Functional derivative $\dfrac{\delta J(y)}{\delta J(x)} = \delta^4(x-y)$ or $\dfrac{\delta}{\delta J(x)} \int J(y)\varphi(y)\, d^4y = \varphi(x)$

$$\langle 0|T\{\varphi(x_1)\}|0\rangle = -i \frac{\delta Z_0[J]}{\delta J(x_1)}\Big|_{J=0}$$

Now $\dfrac{\delta Z_0[J]}{\delta J(x_1)} = \dfrac{\delta}{\delta J(x_1)} \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\, d^4x\, d^4y\right]$

$$= \left\{-\frac{1}{2}\int D(x_1-y)J(y)\, d^4y - \frac{1}{2}\int J(x)D(x-x_1)\, d^4x\right\} \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\, d^4x\, d^4y\right]$$

$$= -\left(\int D(x-x_1)J(x)\, d^4x\right) \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\, d^4x\, d^4y\right]$$

This vanishes at $J=0$ so $\langle 0|T\{\varphi(x_1)\}|0\rangle = 0$ as we would expect

Also $\dfrac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)} = \dfrac{\delta}{\delta J(x_1)}\left\{-\left(\int D(x-x_2)J(x)\, d^4x\right) \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\, d^4x\, d^4y\right]\right\}$

$$= \left\{-D(x_1-x_2) + \left(\int D(x-x_2)J(x)\, d^4x\right)\left(\int D(x-x_1)J(x)\, d^4x\right)\right\} \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\, d^4x\, d^4y\right]$$

Then $\langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle = -\dfrac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0} = D(x_1-x_2)$

Similarly $\langle 0|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)|0\rangle = i\dfrac{\delta^3 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\Big|_{J=0} = 0$

and $\langle 0|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = \dfrac{\delta^4 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big|_{J=0} = D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3)$

Path integrals in QFT - interacting fields

Consider a scalar field φ with $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \mathcal{L}_{int}$
where $\mathcal{L}_{int}$ is the interaction term in the Lagrangian.

For example, in φ^4 theory, $\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$.

Generating functional: $Z[J] = \int D\varphi \exp\left[i\int d^4x\,(\mathcal{L}(\varphi) + J(x)\varphi(x))\right]$
or $Z[J] = \int D\varphi \exp\left[iS + i\int d^4x\, J\varphi\right]$ where action $S = \int \mathcal{L}\, d^4x$

As for the free field case, we can normalize this by diving by $\int D\varphi\, e^{iS}$.

When $\mathcal{L}_{int} \to 0$, $Z[J] \to Z_0[J]$

It can be shown that $Z[J] = N\exp\left[i\int \mathcal{L}_{int}\left(\frac{\delta}{\delta J}\right) d^4x\right] Z_0[J]$

or $Z[J] = N\exp\left[i\int \mathcal{L}_{int}\left(\frac{\delta}{\delta J(z)}\right) d^4z\right] \exp\left[-\frac{1}{2}\int J(x)\,D(x-y)\,J(y)\,d^4x\,d^4y\right]$

where $N^{-1} = \left\{\exp\left[i\int \mathcal{L}_{int}\left(\frac{\delta}{\delta J(z)}\right) d^4z\right] \exp\left[-\frac{1}{2}\int J(x)\,D(x-y)\,J(y)\,d^4x\,d^4y\right]\right\}\Big|_{J=0}$

For example, in φ^4 theory $\mathcal{L}_{int}\left(\frac{\delta}{\delta J(z)}\right) = -\frac{\lambda}{4!}\left(\frac{\delta}{\delta J(z)}\right)^4$

Green's function $G(x_1, x_2, \ldots, x_n) = (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\Big|_{J=0}$

Path integrals and spinor fields

Spinor fields obey anticommutation relations. Thus we need to introduce anticommuting numbers (Grassmann algebra) in path-integral approach

For such numbers ζ, θ, we have $\{\zeta, \theta\} = \zeta\theta + \theta\zeta = 0$ or $\zeta\theta = -\theta\zeta$

Also $\zeta^2 = \theta^2 = 0$ and a polynomial has the form $f(\theta) = a + b\theta$ since higher terms vanish

Consider the free Dirac field with Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$

The generating functional is

$$Z_0[\eta, \bar{\eta}] = N\int D\bar{\psi}\, D\psi \exp\left[i\int (i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \bar{\eta}\psi + \bar{\psi}\eta)\, d^4x\right]$$

where $\bar{\eta}(x)$ is a source for $\psi(x)$ and $\eta(x)$ a source for $\bar{\psi}(x)$

and $\eta, \bar{\eta}$ are anticommuting

Then $Z_0[\eta, \bar{\eta}] = \exp\left[-\int \bar{\eta}(x)\, S(x-y)\, \eta(y)\, d^4x\, d^4y\right]$

with $S(x-y)$ the propagator $(i\gamma^\mu\partial_\mu - m)\, S(x-y) = i\delta^4(x-y)$ and $S(p) = \dfrac{i}{\not{p} - m}$

$$S(x-y) = \langle 0 | T\{\psi(x)\bar{\psi}(y)\} | 0 \rangle = -\left.\frac{\delta^2 Z_0[\eta, \bar{\eta}]}{\delta\bar{\eta}(x)\, \delta\eta(y)}\right|_{\eta=\bar{\eta}=0}$$

So we get the same result as in the canonical formalism.

The propagator $S(p)$ is the inverse of the operator in the Lagrangian (times i)

Path integrals - interacting spinor fields

Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \mathcal{L}_{int}$ with $\mathcal{L}_{int}$ the interaction term

The generating functional is then

$$Z[\eta,\bar{\eta}] = \exp\left[i\int \mathcal{L}_{int}\left(\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}}\right)d^4x\right] Z_0[\eta,\bar{\eta}]$$

Path integrals and gauge fields

Consider a free gauge field with Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

The generating functional is

$$Z[J] = \int DA^\mu \exp\left[i\int\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right)d^4x\right]$$ with source $J^\mu(x)$

Again, need gauge-fixing term. $-\frac{1}{2}(\partial_\mu A^\mu)^2$ in Lagrangian

Further complications for non-Abelian gauge fields (e.g. gluons in QCD) due to self interactions. Then the analog of $F^{\mu\nu}$ is

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu$$