Regularization

We estabilished Feynman Rules for QED and QCD

⇒ Able to make perturbative calculations of Xsec for arbitrary processes

* We have seen tree-level processes in QCD with quarks and gluons

⇒ lowest-order calculations reproduce the parton-model results, however, the dynamical effects of QCD do not appear at tree level.

⇒ It is essential to deal with higher order corrections in perturbation theory i.e. loops and more legs!

* loop contributions to amplitudes diverge
Renormalization program needed.

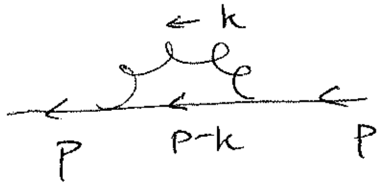
* Intermediate stage: Regularization

Renormalization: divergences are subtracted at in the final physical answer on the basis of renormalization.

We still require, at intermediate stage, that divergent integrals are mathematically manageable. The procedure that makes divergent integrals tentatively finite by introducing a suitable convergence device is generically called regularization.

Regularization: pure mathematical technique, not unique, has no physical consequences.

specific example of diverging diagrams ③



quark self energy $\Sigma_{ij}(p)$

quark propagator at all orders $\tilde{S}_{ij}(p)$

→ full propagator which includes all the radiative corrections

$$\tilde{S}_{ij}(p) = \frac{\delta_{ij}}{m - \not{p} - \Sigma(p)} \qquad 1)$$

$$\Sigma_{ij}(p) = \delta_{ij}\, \Sigma(p)$$

$$\tilde{S}_{ij}(p) = i \int d^4x\, e^{-ip\cdot x} \langle 0 | T(\psi_i(x)\, \bar{\psi}_j(0)) | 0 \rangle_c$$

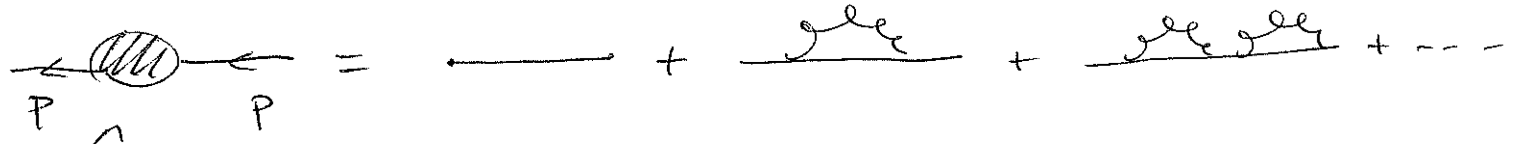
c = connected pieces.

$\Sigma(p)$ self energy part is one-particle irriducible, the propagator is not:

1) Can be expanded in powers of $\Sigma(p)$

$$\tilde{S}_{ij}(p) = \delta_{ij} \Big\{ \tilde{S}_0(p) + \tilde{S}_0(p)\, \Sigma(p)\, \tilde{S}_0(p) + \tilde{S}_0(p)\, \Sigma(p)\, \tilde{S}_0(p)\, \Sigma(p)\, \tilde{S}_0(p) + \dots \Big\}$$

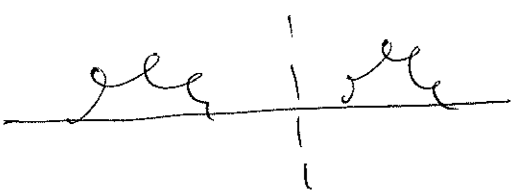
$$\tilde{S}_0(p) = \frac{1}{m - \not{p}}$$ tree-level propagator



$\tilde{S}_{ij}(p)$

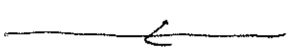
<u>one-particle irriducible</u> (1PI): it's a diagram that cannot become 2 nontrivial diagrams by cutting a single line.

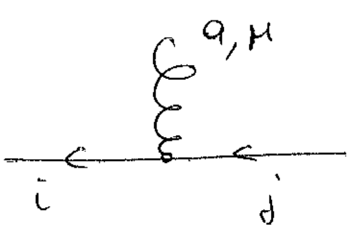
 is 1PI

 is not 1PI

<u>Feynmal rules</u>

$\longrightarrow \int \frac{d^4k}{(2\pi)^4 i}$


$\rightarrow \frac{1}{m - \not{p}}$



$g_s \gamma_\mu T^a_{ij}$

Quark self energy to order g_s^2

$$\Sigma_{ij}(p) = \int \frac{d^4k}{(2\pi)^4 i} \, g_s \gamma_\mu T^a_{il} \, \frac{\delta_{ln}}{m - \not{p} + \not{k}} \, g_s \gamma_\nu T^b_{nj} \, \frac{\delta_{ab}}{k^2} \, d^{\mu\nu}(k)$$

$$d_{\mu\nu}(k) = g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}$$

Color factors

$$T^a_{il}\,\delta_{ln}\,T^b_{nj}\,\delta^{ab} = T^a_{in} T^a_{nj} = \left(T^a T^a\right)_{ij} = \delta_{ij} C_F$$

$C_F = \frac{N^2-1}{2N}$ ⇒ casimir of the fundamental representation of SU(3).

$N = 3$

It labels all the irriducible representations of SU(3).

$$\Sigma(p) = g_s^2 C_F \int \frac{d^4k}{(2\pi)^4 i} \, \frac{\gamma_\mu (m + \not{p} - \not{k}) \gamma_\nu}{k^2\left[m^2 - (p-k)^2\right]} \, d^{\mu\nu}(k)$$

For simplicity, let's consider the Feynman gauge ⇒ $\alpha = 1$ ⇒ $d^{\mu\nu}(k) = g^{\mu\nu}$

$$\Sigma(p) = g_s^2 C_F \int \frac{d^4k}{(2\pi)^4 i} \, \frac{\gamma_\mu (m + \not{p} - \not{k}) \gamma^\mu}{k^2\left(m^2 - (p-k)^2\right)}$$

The 4-dimensional integral in $\Sigma(p)$ is linearly divergent. By doing simple power counting we see that

$$\int \frac{d^4k}{k^2} \frac{\not{k}}{k^2} \sim \lim_{k\to\infty} k$$

The divergence comes from the high-momentum region $|k| \to \infty$. We need to regularize it, that is, we must write it as a suitable limit of a convergent integral.

<u>Cut-off method</u>

Simplest method: the high-momentum region is cut off in the divergent integrals.

Cons: · it breaks translation invariance $\Rightarrow$ a shift in the momentum of the integral changes the result.

· breaks gauge invariance.

$\Rightarrow$ not good for gauge theories

Pauli - Villars

The integrand propagator is replaced by

$$\frac{1}{m^2-k^2} - \frac{1}{M^2-k^2} = \frac{M^2-m^2}{(m^2-k^2)(M^2-k^2)}$$

which reduces to the original propagator when $M \to \infty$

pros: translation and Lorentz invariance maintained. Gauge invariance in QED is preserved. Can be applied to massless QCD only

Cons: it does not maintain gauge invariance in massive Yang-Mills gauge theorres (like QCD with quark masses $\neq 0$)
Not good for the SM!

Analitical regulerization

$$\frac{1}{(m^2-k^2)} \longrightarrow \frac{1}{(m^2-k^2)^\alpha}$$

$\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 1$

In the limit $\alpha \to 1$ the original propagator is recovered.

Pros: extensively used for the proof of renormalizability of a theory.

Cons: Violates gauge invariance $\Rightarrow$ not good for QCD.

Lattice regularization

Here the space-time is discretized. That is, the Minkowski space is made of small cells of size a.

$\Rightarrow$ in the x-space or coordinate space the short-distance contribution to the space-time integration is eliminated.

In the momentum space, this means that we are cutting off the high-momentum region $\Rightarrow$ convergent momentum integral.

pros: good for non-perturbative calculations, e.g., configuration integrals in the functional integrals in QFT

Dimensional Regularization (DR)

A divergent multiple integral is made convergent by reducing the number of multiple integrals.

For example:

divergent 4-dim integral $\int d^4k \frac{k}{k^2 k^2}$

would be finite if the space-time were 2-dim!

Therefore, in dimensional regularization

$$\int d^4k \rightarrow \int d^Dk \qquad D < 4$$

$\Rightarrow$ we obtain the result of the integral in terms of analytic expressions as functions of D.

pros: in dimensional reg. or DR nothing is violated: gauge, Lorentz, unitarity invariant

cons: the space-time is not 4-dim.
Care must be given to the algebra in D-dim.