4-vectors

In three-dimensional space we define a position vector $\vec{x}$ (or $\vec{r}$) as $\vec{x} = (x, y, z)$ [or $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$]

In special relativity space and time are considered together

Define a 4-vector $x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

where c is the speed of light in vacuum

Note that the index μ takes the values $\mu = 0, 1, 2, 3$

x^μ is called a contravariant vector (index upstairs)

We also define a covariant vector x_μ (index downstairs)

$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$

So $x^0 = x_0$, $x^1 = -x_1$, $x^2 = -x_2$, $x^3 = -x^3$

In relativity the squared length $x^2+y^2+z^2$ is not invariant under Lorentz transformations → contraction seen by moving observer

However $c^2t^2-x^2-y^2-z^2$ is invariant

Note that $x^\mu x_\mu = x^0x_0+x^1x_1+x^2x_2+x^3x_3 = c^2t^2-x^2-y^2-z^2$

where we used the summation convention that repeated indices are summed over, i.e. $x^\mu x_\mu = \sum_{\mu=0}^{3} x^\mu x_\mu$

These considerations apply to any four-vector A

A^μ is contravariant vector $A^\mu = (A^0, A^1, A^2, A^3)$ with $A^1=A_x, A^2=A_y, A^3=A_z$

A_μ is covariant vector $A_\mu = (A_0, A_1, A_2, A_3) = (A^0, -A^1, -A^2, -A^3)$

and $A^2 = A\cdot A = A^\mu A_\mu = A^0A_0+A^1A_1+A^2A_2+A^3A_3 = (A^0)^2-(A^1)^2-(A^2)^2-(A^3)^2$

is the dot (scalar) product of the vector with itself, an invariant

For brevity we can also write $A^\mu = (A^0, \vec{A})$ where $\vec{A} = (A^1, A^2, A^3)$ is a 3-vector and $A_\mu = (A^0, -\vec{A})$

e.g. $x^\mu = (ct, \vec{x})$ where $\vec{x} = (x^1, x^2, x^3) = (x, y, z)$

and $x_\mu = (ct, -\vec{x})$

symmetric $g^{\mu\nu} = g^{\nu\mu}$

$A^\mu = g^{\mu\nu} A_\nu$ where $g^{\mu\nu}$ is the metric tensor

In flat spacetime (special relativity) $g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

So $A^0 = g^{0\nu} A_\nu = g^{00} A_0 + g^{01} A_1 + g^{02} A_2 + g^{03} A_3$
$= 1 \cdot A_0 + 0 \cdot A_1 + 0 \cdot A_2 + 0 \cdot A_3 = A_0$

$A^1 = g^{1\nu} A_\nu = \cdots = -A_1$, etc.

Also $A_\mu = g_{\mu\nu} A^\nu$ with $g_{\mu\nu} = g^{\mu\nu}$

Also $A^2 = A^\mu A_\mu = g^{\mu\nu} A_\nu A_\mu = g^{\mu\nu} A_\mu A_\nu$
and $A^2 = A^\mu A_\mu = A^\mu g_{\mu\nu} A^\nu = g_{\mu\nu} A^\mu A^\nu$

$A \cdot B = A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B}$
$= A_\mu B^\mu = g^{\mu\nu} A_\mu B_\nu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3$

<u>Energy-momentum</u> 4-vector $p^\mu = (p^0, \vec{p}) = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right)$

$p_\mu = (p_0, -\vec{p}) = \left(\frac{E}{c}, -\vec{p}\right) = \left(\frac{E}{c}, -p_x, -p_y, -p_z\right)$

Then $p^2 = p \cdot p = p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2$ a relativistic invariant

$E = \sqrt{\vec{p}^{\,2} c^2 + m^2 c^4}$ Also $E = \gamma m c^2$ where the time dilation factor is $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Invariant differential spacetime squared length $ds^2 = dx^\mu dx_\mu = g_{\mu\nu} dx^\mu dx^\nu$

We define $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$

Then $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$ and $\partial^\mu = g^{\mu\nu}\partial_\nu$

Here $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the gradient

The d'Alembertian is $\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian. So $\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

More index gymnastics $A^\mu B_\mu = A^\nu B_\nu$ since μ, ν are dummy indices when repeated

$A_\mu = g_{\mu\nu} A^\nu$ and $A^\nu = g^{\nu\rho} A_\rho$ Then $A_\mu = g_{\mu\nu} g^{\nu\rho} A_\rho = g_\mu{}^\rho A_\rho$ where $g_\mu{}^\rho = \begin{cases} 1 & \text{if } \mu=\rho \\ 0 & \text{if } \mu\neq\rho \end{cases}$

So $A_\mu = g_\mu^\rho A_\rho$ and $A^\mu = g_\rho^\mu A^\rho$

Also $g^{\mu\rho} g_{\nu\rho} = g_\nu^\mu$ and $g^{\mu\rho} g_\rho^\nu = g^{\mu\nu}$