HW7

PHYS4500: Quantum Field Theory

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Problem 1.

We are to show that

$$S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{1.1}$$

is a Green's function for the position-space Dirac operator; that is, we want to show:

$$(i\gamma^\mu \partial_\mu - m)\, S(x-y) = i\delta^4(x-y). \tag{1.2}$$

First, we can make the simplification we made in class where we note that since $(\not{p}+m)(\not{p}-m) = p^2 - m$, we can rewrite the denominator such that we cancel the $\not{p}+m$ on top and get (also taking $\epsilon \to 0$):

$$S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{\not{p}-m} e^{-ip\cdot(x-y)}. \tag{1.3}$$

Now, taking the 4-gradient with respect to x:

$$i\gamma^\mu \partial_\mu S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} i \frac{i\gamma^\mu}{\not{p}-m} (-i) p_\mu e^{-ip\cdot(x-y)} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i\not{p}}{\not{p}-m} e^{-ip\cdot(x-y)}.$$

Next,

$$mS(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{im}{\not{p}-m} e^{-ip\cdot(x-y)},$$

so

$$(i\gamma^\mu \partial_\mu - m)\, S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}-m)}{\not{p}-m} e^{-ip\cdot(x-y)}.$$

Technically, since these are matrices, we can't just "cancel" them, but rather we could multiply top and bottom by $\not{p}+m$, after which we will get $p^2 - m^2$, a scalar, which we can then cancel. Either way, we get

$$(i\gamma^\mu \partial_\mu - m)\, S(x-y) = i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)} = i\delta^4(x-y),$$

as expected.

Problem 2.

a) First:

$$\begin{aligned}
\not{p}\not{q} + \not{q}\not{p} &= \gamma^\mu p_\mu \gamma^\nu q_\nu + \gamma^\nu q_\nu \gamma^\mu p_\mu \\
&= p_\mu q_\nu (\gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu) \\
&= 2g^{\mu\nu} p_\mu q_\nu = \boxed{2p\cdot q}
\end{aligned}$$

b)

$$\gamma^\mu \gamma_\mu = \frac{1}{2}(\gamma^\mu \gamma_\mu + \gamma^\nu \gamma_\nu) = \frac{1}{2} g_{\mu\nu}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g_{\mu\nu} g^{\mu\nu} = \boxed{4.}$$

c)

$$\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\mu &= \gamma_\mu (\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu - \gamma^\mu \gamma^\nu), \\
&= \gamma_\mu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) \\
&= 2g^\nu - \gamma_\mu \gamma^\mu \gamma^\nu = 2\gamma^\nu - 4\gamma^\nu = \boxed{-2\gamma^\nu.}
\end{aligned}$$

Since the components of the 4-momentum are just scalars, we can move them around however we want to easily say

$$\gamma_\mu \not{p} \gamma^\mu = \gamma_\mu \gamma^\nu \gamma^\mu p_\nu = -2\gamma^\nu p_\nu = \boxed{-2\not{p}.}$$

Problem 3. (20.1a)

We are to show that

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0. \tag{3.1}$$

First, if $\mu = \nu$, then we have a square of a gamma matrix which is ± 1, so we can say

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = \pm\mathrm{Tr}[\gamma^5].$$

We calculated γ^5 in the test:

$$\mathrm{Tr}[\gamma^5] = \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0.$$

If $\mu \neq \nu$, then we can expand out γ^5:

$$\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = i\mathrm{Tr}[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\gamma^\nu].$$

Now, the two indices are different, and using the main anti-commutation relation for the gammas, we can cycle γ^μ through until it reaches the gamma matrix it matches. For instance, if $\mu = 1$, we use the anti-commutation relation twice to cycle it through so that it lays right next to γ^1. Since it therefore is different from the ones it moves through, we only pick up negatives. Then, we have its square, basically, which is also just ± 1. What we have done, then, is eliminate $\gamma^\mu\gamma^\nu$ as well as two of the gammas from γ^5, and picked up a ± 1. The two leftover gammas must necessary be different, and again using the main anti-commutation relation, this trace is zero.

These two cases exhaust all possible index combinations, so we can safely say

$$\boxed{\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0.}$$