Photon field quantization in Lorenz gauge

$\partial_\mu A^\mu = 0$, $\pi^\mu = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$

Try commutation relations $[A^\mu(\vec{x},t), \pi^\nu(\vec{y},t)] = i g^{\mu\nu} \delta^3(\vec{x}-\vec{y})$

(Covariant quantization) $[A^\mu(\vec{x},t), A^\nu(\vec{y},t)] = 0$, $[\pi^\mu(\vec{x},t), \pi^\nu(\vec{y},t)] = 0$

However there is a problem: π^0 still vanishes so the first equation cannot hold for A^0, π^0

so we need to change the Lagrangian → add a term that will still give Maxwell equations for Lorenz constraint $\partial_\mu A^\mu = 0$

try $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$ where $-\frac{1}{2}(\partial_\mu A^\mu)^2$ is gauge-fixing term

Then $\partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \dfrac{\partial \mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu \left(-\partial^\mu A^\nu + \partial^\nu A^\mu - g^{\mu\nu} \partial_\rho A^\rho \right) = 0 \Rightarrow \partial_\mu \partial^\mu A^\nu = 0$

Klein-Gordon eq.

A more general gauge-fixing term would be $-\frac{\xi}{2} (\partial_\mu A^\mu)^2$

The choice $\xi = 1$ is called the Feynman gauge.

Now $\pi^0 = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu$ still vanishes in Lorenz gauge

Resolution: treat the Lorenz condition not as an operator identity but as a vanishing expectation value for states: $\langle \psi | \partial_\mu A^\mu | \psi \rangle = 0$

Then $\pi^0 = -\partial_\mu A^\mu$ can satisfy commutation relations

Lorenz-gauge $A^{\mu}(x)=\int \frac{d^3p}{(2\pi)^3(2p^0)^{1/2}} \sum_{\lambda=0}^{3}\left[\varepsilon^{\mu(\lambda)}(p)a^{(\lambda)}(p)e^{-ip\cdot x}+\varepsilon^{*\mu(\lambda)}(p)a^{(\lambda)\dagger}(p)e^{ip\cdot x}\right]$

Four independent solutions $(\lambda=0,1,2,3)$ for ε^{μ}

In the frame $p^{\mu}=(p,0,0,p)$ we choose as a basis

$$\varepsilon^{(0)}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \varepsilon^{(1)}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \varepsilon^{(2)}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \varepsilon^{(3)}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We see that $p^{\mu}\varepsilon_{\mu}^{(1)}=0$ and $p^{\mu}\varepsilon_{\mu}^{(2)}=0$ so these are transverse-physical-photons

but $p^{\mu}\varepsilon_{\mu}^{(0)}\neq 0$ and $p^{\mu}\varepsilon_{\mu}^{(3)}\neq 0$ $\varepsilon^{(0)}$ are timelike photons and $\varepsilon^{(3)}$ are longitudinal

$\varepsilon^{(0)}$ and $\varepsilon^{(3)}$ are unphysical

conjugate momentum $\pi^{\mu}=\frac{\partial\mathcal{L}}{\partial\dot{A}_{\mu}}=F^{\mu 0}-g^{\mu 0}\partial_{\gamma}A^{\gamma}\Rightarrow \pi^{0}=-\dot{A}^{0}+\vec{\nabla}\cdot\vec{A}$ and $\pi^{i}=\partial^{i}A^{0}-\dot{A}^{i}$

Then commutation relations for $a(p)$ and $a^{\dagger}(p)$ become

$$[a^{(\lambda)}(p),a^{(\lambda')\dagger}(p')]=-(2\pi)^3\delta^3(\vec{p}-\vec{p}')\,g^{\lambda\lambda'} \quad \text{where } \lambda,\lambda'=0,1,2,3$$

and $[a^{(\lambda)}(p),a^{(\lambda')}(p')]=0$, $[a^{(\lambda)\dagger}(p),a^{(\lambda')\dagger}(p')]=0$

It can be shown that the contributions of the timelike and longitudinal photons cancel out in the Hamiltonian

→ only the physical transverse photons contribute

In discussing gauge transformations $\psi(x) \to e^{i\theta(x)}\psi(x)$, $A_\mu \to A_\mu - \frac{1}{q}\partial_\mu\theta$

we introduced the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$

Note that under gauge transformation

$$D_\mu\psi \to \left[\partial_\mu + iq\left(A_\mu - \frac{1}{q}\partial_\mu\theta\right)\right](e^{i\theta}\psi) = \cancel{i\partial_\mu\theta e^{i\theta}\psi} + e^{i\theta}\partial_\mu\psi + iqA_\mu e^{i\theta}\psi \cancel{- i\partial_\mu\theta e^{i\theta}\psi}$$

$$= e^{i\theta}(\partial_\mu + iqA_\mu)\psi = e^{i\theta}D_\mu\psi$$

Invariance under $\psi \to e^{i\theta}\psi$ is called $U(1)$ gauge invariance

In general can consider $\psi \to U\psi$ where U is a unitary matrix $U^\dagger U = 1$

In the case of QED, with $U(1)$, $U = e^{i\theta}$ is a unitary 1×1 matrix

This generalizes to higher groups $U(N)$ for $N\times N$ matrices

If we impose $\det U = 1$ (subtract overall phase) then $SU(N)$ group (special unitary)

In general $U = e^{iH}$ with H Hermitian: $H^\dagger = H$

<u>SU(2) for weak interactions</u> $H = \vec{\sigma}\cdot\vec{\theta}$ where $\vec{\sigma}$ are the Pauli spin matrices

So $\psi \to e^{i\vec{\sigma}\cdot\vec{\theta}}\psi$ $SU(2)$ transformation

<u>SU(3) for strong interactions Quantum Chromodynamics</u> $H = \frac{1}{2}\lambda^a\theta^a$ where λ^a are eight Gell-Mann matrices

So $\psi \to e^{\frac{i}{2}\lambda^a\theta^a}\psi$ $SU(3)$ transformation

In general, for $SU(N)$: $H = T^a \theta^a$ and $U = e^{iT^a\theta^a}$

where T^a are the generators of the Lie group

We have $[T^a, T^b] = i f^{abc} T^c$ where f^{abc} are the structure constants

Then gauge field tensor (generalization of $F^{\mu\nu}$ for photons) is

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu$$

The last term shows that e.g. gluons self-interact

<u>Gravity as a gauge theory</u> (general relativity)

A^μ (gauge transformation) corresponds to $\Gamma^\mu_{\lambda\nu} = \frac{1}{2}\left(\partial_\nu g^\mu_\lambda + \partial_\lambda g^\mu_\nu - \partial^\mu g_{\lambda\nu}\right)$ (coordinate transformation) connection coefficient or Christoffel symbol

and $D_\mu \psi = \partial_\mu \psi + iqA_\mu \psi$ corresponds to $D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda$

Finally, $G_a^{\mu\nu}$ corresponds to Riemann curvature tensor $R^\sigma_{\lambda\mu\nu}$

with $R^\sigma_{\lambda\mu\nu} = \partial_\nu \Gamma^\sigma_{\lambda\mu} - \partial_\mu \Gamma^\sigma_{\lambda\nu} + \Gamma^\rho_{\lambda\mu}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\sigma_{\rho\mu}$

Einstein equations of general relativity $\quad R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi T^{\mu\nu}$

with $R^{\mu\nu}$ the Ricci tensor and R the Ricci scalar