponential. For the first term, since we are differentiating with respect to x, we need a $\frac{1}{2}(x - \langle x\rangle)^2$ in the exponential (with the other constants as well, of course), and the second term is easy since $\langle p\rangle$ is not a function of x - it's just a number. With this:

$$\boxed{\psi = Ae^{-\alpha(x-\langle x\rangle)^2/2\hbar + i\langle p\rangle x/\hbar} = Ae^{-\alpha(x-\langle x\rangle)^2/2\hbar}e^{i\langle p\rangle x/\hbar}.} \tag{2.14}$$

The first exponential has a square of x, which is Gaussian. The second exponential is a "wiggle" factor, but the point is that we still get a Gaussian-looking function.

Problem 3. (3.3)

First we assume that $\langle h|\hat{Q}|h\rangle = \langle h|\hat{Q}^\dagger|h\rangle$ (i.e. $\hat{Q}$ is Hermitian). This doesn't actually say much about the function $h(x)$, since are are already assuming it is in Hilbert space. But the stronger condition that operators must satisfy is that $\langle f|\hat{Q}|g\rangle = \langle f|\hat{Q}^\dagger|g\rangle$ for any *two* functions $f(x)$ and $g(x)$. To show this, let's first consider the case where $h(x) = f(x) + g(x)$:

$$\begin{aligned}\langle f+g|\hat{Q}|f+g\rangle &= \int (f^* + g^*)[\hat{Q}(f+g)]\,\mathrm{d}x\\ &= \int [f^*(\hat{Q}g) + g^*(\hat{Q}g) + f^*(\hat{Q}g) + g^*(\hat{Q}f)]\,\mathrm{d}x.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle f+g|\hat{Q}^\dagger|f+g\rangle &= \int [\hat{Q}(f^* + g^*)](f+g)\,\mathrm{d}x\\ &= \int [(\hat{Q}f^*)f + (\hat{Q}g^*)g + (\hat{Q}f^*)g + (\hat{Q}g^*)f]\,\mathrm{d}x.\end{aligned}$$

Since we started with $\langle h|\hat{Q}|h\rangle = \langle h|\hat{Q}^\dagger|h\rangle$, these two equations must be equal, or better, their integrands must be equal

$$f^*(\hat{Q}f) + g^*(\hat{Q}g) + f^*(\hat{Q}g) + g^*(\hat{Q}f) = (\hat{Q}f^*)f + (\hat{Q}g^*)g + (\hat{Q}f^*)g + (\hat{Q}g^*)f.$$

But we can eliminate the first two terms on each side since they are equal by our assumption, so we have

$$f^*(\hat{Q}g) + g^*(\hat{Q}f) = (\hat{Q}f^*)g + (\hat{Q}g^*)f.$$

If we now let $h(x) = f(x) + ig(x)$,

$$\begin{aligned}\langle f+ig|\hat{Q}|f+ig\rangle &= \int (f^* - ig^*)[\hat{Q}(f+ig)]\,\mathrm{d}x\\ &= \int [f^*(\hat{Q}g) + g^*(\hat{Q}g) + if^*(\hat{Q}g) - ig^*(\hat{Q}f)]\,\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\langle f+ig|\hat{Q}^\dagger|f+ig\rangle &= \int [\hat{Q}(f^*-ig^*)](f+ig)\,\mathrm{d}x \\ &= \int [(\hat{Q}f^*)f + (\hat{Q}g^*)g + i(\hat{Q}f^*)g - i(\hat{Q}g^*)f]\,\mathrm{d}x.\end{aligned}$$

Again, these must be equal:

$$f^*(\hat{Q}g) + g^*(\hat{Q}g) + if^*(\hat{Q}g) - ig^*(\hat{Q}f) = (\hat{Q}f^*)f + (\hat{Q}g^*)g + i(\hat{Q}f^*)g - i(\hat{Q}g^*).$$

The first two terms on each side cancel again, so we have (canceling the i's):

$$f^*(\hat{Q}g) - g^*(\hat{Q}f) = (\hat{Q}f^*)g - (\hat{Q}g^*).$$

If we now add to this the result from the previous $h(x) = f(x) + g(x)$, then we get

$$2f^*(\hat{Q}g) = 2(\hat{Q}f^*)g, \tag{3.1}$$

or

$$\boxed{\langle f|\hat{Q}|g\rangle = \langle f|\hat{Q}^\dagger|g\rangle\,.}$$

Problem 4. (3.5)

a) First, for a real x,

$$\langle f|x|f\rangle = \int f^*(xf)\,\mathrm{d}x = \int (xf)^* f\,\mathrm{d}x = \langle f|x^\dagger|f\rangle\,, \tag{4.1}$$

so $x^\dagger = x$, meaning it's Hermitian.

Looking next at i:

$$\langle f|i|f\rangle = \int f^*(if)\,\mathrm{d}x = \int (-if)^* f\,\mathrm{d}x = \langle f|i^\dagger|f\rangle\,, \tag{4.2}$$

so $i^\dagger = -i$.

Finally for $\mathrm{d}/\mathrm{d}x$:

$$\left\langle f \left| \frac{\mathrm{d}}{\mathrm{d}x} \right| f \right\rangle = \int f^* \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f^* f\Big|_{\infty}^{\infty} - \int \frac{\mathrm{d}f^*}{\mathrm{d}x} f\,\mathrm{d}x = \int \left(-\frac{\mathrm{d}f}{\mathrm{d}x}\right)^* f\,\mathrm{d}x = \left\langle f \left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\dagger \right| f \right\rangle, \tag{4.3}$$

so $(\mathrm{d}/\mathrm{d}x)^\dagger = -\mathrm{d}/\mathrm{d}x$.

b) Using Griffiths' notation will be a little more illuminating than the notation I've been using (which is because I've been doing QFT stuff, and they tend to use that notation there more). Starting with $\langle f|(\hat{Q}\hat{R})f\rangle$, this means that $\hat{R}$ acts on f first, then $\hat{Q}$, so to move the two operators to act on the bra instead, we have to start with the outer $\hat{Q}$ the move the inner $\hat{R}$, meaning

$$\langle f|(\hat{Q}\hat{R})f\rangle = \langle(\hat{R}^\dagger\hat{Q}^\dagger)f|f\rangle\,. \tag{4.4}$$

But as Hermitian operators, this means that we must have

$$\langle(\hat{R}^\dagger\hat{Q}^\dagger)f|f\rangle = \langle(\hat{Q}\hat{R})^\dagger f|f\rangle\,, \tag{4.5}$$

meaning that

$$(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger. \tag{4.6}$$

c) With the normalization scheme we used in class, we have that

$$\hat{a}_+ = \frac{1}{\sqrt{2m}}\left(\hat{p} + im\omega\hat{x}\right). \tag{4.7}$$

Taking the Hermitian conjugate, the constant out front doesn't change since it's real, and we know that $\hat{x}$ and $\hat{p}$ are Hermitian, so the only thing that changes is that the i picks up a negative sign:

$$(\hat{a}_+)^\dagger = \frac{1}{\sqrt{2m}}\left(\hat{p} - im\omega x\right). \tag{4.8}$$

Incidentally, this says that

$$(\hat{a}_+)^\dagger = \hat{a}_-. \tag{4.9}$$

Problem 5. (3.18)

For all of these, the last term involving the rate of change of the operator will be zero since none of the operators depend on time.

a) Starting with $Q = 1$, the entire right-hand side is zero since $[\hat{H}, 1] = \hat{H} - \hat{H} = 0$ and 1 obviously has no time dependence, so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle 1\rangle) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\psi|\psi\rangle\right) = 0. \tag{5.1}$$

This is something we proved before - the normalization of the wave-function is independent of time, so we can normalize it at the most convenient time (usually $t = 0$), and we are set forever.

b) For $\hat{Q} = \hat{H}$, it obviously commutes with itself and since it almost never has time dependence, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle H\rangle = 0, \tag{5.2}$$

which is just conservation of energy! Technically, it's that measurements of the energy on any given system are guaranteed to give back the same energy; it cannot change with time so it cannot somehow gain a different energy sometime later.

c) For $\hat{Q} = \hat{x}$, we first need to look at the commutator

$$[\hat{H}, x] = \left[\left(\frac{p^2}{2m} + V\right), x\right] = \frac{1}{2m}[p^2, x], \tag{5.3}$$

since x obviously commutes with $V(x)$. Now,

$$[p^2, x] = p^2x - xp^2 = p\cdot px - xp\cdot p = p[p,x] - pxp - [x,p]p + pxp = -p[x,p] - [x,p]p = -2i\hbar p, \tag{5.4}$$

so

$$[\hat{H}, x] = -\frac{i\hbar p}{m}. \tag{5.5}$$

Plugging this in:

$$\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\left\langle\left(-\frac{i\hbar p}{m}\right)\right\rangle = \frac{\langle p\rangle}{m}, \tag{5.6}$$

or

$$m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \langle p\rangle\,. \tag{5.7}$$

This is just what we had before when doing all the problems with determining the expectation values for various wavefunctions!

d) For $\hat{Q} = \hat{p}$, $[p^2, p] = p^3 - p^3 = 0$ and (using a test function f)

$$\left[V, -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right] f = -i\hbar V\frac{\mathrm{d}f}{\mathrm{d}x} + i\hbar\frac{\mathrm{d}}{\mathrm{d}x}(Vf) = i\hbar\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{5.8}$$

So, plugging everything in:

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\left(-\hbar\frac{\mathrm{d}V}{\mathrm{d}x}\right) = -\left\langle\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle. \tag{5.9}$$

This is the same as Equation (1.28) in Griffiths, which is **Ehrenfest's theorem**.