Interactions and perturbation theory

Split the Hamiltonian into two parts: a "free" term H_0 and an "interaction" term H_{int}. So $H = H_0 + H_{int}$

If H_{int} is small then we can treat it as a perturbation

$H = \int \mathcal{H} d^3x = \int \mathcal{H}_0 d^3x + \int \mathcal{H}_{int} d^3x$ since density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$

We can also write the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

Examples: (1) φ^4 theory: $\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$ and $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$

Then $\mathcal{H} = \pi\dot{\varphi} - \mathcal{L}$ with $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{\partial\mathcal{L}_0}{\partial\dot{\varphi}} = \dot{\varphi}$

Then $\mathcal{H} = \dot{\varphi}^2 - \mathcal{L} = \dot{\varphi}^2 - \mathcal{L}_0 - \mathcal{L}_{int} \Rightarrow \mathcal{H}_0 = \dot{\varphi}^2 - \mathcal{L}_0$ and $\mathcal{H}_{int} = -\mathcal{L}_{int} = \frac{\lambda}{4!}\varphi^4$

If the dimensionless coupling constant is small $\lambda \ll 1$ then perturbation theory can work well

(2) QED: $\mathcal{L}_{int} = -q\bar{\psi}\gamma^\mu\psi A_\mu$ and $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\bar{\psi}\gamma^0$

Then $\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \pi\dot{\psi} - \mathcal{L}_0 - \mathcal{L}_{int} \Rightarrow \mathcal{H}_{int} = -\mathcal{L}_{int} = q\bar{\psi}\gamma^\mu\psi A_\mu$

<u>In general</u>, $\mathcal{H}_{int} = -\mathcal{L}_{int}$ when $\mathcal{L}_{int}$ does not contain derivatives of the fields

Heisenberg representation (picture): operators change (depend on time)
but states are constant (time-independent)

Schrodinger representation (picture): operators are constant (time-independent)
but states change (depend on time)

Interaction picture: time dependence in both operators and states

In QFT the operators are the fields, so in Heisenberg rep. they depend on $\vec{x}$ and t, while in Schrodinger rep. they depend only on $\vec{x}$.

If $|\varphi_H\rangle$ is a ket in Heisenberg rep. and $|\varphi_S\rangle$ in Schrodinger rep. then $|\varphi_H\rangle = e^{iHt}|\varphi_S\rangle$ where H is the Hamiltonian.

For operator $\hat{A}$ we have $\hat{A}_H = e^{iHt}\hat{A}_S e^{-iHt}$ and $i\dfrac{d\hat{A}_H}{dt} = [\hat{A}_H, \hat{H}]$

If $H = H_0 + H_{int}$ then in interaction picture $|\varphi_I\rangle = e^{iH_0 t}|\varphi_S\rangle$
and $\hat{A}_I = e^{iH_0 t}\hat{A}_S e^{-iH_0 t}$

Also $\hat{H}_{int}|\varphi_I\rangle = i\dfrac{d}{dt}|\varphi_I\rangle$ and $i\dfrac{d\hat{A}_I}{dt} = [\hat{A}_I, \hat{H}_0]$

In Schrodinger rep. $|\varphi(t_f)\rangle = e^{-iH(t_f-t_i)}|\varphi(t_i)\rangle$

The amplitude for the process $|\varphi(t_i)\rangle \to |\varphi'(t_f)\rangle$ is $\langle\varphi'(t_f)|e^{-iH(t_f-t_i)}|\varphi(t_i)\rangle$

The evolution operator $e^{-iH(t_f-t_i)}$ in the limit $t_f-t_i\to\infty$ is the <u>S-matrix</u>

Note that S is unitary $S^\dagger S = 1$ unitarity $\to$ conservation of probability

Can also define <u>T-matrix</u> via $S = 1 + iT$

Then $SS^\dagger = 1 \Rightarrow (1+iT)(1-iT^\dagger) = 1 \Rightarrow TT^\dagger = i(T^\dagger - T)$

<u>Quantum scalar field</u> $\varphi(x^\mu)$

In Heisenberg representation $\varphi_H(\vec{x},t) = e^{iHt}\varphi_S(\vec{x})e^{-iHt}$

At time t_0, $\varphi_S(\vec{x},t_0) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{(2p^0)^{1/2}}\left[a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right]$

Then for $t\neq t_0$, $\varphi_H(\vec{x},t) = e^{iH(t-t_0)}\varphi_S(\vec{x},t_0)e^{-iH(t-t_0)}$

Consider φ^4 theory with $\mathcal{H}_{int} = \frac{\lambda}{4!}\varphi^4$ and $H = H_0 + \mathcal{H}_{int}$ As $\lambda\to 0$, $H \to H_0$

Then $\lim_{\lambda\to 0}\varphi_H(\vec{x},t) = e^{iH_0(t-t_0)}\varphi_S(\vec{x},t_0)e^{-iH_0(t-t_0)} = \varphi_I(\vec{x},t)$ interaction-picture field

$$\varphi_I(\vec{x},t) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{(2p^0)^{1/2}}\left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right]$$ with $x^0 = t - t_0$

expression for free field

Express φ_H using φ_I: $\varphi_H(\vec{x},t) = e^{iH(t-t_0)}\,\varphi_S(\vec{x},t_0)\,e^{-iH(t-t_0)}$

But $\varphi_S(\vec{x},t_0) = e^{-iH_0(t-t_0)}\,\varphi_I(\vec{x},t)\,e^{iH_0(t-t_0)}$

Thus $\varphi_H(\vec{x},t) = e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}\,\varphi_I(\vec{x},t)\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$

If we define the unitary operator $U(t,t_0) = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$ time-evolution operator or interaction-picture propagator

then $\varphi_H(\vec{x},t) = U^\dagger(t,t_0)\,\varphi_I(\vec{x},t)\,U(t,t_0)$

$$i\frac{\partial U(t,t_0)}{\partial t} = e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} = e^{iH_0(t-t_0)}H_{int}\,e^{-iH_0(t-t_0)}\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$$

$$\Rightarrow i\frac{\partial U(t,t_0)}{\partial t} = H_I(t)\,U(t,t_0) \quad \text{where } H_I(t) = e^{iH_0(t-t_0)}H_{int}\,e^{-iH_0(t-t_0)} = \int \frac{\lambda}{4!}\varphi_I^4\, d^3x$$

solution is $U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt'\,H_I(t')\right]\right\}$

$$= 1 - i\int_{t_0}^{t} dt_1\,H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1\,dt_2\,T\{H_I(t_1)H_I(t_2)\} + \cdots$$

$$= 1 - i\int_{t_0}^{t} dt_1\,H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\,H_I(t_1)H_I(t_2) + \cdots$$

where the time-ordered product T enforces $t_1 > t_2 > \cdots$

For a product of two fields: $T\{\varphi(x_1)\varphi(x_2)\} = \begin{cases}\varphi(x_1)\varphi(x_2) & \text{if } t_1 > t_2\\ \varphi(x_2)\varphi(x_1) & \text{if } t_2 > t_1\end{cases}$

or $T\{\varphi(x_1)\varphi(x_2)\} = \theta(x_1^0 - x_2^0)\,\varphi(x_1)\varphi(x_2) + \theta(x_2^0 - x_1^0)\,\varphi(x_2)\varphi(x_1)$