HW5

PHYS4210: Quantum Mechanics

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Problem 1.

The normalized spherical harmonics are given by

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\phi}P_\ell^m(\cos\phi), \tag{1.1}$$

where

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_\ell(x), \tag{1.2}$$

and

$$P_\ell(x) = \frac{1}{2^\ell l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell. \tag{1.3}$$

Starting with $m=\ell=0$, we can tell pretty easily that $P_0(\cos\theta)=1$, and so too will $P_0^0(\cos\theta)$. Therefore,

$$\boxed{Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}}.} \tag{1.4}$$

For $\ell=1$ and $m=0$,

$$P_1(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) = x \to \cos\theta, \tag{1.5}$$

$$\to P_1^0 = \cos\theta, \tag{1.6}$$

$$\boxed{Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta.}. \tag{1.7}$$

For $\ell=m=1$:

$$P_1(x) = x \to \cos\theta \tag{1.8}$$

$$P_1^1(x) = (-1)\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}(x) = -\sqrt{1-x^2} \to -\sqrt{1-\cos^2\theta} = -\sin\theta, \tag{1.9}$$

$$\boxed{Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta.} \tag{1.10}$$

For $\ell=2$ and $m=0$:

$$\begin{aligned}
P_2(x) &= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 && (1.11)\\
&= \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}x}[4x(x^2-1)] && (1.12)\\
&= \frac{1}{8}[4(x^2-1)+8x^2] && (1.13)\\
&= \frac{1}{8}(12x^2-4) && (1.14)\\
&\to \frac{1}{2}(\cos^2\theta-1). && (1.15)\\
P_2^0(\cos\theta) &= \frac{1}{2}(\cos^2\theta-1). && (1.16)
\end{aligned}$$

$$\boxed{Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(\cos^2\theta-1),} \tag{1.17}$$

where I brought the 1/2 from $P_2(x)$ inside the square root. For $\ell = 2$ and $m = 1$,

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \to \frac{1}{2}(3\cos^2\theta - 1), \tag{1.18}$$

$$P_2^1(x) = -\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(3x^2-1)\right] \tag{1.19}$$

$$\to -\sin\theta \cdot 3x \to -3\sin\theta\cos\theta, \tag{1.20}$$

$$Y_2^1(\theta,\phi) = -3\sqrt{\frac{5}{4\pi}\frac{1}{3!}}e^{i\phi}\sin\theta\cos\theta, \tag{1.21}$$

$$\boxed{Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta.} \tag{1.22}$$

Lastly, for $\ell = 2$ and $m = -2$, we need to use the footnote 5 on page 135 for the definition of the associated Legendre functions for a negative m:

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \to \frac{1}{2})c\cos^2\theta - 1) \tag{1.23}$$

$$P_2^2(x) = (-1)^2(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{1}{2}(3x^2-1)\right] \tag{1.24}$$

$$= 3x^2 \to 3\sin^2\theta, \tag{1.25}$$

$$P_2^{-2}(x) = (-1)^2\frac{1}{4!}\cdot 3x^2 \to \frac{1}{8}\sin^2\theta, \tag{1.26}$$

$$Y_2^{-2}(\theta,\phi) = \frac{1}{8}\sqrt{\frac{5}{4\pi}\cdot 4!}e^{-2i\phi}\sin^2\theta \tag{1.27}$$

$$\boxed{Y_2^{-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}e^{-2i\phi}\sin^2\theta.} \tag{1.28}$$

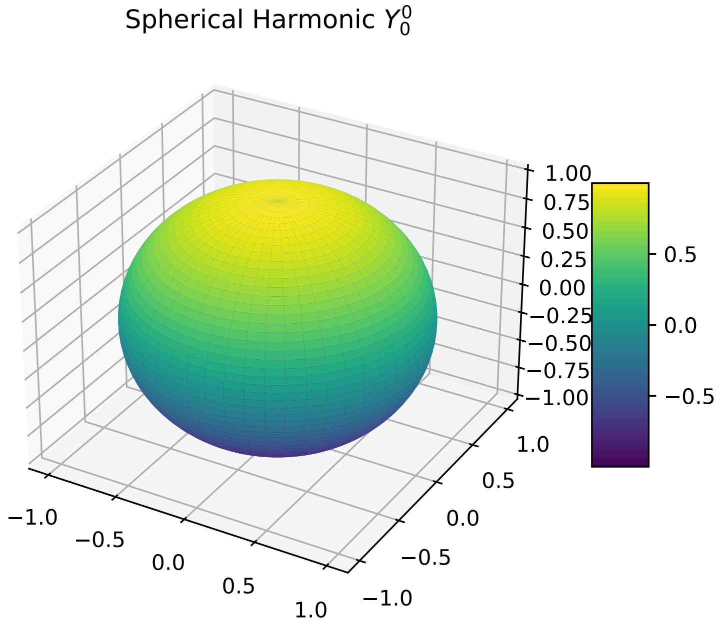

Figure 1: Spherical harmonic $Y_0^0(\theta, \phi)$

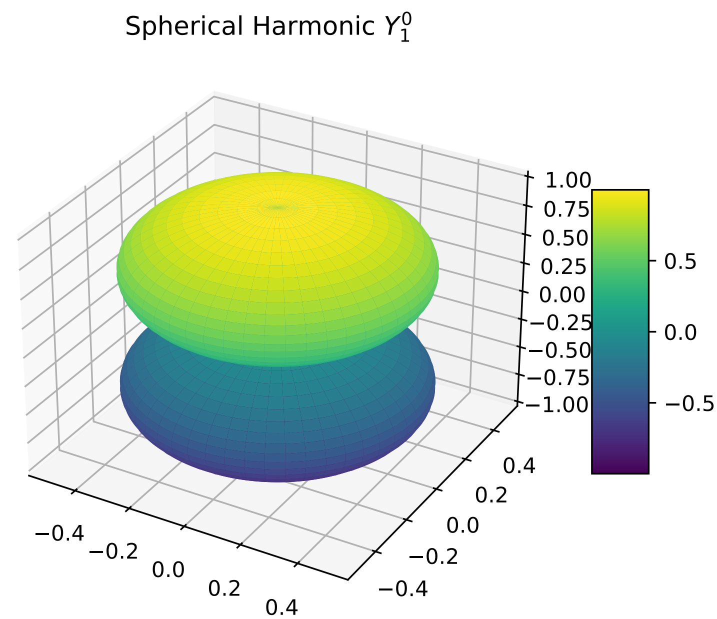

Figure 2: Spherical harmonic $Y_1^0(\theta, \phi)$

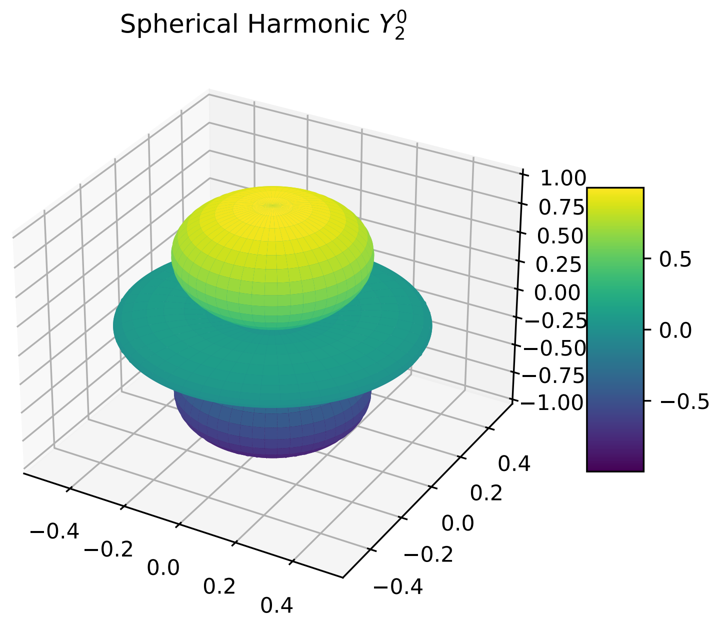

Figure 3: Spherical harmonic $Y_2^0(\theta, \phi)$

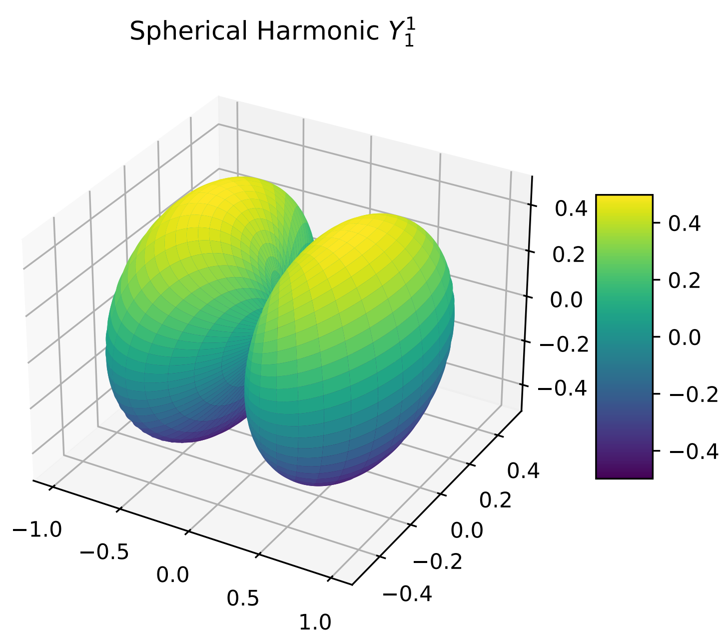

Figure 4: Spherical harmonic $Y_1^1(\theta, \phi)$

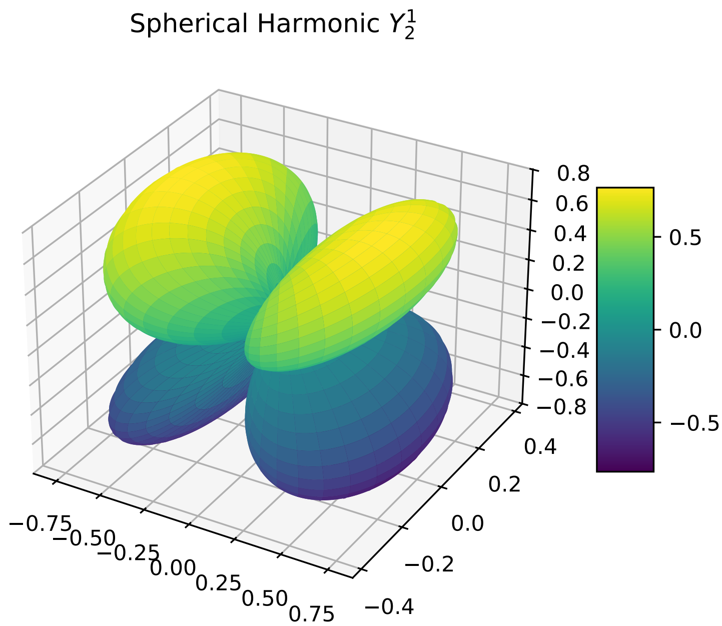

Figure 5: Spherical harmonic $Y_2^1(\theta, \phi)$

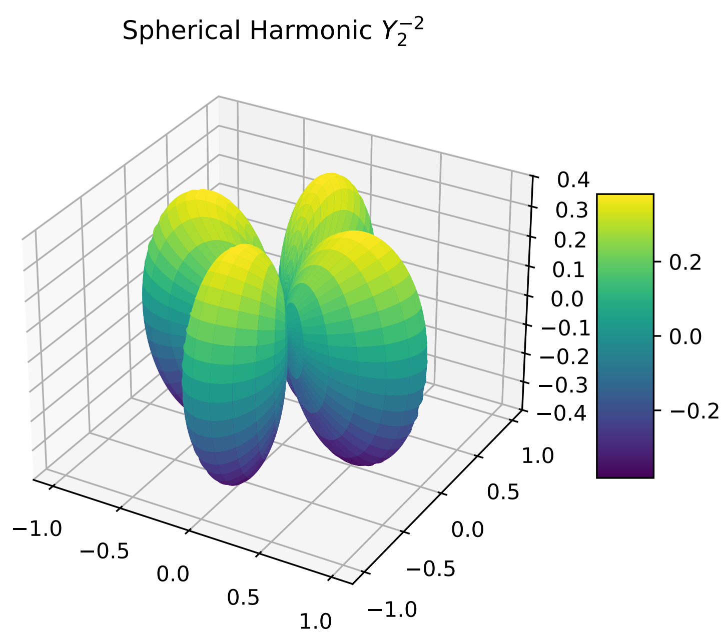

Figure 6: Spherical harmonic $Y_2^{-2}(\theta, \phi)$