HW12
PHYS4500: Quantum Field Theory

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Problem 1.

We are to show that the Dirac Lagrangian

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - \overline{\psi} M \psi \tag{1.1}$$

with

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} m_a & 0 \\ 0 & m_b \end{pmatrix} \tag{1.2}$$

simplifies to the sum of the individual Dirac Lagrangians for the individual spinors ψ_a and ψ_b. First, we know that the quantity $\gamma^\mu \partial_\mu$ is a matrix in spin-space, call it Γ_{ij}, where the i and j indices run from $1-4$ in spin space. Then, we can explicitly write out the indices in order to manipulate them more easily:

$$\overline{\psi}\gamma^\mu \partial_\mu \psi \to \begin{pmatrix} \overline{\psi}_a & \overline{\psi}_b \end{pmatrix}_i \Gamma_{ij} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}_j \tag{1.3}$$

$$= \begin{pmatrix} \overline{\psi}_{a,i}\Gamma_{ij} & \overline{\psi}_{b,i}\Gamma_{ij} \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}_j \tag{1.4}$$

$$= \overline{\psi}_{a,i}\Gamma_{ij}\psi_{a,j} + i\overline{\psi}_{b,i}\Gamma_{ij}\psi_{b,j}, \tag{1.5}$$

or, since our indices are in valid matrix multiplication order, we can remove them to say:

$$\overline{\psi}\gamma^\mu \partial_\mu \psi = \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \overline{\psi}_b \gamma^\mu \partial_\mu \psi_b. \tag{1.6}$$

For the mass part:

$$\overline{\psi} M \psi = \begin{pmatrix} \overline{\psi}_a & \overline{\psi}_b \end{pmatrix} \begin{pmatrix} m_a & 0 \\ 0 & m_b \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} m_a\overline{\psi}_a & m_b\overline{\psi}_b \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = m_a \overline{\psi}_a \psi_a + m_b \overline{\psi}_b \psi_b. \tag{1.7}$$

Thus, both parts have nicely separated out, so our new Lagrangian is:

$$\mathcal{L} = i\overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + i\overline{\psi}_b \gamma^\mu \partial_\mu \psi_b - m_a \overline{\psi}_a \psi_a - m_b \overline{\psi}_b \psi_b, \tag{1.8}$$

as desired.

Problem 2. (23.1)

We start by considering a doublet ψ and a generic 2×2 matrix M that acts on the doublet.

a) First, we consider M such that we have $\overline{\psi}'\psi' = \overline{\psi}\psi$. In order for this to be satisfied, M must be unitary. This can be easily shown by simply writing out the transformations explicity:

$$\overline{\psi}'\psi' = \overline{\psi} M^\dagger M \psi. \tag{2.1}$$

For this to be equal to $\overline{\psi}\psi$, we must have $M^\dagger M = 1$, which is exactly what the unitary condition requires. Thus, M must be unitary.

b) Now we are to show the determinant of these M matrices must be 1. First, we know the matrix identity $\det AB = (\det A)(\det B)$. For us, we can choose $A = M^\dagger$ and $B = M$ so $\det M^\dagger M = \det 1 = 1 = (\det M^\dagger)(\det M)$. Now, we must show that $\det M^\dagger = (\det M)^*$. Consider an arbitrary complex 2×2 matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.2}$$

where a, b, c and d are complex numbers. Now,

$$U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \tag{2.3}$$

It is quite easy to see that $\det U = ad - bc$ and that $\det U^\dagger = a^* d^* - b^* c^* = (ad - bc)^* = (\det U)^*$. Thus,

$$1 = (\det M)^*(\det M) \to |\det M|^2 = 1 \to |\det M| = 1. \tag{2.4}$$

c) For a matrix M with $\det M = e^{i\alpha}$, we can define a new matrix $M_{\text{new}} = e^{-i\alpha/2} M$. Using the same matrix identity as in the previous part, we have that $\det M_{\text{new}} = (\det e^{-i\alpha/2})(\det M)$. Now,

$$\det e^{-i\alpha/2} = \det \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} = e^{-i\alpha}. \tag{2.5}$$

Therefore, we have that $\det M_{\text{new}} = e^{-i\alpha} e^{i\alpha} = 1$.

f) We have used before that elements of $SU(2)$ can be expressed as the exponential of the algebra of the group, something like (in our notation from class)

$$M = e^{-ig\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}/2}. \tag{2.6}$$

We can easily show that this satisfies the $\det M = 1$ constraint meaning M is an element of $SU(2)$ (obviously it is unitary). With the matrix identity that $\det e^A = e^{\text{Tr}[A]}$, we have that

$$\det M = e^{-\frac{i}{2} g \text{Tr}[\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}]}. \tag{2.7}$$

The trace deals with elements down the diagonal. Therefore, the trace is effectively $\text{Tr}[\sigma_z \lambda_z] \to \lambda_z \text{Tr}[\sigma_z]$ since only the third/z Pauli matrix contains elements that are on the main diagonal. Now

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.8}$$

so it's trace is obviously zero, meaning

$$\det M = e^0 = 1, \tag{2.9}$$

as expected of an element of $SU(2)$.