Two-point function with interaction

$$\langle 0 | T\{\varphi(x)\varphi(y)\exp[-i\int_{-\infty}^{+\infty} dt\, H_I(t)]\} | 0 \rangle$$

$$= \langle 0 | T\{\varphi(x)\varphi(y) + \varphi(x)\varphi(y)(-i)\int_{-\infty}^{+\infty} dt\, H_I(t) + \cdots\} | 0 \rangle$$

$$= D(x-y) + \langle 0 | T\{\varphi(x)\varphi(y)(-i)\int_{-\infty}^{+\infty} dt\, H_I(t)\} | 0 \rangle + \cdots$$

Consider φ^4 theory. Then the second term (with H_I, interaction) is

$$\langle 0 | T\{\varphi(x)\varphi(y)(-i)\int dt \int d^3z\, \frac{\lambda}{4!}\varphi^4\} | 0 \rangle$$

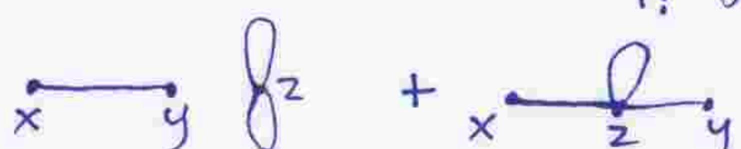
$$= \langle 0 | T\{\varphi(x)\varphi(y)\frac{(-i\lambda)}{4!}\int d^4z\, \varphi(z)\varphi(z)\varphi(z)\varphi(z)\} | 0 \rangle$$

We can apply Wick's theorem to these six φ operators.

Many terms but many of them are the same. In the end we get

$$\langle 0 | T\{\varphi(x)\varphi(y)\frac{(-i\lambda)}{4!}\int d^4z\, \varphi^4_{(z)}\} | 0 \rangle = 3\frac{(-i\lambda)}{4!}D(x-y)\int d^4z\, D(z-z)D(z-z)$$

$$+ 12\frac{(-i\lambda)}{4!}\int d^4z\, D(x-z)D(y-z)D(z-z)$$

Feynmandiagrams 

propagators represent creation, propagation, and annihilation of particles

The interaction also brings vertices where four lines meet: $-i\lambda\int d^4z$ for each vertex

Feynman rules for φ^4 theory

$$\langle 0|T\{\varphi(x)\varphi(y)\exp[-i\int_{-\infty}^{+\infty}dt\, H_I(t)]\}|0\rangle = \langle 0|T\{\varphi(x)\varphi(y)\exp[-i\int d^4z \frac{\lambda}{4!}\varphi^4]\}|0\rangle$$

$$= \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle + \langle 0|T\{\varphi(x)\varphi(y)\left(\frac{-i\lambda}{4!}\right)\int d^4z\, \varphi(z)\varphi(z)\varphi(z)\varphi(z)\}|0\rangle + O(\lambda^2)$$

$$= D(x-y) + 3\left(\frac{-i\lambda}{4!}\right)D(x-y)\int d^4z\, D(z-z)D(z-z) + 12\left(\frac{-i\lambda}{4!}\right)\int d^4z\, D(x-z)D(y-z)D(z-z) + O(\lambda^2)$$

Corresponding Feynman diagrams

So we can write Feynman rules of these diagrams in φ^4 theory

1) For each propagator  we write $D(x-y)$

2) For each vertex  we write $-i\lambda\int d^4z$

Also account for symmetry factors and divide by them

These are position-space Feynman rules

Only connected diagrams contribute to S matrix

The number of lines that meet in each vertex is determined by the interaction: 4 lines in φ^4 theory (3 lines in φ^3 theory)

The vertex factor $-i\lambda$ is amplitude for emission/absorption of particles at a vertex and $\int d^4z$ means that we sum over all points where this can occur → superposition principle (add amplitudes)

As we saw before, the Feynman propagator can be written as

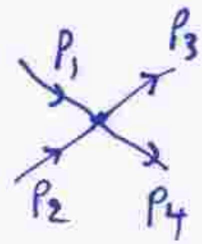
$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

We can use this to write Feynman rules in momentum space for φ^4 theory

1) For each propagator $\xrightarrow[p]{}$ we write $\dfrac{i}{p^2-m^2+i\varepsilon}$
2) For each vertex  we write $-i\lambda$
3) Impose momentum conservation at each vertex
4) Integrate over each undetermined loop momentum $\int \frac{d^4p}{(2\pi)^4}$

Also divide by symmetry factors

Note that for the diagram



we have $\int d^4z \, e^{-ip_1\cdot z} e^{-ip_2\cdot z} e^{ip_3\cdot z} e^{ip_4\cdot z} = (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$

so 4-momentum is conserved at each vertex

Each rule provides an amplitude for that part of the process.
The momentum integrations follow from the superposition principle.
The sum over all relevant diagrams gives $i\mathcal{M}$ where $\mathcal{M}$ is the amplitude for the process

Feynman rules for spinor fields

The Feynman propagator for the Dirac spinor field is

$$S(x-y) = \langle 0 | T\{\psi(x)\bar{\psi}(y)\} | 0 \rangle$$

where $T\{\psi(x)\bar{\psi}(y)\} = \begin{cases} \psi(x)\bar{\psi}(y) & \text{if } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & \text{if } y^0 > x^0 \end{cases}$

Also $T\{\psi(x)\bar{\psi}(y)\} = :\psi(x)\bar{\psi}(y): + S(x-y)$

The Feynman propagator can also be written as

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)} \qquad \text{where } \not{p} = \gamma^\mu p_\mu$$

Note that $S(x-y)$ is a 4×4 matrix since it involves the Dirac matrices

We also note that $(\not{p}+m)(\not{p}-m) = \not{p}\not{p} - m^2 = \gamma^\mu p_\mu \gamma^\nu p_\nu - m^2 = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu p_\nu - m^2$

$\Rightarrow (\not{p}+m)(\not{p}-m) = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}p_\mu p_\nu - m^2 = g^{\mu\nu}p_\mu p_\nu - m^2$ (since $p_\mu p_\nu$ symmetric under $\mu \leftrightarrow \nu$)

$\Rightarrow (\not{p}+m)(\not{p}-m) = p^2 - m^2$ Then $S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p}-m} e^{-ip\cdot(x-y)}$

In momentum space $S(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$ or $S(p) = \frac{i}{\not{p}-m}$

Note that $S(x-y)$ is a Green's function of the Dirac operator:

$(i\gamma^\mu\partial_\mu - m)S(x-y) = i\delta^4(x-y)$ or $(i\not{\partial}-m)S(x-y) = i\delta^4(x-y)$ or $(\not{p}-m)S(x-y) = i\delta^4(x-y)$

Feynman rules for gauge fields

The Feynman propagator for the photon is

$$D_{\mu\nu}(x-y) = \langle 0 | T\{A_\mu(x) A_\nu(y)\} | 0 \rangle$$

In covariant quantization (Lorentz gauge) of the gauge field, this is a generalization of the scalar field case.

Then $D_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{(-i) g_{\mu\nu}}{p^2+i\varepsilon} e^{-ip\cdot(x-y)}$ in Feynman gauge

In momentum space Feynman-gauge propagator $D_{\mu\nu}(p) = \frac{-i\, g_{\mu\nu}}{p^2+i\varepsilon}$ (Note that in general Lorentz gauge $D_{\mu\nu}(p) = \frac{-i}{p^2+i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$)

Feynman rules for Quantum Electrodynamics (QED) in momentum space

1) Dirac propagators (electrons/positrons, etc) $\xrightarrow{p}$ $\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$
2) Dirac propagators for photons $\sim\sim$ $\frac{-i g_{\mu\nu}}{p^2+i\varepsilon}$
3) Vertex factors $-ie\gamma^\mu$
4) External lines for fermions: electrons incoming $\xrightarrow{p}$ $u(p)$ outgoing $\xrightarrow{p}$ $\bar{u}(p)$
   positrons incoming $\xleftarrow{p}$ $\bar{v}(p)$ outgoing $\xleftarrow{p}$ $v(p)$
5) External lines for photons: incoming $\sim\sim$ $\varepsilon_\mu(p)$ outgoing $\sim\sim$ $\varepsilon_\mu^*(p)$

Also conserve 4-momentum at each vertex and integrate $\int \frac{d^4p}{(2\pi)^4}$ over all internal (undetermined) momenta