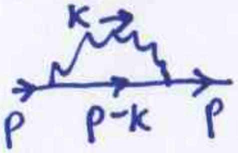
# Electron self-energy at one loop and renormalization




$$= \Sigma(p) = \frac{e^2}{8\pi^2\varepsilon}(\not{p} - 4m) + O(\varepsilon^0)$$

electron propagator $\xrightarrow[p]{} = S_0(p) = \dfrac{i(\not{p}+m)}{p^2-m^2} = \dfrac{i}{\not{p}-m}$

dressed propagator 

$$S(p) = S_0(p) + S_0(p)\,i\Sigma(p)S_0(p) + S_0(p)\,i\Sigma(p)S_0(p)\,i\Sigma(p)S_0(p) + \ldots$$

$$\Rightarrow S(p) = S_0(p)\left[1 + i\Sigma(p)S_0(p) + i\Sigma(p)S_0(p)\,i\Sigma(p)S_0(p) + \ldots\right]$$

$$= S_0(p)\left[1 - i\Sigma(p)S_0(p)\right]^{-1} = \left[S_0^{-1}(p) - i\Sigma(p)\right]^{-1} = \frac{1}{S_0^{-1}(p) - i\Sigma(p)}$$

$$\Rightarrow S(p) = \frac{1}{-i(\not{p}-m) - i\Sigma(p)} = \frac{i}{\not{p} - m + \Sigma(p)}$$

Thus $S^{-1}(p) = -i(\not{p}-m) - i\Sigma(p) = -i(\not{p}-m) - i\dfrac{e^2}{8\pi^2\varepsilon}(\not{p}-4m) + O(\varepsilon^0)$

We choose the Minimal Subtraction (MS) scheme where we drop all constant, i.e. $O(\varepsilon^0)$ terms. Another popular choice is modified MS ($\overline{MS}$) scheme where in addition to $1/\varepsilon$ terms we also keep $\ln(4\pi) - \gamma_E$ terms.

We will add counterterms to the Lagrangian to deal with the infinities and thus renormalize the electron field and the electron mass.

In MS scheme $S^{-1}(p) = -i(\not{p}-m) - i\frac{e^2}{8\pi^2\varepsilon}(\not{p}-4m) = -i\not{p}\left(1+\frac{e^2}{8\pi^2\varepsilon}\right) + im\left(1+\frac{e^2}{2\pi^2\varepsilon}\right)$

The QED Lagrangian is $\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{interaction} + \mathcal{L}_{gauge}$

where $\mathcal{L}_{Dirac} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$ and $\mathcal{L}_{interaction} = -q\bar{\psi}\gamma^\mu\psi A_\mu$

and $\mathcal{L}_{gauge} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$

We add counterterms $iB\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi$ to $\mathcal{L}_{Dirac}$ with $B = -\frac{e^2}{8\pi^2\varepsilon}$ and $M = -\frac{me^2}{2\pi^2\varepsilon}$

thus getting $\mathcal{L}_{Dirac} + \mathcal{L}_{counterterms} = i(1+B)\bar{\psi}\gamma^\mu\partial_\mu\psi - (m+M)\bar{\psi}\psi = iZ_\psi\bar{\psi}\gamma^\mu\partial_\mu\psi - Z_\psi m_b\bar{\psi}\psi$

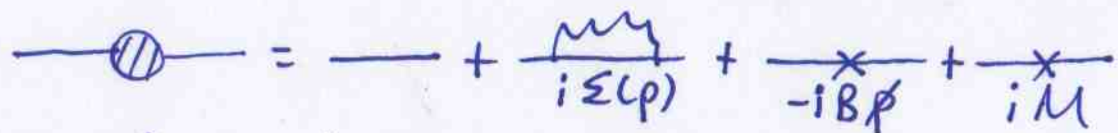
where $Z_\psi = 1+B = 1-\frac{e^2}{8\pi^2\varepsilon}$ and $\underset{\text{(bare mass)}}{m_b} = Z_\psi^{-1}(m+M) = \left(1-\frac{e^2}{8\pi^2\varepsilon}\right)^{-1}\left(m-\frac{me^2}{2\pi^2\varepsilon}\right)$

$$\Rightarrow m_b = \left(1+\frac{e^2}{8\pi^2\varepsilon}+\dots\right)m\left(1-\frac{e^2}{2\pi^2\varepsilon}\right) = m\left(1-\frac{3e^2}{8\pi^2\varepsilon}\right) = m+\Delta m$$

bare field $\psi_b = \sqrt{Z_\psi}\,\psi$ The observed mass is $m$, while $m_b$ is infinite.

Then bare Dirac Lagrangian is $\mathcal{L}_{b\,Dirac} = \mathcal{L}_{Dirac} + \mathcal{L}_{counterterms} = i\bar{\psi}_b\gamma^\mu\partial_\mu\psi_b - m_b\bar{\psi}_b\psi_b$

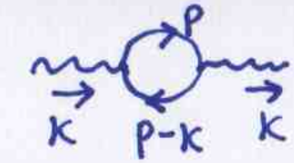
We have renormalized the field and the mass of the electron.

Also, schematically, 

at one loop

$$S^{-1}_{ren}(p) = -i(\not{p}-m) - \frac{ie^2}{8\pi^2\varepsilon}(\not{p}-4m) - iB\not{p} + iM = -i(\not{p}-m)$$

## Photon self-energy diagram (vacuum polarization)




$= \Pi^{\mu\nu}(k)$

$$i\Pi^{\mu\nu}(k) = \int \frac{d^n p}{(2\pi)^n} (-1) \frac{\mathrm{tr}[(-ie\gamma^\mu)\, i(\not p + m)(-ie\gamma^\nu)\, i(\not p - \not k + m)]}{(p^2-m^2)\,[(p-k)^2-m^2]}$$

$$\Rightarrow \Pi^{\mu\nu}(k) = \frac{ie^2}{(2\pi)^n}\int_0^1 dz \int d^n p \, \frac{\mathrm{tr}[\gamma^\mu(\not p+m)\gamma^\nu(\not p-\not k+m)]}{[(p-k)^2 z - m^2 z + (p^2-m^2)(1-z)]^2}$$

$$= \frac{ie^2}{(2\pi)^n}\int_0^1 dz \int d^n p \, \frac{\mathrm{tr}[\gamma^\mu(\not p+m)\gamma^\nu(\not p-\not k+m)]}{[(p-kz)^2 + k^2 z(1-z) - m^2]^2}$$

Now set $p' = p - kz$

Then $$\Pi^{\mu\nu}(k) = \frac{ie^2}{(2\pi)^n}\int_0^1 dz \int d^n p' \, \frac{\mathrm{tr}[\gamma^\mu(\not p' + \not k z + m)\gamma^\nu(\not p' + \not k z - \not k + m)]}{[p'^2 + k^2 z(1-z) - m^2]^2}$$

The odd terms in $\not p'$ in the trace give zero contribution to the integral.
The remaining terms are: $\mathrm{tr}[\gamma^\mu \not p' \gamma^\nu \not p' + \gamma^\mu(\not k z + m)\gamma^\nu(\not k(z-1)+m)]$

$$= \mathrm{tr}[\gamma^\mu p'_\rho \gamma^\rho \gamma^\nu p'_\sigma \gamma^\sigma + \gamma^\mu k_\rho \gamma^\rho z \gamma^\nu \gamma^\sigma k_\sigma (z-1) + m^2 \gamma^\mu \gamma^\nu]$$

$$= (p'_\rho p'_\sigma - k_\rho k_\sigma z(1-z))\,\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) + m^2\,\mathrm{tr}(\gamma^\mu\gamma^\nu)$$

$$= (p'_\rho p'_\sigma - k_\rho k_\sigma z(1-z))\, 4(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu}) + 4m^2 g^{\mu\nu}$$

$$= 4[p'^\mu p'^\nu - g^{\mu\nu}p'^2 + p'^\mu p'^\nu - k^\mu k^\nu (1-z)z + g^{\mu\nu}k^2 z(1-z) - k^\mu k^\nu z(1-z) + m^2 g^{\mu\nu}]$$

$$= 4[2p'^\mu p'^\nu - 2k^\mu k^\nu z(1-z) - g^{\mu\nu}(p'^2 - k^2 z(1-z) - m^2)]$$

So the trace is $8p'^{\mu}p'^{\nu} - 4g^{\mu\nu}p'^2 + 4\left[-2k^{\mu}k^{\nu}z(1-z) + g^{\mu\nu}(k^2z(1-z)+m^2)\right]$

Then $$\Pi^{\mu\nu}(k) = \frac{ie^2}{(2\pi)^n}\int_0^1 dz \left\{ \frac{8i\pi^{n/2}}{\Gamma(2)}(k^2z(1-z)-m^2)^{\frac{n}{2}-2}\frac{1}{2}g^{\mu\nu}(k^2z(1-z)-m^2)\Gamma(2-1-\tfrac{n}{2}) \right.$$

$$-4g^{\mu\nu}\frac{i\pi^{n/2}}{\Gamma(2)}(k^2z(1-z)-m^2)^{\frac{n}{2}-2}\frac{4}{2}(k^2z(1-z)-m^2)\Gamma(2-1-\tfrac{n}{2})$$

$$\left. +4\left[-2k^{\mu}k^{\nu}z(1-z)+g^{\mu\nu}(k^2z(1-z)+m^2)\right] i\pi^{\frac{n}{2}}\frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)}(k^2z(1-z)-m^2)^{\frac{n}{2}-2}\right\}$$

$$\Rightarrow \Pi^{\mu\nu}(k) = -\frac{4e^2}{2^n\pi^{n/2}}\int_0^1 dz\,(k^2z(1-z)-m^2)^{\frac{n}{2}-2}\left\{ \begin{array}{l} -g^{\mu\nu}(k^2z(1-z)-m^2)\Gamma(1-\frac{n}{2}) \\ +\left[-2k^{\mu}k^{\nu}z(1-z)+g^{\mu\nu}(k^2z(1-z)+m^2)\right]\Gamma(2-\frac{n}{2}) \end{array}\right\}$$

Since $\Gamma(2-\frac{n}{2}) = (1-\frac{n}{2})\Gamma(1-\frac{n}{2}) = (-1+\frac{\varepsilon}{2})\Gamma(-1+\frac{\varepsilon}{2})$ this becomes

$$\Pi^{\mu\nu}(k) = -\frac{4e^2}{2^{\cancel{4-\varepsilon}}\pi^{2-\frac{\varepsilon}{2}}}\int_0^1 dz\,(k^2z(1-z)-m^2)^{-\frac{\varepsilon}{2}}\left\{\left[-2g^{\mu\nu}k^2z(1-z)+2k^{\mu}k^{\nu}z(1-z)\right]\cdot\Gamma(-1+\tfrac{\varepsilon}{2})(1+O(\varepsilon))\right\}$$

But $\Gamma(-1+\frac{\varepsilon}{2}) = -\frac{2}{\varepsilon} + O(\varepsilon^0)$

Then $$\Pi^{\mu\nu}(k) = \frac{-e^2}{4\pi^2}\int_0^1 dz\, z(1-z)\,2(k^{\mu}k^{\nu}-g^{\mu\nu}k^2)\frac{(-2)}{\varepsilon} + O(\varepsilon^0)$$

$$= \frac{e^2}{\pi^2\varepsilon}(k^{\mu}k^{\nu}-g^{\mu\nu}k^2)\int_0^1 dz\, z(1-z) + O(\varepsilon^0)$$

$$= \frac{e^2}{\pi^2\varepsilon}(k^{\mu}k^{\nu}-g^{\mu\nu}k^2)\left(\frac{z^2}{2}-\frac{z^3}{3}\right)\Big|_0^1 + O(\varepsilon^0) \Rightarrow \Pi^{\mu\nu}(k) = \frac{e^2}{6\pi^2\varepsilon}(k^{\mu}k^{\nu}-g^{\mu\nu}k^2)+O(\varepsilon^0)$$