HW5
PHYS4500: Quantum Field Theory

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September 15, 2024

Problem 1. (19.4)

We are looking at the Proca Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2 A^\mu A_\mu. \tag{1.1}$$

a) The derivates get a little crazy here, so to make it as easy as possible, let's rewrite the Lagrangian with all the indices lowered, since that's how the derivatives are taken:

$$\mathcal{L} = -\frac{1}{4}g^{\mu\lambda}g^{\nu\rho}F_{\lambda\rho}F_{\mu\nu} + \frac{1}{2}M^2 g^{\mu\nu}A_\nu A_\mu.$$

We can pretty easily find, quoting previous results, that

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = M^2 A^\nu,$$

but the other term is not quite as easy:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}g^{\mu\lambda}g^{\nu\rho}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial_\lambda A_\rho \partial_\mu A_\nu - \partial_\lambda A_\rho \partial_\nu A_\mu - \partial_\rho A_\lambda \partial_\mu A_\nu + \partial_\rho A_\lambda \partial_\nu A_\mu\right).$$

Now, in each term, we have something like $(\partial_a A_b)(\partial_c A_d)$. This total term is equivalent to a term with the indices swapped within each individual term in parentheses, meaning this is equivalent to $(\partial_b A_a)(\partial_d A_c)$. Therefore, for our Euler-Lagrange EQ term, we have that the first and fourth terms are really identical, and the second and third terms are also identical:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}g^{\mu\lambda}g^{\nu\rho}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial_\lambda A_\rho \partial_\mu A_\nu - \partial_\rho A_\lambda \partial_\mu A_\nu\right).$$

Looking at the following term:

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial_\lambda A_\rho \partial_\mu A_\nu\right) &= \partial_\mu A_\nu \frac{\partial(\partial_\lambda A_\rho)}{\partial(\partial_\mu A_\nu)} + \partial_\lambda A_\rho \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\mu A_\nu)}, \\
&= \partial_\mu A_\nu \delta^\mu_\lambda \delta^\nu_\rho + \partial_\lambda A_\rho = 2\partial_\rho A_\lambda.
\end{aligned}$$

So,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -g^{\mu\lambda}g^{\nu\rho}(\partial_\lambda A_\rho - \partial_\rho A_\lambda) = -F^{\mu\nu}.$$

Plugging this all in, we have that:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \boxed{\partial_\mu F^{\mu\nu} + M^2 A^\nu = 0.}$$

b) Plugging the solution $A^\mu(x) = \epsilon^\mu(\mathbf{p})e^{-ip\cdot x}$ into the Euler-Lagrange equation above (and noting that, as in a previous HW, we are looking at a free field, so any 4-gradients of momentum are 0):

$$\begin{aligned}
\partial_\mu\left[\partial^\mu(\epsilon^\nu e^{-ip\cdot x}) - \partial^\nu(\epsilon^\mu e^{-ip\cdot x})\right] + M^2\epsilon^\nu e^{-ip\cdot x} &= 0, \\
\partial_\mu\left(-ip^\mu e^\nu e^{-ip\cdot x} + ip^\nu e^\mu e^{-ip\cdot x}\right) + M^2\epsilon^\nu e^{-ip\cdot x} &= 0, \\
-p^\mu p_\mu \epsilon^\nu e^{-ip\cdot x} + p^\nu \epsilon^\mu p_\mu e^{-ip\cdot x} + M^2\epsilon^\nu e^{-ip\cdot x} &= 0.
\end{aligned}$$

The first and last terms are identical, since $p^\mu p_\mu = M^2$, so all we are left with is:

$$p^\nu \epsilon \cdot p e^{-ip\cdot x} = 0.$$

Since neither the exponential nor the momentum are zero, we must have that

$$\boxed{e \cdot p = 0}.$$

As noted in the book, this is independent of the mass, meaning that the above equation is satisfied by massive vector fields in addition to the photon.

Problem 2.

We are looking to show that with $A^\mu = (V, \mathbf{A})$, we can derive the stress-energy tensor $F^{\mu\nu}$ using its definition and the equations for the electric and magnetic fields in terms of the potentials:

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \quad \text{and} \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.$$

The field-strength tensor is defined like:

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu.$$

First, we can quickly note that the tensor is fully anti-symmetric, meaning we really only need to find half of the components, and the other half will just be the same with a factor of -1. Additionally, all the diagonal components will be 0. First,

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla} V = -\mathbf{E},$$

where turning ∂^i into the 3-gradient picked up a minus since ∂^μ transforms like a *co*variant 4-vector, so its spatial indices have minus signs. From this, we know as well that $F^{i0} = \mathbf{E}$.

Next,

$$\begin{aligned}
F^{12} &= \partial^1 A^2 - \partial^2 A^1 = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}, \\
F^{13} &= \partial^1 A^3 - \partial^3 A^1 = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \\
F^{23} &= \partial^2 A^3 - \partial^3 A^2 = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}.
\end{aligned}$$

We can recognize these as components of a 3-vector as a result of a cross product. The two vectors in question are $\boldsymbol{\nabla}$ and $\mathbf{A}$ whose cross product is

$$\boldsymbol{\nabla} \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}}.$$

Thus, we can identify

$$\begin{aligned}
F^{12} &= -(\boldsymbol{\nabla} \times \mathbf{A})_z = -B_z, \\
F^{13} &= (\boldsymbol{\nabla} \times \mathbf{A})_y = B_y, \\
F^{23} &= -(\boldsymbol{\nabla} \times \mathbf{A})_x = -B_x.
\end{aligned}$$

The terms with indices flipped identical with a minus sign, so we now have all the components:

$$\boxed{F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.}$$

Problem 3.

The 4-current is defined like $j^\mu = (\rho, \mathbf{j})$, and the Euler-Lagrange equation for the QED Lagrangian gives us

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{3.1}$$

For $\nu = 0$, we have

$$\partial_\mu F^{\mu 0} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \rho,}$$

which is Gauss's Law. Moving to $\nu = i$:

$$\begin{aligned} i = 1 \quad &\rightarrow \quad \partial_\mu F^{\mu 1} = -\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{\partial E_x}{\partial t} + (\boldsymbol{\nabla} \times \mathbf{B})_x, \\ i = 2 \quad &\rightarrow \quad \partial_\mu F^{\mu 2} = -\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} = -\frac{\partial E_y}{\partial t} + (\boldsymbol{\nabla} \times \mathbf{B})y, \\ i = 3 \quad &\rightarrow \quad \partial_\mu F^{\mu 3} = -\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{\partial E_z}{\partial t} + (\boldsymbol{\nabla} \times \mathbf{B})_z \end{aligned}$$

We can rewrite this as a vector since each component only carries its own subscript:

$$\partial_\mu F^{\mu i} = \boxed{\boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j},}$$

which is the Ampere-Maxwell law.