Path-integral formalism in Quantum Mechanics

Wavefunction $\psi(q,t) = \langle q | \psi_S \rangle = \langle q | e^{-iHt} | \psi_H \rangle$ or simply $\langle q,t | \psi \rangle$
where $|q,t\rangle = e^{iHt}|q\rangle$

Then $\langle q_f, t_f | \psi \rangle = \int \langle q_f, t_f | q_i, t_i \rangle \langle q_i, t_i | \psi \rangle \, dq_i$ (since $\int |q_i, t_i\rangle\langle q_i, t_i| \, dq_i = 1$)

or $\langle q_f, t_f | \psi \rangle = \int K(q_f, t_f; q_i t_i) \langle q_i, t_i | \psi \rangle \, dq_i$ or $\psi(q_f, t_f) = \int K(q_f, t_f; q_i, t_i) \psi(q_i, t_i) \, dq_i$

where K is the propagator $K(q_f, t_f; q_i, t_i) = \langle q_f, t_f | q_i, t_i \rangle$

Also $\langle q_f, t_f | q_i, t_i \rangle = \int \langle q_f, t_f | q_n, t_n \rangle \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \cdots \langle q_1, t_1 | q_i, t_i \rangle \, dq_1 \cdots dq_{n-1} \, dq_n$

where we have split the interval from t_i to t_f 

sum over paths

Then $\langle q_{m+1}, t_{m+1} | q_m, t_m \rangle = \langle q_{m+1} | e^{-iHt_{m+1}} e^{iHt_m} | q_m \rangle = \langle q_{m+1} | e^{-iH\tau} | q_m \rangle$

(with $\tau = t_{m+1} - t_m$) $= \langle q_{m+1} | 1 - iH\tau + O(\tau^2) | q_m \rangle = \langle q_{m+1} | q_m \rangle - i\tau \langle q_{m+1} | H | q_m \rangle + O(\tau^2)$

$= \delta(q_{m+1} - q_m) - i\tau \langle q_{m+1} | H | q_m \rangle + O(\tau^2) = \frac{1}{2\pi} \int dp \, e^{ip\Delta q} - i\tau \langle q_{m+1} | \frac{\hat{p}^2}{2m} + U(q) | q_m \rangle + O(\tau^2)$

with $\Delta q = q_{m+1} - q_m$

Now $\langle q_{m+1} | \frac{\hat{p}^2}{2m} | q_m \rangle + \langle q_{m+1} | U(q) | q_m \rangle = \int \langle q_{m+1} | p' \rangle \langle p' | \frac{\hat{p}^2}{2m} | p \rangle \langle p | q_m \rangle \, dp \, dp'$

$+ U\left(\frac{q_{m+1} + q_m}{2}\right) \langle q_{m+1} | q_m \rangle = \int \frac{1}{\sqrt{2\pi}} e^{ip' q_{m+1}} \frac{p^2}{2m} \delta(p - p') \frac{1}{\sqrt{2\pi}} e^{-ipq_m} \, dp \, dp' + U\left(\frac{q_{m+1} + q_m}{2}\right) \delta(q_{m+1} - q_m)$

$$= \frac{1}{2\pi}\int e^{i(p'q_{m+1}-pq_m)} \frac{p^2}{2m}\delta(p-p')\,dp\,dp' + \frac{1}{2\pi}\int dp\, e^{ip(q_{m+1}-q_m)}\, U\left(\frac{q_{m+1}+q_m}{2}\right)$$

$$= \frac{1}{2\pi}\left\{\int dp\, e^{ip(q_{m+1}-q_m)}\frac{p^2}{2m} + \int dp\, e^{ip\Delta q}\, U(\bar{q}_m)\right\}$$ where $\bar{q}_m = \frac{q_{m+1}+q_m}{2}$

$$= \frac{1}{2\pi}\int dp_m\, e^{ip_m\Delta q_m}\left(\frac{p_m^2}{2m}+U(\bar{q}_m)\right) = \frac{1}{2\pi}\int dp_m\, e^{ip_m\Delta q_m} H(p_m,\bar{q}_m)$$

Then $K(q_f,t_f;q_i,t_i) = \langle q_f,t_f|q_i,t_i\rangle = \int \prod_{m=1}^{n} dq_m \prod_{m=0}^{n}\frac{dp_m}{2\pi} \exp\left\{\sum_{m=0}^{n} i\left(p_m\Delta q_m - \tau H(p_m,\bar{q}_m)\right)\right\}$

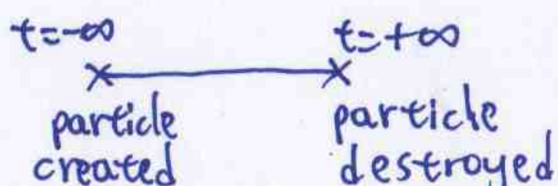
continuum limit $\rightarrow \quad = \int Dq\,Dp\, \exp\left[i\int_{t_i}^{t_f} dt\,(p\dot{q}-H(p,q))\right] = \int Dq\, \exp\left[i\int_{t_i}^{t_f} L(q,\dot{q})\,dt\right]$

$\Rightarrow \langle q_f,t_f|q_i,t_i\rangle = \int Dq\, e^{iS}$ with action $S = \int_{t_i}^{t_f} L(q,\dot{q})\,dt$
(Dirac, Feynman)

$Z[J]$: Vacuum-to-vacuum transition amplitude in the presence of a source $J(t)$ of particle creation $\quad Z[J] \propto \langle 0,\infty|0,-\infty\rangle$

Modify Lagrangian $L \rightarrow L + J(t)q(t)$ (Schwinger)

$$Z[J] = \int Dq\, \exp\left[i\int_{-\infty}^{+\infty}(L+Jq)\,dt\right]$$

diagramatically 

Path integrals in Quantum Field Theory

Vacuum-to-vacuum transition amplitude for scalar field $\varphi(x)$ is

$$Z[J] = \int \mathcal{D}\varphi \exp\left[i \int d^4x \left(\mathcal{L}(\varphi) + J(x)\varphi(x) \right) \right] \quad \text{with source } J(x)$$

For a free particle $Z_0[J] = \int \mathcal{D}\varphi \exp\left[i \int d^4x \left(\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + J\varphi \right) \right]$

Since $\int \partial^\mu\varphi\,\partial_\mu\varphi\, d^4x = \underbrace{\int \partial_\mu(\varphi\,\partial^\mu\varphi)\, d^4x}_{\text{vanishes if } \varphi \to 0 \text{ at } \infty} - \int \varphi\,\partial^\mu\partial_\mu\varphi\, d^4x = -\int \varphi\,\partial^\mu\partial_\mu\varphi\, d^4x$

we have $Z_0[J] = \int \mathcal{D}\varphi \exp\left[-i \int d^4x \left(\frac{1}{2}\varphi\,\partial^\mu\partial_\mu\varphi + \frac{m^2}{2}\varphi^2 - J\varphi \right) \right]$

Now let $\varphi \to \varphi + \varphi_0$ where φ_0 satisfies $\partial^\mu\partial_\mu\varphi_0 + m^2\varphi_0 = J$

Then $Z_0[J] = \int \mathcal{D}\varphi \exp\Big[-i \int d^4x \Big(\frac{1}{2}\varphi\,\partial^\mu\partial_\mu\varphi + \frac{1}{2}\varphi\,\partial^\mu\partial_\mu\varphi_0 + \frac{1}{2}\varphi_0\,\partial^\mu\partial_\mu\varphi + \frac{1}{2}\varphi_0\,\partial^\mu\partial_\mu\varphi_0$
$+ \frac{m^2}{2}(\varphi+\varphi_0)^2 - J(\varphi+\varphi_0) \Big) \Big]$

$\Rightarrow Z_0[J] = \int \mathcal{D}\varphi \exp\left[-i \int d^4x \left(\frac{1}{2}\varphi\,\partial^\mu\partial_\mu\varphi + \frac{1}{2}m^2\varphi^2 - \frac{1}{2}J\varphi_0 \right) \right]$

But $\varphi_0(x) = i\int D(x-y)J(y)\,d^4y$ with $D(x-y)$ the propagator $(\partial^\mu\partial_\mu + m^2)D(x-y) = -i\delta^4(x-y)$

Then $Z_0[J] = \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\,d^4x\,d^4y \right] \underbrace{\int \mathcal{D}\varphi \exp\left[-\frac{i}{2}\int (\varphi\,\partial^\mu\partial_\mu\varphi + m^2\varphi^2)\,d^4x \right]}_{\text{does not depend on } J \;\equiv\; N}$

Then $Z_0[J] = N \exp\left[-\frac{1}{2}\int J(x)D(x-y)J(y)\,d^4x\,d^4y \right]$

$$Z_0[J] = N\left\{ 1 - \frac{1}{2}\int J(x)\,D(x-y)\,J(y)\,d^4x\,d^4y + \frac{1}{2!}\left(\frac{1}{2}\right)^2\left[\int J(x)\,D(x-y)\,J(y)\,d^4x\,d^4y\right]^2 + \cdots \right\}$$

(can normalize $N \to 1$)

Now $$-\frac{1}{2}\int J(x)\,D(x-y)\,J(y)\,d^4x\,d^4y = \frac{-i}{2(2\pi)^4}\int J(x)\,\frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}\,J(y)\,d^4p\,d^4x\,d^4y$$

$$= -\frac{i}{2}(2\pi)^4\int \frac{J(-p)\,J(p)}{p^2-m^2+i\varepsilon}\,d^4p \quad \left(\text{where } J(x) = \int J(p)\,e^{-ipx}\,d^4p\right)$$

$$= -\frac{1}{2}(2\pi)^4\int J(-p)\,D(p)\,J(p)\,d^4p \quad \text{since } D(p) = \frac{i}{p^2-m^2+i\varepsilon}$$

This is represented by the diagram $\frac{1}{2}\,\underset{J}{\times}\!\longrightarrow\!\underset{J}{\times}$

if $\underset{p}{\longrightarrow}$ is $\frac{1}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon} = \frac{1}{(2\pi)^4}D(p)$ and $\underset{J}{\times}\!\underset{p}{\longleftarrow}$ is $i(2\pi)^4 J(p)$

Then $$Z_0[J] = 1 + \frac{1}{2}\;\times\!\longrightarrow\!\times\; + \frac{1}{2!}\left(\frac{1}{2}\right)^2 \begin{matrix}\times\!\longrightarrow\!\times\\ \times\!\longrightarrow\!\times\end{matrix} + \cdots$$

So $Z_0[J]$ is a generating functional for the Green's functions

$$G(x_1, x_2, \ldots, x_n) = \langle 0|T\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\}|0\rangle$$

$$(\text{and } D(x-y) = \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle)$$

We have $$\langle 0|T\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\}|0\rangle = (-i)^n\,\frac{\delta^n Z_0[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}$$