Quantum Chromodynamics (QCD)

We consider $SU(3)$ local gauge transformations $\psi(x) = \begin{pmatrix} \psi_r(x) \\ \psi_b(x) \\ \psi_g(x) \end{pmatrix}$

$\psi(x) \to e^{i\frac{\lambda^a}{2}\theta^a(x)}\psi(x)$ where λ^a are eight Gell-Mann matrices

Thus we have three color charges and eight gluons (gauge bosons)

Each quark flavor $(u,d;c,s;t,b)$ can come in three colors.

Note that in general for $SU(N)$ there are N^2-1 generators of the Lie group

For $SU(3) \to N_c = 3$ so there are $3^2-1=8$ generators – the λ matrices which are 3×3 matrices

$[\lambda^a, \lambda^b] = 2i f^{abc}\lambda^c$ where f^{abc} are the structure constants

The covariant derivative is $D_\mu = \partial_\mu + \frac{ig_s}{2}\sum_{a=1}^{8}\lambda^a G^a_\mu = \partial_\mu + ig_s\frac{\lambda^a}{2}G^a_\mu$

where G^a_μ denote the eight gluon gauge fields (which are massless)

Under the gauge transformation, $G^a_\mu \to G^a_\mu - \frac{1}{g_s}\partial_\mu\theta^a - f_{abc}\theta^b G^c_\mu$

The gluon field tensor is $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s f_{abc} G^b_\mu G^c_\nu$

The QCD Lagrangian (not full yet) is $\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}G^{\mu\nu}_a G^a_{\mu\nu} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{2}g_s\bar{\psi}\gamma^\mu\lambda^a\psi G^a_\mu - \frac{1}{4}G^{\mu\nu}_a G^a_{\mu\nu}$

The λ matrices are $\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c$ The structure constants f^{abc} are completely antisymmmetric $f^{abc} = -f^{bac} = -f^{acb}$ most are zero

$$f^{123} = 1, \quad f^{147} = f^{246} = f^{257} = f^{345} = f^{516} = f^{637} = \frac{1}{2}, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

We can write the generators of SU(3) as $T^a = \frac{\lambda^a}{2}$

Then $[T^a, T^b] = i f^{abc} T^c$

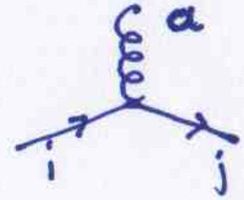
Also $\mathrm{tr}(T^a T^b) = \frac{1}{2} \delta_{ab}$, $T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$

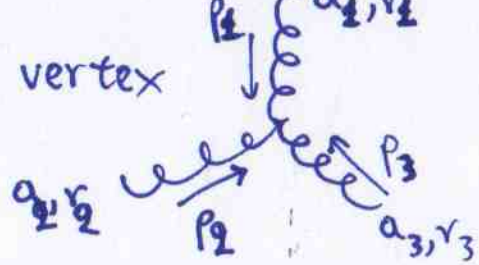
$T^a_{ij} T^a_{jl} = \frac{1}{2}\left(\delta_{il} N_c - \frac{1}{N_c}\delta_{il}\right) = \frac{N_c^2 - 1}{2N_c}\delta_{il} = C_F \delta_{il}$ where $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$

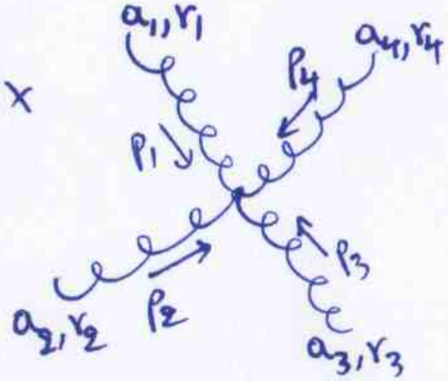
$f^{abc} T^b T^c = \frac{i}{2} C_A T^a$ with $C_A = N_c$, $f^{acd} f^{bcd} = C_A \delta^{ab}$

$f^{ade} f^{bef} f^{cfd} = \frac{N_c}{2} f^{abc}$, $T^d T^a T^d = \left(C_F - \frac{C_A}{2}\right) T^a$

Feynman rules for QCD

quark-gluon vertex  $-i\frac{g_s}{2}\lambda^a_{ji}\gamma^\mu$

three-gluon vertex  $-g_s f^{a_1a_2a_3}\left[g^{r_1r_2}(p_1-p_2)^{r_3}+g^{r_2r_3}(p_2-p_3)^{r_1}+g^{r_3r_1}(p_3-p_1)^{r_2}\right]$

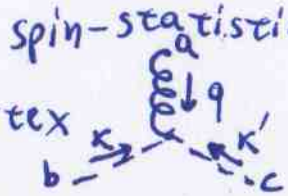
four-gluon vertex 

$$-ig_s^2\Big[f_{a_1a_2c}f_{a_3a_4c}\left(g^{r_1r_3}g^{r_2r_4}-g^{r_1r_4}g^{r_2r_3}\right)$$
$$+f_{a_1a_3c}f_{a_4a_2c}\left(g^{r_1r_4}g^{r_3r_2}-g^{r_1r_2}g^{r_3r_4}\right)$$
$$+f_{a_1a_4c}f_{a_2a_3c}\left(g^{r_1r_2}g^{r_4r_3}-g^{r_1r_3}g^{r_4r_2}\right)\Big]$$

In <u>covariant</u> gauges

→ We also need to introduce 8 "ghost" fields to address unphysical components (extra degrees of freedom) of gluon fields in closed loops 

ghosts only appear in closed loops and have unphysical spin-statistics (spin 0 but Fermi-Dirac)

ghost propagator $a \; \underset{\vec{k}}{- - -} \; b$ $\quad \frac{i\delta_{ab}}{k^2}$ $\quad$ ghost-gluon vertex  $g_s f_{abc} k_\mu$

Feynman rules for QCD

quark propagator $\quad i \xrightarrow[p]{} j \qquad \dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\,\delta_{ij}$

gluon propagator $\quad \nu,a$  $\mu,b \qquad \dfrac{-i\,g_{\mu\nu}}{p^2+i\varepsilon}\,\delta_{ab}$ Feynman gauge

$$\frac{-i}{p^2+i\varepsilon}\left(g_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2+i\varepsilon}\right)\delta_{ab}$$ general covariant (Lorenz) gauge

$$\frac{i}{p^2+i\varepsilon}\left[-g_{\mu\nu}+\frac{p_\mu n_\nu+n_\mu p_\nu}{n\cdot p}-n^2\frac{p_\mu p_\nu}{(n\cdot p)^2}\right]\delta_{ab}$$ physical (axial) gauge - no ghosts

The complete and detailed QCD Lagrangian is (with sum of quark flavors f)

$$\mathcal{L}_{QCD}=\sum_f\left[i\bar{\psi}_f\gamma^\mu D_\mu\psi_f-m\bar{\psi}_f\psi_f\right]-\frac{1}{4}G_a^{\mu\nu}G^a_{\mu\nu}+\mathcal{L}_{\text{gauge-fixing}}+\mathcal{L}_{\text{ghost}}$$

$$=\sum_{f=1}^{n_f=6}\sum_{i,j=1}^{N_c=3}\left[i\bar{\psi}_{f,j}\gamma^\mu D_\mu\psi_{f,i}-m\bar{\psi}_{f,i}\psi_{f,i}\right]-\frac{1}{4}\sum_{a=1}^{N_c^2-1=8}G_a^{\mu\nu}G^a_{\mu\nu}+\mathcal{L}_{\text{gauge-fixing}}+\mathcal{L}_{\text{ghost}}$$

where $D_\mu=\partial_\mu+i\dfrac{g_s}{2}\displaystyle\sum_{a=1}^{N_c^2-1=8}\lambda^a G^a_\mu$, $G^a_{\mu\nu}=\partial_\mu G^a_\nu-\partial_\nu G^a_\mu-g_s f_{abc}G^b_\mu G^c_\nu$

$\mathcal{L}_{\text{gauge-fixing}}=-\dfrac{\xi}{2}\displaystyle\sum_{a=1}^{N_c^2-1=8}(\partial_\mu G^\mu_a)^2$ in covariant gauge ($\xi=1$ in Feynman gauge) $\quad \mathcal{L}_{\text{gauge-fixing}}=-\dfrac{\xi}{2}\displaystyle\sum_{a=1}^{N_c^2-1=8}(n\cdot G_a)^2$ axial gauge with $\xi\to\infty$

and $\mathcal{L}_{\text{ghost}}=\displaystyle\sum_{a,b,d=1}^{N_c^2-1=8}\left[\partial_\mu\bar{c}_a\partial^\mu c_a-g_s f_{abd}(\partial^\mu\bar{c}_a)G^b_\mu c_d\right]$ with $c(x)$ the ghost fields