HW6

PHYS4210: Quantum Mechanics

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Problem 1. (4.1a)

a) We are to work out all of the anti-commutators for each component of the position and momentum operators. We can first qualitatively consider the commutators of the position operators amongst themselves. They are just numbers, there are no derivatives or anything involved with them, and we know that numbers commute, so we can immediately say that

$$[r_i, r_j] = 0. \tag{1.1}$$

Next, let's consider the commutators of the momentum operators among each other:

$$[p_i, p_j]f = -\hbar^2 \left(\frac{\partial^2 f}{\partial r_i \partial r_j} - \frac{\partial^2 f}{\partial r_j \partial r_i} \right). \tag{1.2}$$

By construction, f lives in Hilbert space (rather, we only care about such functions), so it is well behaved, and we are able to switch the order of the derivatives. Hence, the two quantities in parentheses are just identical, so we can say

$$[p_i, p_j] = 0. \tag{1.3}$$

Next, we have the commutator of the position components with the momentum components:

$$[r_i, p_j]f = -i\hbar \left(r_i \frac{\partial f}{\partial r_j} - \frac{\partial}{\partial r_j}[r_i * f] \right). \tag{1.4}$$

If $i \neq j$, we can just pull r_j out of the second derivative:

$$[r_i, p_j]f = i\hbar r_i \left(\frac{\partial f}{\partial r_j} - \frac{\partial f}{\partial r_j} \right). \tag{1.5}$$

But this is zero. The other case is if $i = j$. This is just $[x, p_x] = i\hbar$ that we have done in class before. To combine the two, then, we can use a Dirac delta and say that

$$[r_i, p_j] = i\hbar\delta_{ij}. \tag{1.6}$$

Of course, $[p_i, r_j] = -[r_i, p_j] = -i\hbar\delta_{ij}$.

b) The "generalized" Ehrenfest theorem, given in Eq (3.73) in Griffiths is

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{i}{\hbar} \left\langle [\hat{H}, \hat{Q}] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle. \tag{1.7}$$

However, the operators never depend on time, so really we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{i}{\hbar} \left\langle [\hat{H}, \hat{Q}] \right\rangle. \tag{1.8}$$

Here we have $Q = r_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle r_i \rangle = \frac{i}{\hbar} \left\langle [\hat{H}, r_i] \right\rangle. \tag{1.9}$$

Looking at the commutator:

$$[\hat{H}, r_i] = \left[\frac{\hat{\mathbf{p}}^2}{2m} + V, r_i\right]. \tag{1.10}$$

The potential is just a number, and r_i is just a number, so we can get rid of V. Further, from the previous part we know that only like components of momentum don't commute with like components of position, so the only non-zero terms are p_i^2:

$$= \frac{1}{2m}[\hat{p}_i^2, r_i] \tag{1.11}$$

$$= \frac{1}{2m}\left(\hat{p}_i\hat{p}_i r_i - r_i\hat{p}_i\hat{p}_i + \hat{p}_i r_i\hat{p}_i - \hat{p}_i r_i\hat{p}_i\right), \tag{1.12}$$

where I added and subtracted the same term for the third and fourth terms. Now,

$$= \frac{1}{2m}\left(\hat{p}_i[\hat{p}_i, r_i] + [\hat{p}_i, r_i]\hat{p}_i\right) \tag{1.13}$$

$$= \frac{1}{2m}\left(-i\hbar\hat{p}_i - i\hbar\hat{p}_i\right) \tag{1.14}$$

$$= -\frac{i\hbar}{m}\hat{p}_i. \tag{1.15}$$

Plugging back in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle r_i\rangle = \frac{i}{\hbar}\left\langle -\frac{i\hbar}{m}\hat{p}_i\right\rangle = \frac{1}{m}\langle\hat{p}_i\rangle. \tag{1.16}$$

Since there were no cross-terms between position or momentum components, this will be the same for all three components, meaning we can generally express it in vector form:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathbf{r}\rangle = \frac{1}{m}\langle\mathbf{p}\rangle.} \tag{1.17}$$

Next, we'll let $\hat{Q} = \hat{p}_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}_i\rangle = \left\langle[\hat{H}, \hat{p}_i]\right\rangle. \tag{1.18}$$

Looking at the commutator:

$$[\hat{H}, \hat{p}_i] = \left[\frac{\hat{\mathbf{p}}^2}{2m} + V, \hat{p}_i\right]. \tag{1.19}$$

We know from before that all momentum components commute among each other, so all we have is

$$= [V, \hat{p}_i]. \tag{1.20}$$

Using a test function f:

$$[V, \hat{p}_i]f = -i\hbar\left(V\frac{\partial f}{\partial r_i} - \frac{\partial}{\partial r_i}[Vf]\right) \tag{1.21}$$

$$= i\hbar f\frac{\partial V}{\partial r_i}. \tag{1.22}$$

$$\rightarrow [V, \hat{p}_i] = i\hbar\frac{\partial V}{\partial r_i}. \tag{1.23}$$

Plugging this back in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}_i\rangle = \frac{i}{\hbar}\left\langle i\hbar\frac{\partial V}{\partial r_i}\right\rangle = \left\langle -\frac{\partial V}{\partial r_i}\right\rangle. \tag{1.24}$$

Again, there are no cross-terms among components, so we can express this in vector form:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\mathbf{p}}\rangle = \langle -\boldsymbol{\nabla} V\rangle.} \tag{1.25}$$

c) This is easy. We know only like components of position and momentum don't compute, and since there are no cross terms, they are each equal to $\hbar/2$, as we very well know, so long as $i = j$:

$$\boxed{\sigma_{r_i}^2\sigma_{\hat{p}_j}^2 \geq \frac{\hbar}{2}\delta_{ij}.} \tag{1.26}$$

Problem 2. (4.12)

The radial wavefunction is

$$R_{n\ell} = \frac{1}{r}\rho^{\ell+1}e^{-\rho}v(\rho), \tag{2.1}$$

where ρ is an nth order polynominal whose coefficients are given by

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)}c_j, \tag{2.2}$$

and $\rho = kr$ where $k = 1/an = 1/3a$ for us with $n = 3$.

Starting with R_{30}, let's start with computing the coefficients for the polynomial $v(\rho)$. Since we are told to not bother normalizing, I'll just set $c_0 = 1$ for simplicity. Therefore,

$$c_1 = \frac{2(1-3)}{(1)(2)} = -2 \tag{2.3}$$

$$c_2 = (-2)\frac{2(1+1-3)}{(2)(3)} = \frac{-2}{6}(-2) = \frac{2}{3}, \tag{2.4}$$

$$c_3 = \left(\frac{2}{3}\right)\frac{2(2+1-3)}{\dots} = 0, \tag{2.5}$$

where we don't care about the denominator in the last term since the numerator is zero. Therefore, our polynominal

$$1 - 2\rho + \frac{2}{3}\rho^2 \to 1 - 2\frac{r}{3a} + \frac{2}{3}\left(\frac{r}{3a}\right)^2 = 1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2. \tag{2.6}$$

Thus,

$$R_{30} = \frac{1}{r}\left(\frac{r}{3a}\right)e^{-r/3a}\left[1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2\right] \tag{2.7}$$

$$\boxed{R_{30} = \frac{1}{3a}e^{-r/3a}\left[1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2\right].} \tag{2.8}$$

Next, we consider R_{31}. The coefficients are, with $c_0 = 1$:

$$c_1 = \frac{2(1+1-3)}{(1)(2+2)} = \frac{-1}{2}, \tag{2.9}$$

$$c_2 = \left(-\frac{1}{2}\right)\frac{2(1+1+1-3)}{\dots} = 0, \tag{2.10}$$

where again we can stop before doing the denominator since the numerator is zero. Thus,

$$R_{31} = \frac{1}{r}\left(\frac{r}{3a}\right)^2 e^{-r/3a}\left[1 - \frac{1}{2}\left(\frac{r}{3a}\right)\right] \tag{2.11}$$

$$\boxed{R_{31} = \frac{r}{9a^2}e^{-r/3a}\left[1 - \frac{1}{6}\left(\frac{r}{a}\right)\right].} \tag{2.12}$$

Lastly, for R_{32}, I'll take $c_0 = 1$ so

$$c_1 = \frac{2(2+1-3)}{\dots} = 0, \tag{2.13}$$

so we only have the constant term. Thus,

$$R_{32} = \frac{1}{r}\left(\frac{r}{3a}\right)^3 e^{-r/3a} \tag{2.14}$$

$$\boxed{R_{32} = \frac{r^2}{27a^3}e^{-r/3a}.} \tag{2.15}$$

Problem 3. (4.15)

a) The ground state of an electron in a hydrogen atom is

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}. \tag{3.1}$$

So,

$$\langle r \rangle = \langle \psi_{100} \,|\, r \,|\, \psi_{100} \rangle = \frac{1}{\pi a^3}\int r e^{-2r/a} \mathrm{d}^3 r. \tag{3.2}$$

Converting to spherical coordinates:

$$\langle r \rangle = \frac{1}{\pi a^3}\int \mathrm{d}\Omega \int_0^\infty r^3 e^{-2r/a}\, \mathrm{d}r. \tag{3.3}$$

The solid angle integral is 4π, and we can use the formula from the back of the book

$$\int_0^\infty x^n e^{-x/a}\, \mathrm{d}x = n! a^{n+1} \tag{3.4}$$

to say that

$$\langle r \rangle = \frac{4}{a^3} 3! \left(\frac{a}{2}\right)^4 4a\left(\frac{6}{16}\right) = \boxed{\frac{3}{2}a^2.} \tag{3.5}$$

Similarly,

$$\langle r^2 \rangle = \frac{4}{a^3}\int r^4 e^{-2r/a}\, \mathrm{d}r = \frac{4}{a^3}4!\left(\frac{a}{2}\right)^5 = 4a^2\left(\frac{24}{32}\right) = \boxed{3a^2.} \tag{3.6}$$

b) The ground state only depends on r, the distance from the center; there is no dependence on anything else, so there is perfect spherical symmetry. Therefore, it easily follows that the expectation value of x must be zero: $\boxed{\langle x \rangle = 0x}$

x^2 is a little different, but still doesn't require integration. Since $r^2 = x^2 + y^2 + z^2$, then $\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle$. But again, by spherical symmetry, all three terms on the right should be equal, so

$$\langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = \boxed{a^2.} \tag{3.7}$$

c) First we must figure out ψ_{211}. We know $\psi_{211} = R_{21} Y_1^1$, and R_{21} is given in Equation (4.83):

$$R_{21} = \frac{c_0}{4a^2} r e^{-r/2a}. \tag{3.8}$$

We must normalize it an find c_0. We know that we can normalize the radial equation separately by

$$\int_0^\infty |R|^2 r^2 \, \mathrm{d}r = 1, \tag{3.9}$$

so for us we have

$$\int_0^\infty |R_{21}|^2 r^2 \, \mathrm{d}r = \frac{c_0^2}{16a^4} \int_0^\infty r^4 e^{-r/a} \, \mathrm{d}r \tag{3.10}$$

$$= \frac{c_0^2}{16a^4} 4! a^5 = c_0^2 \frac{3}{2} a = 1, \tag{3.11}$$

so

$$c_0 = \sqrt{\frac{2}{3a}}. \tag{3.12}$$

Next, we can use one of the tables in the book to find that

$$Y_1^1 = -\left(\frac{3}{8\pi}\right) \sin\theta e^{i\phi}. \tag{3.13}$$

So, the total wavefunction is

$$\psi_{211} = \sqrt{\frac{2}{3a}} \frac{1}{4a^2} r e^{-r/2a} \cdot -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \tag{3.14}$$

$$= -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{i\phi}. \tag{3.15}$$

This carries an angular dependence, so it isn't perfectly spherically symmetric, so we must use the fact that $x = r \sin\theta \cos\phi$ to find that

$$\langle x^2 \rangle = \langle \psi_{211} \,|\, r^2 \sin^2\theta \cos^2\phi \,|\, \psi_{211} \rangle = \frac{1}{64\pi a^5} \int_0^\pi \sin^5\theta \, \mathrm{d}\theta \int_0^{2\pi} \cos^2\phi \, \mathrm{d}\phi \int r^6 e^{-r/a} \, \mathrm{d}r. \tag{3.16}$$

I'll just use Mathematica for the angular integrals:

$$\int_0^\pi \sin^5\theta\,\mathrm{d}\theta = \frac{16}{15} \quad \text{and} \quad \int_0^{2\pi} \cos^2\phi\,\mathrm{d}\phi = \pi, \tag{3.17}$$

so

$$\langle x^2\rangle = \frac{1}{64\pi a^5}\left(\frac{16}{15}\right)\cdot\pi\cdot 6!a^7 = \frac{a^2}{60}720 = \boxed{12a^2.} \tag{3.18}$$

Problem 4. (4.21)

We are considering the raising and lowering operators for angular momentum, where

$$L_+ f_\ell^m = A_\ell^m f_\ell^{m+1} \quad \text{and} \quad L_- f_\ell^m = B_\ell^m f_\ell^{m-1}, \tag{4.1}$$

where A_ℓ^m and B_ℓ^m are undetermined functions of ℓ and m and

$$L_\pm \equiv L_x \pm iL_y. \tag{4.2}$$

First, we are to show that $L_+ = (L_-)^\dagger$ (and vice versa). This is straightforward:

$$\langle f|L_+|g\rangle = \int f^*[(L_x + iL_y)g]\,\mathrm{d}x = \int [(L_x - iL_y)f]^* g\,\mathrm{d}x = \left\langle f \,\middle|\, (L_-)^\dagger \,\middle|\, g\right\rangle, \tag{4.3}$$

so, evidently, $L_+ = (L_-)^\dagger$.
Next, what we can do is use this fact so that

$$\langle f_\ell^m|L_-L_+|f_\ell^m\rangle = \left\langle f_\ell^m \,\middle|\, (L_+)^\dagger L_+ \,\middle|\, f_\ell^m\right\rangle = |A_\ell^m|^2\,\langle f_\ell^{m+1}|f_\ell^{m+1}\rangle = |A_\ell^m|^2. \tag{4.4}$$

We can use Equation (4.112) from the book to get that

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z \tag{4.5}$$

so that

$$\begin{aligned} |A_\ell^m|^2 = \langle f_\ell^m|L_-L_+|f_\ell^m\rangle &= \left\langle f_\ell^m \,\middle|\, L^2 - L_z^2 - \hbar L_z \,\middle|\, f_\ell\right\rangle && (4.6)\\ &= [\hbar^2\ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m]\,\langle f_\ell^m|f_\ell^m\rangle && (4.7)\\ |A_\ell^m|^2 &= [\hbar^2\ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m]. && (4.8)\end{aligned}$$

So,

$$\boxed{A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}.} \tag{4.9}$$

For the other case, we have

$$\langle f_\ell^m|L_+L_-|f_\ell^m\rangle = |B_\ell^m|^2, \tag{4.10}$$

and the only difference between L_-L_+ (from before) and L_+L_- (now) is a relative minus on the $\hbar L_z$ term in Equation (4.112) in Griffiths. Therefore, we have a plus instead in the third term in brackets in Equation (4.8) (in this document) meaning

$$\boxed{B_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m-1)}.} \tag{4.11}$$

More succinctly, we could say

$$L_\pm f_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\,f_\ell^{m\pm1}. \tag{4.12}$$

Problem 5. (4.22a)

We are considering the commutators of the z angular momentum operator with the position and linear momentum operators. $\hat{L}_z$ is defined like

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{5.1}$$

Looking first at the x operator:

$$\begin{aligned}
[\hat{L}_z, x] &= (x\hat{p}_y x - y\hat{p}_x x) - (xx\hat{p}_y - xy\hat{p}_x) &(5.2)\\
&= xy\hat{p}_x - y\hat{p}_x x &(5.3)\\
&= xy\hat{p}_x - y\hat{p}_x x - yx\hat{p}_x + yx\hat{p}_x &(5.4)\\
&= y[x, \hat{p}_x] = \boxed{i\hbar y.} &(5.5)
\end{aligned}$$

In the second line, we canceled the first term in the first parentheses with the first term in the second parentheses since $[\hat{p}_y, x] = 0$. Then in the third line I added and subtracted the same term, so that the first and third terms cancel and we are left with a commutator we know.

Next,

$$\begin{aligned}
[\hat{L}_z, y] &= (x\hat{p}_y y - y\hat{p}_x y) - (yx\hat{p}_y - yy\hat{p}_x) &(5.6)\\
&= x\hat{p}_y y - yx\hat{p}_y &(5.7)\\
&= x\hat{p}_y y - yx\hat{p}_y - xy\hat{p}_y + xy\hat{p}_y &(5.8)\\
&= x[\hat{p}_y, y] = \boxed{-i\hbar x.} &(5.9)
\end{aligned}$$

When we do $[\hat{L}_z, z]$, we notice that there are no $\hat{p}_z$ operators, so z will commute with all them, and of course it'll commute with the other position operators, so since it commutes with everything, then we can say $\boxed{[\hat{L}_z, z] = 0.}$

Looking next at the linear momentum operators:

$$\begin{aligned}
[\hat{L}_x, \hat{p}_x] &= (x\hat{p}_y\hat{p}_x - y\hat{p}_x\hat{p}_x) - (\hat{p}_x x\hat{p}_y - \hat{p}_x y\hat{p}_x) &(5.10)\\
&= x\hat{p}_y\hat{p}_x - \hat{p}_x x\hat{p}_y - x\hat{p}_x\hat{p}_y + x\hat{p}_x\hat{p}_y &(5.11)\\
&= [x, \hat{p}_x]\hat{p}_y = \boxed{i\hbar\hat{p}_y.} &(5.12)
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\hat{L}_x, \hat{p}_y] &= (x\hat{p}_y\hat{p}_y - y\hat{p}_x\hat{p}_y) - (\hat{p}_y x\hat{p}_y - \hat{p}_y y\hat{p}_x) &(5.13)\\
&= \hat{p}_y y\hat{p}_x - y\hat{p}_x\hat{p}_y - y\hat{p}_y\hat{p}_x + y\hat{p}_y\hat{p}_x &(5.14)\\
&= [\hat{p}_y, y]\hat{p}_x = \boxed{-i\hbar\hat{p}_x}. &(5.15)
\end{aligned}$$

No z's at all appear in $\hat{L}_z$, and we know that different components of position and linear momentum commute, so like before, $\hat{p}_z$ commutes with everything in $\hat{L}_z$ so $\boxed{[\hat{L}_z, \hat{p}_z] = 0.}$

Problem 6. (4.30)

a) Spinors are states in Hilbert space, so they must be normalized:

$$\chi^\dagger\chi = |A|^2[|3Ai|^2 + (4A)^2] = 25|A|^2 = 1, \tag{6.1}$$

so

$$\boxed{A = \frac{1}{5}.} \tag{6.2}$$

b) We first measure the $\langle \hat{S}_x \rangle$:

$$\langle \hat{S}_x \rangle = \left\langle \chi \middle| \hat{S}_x \middle| \chi \right\rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.3}$$

$$= \frac{\hbar}{50}\begin{pmatrix} 4 & -3i \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.4}$$

$$= \frac{\hbar}{50}(12i - 12i) = \boxed{0.} \tag{6.5}$$

Next,

$$\langle \hat{S}_y \rangle = \left\langle \chi \middle| \hat{S}_y \middle| \chi \right\rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.6}$$

$$= \frac{\hbar}{50}\begin{pmatrix} 4i & -3 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.7}$$

$$= \frac{\hbar}{50}(-12 - 12) = \boxed{-\frac{12\hbar}{25}.} \tag{6.8}$$

Lastly,

$$\langle \hat{S}_z \rangle = \left\langle \chi \middle| \hat{S}_z \middle| \chi \right\rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.9}$$

$$= \frac{\hbar}{50}\begin{pmatrix} -3i & -4 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.10}$$

$$= \frac{\hbar}{50}(9 - 16) = \boxed{-\frac{7\hbar}{50}.} \tag{6.11}$$

c) To get the standard deviations, we need $\langle \hat{S}_i^2 \rangle$. First, we need

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{6.12}$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{6.13}$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{6.14}$$

$$\tag{6.15}$$

So,

$$\langle \hat{S}_i^2 \rangle = \frac{\hbar^2}{4}\langle \chi | \chi \rangle = \frac{\hbar^2}{4} \tag{6.16}$$

no matter what the spinor χ happens to be and for all three components. So,

$$\sigma_{S_x} = \sqrt{\frac{\hbar^2}{4} - 0} = \boxed{\frac{\hbar}{2}}. \tag{6.17}$$

$$\sigma_{S_y} = \sqrt{\frac{\hbar^2}{4} - \hbar^2\frac{144}{625}} = \hbar\sqrt{\frac{625-576}{2500}} = \hbar\sqrt{\frac{49}{2500}} = \boxed{\frac{7\hbar}{50}}. \tag{6.18}$$

$$\sigma_{S_z} = \sqrt{\frac{\hbar^2}{4} - \hbar^2\frac{49}{2500}} = \hbar\sqrt{\frac{625-49}{2500}} = \hbar\sqrt{\frac{576}{2500}} = \boxed{\frac{12\hbar}{25}}. \tag{6.19}$$

d) We now check if this satisfies the uncertainty principle with Equation (4.100) in Griffiths:

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7\hbar}{50} \geq \frac{\hbar}{2}\frac{7\hbar}{50} \checkmark \tag{6.20}$$

$$\sigma_{S_y}\sigma_{S_z} = \frac{7\hbar}{50}\frac{12\hbar}{25} \geq 0 \checkmark \tag{6.21}$$

$$\sigma_{S_z}\sigma_{S_x} = \frac{12\hbar}{25}\frac{\hbar}{2} \geq \frac{\hbar}{2}\frac{12\hbar}{25} \checkmark \tag{6.22}$$

For the non-zero ones, we are right at the limit!