Test 3
PHYS4500: Quantum Field Theory

Casey Hampson

November 10, 2024

Problem 1.

The gauge transformation that the gauge fields must satisfy in Yang-Mills theories is

$$\boldsymbol{\sigma} \cdot \mathbf{A}_\mu \to U\boldsymbol{\sigma} \cdot \mathbf{A}_\mu U^{-1} + \frac{i}{q}(\partial_\mu U)U^{-1}. \tag{1.1}$$

If we assume a small λ, then we can express an arbitary $SU(2)$ transformation like

$$U = e^{iq\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}} \to 1 + iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}. \tag{1.2}$$

Additionally, as an element of $SU(2)$, U is unitary, meaning

$$U^{-1} = U^\dagger = 1 - iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}. \tag{1.3}$$

Now, we just replace all occurrenced of U in Equation (1.1) with the small λ version:

$$\boldsymbol{\sigma} \cdot \mathbf{A}_\mu \to (1 + iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda})\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}(1 - iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}) + \frac{i}{q}[\partial_\mu(1 + iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda})](1 - iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}) \tag{1.4}$$

Looking at just the first term, we have

$$\to iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda} + iq(\boldsymbol{\sigma} \cdot \boldsymbol{\lambda})(\boldsymbol{\sigma} \cdot \mathbf{A}_\mu) - iq(\boldsymbol{\sigma} \cdot \mathbf{A}_\mu)(\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}) + q^2(\boldsymbol{\sigma} \cdot \boldsymbol{\lambda})(\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}). \tag{1.5}$$

Based on the hint given in the problem, we know that the last term will be zero since it involves a cross product of the same vector ($\boldsymbol{\lambda}$) with itself. Thus,

$$\to iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda} + iq(\boldsymbol{\lambda} \cdot \mathbf{A}_\mu + i\boldsymbol{\sigma} \cdot \boldsymbol{\lambda} \times \mathbf{A}_\mu) - iq(\boldsymbol{\lambda} \cdot \mathbf{A}_\mu + i\boldsymbol{\sigma} \cdot \mathbf{A}_\mu \times \boldsymbol{\lambda}). \tag{1.6}$$

The dot product cancels, and we can switch the order of the cross product at the cost of a minus sign so that

$$\to \boldsymbol{\sigma} \cdot \mathbf{A}_\mu - 2q\boldsymbol{\sigma} \cdot \boldsymbol{\lambda} \times \mathbf{A}_\mu \to \boldsymbol{\sigma} \cdot (\mathbf{A}_\mu - 2q\boldsymbol{\lambda} \times \mathbf{A}_\mu). \tag{1.7}$$

For the second term, the derivative will only act on the $\boldsymbol{\lambda}$:

$$\to \frac{i}{q}[\partial_\mu(1 + iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda})](1 - iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}) \tag{1.8}$$

$$\to \frac{i}{q}[iq\boldsymbol{\sigma} \cdot (\partial_\mu\boldsymbol{\lambda})](1 - iq\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}) \tag{1.9}$$

$$\to -\boldsymbol{\sigma} \cdot (\partial_\mu\boldsymbol{\lambda}) + [\boldsymbol{\sigma} \cdot (\partial_\mu\boldsymbol{\lambda})(\boldsymbol{\sigma} \cdot \boldsymbol{\lambda})]. \tag{1.10}$$

Just as before, the last term sill be zero. So, putting it all together and pulling out a $\boldsymbol{\sigma}$:

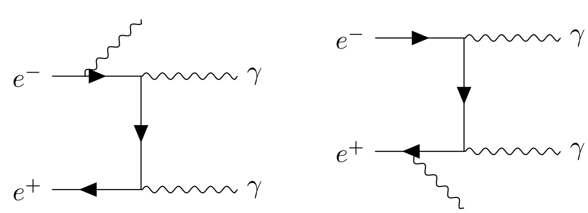
$$\boldsymbol{\sigma} \cdot \mathbf{A}_\mu \to \boldsymbol{\sigma} \cdot [\mathbf{A}_\mu - \partial_\mu\boldsymbol{\lambda} - 2q\boldsymbol{\lambda} \times \mathbf{A}_\mu]. \tag{1.11}$$

Since we now have pulled out a $\boldsymbol{\sigma}$ on both sides, we can consider only the transformation on the field:

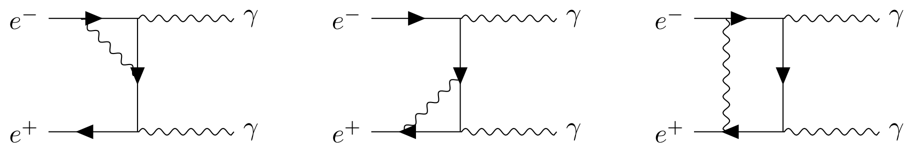
$$\boxed{\mathbf{A}_\mu \to \mathbf{A}_\mu - \partial_\mu\boldsymbol{\lambda} - 2q\boldsymbol{\lambda} \times \mathbf{A}_\mu.} \tag{1.12}$$

Problem 2.

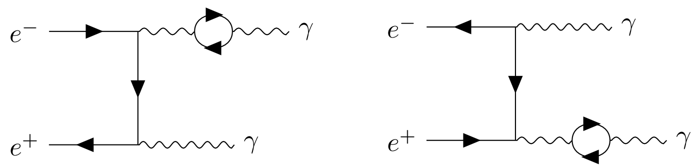
There are two diagrams with a real emission of a photon from a fermion line:



Then there are two vertex correction diagrams along with a “box” diagram:



Then there are two vacuum polarization diagrams:



There are also all of these diagrams but with the two final state photons switched, since they are identical particles and there is no way to tell which photon came from which vertex.

Problem 3.

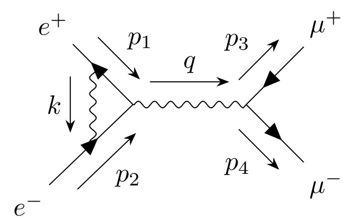


Figure 1: An vertex correction diagram at 1-loop for $e^+ + e^- \to \mu^+\mu^-$.

(I have no idea why this software is making the e^- line longer than the other, but this is the right diagram) Figure 1 contains the diagram that we are evaluating. Starting off:

$$i\mathcal{M} = \bar{v}_1 i\Lambda^\mu u(2) \left(\frac{-ig_{\mu\nu}}{q^2}\right) \bar{u}_4(-ie\gamma^\nu)v_3, \tag{3.1}$$

where $i\Lambda$ is the vertex correction amplitude:

$$i\Lambda^\mu = \int \frac{\mathrm{d}^n k}{(2\pi)^n}(-ie\gamma^\rho)\left(\frac{i(\not{p}_2 + \not{k} + m_e)}{(p_2+k)^2 - m_e^2}\right)(-ie\gamma^\mu)\left(\frac{i(\not{p}_1 - \not{k} + m_e)}{(p_1-k)^2 - m_e^2}\right)(-ie\gamma^\lambda)\left(\frac{-ig_{\rho\lambda}}{k^2}\right) \tag{3.2}$$

$$= -e^3 \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\gamma^\rho(\not{p}_2 + \not{k} - m_e)\gamma^\mu(\not{p}_1 - \not{k} - m_e)\gamma_\rho}{[(p_2+k)^2 - m_e^2][(p_1-k)^2 - m_e^2]k^2}. \tag{3.3}$$

So, the total amplitude is

$$i\mathcal{M} = \frac{ie^4}{(p_1+p_2)^2}\bar{v}_1 \left(\int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\gamma^\rho(\not{p}_2 + \not{k} - m_e)\gamma^\mu(\not{p}_1 - \not{k} - m_e)\gamma_\rho}{[(p_2+k)^2 - m_e^2][(p_1-k)^2 - m_e^2]k^2}\right) u_2 \times (\overline{u}_4\gamma_\mu v_3). \tag{3.4}$$

Problem 4.

To do the integral, we need to employ Feynman parametrization, which has the general form:

$$\prod_{i=1}^{n} \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \int_0^1 \left(\prod_{i=1}^{n} \mathrm{d}x_i \; x_i^{\alpha_i - 1} \right) \frac{\delta(1 - x}{\left(\sum_{i=1}^{n} x_i A_i\right)^{\alpha}} \tag{4.1}$$

where $\alpha = \sum_i \alpha_i$ and $x = \sum_i x_i$. For our case, we have something like

$$\frac{1}{ABC} = 2 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \frac{\delta(1 - x - y - z)}{(Ax + By + Cz)^3} \tag{4.2}$$

$$= 2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{1}{[Ax + By + (1 - x - y)C]^3}. \tag{4.3}$$

Therefore,

$$\frac{1}{[(p_2 + k)^2 - m_e^2][(p_1 - k)^2 - m_e^2]k^2} = 2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \; \frac{1}{[(p_2 + k)^2 x + (p_1 - k)^2 y - m_e^2(x + y) + (1 - x - y)k^2]^3}, \tag{4.4}$$

meaning that our entire amplitude is given by

$$i\mathcal{M} = \frac{2ie^4}{(p_1 + p_2)^2} \overline{u}_4 \gamma_\mu v_3 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\gamma^\rho (\not{p}_2 + \not{k} - m_e) \gamma^\mu (\not{p}_1 - \not{k} - m_e) \gamma_\rho}{[(p_2 + k)^2 x + (p_1 - k)^2 y - m_e^2(x + y) + (1 - x - y)k^2]^3}. \tag{4.5}$$

Problem 5.

The generating functional for non-interacting spinor fields is given by

$$Z_0[\eta, \bar{\eta}] = \exp\left\{ - \int \mathrm{d}^4x \mathrm{d}^4y \; \bar{\eta}(x) S(x - y) \eta(y) \right\}, \tag{5.1}$$

which, as a generating functional, has

$$S(x - y) = \langle 0 \,|\, T\{\psi(x)\overline{\psi}(y)\} \,|\, 0 \rangle = - \left. \frac{\delta^2 Z_0[\eta, \bar{\eta}]}{\delta\bar{\eta}(x)\delta\eta(y)} \right|_{\eta = \bar{\eta} = 0}. \tag{5.2}$$

Doing the differentiation, we note that the x and y in the exponential are dummy, so to differentiate with respect to $\eta(y)$, for instance, we should switch them to primes:

$$\frac{\delta}{\delta\eta(y)} Z_0[\eta, \bar{\eta}] = \frac{\delta}{\delta\eta(y)} \exp\left\{ - \int \mathrm{d}^4x' \mathrm{d}^4y' \; \bar{\eta}(x') S(x' - y') \eta(y') \right\}, \tag{5.3}$$

$$= \left(- \int \mathrm{d}^4x' \bar{\eta}(x) S(x' - y) \right) Z_0[\eta, \bar{\eta}]. \tag{5.4}$$

Doing the next functional derivative:

$$\frac{\delta^2}{\delta\bar{\eta}(x)\eta(y)} = -S(x - y) Z_0[\eta, \bar{\eta}] + \left(- \int \mathrm{d}^4x' \bar{\eta}(x) S(x' - y) \right) \left(- \int \mathrm{d}^4y' S(x - y') \eta(y) \right) Z_0[\eta, \bar{\eta}]. \tag{5.5}$$

Now, obviously $Z[0, 0] = 1$ since the exponent will be zero. Also, since each term in parentheses has a surviving Grassman source, they also vanish, leaving:

$$\left.\frac{\delta^2 Z_0[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\eta(y)}\right|_{\eta=\bar{\eta}=0} = -S(x-y) \tag{5.6}$$

or, considering the $(-i)^n$ factor:

$$\boxed{S(x-y) = -\left.\frac{\delta^2 Z_0[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\eta(y)}\right|_{\eta=\bar{\eta}=0},} \tag{5.7}$$

as expected.