We introduce a covariant derivative $D_\mu = \partial_\mu + iq\vec{\sigma}\cdot\vec{A}_\mu$

Under the gauge transformation $D_\mu\psi \to [\partial_\mu + iq(U\vec{\sigma}\cdot\vec{A}_\mu U^{-1} + \frac{i}{q}(\partial_\mu U)U^{-1}](U\psi)$

$= (\partial_\mu U)\psi + U\partial_\mu\psi + iqU\vec{\sigma}\cdot\vec{A}_\mu\psi - (\partial_\mu U)\psi = U(\partial_\mu + iq\vec{\sigma}\cdot\vec{A}_\mu)\psi = UD_\mu\psi$

Thus the extended Lagrangian is $i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^\mu\vec{\sigma}\cdot\vec{A}^\mu\psi$

$$= i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi$$

A mass term for the gauge fields $\frac{1}{2}m_A^2\vec{A}^\mu\cdot\vec{A}_\mu$ is not locally gauge invariant so we set $m_A = 0$

We can add a kinetic term $-\frac{1}{4}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}$ where $\vec{F}^{\mu\nu} = \partial^\mu\vec{A}^\nu - \partial^\nu\vec{A}^\mu + 2q\vec{A}^\mu\times\vec{A}^\nu$

Note the self-interaction term for gauge fields

In fact, $2q\vec{A}^\mu\times\vec{A}^\nu = q f_{ijk}A_j^\mu A_k^\nu$ where f_{ijk} are the structure constants of the Lie group $SU(2)$, since the generators of $SU(2)$ are the Pauli spin matrices $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$ and $[\sigma_i, \sigma_j] = if_{ijk}\sigma_k \Rightarrow f_{ijk} = 2\varepsilon_{ijk}$

This term is gauge invariant. Note that $U = e^{i\vec{\sigma}\cdot\vec{\theta}} = 1 + i\vec{\sigma}\cdot\vec{\theta} + \dots = 1 + iq\vec{\sigma}\cdot\vec{\lambda} + \dots$

Then for small $\vec{\lambda}$ we can write the gauge transformation as

$$\vec{A}_\mu \to \vec{A}_\mu - \partial_\mu\vec{\lambda} - 2q\vec{\lambda}\times\vec{A}_\mu$$

Electroweak theory (Glashow-Weinberg-Salam model)

Consider a massless Dirac spinor ψ and write it as $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$ in terms of right-handed (positive helicity) and left-handed (negative helicity) spinors [the helicity $\vec{\sigma}\cdot\hat{p}$ is the component of spin in direction of momentum]

Then $\psi_L = \frac{(1-\gamma_5)}{2}\psi$ and $\psi_R = \frac{(1+\gamma_5)}{2}\psi$ where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in chiral representation

Then the Dirac Lagrangian is $i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\bar{\psi}_L\gamma^\mu\partial_\mu\psi_L$

The electron field (and muon and tau) have both L and R components but the neutrinos only have L.

Consider the doublet $L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ and assign weak isospin $I_W = \frac{1}{2}$ with third component $I_W^3(\nu_{eL}) = +\frac{1}{2}$ and $I_W^3(e_L) = -\frac{1}{2}$

We also have a singlet $R = e_R$ for which the weak isospin is 0.

We thus write the Lagrangian as $i\bar{L}\gamma^\mu\partial_\mu L + i\bar{R}\gamma^\mu\partial_\mu R$

This is invariant under the $\underset{SU(2)}{\text{transformation}}$ $L \to e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}L$ and $R \to R$

as well as under the $U(1)$ transformation $R \to e^{ib}R$ and $L \to e^{ib/2}L$ (weak hypercharge Y_W)

The relation between electric charge Q, third component of weak isospin I_w^3, and weak hypercharge Y_w for each particle is

$$Q = I_w^3 + \frac{Y_w}{2}$$

So for e_L: $\quad -1 = -\frac{1}{2} + \frac{Y_w}{2} \Rightarrow Y_w = -1$

for ν_{eL}: $\quad 0 = \frac{1}{2} + \frac{Y_w}{2} \Rightarrow Y_w = -1$

$\Big\}$ so $Y_w = -1$ for $L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$

For e_R: $\quad -1 = 0 + \frac{Y_w}{2} \Rightarrow Y_w = -2$

The Lagrangian $\mathcal{L} = i\bar{L}\gamma^\mu D_\mu L + i\bar{R}\gamma^\mu D_\mu R = i\bar{e}_L\gamma^\mu D_\mu e_L + i\bar{\nu}_{eL}\gamma^\mu D_\mu \nu_{eL} + i\bar{e}_R\gamma^\mu D_\mu e_R$ is invariant under the $SU(2)\otimes U(1)$ local gauge transformation.

weak isospin $\otimes$ hypercharge

We have three gauge fields $\vec{W}^\mu$ for $SU(2)$ and one gauge field B^μ for $U(1)$

The covariant derivative for $SU(2)$ is $D_\mu^{SU(2)} e_L = \partial_\mu e_L + \frac{i}{2} g \vec{\sigma}\cdot\vec{W}_\mu e_L$

with g the $SU(2)$ coupling $\quad$ and $D_\mu^{SU(2)} \nu_{eL} = \partial_\mu \nu_{eL} + \frac{i}{2} g \vec{\sigma}\cdot\vec{W}_\mu \nu_{eL}$

For $U(1)$ we have $D_\mu^{U(1)} e_L = \partial_\mu e_L - \frac{i}{2} g' B_\mu e_L$, $D_\mu^{U(1)} \nu_{eL} = \partial_\mu \nu_{eL} - \frac{i}{2} g' B_\mu \nu_{eL}$

with g' the $U(1)$ coupling $\quad$ and $D_\mu^{U(1)} e_R = \partial_\mu e_R - i g' B_\mu e_R$

Then the full covariant derivative for $SU(2)\otimes U(1)$ is

$$D_\mu e_L = \partial_\mu e_L + \frac{i}{2} g \vec{\sigma}\cdot\vec{W}_\mu e_L - \frac{i}{2} g' B_\mu e_L \quad \text{(same for } \nu_{eL}\text{)}$$

and $D_\mu e_R = \partial_\mu e_R - i g' B_\mu e_R$

Including the kinetic terms for the gauge fields, the Lagrangian becomes

$$\mathcal{L}' = i\bar{e}_L \gamma^\mu D_\mu e_L + i\bar{\nu}_{eL}\gamma^\mu D_\mu \nu_{eL} + i\bar{e}_R \gamma^\mu D_\mu e_R - \frac{1}{4}\vec{W}^{\mu\nu}\cdot\vec{W}_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}$$

where $\vec{W}^{\mu\nu} = \partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu + g\vec{W}^\mu \times \vec{W}^\nu$ and $B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$

At this point all spinor and gauge fields are massless.

Then we introduce a scalar Higgs field $\varphi = \begin{pmatrix}\varphi^+ \\ \varphi^0\end{pmatrix}$ with φ^+, φ^0 complex with weak isospin $I_W = \frac{1}{2}$ and weak hypercharge $Y_W = 1$

The covariant derivative is $D_\mu \varphi = \partial_\mu \varphi + \frac{i}{2} g\vec{\sigma}\cdot\vec{W}_\mu \varphi + \frac{i}{2} g' B_\mu \varphi$

The Lagrangian terms involving φ are

$$\mathcal{L}_\varphi = (D_\mu\varphi)^\dagger D^\mu \varphi - m^2 \varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 - G(\bar{L}\varphi R + \bar{R}\varphi^\dagger L)$$

where $\bar{L}\varphi R + \bar{R}\varphi^\dagger L = \bar{\nu}_{eL} e_R \varphi^+ + \bar{e}_L e_R \varphi^0 + \bar{e}_R \nu_{eL}\varphi^- + \bar{e}_R e_L \bar{\varphi}^0$