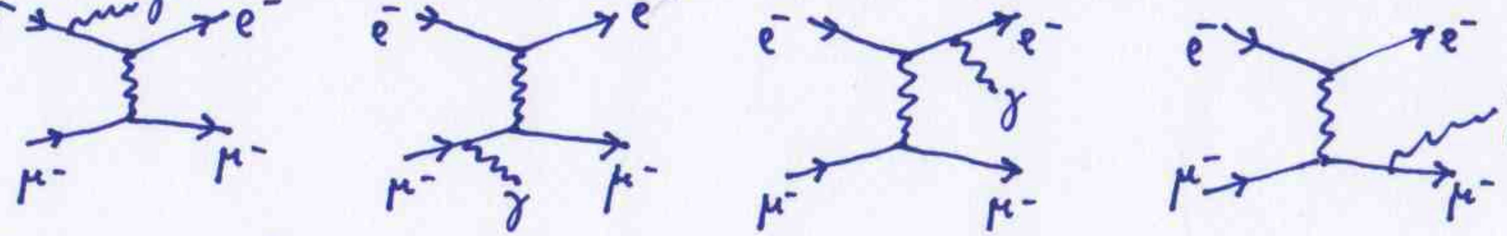
Next-to-leading-order (NLO) diagrams for $e^-(p_1)+\mu^-(p_2) \to e^-(p_3)+\mu^-(p_4)$

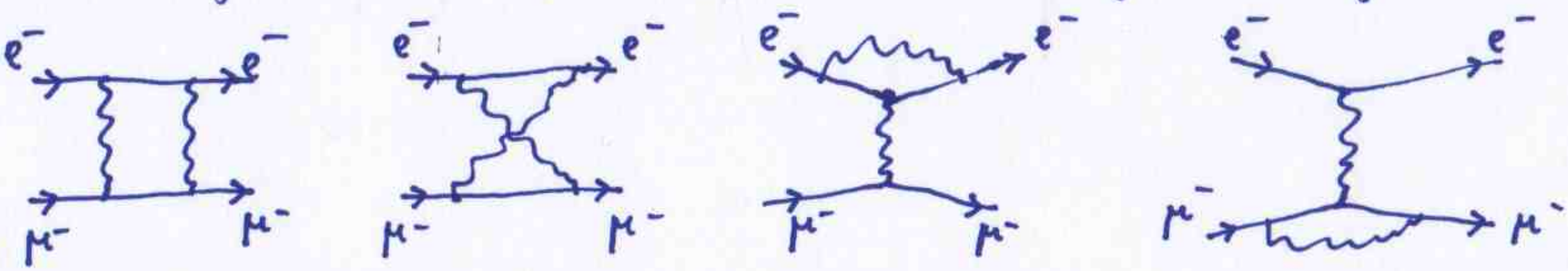
Tree diagrams (real emission)

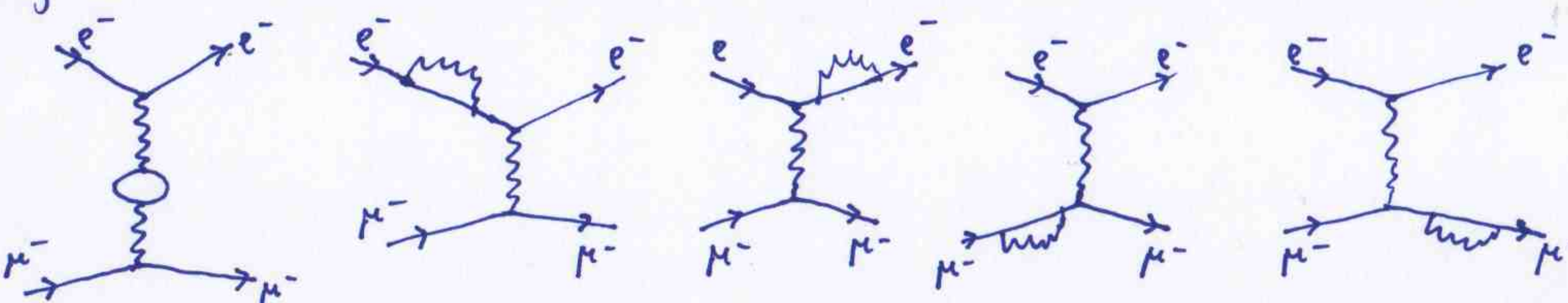


In tree diagrams the internal momenta are determined by the momenta of the external lines through momentum conservation.

The NLO tree diagrams for $e^-+\mu^- \to e^-+\mu^-$ are the leading-order diagrams for $e^-+\mu^- \to e^-+\mu^-+\gamma$

Loop diagrams





In loop diagrams we have undetermined internal momenta that can take arbitrary values and need to be integrated over.

Virtual corrections

Ultraviolet and infrared divergences

NLO and higher-order diagrams involve divergences (singularities)

<u>Ultraviolet</u> (UV) divergences appear in virtual corrections when the loop momentum goes to infinity: $\int \frac{d^4k}{k^m}$ diverges if $m \leq 4$

Can "regularize" the infinities by putting a cutoff on the momentum or by integrating in $n = 4 - \varepsilon$ dimensions (dimensional regularization)
- infinities appear as $\frac{1}{\varepsilon}$ with $\varepsilon \to 0$

<u>Renormalization</u>: absorb infinities in redefinitions of charge, mass

<u>Infrared</u> divergences appear in virtual diagrams in $\frac{1}{k^2}$ propagator terms as $k \to 0$, and they also appear in real emission diagrams as <u>soft</u> or <u>collinear</u> divergences.

We have a <u>soft</u> divergence when additional particle is soft (zero energy) and a <u>collinear</u> divergence when it is collinear (parallel) to another particle:

$$\frac{1}{(p-k)^2} = \frac{-1}{2p\cdot k} = \frac{-1}{2E_pE_k(1-\cos\theta)} \text{ diverges if } E_{p,k} = 0 \text{ (soft) or } \theta = 0 \text{ (collinear)}$$

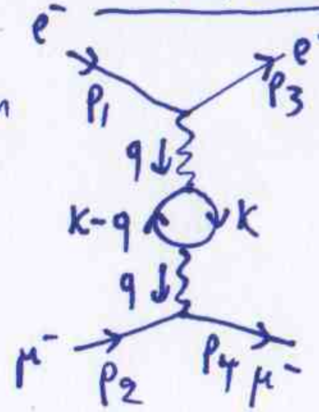
<u>Bloch-Nordsieck</u> theorem: infrared divergences in QED cancel when summing over final states

<u>Kinoshita-Lee-Nauenberg (KLN)</u> theorem: infrared divergences in the Standard Model cancel when summing over initial and final states.

The infrared divergences cancel between contributions from virtual and real corrections.

Ultraviolet divergences

Consider the diagram



Amplitude $\mathcal{M}$

The loop momentum k is not determined.

$$p_1 + p_2 = p_3 + p_4 \quad \text{and} \quad q = p_1 - p_3 = p_4 - p_2$$

So need to integrate $\int \frac{d^4k}{(2\pi)^4}$

$$i\mathcal{M} = \bar{u}(p_3)(-ie\gamma^\mu)u(p_1)\,(-i)\frac{g_{\mu\rho}}{q^2}\,(-1)\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{tr}[i(\not{k}-\not{q}+m)(-ie\gamma^\sigma)i(\not{k}+m)(-ie\gamma^\rho)]}{((k-q)^2-m^2)\,(k^2-m^2)}(-i)\frac{g_{\sigma\nu}}{q^2}\bar{u}(p_4)(-ie\gamma^\nu)u(p_2)$$

where we used an additional Feynman rule that for a closed fermion loop we multiply by a factor of (-1) and take the trace. Then

$$\mathcal{M} = ie\bar{u}(p_3)\gamma^\mu u(p_1)\frac{1}{q^2}\int\frac{d^4k}{(2\pi)^4}\frac{e^2\,\mathrm{tr}[(\not{k}-\not{q}+m)\gamma_\nu(\not{k}+m)\gamma_\mu]}{((k-q)^2-m^2)(k^2-m^2)}\frac{1}{q^2}e\bar{u}(p_4)\gamma^\nu u(p_2)$$

or

$$\mathcal{M} = \frac{ie^4}{(p_1-p_3)^4}\bar{u}(p_3)\gamma^\mu u(p_1)\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{tr}[(\not{k}-\not{p}_1+\not{p}_3+m)\gamma_\nu(\not{k}+m)\gamma_\mu]}{[(k-p_1+p_3)^2-m^2](k^2-m^2)}\bar{u}(p_4)\gamma^\nu u(p_2)$$

$$\Rightarrow \mathcal{M} = \frac{4ie^4}{(2\pi)^4(p_1-p_3)^4}\bar{u}(p_3)\gamma^\mu u(p_1)\int d^4k\,\frac{[(k_\mu-p_{1\mu}+p_{3\mu})k_\nu-(k-p_1+p_3)\cdot k\,g_{\mu\nu}+(k_\nu-p_{1\nu}+p_{3\nu})k_\mu+m^2g_{\mu\nu}]}{[(k-p_1+p_3)^2-m^2](k^2-m^2)}\cdot\bar{u}(p_4)\gamma^\nu u(p_2)$$

The $\int d^4k$ integral diverges as $|\vec{k}|\to\infty$

$$\mathcal{M}=\frac{4ie^4}{(2\pi)^4(p_1-p_3)^4}\int\frac{d^4k}{[(k-p_1+p_3)^2-m^2](k^2-m^2)}\Big\{\bar{u}(p_3)(\not{k}-\not{p}_1+\not{p}_3)u(p_1)\bar{u}(p_4)\not{k}u(p_2)$$
$$-(k-p_1+p_3)\cdot k\;\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)$$
$$+\bar{u}(p_3)\not{k}u(p_1)\bar{u}(p_4)(\not{k}-\not{p}_1+\not{p}_3)u(p_2)$$
$$+m^2\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)\Big\}$$

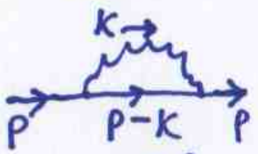
But $\bar{u}(p_3)\not{p}_3=m\bar{u}(p_3)$, $\bar{u}(p_4)\not{p}_4=m\bar{u}(p_4)$,
$\not{p}_2u(p_2)=mu(p_2)$, $\not{p}_1u(p_1)=mu(p_1)$ and $-p_1+p_3=-p_4+p_2$

Then
$$\mathcal{M}=\frac{4ie^4}{(2\pi)^4(p_1-p_3)^4}\int\frac{d^4k}{[(k-p_1+p_3)^2-m^2](k^2-m^2)}\Big\{\bar{u}(p_3)\not{k}u(p_1)\bar{u}(p_4)\not{k}u(p_2)\cdot 2$$
$$+\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)\left[m^2-(k-p_1+p_3)\cdot k\right]\Big\}$$

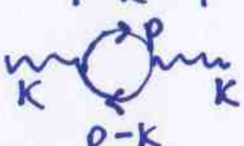
UV divergent as $k\to\infty$

We also have UV divergences in the other loop diagrams.

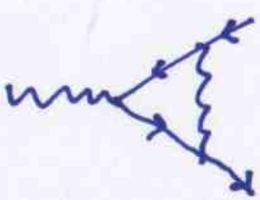
These divergences can be traced to primitively divergent diagrams



fermion self-energy diagram → fermion field & mass charge renormalization



photon self-energy diagram "vacuum polarization" (charge screening)
→ electric charge renormalization & photon field



vertex diagram → electric charge renormalization

Add counter-terms in the Lagrangian to cancel infinities