# HW4

# PHYS4500: Quantum Field Theory

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## Problem 1. (12.6)

a) We are to argue that

$$\hat{\phi}(\mathbf{x}, 0)\,|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2p^0}}\,|\mathbf{p}\rangle \tag{1.1}$$

for a complex scalar field. The book gives a 1-particle Fock state with momentum $\mathbf{p}$ in terms of the creation operator on the vacuum state as

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})\,|0\rangle\,, \tag{1.2}$$

with no normalization constant. Our convention is that

$$|\mathbf{p}\rangle = \sqrt{2p^0}a^\dagger(\mathbf{p})\,|0\rangle\,, \tag{1.3}$$

so I'll get the answer that the book wants then show what the answer would be with the normalization constant in our convention.

To evaluate the field operator on the vacuum state, let's write out $\hat{\phi}(\mathbf{x}, 0)$ in terms of the creation and annihilation operators:

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}}\left[a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{ip\cdot x}\right].$$

Now, at $t = 0$,

$$e^{-ip\cdot x} = e^{-ip^0 x_0}e^{-ip^i x_i} = e^{-ip^0 t}e^{i\mathbf{p}\cdot\mathbf{x}} \to e^{i\mathbf{p}\cdot\mathbf{x}},$$

so

$$\hat{\phi}(\mathbf{x}, 0) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}}\left[a(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right].$$

Letting this act on the vacuum state:

$$\hat{\phi}(\mathbf{x}, 0)\,|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}}\left[a(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right]|0\rangle\,.$$

The first term will look like $a(\mathbf{p})\,|0\rangle$, which is the annihilation operator acting on the vacuum state, which is zero, so all we have left is

$$\hat{\phi}(\mathbf{x}, 0)\,|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}} b^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\,|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}} e^{-i\mathbf{p}\cdot\mathbf{x}} b^\dagger(\mathbf{p})\,|0\rangle\,,$$

but from the book's convention this is

$$\hat{\phi}(\mathbf{x}, 0)\,|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2p^0}}\,|\mathbf{p}\rangle\,,$$

as expected. In our notation, we would need an extra factor of $\sqrt{2p^0}$ in the numerator and hence also the denominator, so we'd have:

$$\hat{\phi}(\mathbf{x}, 0)\,|0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{2p^0}\,|\mathbf{p}\rangle\,.$$

b) Qualitatively, since $\hat{\phi}(\mathbf{x},0)\left|0\right\rangle$ (roughly) creates an anti-particle at position $\mathbf{x}$, the naive assumption would be that $\hat{\phi}^\dagger(\mathbf{x},0)\left|0\right\rangle$ creates a particle at position $\mathbf{x}$. Expanding out the Hermitian conjugate of the complex scalar field in terms of creation and annihilation operators:

$$\hat{\phi}^\dagger(\mathbf{x},t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}}\left[b(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{ip\cdot x}\right],$$

or, when $t = 0$,

$$\hat{\phi}^\dagger(\mathbf{x},t) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}}\left[b(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right].$$

Letting this act on the vacuum state, we get a similar case where the term $b(\mathbf{p})\left|0\right\rangle = 0$, since $b(\mathbf{p})$ is the anti-particle annihilation operator. Hence (in the book's convention),

$$\hat{\phi}^\dagger(\mathbf{x},t)\left|0\right\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}} a^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\left|0\right\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}} e^{-i\mathbf{p}\cdot\mathbf{x}}\left|\mathbf{p}\right\rangle.$$

Since

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3\sqrt{2p^0}} e^{-i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{x}),$$

we can make the interpretation that having this act on a 1-particle Fock state $\left|\mathbf{p}\right\rangle$ gives a state of definite position $\left|\mathbf{x}\right\rangle$, because the delta function delivers a spike there. However, as described in the problem in the book, we have an extra factor of $1/\sqrt{2p^0}$, so this is not a perfect spike at $\mathbf{x}$, and hence not technically a perfect state of position $\left|\mathbf{x}\right\rangle$. But, the idea of the problem was to get a *rough* idea of what the field operator does, and we have found that *roughly*

$$\hat{\phi}^\dagger(\mathbf{x},t)\left|0\right\rangle \sim \left|\mathbf{x}\right\rangle.$$

## Problem 2. (18.5a)

The momentum density is given by

$$\pi^i = T^{0i} = \left[\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_n)}\partial^i\psi_n\right] - g^{0i}\mathcal{L}, \tag{2.1}$$

and the Dirac Lagrangian is given by

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi. \tag{2.2}$$

For the brackets in the formula for the momentum density, we will have a sum over the normal field $\psi$ and the adjoint field $\overline{\psi}$. Before doing either, it will be helpful to rewrite the Lagrangian and split the time and spatial components of the 4-gradient:

$$\mathcal{L} = i\overline{\psi}\gamma^0\partial_0\psi + i\overline{\psi}\gamma^i\partial_i\psi - m\overline{\psi}\psi.$$

Now, looking at $\psi$ first:

$$\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\partial^i\psi = i\overline{\psi}\gamma^0\partial^i\psi.$$

Now for $\overline{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \overline{\psi})}\partial^i \overline{\psi} = 0,$$

since there are no derivatives of the adjoint field present in the Dirac Lagrangian. Hence,

$$T^{0i} = i\overline{\psi}\gamma^0 \partial^i \psi - g^{0i}\left(i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi\right).$$

However, $g^{0i}$ is always zero, since $i$ is a spatial index and is never zero, and $g^{\mu\nu} = 0$ when $\mu \neq \nu$. Hence,

$$T^{0i} = i\overline{\psi}\gamma^0 \partial^i \psi.$$

## Problem 3. (19.3)

a) We are looking at a complex scalar Lagrangian

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi. \tag{3.1}$$

We are considering a local transformation where $\phi \to \phi' = e^{-iq\theta(x^\mu)}\phi$ and $\phi^* \to \phi^{*\prime} = \phi^* e^{iq\theta(x^\mu)}$. Such a transformation, as we saw earlier, allowed us to change the 4-derivative into the *covariant* derivative which included a term involving a gauge field $A_\mu$: $D_\mu = \partial_\mu + iqA_\mu$. Our complex scalar Lagrangian is thus

$$\mathcal{L} = D_\mu \phi D^\mu \phi^* - m^2 \phi^* \phi. \tag{3.2}$$

Expanding out:

$$\begin{aligned}
\mathcal{L} &= (\partial_\mu + iqA_\mu)\phi(\partial^\mu - iqA^\mu)\phi^* - m^2\phi^*\phi, \\
&= (\partial_\mu \phi + iqA_\mu \phi)(\partial^\mu \phi^* - iqA^\mu \phi^*) - m^2 \phi^* \phi, \\
&= \partial_\mu \phi \partial^\mu \phi^* - iq\partial_\mu \phi A^\mu \phi^* + iqA_\mu \phi \partial^\mu \phi^* + q^2 A_\mu A^\mu \phi^* \phi - m^2 \phi^* \phi, \\
&= \partial_\mu \phi \partial^\mu \phi^* - iq\left(\partial_\mu \phi A^\mu \phi^* - A_\mu \phi \partial^\mu \phi^*\right) + q^2 A_\mu A^\mu \phi^* \phi - m^2 \phi^* \phi.
\end{aligned}$$

Looking just at the term in perentheses, we can simplify a little bit:

$$= -iq(A_\mu \phi^* \partial^\mu \phi - A_\mu \phi \partial^\mu \phi) = -iqA_\mu(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*).$$

But the term in parentheses (with the $-iq$ outside) is the conserved current for a complex scalar field under phase ($U(1)$) symmetry, so we can say that this term is just $A_\mu j^\mu$:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* + A_\mu j^\mu + q^2 A_\mu A^\mu \phi^* \phi - m^2 \phi^* \phi.$$

So we have two additional terms: a "mass" term for the gauge field $A_\mu$ which is the third term, and an interaction term given by the second term, which, in the context of EM, can be interpreted as the interaction of a charged scalar field with the photon!

b) Now, if the field is very weak, that means the quantity $A_\mu A^\mu$ is very small compared to the rest of the terms, so we can neglect it, meaning our Lagrangian is (expanding out the 4-current so we can more easily apply the Euler-Lagrange equations):

$$\mathcal{L} = \partial^\mu \phi \partial_\mu \phi^* - iqA^\mu(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) - m^2 \phi^* \phi.$$

The first term in the EL equations for $\phi^*$ is (which will give us the equation of motion for $\phi$):

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = \partial_\mu \left( \partial^\mu \phi + iqA^\mu \phi \right) = \partial_\mu \partial^\mu \phi + iq\phi \partial_\mu A^\mu + iqA^\mu \partial_\mu \phi.$$

The second term is:

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -iqA^\mu \partial_\mu \phi - m^2 \phi.$$

So,

$$\begin{aligned} &\partial_\mu \partial^\mu \phi + iq\phi \partial_\mu A^\mu + iqA^\mu \partial_\mu \phi + iqA^\mu \partial_\mu \phi + m^2 \phi = 0 \\ \to \ &\boxed{\left( \partial_\mu \partial^\mu + iq\partial_\mu A^\mu + 2iqA^\mu \partial_\mu + m^2 \right) \phi = 0.} \end{aligned}$$