HW15
PHYS4500: Quantum Field Theory

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Problem 1.

First, we found, from renormalization that

$$Z_G = 1 + \frac{g_s^2}{24\pi^2\epsilon}(5C_A - 2n_f), \tag{1.1}$$

$$Z_\psi = 1 - \frac{g_s^2}{8\pi^2\epsilon}C_F, \quad \text{and} \tag{1.2}$$

$$Z_L = 1 - \frac{g_s^2}{8\pi^2\epsilon}(C_F + C_A). \tag{1.3}$$

We are to show the relation

$$g_{s,b} = Z_L Z_\psi^{-1} Z_G^{-1/2} g_s \mu^{\epsilon/2}. \tag{1.4}$$

To make things look a little cleaner, I will make the substitution

$$\frac{g_s^2}{48\pi^2\epsilon} \to x, \tag{1.5}$$

since this is a common factor in every term. With this,

$$\frac{g_{s,b}}{g_s\mu^{\epsilon/2}} = \left[1 - 6x\left(\frac{4}{3} + C_A\right)\right](1-8x)^{-1}\left[1 + 2x\left(5C_A - 2n_f\right)\right]^{-1/2} \tag{1.6}$$

$$= \left[1 - 2x\left(4 + 3C_A\right)\right](1+8x)\left[1 - x\left(5C_A - 2n_f\right)\right] \tag{1.7}$$

after expanding to $\mathcal{O}(x)$, or $\mathcal{O}\left(\epsilon^{-1}\right)$. In doing our multiplications of these terms, we can also drop terms of $\mathcal{O}(x^2)$ and higher. With this,

$$\frac{g_{s,b}}{g_s\mu^{\epsilon/2}} = [1 + 8x - 2x(4+3C_A)][1 - x(5C_A - 2n_f)] \tag{1.8}$$

$$= (1 - 6C_A)[1 - x(5C_A - 2n_f)] \tag{1.9}$$

$$= 1 - x(5C_A - 2n_f) - 6xC_A \tag{1.10}$$

$$= 1 - x(5C_A - 2n_f + 6C_A) \tag{1.11}$$

$$= 1 - x(11C_A - 2n_f). \tag{1.12}$$

So, un-substituting for x we find:

$$\boxed{g_{s,b} = \left[1 - \frac{g_s^2}{48\pi^2\epsilon}\left(11C_A - 2n_f\right)\right] g_s\mu^{\epsilon/2}.} \tag{1.13}$$

Problem 2.

We know that we can choose a QCD scale Λ such that

$$\ln \Lambda^2 = \ln \mu_0^2 - \frac{4\pi}{\beta_0 \alpha_s(\mu_0)}, \tag{2.1}$$

where

$$\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f \quad \rightarrow \quad \frac{33}{3} - \frac{10}{3} = \frac{23}{3} \tag{2.2}$$

when we take $n_f = 5$ and $C_A = N_c = 3$. With this, then, we can solve for Λ

$$\rightarrow \Lambda^2 = \mu_0^2 \exp\left(-\frac{12\pi}{23\alpha_s(\mu_0)}\right) \tag{2.3}$$

$$\rightarrow \Lambda = \mu_0 \sqrt{\exp\left(-\frac{12\pi}{23\alpha_s(\mu_0)}\right)}. \tag{2.4}$$

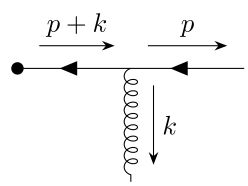
For an initial choice of $\mu_0 = m_Z = 91.1876\,\text{GeV}$, we have that $\alpha_s(m_Z) = 0.1179$ meaning that our QCD scale is, after plugging into a calculator, $\boxed{\Lambda = 0.087\,31\,\text{GeV} = 87.31\,\text{MeV}.}$ This roughly matches the general scale of $\Lambda \sim 200\,\text{MeV}$. Now, once we know this energy scale, we can use the relation

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}} = \frac{12\pi}{23 \ln \frac{\mu^2}{\Lambda^2}} \tag{2.5}$$

to determine the strong coupling constant at a new energy scale μ. Choosing now $\mu = m_t = 172.5\,\text{GeV}$, we find that $\boxed{\alpha_s(m_t) = 0.1080.}$ As expected of the strong coupling constant, it has decreased at a higher energy scale.

Problem 3.

To derive the eikonal rule for an outgoing antiquark, we are considering the diagram



Following the familiar Feynman rules, we find that since the "time" direction of the propagator is opposite the direction of its momentum, then the propagator will pick up a minus in the momentum (since the momentum in the denominator is squared, it doesn't matter there, only the numerator):

$$\rightarrow \quad \frac{i(-\not{p} - \not{k} + m)}{(p+k)^2 - m^2}\left(-ig_s T^a \gamma^\mu\right) v(p) = g_s T^a \frac{-\not{p} + m}{2pk} \gamma^\mu v(p). \tag{3.1}$$

Looking at the numerator of the propagator and rest of the expression to the right, we find

$$\begin{aligned} \left(-\not{p} + m\right) \gamma^\mu v(p) &= \left(-p_v \gamma^\nu \gamma^\mu + m\gamma^\mu\right) v(p) && (3.2)\\ &= \left[-p_v \left(2g^{\nu\mu} - \gamma^\mu \gamma^\nu\right) + m\gamma^\mu\right] v(p) && (3.3)\\ &= \left[-2p^\mu + \gamma^\mu \left(\not{p} + m\right)\right] v(p). && (3.4) \end{aligned}$$

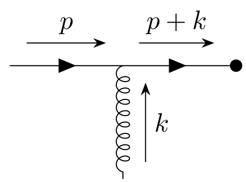
But we can invoke the Dirac equation for antiparticles to say $(\not{p} + m)v(p) = 0$, so all we have left in the amplitude is

$$\rightarrow \quad g_s T^a \frac{-2p^\mu}{2pk} = g_s T^a \frac{-v^\mu}{vk}. \tag{3.5}$$

Essentially, then, the eikonal rule for the outgoing anti-quark is:

$$\boxed{\frac{-v^\mu}{v \cdot k}.} \tag{3.6}$$

We next consider the case of an incoming quark:



Following a similar method, we have that the "time" direction of the propagator is the same as its momentum direction, so we use the normal expression for the propagator.

$$\rightarrow \quad \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \left(-i g_s T^a \gamma^\mu\right) u(p) = g_s T^a \frac{\not{p} + m}{2pk} \gamma^\mu u(p). \tag{3.7}$$

As before, we look at just the numerator, the gamma matrix, and the spinor:

$$(\not{p} + m)\gamma^\mu u(p) = \left(p_\nu \gamma^\nu \gamma^\mu + m\gamma^\mu\right) u(p) \tag{3.8}$$
$$= \left[p_\nu \left(2g^{\nu\mu} - \gamma^\mu \gamma^\nu\right) + m\gamma^\mu\right] u(p) \tag{3.9}$$
$$= \left[2p^\mu - \big(\not{p} - m\big)\right] u(p). \tag{3.10}$$

by virtue of the Dirac equation for regular particles, we have that $(\not{p} - m)u(p) = 0$, so our amplitude is

$$\rightarrow \quad = g_s T^a \frac{2p^\mu}{2pk} u(p) = g_s T^a \frac{v^\mu}{v \cdot k}, \tag{3.11}$$

so the eikonal rule for the incoming quark is

$$\boxed{\frac{v^\mu}{v \cdot k}.} \tag{3.12}$$