HW3
PHYS4500: Quantum Field Theory

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Problem 1. (4.3)

We are told that a particle of mass m moves in the x-direction under some potential $U = bx$.

a) The kinetic energy is simply the classical $T = \frac{1}{2}m\dot{x}^2$, and the potential is given, so our Lagrangian is:

$$L = \frac{1}{2}m\dot{x}^2 - bx. \tag{1.1}$$

b) The first term in the Euler-Lagrange equation is

$$\frac{\partial L}{\partial x} = -b,$$

and the second term is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}[m\dot{x}] = m\ddot{x}.$$

So, the full Euler-Lagrange equation is:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = -b - m\ddot{x} = 0,$$
$$\rightarrow \boxed{m\ddot{x} + b = 0.}$$

c) To solve, we first get just $\ddot{x}$ on one side:

$$\ddot{x} = -\frac{b}{m}.$$

Then we just integrate:

$$\dot{x} = -\frac{b}{m}\int \mathrm{d}t = -\frac{b}{m}t + \dot{x}_0,$$
$$x = \int\left(-\frac{b}{m}t + \dot{x}_0\right)\mathrm{d}t = -\frac{b}{2m}t^2 + \dot{x}_0 t + x_0,$$
$$\rightarrow \boxed{x(t) = -\frac{b}{2m}t^2 + \dot{x}_0 t + x_0.}$$

Problem 2. (5.5)

We are given the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + a + b\phi - \frac{1}{2}m^2\phi^2. \tag{2.1}$$

a) The Euler-Lagrange equation contains only derivatives of the Lagrangian, meaning that any pure constants, like a, have no effect on the resulting equation(s) of motion, so we might as well remove it to get

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + b\phi - \frac{1}{2}m^2\phi^2. \tag{2.2}$$

b) By shifting the field like $\phi \rightarrow \phi' = \phi + d$ where d is some constant, we can eliminate the $b\phi$ term using the mass term, in which there is a ϕ^2. Leaving it as d, we have:

$$\frac{1}{2}m^2(\phi + d)^2 = \frac{1}{2}m^2(\phi^2 + 2\phi d + d^2).$$

Again, since constants have no effect on the resulting equations of motion, we might as well get rid of the d^2 term. Expanding:

$$= \frac{1}{2}m^2\phi^2 + m^2 d\phi.$$

Plugging this into the Lagrangian Density (and recognizing that the extra factor of d might as well disappear inside the 4-derivatives as well):

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + b\phi - \frac{1}{2}m^2\phi^2 - m^2 d\phi.$$

The first and third terms are the Klein-Gordon Lagrangian, so we just need to choose d such that the second and fourth terms cancel. This choice is

$$d = \frac{b}{m^2}$$

So, by shifting the field $\phi' = \phi + \frac{b}{m^2}$, we can eliminate the $b\phi$ term and recover the original Klein-Gordon field Lagrangian.

Problem 3. (6.2b)

We are given the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2. \tag{3.1}$$

The energy density is given as:

$$\begin{aligned}\mathcal{H} \equiv T^{00} &= \left[\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_n)}\partial^0\phi_n\right] - g^{00}\mathcal{L},\\ &= \left[\sum_n \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}\dot{\phi}_n\right] - \mathcal{L}.\end{aligned}$$

Writing the Lagrangian in a more suggestive way:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_0\phi_1\partial^0\phi_1 + \frac{1}{2}\partial_i\phi_1\partial^i\phi_1 + \frac{1}{2}\partial_0\phi_2\partial^0\phi_2 + \frac{1}{2}\partial_i\phi_2\partial^i\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2,\\ &= \frac{1}{2}\left(\dot{\phi}_1\right)^2 + \frac{1}{2}\left(\dot{\phi}_2\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi_1\right)^2 - \frac{1}{2}\left(\boldsymbol{\nabla}\phi_2\right)^2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2.\end{aligned}$$

Now, we compute

$$\begin{aligned}&\mathcal{H} = \dot{\phi}_1^2 + \dot{\phi}_2^2 - \left[\frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 - \frac{1}{2}\boldsymbol{\nabla}\phi_1^2 - \frac{1}{2}\boldsymbol{\nabla}\phi_2^2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2\right],\\ &\boxed{\mathcal{H} = \frac{1}{2}\left[\dot{\phi}_1^2 + \dot{\phi}_2^2 + (\boldsymbol{\nabla}\phi_1)^2 + (\boldsymbol{\nabla}\phi_2)^2 + m_1^2\phi_1^2 + m_2\phi_2^2 + 2\lambda\phi_1^2\phi_2^2\right].}\end{aligned}$$

Or, maybe slightly more compactly (and more generally):

$$\boxed{\mathcal{H} = \frac{1}{2}\sum_{n=1}^{2}\left[\dot{\phi}_n^2 + (\boldsymbol{\nabla}\phi_n)^2 + (m_n\phi_n)^2\right] + 2\lambda\prod_{n=1}^{2}\phi_n^2.}$$

Problem 4.

The expression for the real scalar field as a Fourier expansion over the creation and annihilation operators is

$$\phi(x^\mu) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2E}}\left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right], \tag{4.1}$$

and the Klein-Gordon equation for a real, free scalar field is

$$(\partial_\mu \partial^\mu + m^2)\phi = 0. \tag{4.2}$$

Starting with the first derivative, since we are looking at the free field, we will have that $\partial_\mu p^\mu$ is zero, so anything which is a function of purely momentum (specifically, the creation/annihilation operators) will have its 4-gradient will be zero:

$$\begin{aligned}
\partial^\mu \phi &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E}} \partial^\mu \left[a(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\right], \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E}} \left[a(p) \partial^\mu e^{-ip\cdot x} + a^\dagger(p) \partial^\mu e^{ip\cdot x}\right], \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E}} \left[-ip^\mu a(p) e^{-ip\cdot x} + ip^\mu a^\dagger(p) e^{ip\cdot x}\right].
\end{aligned}$$

Doing the second derivative:

$$\begin{aligned}
\partial_\mu \partial^\mu \phi &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E}} \left[-p_\mu p^\mu a(p) e^{-ip\cdot x} - p_\mu p^\mu a^\dagger(p) e^{ip\cdot x}\right], \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E}} (-m^2) \left[a(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\right],
\end{aligned}$$

where I've used that $p^2 = m^2$. So,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E}} (m^2 - m^2) \left[a(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x}\right] = 0,$$

as expected.