Dimensional regularization

Work in n dimensions with $n = 4 - \varepsilon$ (and $\varepsilon \to 0$)

Consider the integral $I(q) = \int \frac{d^n p}{(p^2 + 2p\cdot q - m^2)^a}$.

$p^\mu = (p^0, \vec{p})$ In polar coordinates in n-dimensions $p^\mu = (p^0, r, \varphi, \theta_1, \theta_2, \dots, \theta_{n-3})$ where $r = |\vec{p}|$

Then $d^n p = dp^0\, r^{n-2}\, dr\, d\varphi\, \sin\theta_1\, d\theta_1\, \sin^2\theta_2\, d\theta_2 \cdots \sin^{n-3}\theta_{n-3}\, d\theta_{n-3}$

(where $-\infty < p^0 < \infty$, $0 \leq r < \infty$, $0 \leq \varphi \leq 2\pi$, $0 \leq \theta_i \leq \pi$)

or $d^n p = dp^0\, r^{n-2}\, dr\, d\varphi \prod_{k=1}^{n-3} \sin^k\theta_k\, d\theta_k$

Using the formula $\int_0^\pi \sin^k\theta_k\, d\theta_k = \sqrt{\pi}\, \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}$ where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$

$\Gamma(\lambda) = (\lambda-1)!$ for integer $\lambda > 0$

we find $I(q) = \int_{-\infty}^{+\infty} dp^0 \int_0^\infty r^{n-2}\, dr \int_0^{2\pi} d\varphi \int_0^\pi \frac{\prod_{k=1}^{n-3} \sin^k\theta_k\, d\theta_k}{(p^2 + 2p\cdot q - m^2)^a}$

$$= \int_{-\infty}^{+\infty} dp^0 \int_0^\infty r^{n-2}\, dr \cdot 2\pi \cdot \sqrt{\pi}\, \frac{\Gamma(1)}{\Gamma(3/2)}\, \sqrt{\pi}\, \frac{\Gamma(3/2)}{\Gamma(2)} \cdots \sqrt{\pi}\, \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{((p^0)^2 - r^2 + 2p\cdot q - m^2)^a}$$

$$\Rightarrow I(q) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{+\infty} dp^0 \int_0^\infty \frac{r^{n-2}\, dr}{[(p^0)^2 - r^2 + 2p\cdot q - m^2]^a}$$

Then $I(q) = (-1)^{2a + \frac{(n-1)}{2}} \pi^{\frac{n-1}{2}} \dfrac{\Gamma\left(a - \frac{(n-1)}{2}\right)}{\Gamma(a)} \displaystyle\int_{-\infty}^{+\infty} \frac{dp_0}{\left[(p^0)^2 - (q^2+m^2)\right]^{a - \frac{(n-1)}{2}}}$

$\Rightarrow I(q) = i\pi^{n/2} \dfrac{\Gamma\left(a - \frac{n}{2}\right)}{\Gamma(a)} \dfrac{1}{(-q^2-m^2)^{a-\frac{n}{2}}}$ or $I(q) = i\pi^{2-\frac{\varepsilon}{2}} \dfrac{\Gamma\left(a-2+\frac{\varepsilon}{2}\right)}{\Gamma(a)} (-q^2-m^2)^{2-\frac{\varepsilon}{2}-a}$

So $\displaystyle\int \frac{d^n p}{(p^2+2p\cdot q-m^2)^a} = i\pi^{n/2} \frac{\Gamma\left(a-\frac{n}{2}\right)}{\Gamma(a)} (-q^2-m^2)^{\frac{n}{2}-a}$

If $q=0$ this becomes $\displaystyle\int \frac{d^n p}{(p^2-m^2)^a} = i\pi^{n/2} \frac{\Gamma\left(a-\frac{n}{2}\right)}{\Gamma(a)} (-m^2)^{\frac{n}{2}-a}$

Also $\displaystyle\int \frac{d^n p \; p^\mu}{(p^2+2p\cdot q-m^2)^a} = -i\pi^{\frac{n}{2}} \frac{\Gamma\left(a-\frac{n}{2}\right)}{\Gamma(a)} q^\mu (-q^2-m^2)^{\frac{n}{2}-a}$

and

$\displaystyle\int d^n p \frac{p^\mu p^\nu}{(p^2+2p\cdot q-m^2)^a} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(a)} (-q^2-m^2)^{\frac{n}{2}-a}\left[q^\mu q^\nu \Gamma\left(a-\frac{n}{2}\right) + \frac{1}{2} g^{\mu\nu}(-q^2-m^2)\Gamma\left(a-1-\frac{n}{2}\right)\right]$

In n dimensions $\underbrace{\gamma^\mu \gamma_\mu = n}_{=4-\varepsilon}$ and $\gamma^\mu \gamma^\nu \gamma_\mu = (2-n)\gamma^\nu = (-2+\varepsilon)\gamma^\nu$

Feynman parameters

useful identities for performing loop integrals

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2} = \int_0^1 dx \int_0^1 dy \, \frac{\delta(1-x-y)}{[xA+yB]^2}$$

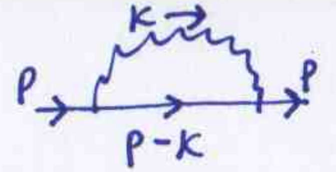
$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{\delta(1-x-y-z)}{[xA+yB+zC]^3} = 2\int_0^1 dx \int_0^{1-x} \frac{dy}{[xA+yB+(1-x-y)C]^3}$$

since $0 < z = 1-x-y < 1$

In general

$$\frac{1}{A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n}} = \frac{\Gamma(a_1+a_2+\dots+a_n)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)} \int_0^1 dx_1 \cdots dx_n \, \frac{x_1^{a_1-1} x_2^{a_2-1} \cdots x_n^{a_n-1} \, \delta(1-x_1-x_2-\dots-x_n)}{[x_1A_1+x_2A_2+\dots+x_nA_n]^{a_1+a_2+\dots+a_n}}$$

Electron self-energy diagram



$= \Sigma(p)$

$$i\,\Sigma(p) = \int \frac{d^n k}{(2\pi)^n} (-ie\gamma^\mu) \frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2} (-ie\gamma^\nu) \frac{(-ig_{\mu\nu})}{k^2} = -e^2 \int \frac{d^n k}{(2\pi)^n} \frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\mu}{[(p-k)^2-m^2]\,k^2}$$

$$= \frac{-e^2}{(2\pi)^n} \int_0^1 dz \int d^n k \frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\mu}{[(p-k)^2 z - m^2 z + k^2(1-z)]^2} = \frac{-e^2}{(2\pi)^n} \int_0^1 dz \int d^n k \frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\mu}{[(k-pz)^2+p^2 z(1-z)-m^2 z]^2}$$

Let $k' = k - pz$. Then $$i\Sigma(p) = -\frac{e^2}{(2\pi)^n} \int_0^1 dz \int d^n k' \frac{\gamma^\mu(\not{p}-\not{k}'-\not{p}z+m)\gamma_\mu}{[k'^2+p^2 z(1-z)-m^2 z]^2}$$

$$\Rightarrow i\Sigma(p) = -\frac{e^2}{(2\pi)^n} \int_0^1 dz\, \gamma^\mu(\not{p}-\not{p}z+m)\gamma_\mu \, i\pi^{n/2} \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \left(p^2 z(1-z)-m^2 z\right)^{\frac{n}{2}-2}$$

$$= \frac{-ie^2}{2^{4-\varepsilon}\pi^{2-\frac{\varepsilon}{2}}} \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dz\, \gamma^\mu[p_\nu\gamma^\nu(1-z)+m]\gamma_\mu \,[p^2 z(1-z)-m^2 z]^{-\frac{\varepsilon}{2}}$$

Next we use $\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma_E + O(\varepsilon)$ where $\gamma_E = 0.5772\ldots$ is the Euler constant and $x^\varepsilon = e^{\varepsilon \ln x} = 1 + \varepsilon \ln x + O(\varepsilon^2)$

Then $$\Sigma(p) = \frac{-e^2}{2^4\pi^2}(1+\varepsilon\ln 2+\ldots)\left(1+\frac{\varepsilon}{2}\ln\pi+\ldots\right)\left(\frac{2}{\varepsilon}-\gamma_E+\ldots\right)\int_0^1 dz\,[(-2+\varepsilon)\not{p}(1-z)+m(4-\varepsilon)] \cdot \left[1-\frac{\varepsilon}{2}\ln(p^2 z(1-z)-m^2 z)+\ldots\right]$$

$$= \frac{-e^2}{16\pi^2}\frac{2}{\varepsilon}\int_0^1 dz\,[-2\not{p}(1-z)+4m]+\ldots = \frac{-e^2}{8\pi^2\varepsilon}\left[-2\not{p}\left(z-\frac{z^2}{2}\right)+4mz\right]\Big|_0^1+\ldots$$

$$\Rightarrow \Sigma(p) = \frac{e^2}{8\pi^2\varepsilon}(\not{p}-4m) + O(\varepsilon^0)$$