HW1

PHYS4210: Quantum Mechanics

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Problem 1. (2.5)

We are given the initial wave function

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)], \tag{1.1}$$

where $\psi_1(x)$ and $\psi_2(x)$ are the $n = 1$ and $n = 2$ stationary states for the infinite square well.

a) First, we need to find A:

$$\langle \Psi(x,0) \,|\, \Psi(x,0) \rangle = A^2 \int_{-\infty}^{\infty} \left(|\psi_1(x)|^2 + |\psi_2(x)|^2 + \psi_1(x)^* \psi_2(x) + \psi_1(x)\psi_2(x)^* \right) \, \mathrm{d}x.$$

We know that the solutions to the TISE form an orthonormal set, so the first two terms are 1 and the second two terms are zero, meaning that:

$$\langle \Psi(x,0) \,|\, \Psi(x,0) \rangle = A^2 \int_{-\infty}^{\infty} (1 + 1 + 0 + 0) \, \mathrm{d}x = 2A^2 = 1,$$

$$\rightarrow \boxed{A = \frac{1}{\sqrt{2}}.}$$

Thus,

$$\boxed{\Psi(x,0) = \frac{1}{\sqrt{2}}[\psi_1(x) + \psi_2(x)].}$$

b) Next, we need to $\Psi(x,t)$ and $|\Psi(x,t)|^2$. To find the former, we know that the general solution is a linear combination of the stationary states:

$$\Psi(x,t) = \sum_n c_n \psi_n(x),$$

so, since we only have $n = 1$ and $n = 2$, we have

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2}} \left[\psi_1(x) e^{-iE_1 t/\hbar} + \psi_2(x) e^{-iE_2 t/\hbar} \right], \\
&= \frac{1}{\sqrt{a}} \left[\sin\left(\frac{\pi x}{a}\right) e^{-iE_1 t/\hbar} + \sin\left(\frac{2\pi x}{a}\right) e^{-iE_2 t/\hbar} \right].
\end{aligned}$$

Now,

$$\begin{aligned}
|\Psi(x,t)|^2 = \frac{1}{a} \Bigg[&\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) e^{iE_1 t/\hbar} e^{-iE_2 t/\hbar} \\
&+ \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) e^{-iE_1 t/\hbar} e^{iE_2 t/\hbar} \Bigg]
\end{aligned}$$

$$\begin{aligned}
|\Psi(x,t)|^2 &= \frac{1}{a} \left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \left(e^{-i(E_2 - E_1)t/\hbar} + e^{i(E_2 - E_1)t/\hbar} \right) \right], \\
&= \frac{1}{a} \left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{E_2 - E_1}{\hbar} t\right) \right].
\end{aligned}$$

We know that for the simple harmonic oscillator the energies are quantized:

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},$$

so

$$\frac{E_2 - E_1}{\hbar} = 3\frac{\pi^2\hbar}{2ma^2} = 3\omega$$

with $\omega \equiv \pi^2\hbar/2ma^2$. Thus,

$$\boxed{|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right].}$$

c) To compute $\langle x\rangle$:

$$\begin{aligned}
\langle x\rangle = \langle \Psi(x,t)\,|\,x\,|\,\Psi(x,t)\rangle &= \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 \,\mathrm{d}x, \\
&= \frac{1}{a}\int_{-a}^{a} x\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\right]\,\mathrm{d}x + \frac{\cos(3\omega t)}{a}\int_{-a}^{a} x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x
\end{aligned}$$

For the first set of integrals, we will have something of the form:

$$\begin{aligned}
\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x &= \frac{1}{2}\int_0^a x\left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right]\,\mathrm{d}x, \\
&= \frac{1}{2}\int_0^a x\,\mathrm{d}x - \frac{1}{2}\int_0^a x\cos\left(\frac{2n\pi x}{a}\right)\,\mathrm{d}x.
\end{aligned}$$

Now, the second term, if we were to do integration by parts, would contain sines for both terms, which, when evaluated at the limits we have, will be zero, so we only have the first integral:

$$\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x = \frac{a^2}{4}.$$

This is independent of n! The second integral we will have to do is

$$\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x = 2\int_0^a x\sin^2\left(\frac{\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x.$$

If we do integration by parts with $u = x$ and $\mathrm{d}v = \sin^2(\pi x/a)\cos(\pi x/a)\mathrm{d}x$, we will need to do a more complicated integral to find v. This integral is:

$$\int \sin^2\left(\frac{\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x \to \begin{bmatrix} w = \sin(\pi x/a), \\ \mathrm{d}w = \dfrac{\pi}{a}\cos(\pi x/a) \end{bmatrix} \to \frac{a}{\pi}\int w^2\,\mathrm{d}w = \frac{a\sin^3(\frac{\pi x}{a})}{3\pi}.$$

So,

$$2\int_0^a x\sin^2\left(\frac{\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x = 2\left\{\frac{a}{\pi}\left[\sin^3\left(\frac{\pi x}{a}\right)\right]_0^a - \frac{a}{3\pi}\int \sin^3\left(\frac{\pi x}{a}\right)\,\mathrm{d}x\right\}.$$

The first term in braces is zero trivially, so we have

$$\begin{aligned}
-\frac{2a}{3\pi}\int_0^a \sin^3\left(\frac{\pi x}{a}\right)\,\mathrm{d}x &= -\frac{a}{6\pi}\int_0^a\left[3\sin\left(\frac{\pi x}{a}\right)-\sin\left(\frac{3\pi x}{a}\right)\right]\,\mathrm{d}x,\\
&= -\frac{a^2}{2\pi^2}\left[\cos\left(\frac{\pi x}{a}\right)\right]_0^a+\frac{a^2}{18\pi^2}\left[\cos\left(\frac{3\pi x}{a}\right)\right]_0^a,\\
&= \frac{a^2}{\pi^2}-\frac{a^2}{9\pi^2}=-\frac{8a^2}{9\pi^2}.
\end{aligned}$$

Thus,

$$\langle x\rangle = \frac{1}{a}\left(\frac{a^2}{4}+\frac{a^2}{4}\right)+\frac{\cos(3\omega t)}{a}\left(-\frac{8a^2}{9\pi^2}\right)=\boxed{\frac{a}{2}-\frac{16a\cos(3\omega t)}{9\pi^2}.}$$

d) Now we need to compute $\langle p\rangle$, but this one is easy:

$$\langle p\rangle = m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t}=\frac{16ma\omega\sin(3\omega t)}{3\pi^2}.$$

But

$$\frac{16ma\omega}{3\pi^2}=\frac{16ma}{3\pi^2}\left(\frac{\pi^2\hbar}{2ma^2}\right)=\frac{8\hbar}{3a},$$

so

$$\boxed{\langle p\rangle = \frac{8\hbar}{3a}\sin(3\omega x).}$$

e) We can get either E_1 or E_2 since our wave function consists only of the $n=1$ and $n=2$ states:

$$E_1=\frac{\pi^2\hbar^2}{2ma^2},\quad \text{and}\quad E_2=\frac{2\pi^2\hbar^2}{ma^2}.$$

The probability of getting any one state is tied to its coefficient, from orthonormality of $\{\psi_n\}$:

$$P(E=E_n)=|c_n|^2.$$

So, since $c_n=1/\sqrt{2}$ for both states, then they both have a probability of 1/2.
Lastly,

$$\langle H\rangle=\sum_n|c_n|^2E_n=\frac{1}{2}(E_1+E_2)=\boxed{\frac{5\pi^2\hbar^2}{4ma^2}.}$$

The expecation value of the total energy is equal to the average of the two individual energies of the states that make up our total wavefunction.

Problem 2. (2.7)

We are given the wave function

$$\Psi(x,0) = \begin{cases} Ax & 0 \le x \le a/2, \\ A(a-x) & a/2 \le x \le a. \end{cases} \tag{2.1}$$

a) First, we normalize:

$$\begin{aligned} \langle \Psi \mid \Psi \rangle = A^2 \int_0^{a/2} x^2 \, \mathrm{d}x + A^2 \int_{a/2}^{a} (a-x)^2 \, \mathrm{d}x = 1, \\ = \frac{A^2}{3} \frac{a^3}{8} - \frac{A^2}{3} \left[(a-x)^3 \right]_{a/2}^{a}, \\ = A^2 \left[\frac{a^3}{24} + \frac{1}{3} \frac{a^3}{8} \right] = \frac{A^2 a^3}{12}, \end{aligned}$$

so we have that

$$\boxed{A = \sqrt{\frac{12}{a^3}} = \frac{2\sqrt{3}}{a\sqrt{a}}.}$$

As a quick sketch:

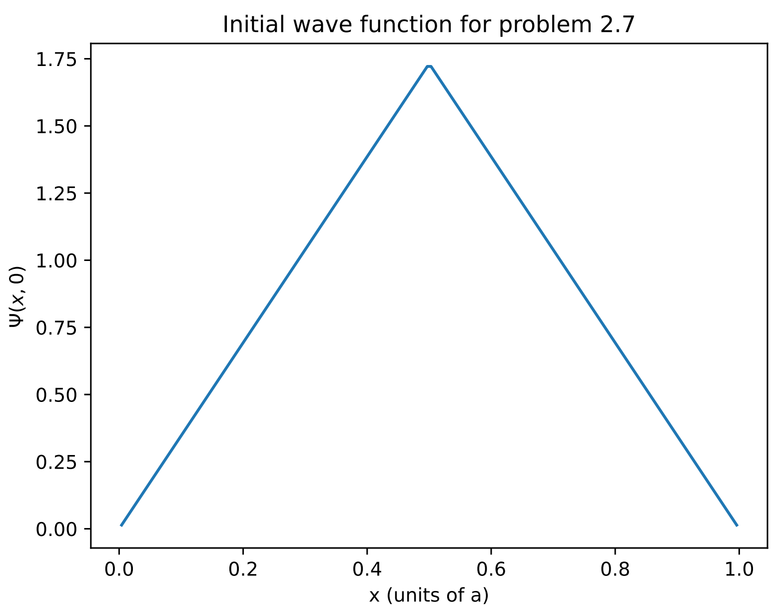

Figure 1: Sketch of the initial wave function Eq. (2.1).

b) The full solution is given by:

$$\Psi(x,t) = \sum_n c_n \psi_n(x),$$

but to find the c_n's, we know by the completeness of the stationary states that we can also express the initial wavefunction as a linear combination of the stationary states:

$$\Psi(x,0) = \sum_n c_n \psi_n(x),$$

where these c_n's would be different, of course, and the stationary states are given by

$$\psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right).$$

We can then invoke tactics from Fourier analysis to find the coefficients:

$$c_n = \int \psi_n \Psi(x,0)\ \mathrm{d}x = \sqrt{\frac{12}{a^3}}\sqrt{\frac{2}{a}}\left[\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x + \int_{a/2}^{a} (a-x)\sin\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x\right].$$

Looking at the first integral:

$$\begin{aligned}
\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x &= \left[-\frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)_0^{a/2}\right] + \frac{a}{n\pi}\int_0^{a/2}\cos\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x,\\
&= -\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{a}\right)\right]_0^{a/2},\\
&= -\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

Now for the second integral:

$$\int_{a/2}^{a} (a-x)\sin\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x = a\int_{a/2}^{a}\sin\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x - \int_{a/2}^{a} x\sin\left(\frac{n\pi x}{a}\right)\ \mathrm{d}x.$$

We can use the results of the previous integral for the second integral here:

$$\begin{aligned}
&= -\frac{a^2}{n\pi}\left[\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a} - \left\{\left[-\frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a} + \frac{a^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{a}\right)\right]_{a/2}^{a}\right\},\\
&= -\frac{a^2}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right] - \left\{-\frac{a^2}{n\pi}\cos(n\pi) + \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right]\right\},\\
&= -\frac{a^2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n\pi}\cos(n\pi) - \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right),\\
&= \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

Adding this with the previous integral gets us:

$$-\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \frac{2a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

So,

$$c_n = \sqrt{\frac{24}{a^4}} \cdot \frac{2a^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{(n+1)/2},$$

where this is only valid for odd n. Therefore,

$$\boxed{\Psi(x,t) = \frac{4\sqrt{6}}{\pi^2}\sqrt{\frac{2}{a}} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n+1)/2}}{n^2} \sin\left(\frac{n\pi x}{a}\right) e^{-i(n^2\pi^2\hbar/2ma^2)t}.}$$

c) We know that such a probability is equal to $|c_n|^2$, so:

$$P(E=E_1) = \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 \approx \boxed{0.9855.}$$

d) The expectation value of the energy is:

$$\langle H \rangle = \sum_n |c_n|^2 E_n = \left(\frac{4\sqrt{6}}{n^2\pi^2}\right)^2 \frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{48\hbar^2}{m\pi^2 a^2} \sum_{n=1,3,5,\ldots} \frac{1}{n^2}. \tag{2.2}$$

I looked up the series, it turns out is is equal to $\pi^2/8$, so:

$$\boxed{\langle H \rangle = \frac{6\hbar^2}{ma^2}.}$$

Problem 3. (2.11)

We will need:

$$\psi_0 = \alpha e^{-\frac{m\omega}{2\hbar}x^2}, \qquad \text{and} \qquad \psi_1 = \alpha\sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2}, \tag{3.1}$$

where

$$\alpha \equiv \sqrt[4]{\frac{m\omega}{\pi\hbar}}. \tag{3.2}$$

To make things easier, we introduce a change of variables with

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x,$$

so

$$\psi_0 = \alpha e^{-\xi^2/2}, \qquad \text{and} \qquad \psi_1 = \alpha\sqrt{2}\xi e^{-\xi^2/2}$$

a) Let's start with $\langle x \rangle$ for ψ_0:

$$\langle x \rangle = \langle \psi_0 \,|\, x \,|\, \psi_0 \rangle = \alpha^2 \int_{-\infty}^{\infty} x e^{-\xi^2}\,\mathrm{d}x = \alpha^2 \left(\frac{\hbar}{m\omega}\right) \int_{-\infty}^{\infty} \xi e^{-\xi^2}\,\mathrm{d}\xi.$$

However, ξ is odd and the exponential is even, so the integrand itself is odd, which, when integrated over a symmetric interval, is zero. Also, since we are looking at individual stationary states, we know that we know that time-dependence won't change them, so we can say:

$$\langle p \rangle = m\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = 0.$$

For $\langle x^2 \rangle$:

$$\langle x^2\rangle = \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\xi^2}\, \mathrm{d}x = 2\alpha^2 \left(\frac{\hbar}{m\omega}\right)^{3/2} \int_0^{\infty} \xi^2 e^{-\xi^2}\, \mathrm{d}\xi.$$

Using the integral table in the back of the book, the integral evaluate to $\sqrt{\pi}/4$, so

$$\langle x^2\rangle = 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{\hbar}{m\omega}\right)^{3/2} \frac{\sqrt{\pi}}{4} = \frac{1}{2}\left(\frac{\hbar}{m\omega}\right)^{-1/2} \left(\frac{\hbar}{m\omega}\right)^{3/2} = \boxed{\frac{\hbar}{2m\omega}.}$$

Now for $\langle p^2\rangle$:

$$\langle p^2\rangle = -\hbar^2\alpha^2 \int_{-\infty}^{\infty} e^{-\xi^2/2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[e^{-\xi^2/2}\right] \mathrm{d}x.$$

The derivative term is:

$$\left(\frac{m\omega}{\hbar}\right) \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[e^{-\xi^2/2}\right] = \left(\frac{m\omega}{\hbar}\right) \frac{\mathrm{d}}{\mathrm{d}\xi}\left[-\xi e^{-\xi^2/2}\right] = \left(\frac{m\omega}{\hbar}\right)\left(-e^{-\xi^2/2} + \xi^2 e^{-\xi^2/2}\right),$$

so

$$\begin{aligned}
\langle p^2\rangle &= -2\hbar^2\alpha^2 \left(\frac{m\omega}{\hbar}\right) \sqrt{\frac{\hbar}{m\omega}} \left[-\int_0^{\infty} e^{-\xi^2}\, \mathrm{d}x + \int_0^{\infty} \xi^2 e^{\xi^2}\, \mathrm{d}x\right], \\
&= -2\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\hbar}{m\omega}} \left(\frac{m\omega}{\hbar}\right) \left(-\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{4}\right), \\
&= \frac{\hbar m\omega}{2}.
\end{aligned}$$

So, for ψ_0, we have

$$\begin{aligned}
\langle x\rangle &= 0, \\
\langle x^2\rangle &= \frac{\hbar}{2m\omega}, \\
\langle p\rangle &= 0, \\
\langle p^2\rangle &= \frac{\hbar m\omega}{2}.
\end{aligned}$$

Thus,

$$\sigma_x\sigma_p = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)\left(\sqrt{\frac{\hbar m\omega}{2}}\right) = \frac{\hbar}{2} \geq \frac{\hbar}{2}. \checkmark$$

Now we turn to $\psi_1(x)$; starting with $\langle x\rangle$:

$$\langle x\rangle = 2\alpha^2 \int_{-\infty}^{\infty} \xi^2 x e^{-\xi^2/2}\, \mathrm{d}x = 2\alpha^2 \left(\frac{\hbar}{m\omega}\right) \int_{-\infty}^{\infty} \xi^3 e^{-\xi^2/2}\, \mathrm{d}\xi.$$

We can again stop here since we have an odd integrand, so both $\langle x\rangle$ and $\langle p\rangle$ are zero. Turning to $\langle x^2\rangle$:

$$\begin{aligned}\langle x^2 \rangle &= 2\alpha^2 \int_{-\infty}^{\infty} x^2 \xi^2 e^{-\xi^2}\, \mathrm{d}x, \\ &= 4\sqrt{\frac{m\omega}{\pi\hbar}} \left(\frac{\hbar}{m\omega}\right)^{3/2} \int_0^{\infty} \xi^4 e^{-\xi^2}\, \mathrm{d}\xi.\end{aligned}$$

From the integral table, we can find that the integral evaluates to $3\sqrt{\pi}/8$, so:

$$\langle x^2 \rangle = \frac{4}{\sqrt{\pi}} \left(\frac{\hbar}{m\omega}\right) \frac{3\sqrt{\pi}}{8} = \frac{3\hbar}{2m\omega}.$$

Now for $\langle p^2 \rangle$:

$$\langle p^2 \rangle = -2\hbar^2\alpha^2 \int_{-\infty}^{\infty} \xi e^{-\xi^2/2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[\xi e^{-\xi^2/2}\right] \mathrm{d}x.$$

The derivative term is:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\xi e^{-\xi^2/2}\right] &= \left(\frac{m\omega}{\hbar}\right) \frac{\mathrm{d}}{\mathrm{d}\xi}\left[-\xi^2 e^{-\xi^2/2} + e^{-\xi^2/2}\right], \\ &= \left(\frac{m\omega}{\hbar}\right)\left(-2\xi e^{-\xi^2/2} + \xi^3 e^{-\xi^2/2} - \xi e^{-\xi^2/2}\right), \\ &= \left(\frac{m\omega}{\hbar}\right)(\xi^3 - 3\xi)e^{-\xi^2/2},\end{aligned}$$

so

$$\begin{aligned}\langle p^2 \rangle &= -4\hbar^2\alpha^2 \left(\frac{m\omega}{\hbar}\right)\sqrt{\frac{\hbar}{m\omega}} \left[\int_0^{\infty} \xi^4 e^{-\xi^2}\, \mathrm{d}\xi - 3\int_0^{\infty} \xi^2 e^{-\xi^2}\, \mathrm{d}\xi\right], \\ &= -4\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{m\omega}{\hbar}} \left(\frac{3\sqrt{\pi}}{8} - \frac{3\sqrt{\pi}}{4}\right), \\ &= \frac{3\hbar m\omega}{2}.\end{aligned}$$

Thus,

$$\sigma_x\sigma_p = \sqrt{\frac{3\hbar}{2m\omega}}\sqrt{\frac{3\hbar m\omega}{2}} = 3\frac{\hbar}{2} \geq \frac{\hbar}{2}. \checkmark$$

b) Starting with ψ_0, we can use the definition of kinetic energy:

$$T = \frac{p^2}{2m} \ \rightarrow \langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar\omega}{4}.$$

Similar for potential:

$$V = \frac{1}{2}m\omega^2 x^2 \ \rightarrow \langle V \rangle = \frac{1}{2}m\omega^2 \langle x^2 \rangle = \frac{\hbar\omega}{4}$$

Their sum is $\hbar\omega/2$, which is exactly what the total energy of the $n = 0$ state is, as expected.

Now for $n = 1$, we have

$$\langle T \rangle = \frac{3\hbar\omega}{4}, \quad \langle V \rangle = \frac{3\hbar\omega}{4},$$

whose sum is $3\hbar\omega/2$, which is also the total energy of the $n = 1$ state using the forula $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

Problem 4. (2.21)

We are given the initial wavefunction

$$\Psi(x, 0) = Ae^{-ax^2}. \tag{4.1}$$

a) Let's find A:

$$\langle \Psi \mid \Psi \rangle = 2A^2 \int_0^\infty e^{-2ax^2}\,\mathrm{d}x = \frac{2A^2}{\sqrt{2a}} \int_0^\infty e^{-u^2}\,\mathrm{d}u = \frac{2A^2}{\sqrt{2a}} \frac{\sqrt{\pi}}{2} = 1,$$

so

$$\boxed{A = \sqrt[4]{\frac{2a}{\pi}}.}$$

b) To find $\Psi(x, t)$, we need to find the coefficient function $\phi(k)$, which can be done using Fourier transforms:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \Psi(x, 0)e^{-ikx}\,\mathrm{d}x = \sqrt[4]{\frac{2a}{\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-(ax^2+ikx)}\,\mathrm{d}x.$$

Using the hint from the book, we can "complete the square" by making the change of variables:

$$u = \sqrt{a}\left(x + \frac{ik}{2a}\right), \quad \text{so} \quad \mathrm{d}x = \frac{1}{\sqrt{a}}\mathrm{d}u.$$

Now, to check,

$$u^2 + \frac{k^2}{4a} = a\left(x^2 - \frac{k^2}{4a^2} + \frac{ikx}{a}\right) + \frac{k^2}{4a} = ax^2 - \frac{k^2}{4a} + ikx + \frac{k^2}{4a} = ax^2 + ikx,$$

which is exactly what we have in our exponential. So,

$$\begin{aligned}
\phi(k) &= \sqrt[4]{\frac{2a}{\pi}} \frac{1}{\sqrt{2a\pi}} \int_{-\infty}^\infty e^{-(u^2+k^2/4a)}\,\mathrm{d}u, \\
&= \sqrt[4]{\frac{2a}{\pi}} \frac{2}{\sqrt{2a\pi}} e^{-k^2/4a} \int_0^\infty e^{-u^2}\,\mathrm{d}u, \\
&= \sqrt[4]{\frac{2a}{\pi}} \frac{2}{\sqrt{2a\pi}} e^{-k^2/4a} \frac{\sqrt{\pi}}{2}, \\
\phi(x) &= \frac{e^{-k^2/4a}}{(2\pi a)^{1/4}}
\end{aligned}$$

Now,

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}\,\mathrm{d}k,\\ &= \frac{1}{(2\pi a)^{1/4}\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-k^2/4a}e^{i(kx-\frac{\hbar k^2}{2m}t)}\,\mathrm{d}k,\\ &= \frac{1}{(2\pi a)^{1/4}\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-[(1/4a+i\hbar t/2m)k^2-ixk]}\,\mathrm{d}x.\end{aligned}$$

Doing the same thing as before, we will have

$$u = \sqrt{\frac{1}{4a}+\frac{i\hbar t}{2m}}\left[k-\frac{ix}{(1/2a+i\hbar t/m)}\right],\quad \text{so}\quad \frac{b^2}{4a} = \frac{-x^2}{(1/a+2i\hbar t/m)},$$

so

$$\begin{aligned}\Psi(x,t) &= \frac{1}{(2\pi a)^{1/4}\sqrt{2\pi}}\frac{1}{\sqrt{(1/4a+i\hbar t/2m)}}\int_{-\infty}^{\infty}e^{-u^2}e^{-x^2/(1/a+2i\hbar t/m)}\,\mathrm{d}x,\\ &\frac{2\sqrt{2}e^{-ax^2/(1+2i\hbar at/m)}}{(2\pi a)^{1/4}\sqrt{\pi}}\frac{1}{\sqrt{(1/a+2i\hbar t/m)}}\int_{0}^{\infty}e^{-u^2}\,\mathrm{d}u,\\ &= \frac{\sqrt{2a}e^{-ax^2/(1+2i\hbar at/m)}}{(2\pi a)^{1/4}}\frac{1}{\sqrt{(1+2i\hbar at/m)}},\\ &\boxed{\Psi(x,t) = \sqrt[4]{\frac{2a}{\pi}}\frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{(1+2i\hbar at/m)}}.}\end{aligned}$$

Using the book's suggestion of $\gamma = \sqrt{1+2i\hbar at/m}$:

$$\boxed{\Psi(x,t) = \sqrt[4]{\frac{2a}{\pi}}\frac{1}{\gamma}e^{-ax^2/\gamma^2}.}$$

c) Squaring this,

$$|\Psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+(2\hbar at/m)^2}}e^{-ax^2/(1+2i\hbar at/m)}e^{-ax^2/(1-2i\hbar at/m)}.$$

Looking just at the exponentials real quick:

$$-ax^2\left(\frac{1-2i\hbar at/m+1+2i\hbar at/m}{1+(2\hbar at/m)^2}\right) = \frac{-2ax^2}{1+(2\hbar at/m)^2} - -2\omega^2x^2,$$

with $\omega = \sqrt{a/[(1+(2\hbar at/m)^2)]}$. So,

$$\boxed{|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}\omega e^{-2\omega^2x^2}.}$$

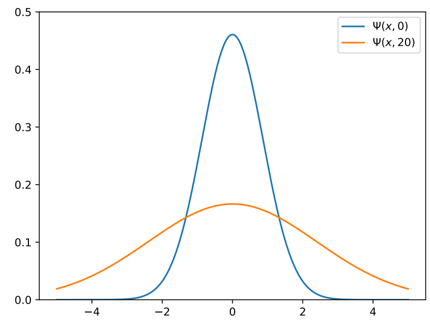

Figure 2: Plot of $|\Psi(x,t)|^2$ for time $t = 0$ and $t = 20$. All constants were taken to be 1 just to get a qualitative picture.

d) Since $\hat{x} = x$ is just a number, we have that:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x.t)|^2 \, \mathrm{d}x$$

Since $\Psi \sim e^{-\xi^2}$, we have an odd integrand so $\langle x \rangle = 0$. Similarly, then, $\langle p \rangle = 0$. Now,

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2\omega^2 x^2} \, \mathrm{d}x.$$

With $u = w\sqrt{2}x$, this is

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}}\omega \left(\frac{1}{\omega\sqrt{2}}\right)^3 \int_{-\infty}^{\infty} u^2 e^{u^2} \, \mathrm{d}u = 2\sqrt{\frac{2}{\pi}}\omega \left(\frac{1}{\omega\sqrt{2}}\right)^3 \frac{\sqrt{\pi}}{4} = \boxed{\frac{1}{4\omega^2}.}$$

Now, $\langle p^2 \rangle$ will be much more challenging:

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} \, \mathrm{d}x.$$

Doing the derivatives and using the equation for Ψ in terms of γ:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} = \sqrt[4]{\frac{2a}{\pi}} \frac{1}{\gamma} \frac{\mathrm{d}}{\mathrm{d}x} \left[-\frac{2ax}{\gamma} e^{-ax^2/\gamma^2} \right] = \sqrt[4]{\frac{2a}{\pi}} \frac{1}{\gamma} \left(\frac{4a^2}{\gamma^4} x^2 - \frac{2a}{\gamma^2} \right) e^{-ax^2/\gamma^2}.$$

So,

$$\langle p^2 \rangle = -\hbar^2 \sqrt{\frac{2}{\pi}} \frac{\sqrt{a}}{\gamma^*\gamma} \int_{-\infty}^{\infty} \left(\frac{4a^2}{\gamma^4} x^2 - \frac{2a}{\gamma^2} \right) e^{-ax^2/(\gamma^2)^*} e^{-ax^2/\gamma^2} \, \mathrm{d}x.$$

Now, let's look at this term:

$$\frac{\sqrt{a}}{\gamma^*\gamma} = \frac{\sqrt{a}}{\sqrt{(1 - 2i\hbar at/m)(1 + 2i\hbar at/m)}} = \frac{\sqrt{a}}{\sqrt{1 + (2\hbar at/m)^2}} = \omega,$$

and the exponentials will look similar as well; they become what we got for Ψ in terms of ω. Thus,

$$
\begin{aligned}
\langle p^2 \rangle &= \frac{2a\hbar^2}{\gamma^2}\sqrt{\frac{2}{\pi}}\omega \int_{-\infty}^{\infty} \left(1 - \frac{2a}{\gamma^2}x^2\right) e^{-2\omega^2 x^2}\ \mathrm{d}x, \\
&= \frac{4a\hbar^2}{\gamma^2}\sqrt{\frac{2}{\pi}}\omega \left[\int_0^{\infty} e^{-2\omega^2 x^2}\ \mathrm{d}x - \frac{2a}{\gamma^2}\int_0^{\infty} x^2 e^{-2\omega^2 x^2}\ \mathrm{d}x\right], \\
&= \frac{4a\hbar^2}{\gamma^2}\sqrt{\frac{2}{\pi}}\omega \left[\left(\frac{1}{\omega\sqrt{2}}\right)\int_0^{\infty} e^{-u^2}\ \mathrm{d}u - \frac{2a}{\gamma^2}\left(\frac{1}{\omega\sqrt{2}}\right)^3 \int_0^{\infty} u^2 e^{-u^2}\ \mathrm{d}u\right], \\
&= \frac{4a\hbar^2}{\gamma^2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{\sqrt{2}}\left(\frac{\sqrt{\pi}}{2}\right) - \frac{2a}{\gamma^2}\frac{1}{2\omega^2\sqrt{2}}\left(\frac{\sqrt{\pi}}{4}\right)\right], \\
&= \frac{2a\hbar^2}{\gamma^2}\left(1 - \frac{a}{2\gamma^2\omega^2}\right).
\end{aligned}
$$

Looking at the term in parentheses:

$$
1 - \frac{a\left[1 + (a\hbar at/m)^2\right]}{2a(1 + 2i\hbar at/m)} = 1 - \frac{(1 + 2i\hbar at/m)(1i2i\hbar at/m)}{2(1 + 2i\hbar at/m)} = 1 - \frac{1 - 2i\hbar at/m}{2} = \frac{1 + 2i\hbar at/m}{2} = \frac{\gamma^2}{2},
$$

So,

$$
\boxed{\langle p^2 \rangle = a\hbar^2.}
$$

Therefore, we have that

$$
\sigma_x = \sqrt{\frac{1}{4\omega^2}} = \frac{1}{2\omega},\ \text{and}\ \sigma_p = \sqrt{a\hbar^2} = \hbar\sqrt{a}.
$$

e) From this, we can say

$$
\sigma_x\sigma_p = \frac{\hbar}{2}\frac{\sqrt{a}}{\omega} = \frac{\hbar}{2}\sqrt{[1 + (2\hbar at/m)^2]}.
$$

The square root will allways be greater than one since the quantity in parentheses will always be positive. We can see that for $t = 0$, this comes exactly to the uncertainty limit.