$$\Sigma_2(p) = 2g_s^2 C_F \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D k}{(2\pi)^D} \, *$$

$$* \left\{ \frac{1}{(k^2+L)^3}\left(\frac{2(1-x)}{D} - 1 - 2x\right) L - \frac{1}{(k^2+L)^2}\left(\frac{2(1-x)}{D} - 1 - x\right) \right\}$$

At this point we use a generalization of a previous result

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+L)^a} = \frac{\Gamma(a - D/2)}{(4\pi)^{D/2}\,\Gamma(a)} L^{D/2-a} \qquad a \in \mathbb{C} \ni' \mathrm{Re}[a] > 0$$

$L = -x(1-x)p^2$. After the integration over k,

The <u>first piece</u> has 3 contributions

$$\int_0^1 dx\,(1-x)\left[\frac{2(1-x)}{D} - 1 - 2x\right] L^{D/2-2} =$$

$$= \int_0^1 dx\, \frac{2}{D}(1-x)^2 L^{D/2-2} - \int_0^1 dx (1-x) L^{D/2-2} - \int_0^1 dx\, 2x(1-x) L^{D/2-2}$$

$$= \int_0^1 dx\, \frac{2}{D} x^{D/2-2}(1-x)^{D/2}(-p^2)^{D/2-2} - \int_0^1 dx\, x^{D/2-2}(1-x)^{D/2-1}(-p^2)^{D/2-2} - 2\int_0^1 dx\, x^{D/2-1}(1-x)^{D/2-1}(-p^2)^{D/2-2}$$

$$B(p,q) = \int_0^1 dx\, x^{p-1}(1-x)^{q-1} \Rightarrow$$

$$= (-p^2)^{D/2-2}\left\{ B(D/2-1, D/2+1)\frac{2}{D} - B(D/2-1, D/2) - 2B(D/2, D/2) \right\}$$

- $B(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$
- $\Gamma(z+1) = z\,\Gamma(z)$

We use these two properties to rewrite our result for the first piece

$$B(D/2-1,\, D/2+1) = \frac{\Gamma(D/2-1)\,\Gamma(D/2+1)}{\Gamma(D)} = \frac{(D/2-1)\,\Gamma(D/2-1)\,\Gamma(D/2+1)}{(D/2-1)\quad \Gamma(D)}$$

$$= \frac{\Gamma(D/2)\quad D/2\quad \Gamma(D/2)}{(D/2-1)\ \Gamma(D)} = \frac{\Gamma(D/2)}{\Gamma(D)}\,\frac{\Gamma(D/2)\,D}{(D-2)}$$

Then we have

$$B(D/2-1,\, D/2) = \frac{\Gamma(D/2-1)\,\Gamma(D/2)}{\Gamma(D-1)} = \frac{(D/2-1)\quad \Gamma(D/2-1)\,\Gamma(D/2)}{(D/2-1)\,\Gamma(D-1)} =$$

$$= \frac{\Gamma(D/2)\quad \Gamma(D/2)\quad (D-1)}{(D/2-1)\,(D-1)\,\Gamma(D-1)} = \frac{\Gamma(D/2)\,\Gamma(D/2)}{\Gamma(D)}\,\frac{(D-1)}{(D/2-1)}$$

$$B(D/2,\, D/2) = \frac{\Gamma(D/2)\ \Gamma(D/2)}{\Gamma(D)}$$

$$B(D/2-1,\, D/2+1) = B(D/2, D/2)\,\frac{D}{D-2}$$

$$B(D/2-1,\, D/2) = B(D/2, D/2)\,\frac{D-1}{D/2-1} = B(D/2, D/2-1)$$

$$= \frac{2(D-1)}{D-2}$$

Therefore, the 1st piece gives

$$\int_0^1 dx\,(1-x)\left[\frac{2(1-x)}{D} - 1 - 2x\right] L^{D/2-2} = (-p^2)^{D/2-2}\left\{ B(D/2, D/2)\frac{2}{D-2}\right.$$

$$\left. - B(D/2, D/2)\,\frac{2(D-1)}{D-2} - 2\,B(D/2, D/2)\right\} =$$

$$= (-p^2)^{D/2-2}\, B(D/2, D/2)\left\{ \frac{2 - 2D + 2 - 2D + 4}{D-2}\right\} =$$

$$= (-p^2)^{D/2-2}\, B(D/2, D/2)\,\frac{(8-4D)}{D-2}$$

This must be multiplied by the rest of the result of the k-integration

$$= (-p^2)^{D/2-2}\, B(D/2; D/2)\left(\frac{8-4D}{D-2}\right)\frac{\Gamma(3-D/2)}{(4\pi)^{D/2}\,\Gamma(3)}$$

The <u>second integral</u>

$$\int_0^1 dx\,(1-x)\left[\frac{2(1-x)}{D}-1-x\right] L^{D/2-2}$$ because

the integral over k gives

$$\int \frac{d^D k}{(2\pi)^D}\frac{1}{(k^2+L)^2} = \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}\,\Gamma(2)}\,L^{D/2-2}$$

Therefore we obtain

$$(-p^2)^{D/2-2}\int_0^1 dx\,(1-x)\left[2\frac{(1-x)}{D}-1-x\right]\left[x(1-x)\right]^{D/2-2} =$$

$$= (-p^2)^{D/2-2}\left\{\int_0^1 \frac{2x}{D}^{D/2-2}(1-x)^{D/2}\,dx - \int_0^1 dx\,x^{D/2-2}(1-x)^{D/2-1} - \int_0^1 dx\,x^{D/2-1}(1-x)^{D/2-1}\right\}$$

$$=(-p^2)^{D/2-2}\left\{\frac{2}{D}B(D/2-1,D/2+1) - B(D/2-1,D/2) - B(D/2,D/2)\right\}$$

The B-functions have been calculated before

$$= (-p^2)^{D/2-2}\left\{\frac{2}{\cancel{D}}\frac{\cancel{D}}{D-2} - 2\frac{(D-1)}{(D-2)} - 1\right\} B(D/2,D/2)$$

With the k integration result we get

$$= (-p^2)^{D/2-2}\,\frac{(6-3D)}{(D-2)}\,B(D/2,D/2)\,\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}\,\Gamma(2)}$$

Adding the two contributions together

$$\frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{(8-4D)}{D-2} \frac{\Gamma(3-D/2)}{\Gamma(3)} - \frac{(6-3D)}{D-2} \frac{\Gamma(2-D/2)}{\Gamma(2)} \right\}$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{(8-4D)}{(D-2)} \frac{(2-D/2)\,\Gamma(2-D/2)}{2\,\Gamma(2)} - \frac{(6-3D)}{(D-2)} \frac{\Gamma(2-D/2)}{\Gamma(2)} \right\}$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{(4-2D)(2-D/2)}{(D-2)\;\Gamma(2)} - \frac{(6-3D)}{(D-2)\,\Gamma(2)} \right\} \Gamma(2-D/2)$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{\cancel{(2-D)}(4-D)}{\cancel{(D-2)}} - \frac{(6-3D)}{(D-2)} \right\} \Gamma(2-D/2)$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ D-4 \quad +3 \right\} \Gamma(2-D/2)$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2)\,(D-1)\,\Gamma(2-D/2) \Rightarrow$$

$$\boxed{\Sigma_2(p) = \frac{2 g_s^2 C_F}{(4\pi)^{D/2}} \not{p}\, (-p^2)^{D/2-2} (D-1)\, B(D/2, D/2)\, \Gamma(2-D/2)}$$

This is exactly equal to $\Sigma(p)$ we calculated at Pag 16 !

We found that $\Sigma_2(\not{p}) = \Sigma_1(\not{p})$ where $\Sigma_1(\not{p})$ was obtained before at pag 16.

Therefore for a covariant gauge with arbitrary α

$$\Sigma(\not{p}) = \alpha \, \frac{2 C_F g_s^2}{(4\pi)^{D/2}} \not{p} (-p^2)^{D/2-2} (D-1) \, B(D/2, D/2) \, \Gamma(2 - D/2)$$

$$= \alpha \, \frac{g_{0s}^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2} \right) \right) + O(\varepsilon)$$

Renormalization schemes (preliminaries)

Renormalization: redefinition of mass and coupling constant together with a re-adjustment of the normalization of Green functions by suitable multiplicative factors that may eliminate possible infinities in the Green functions.

Renormalization is not unique: divergent pieces in the Green functions are not uniquely defined.
$\Rightarrow$ ambiguity in the finite piece of the Green func.

How do we remove this ambiguity?

→ specify how the divergent piece is defined so that it can be consistently subtracted.

→ the subtraction prescription is called Renormalization scheme.

→ Different renormalization schemes are always connected by a finite renormalization.

Let's consider $\Sigma(p)$ which we have computed:

$$\Sigma(p) = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \not{p}\left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right) + O(\varepsilon)$$

if we substitute this into Eq (1) at pag 3 of these notes,

$$\tilde{S}_{ij}(p) = \frac{\delta_{ij}}{m - \not{p} - \Sigma(p)}$$

and we set $m = 0$ for simplicity, we obtain

$$\tilde{S}_{ij}(p) = -\frac{\delta_{ij}}{\not{p}} \frac{1}{1 + \sigma(p^2)}$$

$$\sigma(p^2) = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right) + O(g_{0s}^4)$$

where all terms of order ε have been set to zero.

- $\tilde{S}_{ij}(p)$ has a pole at $\not{p} = 0$
- massless quark stays massless after the inclusion of 1-loop corrections (this is generally true for massless quarks at all orders in perturbation theory)

We renormalize the quark propagator by a multiplicative factor Z_2

Z_2 = quark-field renormalization constant.

$$\tilde{S}_{R\,ij}(p) = Z_2^{-1}\,\tilde{S}_{ij}(p)$$

$\downarrow$

renormalized (finite) quark propagator

Z_2 can be expanded in g_{os} powers

$$Z_2 = 1 - z_2 + O(g_{os}^4)$$

$\downarrow$

g_{os}^2 – term (divergent)

Substituting this into $\tilde{S}_{ij}(p)$ gives us

$$\tilde{S}_{R\,ij}(p) = -\frac{\delta_{ij}}{\not{p}}\,\frac{1}{1+\sigma(p^2)-z_2}$$

where we keep only the g_{os}^2 terms. In fact

$$Z_2\,\sigma(p^2) = \sigma(p^2) + O(g_{os}^4)$$

Note that $\tilde{S}_{R\,ij}(p)$ should have the renormalized version of g_{os}, but at this perturbative order there is no effect on g_{os}. Therefore, we'll keep using g_{os} for now.

- $\tilde{S}_{Rij}(p)$ should be free of divergences $\Rightarrow$

$\sigma(p^2) - Z_2$ must be finite, and the divergences in $\sigma(p^2)$ should be cancelled by Z_2

This requirement determines Z_2 up to a finite additive constant.

$\Rightarrow$ we need an extra requirement which sets up a renormalization scheme (prescription).

As discussed before, there are several renormalization schemes depending on this prescription.

Let's see a few examples.

1. On-shell subtraction

Z_2 is determined on the mass shell of quarks by imposing the condition

$$\tilde{S}_{R_{ij}}(p) \sim \frac{\delta_{ij}}{m - \not{p}} \quad \text{for } \not{p} \sim m$$

this is traditionally used in QED. In our case $m=0$ and so $Z_2 = \sigma(0)$. $\sigma(0)$ is not well-defined in this example because for the massless quark the singularity is in $\sigma(p^2)$.

2. Off-shell subtraction

At an unphysical (off-shell) value of p^2, say $p^2 = -\lambda^2$ with $-\lambda^2 < 0$, we require that $\tilde{S}_{Rij}(p)$ be of the form of the free (massless) propagator

$$\tilde{S}_{Rij}(p) \sim -\frac{\delta_{ij}}{\not{p}} \quad \text{for } p^2 \sim -\lambda^2$$

This condition determines z_2 such that

$$z_2 = \sigma(-\lambda^2) = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{\lambda^2}{4\pi\mu^2}\right)\right)$$

and the renormalized propagator reads

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{\not{p}} \left(1 - \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \ln\left(-\frac{p^2}{\lambda^2}\right)\right)^{-1}$$

This scheme is also called momentum-space subtraction scheme. (MOM)

3. Minimal subtraction (MS) ('t Hooft)

This is specific to DR. We only eliminate the $1/\varepsilon$ pole in the DR expression of the Green functions. This scheme is very economical and often used in QCD and other gauge theories. The requirement imposes that

$$Z_2 = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \frac{1}{\varepsilon}$$

Therefore, the renormalized propagator is

$$\tilde{S}_{R_{ij}}(p) = -\frac{\delta_{ij}}{\not p}\left\{1 - \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F\left(\gamma_E - 1 + \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right)\right\}^{-1}$$

- renormalization constants → simple expression
- Green functions → complicated

Z_2 independent of mass parameters → easy to define renormalization group functions.

The $\tilde{S}_{R_{ij}}(p)$ above can be converted in the off-shell subtraction (MOM) by setting

$$\Lambda^2 = 4\pi e^{1-\gamma_E} \mu^2$$