Then $N(p)\,a(p)|n(p)\rangle = \left([N(p), a(p)] + a(p)N(p)\right)|n(p)\rangle$

$$= \left(-a(p) + a(p)N(p)\right)|n(p)\rangle = -a(p)|n(p)\rangle + a(p)\,n(p)|n(p)\rangle$$

$$= (n(p)-1)\,a(p)|n(p)\rangle$$

So $a(p)|n(p)\rangle$ is an eigenstate of $N(p)$ with eigenvalue $n(p)-1$

Also $N(p)\,a^\dagger(p)|n(p)\rangle = (n(p)+1)\,a^\dagger(p)|n(p)\rangle$

So $a^\dagger(p)|n(p)\rangle$ is an eigenstate of $N(p)$ with eigenvalue $n(p)+1$

We identify $N(p)$ as a <u>particle-number</u> operator while $a(p)$ and $a^\dagger(p)$ are <u>annihilation</u> and <u>creation</u> operators.

Hamiltonian $H = \displaystyle\int \frac{d^3p}{(2\pi)^3}\frac{p^0}{2}\left[a^\dagger(p)a(p) + a(p)a^\dagger(p)\right] = \int \frac{d^3p}{(2\pi)^3} p^0 \left(N(p) + \frac{1}{2}\right)$

ground state (vacuum) $|0\rangle$ and $a(p)|0\rangle = 0$

so $N(p)|0\rangle = a^\dagger(p)a(p)|0\rangle = 0 \rightarrow$ vacuum contains no particles

Also $a^\dagger(p)|0\rangle$ are one-particle states with momentum p

We can ignore the zero-point energy and redefine

$$H = \int \frac{d^3p}{(2\pi)^3} p^0 N(p) = \int \frac{d^3p}{(2\pi)^3} p^0 a^\dagger(p) a(p)$$

$$\langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3}\, p^0 \langle 0|a^\dagger(p)\, a(p)|0\rangle = 0 \quad \text{since } a(p)|0\rangle = 0$$

<u>Normal ordering</u>: put all annihilation operators to the right of all creation operators. Denote it by $::$ So $:aa^\dagger: = a^\dagger a$

If $\varphi^+(x) = \int \frac{d^3p}{(2\pi)^3 (2p^0)^{1/2}}\, a(p)\, e^{-ip\cdot x}$ and $\varphi^-(x) = \int \frac{d^3p}{(2\pi)^3 (2p^0)^{1/2}}\, a^\dagger(p)\, e^{ip\cdot x}$

then $:\varphi(x)\varphi(y): = :(\varphi^+(x)+\varphi^-(x))(\varphi^+(y)+\varphi^-(y)):$

$$= :\varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y): = \varphi^+(x)\varphi^+(y) + \varphi^-(y)\varphi^+(x) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y)$$

Single-particle states $|p\rangle = (2p^0)^{1/2} a^\dagger(p)|0\rangle$

Then $\langle p|p'\rangle = 2(p^0p'^0)^{1/2}\langle 0|a(p)a^\dagger(p')|0\rangle = 2(p^0p'^0)^{1/2}\langle 0|[a(p),a^\dagger(p')] + a^\dagger(p')a(p)|0\rangle$

$$= 2(p^0p'^0)^{1/2}\big(\langle 0|[a(p),a^\dagger(p')]|0\rangle + \underbrace{\langle 0|a^\dagger(p')a(p)|0\rangle}_{=0 \text{ since } a(p)|0\rangle = 0}\big) = 2(p^0p'^0)^{1/2}\langle 0|(2\pi)^3\delta^3(\vec p - \vec p\,')|0\rangle$$

$= 2p^0(2\pi)^3\delta^3(\vec p - \vec p\,')\langle 0|0\rangle = 2p^0(2\pi)^3\delta^3(\vec p - \vec p\,')$ since $\langle 0|0\rangle = 1$

<u>1-particle wavefunction</u> $\psi(x) = \langle 0|\varphi(x)|p\rangle = \langle 0|\int \frac{d^3p'}{(2\pi)^3(2p'^0)^{1/2}}[a(p')e^{-ip'\cdot x} + \overbrace{a^\dagger(p')e^{ip'\cdot x}}^{=0 \text{ since } \langle 0|a^\dagger = 0}]|p\rangle$

$= \langle 0|\int \frac{d^3p'}{(2\pi)^3(2p'^0)^{1/2}} a(p')e^{-ip'\cdot x}|p\rangle = \int \frac{d^3p'}{(2\pi)^3(2p'^0)^{1/2}} \frac{1}{(2p'^0)^{1/2}}\langle p'|e^{-ip'\cdot x}|p\rangle = e^{-ip\cdot x}$ since $\langle 0|a = \frac{1}{(2p'^0)^{1/2}}\langle p'|$

operator $\varphi(x)$ acting on the vacuum creates a particle at $\vec x$

<u>Bose-Einstein statistics</u> These particles are <u>bosons</u>. No restriction on $n(p)$ - follows from commutation relations

Complex scalar field

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi^* \varphi \qquad \begin{array}{l} \partial_\mu \partial^\mu \varphi + m^2 \varphi = 0 \\ \text{and } \partial_\mu \partial^\mu \varphi^* + m^2 \varphi^* = 0 \end{array}$$

$$\varphi(x) = \int \frac{d^3 p}{(2\pi)^3 (2p^0)^{1/2}} \left[a(p) e^{-ip\cdot x} + b^\dagger(p) e^{ip\cdot x} \right]$$

$$\varphi^\dagger(x) = \int \frac{d^3 p}{(2\pi)^3 (2p^0)^{1/2}} \left[b(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x} \right] \qquad (\text{note that } \varphi^\dagger = \varphi^*)$$

Commutation relations $[a(p), a^\dagger(p')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,') \qquad [b(p), b^\dagger(p')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,')$

$[a(p), a(p')] = 0, \; [a^\dagger(p), a^\dagger(p')] = 0, \; [b(p), b(p')] = 0, \; [b^\dagger(p), b^\dagger(p')] = 0$

Hamiltonian $H = \int \frac{d^3 p}{(2\pi)^3} p^0 \left(a^\dagger(p) a(p) + b^\dagger(p) b(p) \right)$

$a^\dagger$ is creation operator for particles and $b^\dagger$ is creation operator for antiparticles

<u>a</u> is annihilation operator for particles and <u>b</u> is annihilation operator for antiparticles

$a|0\rangle = 0$ and $b|0\rangle = 0$ Also $a^\dagger(p)|0\rangle$ are one-particle states with momentum p

and $b^\dagger(p)|0\rangle$ are one-antiparticle states with momentum p

Spin-0 and same mass m for particles and antiparticles.

charge (integral of ρ) $Q = i \int \left(:\varphi^\dagger \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi^\dagger}{\partial t} \varphi: \right) d^3x = \int \frac{d^3 p}{(2\pi)^3} \left(a^\dagger(p) a(p) - b^\dagger(p) b(p) \right)$ <u>not</u> positive definite

so numbers of particles minus number of antiparticles, hence antiparticles have opposite charge.

Dirac spinor field

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \quad \text{with } \bar{\psi} = \psi^\dagger\gamma^0 \text{ the adjoint spinor}$$

Euler-Lagrange equations $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi} \Rightarrow \partial_\mu(i\bar{\psi}\gamma^\mu) = -m\bar{\psi} \Rightarrow i\partial_\mu\bar{\psi}\,\gamma^\mu = -m\bar{\psi}$

and $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \Rightarrow \partial_\mu(0) = i\gamma^\mu\partial_\mu\psi - m\psi \Rightarrow i\gamma^\mu\partial_\mu\psi = m\psi$

Thus we get the Dirac equation for the spinor ψ and the adjoint spinor $\bar{\psi}$

Conjugate momentum $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger\gamma^0\gamma^0 = i\psi^\dagger(x)$

since $(\gamma^0)^2 = 1$

Hamiltonian density $\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^\dagger\dot{\psi} - i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi$

or $\mathcal{H} = i\psi^\dagger\dot{\psi} - \bar{\psi}\big(\underbrace{i\gamma^\mu\partial_\mu\psi - m\psi}_{=0 \text{ Dirac eq.}}\big) = i\psi^\dagger\dot{\psi}$

$H = \int d^3x\,\mathcal{H}$ still not positive definite

This will be resolved next on quantization via use of anticommutation relations