Fourier expansion for Dirac field: $\psi(x)=\int\frac{d^3p}{(2\pi)^3(2p^0)^{1/2}}\sum_{d=1,2}\left[a_d(p)\,u^{(d)}(p)e^{-ip\cdot x}+b_d^\dagger(p)\,v^{(d)}(p)e^{ip\cdot x}\right]$

and $\bar\psi(x)=\int\frac{d^3p}{(2\pi)^3(2p^0)^{1/2}}\sum_{d=1,2}\left[a_d^\dagger(p)\,\bar u^{(d)}(p)\,e^{ip\cdot x}+b_d(p)\,\bar v^{(d)}(p)\,e^{-ip\cdot x}\right]$

Operators a annihilate and $a^\dagger$ create particles

Operators b annihilate and $b^\dagger$ create antiparticles

$a|0\rangle=0$ and $b|0\rangle=0$ $\quad a^\dagger|0\rangle$ are one-particle states and

$b^\dagger|0\rangle$ are one-antiparticle states

Then $H=\int d^3x\, i\psi^\dagger\dot\psi=\int d^3x\, i\bar\psi\gamma^0\dot\psi=\int d^3x\, i\int\frac{d^3p}{(2\pi)^3(2p^0)^{1/2}}\sum_{d=1,2}\left[a_d^\dagger(p)\bar u^{(d)}(p)e^{ip\cdot x}+b_d(p)\bar v^{(d)}(p)e^{-ip\cdot x}\right]\gamma^0$

$\cdot\int\frac{d^3q}{(2\pi)^3(2q^0)^{1/2}}\sum_{d'=1,2}\left[a_{d'}(q)\,u^{(d')}(q)(-iq^0)e^{-iq\cdot x}+b_{d'}^\dagger(q)\,v^{(d')}(q)(iq^0)e^{iq\cdot x}\right]$

$$=\int\frac{d^3x\,d^3p\,d^3q\,q^0}{(2\pi)^6\,2\sqrt{p^0q^0}}\sum_{d,d'=1,2}\Big[a_d^\dagger(p)u^{\dagger(d)}(p)a_{d'}(q)u^{(d')}(q)\,e^{i(p-q)\cdot x}-a_d^\dagger(p)u^{\dagger(d)}(p)b_{d'}^\dagger(q)v^{(d')}(q)\,e^{i(p+q)\cdot x}$$
$$+b_d(p)v^{\dagger(d)}(p)\,a_{d'}(q)u^{(d')}(q)e^{-i(p+q)\cdot x}-b_d(p)v^{\dagger(d)}(p)\,b_{d'}^\dagger(q)v^{(d')}(q)e^{-i(p-q)\cdot x}\Big]$$

$$=\int\frac{d^3p}{(2\pi)^3 2}\sum_{d,d'=1,2}\Big[a_d^\dagger(p)u^{\dagger(d)}(p)a_{d'}(p)u^{(d')}(p)+a_d^\dagger(p)u^{\dagger(d)}(p)b_{d'}^\dagger(-p)v^{(d')}(-p)-b_d(p)v^{\dagger(d)}(p)a_{d'}(-p)u^{(d')}(-p)-b_d(p)v^{\dagger(d)}(p)b_{d'}^\dagger(p)v^{(d')}(p)\Big]$$

(where we used $\int\frac{d^3x}{(2\pi)^3}e^{i(\vec p-\vec q)\cdot x}=\delta^3(\vec p-\vec q)$)

$=\int\frac{d^3p}{(2\pi)^3}p^0\sum_{d=1,2}\left[a_d^\dagger(p)a_d(p)-b_d(p)b_d^\dagger(p)\right]$ where we used $u^{\dagger(d)}(p)u^{(d')}(p)=2p^0\delta^{dd'}$, $v^{\dagger(d)}(p)v^{(d')}(p)=2p^0\delta^{dd'}$

$u^{\dagger(d)}(p)v^{(d')}(-p)=0$, $v^{\dagger(d)}(p)u^{(d')}(-p)=0$

Dirac field Hamiltonian $H=\int \frac{d^3p}{(2\pi)^3}\, p^0 \sum_{d=1,2}\left[a_d^\dagger(p)a_d(p)-b_d(p)b_d^\dagger(p)\right]$

But this would not be positive definite after normal ordering if the operators satisfy commutation relations because of the minus sign in front of $b^\dagger b$. To resolve this problem the operators must satisfy anticommutation relations $\{a_d(p), a_{d'}^\dagger(p')\}=(2\pi)^3\delta^3(\vec{p}-\vec{p}')\delta_{dd'}$ where the anticommutator is defined by $\{a,a^\dagger\}=aa^\dagger+a^\dagger a$

$$\{b_d(p), b_{d'}^\dagger(p')\}=(2\pi)^3\delta^3(\vec{p}-\vec{p}')\delta_{dd'}$$

$$\{a_d(p),a_{d'}(p')\}=0,\quad \{a_d^\dagger(p),a_{d'}^\dagger(p')\}=0,\quad \{b_d(p),b_{d'}(p')\}=0,\quad \{b_d^\dagger(p),b_{d'}^\dagger(p')\}=0$$

Then, after normal ordering, $H=\int\frac{d^3p}{(2\pi)^3}\,p^0\sum_{d=1,2}\left[a_d^\dagger(p)a_d(p)+b_d^\dagger(p)b_d(p)\right]$ which is positive definite.

Also $\{b_d^\dagger(p),b_d^\dagger(p)\}=0 \Rightarrow b_d^\dagger(p)b_d^\dagger(p)=0$ and thus $b_d^\dagger(p)b_d^\dagger(p)|0\rangle=0$

so we cannot have two Dirac-spin$\frac{1}{2}$-particles in the same state.

This is the Pauli exclusion principle $\rightarrow$ fermions obey Fermi-Dirac statistics

Also charge $Q=\int d^3x :\psi^\dagger(x)\psi(x): = \int\frac{d^3k}{(2\pi)^3}\sum_{d=1,2}\left(a_d^\dagger(p)a_d(p)-b_d^\dagger(p)b_d(p)\right)$

$a^\dagger$ creates particles (e.g. electrons) and $b^\dagger$ antiparticles (e.g. positrons) with opposite charge

Finally, ψ and $\psi^\dagger$ also satisfy anticommutation relations (equal-time)

$\{\psi_a(\vec{x},t)\,\psi_b^\dagger(\vec{x}',t)\}=\delta^3(\vec{x}-\vec{x}')\delta_{ab}$ with $a,b=1,..,4$ and $\{\psi_a(\vec{x},t),\psi_b(\vec{x}',t)\}=0$, $\{\psi_a^\dagger(\vec{x},t),\psi_b^\dagger(\vec{x}',t)\}=0$

Local gauge invariance

The Dirac Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$ with $\bar{\psi} = \psi^\dagger\gamma^0$
is invariant under a global gauge transformation $\psi \to e^{i\theta}\psi$
where θ is constant, i.e. a global phase $\psi^\dagger \to \psi^\dagger e^{-i\theta}$

check: $\mathcal{L} \to i\bar{\psi}e^{-i\theta}\gamma^\mu\partial_\mu(e^{i\theta}\psi) - m\bar{\psi}e^{-i\theta}e^{i\theta}\psi = \mathcal{L}$

Next, try a local gauge transformation $\psi(x) \to e^{i\theta(x)}\psi(x)$
i.e. $\theta(x)$ is a local phase, different at each point

Then $\mathcal{L} \to i\bar{\psi}(x)e^{-i\theta(x)}\gamma^\mu\partial_\mu(e^{i\theta(x)}\psi(x)) - m\bar{\psi}(x)e^{-i\theta(x)}e^{i\theta(x)}\psi(x)$

$= i\bar{\psi}(x)e^{-i\theta(x)}\gamma^\mu\left(i\partial_\mu\theta(x)\,e^{i\theta(x)}\psi(x) + e^{i\theta(x)}\partial_\mu\psi(x)\right) - m\bar{\psi}(x)\psi(x)$

$= -\bar{\psi}(x)\gamma^\mu\partial_\mu\theta(x)\,\psi(x) + i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\bar{\psi}(x)\psi(x) = -\partial_\mu\theta(x)\,\bar{\psi}(x)\gamma^\mu\psi(x) + \mathcal{L}$

So we have an additional term $\to$ Dirac Lagrangian <u>not</u> invariant under local gauge transformation

Try to find a Lagrangian that is invariant. Write $\theta(x) = q\,\lambda(x)$
and add a term $-q\bar{\psi}\gamma^\mu\psi A_\mu$ to the Dirac Lagrangian
where the new 4-vector field A_μ transforms as $A_\mu \to A_\mu - \partial_\mu\lambda$

Then new Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^\mu\psi A_\mu$ is invariant under local gauge transformations

Additional terms for A^μ in the Lagrangian would be a "kinetic" term $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

and a "mass" term $\frac{1}{2}m_A^2 A^\mu A_\mu$

Kinetic term is locally gauge invariant but mass term is not

So we set $m_A = 0$ → photon is massless and electromagnetic force has infinite range

Identify A^μ as the e&m 4-potential and q as the electric charge of the particle

Then the Lagrangian for Quantum Electrodynamics (QED) is

$$\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

If we introduce covariant derivative $D_\mu = \partial_\mu + iqA_\mu$

then $\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$