HW1
PHYS4500: Quantum Field Theory

Casey Hampson

August 18, 2024

Question 1. (2.2)

We are given $p^\mu = \begin{bmatrix} 5m & -4m & 0 & 0 \end{bmatrix}^\intercal$.

a) We know that the 4-momentum has the following property:

$$p_\mu p^\mu = M^2, \tag{1}$$

so

$$p_\mu p^\mu = (5m)^2 - (-4m)^2 = 25m^2 - 16m^2 = 9m^2 = M^2 \tag{2}$$
$$\rightarrow \boxed{M = 3m}. \tag{3}$$

c) This simple, since $p^0 = E$ and we just solved for M:

$$K = E - M = 5m - 3m = 2m. \tag{4}$$
$$\rightarrow \boxed{K = 2m}. \tag{5}$$

d) We apply the Lorentz transformation $p^{\mu\prime} = \Lambda^{\mu\prime}_{\mu} p^\mu$:

$$\begin{bmatrix} p^{0\prime} \\ p^{1\prime} \\ p^{2\prime} \\ p^{3\prime} \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix}. \tag{6}$$

We must determine the value of γ:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{4}{5}\right)^2}} = \frac{1}{\sqrt{1 - \frac{16}{25}}} = \frac{1}{\sqrt{\frac{9}{25}}} = \frac{5}{3}. \tag{7}$$

Now,

$$p^{0\prime} = \gamma p^0 - \beta\gamma p^1 = \frac{5}{3}(5m) - \frac{4}{5}\frac{5}{3}(-4m) = \frac{25}{3}m + \frac{16}{3}m = \frac{41}{3}m, \tag{8}$$
$$p^{1\prime} = -\beta\gamma p^0 + \gamma p^1 = -\frac{4}{5}\frac{5}{3}(5m) + \frac{5}{3}(-4m) = -\frac{20}{m} - \frac{20}{m} = -\frac{40}{3}m, \tag{9}$$
$$p^{2\prime} = p^2 = 0, \tag{10}$$
$$p^{3\prime} = p^3 = 0. \tag{11}$$

So,

$$\boxed{p^{\mu\prime} = \begin{bmatrix} 41m/3 \\ -40m/3 \\ 0 \\ 0 \end{bmatrix}}. \tag{12}$$

Question 2. (3.2)

We are given $A^\mu = \begin{bmatrix} a(t^2 - x^2) & bx^2 & cx & 0 \end{bmatrix}^\intercal$.

a) Using the metric tensor:

$$A_\mu = g_{\mu\nu} A^\nu. \tag{13}$$

This really just involves a negating of the sign on all the spatial terms, so we have:

$$A_\mu = \begin{bmatrix} a(t^2 - x^2) & -bx^2 & -cx & 0 \end{bmatrix}. \tag{14}$$

b) Since the indicies are different, this quantity is a rank-2 tensor:

$$\partial_\mu A_\nu = T_{\mu\nu} = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t}\left[a(t^2-x^2)\right] & \frac{1}{c}\frac{\partial}{\partial t}\left[bx^2\right] & \frac{1}{c}\frac{\partial}{\partial t}\left[cx\right] & 0 \\ \frac{\partial}{\partial x}\left[a(t^2-x^2)\right] & \frac{\partial}{\partial x}\left[bx^2\right] & \frac{\partial}{\partial x}\left[cx\right] & 0 \\ \frac{\partial}{\partial y}\left[a(t^2-x^2)\right] & \frac{\partial}{\partial y}\left[bx^2\right] & \frac{\partial}{\partial y}\left[cx\right] & 0 \\ \frac{\partial}{\partial z}\left[a(t^2-x^2)\right] & \frac{\partial}{\partial z}\left[bx^2\right] & \frac{\partial}{\partial z}\left[cx\right] & 0 \end{bmatrix} = \begin{bmatrix} 2at/c & 0 & 0 & 0 \\ -2ax & 2bx & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

As a note about the problem's construction, the constants in A^μ should have been chosen such that there were not 2 c's. The c in the top left matrix element is the speed of light, while the one in the second row and third column is just a constant.

c) This time we have a contraction, so it will just be a scalar:

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial}{\partial t}\left[a(t^2-c^2)\right] + \frac{\partial}{\partial x}\left[bx^2\right] + \frac{\partial}{\partial y}\left[cx\right] \tag{16}$$

$$= \boxed{\frac{2at}{c} + 2bx} \tag{17}$$

Question 3. (5.4)

We are given a plane-wave solution $\phi(x^\mu) = A\exp(-ik_\mu x^\mu)$ to the Klein-Gordon Equation, where A is a constant.

a) In order to show eventually that the phase speed is greater than c, I'll reintroduce c into the Klein-Gordon equation, but leave $\hbar = 1$:

$$\partial_\mu \partial^\mu \phi + m^2c^2\phi = 0. \tag{18}$$

We can simply plug in our plane wave solution into the Klein-Gordon equation and determine what the components of k^μ must be, then solve for the phase speed. First,

$$\partial_\mu \partial^\mu \left(A\exp(-ik_\mu x^\mu)\right) = -Ak_\mu k^\mu \exp(-ik_\mu x^\mu), \tag{19}$$

so we have that

$$-Ak_\mu k^\mu \exp(-ik_\mu x^\mu) + m^2c^2A\exp(-ik_\mu x^\mu) = 0, \tag{20}$$

$$A\exp(-ik_\mu x^\mu)\left(-k_\mu k^\mu + m^2c^2\right) = 0. \tag{21}$$

Since we don't want the trivial solution where $A = 0$ and since the exponential is never zero,

$$-k_\mu k^\mu + m^2c^2 = 0, \tag{22}$$

$$-\frac{\omega^2}{c^2} + |\boldsymbol{k}|^2 + m^2c^2 = 0, \tag{23}$$

$$\omega^2 = |\boldsymbol{k}|^2c^2 + m^2c^4, \tag{24}$$

$$\omega = \sqrt{|\boldsymbol{k}|^2c^2 + m^2c^4}. \tag{25}$$

Now, by the definition of the phase speed $v_p = \omega/|\boldsymbol{k}|$, we have

$$\omega/|\boldsymbol{k}| \equiv \omega/k = \sqrt{c^2 + \frac{m^2c^4}{k^2}} = c\sqrt{1 + \frac{m^2c^2}{k^2}}. \tag{26}$$

Since the fraction in the exponent will always be positive, the quantity in the square root must always be greater than one, meaning we can safely say that $v_p > c$, as expected.

b) Here, we will take our original expression for ω and simply differentiate with respect to $|\boldsymbol{k}| \equiv k$:

$$v_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}\left[\sqrt{k^2c^2 + m^2c^4}\right], \tag{27}$$

$$= \frac{1}{2}\left(k^2c^2 + m^2c^2\right)^{-1/2} \cdot 2kc^2, \tag{28}$$

$$= \frac{kc^2}{\sqrt{k^2c^2 + m^2c^2}}, \tag{29}$$

$$= \frac{kc}{\sqrt{k^2 + m^2}}. \tag{30}$$

Here, the coefficient of c is always less than 1 due to the additive factor of m^2 in the square root in the denominator. This time, then, $\mathrm{d}\omega/\mathrm{d}k < c$, as expected.