Notes
PHYS3110: Directed Methods

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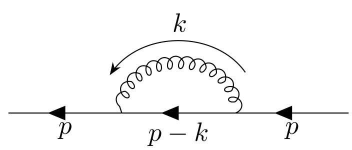
Renormalization and Regularization

- From previous classes, we are now able to make perturbative calculations in QED and QCD at leading order.
- This accurately discovers and treats the parton model to some extent, but we are still missing a lot of things from QED.
- For instance, in Drell-Yan, in which we have an electron probing a proton through a virtual photon, the leading order contribution is pure QCD, despite containing quarks and despite having a proton involved, both of which are highly QCD-driven objects.
- Loop diagrams are the first step into exploring this.

- But it is not so simple. Loop diagrams introduce annoying integrals, which is made worse by the fact that those integrals diverse for many values of momentum (mostly high ones, as one would expect).
- This idea of **renormalization** is needed in order to handle these infinities. First, however, the integrals, as mentioned, are quite hard to begin with, so a process called **regularization** is used.
- Regularization is a pure mathematical process by which the integral is essentially turned into a more simple object. Slightly more formally, the integral is turned into a function whose limit diverges, and treating such an object is easier than the raw integral itself.
- There are a number of different regularization techniques; there is no unique process, as it is purely mathematical. There are pros and cons with all of them.
- After the has been achieved, we are ready to renormalize. In short, renormalization is a process by which the infinities are systematically subtracted away or removed from the final physical answer.
- An example of a diverging diagram is the following:



- This is called the "quark self-energy", and it is denoted by $\Sigma_{ij}(p)$. The modification of the tree level quark propagator to contain *all* radiative corrections (including this one) is given by $\tilde{S}_{ij}(p)$, where

$$\tilde{S}_{ij}(p) = \frac{\delta_{ij}}{m - \not{p} + \Sigma(p)}, \tag{0.1}$$

where

$$\Sigma_{ij}(p) = \delta_{ij}\Sigma(p) \tag{0.2}$$

defines $\Sigma(p)$. We can also represent the total quark propagator as the following Fourier transform:

$$\tilde{S}_{ij} = i\int \mathrm{d}^4x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \langle 0|\, T\overline{\psi}_i(x)\psi_j(0)\,|0\rangle \tag{0.3}$$

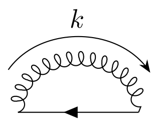
- It can be shown that we can expand this propagator in terms of powers of $\Sigma(p)$:

$$\tilde{S}_{ij} = S_{ij}\left\{\tilde{S}_0 + \tilde{S}_0\Sigma(p)\tilde{S}_0 + \tilde{S}_0\Sigma(p)\tilde{S}_0\Sigma(p)\tilde{S}_0 + \dots\right\} \tag{0.4}$$

- We can identity the first term of this expansion as the ordinary propagator with no corrections, and further terms as those with corrections.

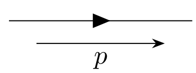
- It will be helpful to define something called 1-particle irreducible diagrams. One of the main things behind regular/renormalization involves classifying together all of these diagrams. Such diagrams are diagrams that cannot be decomposed into two new diagrams by making a "cut". Our Feynman diagram above is 1PI, but if there were a second loop to the right, we could make a cut between the two and create two new, 1-loop diagrams, and hence, such a diagram would not be 1PI.

- Now we can move on to calculating our loop diagram. First, we need to know the following Feynman rules:
  - A loop incurs an integral over the loop's momenta:



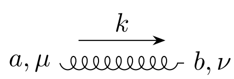
$$\rightarrow \int \frac{\mathrm{d}x^4}{(2\pi)^4 i}$$

  - Normal fermion propagator:



$$\rightarrow \frac{1}{m - \not{p}}$$

  - Normal gluon propagator:

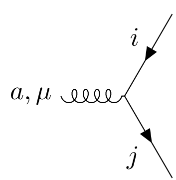


$$\rightarrow \frac{\delta_{ab}}{k^2} d^{\mu\nu}(k)$$

    where $d^{\mu\nu}(k)$ relies on the choice of gauge, but is generally described as

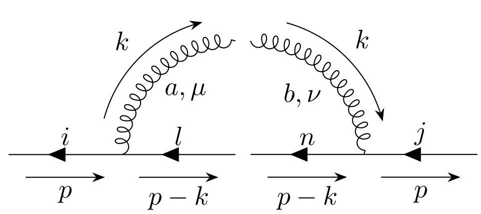
$$d^{\mu\nu}(k) = g^{\mu\nu} - (1 - \alpha)\frac{k^\mu k^\nu}{k^2}. \tag{0.5}$$

  - The QCD fermion/fermion/gluon vertex:



$$\rightarrow \frac{1}{m - \not{p}}$$

- To evaluate the amplitude, we can first imagine cutting a line straight through the center to split the diagram in half, thus giving two vertices, basically.



- First, we have the loop integral:

$$\Sigma_{ij}(p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i} \ \cdots .$$

- Next, we have the vertices:

$$\Sigma_{ij}(p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i}\ (g_s \gamma^\mu T^a_{il})(g_s \gamma^\nu T^b_{nj}) \dots \tag{0.6}$$

- Then, the fermion propagator in between the vertices. However, since during the propagation, the quark doesn't change, i.e. there is no difference between l and n, we need to have a delta function there:

$$\Sigma_{ij}(p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i}\ (g_s \gamma^\mu T^a_{il}) \frac{\delta_{ln}}{m - (\not{p} - \not{k})} (g_s \gamma^\nu T^b_{nj}) \dots \tag{0.7}$$

- Lastly, then, we have the gluon progagator:

$$\Sigma_{ij}(p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i}\ (g_s \gamma^\mu T^a_{il}) \frac{\delta_{ln}}{m - \not{p} + \not{k}} (g_s \gamma^\nu T^b_{nj}) \frac{\delta_{ab}}{k^2} d_{\mu\nu}(k) \tag{0.8}$$

- In the Feynman gauge, we have that $\alpha = 1$, so $d_{\mu\nu}(k) = g_{\mu\nu}$, so

$$\Sigma_{ij}(p) = g_s^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i}\ \gamma^\mu \frac{1}{m - \not{p} + \not{k}} \gamma^\nu \frac{g_{\mu\nu}}{k^2} \left(T^a_{il} \delta_{ln} T^b_{nj} \delta_{ab}\right) \tag{0.9}$$

- The color factor is:

$$T^a_{il} \delta_{ln} T^b_{nj} \delta_{ab} = T^a_{in} T^a_{nj} = (T^a T^a)_{ij} = \delta_{ij} C_F, \tag{0.10}$$

  where C_F is called **Casimir of the fundamental**, and it is defined as:

$$C_F = \frac{N^2 - 1}{N}, \tag{0.11}$$

  where N is the dimensionality of the representation, which for us, in the fundamental representation, is 3. We can also now remove the delta function by virtue of (0.2), so we have

$$\Sigma(p) = g_s^2 C_F \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i}\ \gamma^\mu \frac{1}{k^2 (m - \not{p} + \not{k})} \gamma_\mu \tag{0.12}$$

- The last thing that we can do is the following: when we have a $\not{p}$ in the denominator, we can "multiply and divide" by it to get:

$$\frac{1}{\not{p}} \cdot \frac{\not{p}}{\not{p}} = \frac{\not{p}}{p^2}, \tag{0.13}$$

  so we have now "moved" the operator to the numerator. Doing this here,

$$\Sigma(p) = g_s^2 C_F \int \frac{\mathrm{d}^4 k}{(2\pi)^4 i}\ \frac{\gamma^\mu \left(m + \not{p} - \not{k}\right) \gamma_\mu}{k^2 [m^2 - (p-k)^2]} \tag{0.14}$$

- We can go no further, however, the integral is over k, so if we just look at the powers of k:

$$\Sigma(p) \approx \int \mathrm{d}^4 k\ \frac{\not{k}}{k^4} \approx \int \mathrm{d}^4 k\ \frac{1}{k^3} = \int \mathrm{d}k \mathrm{d}\Omega\ k^3 \frac{1}{k^3} = \int_0^\infty \mathrm{d}k \int \mathrm{d}\Omega. \tag{0.15}$$

- However, the k integral diverges, here, obviously. The way we deal with this is to regularize it, the aforementioned intermediate step. Again, what this does is to make our integral into some convergent function, that diverges only as we take the limit.

- There are a number of methods to do this, we will briefly cover them here:

Regularization Methods

- Cut-off Method

  This method involves simply cutting the integral off at a finite, but still suitably large, momentum in the divergent integrals.

  Pros: It is a very easy method.

  Cons: It breaks a lot of invariances, most importantly translation invariance and gauge invariance. Hence, it is not good for gauge theories.

- Pauli-Villars

  Here, we replace the propagator in the integrand with

$$\frac{1}{m^2 - k^2} - \frac{1}{M^2 - k^2} = \frac{M^2 - m^2}{(m^2 - k^2)(M^2 - k^2)}, \tag{0.16}$$

  which reduces to our original propagator in the limit as $M \to \infty$.

  Pros: This time, translation invariance is kept. Additionally, Lorentz invariance and QED gauge invariance are kept, as well as in massless QCD

  Cons: In massive Yang-Mils gauge theories, gauge invariance is broken. So, this isn't great for the standard model.

- Analytical Regularization

  Here, we put a power on the propagator:

$$\frac{1}{m^2 - k^2} \to \frac{1}{(m^2 - k^2)^\alpha}, \tag{0.17}$$

  where $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 1$. In the limit as $\alpha \to 1$, our original propagator is recovered.

  Pros: This method works very nicely and is used very often to prove whether a theory is renormalizable.

  Cons: Violates gauge invariance, so it isn't great for QCD.

- Lattice Regularization

  This involves discretizing space-time into a lattice, i.e. our Minkowski space is now made of cells of size a. Now, in the configuration space, the arbitrarily small short-distance contribution to the Fourier transform integral is eliminated by capping this short-distance contribution, thus stopping our momentum space region from diverging.

  Pros: Good for non-perturbative calculations (lattice QCD is a good and prominent example).

- Dimensional Regularization (DR)

  This involves handling our divergent multiple-integral by reducing the number of dimensions such that the integrals no longer diverge. For instance, our divergent 4-dimensional integral would not be divergent in 2 dimensions. So, what we do is

$$\int \mathrm{d}^4 k \to \int \mathrm{d}^D k, \ \text{where} D < 4. \tag{0.18}$$

  However, D can be something like $D = 4 - \epsilon$, where ϵ is very small. This makes things very challenging, but it works very well (see pros)

  Pros: Nothing is violated!!

  Cons: Quite challenging: since the space is no longer 4-dimensional, we must be very careful about our algebra.

Dimensional Regularization

Green's Function

This is a super brief coverage of Green's functions and their application into particle physics.

- In the context of QFT, we use Green's functions to compute the expectation value of a time-ordered set of fields:
$$\hat{\phi}(x) = \langle \mathcal{O}| \, T[\phi(x_1)\phi(x_2)\ldots\phi(x_n)] \, |\mathcal{O}\rangle \,. \tag{0.19}$$
This is called the **correlator**.

- The simplest example is the Dirac Propagator:
$$\langle \mathcal{O}| \, \psi_a(x)\overline{\psi}_b(y) \, |\mathcal{O}\rangle \,,$$
where we have a particle going from point x to point y at some later time. There is also the other form:
$$\langle \mathcal{O}| \, \overline{\psi}_b(y)\psi_a(x) \, |\mathcal{O}\rangle \,.$$

- Now we know the solutions of the Dirac equation (expanded as a Fourier transform):
$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E_p}} \sum_s \left(a^{(s)}u^{(s)}(p)e^{-ip\cdot x} + b^{(s)\dagger}v^{(s)}(p)e^{ip\cdot x} \right),$$
where the a's and b's are creation and annihilation operators that satisfy certain commutation relations.

- Now we can calculate the two amplitudes above for particles going from point x to point y:
$$S_R^{ab}(x-y) = \Theta(x^0 - y^0) \, \langle \mathcal{O}| \left\{ \psi_a(x), \overline{\psi}_b(y) \right\} |\mathcal{O}\rangle \,. \tag{0.20}$$
This is called the "retarted" Green's function. We can also define
$$S_R(x-y) = (i\partial\!\!\!/ - m)D_R(x-y). \tag{0.21}$$
WHAT IS GOING ON HERE!

Consequences of using DR

- Returning back to Dimensional Regularization, there are numerous consequences with going to a different dimensionality. The first is that our indices μ no longer go from $0\ldots3$, but rather $0\ldots D-1$. This means that
$$\begin{aligned} g^{\mu\nu}g_{\mu\nu} &= D, \\ &= \gamma^\mu\gamma_\mu = D, \\ &= \gamma^\mu\gamma_\nu\gamma_\mu = (2-D)\gamma_\nu. \end{aligned}$$

- Additionally, in our integral, we had a factor of $1/(2\pi)^4$, but this would instead have to be $1/(2\pi)^D$.

- The alternative consequence is that the traces of the gamma matrices would be really wacky:
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 2^{D/2}\gamma^{\mu\nu}.$$

- So, the convention we use is to "take" the first consequence; i.e. we keep the $1/(2\pi)^D$ factor, but retain the normal 4-dim gamma traces.
    - I could not even begin to explain how/why this works...

Continuing the Calculation

- Back to the self-energy; let's take, for simplicity, $m = 0$ for our quark. Recall that our self-energy amplitude was

$$\Sigma(p) = g_s^2 C_F (2-D) \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\not{k}^2 - \not{p}^2}{k^2 (k-p)^2}.$$

- Now, if this were any space of dimension less than 3, this integral would surely not diverge, but how do we try and go about this; what the hell do we do with a $\mathrm{d}^4 k$?

- First, we use something called **Feynman Parametrization**, where we turn something of the form $1/AB$ into an integral. Specifically:

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{[xA + (1-x)B]^2}. \tag{0.22}$$

It is trivial to show that this integral gives back correctly our $1/AB$. This can also be generalized for any number of factors.

- Doing this for our integral:

$$\Sigma(p) = g_s^2 C_F (2-D) \int \frac{\mathrm{d}^D k}{(2\pi)^D} (\not{k} - \not{p}) \int_0^1 \frac{\mathrm{d}x}{[x(k-p)^2 + (1-x)k^2]^2}.$$

Now, apparently (not sure why!) we can swap the two integral signs (and do a little rearranging) to get

$$\Sigma(p) = g_s^2 C_F (2-D) \int \frac{\mathrm{d}^D k}{(2\pi)^D} \int_0^1 \frac{\not{k} - \not{p}}{[(k-xp)^2 + x(1-x)p^2]^2}.$$

- Since DR preserves translational symmetry, we are allowed to simply make a shift in the momentum and be safe knowing that nothing changes with the underlying physics (I guess we just take this translational symmetry as an axiom?). So, let's do a transformation $k' = k - xp$:

$$\Sigma(p) = g_s^2 C_F (2-D) \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k'}{(2\pi)^D} \frac{\not{k}' - (1-x)\not{p}}{[k' + x(1-x)p^2]^2}.$$

- We now have a sum of two functions. The first function is a function of k', however, we can make the important note that it is an odd function. Since we are integrating momentum over all space ideally, it is technically a symmetric limit, meaning that the entire integral will evaluate to zero. So all we have left is:

$$\Sigma(p) = g_s^2 C_F (D-2)\not{p} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D k'}{(2\pi)^D} \frac{1}{[k'^2 + x(1-x)p^2]^2}.$$

- We can now do what is called a **Wick Rotation**. The basic idea is that this integral would be much easier to handle in Euclidean space (for some reason), but obviously we are not in Euclidean space. The main thing we are looking to eliminate is the fact that in Minkowski space we have that $k'^2 = (k'^0)^2 - |\mathbf{k}'|^2$, but we want everything the same sign: $k'^2 = -(k'^0)^2 - |\mathbf{k}'|^2$. To do this, we can make a complex rotation of the time component of this momentum to add a factor of i. So, let's do the following:

$$k_0' = iK_0, \; K \in \mathbb{R}, \quad \text{and} \quad \mathbf{k}' = \mathbf{K}.$$

Now,

$$k'^2 = -K_0^2 - |\mathbf{K}|^2, \quad \text{and} \quad \mathrm{d}^D k \to i d^D K.$$

So,

$$\Sigma(p) = g_s^2 C_F (D-2)\not{p} \int_0^1 \mathrm{d}x (1-x) \int \frac{\mathrm{d}^D K}{(2\pi)^D} \frac{1}{(K^2 + L)^2}.$$

where $L \equiv -x(1-x)p^2$

- Now that it is Euclidean, we know that it will converge whenever $L > 0$, meaning we want to keep $p^2 < 0$, or space-like.
- To solve this integral, if it were simple 3-dim space, we would use spherical coordinates by saying $d^3\mathbf{k} = d\Omega_3 |\mathbf{k}|^2 d|\mathbf{k}|$, where Ω_3 is the 3-dim solid angle.
- To generalize this to D dimensions, let's look at the coordinate transformations for a few dimensions and we will find a sort of recursion relation. For 3-dim, we have

$$\begin{cases} x^1 &= \cos\theta_1, \\ x^2 &= \sin\theta_1 \cos\theta_2, \\ x^3 &= \sin\theta_1 \sin\theta_2, \end{cases}$$

  where we have three coordinates but need only two angles, and we have taken $|\mathbf{r}| = 1$ for simplicity.
- Now let's visualize this: the first transformation to x^1 grabs the z-axis, which is fine, since z is one of our coordinates. However, going the other way and using sine only projects us onto the 2-dim x-y plane, meaning we need a second angle to project onto the x and y axes themselves.
- 4-dim looks like:

$$\begin{cases} x^1 &= \cos\theta_1, \\ x^2 &= \sin\theta_1 \cos\theta_2, \\ x^3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\ x^4 &= \sin\theta_2 \sin\theta_2 \sin\theta_3. \end{cases}$$

- Here, x^1 remains the same, as we just project onto that axis (maybe not called z anymore). Then, the sine projection will project it onto a 3-dim plane! This time, the cosine projects it onto our z-axis, but the sine one will project again onto the 2-dim x-y plane, after which we can project to the x and y axes themselves.
- We can already see a pattern emerging: cosine projects onto that dimensions "z" axis, then the sine projects onto the $D-1$ "plane". We continue this all the way until we project onto the 2-dim plane, where the sine projects onto the y axis, and we are done.
- So the recursion relation is that for D-dim, we need $D-1$ angles, and we have

$$\begin{cases} x_1 &= \cos\theta_1, \\ &\vdots \\ x_{D-1} &= \sin\theta_1 \ldots \sin\theta_{D-2} \cos\theta_{D-1}, \\ x_D &= \sin\theta_1 \ldots \sin\theta_{D-2} \sin\theta_{D-1}. \end{cases}$$

  Now

$$\begin{aligned} K_0 &= |K| \cos\theta_1, \\ &\vdots, \\ K_{D-1} &= |K| \sin\theta_1 \ldots \sin\theta_{D-1}, \end{aligned}$$

  and

$$d^D K = d\Omega_D |\mathbf{K}|^{D-1} d|\mathbf{K}|,$$

  where

$$d\Omega_D = \prod_{\ell=1}^{D-1} \sin^{D-1-\ell} \theta_\ell \, d\theta_\ell.$$

- We now have that the integral over the D-dim solid angle is

$$\int \mathrm{d}\Omega_D = \int_0^\pi \sin^{D-2}\theta_1 \ \mathrm{d}\theta_1 \ \ldots \int_0^\pi \sin\theta_{D-2} \ \mathrm{d}\theta_{D-2} \int_0^\pi \ \mathrm{d}\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}.$$

- To show this, we can look at a unit sphere in D-dim. First, we look at the normal 1-dim Gaussian integral:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} \ \mathrm{d}x \ \rightarrow \left(\sqrt{\pi}\right)^D = \left(\int_{-\infty}^{\infty} x^{-x^2} \ \mathrm{d}^D x\right).$$

In D-dim, we can take this integral over each independent coordinate:

$$(\sqrt{\pi})^D = \int_{-\infty}^{\infty} e^{-\left(\sum_i^D x_i^2\right)} \ \mathrm{d}^D x.$$

We can now switch to polar coordinates, where defining $x \equiv |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_D^2}$ we get our volume differential to be

$$\mathrm{d}^D x = x^{D-1} \mathrm{d}\Omega_D \mathrm{d}x.$$

Now, using our definition of x:

$$(\sqrt{\pi})^D = \int \mathrm{d}\Omega_D \int x^{D-1} e^{-x^2} \ \mathrm{d}x.$$

Doing a change of variable to x^2, we get:

$$= \int \mathrm{d}\Omega_D \frac{1}{2} \int_0^\infty (x^2)^{(D/2)-1} e^{x^2} \ \mathrm{d}x^2.$$

This looks just like the Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \ \mathrm{d}x, \quad \text{for} \quad z \in \mathbb{C} \text{ with } \mathrm{Re}[z] > 0,$$

so

$$(\sqrt{\pi})^D = \frac{1}{2}\Gamma(D/2) \int \mathrm{d}\Omega_D,$$

meaning

$$\boxed{\int \mathrm{d}\Omega_d = \frac{2\pi^{D/2}}{\Gamma(D/2)}.}$$

- With this, we can now find the total integral:

$$\int \frac{\mathrm{d}^D K}{(2\pi)^D} \frac{1}{(K^2 + L)^2} = \frac{B\left(\frac{L}{2},\ 2 - \frac{D}{2}\right) L^{(D/2)-2}}{(4\pi)^{D-2} \Gamma\left(\frac{D}{2}\right)},$$

where one representation of this **beta function** B is:

$$B(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} \ \mathrm{d}t.$$

- At last, then, we can return to the quark self-energy and insert this result:

$$\sigma(p) = g_s^2 C_F (D-2) \not{p} \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} (-p^2)^{D/2-2},$$

or, in terms of the Beta function:

$$\sigma(p) = \frac{2 C_F g_s^2}{(4\pi)^{D/2}} \not{p} (-p^2)^{D/2-2} (D-1) B\left(\frac{D}{2}, \frac{D}{2}\right) \Gamma\left(2 - \frac{D}{2}\right).$$

- We know have an analytic function in terms of only D!