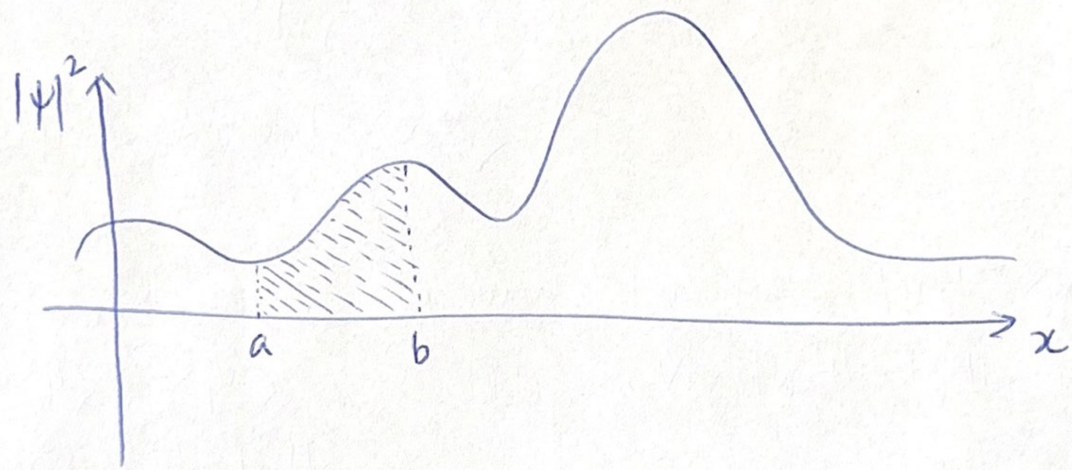
Born's interpretation of wave function

$\int_a^b |\Psi(x,t)|^2 \, dx$ gives the probability of finding a particle between $x=a$ and $x=b$ at time t.



Since the particle has to be somewhere

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1 \quad \text{—— } (2.1)$$

Any solution of Schrödinger equation must also satisfy eq^n 2.1

If $\Psi(x,t)$ is a solution of Schrödinger equation, $A\Psi(x,t)$ is also a solution, given A is a constant. It is then possible to choose A in such a way that eq^n (2.1) is satisfied. This process is called normalization of wave functions.

Physically realizable states correspond to square integrable solutions of Schrödinger equation.

Non-square integrable solution, i.e. $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \infty$

are not normalizable as no constant multiplier will make it equal to 1.

Similarly, trivial solution $\psi = 0$ is also not normalizable as no constant multiplier will make it equal to 1.

Theorem: A wave function normalized at $t=0$ will stay normalized for all future times.

Proof: Schrödinger equation is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi \quad \text{——} \quad (2.2)$$

Taking complex conjugate

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* \quad \cancel{\text{——} (2.3)}$$

$$i\hbar \frac{\partial \psi^*}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} - V\psi^* \quad \text{——} \quad (2.3)$$

Now $\psi^* \times$ eqn (2.2) $+ \psi \times$ eqn (2.3) will yield

$$i\hbar\left(\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right)$$

$$+ \cancel{V(\psi^*\psi - \psi\psi^*)} \rightarrow 0$$

or $$\frac{\partial \psi\psi^*}{\partial t} = \frac{i\hbar}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)$$

or $$\frac{\partial |\psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)\right]$$

[Hint : work backward - apply product rule and make cancellations]

Now, integrating both sides w.r.t x

$$\int_{-\infty}^{\infty}\frac{\partial |\psi|^2}{\partial t}dx = \frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)\Bigg|_{-\infty}^{\infty}$$

or $$\frac{d}{dt}\int_{-\infty}^{\infty}|\psi|^2 dx = 0$$

Hence, the integral $\int_{-\infty}^{\infty}|\psi|^2 dx$ is constant and independent of time.

if ψ is normalized at time $t=0$, it will stay normalized. This is a remarkable property of Schrödinger equation and is the reason why its statistical interpretation is possible.

Problem 1.4 of Griffiths and Schroeter

At time $t=0$, a particle is represented by wave function

$$\psi(x,0) = \begin{cases} A(x/a), & 0 \leq x \leq a \\ A(b-x)/(b-a), & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$

where A, a and b are (positive) constants,

(a) Normalize ψ (that is, find A, in term of a and b)

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1$$

or $$\frac{|A|^2}{a^2} \int_0^a x^2 \, dx + \frac{|A|^2}{(b-a)^2} \int_a^b (b-x)^2 \, dx = 1$$

or $$|A|^2 \left[\frac{1}{a^2} \frac{a^3}{3} + \frac{1}{(b-a)^2} \left\{ \int_a^b (b^2 - 2bx + x^2) \, dx \right\} \right] = 1$$

or $$|A|^2 \left[\frac{a}{3} + \frac{1}{(b-a)^2} \left\{ b^2 x \Big|_a^b - 2b \frac{x^2}{2} \Big|_a^b + \frac{x^3}{3} \Big|_a^b \right\} \right] = 1$$

or $$|A|^2 \left[\frac{a}{3} + \frac{1}{(b-a)^2} \left\{ b^2(b-a) - 2b \left(\frac{b^2}{2} - \frac{a^2}{2} \right) + \left(\frac{b^3}{3} - \frac{a^3}{3} \right) \right\} \right] = 1$$

or $$|A|^2 \left[a/3 + \frac{1}{(b-a)^2} \left\{ b^2(b-a) - 2b \frac{(b+a)(b-a)}{2} + \frac{(b-a)(b^2+a^2+ab)}{3} \right\} \right] = 1$$

or $|A|^2\left[\frac{a}{3}+\frac{b^2}{b-a}-\frac{b(b+a)}{(b-a)}+\frac{b^2+a^2+ab}{3(b-a)}\right]=1$

or $|A|^2\left[\frac{a}{3}+\frac{3b^2-3b(b+a)+b^2+a^2+ab}{3(b-a)}\right]=1$

or $|A|^2\left[\frac{a}{3}+\frac{\cancel{3b^2}-\cancel{3b^2}-3ab+b^2+a^2+ab}{3(b-a)}\right]=1$

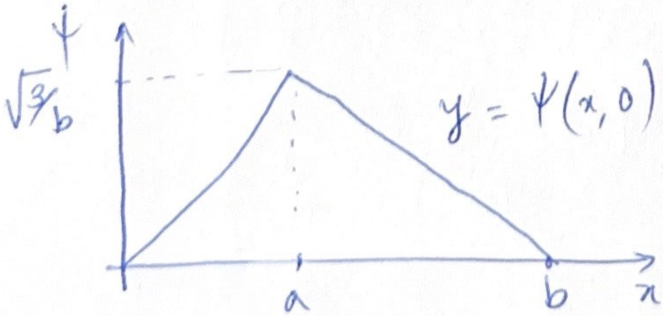
or $|A|^2\left[\frac{a}{3}+\frac{b^2-2ab+a^2}{3(b-a)}\right]=1$

or $|A|^2\left[\frac{a}{3}+\frac{b-a}{3}\right]=1$

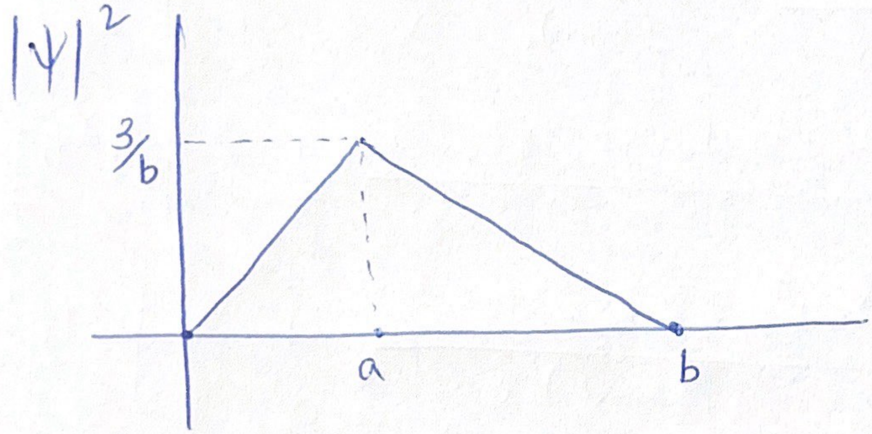
or $|A|^2\times\frac{b}{3}=1$

$|A|=\sqrt{\frac{3}{b}}$

(b) Sketch $\psi(x,0)$, as a function of x.



(c) where is the particle most likely to be found, at $t=0$?



Answer: At $x=a$, with probability $3/b$

(d) what is the probability of finding the particle to the left of a? Check your result with the limiting cases $b=a$ and $b=2a$.

$$P = \int_{-\infty}^{a} |\Psi(x,0)|^2 \, dx$$

$$= \cancelto{0}{\int_{-\infty}^{0} |\Psi(x,0)|^2 \, dx} + \int_{0}^{a} |\Psi(x,0)|^2 \, dx$$

$$= \frac{3}{b} \int_{0}^{a} \frac{x^2}{a^2} \, dx$$

$$= \frac{3}{b} \cdot \frac{1}{a^2} \cdot \frac{a^3}{3}$$

$$= \frac{a}{b}$$

For $b=a$, $P=1$
For $b=2a$, $P=\frac{1}{2}$
$\Big\}$ Both make sense.

(e) What is the expectation value of x?

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi|^2 \, dx$$

$$= \int_0^a x \cdot \frac{3}{b} \frac{x^2}{a^2} \, dx + \int_a^b x \cdot \frac{3}{b} \frac{(b-x)^2}{(b-a)^2} \, dx$$

$$= \frac{3}{a^2 b} \int_0^a x^3 \, dx + \frac{3}{b(b-a)^2} \int_a^b x (b-x)^2 \, dx$$

Integrate and simplify ________________