Least-action principle and Lagrangians

We start with classical mechanics and consider N particles with generalized coordinates $q_i(t)$, where $i=1,\dots,N$, and their derivatives $\dot{q}_i = \frac{dq_i}{dt}$

Define the Lagrangian $L(q_1,\dots,q_N;\dot{q}_1,\dots,\dot{q}_N;t) = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{q}_i^2 - U(q_1,\dots,q_N)$

i.e. kinetic minus potential energy   simply denote it as $L(q,\dot{q},t)$

Principle of least action (Hamilton's principle)

The action $S = \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$ has a minimum (extremum) value for the physical trajectory of a particle between $q(t_1)$ and $q(t_2)$

Let $q(t)$ be the function for which S is a minimum, and consider variation of the path: $q(t) \to q(t) + \delta q(t)$ Then $\delta S = 0$

But $\delta S = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)\,dt - \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt = \delta \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right) dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q\,dt + \underbrace{\left(\frac{\partial L}{\partial \dot{q}}\delta q \right)\Bigg|_{t_1}^{t_2}}_{0 \text{ since } \delta q(t_1)=\delta q(t_2)=0} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q\,dt$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\,dt$$

Then $\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$ Euler-Lagrange equation

For N particles there are N Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad \text{with } i = 1, \dots, N$$

With $L = \frac{1}{2} m\dot{q}^2 - U(q)$ we have $\frac{d}{dt}(m\dot{q}) = -\frac{\partial U}{\partial q} \Rightarrow m\frac{d\dot{q}}{dt} = -\frac{\partial U}{\partial q}$

$\Rightarrow m\frac{dv}{dt} = -\frac{\partial U}{\partial q} \Rightarrow ma = F$ where $F = -\frac{\partial U}{\partial q} = \frac{\partial L}{\partial q}$ So we get Newton's second law of motion

Also define conjugate momentum $p = \frac{\partial L}{\partial \dot{q}}$ hence $p = m\dot{q} = mv$ as expected

For N particles, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and $F_i = -\frac{\partial U}{\partial q_i} = \frac{\partial L}{\partial q_i}$ with $i = 1, \dots, N$

so $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \Rightarrow \frac{d}{dt} p_i = F_i \Rightarrow F_i = \dot{p}_i$ Newton's second law

The Hamiltonian is $H(p_1, \dots, p_N; q_1, \dots, q_N) = \sum_{i=1}^{N} p_i \dot{q}_i - L$

substituting L above this gives $\sum_{i=1}^{N} \frac{1}{2} m_i \dot{q}_i^2 + U$ which is kinetic plus potential energy

Example: simple harmonic oscillator $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

Then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}(m\dot{x}) = -kx \Rightarrow m\ddot{x} = -kx$ with $\ddot{x} = d^2x/dt^2 = d\dot{x}/dt$

Also $p = \partial L/\partial \dot{x} = m\dot{x}$ and $F = \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = -kx$ so $F = \dot{p} = dp/dt$

Action and Lagrangian for a free relativistic particle

In relativity the action must be invariant under Lorentz transformations so it must be a scalar

Action $S = -A\int_a^b ds$ over interval ds along the world line where A is a constant to be determined

But also $S = \int_{t_1}^{t_2} L\,dt$ and $ds = c\,dt_p = c\,dt\sqrt{1-\frac{v^2}{c^2}}$ with t_p the proper time

Thus $S = -A\int_{t_1}^{t_2} c\sqrt{1-\frac{v^2}{c^2}}\,dt = \int_{t_1}^{t_2} L\,dt \Rightarrow L = -Ac\sqrt{1-\frac{v^2}{c^2}}$

We can determine A by considering the non-relativistic limit $\frac{v}{c} \ll 1$

As $\frac{v}{c} \to 0$, $L \to -Ac\left(1-\frac{v^2}{2c^2}\right) = -Ac + \frac{Av^2}{2c}$ The term $-Ac$ is a constant while the term $\frac{Av^2}{2c}$ gives the non-relativistic expression $\frac{1}{2}mv^2$ if $A = mc$

Thus we find $L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$ and $S = -mc^2\int_{t_1}^{t_2}\sqrt{1-\frac{v^2}{c^2}}\,dt$

Note that $\dot{x}^\mu = \frac{d}{dt}(ct, x, y, z) = (c, \dot{x}, \dot{y}, \dot{z})$

so $\dot{x}^\mu \dot{x}_\mu = c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 = c^2 - v^2 = c^2\left(1-\frac{v^2}{c^2}\right) \Rightarrow \sqrt{\dot{x}^\mu \dot{x}_\mu} = c\sqrt{1-\frac{v^2}{c^2}}$

So $S = -mc\int_{t_1}^{t_2}\sqrt{\dot{x}^\mu \dot{x}_\mu}\,dt = -mc\int_{t_1}^{t_2}\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,dt$

Least-action principle and Lagrangians for fields

In going from a particle-centered formalism, with position $\vec{x}(t)$, to a field formalism, with field $\varphi(x^\mu) = \varphi(\vec{x}, t)$, we essentially "replace" x by φ and t by x^μ

The action $S = \int L\, dt$ with $L = \int \mathcal{L}(\varphi, \partial_\mu \varphi) d^3x$

where $\mathcal{L}$ is the Lagrangian density $\rightarrow$ we will simply call it Lagrangian

Then we have $S = \int \mathcal{L}(\varphi, \partial_\mu \varphi)\, d^4x$

Least-action principle: Consider the variation $\varphi(x^\mu) \rightarrow \varphi(x^\mu) + \delta\varphi(x^\mu)$
Then $\delta S = 0$

But $\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right] d^4x$

$$= \int \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi\, d^4x + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \Big|_{\text{boundary}}}_{0 \text{ since } \delta\varphi = 0 \text{ on boundary}} - \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi\, d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi\, d^4x$$

Then $\delta S = 0 \Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi}$ Euler-Lagrange equation for fields

For N fields there are N Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_i} \quad \text{with } i=1,\dots,N \quad \left(\begin{array}{l}\text{compare} \\ \text{with particles}\end{array} \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \right)$$

We also define conjugate momenta

$$\pi_i(x^\mu) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_i)} \quad \text{or} \quad \pi_i(x^\mu) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i(x^\mu)} \quad \text{where } \dot{\varphi}_i = \partial_0 \varphi_i$$

Use natural units: $\hbar = c = 1$ from now on

Klein-Gordon field $\varphi(x^\mu)$
a single real scalar field

Lagrangian $$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi}_{\text{"kinetic term"}} - \underbrace{\frac{1}{2} m^2 \varphi^2}_{\text{"mass term"}}$$

Then $$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi} \Rightarrow \partial_\mu(\partial^\mu \varphi) = -m^2 \varphi \Rightarrow \partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$$

Klein-Gordon equation

Complex scalar field
$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$
$$\varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$$

$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi^* \varphi$ Then $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi} \Rightarrow \partial_\mu \partial^\mu \varphi^* = -m^2 \varphi^*$ and $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi^*}$

$$\Rightarrow \partial_\mu \partial^\mu \varphi = -m^2 \varphi$$