HW7
PHYS4500: Quantum Field Theory

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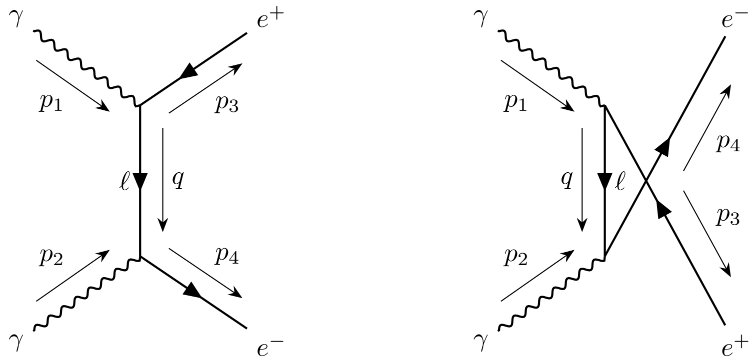
Problem 1. (20.6)

There are two Feynman diagrams here, and the reason (which was given in the book) is that even though we can differentiate between the two final-state particles, it isn't an elastic process, meaning that the photons are annihilated somewhere in the middle and then we get an electron-positron pair. As such, it is just as likely for the electron to come from the first photon than from the second photon, meaning we have to consider both, which involves inclusion of both the t-channel and u-channel diagrams. If it were elastic, like e^+/e^- scattering, then the electron can only come from the vertex the initial electron goes into, so in that case there'd be no u-channel diagram.

The two Feynman diagrams are:



With electron-electron scattering, we kept the locations of the p_3 and p_4 momenta the same, but this time since we can distinguish the two final state particles, they keep the same momentum, so we also switch the momentum labels. Using the Feynman rules, we can pretty easily write down the amplitude for the t-channel diagram. For notational simplicity, I will define $u_i = u^{(s_i)}(p_i)$:

$$\begin{aligned} i\mathcal{M}_t &= \bar{u}_4(-ie\gamma_\mu)\left[\frac{i(\not{q}+m)}{q^2-m^2}\right](-ie\gamma_\nu)v_3(\epsilon_2^\mu\epsilon_1^\nu), \\ &= -ie^2\left[\bar{u}_4\not{\epsilon}_2\frac{\not{p}_1-\not{p}_3+m}{(p_1-p_3)^2-m^2}\not{\epsilon}_1 v_3\right]. \end{aligned}$$

For the other diagram, it is identical except that the polarization vectors become associated with the opposite vertex, so they are in effect switched. Additionally, the virtual lepton momentum is determind by $p_1 - p_4$ now, so:

$$i\mathcal{M}_u = -ie^2\left[\bar{u}_4\not{\epsilon}_1\frac{\not{p}_1-\not{p}_4+m}{(p_1-p_4)^2-m^2}\not{\epsilon}_2 v_3\right].$$

There is no anti-symmetrization, because we are unable to twist lines around and go from one diagram to the other. Summing the two contributions and multiplying by $-i$:

$$\mathcal{M} = -e^2\left[\bar{u}_4\not{\epsilon}_2\frac{\not{p}_1-\not{p}_3+m}{(p_1-p_3)^2-m^2}\not{\epsilon}_1 v_3 + \bar{u}_4\not{\epsilon}_1\frac{\not{p}_1-\not{p}_4+m}{(p_1-p_4)^2-m^2}\not{\epsilon}_2 v_3\right]. \tag{1.1}$$

The book writes matrix elements as $-i\mathcal{M}$, so that's why the book's one is positive and this is negative. It doesn't really matter, though, because the important quantity, the cross section, involves the square of the amplitude.

Problem 2.

The amplitude squared for electron-muon scattering that we got in class was

$$|\mathcal{M}|^2 = \frac{8e^4}{t^2}[p_1\cdot p_2 p_3\cdot p_4 + p_1\cdot p_4 p_2\cdot p_3 - m_e^2 p_2\cdot p_4 - m_\mu^2 p_1\cdot p_3 + 2m_e^2 m_\mu^2]. \tag{2.1}$$

First, we can write

$$s = (p_1 + p_2)^2 = m_e^2 + m_\mu^2 + 2p_1 \cdot p_2,$$

so

$$p_1 \cdot p_2 = \frac{s - m_e^2 - m_\mu^2}{2}.$$

But, since $s = (p_3 + p_4)^2$ from momentum conservation, then this is the same for $p_3 \cdot p_4$. This will also be the case for p_1/p_4 and p_2/p_3, since those momenta pairs are associated with both the muon and the electron, but not p_1/p_3 and p_2/p_4, since those are each associated with only one. We can make some simplifications in the amplitude squared:

$$|\mathcal{M}|^2 = \frac{8e^4}{t^2}[(p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2 - m_e^2 p_2 \cdot p_4 - m_\mu^2 p_1 \cdot p_3 + 2m_e^2 m_\mu^2].$$

Now,

$$\begin{aligned}
(p_1 \cdot p_2)^2 &= \frac{1}{4}(s - m_e^2 - m_\mu^2)^2 = \frac{1}{4}(s^2 + m_e^4 + m_\mu^4 - 2sm_e^2 - 2sm_\mu^2 + 2m_e^2 m_\mu^2), \\
(p_1 \cdot p_4)^2 &= \frac{1}{4}(m_e^2 + m_\mu^2 - u)^2 = \frac{1}{4}(u^2 + m_e^4 + m_\mu^4 - 2um_e^2 - 2um_\mu^2 + 2m_e^2 m_\mu^2).
\end{aligned}$$

So,

$$(p_1 \cdot p_2)^2 + (p_1 \cdot p_4)^2 = \frac{1}{4}[s^2 + u^2 + 2m_e^4 + 2m_\mu^4 - 2(s + u)(m_e^2 + m_\mu^2) + 4m_e^2 m_\mu^2].$$

For the other terms:

$$t = (p_1 - p_3)^2 = 2m_e^2 - 2p_1 \cdot p_3 \quad \rightarrow \quad p_1 \cdot p_3 = \frac{2m_e^2 - t}{2}, \quad \text{and} \quad p_2 \cdot p_4 = \frac{2m_\mu^2 - t}{2},$$

so

$$m_e^2 p_2 \cdot p_4 = \frac{1}{2}(2m_e^2 m_\mu^2 - m_e^2 t) \quad \text{and} \quad m_\mu^2 p_1 \cdot p_3 = \frac{1}{2}(2m_e^2 m_\mu^2 - m_\mu^2 t),$$

hence

$$-m_e^2 p_2 \cdot p_4 - m_\mu^2 p_1 \cdot p_3 = \frac{1}{4}[-8m_e^2 m_\mu^2 + 2t(m_e^2 + m_\mu^2)].$$

The quantity in the brackets in the amplitude squared is therefore

$$\begin{aligned}
&\frac{1}{4}[s^2 + u^2 + 2m_e^4 + 2m_\mu^4 - 2(s + u)(m_e^2 + m_\mu^2) + 4m_e^2 m_\mu^2 - 8m_e^2 m_\mu^2 + 2t(m_e^2 + m_\mu^2) + 8m_e^2 m_\mu^2], \\
&= \frac{1}{4}[s^2 + u^2 + 2m_e^4 + 2m_\mu^4 + 4m_e^2 m_\mu^2 - 2(s - t + u)(m_e^2 + m_\mu^2)].
\end{aligned}$$

A nice property of the Mandelstam variables is that the sum of the Mandelstam variables is the sum of all involved masses squared:

$$s + t + u = 2m_e^2 + 2m_\mu^2,$$

so the last quantity in the brackets becomes:

$$2(s + t + u - 2t)(m_e^2 + m_\mu^2) = 4(m_e^2 + m_\mu^2)^2 - 4t(m_e^2 + m_\mu^2) = 4m_e^4 + 4m_\mu^4 + 8m_e^2 m_\mu^2 - 4t(m_e^2 + m_\mu^2).$$

and our total expression is now

$$= \frac{1}{4}[s^2 + u^2 - 2m_e^4 - 2m_\mu^4 - 4m_e^2 m_\mu^2 + 4t(m_e^2 + m_\mu^2)].$$

With this, the amplitude squared in terms of the Lorentz-invariant Mandelstam variables and the masses is given by

$$|\mathcal{M}|^2 = \frac{2e^4}{t^2}[s^2 + u^2 - 2m_e^4 - 2m_\mu^4 - 4m_e^2 m_\mu^2 + 4t(m_e^2 + m_\mu^2)]. \tag{2.2}$$

We found in class that we can write the differential scattering cross section like

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_e^2, m_\mu^2)},$$

so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{e^4}{8\pi t^2 \lambda(s, m_e^2, m_\mu^2)}[s^2 + u^2 - 2m_e^4 - 2m_\mu^4 - 4m_e^2 m_\mu^2 + 4t(m_e^2 + m_\mu^2)].$$

In the limit where $m_e \to 0$, we get

$$|\mathcal{M}|^2 = \frac{2e^4}{t^2}[s^2 + u^2 - 2m_\mu^2 + 4tm_\mu^2]. \tag{2.3}$$

In the cross section, λ can simplify enough that it looks nice to write:

$$\lambda(s, 0, m_\mu^2) = (s - m_\mu^2)^2,$$

so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{e^4}{8\pi t^2}\frac{s^2 + u^2 - 2m_\mu^2 + 4tm_\mu^2}{(s - m_\mu^2)^2}. \tag{2.4}$$

I don't think this is really simplifiable. Of course, if we take the super high-energy limit where the muon mass also vanishes, it becomes a very nice expression, but that is not what was asked in the problem.

Problem 3. (14.2)

If the decay rate for a particle is $\Gamma = 10\,\mathrm{MeV} = 10^{-2}\,\mathrm{GeV}$, then its average lifetime is given by the inverse of the decay rate:

$$\tau = \frac{1}{\Gamma} = 10^2\,\mathrm{GeV}^{-1}. \tag{3.1}$$

This is in natural units. We know that from our prescription of $\hbar = 1$, this means that we have

$$6.582 \times 10^{-25}\,\mathrm{GeV\,s} = 1, \quad \text{or} \quad 1\,\mathrm{GeV}^{-1} = 6.582 \times 10^{-25}\,\mathrm{s},$$

where we have now related seconds and inverse energy. Now, we can say that

$$\tau = 10^2\,\mathrm{GeV}^{-1} = \boxed{6.582 \times 10^{-23}\,\mathrm{s}.}$$

Next, the half-life is given by $t_{1/2} = \tau \ln 2$, so all we need to do is multiply by $\ln 2$:

$$t_{1/2} = \ln 2 \times 6.582 \times 10^{-23}\,\mathrm{s} = \boxed{4.562 \times 10^{-23}\,\mathrm{s}.}$$

Problem 4. (14.4)

Just like before, we can use the fact that $\hbar = 1$ to say that

$$6.582 \times 10^{-25}\,\mathrm{GeV\,s} = 1 \quad \rightarrow \quad 1\,\mathrm{s} = 1.519 \times 10^{24}\,\mathrm{GeV}^{-1}.$$

Additionally, from our prescription of $c = 1$, we have that

$$3.00 \times 10^{8}\,\mathrm{m\,s}^{-1} = 1 \quad \rightarrow \quad 3.00 \times 10^{8}\,\mathrm{m} = 1\,\mathrm{s}.$$

Therefore,

$$\begin{aligned} 3.00 \times 10^{8}\,\mathrm{m} &= 1.519 \times 10^{24}\,\mathrm{GeV}^{-1}, \\ 1\,\mathrm{m}^2 &= 2.564 \times 10^{31}\,\mathrm{GeV}^{-2}, \end{aligned}$$

so

$$\boxed{1\,\mathrm{b} = 2.564 \times 10^{3}\,\mathrm{GeV}^{-2}.}$$