PHYS4500: Quantum Field Theory Notes

Casey Hampson

December 12, 2024

1 Intro

This section will comprise a review of special relativity, particularly notation and whatnot.

- In relativity, we know that space and time must be on equal footing, hence instead of ordinary 3-vectors that we deal with normally, we instead have the 4-vector: x^μ.
- Specifically, this 4-vector is called a *contravariant* 4-vector. Often, when we say "4-vector", we mean the contravariant 4-vector.
- The position 4-vector is given by:

$$x^\mu = (ct, x, y, z)^\intercal = (x^0, x^1, x^2, x^3)^\intercal \tag{1.1}$$

- The "other" type of 4-vector is called the *covariant* 4-vector, given with the greek index in the subscript rather than the superscript: x_μ. This is given by contracting the normal (covariant) 4-vector with the Minkowski metric, the metric for the space-time in which we are working:

$$x_\mu = g_{\mu\nu} x^\nu. \tag{1.2}$$

- Since the metric is a rank-2 tensor, we can express it as an ordinary 4×4 matrix (since we are working in 4-d spacetime):

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.3}$$

- Two interesting things to note are that 1) the metric is symmetric, meaning $g_{\mu\nu} = g_{\nu\mu}$, and 2) it is equal to its inverse, meaning $g_{\mu\nu} = g^{\mu\nu}$.
- One important quantity we would be interested in in classical mechanics is the distances of things. Particularly, we were interested in the quantity $\mathbf{x} \cdot \mathbf{x}$ or $|\mathbf{x}|^2$, where $|\mathbf{x}|$ is the length of something. This quantity is always invariant under rotations and other changes of basis *in classical mechanics*.
- In special relativity, since we can now boost to super fast reference frames (we could do this before, but we didn't know that it would be weird), we have to take time into account, and this quantity is no longer an invariant under reference frame changes.
- Now, we are looking at the dot product of the 4-vector with itself, incorporating time:

$$x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu = c^2 t^2 - x^2 - y^2 - z^2 = \text{inv}. \tag{1.4}$$

- It is this quantity that is invariant under Lorentz transformations.
- Just as we used the 4-position to have distance and time on the same footing, we also have a similar case for momentum and energy: we call it the 4-momentum:

$$p^\mu = (E/c, p_x, p_y, p_z)^\intercal. \tag{1.5}$$

- The invariant quantity here is quite special:

$$p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2, \tag{1.6}$$

from the "Pythagorean relation", given by:

$$E = \sqrt{(\vec{p}c)^2 + (mc^2)^2}. \tag{1.7}$$

- There is the interesting effect of this being that we now have the "ability" to have massless particles; or rather, they just make a lot more sense. They can also have energy and momentum.
- Additionally, we can make the following definitions:

$$p_\mu p^\mu \begin{cases} > 0 \rightarrow \text{“time-like”} \\ = 0 \rightarrow \text{“light-like”} \\ > 0 \rightarrow \text{“space-like”}. \end{cases} \tag{1.8}$$

- All massive particules, naturally, are time-like, since they have a positive mass, and the quantity $p_\mu p^\mu = m^2c^2$ will hence be positive. The photon, as a massless particle, is light-like
- There are no space-like particles, but it is still important to examine when considering something like the separation of two events in space-time. It is possible to have them be space-like separated, where they cannot influence each other; i.e. no light/information can travel between the two events before they happen.
- We may also be interested in infinitesimals: $\mathrm{d}x^\mu$. We can form an invariant infinitesimal quantity like before: $\mathrm{d}x_\mu \mathrm{d}x^\mu = \text{inv}$.
- Lastly, derivatives are going to be very important. We define a derivative in special relativity as the following:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right) = \left(\frac{1}{c}\frac{\partial}{\partial t},\ \boldsymbol{\nabla} \right). \tag{1.9}$$

- There is also the special relativity version of the Laplacian, called the **d'Alembertian**:

$$\Box \equiv \partial_\mu \partial^\mu = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2},\ \frac{\partial^2}{\partial x^2},\ \frac{\partial^2}{\partial y^2},\ \frac{\partial^2}{\partial z^2} \right) \tag{1.10}$$

- Now we can turn to Lorentz transformations, which involve boosting to a frame S' that is moving at some speed v with respect to the originalf frame S.
- We have the following relations:

$$\begin{cases} t' = \gamma\left(t - \frac{v}{c^2}\right) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases} = \begin{cases} x^{0\prime} = \gamma\left(x^0 - \frac{v}{c}\right) \\ x^{1\prime} = \gamma\left(x^1 - \frac{v}{c}x^0\right) \\ x^{2\prime} = x^2 \\ x^{3\prime} = x^3. \end{cases} \tag{1.11}$$

- We can simplify this by representing this transformation as a rank-2 tensor:

$$x^{\mu\prime} = \Lambda^{\mu\prime}_{\mu} x^\mu. \tag{1.12}$$

- We can again, as with the metric, represent this as a 4×4 matrix:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.13}$$

- In fact, this is really a more mathematically correct definition of the 4-vector: an object that transforms in the above way due to a Lorentz Transformation.
- Should we be interested in transforming a rank-2 tensor, we need another transformation matrix:

$$T^{\mu\prime\nu\prime} = \Lambda^{\mu\prime}_{\nu} \Lambda^{\nu\prime}_{\nu} T^{\mu\nu}. \tag{1.14}$$

- The last thing we can look at is collisions, and the basics with the kinematics involved.
- Let's consider a collision $A + B \to 1 + 2$. We know from simple energy/momentum conservation that:

$$P_a^\mu + P_b^\mu = P_1^\mu + P_2^\mu. \tag{1.15}$$

- Now, these individual 4-vectors themselves don't mean much, and the quantity on either (or both) sides can change depending on the frame we look at. We know, though, that we can form Lorentz invariant scalars by squaring both sides.
- In such a case as this, with a $2 \to 2$ reaction, such squares have special names: the **Mandelstam variables**:

$$s = (P_a^\mu + P_b^\mu)^2 = (P_1^\mu + P_2^\mu)^2 \tag{1.16}$$
$$t = (P_a^\mu - P_1^\mu)^2 = (P_b^\mu - P_2^\mu)^2 \tag{1.17}$$
$$u = (P_a^\mu - P_2^\mu)^2 = (P_b^\mu - P_1^\mu)^2 \tag{1.18}$$

- Now, we can choose whatever frame we want to calculate these values, and since they are Lorentz invariant values, they will be the same no matter what frame.
- Because of this, we should just choose the frame that makes the calculation the easiest. Often this is the *center-of-mass* frame, where the total momentum $\sum \mathbf{p} = 0$. This gives us

$$P_a^\mu + P_b^\mu = \left(E_a^{CM} + E_b^{CM}, \mathbf{0}\right), \tag{1.19}$$

  where we have adopted natural units now, which means that we have set $\hbar = c = 1$ for simplicity.
- Now, it is super easy to calculate s, for instance:

$$s = (P_a^\mu + P_b^\mu)^2 = (E_a^{CM} + E_b^{CM})^2 \tag{1.20}$$

- s specifically is a pretty important variable. From the above relation, we have that

$$\sqrt{s} = E_a^{CM} + E_b^{CM}, \tag{1.21}$$

  which is the sum of the center-of-mass energies of the two incoming particles, which is a pretty universal and useful variable. This will be used later (probably, I don't know).
- Lastly, we have a neat identity whose proof is relatively simple:

$$s + t + u = \sum_i m_i^2, \tag{1.22}$$

  for all the masses in the process.

2 The Klein Gordon Equation

Now to some more cool/new stuff.

- The Schrödiger Equation (SE) did very well for atomic/non-relativistic physics, where speeds weren't necessarily all that fast:

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{2.1}$$

  where $\hat{H}$ is the Hamiltonian operator

$$\hat{H} = -\frac{\hbar}{2m}\boldsymbol{\nabla}^2 + V. \tag{2.2}$$

- Here, we have used the classical/Newtonian Hamiltonian but applied the quantum prescription that $p \to -i\hbar\boldsymbol{\nabla}$ and let them become operators that act on a wave function.
- Additionally, there is the prescription that $E \to i\hbar\dfrac{\partial}{\partial t}$.
- These are motivated (but not necessarily derived; I believe the correct terminology for the derivation has to do with canonical quantization or something) by Noether's theorem and the idea of symmetries.
- If there is space-translational symmetry, it implies conservation of linear momentum, which is why we have a spatial derivative taking the place of the momentum.
- Similarly, for time-translational symmetry, we have conservation of energy, which is why we have a time derivative taking the place of the energy.
- However, the point of all of this is that these are quantum prescriptions, which are good, but they are only prescriptions on the inherently non-relativistic Newtonian version of the Hamiltonian.
- Additionally, the SE is manifestly not relativistic due to its different treatment of space and time by having different orders in their derivatives.
- This just won't work.
- So let's try and fix it by first just starting with a free particle, meaning there is no potential energy function.
- Instead of using the non-relativistic Hamiltonian, let's use the relativistic energy-momentum relation $p_\mu p^\mu = m^2$.
- Additionally, let's change the prescriptions to be relativistic, since before, specifically the momentum one, just resulted in the classical gradient. This new prescription will be $p^\mu \to i\hbar\partial^\mu$.
- By doing this, plugging into our relation, and letting it act on some wave function ϕ (a different letter for differentiating between this and the non-relativistic version):

$$p_\mu p^\mu \phi = m^2 \phi \tag{2.3}$$

$$-\partial_\mu \partial^\mu \phi = m^2 \phi \tag{2.4}$$

$$\boxed{(\partial_\mu \partial^\mu + m^2)\phi = 0} \tag{2.5}$$

- This is the **Klein-Gordon Equation**. Interestingly, Schrödinger came up with this before his SE, but since it didn't work for the hydrogen atom (due to spin), he abandoned it. Klein and Gordon then, a little while later, independently came up with this equation again and published anyway before spin (and the other problems) were analyzed.
- The solutions to this equation are plane-wave solutions:

$$\phi(x^\mu) = A \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right) \tag{2.6}$$

- Now, from the non-rel. case, we had that for a wavefunction ψ that obeyed the SE, the Born interpretation meant that $|\psi|^2 = \psi^*\psi$ gives the probability density. This value is always positive definite, which is a good thing.
- On the other hand, $|\phi|^2$ is not always positive. This is one of the reasons Schrödinger held back on publishing this equation as well, because this was not understood at the time.

- Back again to the SE, we had that the wave function satisfied a *probability current*

$$\mathbf{j} = -\frac{i\hbar}{2m}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right), \tag{2.7}$$

which is analogous to the electric current, in the sense that it must be conserved, i.e. it satisfies some *continuity equation*:

$$\frac{\partial\psi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0. \tag{2.8}$$

- To make this relativistic so it works in our theory, we need to consider a temporal component in some way. So, let's define

$$\rho \equiv -\frac{i\hbar}{2m}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right). \tag{2.9}$$

- We can now define a new 4-vector current

$$j^\mu \equiv (\rho, \mathbf{j}) = \frac{i\hbar}{2m}\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right), \tag{2.10}$$

- Which satisfies the continuity equation

$$\partial_\mu j^\mu = 0. \tag{2.11}$$

- Let's show that this works:

$$\partial_\mu j^\mu = \partial_\mu\left[\frac{i\hbar}{2m}\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)\right] \tag{2.12}$$

$$= \frac{i\hbar}{2m}\left[\partial_\mu\phi^*\partial^\mu\phi + \phi^*\partial_\mu\partial^\mu\phi - \partial_\mu\phi\partial^\mu\phi^* - \phi\partial_\mu\partial^\mu\phi^*\right]. \tag{2.13}$$

The first and third terms cancel since they are the same, and by using the Dirac equation and its complex conjugate, this becomes

$$\partial_\mu j^\mu = \frac{i\hbar}{2m}\left[-\phi^* m^2\phi + \phi m^2\phi^*\right] = 0. \tag{2.14}$$

3 The Dirac Equation

- There are a number of issues with the KG Equation. First, as mentioned before, $|\phi|^2$ is not positive-definite always, and negative probabilities don't make sense.
- This is resolved later by treating the KG no longer as a single-particle equation but as a field equation. This realization occurs since the number of particles in any given process is no longer constant; we can create particle/anti-particle comboes from the vacuum.
- The other main problem that Dirac saw was the negative energy states, which came from:

$$p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2 \tag{3.1}$$

$$\rightarrow E = \pm\sqrt{\mathbf{p}^2c^2 + m^2c^4}. \tag{3.2}$$

- To remedy this, he sought to find an equation that was first order in derivatives. Starting again from the energy-momentum relation, he wanted to factorize it. However, since $p_\mu p^\mu$ isn't an ordinary square, there must be some factor, call it γ, that should be a 4-vector:

$$(p_\mu p^\mu - m^2) = (\gamma_\mu p^\mu - m)(\gamma_\nu p^\nu + m) = 0, \tag{3.3}$$

$$\gamma_\mu p^\mu\gamma_\nu p^\nu + m\gamma_\mu p^\mu - m\gamma_\nu p^\nu - m^2 = 0, \tag{3.4}$$

$$\gamma_\mu p^\mu\gamma_\nu p^\nu - m^2 = 0. \tag{3.5}$$

Now, this is basically the same as the original energy-momentum relation, as long as the first term is equal to p^2:

$$\gamma_\mu p^\mu \gamma_\nu p^\nu = p_\mu p^\mu \tag{3.6}$$

Applying the quantum prescription (in natural units) and flipping upper to lower indices and vice versa (for consistency later on):

$$\gamma^\mu (i\partial_\mu)\gamma^\nu (i\partial_\nu) = (i\partial^\mu)(i\partial_\mu), \tag{3.7}$$

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = g^{\mu\nu}\partial_\mu \partial_\nu, \tag{3.8}$$

Now, since $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, we can say the following:

$$\frac{1}{2}\left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \gamma^\mu \gamma^\nu \partial_\nu \partial_\mu\right) = g^{\mu\nu}\partial_\mu \partial_\nu, \tag{3.9}$$

and since we can mess around with indices in any given term here (all the indices would contract, so they really are dummy indices), by switching the μ's and ν's in the second term in parentheses, we can say

$$g^{\mu\nu}\partial_\mu \partial_\nu = \frac{1}{2}\left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \gamma^\nu \gamma^\mu \partial_\mu \partial_\nu\right), \tag{3.10}$$

$$g^{\mu\nu}\partial_\mu \partial_\nu = \frac{1}{2}\left\{\gamma^\mu,\ \gamma^\nu\right\}\partial_\mu \partial_\nu. \tag{3.11}$$

Lastly, since the coefficients on each side have to match, we can say

$$\boxed{\left\{\gamma^\mu,\ \gamma^\nu\right\} = 2g^{\mu\nu}}. \tag{3.12}$$

- The above relation is very useful for determining the form of the these matrices. From here, we can say
  - For $\mu = \nu = 0$:

$$\left\{\gamma^0,\ \gamma^0\right\} = \gamma^0\gamma^0 + \gamma^0\gamma^0 = 2\left(\gamma^0\right)^2 = 2g^{\mu\nu} \tag{3.13}$$

$$\left(\gamma^0\right)^2 = g^{\mu\nu}. \tag{3.14}$$

  - For $\mu = \nu = i$ where $i = 1, 2, 3$:

$$\left(\gamma^i\right)^2 = -1 \tag{3.15}$$

  - And for $\mu \neq \nu$:

$$\left\{\gamma^\mu,\ \gamma^\nu\right\} = 0. \tag{3.16}$$

- From these relations, it is clear that the gamma's cannot be numbers; instead, Dirac found, they are 4×4 matrices. In the standard representation (multiple sets satisfy the above relations), we have:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.17}$$

- Here, we have adopted the convention where each element in the 2×2 matrix is itself a 2×2 matrix, where 1 is the identity and 0 is the matrix of zeros. In this convention, the other gammas can be very nicely represented in terms of the Pauli spin matrices:

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{3.18}$$

- Back again to the factorization of the energy-momentum relation, if we take one of the factors (the one with the negative mass term), we can apply the (relativistic) quantum prescription and have it act on a **Dirac spinor** ψ, we get:

$$\gamma^\mu p_\mu - m = 0, \tag{3.19}$$
$$i\gamma^\mu \partial_\mu - m = 0, \tag{3.20}$$
$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi = 0}. \tag{3.21}$$

- We can also adopt to the *Feynman slash* notation, which defines $\not{A} \equiv \gamma^\mu A_\mu$ to get:

$$\boxed{\left(i\not{\partial} - m\right)\psi = 0}. \tag{3.22}$$

- Equations (3.21) and (3.22) are the **Dirac Equation**.

- The spinor ψ, from this equation, must be a 4-component object, since the gammas are 4×4 matrices. However, the spinor is not a 4-vector, as it does not transform the same under Lorentz transformations (we will prove this later).

- It will be helpful to also define the **adjoint spinor**:

$$\overline{\psi} = \psi^\dagger \gamma^0, \tag{3.23}$$

as well as the properties

$$\gamma^{0\dagger} = \gamma^0, \qquad \gamma^{i\dagger} = -\gamma^i. \tag{3.24}$$

- Now if we just take the Hermitian conjugate of the Dirac equation (written in a more appealing way to do Hermitian conjugates):

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi &= m\psi \\ \rightarrow (i\gamma^\mu \partial_\mu \psi)^\dagger &= m\psi^\dagger \\ -i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} &= m\psi^\dagger. \end{aligned}$$

Now, since the time and spatial components of the gamma matrices are Hermitian and anti-Hermitian respectively, we need to split them up, so we get

$$-i\partial_0 \psi^\dagger \gamma^{0\dagger} - i\partial_i \psi^\dagger \gamma^{i\dagger} = m\psi^\dagger.$$

What we can do here is first insert $\gamma^0\gamma^0$ in between the $\psi^\dagger$ and $\gamma^{i\dagger}$, since $\left(\gamma^0\right)^2 = 1$. Additionally, let's also multiply on the right by γ^0 on both sides. Lastly, in this one step, we will also use the propertes in Eq. (3.24):

$$-i\partial_0 \psi^\dagger \gamma^0 \gamma^0 + i\partial_i \psi^\dagger \gamma^0 \gamma^0 \gamma^i \gamma^0 = m\psi^\dagger \gamma^0.$$

. In each term, we have $\psi^\dagger \gamma^0$, which is just the definition of the adjoint spinor as in Eq. (3.23):

$$-i\partial_0 \overline{\psi} \gamma^0 + i\partial_i \overline{\psi} \gamma^0 \gamma^i \gamma^0 = m\overline{\psi}.$$

We also know that the time and spatial gamma matrices anti-commute, so we can flip the γ^i with either of the γ^0's and pick up a minus sign:

$$\begin{aligned} -i\partial_0 \overline{\psi} \gamma^0 - i\partial_i \overline{\psi} \gamma^0 \gamma^0 \gamma^i &= m\overline{\psi}, \\ -i\partial_0 \overline{\psi} \gamma^0 - i\partial_i \overline{\psi} \gamma^i &= m\overline{\psi}, \end{aligned}$$

$$\rightarrow \boxed{-i\partial_\mu \overline{\psi} \gamma^\mu = m\overline{\psi}.} \tag{3.25}$$

This is the **adjoint Dirac equation**.

- We can also assemble the **4-current** or the **probability current** $j^\mu = \overline{\psi}\gamma^\mu\psi$, which is a conserved quantity (we will look at these 4-currents later):

$$\partial_\mu j^\mu = \partial_\mu \overline{\psi}\gamma^\mu\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi.$$

  Using the Dirac equation (Eq. (3.21)) and adjoint Dirac equation (Eq. (3.25)), the two terms become:

$$\partial_\mu j^\mu = im\overline{\psi}\psi - im\overline{\psi}\psi = 0,$$

  hence j^μ is conserved.

- The analog for the probability $|\Psi|^2$ in the Schrödinger equation is the "time" component of the 4-current $j^0 = \rho$, so now we have that

$$\rho = j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0.$$

  This is positive definite; we have solved the negative probability problem!

- Unfortunately, though, the problem of negative *energy* still remains, and we will come to that soon. For now, we will continue interpreting this Dirac equation now that we have got some of the formalism down.

- First, the general understanding nowadays is that the negative energy states are anti-particles, and the four components of the Dirac spinor ψ correspond to the spin up/down particles and their anti-particles.

- At the time of the Dirac equations founding, though, anti-particles were not found yet, so they had a different train of thought.

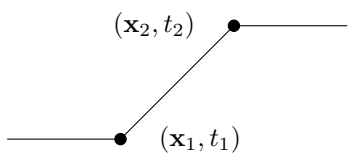
- For instance, Dirac had the idea that we were in a sea of completely negative energy states that were always completely filled, and because of the Pauli Exclusion Principle, all we observe are the other positive energy states.

- Now, if we sent a photon into this "sea", it would knock out one of the negative energy state and give it positive energy, and we would observe that as an electron. There would be a vacancy then in the "sea", which, since we "made" an electron, would be a vacancy of negative charge, or positive charge. This corresponds to the "positron".

- This thinking (by Dirac) led to the discovery of the positron by 1932 by Carl Anderson at Caltech.

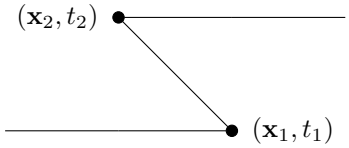
- There is another view of why negative energy states should exist, and it comes as a direct and manifest consequence of including special relativity in the theory, which is super neat.

- Let's imagine a particle traveling through space, and let's imagine two events/transformations that particle undergoes at two points in space time, call them $(\mathbf{x}_1, t_1)$ and $(\mathbf{x}_2, t_2)$ respectively. At the first space-time point, our particle transforms from an initial state $|\psi\rangle$ to a new state $|\psi'\rangle$, and at the second point, transforms back to $|\psi\rangle$. In our reference frame S, we might observe this trajectory to be something like:



Time here goes from left to right. So, we observe one particle transform once, then transform again.

- Now, what happens if we boost to a different reference frame S', such that we see something like the following:



(Again, time flows left to right.) In this reference frame, however, what we see is particle $|\psi\rangle$ traveling, then suddenly a $|\psi\rangle$ and $|\psi'\rangle$ pair produce, where the $|\psi'\rangle$ and the original $|\psi\rangle$ annihilate, leaving only the new $|\psi\rangle$ left. In both cases, we have a single particle enter and exit, but just by changing reference frames, we realize that the idea of anti-particles must be present.

3.1 Plane Wave Solutions to the Dirac Equation for a Free Particle

- For a free particle, again, this means that we will have no potential, and so we will have a similar situation to the KG solution in Eq. (2.6):

$$\psi(x^\mu) = e^{ix^\mu p_\mu} u(p), \tag{3.26}$$

where we have a similar time-dependent exponential factor, but also a new factor $u(p)$ to preserve the four dimensions of the spinor.

- We won't derive these solutions here, but the positive energy solutions are

$$u^{(1)}(p) = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \tag{3.27}$$

$$u^{(2)}(p) = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ -\frac{cp_z}{E+mc^2} \end{pmatrix}. \tag{3.28}$$

- These satisfy the normalization condition

$$u^\dagger u = \frac{2E}{c}. \tag{3.29}$$

- In momentum space, i.e. by undoing the quantum momentum prescription $\hat{p} \to i\partial_\mu$, we have that the Dirac equation becomes

$$(\not{p} - m)\psi = 0, \tag{3.30}$$

and also that

$$(\not{p} - m)u = 0. \tag{3.31}$$

- The negative energy solutions look like:

$$\psi(x^\mu) = e^{ix^\mu p_\mu} v(p), \tag{3.32}$$

where the $v(p)$ spinors are given by

$$v^{(1)}(p) = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} \frac{c(p_x - ip_y)}{E+mc^2} \\ -\frac{cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}, \tag{3.33}$$

$$v^{(2)}(p) = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x + ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}, \tag{3.34}$$

where these satisfy the same normalization condition and satisfy the adjoint Dirac equation

$$(\not{p} + m)v = 0. \tag{3.35}$$

- The last thing to look at quickly here is a particle's rest frame, in which all the components of momentum are zero, and hence all the components of the spinor are zero except for the single component with a 1 in it. In such a case, we can make the following connection:

$$\psi = \begin{pmatrix} e^- \text{ spin up} \\ e^- \text{ spin down} \\ e^+ \text{ spin down} \\ e^+ \text{ spin up} \end{pmatrix}.$$

It is important to notice that the spin states for positron seem reversed compared to those for the electron. This is intentional!

3.2 The Chiral/Weyl Representation

- We now turn our attention to a different representation of the gamma matrices, since there is more than just one combination of them that yields the commutation relation in Eq. (3.12). This representation we will call the **Chiral** or **Weyl** representation:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{3.36}$$

Notice that the "spatial" gammas have picked up an overall factor of -1. In many places in the literature, this is not the case; i.e. the spatial gammas in the chiral representation are the same as in the standard representation. We choose this convention as it will be more intuitive later.

- Let's also express the Dirac spinor ψ as a two component object:

$$\psi = \begin{pmatrix} \phi_R \\ \psi_L \end{pmatrix}, \tag{3.37}$$

where R stands for "right-handed" and L stands for "left-handed". This is in terms of **helicity**, which is the component of spin in the direction of the particle's motion.

- Now, in this representation, we can slightly more easily describe how a spinor transforms under Lorentz transformations: the two components transform as the fundamental and adjoint/anti-fundamental (this may or may not be the correct word(s)) representations of $SL(2, C)$, the special linear group in two-dimensions (this is not super important). A general transformation looks like:

$$\psi \to \psi' = \begin{pmatrix} e^{\frac{i}{2}\boldsymbol{\sigma}(\boldsymbol{\theta} - \mathbf{i}\boldsymbol{\varphi})} & 0 \\ 0 & e^{\frac{i}{2}\boldsymbol{\sigma}(\boldsymbol{\theta} + \mathbf{i}\boldsymbol{\varphi})} \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}, \tag{3.38}$$

where $\boldsymbol{\theta}$ is a vector of rotation parameters, $\boldsymbol{\varphi}$ is a vector of boost parameters, and $\boldsymbol{\sigma}$ is the familiar vector of Pauli spin matrices.

- Now, back to the idea of helicity. For massive particles, in one reference frame we may observe the helicity to be one value, but if we boost to a new frame in which we are moving faster than the particle, we will observe the particle to be moving in the other direction. However, the spin will obviously still appear the same, meaning the particle's helicity has reversed.
- This is not the case for massless particles, as we cannot boost to a reference frame that is moving faster than the particle, and as such, cannot view it as appearing to move in an opposite direction (while the spin remains the same). Hence, the helicity of a massless particle is constant, and it is often called **chirality** in this case.
- Now, for such massless particles, we only have one term remaining in the momentum space Dirac equation:

$$\begin{gathered}\gamma^\mu p_\mu \psi = 0, \\ \gamma^0 p_0 \psi + \gamma^i p_i \psi = 0, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_0 \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} p_i \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = 0,\end{gathered}$$

where in the second line we don't have minus signs because we are keeping the momentum indices lowered. Should we raise them (which we in the next step), we will only insert the minus sign then. Here, we really only just expanded the implicit sum. Simplifying:

$$\to \left[\begin{pmatrix} 0 & p_0 \\ p_0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\boldsymbol{\sigma} \cdot \mathbf{p} \\ \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \end{pmatrix} \right] \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = 0,$$

where now that we have made the momentum an ordinary vector (whose components do not have a minus sign), we insert the minus sign. We brought it back out again to have the relative minus between the two matrices in brackets though, but this is just how we chose to simplify. Continuing:

$$\to \begin{pmatrix} 0 & p_0 + \boldsymbol{\sigma} \cdot \mathbf{p} \\ p_0 - \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}.$$

This leads us to a system of two equations:

$$\begin{gathered}\begin{cases} (p_0 + \boldsymbol{\sigma} \cdot \mathbf{p}) \phi_L = 0, \\ (p_0 - \boldsymbol{\sigma} \cdot \mathbf{p}) \phi_R = 0, \end{cases} \\ \begin{cases} \boldsymbol{\sigma} \cdot \mathbf{p} \phi_L = -p_0 \phi_L, \\ \boldsymbol{\sigma} \cdot \mathbf{p} \phi_R = p_0 \phi_R, \end{cases} \\ \begin{cases} \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \phi_L = -\phi_L, \\ \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \phi_R = \phi_R. \end{cases}\end{gathered} \tag{3.39}$$

These are the **Weyl equations**, and they describe massless particles. The operator $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$ is the helicity operator, and we can see that for massless particles, left-handed states have a helicity of -1, and right-handed states have a helicity of $+1$.

4 Lagrangians and the Principle of Least Action

- For a system of N particles, we can label the positions and velocities with q_i and $\dot{q}_i$, respectively, where q is a "generalized" coordinate that can be cartesian, polar, etc, so long as it completely describes the system, and $i = 1, 2, \ldots, N$.

- We can define the **Lagrangian**:
$$L = L(\{q_i, \dot{q}_i\},\ t) = T - V, \tag{4.1}$$
where T and V are the kinetic and potential energies, and $\{q_i, \dot{q}_i\} = \{q_1, q_2, \ldots, q_N,\ \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N\}$.

- The **action** is defined as
$$S \equiv \int_{t_1}^{t_2} L \, \mathrm{d}t, \tag{4.2}$$
and the **principle of least action** by Hamilton states that for a real/physical trajectory/path of a particle, this action is extremized, or more specifically, it is minimized. What that means is that if we introduce an infinitesimal perturbation $q(t) \to q'(t) = q(t) + \delta q(t)$, then the action should remain extremized; i.e. it shouldn't change. Here, δ is small *variation*, similar to the differential.

- Formally:
$$\delta S = \int_{t_1}^{t_2} \delta L \, \mathrm{d}t. \tag{4.3}$$

- Additionally, we want to enforce that t'his variation leaves the endpoints the same: $q'(t_1) = q(t_1)$,and $q'(t_2) = q(t_2)$.

- Now, we can expand out the variation similar to the derivative chain rule:
$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \, \mathrm{d}t.$$
Doing integration by parts on the second term:
$$\delta S = \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q \, \mathrm{d}t + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta \, \mathrm{d}t.$$
Since we want the endpoints to be unaffected by our perturbation, the middle term vanishes. So,
$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \, \mathrm{d}t = 0.$$
Now, we want this to be valid for *any* variation δq, meaning we can't put any restrictions on it. Therefore, the only way we can have the integral evaluate to zero is for the quantity in brackets to be zero:
$$\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0.} \tag{4.4}$$

- Eq. (4.4) is called the **Euler-Lagrange Equation**, or in the case of multiple particles, the **Euler-Lagrange Equations**.

- To show that this works, we can use a "default" Lagrangian and see that we get Newton's second law from it. Using
$$L = \frac{1}{2} m \dot{q}^2 - V(q),$$
the Euler-Lagrange equation becomes
$$-\frac{\mathrm{d}V}{\mathrm{d}q} - \frac{\mathrm{d}}{\mathrm{d}t} \left[m \dot{q} \right] = -\frac{\mathrm{d}V}{\mathrm{d}q} - m\ddot{x},$$
$$\to\ -\frac{\mathrm{d}V}{\mathrm{d}q} = m\ddot{q}.$$
Since $\ddot{q} = a$ and we know that for conservative forces $F = -\boldsymbol{\nabla} V$, we retrieve $F = ma$, as expected.

- We can also define the **conjugate momentum**, which can be found by:

$$p = \frac{\partial L}{\partial \dot{q}}, \tag{4.5}$$

which is the quantity in parentheses in Eq. (4.4). For our default Lagrangian, we have that

$$p = m\dot{q} = mv,$$

which is also expected. It also allows us to find the more general version of Newton's 2nd law:

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}.$$

- The Hamiltonian is also closely related; in fact, it can be derived from the Lagrangian like so:

$$H = H(q,p) = p\dot{q} - L = \frac{\partial L}{\partial \dot{q}}\dot{q} - L. \tag{4.6}$$

This Hamiltonian is a function of position and momentum, rather than position and its time derivative (i.e. velocity). From this definition we can retrieve

$$H = (m\dot{q})\dot{q} - \left(\frac{1}{2}m\dot{q}^2 - V\right) = \frac{1}{2}m\dot{q}^2 + V,$$

which is the kinetic plus the potential energy, as expect.

- Turning to relativity, let's write the Lagrangian for a free (so $V = 0$) relativistic particle. We need our Lagrangian (and hence our action) to be invariant under Lorentz transformations. So, we can say that our Lagrangian should be proportional to the integral over the space-time interval $\mathrm{d}s$, since $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ is a relativistic invariant.

$$S = \int_{t_1}^{t_2} L \,\mathrm{d}t = -A\int_a^b \mathrm{d}s,$$

where A is some constant we want to find; it'll give us the Lagrangian. We can just look at the time component, or more specifically, the *proper* time τ:

$$S = -A\int_{\tau_1}^{\tau_2} c\,\mathrm{d}\tau = -A\int_{t_1}^{t_2} c\sqrt{1-\beta^2}\,\mathrm{d}t$$

Now that we have our integral back in terms of normal time, we can compare to the original equation for the action and find an equation for the Lagrangian:

$$L = -Ac\sqrt{1-\beta^2}.$$

To find A now, we can take the classical limit and compare to what we expect to get from Newtonian physics. This limit is $\beta << 1$, so we can Taylor expand the square root to get:

$$L = -Ac\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = -Ac + A\frac{1}{2}\frac{v^2}{c}.$$

The first term is simply a constant, and since the Euler-Lagrange equation contains only derivates of the Lagrangian, we might as well just get rid of that term:

$$L = A\frac{1}{2}\frac{v^2}{c}.$$

We expect, for a free classical particle, the Lagrangian to just be its kinetic energy $L = \frac{1}{2}mv^2$. So, we just set the two equal and solve for A:

$$A\frac{1}{2}\frac{v^2}{c} = \frac{1}{2}mv^2,$$

so $A = mc$. Thus, the Lagrangian for a free relativistic particle is

$$\boxed{L = -mc^2\sqrt{1-\beta^2}.}$$

- We can write this a little differently by defining

$$\dot{x}^\mu = \begin{pmatrix} t & \dot{x} & \dot{y} & \dot{z} \end{pmatrix}^\intercal, \tag{4.7}$$

so that we can form the relativistically-invariant quantity

$$\dot{x}_\mu \dot{x}^\mu = c^2 - v^2. \tag{4.8}$$

Now we can say

$$L = -mc\sqrt{\dot{x}_\mu \dot{x}^\mu},$$

so the action is

$$S = -mc\int_{t_1}^{t_2} \sqrt{g_{\mu\nu}\dot{x}^\nu \dot{x}^\mu}\ \mathrm{d}t.$$

- Bringing this to fields, we have that $q \to \phi(x^\mu)$ and $\dot{q} \to \partial_\mu \phi(x^\mu)$.
- Although, we have the problem that space and time are not treated equally, as the action only integrates over time. So, we can define

$$L = \int \mathcal{L}\ \mathrm{d}^3x, \tag{4.9}$$

where $\mathcal{L}$ is called the **Lagrangian density**. In particle physics, we almost solely work with this quantity, so we often just call it the Lagrangian, and we will do so in this course.

- We can now derive the analog of the Euler-Lagrange Equation(s) for fields. Again considering an infinitesimal variance in the field $\phi \to \phi' = \phi + \delta\phi$, we can, using the chain rule analog for variations, find the variation in the action as

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \right)\ \mathrm{d}^4x.$$

Doing integration by parts on the second term, we find

$$\delta S = \int \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi\ \mathrm{d}^4x + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right]_{t_1}^{t_2} - \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi\ \mathrm{d}^4x.$$

Again, the middle term will vanish due to our restriction that the perturbation be zero at the end points, so

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta\phi\ \mathrm{d}^4x = 0,$$

so

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0.} \tag{4.10}$$

This is the field version of the Euler-Lagrange equation.

- Similarly to the classical single-particle case, we can define a conjugate momentum

$$\pi(x^\mu) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}, \tag{4.11}$$

where we are only taking the time derivative rather than the full 4-derivative. This also keeps the conjugate momentum a scalar, as otherwise it'd be a 4-vector.

- We can derive a Lagrangian density for the Klein-Gordon field such that when applying the Euler-Lagrange (EL) equations, we get back the Klein-Gordon equation. This is seemingly a little strange, but the Lagrangian has a whole slew of other important things that revolve around the Lagrangian. This Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{4.12}$$

The first term in the EL equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi.$$

The second term is a little harder, as it is a bit nuanced (but only as a consequence of hiding summations in the Einstein summation convention). First, let's rewrite the Lagrangian to have all covariant derivatives by introducing the metric:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\nu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2.$$

Now when go to solve the second term in the EL equation, the presence of the index implies we need to sum over all the covariant 4-derivatives in the Lagrangian. So, the second term is really

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) = \frac{1}{2}\partial_\mu \left(g^{\mu\nu} \partial_\nu \phi \right) + \frac{1}{2}\partial_\nu \left(g^{\mu\nu} \partial_\nu \phi \right) = \partial_\mu \partial^\mu \phi = \Box \phi.$$

Plugging these into the full EL equation:

$$\boxed{\left(\Box + m^2 \right) \phi = 0,}$$

which is the KG equation, as expected.

4.1 Noether's Theorem

- Going back to classical stuff for a moment: imagine we make an infinitesimal transformation in the coordinates of our system by some function f, defined by some parameter ϵ such that when $\epsilon = 0$, we have no transformation. We are looking at something like $q_i' = f(q_i, \epsilon)$, with $q_i = f(q_i, 0)$ for no transformation. Assessing the change in the Lagrangian with respect to this parameter is:

$$\frac{\mathrm{d}L}{\mathrm{d}\epsilon} = \sum_i \left[\frac{\partial L}{\partial q_i'} \frac{\mathrm{d}q_i'}{\mathrm{d}\epsilon} + \frac{\partial L}{\partial \dot{q}_i'} \frac{\mathrm{d}\dot{q}_i'}{\mathrm{d}\epsilon} \right].$$

Now, if we are to evaluate this at $\epsilon = 0$, we have

$$\left. \frac{\mathrm{d}L}{\mathrm{d}\epsilon} \right|_0 = \sum_i \left[\frac{\partial L}{\partial q_i'} \frac{\mathrm{d}q_i'}{\mathrm{d}\epsilon} + \frac{\partial L}{\partial \dot{q}_i'} \frac{\mathrm{d}\dot{q}_i'}{\mathrm{d}\epsilon} \right]_0 = \sum_i \left\{ \frac{\partial L}{\partial q_i} \left[\frac{\mathrm{d}q_i'}{\mathrm{d}\epsilon} \right]_0 + \frac{\partial L}{\partial \dot{q}_i} \left[\frac{\mathrm{d}\dot{q}_i'}{\mathrm{d}\epsilon} \right] \right\},$$

where, when we evaluate the derivatives of L with respect to the primed coordinates at $\epsilon = 0$, we recall that $q_i = f(q_i, 0)$, so such evaluation essentially entails a dropping of the prime. The other

evaluation, the one of the derivative of the coordinate with respect to ϵ is not trivial. If we now use the EL equations and apply an reverse product rule, we get that:

$$\left.\frac{\mathrm{d}L}{\mathrm{d}\epsilon}\right|_0 = \sum_i \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\left[\frac{\mathrm{d}q_i'}{\mathrm{d}\epsilon}\right]_0 + \frac{\partial L}{\partial \dot{q}_i}\left[\frac{\mathrm{d}\dot{q}_i'}{\mathrm{d}\epsilon}\right]\right\} = \sum_j \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial L}{\partial \dot{q}_i}\left[\frac{\mathrm{d}q_i'}{\mathrm{d}\epsilon}\right]_0\right\}.$$

Now, we want our Lagrangian to be invariant/symmetric under such coordinate transformations, so ideally, its derivative with respect to the parameter characterizing that transformation should be zero. Again, the derivative of the coordinate with respect to ϵ remains, as it depends on the coordinate and the nature fo the parameter. Thus, the quantity in brackets must be equal to a constant:

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\left[\frac{\mathrm{d}q_i'}{\mathrm{d}\epsilon}\right]_0 = \text{const.} \tag{4.13}$$

While the Lagrangian itself should remain invariant under such a transformation, the coordinates themselves obviously change, so we are keeping the evaluation at $\epsilon = 0$ to enforce that this conserved quantity itself remain invariant for the unmodified coordinates, as those are the ones that the EL equations use, and it is the ones that the system actually follows.

- Let's consider a specific Lagrangian such as

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{z}^2 - mgz.$$

Making the transformation $x \to x' = x + \epsilon$, we immediately note that x itself does not appear and $\dot{x} = x$, so ϵ appears nowhere in the Lagrangian, and we can confirm that $\dfrac{\mathrm{d}L}{\mathrm{d}\epsilon} = 0$ here. Now using Eq. (4.13),

$$\frac{\partial L}{\partial \dot{x}}\left[\frac{\mathrm{d}x'}{\mathrm{d}\epsilon}\right] = m\dot{x},$$

which is x-component of linear momentum.

- What we have found is that for space translations, the corresponding conserved quantity is linear momentum. We can also find that for a time translation, we get something of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right] = 0,$$

where the quantity in brackets in the right-most expression is the Hamiltonian. Hence, for time-translational symmetry, the corresponding conserved quantity is energy.

- This is the essence of **Noether's Theorem**: for every symmetry in a system there is a corresponding conserved quantity given by Eq. (4.13). We are also able to find that for rotational symmetry, the corresponding conserved quantity is angular momentum.

- We now turn to **currents**, which are another way to express conserved quantities. From Physics II, we know that both charge density ρ and current density $\mathbf{J}$ are conserved. We can express this cleanly by placing them in a 4-vector where the charge density is the time component and the current density is/are the spatial components. Then, we have that

$$\partial_\mu J^\mu = 0$$

corresponds to what we found in Physics II:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\oint \mathbf{J}\cdot\mathrm{d}\mathbf{A}.$$

- In general, then, we are looking for conserved *currents* j^μ which give us our conservation laws.

- Making the jump to fields, we can easily show that Eq. (4.13) in terms of fields is

$$\left.\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\epsilon}\right|_0 = \partial_\mu \left[\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\left.\frac{\mathrm{d}\phi_n'}{\mathrm{d}\epsilon}\right|_0\right]. \tag{4.14}$$

This is different than the classical case, because we can transform both the fields themselves like we did with our derivation of the field version of the EL equation, and also the coordinates themselves. Field transformations would ideally leave the Lagrangian invariant, but transforming the coordinates, which the fields are a function of, would generally change the Lagrangian's value. Thus, in general, we cannot make the same assumption that $\mathrm{d}\mathcal{L}/\mathrm{d}\epsilon = 0$ like in the classical case. But, using the chain rule, we can write

$$\left.\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\epsilon}\right|_0 = \frac{\partial\mathcal{L}}{\partial x^\mu}\left.\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon}\right|_0 = (\partial_\mu\mathcal{L})\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon} = \partial_\mu\left(\mathcal{L}\left.\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon}\right|_0\right) - \mathcal{L}\left[\partial_\mu\left(\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon}\right)\right]_0.$$

It turns out that for most if not all transformations of interest, this last term will be zero; i.e. our transformations will have the property that $\partial_\mu(\mathrm{d}x^{\mu\prime}\epsilon) = 0$. Now we can subtract this from Eq. (4.14) to get zero:

$$0 = \partial_\mu\left\{\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\left[\frac{\mathrm{d}\phi_n'}{\mathrm{d}\epsilon}\right]_0 - \mathcal{L}\left[\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon}\right]_0\right\}, \tag{4.15}$$

where the quantity in the braces can be identified with our conserved current j^μ.

- Let's now look at a coordinate transformation $x^\nu \to x^{\nu\prime} = x^\nu + \epsilon^\nu$. This transformation now contains both time and position displacement. For the Lagrangians that we have seen so far, we never see any actual coordinates, only the fields which are functions of coordinates, so the value of the Lagrangian will change due to this transformation (as discussed above). Now,

$$\partial_\mu\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon^\nu} = \partial_\mu\delta^\mu_\nu = 0,$$

which confirms our previous assumption. Our conserved current also picks up an additional covariant index due to the derivative with respect to the contravariant ϵ^μ (recall that derivatives with respect to contravariant 4-vectors behave as covariant 4-vectors). The derivative with respect to the field is (dropping the n subscript for simplicity):

$$\left.\frac{\mathrm{d}\phi'}{\mathrm{d}\epsilon^\nu}\right|_0 = \frac{\partial\phi}{\partial x^\mu}\left[\frac{\mathrm{d}x^{\mu\prime}}{\mathrm{d}\epsilon^\nu}\right] = \frac{\partial\phi}{\partial x^\mu}\delta^\mu_\nu = \frac{\partial\phi}{\mathrm{d}x^\nu} = \partial_\nu\phi.$$

Plugging into our formula for the conserved current T^μ_ν (Eq. (4.15)):

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}.$$

It will be easier to work with if we raise the ν index, which turns the Kronecker delta unto the metric (as it stands, it's the identity matrix; by raising one of the indices, we are reversing the sign on the spatial components, and thus, we get the metric!):

$$T^{\mu\nu} = \left[\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\partial^\nu\phi_n\right] - g^{\mu\nu}\mathcal{L}. \tag{4.16}$$

- This is the **Stress-Energy tensor**, and it describes a number of quantities:

a) T^{00} is the field's energy density.

b) T^{0i} is the field's ith momentum density.

To show (a):

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0)\phi}\partial^0\phi - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L},$$

Hamiltonian *density*, which is just like the Lagrangian density in that

$$H = \int \mathcal{H}\, \mathrm{d}x^3. \tag{4.17}$$

- Looking at the KG field

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2,$$

we can compute the energy density

$$\begin{aligned}\mathcal{H} &= \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2\right) \\ &= (\partial_0\phi)^2 - \frac{1}{2}\left[(\partial^0\phi)^2 - (\boldsymbol{\nabla}\phi)^2\frac{1}{2}m^2\phi^2\right], \\ &= \left[(\partial_0\phi)^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right].\end{aligned}$$

This quantity is positive definite! We have now solved the negative energy problem too!

5 Quantization of Free Fields

- It can be shown that we can do a Fourier Expansion for a real scalar field ϕ like so:

$$\phi(x^\mu) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}(2\pi)\delta\left(p^2 - m^2\right)\Theta\left(p^0\right)\left[a\left(p^\mu\right)e^{-ip^\mu x_\mu} + a^\dagger\left(p^\mu\right)e^{ip^\mu x_\mu}\right](2p_0)^{1/2}. \tag{5.1}$$

Here, Θ is the step function, ensuring that the energy is positive, and the delta function handles the on-shell condition. The a's and $a^\dagger$'s are annihilation operators, respectively.

- If we integrate over p_0, we only have to focus on the delta function:

$$\delta\left(p^2 - m^2\right) = \delta\left((p^0)^2 - \mathbf{p}^2 - m^2\right),$$

so the integral sends $(p^0)^2 \to \mathbf{p}^2 + m^2 = E^2$, but again, this is just the mass-shell condition, so this doesn't really tell is much. Our integral is now:

$$\phi(x^\mu) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E}}\left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right]. \tag{5.2}$$

- In non-relativistic QM, we have that position and momentum are operators and time is a parameter. The position and momentum operators satisfy certain commutation relations, namely:

$$[\hat{x}, \hat{p}_x] = i, \qquad \text{and} \qquad [\hat{x}, \hat{x}] = [\hat{p}_x, \hat{p}_x] = 0. \tag{5.3}$$

- In QFT, we now need space and time on the same footing, so they both become parameters, and we promote the field itself to be the operator. The momentum operator here is the field version of the conjugate momentum $\hat{\pi}$, and these operators also satisfy certain commutation relations:

$$\left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)\right] = i\delta^3(\mathbf{x} - \mathbf{y}), \qquad \text{and} \qquad \left[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{y}, t)\right] = [\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)] = 0. \tag{5.4}$$

These are called the **equal time commutation relations**, since, clearly, the (anti-)commutators are taken for an equal time t.

- We can also find the commutation relations for the creation and annilihation operators by doing an inverse Fourier transform:

$$a(p) = \int \frac{\mathrm{d}^3x}{\sqrt{2E}} i \left[e^{ip\cdot x} \partial_0 \phi(x) - \left(\partial_0 e^{ip\cdot x}\right) \phi(x) \right], \tag{5.5}$$

and we find that

$$\left[a(p), a^\dagger(p)\right] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'), \qquad \text{and} \qquad [a(p), a(p')] = \left[a^\dagger(p), a^\dagger(p')\right] = 0. \tag{5.6}$$

- From here, we can define the **number operator** $N(p) \equiv a^\dagger(p)a(p)$ whose eigenvalues are $n(p)$, the number of particles at momentum p, and whose eigenkets are $|n(p)\rangle$ such that

$$N(p)\,|n(p)\rangle = n(p)\,|n(p)\rangle\,.$$

- From this, we can show that $a(p)$ is the annihilation operator and $a^\dagger(p)$ is the creation operator by considering applying $a(p)$ to an eigenket of the number operator, then applying the number operator:

$$N(p)a(p)\,|n(p)\rangle = \left([N(p), a(p)] + a(p)N(p)\right)|n(p)\rangle\,.$$

It can be shown that the above commutator is $-a(p)$, so

$$\begin{aligned} N(p)a(p)\,|n(p)\rangle &= -a(p)\,|n(p)\rangle + a(p)N(p)\,|n(p)\rangle\,, \\ &= (n(p) - 1)a(p)\,|n(p)\rangle\,. \end{aligned}$$

So, we see that we get one less particle when we apply the $a(p)$ operator; hence, it is the annihilation operator. We can find similarly that $a^\dagger(p)$ is the creation operator.

- These are analogous to the raising and lowering operators for the simple harmonic oscillator in QM.

- It is a bit length, so we won't show it, but by using the defintion of the fields in terms of the Fourier expension of the creation and annihilation operators we can express the Hamiltonian as

$$\begin{aligned} H &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{p^0}{2} \left[a^\dagger(p)a(p) + a(p)a^\dagger(p)\right], \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^0 \left[N(p) + \frac{1}{2}\right]. \end{aligned}$$

- However, the 1/2 part leads to an infinity since we are integrating over all momenta. So, as good physicist, since it is unphysical, we just ignore it! (Shortly we will give a *slightly* more technical reason as to why we should remove it.) Hence, the Hamiltonian operator is given as:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^0 N(p). \tag{5.7}$$

- This makes sense; we multiply the number of particles from $N(p)$ by the energy from p^0 to get the total energy.

- Now let's make some more definitions. The **ground state** or **vacuum state** is given by the ket $|\mathcal{O}\rangle$, however in particle physics we just use a 0 rather than a calligraphic $\mathcal{O}$. It will be very apparent as to whether we are looking at just a 0 ket or the vacuum state ket.

- Obviously, applying the annihilation operator should return 0: $a(p)\,|0\rangle = 0$, and so should applying the number operator: $N(p)\,|0\rangle = 0$.

- On the other hand, applying the creation operator $a^\dagger |0\rangle$ gives a 1-particle state. We will explore this in just a little bit.

- The vacuum expectation value of the energy is given by the braket

$$\langle 0| H |0\rangle = \langle 0| \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p^0 N(p) |0\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p^0 \langle 0| a^\dagger(p) a(p) |0\rangle = 0,$$

since $N(p)|0\rangle = 0$, meaning the expectation value for the energy in vacuum is zero, as expected. We could also have stated that $(a(p)|0\rangle)^\dagger = \langle 0| a^\dagger(p)$, so this bit is also zero.

- We now introduce the aforementioned "technical" way to get rid of the $1/2$ that led to infinity: **normal ordering**. Normal ordering is a convention that states that all creation operators should come to the left of all annihilation operators, so we should always have terms like $a^\dagger a$ and never have any terms like $aa^\dagger$. So, the Hamiltonian

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \frac{p^0}{2} \left[a^\dagger a = a a^\dagger\right] \to \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \frac{p^0}{2} \left[a^\dagger a + a^\dagger a\right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, p^0 N(p),$$

as we found. It will become increasingly clear that a significant part of our job as particle physicists is figuring out ways to systematically "sweep the infinities under the rug".

- The notation for this is the colon ":" that we place once before the operators we want to normal order and again after, meaning that $: aa^\dagger := a^\dagger a$.

- Now let's consider a 1-particle state with momentum p, which we get from applying the creation operator to the vacuum state along with a factor based on convention:

$$|p\rangle = \sqrt{2p^0} a^\dagger(p) |0\rangle .$$

- The corresponding bra is given by

$$\langle p| = \sqrt{2p^0} \langle 0| a(p).$$

- Therefore, the braket $\langle p|p'\rangle$ is

$$\begin{aligned} \langle p|p'\rangle &= \sqrt{2p^0}\sqrt{2p^{0\prime}} \langle 0| a(p) a^\dagger(p') |0\rangle , \\ &= 2\sqrt{EE'} \langle 0| \left[a(p), a^\dagger(p)\right] + a^\dagger(p) a(p) |0\rangle . \end{aligned}$$

The second term will be zero because $a(p)|0\rangle = 0$, so

$$\langle p|p'\rangle = 2\sqrt{EE'} \langle 0| \left[a(p), a^\dagger(p)\right] |0\rangle .$$

But we know what the commutator of these operators is from Eq. (5.6):

$$\langle p|p'\rangle = 2E(2\pi)^3 \delta^3(p - p') \langle 0|0\rangle .$$

By normalization we have that $\langle 0|0\rangle = 1$, so:

$$\langle p|p'\rangle = 2E(2\pi)^3 \delta^3(p - p'). \tag{5.8}$$

- We can now use this result to determine a 1-particle wavefunction as in non-relativistic QM. We normally wouldn't ever do this, but it is a good test of our machinery.

- It turns out that we can do so by the following:

$$\psi(x) = \langle 0| \phi(x) |p\rangle . \tag{5.9}$$

Expanding:

$$\psi(x) = \left\langle 0 \left| \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2p^{0\prime}}} \left[a(p')e^{-ip'\cdot x} + a^\dagger(p')e^{ip'\cdot x}\right] \right| p \right\rangle$$

Since the vacuum state $\langle 0|$ doesn't depend on anything, we can just bring it inside the integral and see that the second term in brackets (the creation term) will be zero since $\langle 0| a^\dagger = 0$:

$$\psi(x) = \left\langle 0 \left| \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E'}} a(p')e^{-ip'\cdot x} \right| p \right\rangle .$$

Since $\langle p| = \sqrt{2E}\langle 0| a(p)$, then $\langle 0| a(p) = \langle p|/\sqrt{2E}$, so

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E'} \left\langle p' \left| e^{-ip'\cdot x} \right| p \right\rangle .$$

But the exponential is just a number, so we can pull it out and use Eq. (5.8):

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E'} e^{-ip'\cdot x} \cdot 2E(2\pi)^3\delta^3(p'-p). \tag{5.10}$$

The integral just makes $p = p'$, so everything nicely cancels and we get:

$$\boxed{\psi(x) = e^{-ip\cdot x}.}$$

This is just a plane-wave solution, exactly what we would expect!

The Complex Scalar Field

- We can combine two fields ψ_1 and ψ_2 into one complex field like so

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad \text{and} \quad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2).$$

Now our Lagrangian is

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi, \tag{5.11}$$

and we have two Euler-Lagrange equations, one for ϕ and the other for ϕ^*, giving us two equations:

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \quad \text{and} \quad (\partial_\mu\partial^\mu + m^2)\phi^* = 0. ComplexScalarELEQs \tag{5.12}$$

- The Fourier expansions for these two fields become

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} \left[a(p)e^{-ip\cdot x} + b^\dagger(p)e^{ip\cdot x}\right], \tag{5.13}$$

$$\phi^\dagger(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} \left[b(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right]. \tag{5.14}$$

- The interpretation here is that $a(p)$ annihilates particles and $b^\dagger(p)$ creates anti-particles, and $b(p)$ annihilates anti-particles and $a^\dagger(p)$ creates particles.
- These a's and b's behave identically as the a's for the real scalar field, and they also do commute between each other.
- In normal ordering, the Hamiltonian operator is

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^0 \left[a^\dagger(p)a(p) + b^\dagger(p)b(p)\right]. \tag{5.15}$$

The Dirac Spinor Field

- The Lagrangian for the Dirac field is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi = \overline{\psi}(\partial\!\!\!/ - m)\psi.$$

- Since we have a spinor and the adjoint spinor fields, we have two Euler-Lagrange equations here as well. The one for the normals spinor field is

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi},$$
$$\partial_\mu\left(i\overline{\psi}\gamma^\mu\right),$$

  so

$$i\overline{\psi}\partial\!\!\!/ + m\overline{\psi} = \overline{\psi}(i\partial\!\!\!/ + m) = 0.$$

  Following a similar process with the adjoint spinor gives us

$$(i\partial\!\!\!/ - m)\psi = 0.$$

  These are the adjoint and normal Dirac equations.

- The conjugate momentum is

$$\pi(x) = d^0\psi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}.$$

  Expanding out the Lagrangian:

$$\mathcal{L} = i\overline{\psi}\gamma^0\partial_0\psi + i\overline{\psi}\gamma^i\partial_i\psi - m\overline{\psi}\psi,$$

  so

$$\pi(x) = i\overline{\psi}\gamma^0 = i\psi^\dagger\gamma^0\gamma^0 = i\psi^\dagger.$$

- From here we can find the Hamiltonian density:

$$\begin{aligned}\mathcal{H} &= \pi\dot{\psi} - \mathcal{L},\\ &= i\psi^\dagger\dot{\psi} - i\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}\psi,\\ &= i\psi^\dagger\dot{\psi} - \overline{\psi}(i\partial\!\!\!/ - m)\psi.\end{aligned}$$

  If our spinor satisfies the Dirac equation, which we obviously want, then since the second term *is* the Dirac equation, it's zero, so

$$\mathcal{H} = i\psi^\dagger\dot{\psi}. \tag{5.16}$$

- We also expand out the spinors in terms of creation and annihilation operators:

$$\psi(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}}\sum_{d=1,2}\left[a_d(\mathbf{p})u^{(d)}(\mathbf{p})e^{-ip\cdot x} + b_d^\dagger(\mathbf{p})v^{(d)}(\mathbf{p})e^{ip\cdot x}\right], \quad \text{and} \tag{5.17}$$

$$\overline{\psi}(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}}\sum_{d=1,2}\left[a_d^\dagger(\mathbf{p})\bar{u}^{(d)}(\mathbf{p})e^{ip\cdot x} + b_d(\mathbf{p})\bar{v}^{(d)}(\mathbf{p})e^{-ip\cdot x}\right]. \tag{5.18}$$

- The a's and b's follow identical commutation relations and interpretations as they did for the complex scalar field.

- The Hamiltonian now, writing Eq. (5.16) in terms of the creation and annihilation operators, is

$$H = \int \mathrm{d}^3x\, i \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} \sum_{d=1,2} \left[a_d^\dagger(\mathbf{p})\bar{u}^{(d)}(\mathbf{p})e^{ip\cdot x} + b_d(\mathbf{p})\bar{v}^{(d)}(\mathbf{p})e^{-ip\cdot x}\right]$$
$$\times \gamma^0 \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2q^0}} \sum_{d'=1,2} \left[(-iq^0)a_{d'}(\mathbf{q})u^{(d')}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}} + (iq^0)b_{d'}^\dagger(\mathbf{q})v^{(d')}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\right],$$

$$H = \int \frac{\mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^3 2\sqrt{p^0q^0}} q^0 \sum_{d,d'=1,2} \left[a_d^\dagger(\mathbf{p})u^{\dagger(d)}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}a_{d'}(\mathbf{q})u^{(d')}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}} + \ldots\right],$$

  where in the evaluation of the time derivatives, we should end up with exponentials containing q^0 and p^0. These, however, cancel later on, so we will avoid writing them for clariry. Additionally, there will be three more terms (hence the $\ldots$); writing them out will be a pain, so for more notational simplification, we will just consider what happens with the first term. Lastly, the simplification of $\bar{u}\gamma^0 u = u^\dagger\gamma^0\gamma^0 u = u^\dagger u$ was made so that we no longer have gammas. We are left with

$$H = \int \frac{\mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^3 2\sqrt{p^0q^0}} q^0 \sum_{d,d'=1,2} \left[a_d^\dagger(\mathbf{p})u^{\dagger(d)}(\mathbf{p})a_{d'}(\mathbf{q})u^{(d')}(\mathbf{q})e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right].$$

  If we now do the x integral, we can use the identity that

$$\int \frac{\mathrm{d}^3x}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}} = \delta^3(\mathbf{p}),$$

  with $\mathbf{p} \to \mathbf{p} - \mathbf{q}$ to say

$$H = \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^3 2\sqrt{p^0q^0}} iq^0 \sum_{d,d'=1,2} \left[a_d^\dagger(\mathbf{p})u^{\dagger(d)}(\mathbf{p})a_{d'}(\mathbf{q})u^{(d')}(\mathbf{q})\right] \delta^3(\mathbf{p}-\mathbf{q}).$$

  We can now kill the q integral, say, where the delta function sends $\mathbf{q} \to \mathbf{p}$:

$$H = \int \frac{\mathrm{d}^3p}{2(2\pi)^3} \sum_{d,d'=1,2} \left[a_d^\dagger(\mathbf{p})u^{\dagger(d)}(\mathbf{p})a_{d'}(\mathbf{p})u^{(d')}(\mathbf{p}) + \ldots\right].$$

  Now, we can show that:

$$u^{\dagger(d)}(\mathbf{p})u^{(d')}(\mathbf{p}) = 2p^0\delta^{dd'},$$

  so,

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, p^0 \sum_{d=1,2} \left[a_d^\dagger(\mathbf{p})a_d(\mathbf{p}) - b_d\mathbf{p}b_d^\dagger\mathbf{p}\right],$$

  where the middle two terms have canceled.

- This is disastrous! This term is not positive definite! Even if we apply normal ordering by using the commutator of the operators, we still have the possibility of getting negative energy solutions, which is not good.

- To fix this, we impose *anti*commutation relations on the operators rather than commutation relations:

$$\left\{a_d(\mathbf{p}), a_{d'}^\dagger(\mathbf{p'})\right\} = \left\{b_d(\mathbf{p}), b_{d'}^\dagger(\mathbf{p'})\right\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p'})\delta^{dd'}, \tag{5.19}$$

  and all other combos are zero. Now, we can reverse the order of the b's in the second term to pick up a plus sign. We end up with a cross term, but this is the same as the 0-point energy from before: it diverges, so we just ignore it! We are at last left with:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, p^0 \sum_{d=1,2} \left[a_d^\dagger(\mathbf{p})a_d(\mathbf{p}) + b_d^\dagger(\mathbf{p})b_d(\mathbf{p})\right]. \tag{5.20}$$

- Imposing anti-commutation relations has another consequence: $a^\dagger$ is the particle creation operator; imposing it twice on the vacuum state gives $a^\dagger a^\dagger \langle 0| = 0$, meaning we cannot have two particles in the same state! This is the **Pauli Exclusion Principle**.

- We also have charge, given by:

$$Q = \int : \psi^\dagger(x)\psi(x) : \, \mathrm{d}^3x = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{d=1,2} \left[a_d^\dagger(\mathbf{p}) a_d(\mathbf{p}) - b_d^\dagger(\mathbf{p}) b_d(\mathbf{p}) \right]. \tag{5.21}$$

  We have the possibility for negative charge here, but that's okay this time, since negatively charged particles exist in nature.

- Lastly, we can promote the spinors fields to operators and get (equal time) *anti*commutation relations:

$$\left\{ \psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t) \right\} = \delta^3(\mathbf{x} - \mathbf{y}) \delta_{ab}, \tag{5.22}$$

  and the others are zero.

- As a final note, this is all called **canonical quantization**.

6 Local Gauge Invariance

- Let's look at the Dirac Lagrangian:

$$\mathcal{L} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi.$$

- Let's consider a transformation of the field like $\psi \to \psi' = e^{i\theta}\psi$, where θ is some angle that is the same everywhere in space-time; this is called a **global transformation**. Similarly, the adjoint field will transform like $\overline{\psi} \to \overline{\psi}' = e^{-i\theta}\overline{\psi}$.

- We can easily find that the Lagrangian does not change under this transformation; the exponentials cancel in the mass term trivially, and we can pull the exponential past the derivative since it's just a constant.

- So, in the QFT lingo, a global phase rotation of the field is a symmetry of the Lagrangian.

  - This is analogous to the SE where we have the fact that we can make a phase rotation of the wavefunction and have it not impact the physics, since we always have $|\psi|^2$.

- Now what happens if we let our rotation angle vary across space-time, i.e. $\theta = \theta(x^\mu)$ (this is called a **local transformation**)? Now, our fields transform like $\psi' = e^{iq\theta(x^\mu)}\psi$ and $\overline{\psi}' = \overline{\psi}e^{-iq\theta(x^\mu)}$, where we have added a constant q for generality (we could've added it before in the global transformation, but since θ was also a constant there, it would've just been absorbed). Again, the mass terms remain invariant trivially, but we cannot pull the exponential through the derivative this time since it isn't a constant. Our Lagrangian at this stage (with the mass term cancellation) looks like

$$\mathcal{L}' = i\overline{\psi}e^{-iq\theta(x^\mu)}\gamma^\mu \partial_\mu e^{iq\theta(x^\mu)}\psi - m\overline{\psi}\psi.$$

  Doing the product rule,

$$\begin{aligned}
\mathcal{L}' &= i\overline{\psi}e^{-iq\theta(x^\mu)}\gamma^\mu \left(\psi \partial_\mu e^{i\theta(x^\mu)} + e^{iq\theta(x^\mu)}\partial_\mu \psi \right) - m\overline{\psi}\psi, \\
&= i\overline{\psi}e^{-iq\theta(x^\mu)}\gamma^\mu e^{iq\theta(x^\mu)}\partial_\mu \psi + i\overline{\psi}e^{-iq\theta(x^\mu)}\gamma^\mu \psi (iq) e^{iq\theta(x^\mu)}\partial_\mu \theta(x^\mu) - m\overline{\psi}\psi.
\end{aligned}$$

  In the first term, the exponentials do cancel, so along with the mass term, that is our original Lagrangian. The second term is an additional term we have picked up (where the exponentials now do cancel), so we have

$$\mathcal{L}' = \mathcal{L} - q\overline{\psi}\gamma^\mu \psi \partial_\mu \theta.$$

- To fix this, let's just add this term to the Lagrangian! We will quantify the change in the phase angle with a new 4-vector A_μ (along with a constant), so our new Lagrangian is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu\psi A_\mu.$$

- This new quantity must transform like $A_\mu \to A'_\mu = A_\mu - \partial_\mu\theta$ in order to keep our Lagrangian invariant. Now we have an invariant Lagrangian under *local* transformations!

- We will soon identify A_μ as the electromagnetic four-potential.

- Now we need a kinetic and mass term for this new field A_μ; they will be provided here without derivation:

$$\mathcal{L} \subset -\frac{1}{2}m_A^2 A^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

  where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the EM stress-energy tensor. I said "without derivation" before, but we notice at least that the mass term is identical to that for the scalar fields, where $A^\mu A_\mu \sim A^2 \sim \phi^2$. The stress-energy tensors are just what we know to be true.

- Now for these terms to remain invariant, it is required that m_A be zero! Since we have already identified this as relating to EM, this means that our new field A_μ describes photons, which are massless! On the other hand, this sort of explains why they must be massless.

- What we have now is the QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu A_\mu\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{6.1}$$

  or, defining the **covariant derivative** $D_\mu \equiv \partial_\mu + iqA_\mu$, we can write this more simply as

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{6.2}$$

Derivation of Maxwell's Equations

- Let's do the EL equation for $\overline{\psi}$:

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\right) = i\gamma^\mu D_\mu\psi - m\psi = 0, \to (i\not{D} - m)\psi = 0.$$

  Or, expanding out the covariant derivative, we get

$$(i\not{\partial} - q\not{A} - m)\psi = 0.$$

  It is basically the same as the original Dirac equation, but now we have an extra term involving our gauge field.

- Doing the same process with ψ in the EL equation, we get the adjoint Dirac equation:

$$\overline{\psi}(i\not{D}\overleftarrow{} + m) = 0$$

- Now looking at the gauge field A_μ:

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + q\overline{\psi}\gamma^\nu\psi = 0.$$

  But we can recognize the second term as the conserved current, so what we have is:

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{6.3}$$

- We can now show the continuity equation using this. First, let's take the 4-gradient of both sides:

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu.$$

Now, we know that $F^{\mu\nu} = -F^{\nu\mu}$, so this is:

$$-\partial_\nu \partial_\mu F^{\nu\mu} = \partial_\nu j^\nu.$$

But $\partial_\nu \partial_\mu$ is fully symmetric, so we can switch those indices freely:

$$-\partial_\mu \partial_\nu F^{\nu\mu} = \partial_\nu j^\nu.$$

However, since the left-hand side is just a scalar, we can freely flip the indices since they are just dummy indices:

$$-\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu.$$

But this is saying that $\partial_\nu \partial_\mu F^{\mu\nu}$ is equal to negative of itself, since the right-hand side is the same. The only way for this to be the case is if it is zero, meaning

$$\partial_\mu j^\mu = 0, \tag{6.4}$$

which is the continuity equation.

- If we say that $A^\mu \equiv [V,\ \mathbf{A}]^\intercal$, where V is the voltage and $\mathbf{A}$ is the magnetic vector potential, then the field strength tensor is:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{6.5}$$

- Now let's consider, from the equation of motion we got for the gauge field:

$$\partial_\mu F^{\mu 0} = j^0.$$

From our definition of the 4-current, $j^0 = \rho$, the charge density. Doing the work, we get:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho. \tag{6.6}$$

This is Gauss's Law for the electric field! Similarly, $j^i = \mathbf{j}$, the current density, so

$$\partial_\mu F^{\mu i} = j^i$$

$$\rightarrow\ \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}, \tag{6.7}$$

which is the Ampere-Maxwell law!

- Next, if we define

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{6.8}$$

then we can express the rest of Maxwell's equations (the homogeneous ones; we just derived the inhomogeneous ones) as

$$\partial_\mu \tilde{F}^{\mu\nu} = j^\nu, \tag{6.9}$$

which give us

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \text{ and} \tag{6.10}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \tag{6.11}$$

which are Gauss's Law of Magnetism and Faraday's Law of Induction, respectively.

- Lastly, we saw briefly before that we can make a *change of gauge*, by which we take $A_\mu \to A_\mu + \partial_\mu \lambda$, and the physics remains the same.
- So, we need to make a choice of our gauge in order to reduce these spurious degrees of freedom. Ideally, we know photons have two transverse polarizations, so we need to reduce 4 degrees of freedom to 2.
- One such gauge choice is the **Lorenz gauge** (not Lorentz!), in which we impose that $\partial_\mu A^\mu = 0$. A "subset" of this gauge choice is the **Coulomb gauge**, in which we further impose that $A^0 = V = 0$. With these two restrictions, we now have 2 degrees of freedom.
  - We have broken Lorentz symmetry here, because we now have to repeatedly Lorentz transform this everytime we change reference frames in order to keep $A^0 = 0$. But this turns out to be fine normally.
- Now, for a free photon, $j^\mu = 0$, so

$$\begin{aligned}\partial \mu F^{\mu\nu} &= \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu, \\ &= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu, \\ &= \partial_\mu \partial^\mu A^\nu = 0.\end{aligned}$$

  Where the second term is zero because, in the Lorenz gauge, $\partial_\mu A^\mu = 0$.
- Analogouly to the Dirac equation, we know have plane-wave solutions that look like:

$$A^\mu(x) = A e^{ip\cdot x} \epsilon(p),$$

  where $\epsilon(p)$ is analogous to the u and v spinors from the Dirac equation.

7 Quantization of Gauge Fields

Quantization of QED Lagrangian

- In the quantization of the free Dirac field, we had the u and v spinors, each with 2 components, or degrees of freedom. Here, with only 1 vector, we'd ordinarily have 4 degrees of freedom, but under the Lorenz gauge we have restricted it to only two, meaning that when we expand the field into the creation and annihilation operators, we only sum over two polarizations:

$$A_\mu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2p^0}} \sum_{\lambda=1,2} \left[\epsilon_\mu^{(\lambda)}(\mathbf{p}) a_\lambda(\mathbf{p}) e^{-ip\cdot x} + \epsilon_\mu^{*(\lambda)}(\mathbf{p}) a_\lambda^\dagger(\mathbf{p}) e^{ip\cdot x} \right]. \tag{7.1}$$

- Photons are bosons, meaning they satisfy commutation relations; the only non-zero ones are:

$$\left[a_\lambda(\mathbf{p}), a_{\lambda'}^\dagger(\mathbf{p}') \right] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta^{\lambda\lambda'}. \tag{7.2}$$

- We can also write the QED Lagrangian for a free photon:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

  so our conjugate momenum is

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = -F^{0i} = E_i. \tag{7.3}$$

  Using these definitions, we can find the equal-time commutation relations for the field and the conjugate momentum:

$$\left[\hat{A}^\mu(\mathbf{x}, t), \hat{\pi}^\mu(\mathbf{y}, t) \right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \left(\delta^{ij} - \frac{p^i p^j}{|\mathbf{p}|^2} \right). \tag{7.4}$$

- We also find that the Hamiltonian is

$$H = \frac{1}{2}\int (|\mathbf{E}|^2 + |\mathbf{B}|^2)\, \mathrm{d}^3x, \tag{7.5}$$

and in terms of operators:

$$\mathcal{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \frac{p^0}{2}\left[a_\lambda(\mathbf{p})a_\lambda^\dagger(\mathbf{p}) + a_\lambda^\dagger(\mathbf{p})a_\lambda(\mathbf{p})\right],$$

or with normal ordering:

$$\mathcal{H} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, p^0 a_\lambda^\dagger(\mathbf{p})a_\lambda(\mathbf{p}). \tag{7.6}$$

This is positive definite, which is a good sign.

Going to the Lorenz Gauge

- All of the above was done in the Coulomb gauge, but we want to be more general, so let's take a step back to the Lorenz gauge, meaning that we no longer have that $A^0 = \pi^0 = 0$, and our commutation relations for the fields becomes:

$$\left[\hat{A}^\mu(\mathbf{x}, t), \hat{\pi}^\mu(\mathbf{y}, t)\right] = ig^{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}), \tag{7.7}$$

and all others are zero, like before.

- However, even in this more general gauge, we end up finding that $\pi^0 = 0$ still is zero, meaning our new commutator won't work; i.e. it just reduces to what we had before. What we can do is add a *gauge fixing* term to the Lagrangian that keeps the physics the same but helps us with this gauge issue:

$$\mathcal{L} \to -\frac{1}{4}F\mu\nu F_{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2.$$

Doing the EL equations for this new Lagrangian, we get:

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) \to \partial_\mu\left(-\partial^\mu A^\nu + \partial^\nu A^\mu - g^{\mu\nu}\partial_\rho A^\rho\right) = 0.$$

But in the Lorenz gauge, $\partial_\mu A^\mu = 0$, so all that remains is $\partial_\mu\partial^\mu A^\nu = 0$, which is just the KG equation for a vector field, as we found before.

- With this, $\pi^0 = -\partial_\mu A^\mu$, but this is still zero....

- Now what we do is reinterpret the Lorenz gauge to no longer say that the quantity $\partial_\mu A^\mu = 0$ itself, but rather that its *expection value* is zero, meaning the quantity itself doesn't have to be zero, only $\langle\psi\,|\,\partial_\mu A^\mu\,|\,\psi\rangle = 0$. With this, then, we have solved our issue.

- When we expand the vector field now into the creation and annihilation operators, we now have to sum over all 4 possible polarizations, rather than two as before when we were restricted. However, we there still will be only two physical ones, meaning we have to go in afterwards and remove the unphysical ones. This is the consequence of choosing a more general gauge: carrying around spurious degrees of freedom. Doing this expansion:

$$A_\mu(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} \sum_{\lambda=0}^{3}\left[\epsilon_\mu^{(\lambda)}(\mathbf{p})a_\lambda(\mathbf{p})e^{-ip\cdot x} + \epsilon_\mu^{*(\lambda)}(\mathbf{p})a_\lambda^\dagger(\mathbf{p})e^{ip\cdot x}\right]. \tag{7.8}$$

- Now let's imagine choosing the simplest case for a vector field with momentum $p^\mu = \begin{pmatrix} p & 0 & 0 & p \end{pmatrix}^\intercal$ and the following basis vectors:

$$\epsilon^{\mu(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{\mu(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{\mu(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^{\mu(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

  With this, since we need $\boldsymbol{\epsilon} \cdot \mathbf{p} = 0$, we know that only $\epsilon^{\mu(2)}$ and $\epsilon^{\mu(3)}$ are the physical solutions.

- It turns out, as well, that these are the only terms that end up contributing to the Hamiltonian, which makes sense.

- With all of this, then, we have that generally:

$$\pi^\mu = F^{\mu 0} - g^{\mu 0}\partial_\nu A^\nu, \quad \pi^0 = -\partial_\mu A^\mu = -\dot{A}_0 + \boldsymbol{\nabla} \cdot \mathbf{A}, \quad \pi^i = \partial^i A^0 - \dot{A}^i,$$

  and that the creation and annihilation operators obey the familiar commutation relations:

$$\left[a_\lambda(\mathbf{p}), a^\dagger_\lambda(\mathbf{p}')\right] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'). \tag{7.9}$$

8 PLACEHOLDER

A Brief Look at Some Group Theory

- Going back to our local transformations, we saw that we could induce a phase transformation onto our field ψ such that $\psi \to \psi' = e^{i\theta}\psi$, and that our Lagrangian (and hence the physics) remained invariant under such a transformation.

- We now recognize that $e^{i\theta}$ is an element of the group $U(1)$, which consists of unitary 1×1 matrices. Now, a 1×1 matrix is just a scalar, so the unitary condition becomes: $U^\dagger U \to U^* U = 1$. It is trivial to check that this is satisfied in our case.

- We also associated such transformations with electromagnetism; the local transformation gave rise to a vector field which, when incorporated into the Lagrangian, introduced terms that led naturally to Maxwell's equations, for instance. Hence, we can say that $U(1)$ invariance leads to electromagnetism.

- Other forces, as we will see eventually, like the weak force and strong force, are associated with different symmetries, namely $SU(2)$ and $SU(3)$ respectively. Since those have more than 1 dimension, they are truly matrices, and therefore we impose that they be "special" (hence the 'S' in the group name), which just means that they have a determinant of one.

- What is also means is that there are going to be more free parameters. In just the complex space, there is only one rotation angle, but intuitively, as you increase the dimensions, you end up with more rotation angles. I mean, in 3-dim, we have 3 rotation angles about each axis. In group theory, we don't really talk about the angles in the space itself, but rather than number of free parameters that each group "control". The *group* of rotations in 3-dim end up with 3 free parameters, which we can associated with the 3 rotation angles.

- It is a little bit more abstract in the cases of these groups, but we find that in general, for $SU(n)$, there are $n^2 - 1$ free parameters, or "rotation angles". $SU(2)$ thus has 3 free parameters and $SU(3)$ has 8.

- Additionally, an element of these higher dimensional groups is no longer just a simple scalar like $e^{i\theta}$. However, we can express an element of the group like e^{iH}, where now H is a matrix (this exponential of a matrix is defined as its power series expansion). It turns out that is must be Hermitian; let's see this:

$$U^\dagger U = e^{-iH^\dagger} e^{iH} = e^{i(H - H^{dagger})} = 1.$$

  The only way for this to be satisfied is if $H = H^\dagger$, which means that it is Hermitian.

- More specifically, we will be looking at **Lie groups**, which are special groups whose free parameters are continuous. A rotation angle is clearly continuous, so these groups are Lie groups. When looking at a Lie group, we can write an element of the group as:

$$e^{iT^a\theta^a},$$

where θ^a is an n^2-1 dimensional vector capturing each free parameter (like the angle of rotation about each axis in 3-dim), and T^a is a similarly sizes vector consisting of the **generators** of the group. These can be calculated in many ways, but their interpretation (and why they are called generators) is that they induce infinitesimal transformations in a particular direction. As an analogy, there would be a generator for each possible rotation in 3-dim, and the x-axis rotation generator would generate an infinitesimal rotation about that axis. In the power series representation, we can sort of see how, when combined with the corresponding rotation angle/free parameter(since, expanding for the 3-dim case, we would have something like $e^{i(T_x\theta_x)}$ if we were only considering x) we would end up getting a full rotation.

- It turns out that for $SU(2)$, these generators are something we have seen before: the Pauli matrices! So, we can generate an $SU(2)$ transformation like so:

$$e^{-\boldsymbol{\sigma}\cdot\boldsymbol{\theta}/2},$$

where we have used normal vector notation, since we know that $SU(2)$ has three free parameters. The factor of $1/2$ there for a nuanced reason: in the "fundamental representation", the generators are the Pauli matrices divived by two. This gets into representation theory, which is a nightmare.

- Similarly, and element of $SU(3)$ can be found like:

$$e^{i\lambda^a\theta^a/2},$$

where the λ's are the Gell-Mann matrices, the analog of the Pauli spin matrices for $SU(3)$. Again, the $1/2$ is in there because in the fundamental representation the generators are the Gell-Mann matrices divided by two.

- Another property of Lie groups is that these generators satisfy certain commutation relations:

$$[T^a, T^a] = if^{abc}T^c,$$

where f^{abc} are called the **structure constants** of the group, and, apart from the obvious fact of there being simply more/less generators in other groups, these sort of differentiate groups from each other and define how they operate.

- Another interesting fact as our first application of this to physics stuff: when we enforce that a local gauge transformation of an element of $SU(n)$ in general, we still get a vector/gauge field as always, but we will also get some more complex terms when it comes to how they transform and interact. For instance, the "field-strength" tensor that arose from local $U(1)$ symmetry can be generalized as:

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu,$$

where now we can see that we have interactions among the gauge fields themselves (g is just a coupling constant, characterizing the strength of this interaction). This term was not present in our $U(1)$ case, and we know that the photon does not couple to itself; it mediates the electromagnetic force, so as a chargless particle, it obviously won't interact with itself!

9 Feynman Rules

Interactions and Perturbation Theory

- It turns out, the derivation of Feynman rules (which are used to evaluate amplitudes in QFT) follows pretty smoothly through a number of different steps, and it starts with a bit of formalism into interactions and some perturbation theory.

- In general, we can split up the Hamiltonian into the interaction term(s), and "the rest": $H = H_0 + H_{\text{int}}$. We can do this similarly for the Hamiltonian density as well as for Lagrangians: $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ and $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$.
- As an example, we considered ϕ^4 theory previous during the test, and we had an interaction term $\mathcal{L}_{\text{int}} = \frac{1}{4}\phi^4$, and the "rest", $\mathcal{L}_0$, was just the ordinary free Klein-Gordon Lagrangian.
- We know that we can write the Hamiltonian density as $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$, but since $\pi = \dot{\phi}$ for this case, we have that $\mathcal{H} = \dot{\phi}^2 - \mathcal{L}_0 - \mathcal{L}_{\text{int}}$. So, $\mathcal{H}_0 = \dot{\phi}^2 - \mathcal{L}_0$ and $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$. Interestingly, the Hamiltonian interaction term is identical to the Lagrangian interaction term with a minus sign.
- In QED, we have that

$$\mathcal{L}_{\text{int}} = -q\overline{\psi}\gamma^\mu\psi A_\mu \quad \text{and} \quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\overline{\psi}\gamma^0,$$

  so

$$\mathcal{H} = \pi\dot{\psi} = \psi^\dagger\dot{\psi} - \mathcal{L}_0 - \mathcal{L}_{\text{int}}.$$

  Thus,

$$\mathcal{H}_0 = \psi^\dagger\dot{\psi} - \mathcal{L}_0, \quad \text{and} \quad \mathcal{H}_{\text{int}} = q\overline{\psi}\gamma^\mu\psi A_\mu.$$

- Now turning to perturbation theory: if the size of the coupling constant (λ in our ϕ^4 theory here, for instance, or q for QED) is very small, we can consider the interaction term as a small perturbation to the full Hamiltonian. From there, we can apply perturbation techniques.
- First, though, let's lay down some formalism:
- **The Schrodinger picture/representation** (SP) is the representation that has states that evolve with time (the wavefunction ψ) and operators that are constant ($\hat{x}$ is just the variable x always).
- **The Heisenberg picture/representation** (HP) is the representation that has states that are constant in time but whose operators do evolve with time.
- **The interaction picture/representation** (IP) is an intermediate position between the two previous representations. We will look at this a little more later, but it is what naturally leads to perturbation theory stuff.
- Now, we have been already basically working within the HP, where our operators are the fields, and they indeed depend on position as well as time: $\hat{\phi}(\mathbf{x}, t)$. If we had been using the SP, we'd only have something like $\hat{\phi}(\mathbf{x})$.
- In this case, the "states" are the kets $|\phi\rangle$, which we will index as $|\phi_S\rangle$ for the SP and $|\phi_H\rangle$ for the HP.
- Intuitively there is only a time-evolution factor different between the two states, and this turns out to be the correct line of thought: the relation between them is given by:

$$|\phi_H\rangle = e^{iHt}\,|\phi_S\rangle\,. \tag{9.1}$$

- The relation between operators is given by, for a generic operator $\hat{A}$:

$$\hat{A}_H = e^{iHt}\hat{A}_S e^{-iHt}. \tag{9.2}$$

- In the SP, we had the Schrodinger Equation which dictated the time evolution of that state. In the HP, there is an analog for the operators who are now the time-dependent quantities:

$$i\frac{\mathrm{d}\hat{A}_H}{\mathrm{d}t} = \left[\hat{A}_H, \hat{H}\right]. \tag{9.3}$$

- Now, in the interaction picture, we consider the place between the two pictures. From Eq. (9.1), we go "halfway" by only considering the "rest" of the Hamiltonian, not the interaction term (and index a state in this picture with $|\phi_I\rangle$):
$$|\phi_I\rangle = e^{iH_0 t} |\phi_S\rangle . \tag{9.4}$$
We have similar relations for the operators:
$$\hat{A}_I = e^{iH_0 i} \hat{A}_S e^{-iH_0 t}, \quad \text{and} \quad i\frac{\mathrm{d}\hat{A}_I}{\mathrm{d}t} = \left[\hat{A}_I, \hat{H}_0\right]. \tag{9.5}$$

- In this picture, we have that
$$\hat{H}_{\text{int}} |\phi_I\rangle = i\frac{\mathrm{d}}{\mathrm{d}t} |\phi_I\rangle . \tag{9.6}$$

- Now, in the SP (where we have time-dependent states), we can evolve a state $|\phi_S(t_i)\rangle$ for some initial time $t = t_i$ to a state $|\phi_S(t_f)\rangle$ at some later time $t = t_f$ using a time evolution operator like so:
$$|\phi_s(t_f)\rangle = e^{-iH(t_f - t_i)} |\phi_S(t_i)\rangle .$$

- We define the **amplitude** for such a generic process as
$$\left\langle \phi'_S(t_f) \right| e^{-iH(t_f - t_i)} \left| \phi_S(t_i) \right\rangle . \tag{9.7}$$

- Essentially, the amplitude is given by sandwiching the time-evolution operator between the the final and initial states.

- If we take the limit as $t_f \to \infty$, then the exponential term becomes what is called the **S-Matrix**.

- It is *unitary*, obviously, since the Hamiltonian is hermitian.[1]

- We can also define something called the **T-Matrix**, which is defined as
$$S = 1 + iT. \tag{9.8}$$

- It is not necessarily unitary; in fact,
$$S^\dagger S = (1 - iT^\dagger)(1 + iT) = 1 + it - iT^\dagger + T^\dagger T = 1 \to T^\dagger T = i(T^\dagger - T) \neq 1.$$
We will (presumably) explore this a little later, it just doesn't hurt to lay down the notation now.

- Bringing in what we have done, we have, again, that our field operators are in the HP, and can therefore be represented in terms of SP operators (that are time-independent) like so:
$$\hat{\phi}_H(\mathbf{x}, t) = e^{iHt} \hat{\phi}_S(\mathbf{x}) e^{-iHt}.$$
The expansion for $\hat{\phi}_S(\mathbf{x})$ is nearly identical to that for the HP operator:
$$\hat{\phi}_S(\mathbf{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2p^0}} \left[a(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right].$$

- Similarly to before, if we have the above case for some initial $t = t_0$ (we can then "label" $\hat{\phi}_S$ like $\hat{\phi}_S(\mathbf{x}, t_0)$, but keep in mind that this doesn't otherwise imply time-dependence, it's just a general label), then for some later t:
$$\hat{\phi}_H(\mathbf{x}, t) = e^{iH(t - t_0)} \hat{\phi}_S(\mathbf{x}, t_0) e^{-iH(t - t_0)}. \tag{9.9}$$

[1] It turns out this unitarity is important for conservation laws.

- Now let's consider what happens to the Hamiltonian when our coupling constant in the interaction term is small, e.g. λ is tiny. At an extreme, we can consider the limit as $\lambda \to 0$, in which case the interaction term vanishes so $H \to H_0$. In such a case:

$$\lim_{\lambda\to\infty} \hat{\phi}_H(\mathbf{x}, t) = e^{iH_0(t-t_0)}\hat{\phi}_S(\mathbf{x}, t_0)e^{-iH_0(t-t_0)}.$$

- But this is just a state the interaction picture as we described before, where we just consider the $\mathcal{H}_0$ term:[2]

$$\hat{\phi}_I(\mathbf{x}, t) = \lim_{\lambda\to\infty} \hat{\phi}_H(\mathbf{x}, t) = e^{iH_0(t-t_0)}\hat{\phi}_S(\mathbf{x}, t_0)e^{-iH_0(t-t_0)}. \tag{9.10}$$

  This field operator can be expanded out in the normal way:

$$\hat{\phi}_I(\mathbf{x}, t) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} \left[a(\mathbf{p})e^{ip\cdot x} + a^\dagger(\mathbf{p})e^{-ip\cdot x}\right], \tag{9.11}$$

  where $x^0 = t - t_0$.

- Let's try to make this more general by expressing $\hat{\phi}_H$ in terms of $\hat{\phi}_I$. First, let's reverse our defintion of $\hat{\phi}_I$:

$$\hat{\phi}_S(\mathbf{x}, t) = e^{-iH(t-t_0)}\hat{\phi}_I(\mathbf{x}, t_0)e^{iH(t-t_0)}$$

  , then plug this into our defition of $\hat{\phi}_H$ from before:

$$\hat{\phi}_H(\mathbf{x}, t) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\hat{\phi}_I(\mathbf{x}, t)e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{9.12}$$

- If we define the IP time-evolution operator, or otherwise called the **IP propagator**, as

$$U(t, t_0) \equiv e^{iH_0(t-t_0)}e^{-iH(t-t_0)}, \tag{9.13}$$

  then we can write this as

$$\hat{\phi}_H(\mathbf{x}, t) = U^\dagger\hat{\phi}_I(\mathbf{x}, t)U. \tag{9.14}$$

- If we consider now the time evolution of this operator, we have:

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = e^{iH_0(t-t_0)}(H - H_0)e^{-iH(t-t_0)}.$$

  But $H - H_0$ is just H_{int}:

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH(t-t_0)}.$$

  If we cleverly insert 1 after the interaction Hamiltonian term, we get

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}e^{iH(t-t_0)}e^{-iH(t-t_0)}.$$

- Now, from earler we define how to relate operators between pictures, what the first three terms are, then, is the interaction Hamiltonian in the interaction picture:

$$H_I(t) \equiv e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)},$$

  and the last two terms are just U, so we have:

$$i\frac{\mathrm{d}U}{\mathrm{d}t} = H_iU. \tag{9.15}$$

  This is remarkably similar to the Schrodinger equation!

[2]The equality is only realized when the coupling constant really is zero; of course, this is never the case, so this can only be an approximation at best.

- Solutions to this equation are tough, since we are working with matrices, so we will just quote the results here:

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} H_I(t')\,\mathrm{d}t'\right]\right\}, \tag{9.16}$$

where $T\{\ldots\}$ is defined as the **time-ordered product**, which essentially means that the product of operators/fields within it are ordered with time decreasing as you go to the right, so later times appear earlier in the product. For example:

$$T\left\{\hat{\phi}(x_1^\mu)\hat{\phi}(x_2^\mu)\right\} = \begin{cases}\hat{\phi}(x_1^\mu)\hat{\phi}(x_2^\mu) & \text{if } t_1 > t_2,\\ \hat{\phi}(x_2^\mu)\hat{\phi}(x_1^\mu) & \text{if } t_2 > t_1.\end{cases}$$

We can write this a bit more formally using the **Heaviside step function**, or colloquially called just the "theta function":

$$T\left\{\hat{\phi}(x_1^\mu)\hat{\phi}(x_2^\mu)\right\} = \Theta(x_1^0 - x_2^0)\hat{\phi}(x_1^\mu)\hat{\phi}(x_2^\mu) + \Theta(x_2^0 - x_1^0)\hat{\phi}(x_2^\mu)\hat{\phi}(x_1^\mu).$$

- If we Taylor expand the solution we would have:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t_1)\,\mathrm{d}t_1 + \frac{(-i)^2}{2!}\int_{t_0}^{t} T\{H_I(t_1)H_I(t_2)\}\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

It can be shown that the second integral can be re-written after time-ordering like so:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t_1)\,\mathrm{d}t_1 - \int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1} H_I(t_1)H_I(t_2)\,\mathrm{d}t_2 + \ldots.$$

The Feynman Propagator

Just to make this clear, since this next section would otherwise be notation heavy: all fields (any operators, really) are still operators, but we remove the hats. All x's and p's and whatnot are 4-vectors, else they are bolded to signify a 3-vector.

- Let's define the **n-point Green's function** as the vacuum expectation value of the time-ordered product of n fields:

$$G(x_1, x_2, \ldots, x_n) = \langle 0 \,|\, T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\} \,|\, 0\rangle \tag{9.17}$$

- Using our definitions from above, we can write this as:

$$\begin{aligned}&= \left\langle 0 \,\middle|\, \left(U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_0)\right)\left(U^\dagger(t_2,t_0)\phi_I(x_2)U(t_2,t_0)\right)\ldots \,\middle|\, 0\right\rangle,\\ &= \left\langle 0 \,\middle|\, U^\dagger(t,t_0)T\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)U(t,t_1)U(t_1,t_2)\ldots U(t_n,-t)\}U(-t,t_0) \,\middle|\, 0\right\rangle.\end{aligned}$$

This simplification is not trivial, but it would take too long to show. Now, the product of the U's can be simplified:

$$= \left\langle 0 \,\middle|\, U^\dagger(t,t_0)T\left\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\exp\left[-i\int_{-t}^{t} H_I(t')\,\mathrm{d}t\right]\right\}U(-t,t_0) \,\middle|\, 0\right\rangle.$$

- We now will make some assumptions/choices. First, let's assume that $t >> t_1 > t_2 > \ldots > t_n > -t$. We will also choose our initial time to be $-t$, and lastly, we will let $t \to \infty$. With this, we have that

$$U(t,t_0) \to U(\infty,-\infty) \quad \text{and} \quad U(-t,t_0) \to U(-t,-t) = 1.$$

- Additionally, let's consider $U(t,t_0)\,|0\rangle$. It's not a creation/annihilation operator, so in a general case, we will retrieve still the vacuum state but with some phase factor: $U(t,t_0)\,|0\rangle = e^{i\theta}\,|0\rangle$. Now,

$$\langle 0\,|\,U(\infty,-\infty)\,|\,0\rangle = \langle 0\,|\,e^{i\theta}\,|\,0\rangle = e^{i\theta}\,\langle 0\,|\,0\rangle = e^{i\theta},$$

  if we choose our states to be normalized.

- Now let's do a little bit of gymnastics:

$$(U(t,t_0)\,|0\rangle)^\dagger = \langle 0|\,U^\dagger(t,t_0) = \langle 0|\,e^{-i\theta} = \frac{\langle 0|}{e^{i\theta}} = \frac{\langle 0|}{\left\langle 0\,\middle|\,T\left\{\exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\right]\right\}\,\middle|\,0\right\rangle}.$$

- From this, then, we can make

$$\boxed{G(x_1,x_2,\ldots,x_n) = \frac{\left\langle 0\,\middle|\,T\left\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\exp\left[\int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\right]\right\}\,\middle|\,0\right\rangle}{\left\langle 0\,\middle|\,T\left\{\exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\right]\right\}\,\middle|\,0\right\rangle}.} \tag{9.18}$$

- This, as we will find out, gives the amplitude for all possible propagations for all n-points.

- Let's consider the 2-point Green's function with no interactions and see what happens. This means that the interaction Hamiltonian terms are 1 (exponential of 0) so

$$G(x,y) = \langle 0\,|\,T\,\{\phi(x)\phi(y)\}\,|\,0\rangle\,,$$

  where we have dropped the I subscript for notational simplicity.

- Now, we can split the field into separate components containing the creation and annihilation operators:

$$\phi(x) = \phi^+(x) + \phi^-(x), \quad \text{where} \tag{9.19}$$

$$\phi^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} a(\mathbf{p})e^{-ip\cdot x} \quad \text{and} \quad \phi^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2p^0}} a^\dagger(\mathbf{p})e^{ip\cdot x}. \tag{9.20}$$

- We first consider the case where $x^0 > y^0$, then

$$\begin{aligned} T\,\{\phi(x)\phi(y)\} &= \phi(x)\phi(y) = (\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y)) \\ &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y). \end{aligned}$$

- Let's normal order this. Fortunately, only the 2nd term changes; it picks up a commutator:

$$= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \left[\phi^+(x),\phi^-(y)\right] + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y),$$

  or, more succinctly:

$$T\,\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + D(x-y),$$

  where $D(x-y)$ is the **Feynman propagator**. We will see why we have "already" found it, and why it is a function of $x-y$ in a little.

- Taking the vacuum expectation value:

$$\langle 0\,|\,T\,\{\phi(x)\phi(y)\}\,|\,0\rangle = \langle 0\,|\,:\phi(x)\phi(y):\,|\,0\rangle + \langle 0\,|\,D(x-y)\,|\,0\rangle\,.$$

- In the definition of normal ordering, we have annihilation operators on the right, and we know that annihilation operators acting on vacuum states are zero, so the first term is zero. In general, *the vacuum expection value of any normal ordered product is zero.* We can then pull the Feynman propagator out and get:

$$G(x,y) = D(x-y)\,\langle 0\,|\,0\rangle = D(x-y).$$

  This is why we already called it the Feynman propagator, because everything else cancels.

- We considered specifically the case when $x^0 > y^0$, but it turns out it is almost identical for the opposite, and we can write, generally,

$$D(x-y) = \Theta(x^0-y^0)\left[\phi^+(x),\phi^-(y)\right] + \Theta(y^0-x^0)\left[\phi^+(y),\phi^-(x)\right].$$

- Let's expand the $+/-$ fields into creation and annihilation operators:

$$D(x-y) = \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6 2\sqrt{p^0q^0}}\big\{\Theta(x^0-y^0)\left[a(\mathbf{p}),a^\dagger(\mathbf{q})\right]e^{-ip\cdot x}e^{iq\cdot y} + \Theta(y^0-x^0)\left[a(\mathbf{p}),a^\dagger(\mathbf{p})\right]e^{-ip\cdot y}e^{iq\cdot x}\big\}.$$

From the commutation relations, we have

$$D(x-y) = \int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6 2\sqrt{p^0q^0}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\left\{\Theta(x^0-y^0)e^{-ip\cdot x}e^{iq\cdot y} + \Theta(y^0-x^0)e^{-ip\cdot y}e^{iq\cdot x}\right\}.$$

We can now kill the q integral, for instance, forcing $\mathbf{p}=\mathbf{q}$ and as a consequence $p^0=q^0$:

$$D(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\left\{\Theta(x^0-y^0)e^{-ip\cdot(x-y)} + \Theta(y^0-x^0)e^{ip\cdot(x-y)}\right\}.$$

It turns out that with some complex integration shenanigans, we can express this integral as:

$$D(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)}.$$

- The $i\epsilon$ at the bottom disappears since we end up taking $\epsilon\to 0$, but for completeness we left it. We now see why we said D was a function of $x-y$. What we also see is that we have essentially found the momentum space form of the Dirac propagator:

$$\boxed{D(p) = \frac{i}{p^2-m^2}.} \tag{9.21}$$

- Now, Green's functions are solutions to linear differential equations in response to a delta function, meaning if we plug Green's function (whatever it happens to be for the problem at hand) to the governing linear differential equation, we get a delta function. In this case, for a scalar field, this differential equation is the Klein-Gordon equation. Doing the derivatives for x, for instance (it doesn't matter which):

$$\begin{aligned}\left[(\partial_\mu)_x(\partial^\mu)_x + m^2\right]G(x,y) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2}(-p^2+m^2)e^{-ip\cdot(x-y)},\\ &= -i\int\frac{\mathrm{d}^4p}{(2\pi)^4}e^{-ip\cdot(x-y)} = -i\delta^4(x-y).\end{aligned}$$

Indeed, we get a delta function!

Wick's Theorem

- We considered the simplest case of a time-ordered product of two fields. Wicks' theorem states it more generally:

$$T\left\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\right\} =: \phi(x_1)\phi(x_2)\dots\phi(x_n) : + \text{ all possible contractions/Feynman propagators.} \tag{9.22}$$

- As an example, let's look at the 4-point time-ordered product. For notational simplicity, $\phi(x_i)\to\phi_i$ and $D(x_i-x_j)\to D_{ij}$:

$$T\left\{\phi_1\phi_2\phi_3\phi_4\right\} =: \phi_1\phi_2\phi_3\phi_4 : + D_{12} : \phi_3\phi_4 : + D_{13} : \phi_2\phi_4 : + D_{14} : \phi_2\phi_3 : + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}.$$

Feynman Rules for ϕ^4 Theory

- The previous stuff was all done for non-interacting scalar fields, but we now bring back in interactions. We saw the form of this already:

$$\left\langle 0 \left| T\left\{\phi(x)\phi(y)\exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\right]\right\}\right| 0\right\rangle.$$

  We can expand the exponential out to first order:

$$\left\langle 0 \left| T\left\{\phi(x)\phi(y)+\phi(x)\phi(y)(-i)\int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\right\}\right| 0\right\rangle.$$

  The first term is of course just the Dirac propagator:

$$= D(x-y)+\left\langle 0 \left| T\left\{\phi(x)\phi(y)(-i)\int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\right\}\right| 0\right\rangle.$$

- Now, in ϕ^4 theory, we have that

$$H_{\text{int}} = -L_{\text{int}} = -\int \mathrm{d}^3z\ \mathcal{L}_{\text{int}} = \int \mathrm{d}^3z\ \left(\frac{\lambda}{4!}\phi^4\right),$$

  where we use z to differentiate it from the x and y we are already using. So, our total interaction term is (writing out the four ϕ's manually)

$$\left\langle 0 \left| T\left\{\phi(x)\phi(y)\frac{(-i\lambda)}{4!}\int \mathrm{d}^4z\phi(z)\phi(z)\phi(z)\phi(z)\right\}\right| 0\right\rangle.$$

  Note that we combined $\int \mathrm{d}t \int \mathrm{d}^3z = \int \mathrm{d}^4z$. Now, we can use Wick's theorem on these six fields, and we will get a ton of terms, but thankfully, since four of the fields are identical, we actually only have two unique terms:

$$3\frac{(-i\lambda)}{4!}D(x-y)\int \mathrm{d}^4z D(z-z)D(z-z) + 12\frac{(-i\lambda)}{4!}D(x-y)\int \mathrm{d}^4z D(y-z)D(z-z).$$

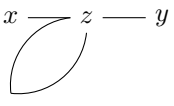
- The 3 and 12 are just combinatorial factors, but we can make a few observations here. The first term represents a particle propagating from point x to y, and (separately) two self-interactions at point z. Really, its *every* point z, since we are integrating over all z. This is sort of the analog for the principle of superposition from QM.

- The second term involves a single interaction of four fields. One thing to note: at every vertex, there are always four lines coming in/out: that's because we are working with a ϕ^4 theory: interactions involve four fields always. We also see mathematically that vertices always involve a factor of $-i\lambda\int \mathrm{d}^4z$.

- We can now write general Feynman Rules for ϕ^4 theory in position-space:

  1. For each propagator, we write $D(x-y)$
  2. For each vertex, we write $-i\lambda\int \mathrm{d}^4z$
  - Account for symmetry factors and divide by them.

  The last bit we won't really look at too heavily, so I didn't give it a number. So, for the following (position-space) Feynman diagram:



we can immediately write down a term equal to the second term above.

- It turns out that only **connected** diagrams contribute to the S-matrix, so this second term (as depicted above) would contribute, but if we were to draw out the first diagram, we would have independent pieces, hence it would be **disconnected**, and not contribute to the final S-matrix.

- Now, position-space is typically not our first pick for solving Feynman diagrams; far more often we are interested in the momentum-space versions. This is because we can impose momentum conservation and all that; we never really care about *where* a particle is, but rather its *momentum/energy*.

- We found before that we can get the momentum-space Feynman propagator by doing some complex integration stuff:

$$D(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}, D(p) \qquad = \frac{i}{p^2 - m^2},$$

  where we took $\epsilon \to 0$. Additionally, we get rid of the full space integration for the vertex factor. To see this, we consider a vertex with momenta p_1 and p_2 incoming, and p_3 and p_4 outgoing. This gives a factor like

$$\mathrm{d}^4 z e^{-ip_1\cdot z} e^{-ip_2\cdot z} e^{ip_3\cdot z} e^{ip_4\cdot z} = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4).$$

  This is just momentum conservation at the vertex.

- So we can now write the Feynman rules in momentum space:

  1. For each propagator we write $\frac{i}{p^2 - m^2}$.
  2. For each vertex we write $-i\lambda$.
  3. Impose conservation of momentum at each vertex.

  - Integrate over each undetermined internal/loop momenta.

  The last step involves loop diagrams and whatnot, which is something that we will not delve into here, so I refrained from giving it a number again.

- We also see that the delta function from the 4-position integral has become its own Feynman rule. Really, we needn't make it its own Feynman rule, momentum conservation should apply at every vertex in every theory ever if it wants to model the real world. But, we need it since we place all of these things in a total mathematical object called the **amplitude**, which we can then square, and this quantity is now proportional to a cross section or decay rate.

Feynan Rules for Spinor Fields

- The Feynman rules for spinor fields are derived in nearly the same way, with a few extra bits due to the more complex nature of spinors. Since it's close enough, we won't run through the full derivations, but rather quote the meaningful steps and the final bits.

- First, we consider the time ordered product of two fields as the Feynman propagator

$$S(x-y) = \left\langle 0 \left| T\left\{\psi(x)\overline{\psi}(y)\right\} \right| 0 \right\rangle.$$

  This is almost identical to the result for scalar fields, but since spinors anti-commute, we have

$$T\left\{\psi(x)\overline{\psi}(y)\right\} = \begin{cases} \psi(x)\overline{\psi}(y) & x^0 > y^0, \\ -\overline{\psi}(y)\psi(x) & y^0 > x^0. \end{cases}$$

  We will still end up with a final result that looks like this:

$$T\left\{\psi(x)\overline{\psi}(y)\right\} =: \psi(x)\overline{\psi}(y) : + S(x-y).$$

- Now, we do all the complex integration shenanigans that we did before, and this nets us

$$S(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}.$$

- Now, $S(x-y)$ is a 4×4 matrix due to the γ^μ.
- Some simple algebra nets us the fact that $(\not{p}+m)(\not{p}-m) = p^2 - m^2$, so after taking $\epsilon \to 0$, we can write that

$$S(p) = \frac{i}{\not{p} - m}.$$

- We can also pretty easily show that this is a Green's function for the momentum-space Dirac operator.

Feynman Rules for Gauge Fields

- We have now that $D \to D_{\mu\nu}$ since our gauge field is a 4-vector. Again, derivations for this are very similar, so we will just quote the result:

$$D_{\mu\nu}(x-y) = \langle 0 \,|\, T\left\{A_\mu(x) A_\nu(y)\right\} \,|\, 0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{(-i) g_{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot(x-y)},$$

  meaning

$$D_{\mu\nu}(p) = \frac{-i g_{\mu\nu}}{p^2}.$$

- However, this is for our Coulomb gauge; for a general Lorenz gauge:

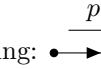
$$D_{\mu\nu}(p) = \frac{-i}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right].$$

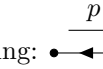
  Here, then, the Coulomb gauge sets $\xi = 1$.

Feynman Rules for QED

- Bringing everything together, we can write down the Feynman rules for QED:

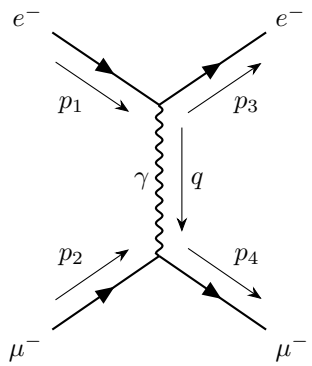
  1. Dirac propagator for photons:  : $\dfrac{-i g_{\mu\nu}}{p^2}$.
  2. Dirac propagator for fermions:  : $\dfrac{i(\not{p}+m)}{p^2 - m^2}$.
  3. Vertex factor: $-ie\gamma^\mu$
  4. External fermion lines:

     Electrons: Incoming:  : $u(\mathbf{p})$, Outgoing:  : $\bar{u}(\mathbf{p})$.

     Positrons: Incoming:  : $\bar{v}(\mathbf{p})$, Outgoing:  : $v(\mathbf{p})$.
  5. Extern photon lines: Incoming:  : $\epsilon_\mu(\mathbf{p})$, Outgoing:  : $\epsilon^*_\mu(\mathbf{p})$.
  6. Enforce momentum conservation at each vertex and integrate over all undetermined momenta (again, this is to ensure the principle of superposition is satisfied).

Electron-Muon Scattering

- As an example, let's apply this to electron muon scattering, where we have $e^-(p_1)+\mu^-(p_2) \to e^-(p_3)+\mu^-(p_4)$. The Feynman diagram for this one looks like:



- To start, we use our Feynman rules to quickly write down that the amplitude for this process is:

$$i\mathcal{M} = \bar{u}^{(s)}(p_3)(-ie\gamma^\mu)u^{(s)}(p_1)\left[\frac{-ig_{\mu\nu}}{q^2}\right]\bar{u}^{(s')}(p_4)(-ie\gamma^\nu)u^{(s')}(p_2),$$

$$\mathcal{M} = \frac{e^2}{(p_1-p_3)^2}[\bar{u}^{(s)}(p_3)\gamma^\mu u^{(s)}(p_1)][\bar{u}^{(s')}(p_4)\gamma_\mu u^{(s')}(p_2)],$$

  where by momentum conservation, the virtual photon momentum q is fixed to be $p_1 - p_3$. I chose to put brackets where I did because each quantity in the brackets is just a scalar.

- Recall, the final quantity we want to calculate is the cross section which is proportional to the amplitude squared

$$|\mathcal{M}|^2 = \frac{e^4}{t^2}[\bar{u}^{(s)}(p_3)\gamma^\mu u^{(s)}(p_1)][\bar{u}^{(s')}(p_4)\gamma_\mu u^{(s')}(p_2)][\bar{u}^{(s)}(p_3)\gamma^\nu u^{(s)}(p_1)]^*[\bar{u}^{(s')}(p_4)\gamma_\nu u^{(s')}(p_2)]^*$$

- It can be shown that the complex conjugate of the quantities in the brackets simply involves a switching of the spinors (and bar-ing the one to the left of the gamma and unbar-ing the one to the right of the gamma), so

$$[\bar{u}(p_i)\gamma^\mu u(p_j)]^* = [\bar{u}(p_j)\gamma^\mu u(p_i)],$$

  so

$$|\mathcal{M}|^2 = \frac{e^4}{t^2}[\bar{u}^{(s)}(p_3)\gamma^\mu u^{(s)}(p_1)][\bar{u}^{(s)}(p_1)\gamma^\nu u^{(s)}(p_3)][\bar{u}^{(s')}(p_4)\gamma_\mu u^{(s')}(p_2)][\bar{u}^{(s')}(p_2)\gamma_\nu u^{(s')}(p_4)].$$

- Now, for a generic process, we will have some unpolarized beam in the initial state and get some wide variety of final states (in terms of spins), so if we want to accurately represent such a process, we need to average over the initial spins since, in principle, each spin has an equal probability of occuring in the beam, then sum over all the final spins to take into account all of the possible final states.

- In effect, this involves summing over *every* spin, then dividing by the number of possible spin configurations in the initial state. Here, since we have two particles with two possible spin states, there are four possible configurations, so we will divide by four.

- To do the spin sums, we will use the *completeness relations*:

$$\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p} + m, \quad \text{and} \quad \sum_s v^{(s)}(p)\bar{v}^{(s)}(p) = \not{p} - m.$$

  Doing this, we get

$$|\mathcal{M}|^2 = \frac{e^4}{t^2}[\bar{u}^{(s)}(p_3)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu u^{(s)}(p_3)][\bar{u}^{(s')}(p_4)\gamma_\mu(\not{p}_2 + m_\mu)\gamma_\nu u^{(s')}(p_4)].$$

- Let's first examine the first term in brackets. If we write out the indices in each component explicitly, we can then move them around however we want:

$$\bar{u}_a^{(s)}(p_3)\left(\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right)_{ab}u_b^{(s)}(p_3)=u_b^{(s)}(p_3)\bar{u}_a^{(s)}(p_3)\left(\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right)_{ab}.$$

  We can now do the spin sums to get:

$$\left(\not{p}_3+m_e\right)_{ba}\left(\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right)_{ab}.$$

  This is matrix multiplication, but with the two final indices as the same thing. Since these are implicitly summed over, what this means is that this quantity is really a trace:

$$[\bar{u}^{(s)}(p_3)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu u^{(s)}(p_3)]=\mathrm{Tr}[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu].$$

- So, our total amplitude squared after the spin sum and initial average is

$$|\mathcal{M}|^2=\frac{e^4}{t^2}\mathrm{Tr}[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu]\mathrm{Tr}[(\not{p}_4+m_\mu)\gamma_\mu(\not{p}_2+m_\mu)\gamma_\nu].$$

- Both of these traces are functionally the same, so we can just examine one at first. We will need a number of trace theorems for gamma matrices, I won't put them here because they are easily findable in literally any QFT book and very easily findable on the internet, it would just waste my time to write them all out here somewhere. So:

$$\mathrm{Tr}[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu]=\mathrm{Tr}[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu+m_e\not{p}_3\gamma^\mu\gamma^\nu+m_e\gamma^\mu\not{p}_1\gamma^\nu+m_e^2\gamma^\mu\gamma^\nu].$$

  We know that the trace of an odd number of gamma matrices is zero, so the middle terms are zero and we have (pulling out the momenta to have just gammas in the traces):

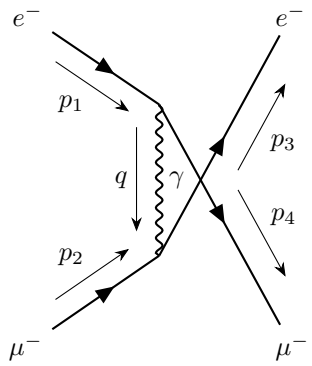
$$\begin{aligned}
&=p_{3,\sigma}p_{1,\rho}\mathrm{Tr}[\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu]+m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\nu],\\
&=p_{3,\sigma}p_{1,\rho}(g^{\sigma\mu}g^{\rho\nu}-g^{\sigma\rho}g^{\mu\nu}+g^{\sigma\nu}g^{\mu\rho})+4m_e^2g^{\mu\nu},\\
&=(p_3^\mu p_1^\nu+p_3^\nu p_1^\mu-p_3\cdot p_1g^{\mu\nu})+4m_e^2g^{\mu\nu},\\
&=p_3^\mu p_1^\nu+p_3^\nu p_1^\mu+g^{\mu\nu}(4m_e^2-p_3\cdot p_1).
\end{aligned}$$

  The other trace is identical except $p_3\to p_4$ and $p_1\to p_2$. Then, contracting the two traces gives a bunch more tedious, annoying algebra, so since I have already worked this out myself a while back, I will just quote the answer:

$$|\mathcal{M}|^2=\frac{8e^4}{t^2}[p_1\cdot p_2p_3\cdot p_4+p_1\cdot p_4p_2\cdot p_3-m_e^2p_2\cdot p_4-m_\mu^2p_1\cdot p_3+2m_e^2m_\mu^2].$$

- This can be further reduced into just the Mandelstam variables, and then we are at last left with fully Lorentz-invariant quantities that are true in any given frame.

- The last thing we want to check before moving on to cross sections is what happens for electron-electron scattering, for instance. The Feynman diagram would look just like that for electron-muon scattering, but there is a catch. Which electron has which momentum in the final state? We could which final state particle had which momentum because they were distinct particles, but electrons are fundamentally indistinguishable, so there's nothing saying that the electron coming from the top vertex can't have momentum p_4. Because of this, we have the **u-channel** diagram, named in this way because the momentum q is determined by momentum conservation to be $q^2=u$.



- Mathematically, all that this modified diagram implies is a switching of $p_3 \leftrightarrow p_4$, so all we really need to do is just make that replacement on the ordinary t-channel diagram, however, it would be almost completely visually indistinguishable, so instead we choose to keep the locations of the final momenta in the same spot then swap the lines of the two final particles.

- This new diagram is an addition to the other one to get the full amplitude, meaning that the total amplitude squared will pick up a cross term:

$$|\mathcal{M}|^2 = |\mathcal{M}_t + \mathcal{M}_u|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + \mathcal{M}_t^* \mathcal{M}_u + \mathcal{M}_t \mathcal{M}_u^*.$$

  It turns out that the since the amplitude will be real, then we can simplify the cross terms to just be

$$|\mathcal{M}|^2 = |\mathcal{M}_t + \mathcal{M}_u|^2 = |\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + 2\mathcal{M}_t^* \mathcal{M}_u.$$

- We could also have put the complex conjugate on the u-channel amplitude, but it doesn't matter, they're equivalent.

10 Scattering Cross Sections and Decay Rates

- Now that we know how to calculate the amplitudes for processes, let's now turn to the aforementioned quantities that are proportional to the amplitudes squared. We will consider scattering cross sections first.

- Let's imagine we have two beams being shot at each other with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, masses m_1 and m_2, and number densities (total number of particles per unit volume) n_1 and n_2. The relative velocity of $\mathbf{v}_2$ from the reference frame of $\mathbf{v}_1$ is given by $\mathbf{v}_{\text{rel}} = \mathbf{v}_1 - \mathbf{v}_2$. If $\mathbf{v}_2$ is zero, this is called a *fixed-target* experiment.

- The quantity we are interested in examining is the number of events per unit volume per unit time. This is, intuitively, proportional to the product of the number densities and the relative velocity. The proportionality constant we will denote with σ, which is the cross section:

$$\frac{\mathrm{d}N}{\mathrm{d}V\,\mathrm{d}t} = \sigma \mathbf{v}_{\text{rel}} n_1 n_2.$$

- The quantity $\mathrm{d}N$ is Lorentz-invariant, since otherwise, it'd violate causality. Because of this, we can choose whatever reference frame we want to evaluate this. We will choose the reference frame of the second beam so that $\mathbf{v}_2 = 0$ and $\mathbf{v}_{\text{rel}} = \mathbf{v}_1$. Additionally, we will have that $p_1 \cdot p_2 = E_1 E_2$, or since $E = \gamma m = m/\sqrt{1 - \mathbf{v}^2}$, we can say:

$$p_1 \cdot p_2 = \frac{m_1 m_2}{\sqrt{1 - \mathbf{v}_{\text{rel}}^2}}$$

- Solving for $\mathbf{v}_{\text{rel}}$, we get (after a little bit of work)

$$|\mathbf{v}_{\text{rel}}| = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}.$$

- We can plug this into our previous number of events equation to get

$$\begin{aligned} \mathrm{d}N &= \sigma \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} n_1 n_2 \mathrm{d}V \mathrm{d}t, \\ &= \sigma \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} \frac{N_1 N_2}{V_0^2} \mathrm{d}V \mathrm{d}t, \\ &= \sigma \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} \frac{N_1 N_2}{V} t, \end{aligned}$$

  where the volume in the denominator in the second line is a constant volume based on the beams, so I indexed it like V_0 to differentiate it from the integration volume V; it won't be affected. As such, we just get a factor of V after the integration, which is the constant V_0, so one cancels, and I drop the subscript again in the third line. Similarly, nothing is dependent on time, so we just pick up a multiplicative t.

- Now the actual derivation for the cross section is a little more tricky, we just did this to sort of visually motivate the form of it. It can be shown that the above expression is proportional to the amplitude squared, so after the necessary work, we arrive at

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i=3}^{n} p_i \right) \prod_{i=3}^{n} \frac{\mathrm{d}^3 \mathbf{p}_i}{(2\pi)^3 2E_i}, \tag{10.1}$$

  for an arbitrary $p_1 + p_2 \to p_3 + p_4 + \ldots + p_n$ process.

- Everything to the right of the first fraction is called the **phase space**, because it essentially states that we are considering every possible final state, all while placing momentum conservation restrictions.

- If we restrict ourselves to just a $2 \to 2$ process, this becomes:

$$\begin{aligned} \mathrm{d}\sigma &= \frac{|\mathcal{M}|^2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^4 \left(p_1 + p_2 - p_3 - p_4 \right) \frac{\mathrm{d}^3 \mathbf{p}_3 \mathrm{d}^3 \mathbf{p}_4}{(2\pi)^3 2E_3 (2\pi)^3 2E_4}, \\ \sigma &= \iint \frac{|\mathcal{M}|^2}{64\pi^2 E_3 E_4} \frac{\delta^4(p_1 + p_2 + p_3 + p_4)}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \mathrm{d}^3 \mathbf{p}_3 \mathrm{d}^3 \mathbf{p}_4. \end{aligned}$$

- To make it so we can actually do these integrals, we can split up the delta function like so:

$$\delta^4(p_1 + p_2 + p_3 + p_4) = \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4).$$

  This way, we can use the 3-dim delta to kill the $\mathbf{p}_4$ integral:

$$\sigma = \int \frac{|\mathcal{M}|^2}{64\pi^2 E_3 E_4} \frac{\delta(E_1 + E_2 - E_3 - E_4)}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \mathrm{d}^3 \mathbf{p}_4.$$

- Let's look at the center of mass frame. In this frame, it can be shown that $\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = (E_1 + E_2)|\mathbf{p}_1|$. Additionally, we will expand out energies into the form $E = \sqrt{\mathbf{p}^2 + m^2}$ so that we have everything that is dependent on $\mathbf{p}_3$ (the energies) written in terms of it. Lastly, in the CM frame, we have that $\mathbf{p}_3 + \mathbf{p}_4 = 0 \to \mathbf{p}_4 = -\mathbf{p}_3$, so

$$\sigma = \frac{1}{64\pi^2 (E_1 + E_2)|\mathbf{p}_1|} \int \frac{|\mathcal{M}|^2 \mathrm{d}^3 \mathbf{p}_3}{\sqrt{\mathbf{p}_3^2 + m_3^2} \sqrt{\mathbf{p}_3^2 + m_4^2}} \delta \left(E_1 + E_2 - \sqrt{\mathbf{p}_3^2 + m_3^2} - \sqrt{\mathbf{p}_3^2 + m_4^2} \right).$$

- From here, we can switch to spherical coordinates with $\mathrm{d}^3\mathbf{p}_3 = |\mathbf{p}_3|^2\mathrm{d}|\mathbf{p}_3|\mathrm{d}\Omega$, then do a couple of suitable u-substitutions, which are a lot of work, but relatively trivial. After all this, we get the rather simple result
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}. \tag{10.2}$$

- Further, if our scattering process happens to be elastic (which has a different meaning in quantum field theory; here, it means the same particles that go in come out), we have that $m_1 = m_3$ (and $m_2 = m_4$), so $|\mathbf{p}_3| = |\mathbf{p}_1|$, so the right-most term is just one:
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s}. \tag{10.3}$$

- Often, we are interested in finding the differential scattering cross section against $\mathrm{d}t$, the Mandelstam variable. This seems a little strange: why t and not s or u? Well, as we just saw, $\mathbf{p}_1$ and $\mathbf{p}_3$ were involved, which leads to t. Also, as we will see, it is quite a nice result.

- First,
$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2E_1E_2 + 2|\mathbf{p}_1||\mathbf{p}_3|\cos\theta,$$
so
$$\mathrm{d}t = 2|\mathbf{p}_1||\mathbf{p}_3|\mathrm{d}(\cos\theta).$$
Multiplying by the differential aziumathal angle:
$$\mathrm{d}t\mathrm{d}\phi = 2|\mathbf{p}_1||\mathbf{p}_3|\mathrm{d}(\cos\theta)\mathrm{d}\phi = 2|\mathbf{p}_1||\mathbf{p}_3|\mathrm{d}\Omega,$$
so
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \int \mathrm{d}\phi \frac{1}{2|\mathbf{p}_1||\mathbf{p}_3|}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}.$$

- Thus, dropping the elastic assumption, we have
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \int \mathrm{d}\phi \frac{|\mathcal{M}|^2}{2|\mathbf{p}_1||\mathbf{p}_3|}\frac{|\mathbf{p}_3|}{|\mathbf{p}_1|}\frac{1}{64\pi^2 s}.$$
The $|\mathbf{p}_3|$ cancels! Further, usually, the kinematics are independent of the azimuthal angle, so we do the integration and get a factor of 2π:
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{64\pi s|\mathbf{p}_1|^2}. \tag{10.4}$$

- This is not Lorentz-invariant, since we have just a raw momentum. It can be shown that this can be rewritten like
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_1^2, m_2^2)},$$
where λ is the "triangle-function", defined like
$$\lambda(s, m_1^2, m_2^2) = (s - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2. \tag{10.5}$$

- In the massless/high-energy limit, this simplifies to
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{16\pi s^2}. \tag{10.6}$$

- Moving to decays, we have, as an example, the muon decaying into an electron and two neutrinos: $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$. Of course, decay rates are probabilistic, and what's more, elementary particles don't have an "age", meaning that a particle that was just created has the exact same probability of decaying in the next instant as does an identical particle that has been around for 1000 years.

- We denote the decay rate with Γ, just as normal decay rates. This is defined by considering a group of $N(t)$ unstable particles at some time t. The rate of change of the number of particles is proportional to the decay rate:

$$\mathrm{d}N = -\Gamma N(t)\,\mathrm{d}t.$$

  This is a simple ordinary differential equation:

$$N(t) = N(0)e^{-\Gamma t}.$$

- The average lifetime is given by $\tau \equiv 1/\Gamma$, and the half-life is given by $t_{1/2} \equiv t \ln 2$. This can be found by considering that $N(t_{1/2}) = N(0)/2$, then solving for $t_{1/2}$.

- Turning now to QFT, let's consider a decay process wherein a particle decays into an arbitrary number of particles: $p \to p_1 + p_2 + \ldots + p_n$. It turns out that the formula for the decay rate is almost identical to that for the scattering cross section:

$$\mathrm{d}\Gamma = \frac{|\mathcal{M}|^2}{2E}(2\pi)^4\delta^4\left(p - \sum_{i=1}^{n}\right)\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i}. \tag{10.7}$$

- Actually, this particular phase space integration is special; it is called the "n-body phase space":

$$\mathrm{d}\Phi^{(n)} = (2\pi)^4\delta^4\left(p - \sum_{i=1}^{n}\right)\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i}. \tag{10.8}$$

- With this, we can more compactly write the differential decay rate:

$$\mathrm{d}\Gamma = \frac{|\mathcal{M}|^2}{2E}\mathrm{d}\Phi^{(n)}. \tag{10.9}$$

11 NLO Calculations

- As we alluded two before, calculations with more vertices (and hence higher powers of the coupling constant) are going to be more challenging to approach. The main issue that we will find is that suddenly lots of things start diverging.

- There are two types of diagrams we will consider when we go to NLO - they are *real emission* diagrams shown in the left of Figure 1, and *loop diagrams* as shown in right of the same figure. This particular loop diagram is called a *vertex correction* diagram, since the structure of an ordinary vertex is still there, but with an extra photon in there. There are also self-energy diagrams and vacuum polarization diagrams as we will see soon enough.

Figure 1: An example of a real emission diagram (left) and a loop diagram (right) at NLO.

- Now, in both scenarios, we get divergences in terms whenever the energy is relatively small - these are called **infrared (IR) divergences**. This gets its name from the relatively lower energy the infrared spectrum has compared to visible light. These come from when we have low energies in propagator terms that go like $\frac{1}{k^2}$.

- Further, in real emission diagrams specifically, we can have **soft** (low energy) or **colinear** emissions, which, in the IR range, give divergences. This is because we end up getting terms like

$$\sim \frac{1}{E(1-\cos\theta)},$$

  which, for colinear emission $\theta = 0$ so $1 - \cos\theta = 1 - 1 = 0$, which is a divergence. Similarly, for soft emission, we also get divergences.

- Loop diagrams are a little different. They carry similar IR divergences for propagator terms, but they also carry divergences in the high energy limit - these are called **ultraviolet (UV) divergences** for a similar reason as for IR divergences. These come from the fact that these have undetermined loop momenta, which requires an integration over all possible momenta. In the high momenta case, we will find divergences in terms where we have something like

$$\int \frac{\mathrm{d}^4 p}{k^m},$$

  which diverges for $m \leq 4$, and we often get terms with an $m \leq 4$.

- There is a nice theorem that states that if we consider (and sum) over all possible final states, including all possible angles of real emission and energies of emitted particles, then the IR divergences will end up canceling in the end, which is great!

- There is no such theorem for UV divergences though, so there is a process by which we take care of these. The first step is called **regularizing** the integral, and there are a number of different methods by which this is done. In general, though, every method attempts to consolidate all of the infinites into singular terms, so that in the second step which we call **renormalization**, we can absorb these terms into redefinitions of charge/mass/coupling constants.

- The method we will be using to do regularization is called **dimensional regularization (DR)**. All of these methods, by the way, are purely mathematical. One step involves motivation from a physical point of view, but apart from that, it is just mathematical tricks. In DR, we recognize that such a diverging integral as above is not divergent in a different dimension, so we just integrate over a different dimension $D = 4 - \epsilon$. After doing all the work, we end up getting isolated divergences represented as poles in ϵ, which we then renormalize (there are a number of renormalization techniques as well, but they are more closely related than the regularization techniques).

- Let's look at an example of vacuum polarizaiton; the Feynman diagram for this process in the reaction $e^- + \mu^- \to e^- + \mu^-$ is given in Figure 2.

- To evaluate this, we really only need one (two, really) Feynman rules that we didn't have before: *for a closed fermion loop, we pick up a factor of minus one and take the trace.*

- We can constrain q nicely to be $p_1 - p_3$, but as for k, it can be anything, there is nothing constraining it, so we must integrate over this. Doing the integation and applying the familiar Feynman rules as well as this new one we get

$$i\mathcal{M} = \bar{u}_3(-ie\gamma^\mu)u_1\left[\frac{-ig_{\mu\nu}}{q^2}\right] \times \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-\mathrm{Tr}[i(\not{k} - \not{q} + m)(-ie\gamma^\rho)i(\not{k} + m)(-ie\gamma^\sigma)]}{[(k-q)^2 - m^2](k^2 - m^2)} \left[\frac{-ig_{\sigma\nu}}{q^2}\right] \bar{u}_4(-ie\gamma^\nu)u_2.$$

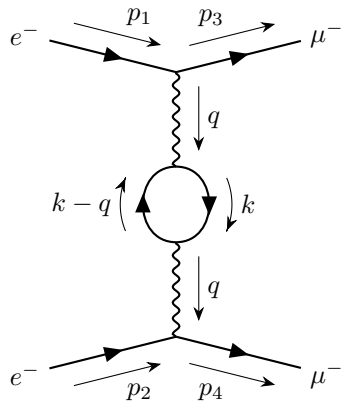


Figure 2: t-channel vacuum polarization diagram.

- Doing some simplifications, we arrive at

$$\mathcal{M} = \frac{ie^4}{(p_1-p_3)^2}[\bar{u}_3\gamma^\mu u_1]\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k}-\not{p}_1+\not{p}_3+m)\gamma_\nu(\not{k}+m)\gamma_\mu]}{[(k-p_1+p_3)^2-m^2](k^2-m^2)}[\bar{u}_4\gamma^\nu u_2].$$

- We could continue with the trace stuff, but we can already see that we are in trouble. We have, in the denominator of the integrand, a factor of order 4, so no matter what's on top we will satisfy the $m \leq 4$ condition, meaning this will diverge no matter what.

Dimensional Regularization

- We start with regularization, and we choose the method of **Dimensional Regularization (DR)**, where we essentially just convert the integral to $n = 4-\epsilon$ dimensions, and take epsilon to zero at the end. The goal is to take the diverging into and turn it into an analytic expression in terms of this ϵ, where the infinites are now just poles like $1/\epsilon$. These are then reabsorbed into the definitions of our constants in the renormalization step.

- Before jumping in to a full calculation, let's lay out all the pieces that we are going to need. Let's consider an integral of the form:

$$I(q) = \int\frac{\mathrm{d}^n p}{(p^2+2p\cdot q-m^2)^\alpha},$$

  where α is any number.

- To start, we move to spherical coordinates in n-dimensions. In any dimensional space-time, we have a single time dimensions and $n-1$ spatial dimensions, so we really are going to spherical in the $n-1$ dimenional space. Regardless - we will have the single time coordinate, the radial coordinate, the azimuthal angle, and the rest of the $n-3$ coordinates we will denote with θ_i as the polar angles.

- So, we have a four-vector that looks like this in spherical n-dimensional space-time:

$$p^\mu = \left(p^0,\ r,\ \phi,\ \theta_1,\ \theta_2,\ \ldots,\ \theta_{n-3}\right).$$

- The 4-dim differential in spherical coordinates is

$$\mathrm{d}^4p = \mathrm{d}p^0\mathrm{d}^3\mathbf{r} = \mathrm{d}p^0 r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

- where the power of r is one less than the number of spatial dimensions. In n-dimensions, each subsequent polar angle picks up an additional factor of sine, so what we have in n-dim is

$$\mathrm{d}^n p = \mathrm{d}p^0 r^{n-2}\mathrm{d}r\mathrm{d}\phi\prod_{k=1}^{n-3}\sin^k\theta_k\mathrm{d}\theta_k.$$

- The θ is the only non-trivial one (usually); it can be shown that

$$\int_0^\pi \sin^k\theta \, \mathrm{d}\theta = \sqrt{\pi}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}, \tag{11.1}$$

  where the Gamma function is defined for a complex z like so:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x} \, \mathrm{d}x. \tag{11.2}$$

  For real integer z, call it ℓ, it turns out that

$$\Gamma(\ell) = (\ell-1)!. \tag{11.3}$$

- With this,

$$I(q) = \int_{-\infty}^\infty \mathrm{d}p^0 \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^\pi \prod_{i=1}^{n-3} \sin^k\theta_k \, \mathrm{d}\theta_k \cdot \frac{r^{n-2}}{(p^2+2p\cdot q - m^2)^\alpha}.$$

- The ϕ integral is trivial; there is no phi dependence anywhere, so that's just 2π. The sine integrals can be treated separately so that

$$\int_0^\pi \prod_{i=1}^{n-3} \sin^k\theta_k \, \mathrm{d}\theta_k = \sqrt{\pi}\frac{\Gamma(1)}{\Gamma(3/2)}\sqrt{\pi}\frac{\Gamma(3/2)}{\Gamma(2)}\sqrt{\pi}\frac{\Gamma(2)}{\Gamma(5/2)}\cdots$$

  What ends up happening here is all the intermediate gamma functions cancel, leaving only the numerator of the very first term and the denominator of the very last term. Additionally, $\Gamma(1) = 1$, so we get that

$$I(q) = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^\infty \mathrm{d}p^0 \int_0^\infty \mathrm{d}r \frac{r^{n-2}}{[(p^0)^2 - r^2 + 2p\cdot q - m^2]^\alpha}.$$

- These two integrals can be done; they are a bit tedious, so we will just quote the result:

$$I(q) = i\pi^{n/2}\frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\alpha}(-q^2-m^2)^{(n/2)-\alpha}. \tag{11.4}$$

- There are two more things that we will need before moving on. The first is that in n-dimensions, gamma matrix identities are a little different. The two that we will use are

  1. $\gamma^\mu\gamma_\mu = n$,
  2. $\gamma^\mu\gamma^\nu\gamma_\mu = (2-n)\gamma^\nu$.

- The other thing is **Feynman parametrization**. This involves turning a product of two propagators into an integral like so:

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{[Ax + (1-x)B]^2}. \tag{11.5}$$

  This can be generalized to any number and any power of propagators:

$$\frac{1}{\prod_{i=1}^n A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_0^1 \prod_{i=1}^n \mathrm{d}x_i \; x_i^{\alpha_i - 1} \frac{\delta(1-x)}{[\sum_{i=1}^n x_i A_i]^\alpha}, \tag{11.6}$$

  where $\alpha \equiv \sum_{i=1}^n \alpha_i$ and $x \equiv \sum_{i=1}^n x_i$.

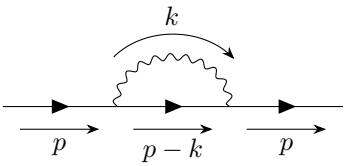


Figure 3: Feynman diagram for the electron (or any fermion, really) self-energy.

The Electron Self-Energy

- We are now ready to tackle one of the aforementioned primitively divergent integrals called the *electron self-energy*, whose Feynman diagram is given in Figure 3.
- When solving for these primitive diagrams, we don't solve for the traditional amplitude since it's usually part of a larger diagram, so we denote this amplitude with a $\Sigma(p)$. With this, since we also omit the Dirac spinors for the external lines since we don't know if they are actually external lines in this larger diagram or whether they are internal. With this we can use our Feynman rules to write down the amplitude

$$\begin{aligned} i\Sigma(p) &= \int \frac{\mathrm{d}^n k}{(2\pi)^n}(-ie\gamma^\mu)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}(-ie\gamma^\nu)\frac{-ig_{\mu\nu}}{k^2}, \\ &= -e^2\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\gamma^\mu(\not{p}-\not{k}+m)}{[(p-k)^2-m^2]k^2}. \end{aligned}$$

- Now we see the motivation for the Feynman parameter stuff. It might have seemed ridiculous to further complicate a simple product in the denominator to an additional integral, but now we see that by doing that, we get a result that is somewhat similar $I(q)$ which we know how to integrate. So, if we take $A = (p-k)^2 - m^2$ and $B = k^2$, we can write this as

$$i\Sigma(p) = -\frac{e^2}{(2\pi)^n}\int \mathrm{d}^4k' \int_0^1 \mathrm{d}x\ \frac{\gamma^\mu(\not{p}-\not{k}+m)}{\left\{\left[(p-k)^2-m^2\right]x + k^2(1-x)\right\}^2}.$$

- We are almost to a form that is like $I(q)$. With a little bit of trivial algebra, we can massage the denominator to look like $[(k-xp)^2 + p^2x(1-x) - m^2x]$, and define a shifted momentum $k' = k - xp$ such that

$$i\Sigma(p) = -\frac{e^2}{(2\pi)^n}\int \mathrm{d}^4k' \int_0^1 \mathrm{d}x\ \frac{\gamma^\mu((1-x)\not{p}-\not{k}'+m)}{[k'^2 + p^2x(1-x) - m^2x]^2}.$$

- Now this is of an identical form as in $I(q)$, except that we have no linear term in k', which is what happens when $q = 0$. Then, if we let $L \equiv -x[(1-x)p^2 - m^2]$, it can take the place of the $-m^2$ in the denominator so we have

$$i\Sigma(p) = -\frac{e^2}{(2\pi)^n}\int \mathrm{d}^4k' \int_0^1 \mathrm{d}x\ \frac{\gamma^\mu((1-x)\not{p}-\not{k}'+m)}{[k'^2 - L]^2}.$$

- If we do a little more work and re-express things in terms of ϵ, we get something like

$$i\Sigma(p) = \frac{-ie^2}{2^{4-\epsilon}\pi^{2-(\epsilon/2)}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 \mathrm{d}x\ \gamma^\mu[\not{p}(1-x)+m]\gamma_\mu[p^2x(1-x)-m^2x].$$

  I had started to work on my own during the lecture and so my form might not be exactly similar, but regardless, this is what you get.

- Now if we do Laurent expansions in terms of ϵ we get something like

$$\Sigma(p) = \frac{-e^2}{16\pi^2}\frac{2}{\epsilon}\int_0^1 \mathrm{d}x\,[-2\not{p}(1 - x_+ 4m)] + \mathcal{O}(\epsilon),$$

  where we have gotten rid of every linear in epsilon, since when we take it zero those terms will go to zero. We can go a little further and say

$$\Sigma(p) = \frac{e^2}{8m^2}\frac{1}{\epsilon}(\not{p} - 4m).$$

- At this stage, we have successfully isolated the diverging terms into an analytical expression in ϵ which contains a single pole. This is what we will renormalize into redefinitions of the constants in the next step.

- Now, our leading-order no-loop propagator we indicate with $S_0(p)$:

$$S_0(p) = \frac{i}{\not{p} - m} \tag{11.7}$$

- In general, though, there can be loops and such. We consider all 1-loop diagrams as the **dressed** propagator $S(p)$. Diagrammatically, this looks like[3]

$S(p) =$  $+\ldots$

- Mathematically,

$$\begin{aligned} S(p) &= S_0(p) + S_0(p)i\Sigma(p)S_0(p) + S_0(p)i\Sigma(p)S_0(p)i\Sigma(p)S_0(p) + \ldots \\ &= S_0(p)[1 + i\Sigma(p)S_0(p) + i\Sigma(p)S_0(p)i\Sigma(p)S_0(p) + \ldots]. \end{aligned}$$

- The quantity in brackets is just the power series expansion for $1/(1-x)$ where $x = i\Sigma(p)S_0(p)$:

$$\begin{aligned} S_0(p) &= [1 - i\Sigma(p)S_0(p)]^{-1} \\ &= [S_0(p) - i\Sigma(p)]^{-1}. \end{aligned}$$

- Plugging in the normal propagator:

$$S(p) = [-i(\not{p} - m) - i\Sigma(p)]^{-1} = \frac{i}{\not{p} - m + \Sigma(p)}. \tag{11.8}$$

- Interestingly, this looks identical to the normal propagator but with a little "correction" that is the self-energy part. If we plug this in we get (note the inverse)

$$S^{-1}(p) = -i(\not{p} - m) - \frac{ie^2}{8\pi^2\epsilon}(\not{p} - 4m) + \mathcal{O}(\epsilon^0), \tag{11.9}$$

  where we have isolated only the $1/\epsilon$ poles and disregarded any linear terms. The constant terms are encapsulated in the $\mathcal{O}(\epsilon^0)$ terms, because as we will see in the next step, what of those terms we consider is dependent on the scheme we choose.

[3]Notice that we don't have a scenario where emits a second photon before absorbing again the first. This is considered a two-loop diagram. More specifically, the dressed propagator considers only **1-particle irreducable** diagrams, which, roughly speaking, incorporates diagrams that can be "cut" into parts that each are valid Feynman diagrams.

- We have now completed the regularization step! We have successfully got a full dressed propagator term with the infinites isolated as $1/\epsilon$ poles.
- Next, we move to renormalization where we basically absorb all of these infinities into the electric charge, the mass, or the fields.
- There are a number of ways in which to do this which correspond to the different schemes. One of the more simple ones is called the **minimal subtraction (MS)** scheme, in which we only look at the divergent terms (hence why we moved all the constant terms into $\mathcal{O}(\epsilon^0)$).
- The **modified minimal subtraction ($\overline{\text{MS}}$)** scheme is similar but also takes into account some other things like a $\ln 4\pi$ and the Euler-Mascheroni constant γ_E. The latter is more commonly used, but the former is a little easier.
- First, let's rewrite the dressed propagator:

$$S^{-1}(p) = -i\not{p}\left(1 + \frac{e^2}{8\pi^2\epsilon}\right) + im\left(1 + \frac{e^2}{2\pi^2\epsilon}\right). \tag{11.10}$$

- Recall, the QED Lagrangian is (with our gauge fixing term since we continue to work in the Feynman gauge)

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{11.11}$$

- What we will now do is add *counterterms* to the Lagrangian that effectively cancel the infinities. We will only consider the Dirac part (the kinetic and mass terms for the spinor fields), and just tack on similar looking terms but with undetermined constants out front:

$$\mathcal{L}_{\text{Dirac}} \to i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi + iB\overline{\psi}\gamma^\mu\partial_\mu\psi - iM\overline{\psi}\psi. \tag{11.12}$$

- If we let

$$B = -\frac{e^2}{8\pi^2\epsilon} \quad \text{and} \quad M = -\frac{me^2}{2\pi^2\epsilon}, \tag{11.13}$$

  then we have

$$\mathcal{L}_{\text{Dirac}} = i\left(1 - \frac{e^2}{8\pi^2\epsilon}\right)\overline{\psi}\gamma^\mu\partial_\mu\psi - \left(m - \frac{me^2}{2\pi^2\epsilon}\right)\overline{\psi}\psi. \tag{11.14}$$

- It turns out that this exactly cancels the infinities! I'm not really 100% but I trust it. Now if we define

$$Z_\psi = 1 + B = 1 - \frac{e^2}{8\pi^2\epsilon} \quad \text{and} \quad m_b = \frac{m + M}{Z_\psi}, \tag{11.15}$$

  we get

$$\mathcal{L}_{\text{Dirac}} = iZ_\psi\overline{\psi}\gamma^\mu\partial_\mu\psi - Z_\psi m_b\overline{\psi}\psi. \tag{11.16}$$

- The b subscript indicates "bare". I'll explain in just a minute.
- Lastly, we define the "bare" field $\psi_b \equiv \sqrt{Z_\psi}\psi$ so that we have

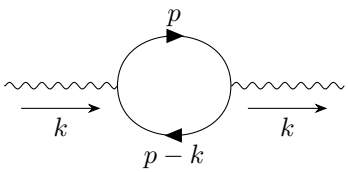
$$\mathcal{L}_{b,\text{Dirac}} = i\overline{\psi}_b\gamma^\mu\partial_\mu\psi_b - m_b\overline{\psi}_b\psi_b. \tag{11.17}$$

- This is the original Lagrangian in terms of bare quantities! We have absorbed the infinities into these bare quantities.

- The reason why these are called "bare" is because they represent the quantities if we were able to get infinitely close to it. We will see the charge renormalization later, but that's the best example. If we get infinitely close to the electron, there are no loops or particles coming from the vacuum to "shield" the charge; we see the *bare* charge, which turns out to be infinity.
- But we are never truly able to measure this bare charge because it would require an infinite amount of energy to get infinitely close.
- So our motivation for doing this is that we can just absorb the infinities into these new bare quantities without any side effects because we aren't able to measure them anyways, so it won't affect anything. This is a little unsettling, hence why people like Dirac and Feynman were so against it, but it works very well so we just accept it.

The Photon Self-Energy / Vacuum Polarization

- The next primitive divergent diagram is the photon self-energy or the vacuum polarization:



- We denote this particle amplitude with an uppercase Π:

$$i\Pi^{\mu\nu}(k) = \int \frac{\mathrm{d}^n p}{(2\pi)^n}(-1)\frac{\mathrm{Tr}[(-ie\gamma^\mu)i(\not{p}+m)(-ie\gamma^\nu)i(\not{p}-\not{k}+m)]}{(p^2-m^2)[(p-k)^2-m^2]} \tag{11.18}$$

$$\Pi^{\mu\nu}(k) = ie^2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n p}{(2\pi)^n}\frac{\mathrm{Tr}[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}-\not{k}+m)]}{[(p-k)^2x - m^2x + (p^2-m^2)(1-x)]^2}. \tag{11.19}$$

- We only wrote a few of the intermediate steps in class, but I'll skip them here and just write the final answer after doing all of the integrations and stuff:

$$\Pi^{\mu\nu}(k) = \frac{e^2}{6\pi^2\epsilon}(k^\mu k^\nu - g^{\mu\nu}k^2) + \mathcal{O}(\epsilon^0). \tag{11.20}$$

- Doing the same stuff as before, we can write the dressed propagator as a sum of all the 1-particle irreducible vacuum polarization diagrams:

$$D_{\mu\nu}(k) = D_{0,\mu\nu} + D_{0,\mu\rho}i\Pi^{\rho\sigma}(k)D_{0,\sigma\nu}(k) + \dots \tag{11.21}$$

$$= -\frac{ig_{\mu\nu}}{k^2} - \frac{ig_{\mu\rho}}{k^2}\frac{ie^2}{6\pi^2\epsilon}(k^\rho k^\sigma - g^{\rho\sigma}k^2)\frac{(-ig_{\sigma\nu})}{k^2} + \dots \tag{11.22}$$

$$= -\frac{ig_{\mu\nu}}{k^2} - \frac{ie^2}{6\pi^2\epsilon}\frac{k_\mu k_\nu}{k^4} + \frac{ie^2}{6\pi^2\epsilon}\frac{g_{\mu\nu}}{k^2} + \dots \tag{11.23}$$

$$= -\frac{ig_{\mu\nu}}{k^2}\left(1 - \frac{e^2}{6\pi^2\epsilon}\right) - \frac{ie^2}{6\pi^2\epsilon}\frac{k_\mu k_\nu}{k^4} + \dots \tag{11.24}$$

- We again add counterterms to the Lagrangian, but this time we are only going to look at the gauge term:

$$\mathcal{L}_{\text{gauge}} \to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 - \frac{1}{4}CF_{\mu\nu}F^{\mu\nu} - \frac{1}{2}N(\partial_\mu A^\mu)^2. \tag{11.25}$$

- We find that

$$C = N = -\frac{e^2}{6\pi^2\epsilon}, \tag{11.26}$$

so

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\left(1 - \frac{e^2}{6\pi^2\epsilon}\right)F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(1 - \frac{e^2}{6\pi^2\epsilon}\right)(\partial_\mu A^\mu)^2. \tag{11.27}$$

- With another renormalization constant $Z_A = 1 + C$:

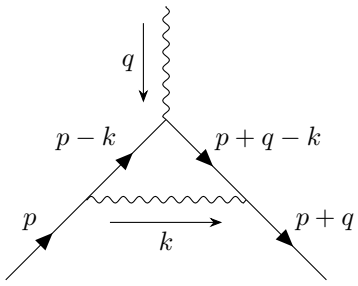
$$\mathcal{L}_{b,\text{gauge}} = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} Z_A (\partial_\mu A^\mu)^2. \tag{11.28}$$

- We can then absorb this new renormalization constant into the gauge field in a similar way to get

$$\mathcal{L}_{b,\text{gauge}} = -\frac{1}{4} F_{b,\mu\nu} F_b^{\mu\nu} - \frac{1}{2} (\partial_\mu A_b^\mu)^2. \tag{11.29}$$

Vertex Correction

- The last thing that we consider is the vertex correction:



- The amplitude is

$$i\Lambda^\mu(p,q) = \int \frac{\mathrm{d}^n p}{(2\pi)^n} (-ie\gamma^\nu) \frac{i(\not{p} + \not{q} - \not{k} + m)}{(p+q-k)^2 - m^2} (-ie\gamma^\mu) \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2} (-ie\gamma^\rho) \frac{-ig_{\rho\nu}}{k^2} \tag{11.30}$$

$$\Lambda^\mu(p,q) = ie^3 \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{\gamma^\nu(\not{p} + \not{q} - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\gamma_\nu}{[(p+q-k)^2 - m^2][(p-k)^2 - m^2]k^2} \tag{11.31}$$

$$= 2ie^3 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{\gamma^\nu(\not{p} + \not{q} - \not{k} + m)\gamma^\mu(\not{p} - \not{k} + m)\gamma_\nu}{\{x[(p+q-k)^2 - m^2] + y[(p-k)^2 - m^2] + (1-x-y)k^2\}^3}. \tag{11.32}$$

- Doing all the steps we arrive at a final answer of

$$\Lambda^\mu(p,q) = -\frac{e^3}{8\pi^2\epsilon}\gamma^\mu + \mathcal{O}(\epsilon^0). \tag{11.33}$$

- Again, we add counterterms to the Lagrangian, this time specifically to the interaction term:

$$\mathcal{L}_{\text{int}} \to -e(1+L)\overline{\psi}\gamma^\mu\psi A_\mu, \tag{11.34}$$

where

$$L = -\frac{e^2}{8\pi^2\epsilon}, \tag{11.35}$$

so

$$\mathcal{L}_{\text{int}} = -e\left(1 - \frac{e^2}{8\pi^2\epsilon}\right)\overline{\psi}\gamma^\mu\psi A_\mu. \tag{11.36}$$

- If we go back and check we will find that the quantity in parentheses is identical to Z_ψ that we defined earlier.[4] With this, then,

$$\mathcal{L}_{b,\text{int}} = -e\overline{\psi}_b\gamma^\mu\psi_b A_\mu. \tag{11.37}$$

- We can multiply and divide by $\sqrt{Z_A}$ and define $e_b \equiv e/\sqrt{Z_A}$ so that we get

$$\mathcal{L}_{b,\text{int}} = -e_b\overline{\psi}_b\gamma^\mu\psi_b A_{b,\mu}. \tag{11.38}$$

- Thus, with all of the primitively divergent diagrams, we have been able to successfully renormalize the divergences into redefinitions of the fundamental quantities in the Lagrangian!

[4] It turns out that this is a/the **Ward identity**, which is quite interesting because it is nothing at all like I thought it was from what we did last semester in Directed Methods.

Dimensional Analysis

- Let's take a step back and look at the dimensions and units of all this stuff. We will find a super interesting result by the end!
- First, in the context of DR, we have that the action is now an integral over n-dimensions:

$$S = \int \mathcal{L} \, \mathrm{d}^n x. \tag{11.39}$$

- With this, then we know that $\mathcal{L}$ must have dimensions of $[\text{length}]^{-1}$ (the brackets denote the dimension of the quantity inside - ordinarily we'd use the letter corresponding to lenghth, like E for energy, but this works).
- But since we are using natural units, $[E] = [\text{length}]^{-1} = [p] = [m]$, so the action must have n mass dimensions. From here on out, we will consider specifically the *mass* dimension of things, so $[m] = 1$, since obviously one mass has one mass dimension.
- Each term in the Lagrangian must therefore have n mass dimensions; let's look at the mass term first: $[m\overline{\psi}\psi] = [m] + [\overline{\psi}] + [\psi] = n$. Again, obviously mass has 1 mass dimension, and the field and its adjoint must have the same mass dimension. With this, then

$$[\psi] = \frac{n-1}{2}.$$

- Turning to the kinetic term for the gauge field, $[F_{\mu\nu}F^{\mu\nu}] = n$ so $[F_{\mu\nu}] = n/2$. With this, then $[\partial_\mu A_\nu] = n/2$. The derivative will take away one of the length dimensions, so A_ν must have an additional length dimension, or one less mass dimension:

$$[A_\mu] = \frac{n-2}{2}. \tag{11.40}$$

- Lastly, let's look at the interaction term:

$$[e\overline{\psi}\psi A_\mu] = n \to [e] + [\overline{\psi}] + [\psi] + [A_\mu] = n \tag{11.41}$$

$$= [e] + 2\left(\frac{n-1}{2}\right) + \left(\frac{n-2}{2}\right) = n \tag{11.42}$$

$$= [e] + \frac{n-4}{2} = n \tag{11.43}$$

$$\to [e] = \frac{4-n}{2} = \frac{\epsilon}{2}. \tag{11.44}$$

- Obviously, in our normal four space-time dimensions, this goes to zero, meaning the electric charge is a constant. However in the context of dimensional regularization this is not the case!
- This is a bit of an issue. To fix this, we introduce an arbitrary "renormalization scale" μ that absorbs this dimensionality, so $[\mu] = [e]^{-1}$. Then,

$$\mathcal{L}_{\text{int}} = -e\mu^{(4-n)/2}\overline{\psi}\gamma^\mu\psi A_\mu, \tag{11.45}$$

  where this e is back to being a real constant.
- However, we have now picked up an arbitrary energy scale that we must set ourselves. We typically choose it to be characteristic of the experiment we are conducting. For instance, we'd set it to the mass of the top quark if we are pair producing them, or if we are just going stuff with leptons we'd set it lower.

- Now, the bare mass of the electron is given as

$$e_b = \frac{e\mu^{\epsilon/2}}{\sqrt{Z_A}} = \frac{e\mu^{\epsilon/2}}{\sqrt{1-\frac{e^2}{6\pi^2\epsilon}}} = e\mu^{\epsilon/2}\left(1+\frac{e^2}{12\pi^2\epsilon}+\dots\right), \tag{11.46}$$

  where we used the taylor expansion for $(1-x)^{-1/2}$.

- We know that the bare mass cannot be a function anything, let alone the bare quark mass, meaning that

$$\frac{\partial e_b}{\partial \mu} = 0. \tag{11.47}$$

  But using our definition for the bare mass we just described tells us that

$$\mu\frac{\partial e}{\partial \mu} \equiv b(e) = -\frac{\epsilon}{2} + \frac{e^3}{12\pi^2} + \dots. \tag{11.48}$$

- Going back to four dimensions, we have that

$$b(e) = \mu\frac{\partial e}{\partial \mu} = \frac{e^3}{12\pi^2} > 0, \tag{11.49}$$

  which essentially says that the electron charge grows when we increase our energy scale! In fact, we can solve the differential equation to find that, if we know the electron charge at some initial energy scale μ_0, then the energy scale at any μ is

$$e^2(\mu) = \frac{e^2(\mu_0)}{1-\frac{e^2(\mu_0)}{6\pi^2}\ln\left(\frac{\mu}{\mu_0}\right)}. \tag{11.50}$$

- Indeed, the electron charge blows up at super high energy scales!

- This is also the reasoning behind the "running coupling" idea, since the coupling is directly proportional to the electric charge (squared):

$$\alpha_{\text{em}} = \frac{e^2}{4\pi} \approx \frac{1}{137} \quad \text{for low energy.} \tag{11.51}$$

12 Path Integral Formulation

Path Integrals in QM

- In the Schrodinger picture in Quantum Mechanics, we have a wave function $\psi(q,t)$ (q is a generic coordinate) which is given by the braket of the generic state $|\psi_S\rangle$ with $\langle q|$ to put it in the position basis:

$$\psi(q,t) = \langle q|\psi_S\rangle\,. \tag{12.1}$$

- We know that in the Heisenberg picture, the wave-functions are not time-dependent; the relation between states in either picture is given by

$$|\psi_S\rangle = e^{-iHt}\,|\psi_H\rangle\,, \tag{12.2}$$

  which means that

$$\psi(q,t) = \langle q|e^{-iHt}|\psi_s| = \langle q,t|\psi_H\rangle\,. \tag{12.3}$$

- From here on out, we will continue to sit in the Heisenberg picture, so any generic wavefunction ψ is really ψ_H.

- Now, some arbitrary final state can be written as $\langle q_f, t_f|\psi\rangle$, where q_f is the final position and t_f is the final time. We can then express it as an integral over all possible intermediate states, like so:

$$\langle q_f, t_f|\psi\rangle = \int \langle q_f, t_f|q_i, t_i\rangle \langle q_i, t_i|\psi\rangle \ \mathrm{d}q_i. \tag{12.4}$$

- This holds due to the completeness relation in Hilbert space. Specifically:

$$\int |x\rangle \langle x| \ \mathrm{d}x = 1. \tag{12.5}$$

  Essentially, then, all we have done is insert 1 into the first final state bracket.

- We can see that the first term is just the propagator, since its the braket of the final state with the initial state:

$$K(q_f, t_f, q_i, t_i) \equiv \langle q_f, t_f|q_i, t_i\rangle . \tag{12.6}$$

- In principle, we can do this (write the propagator) for any arbitrary number of intermediate states:

$$\langle q_f, t_f|q_i, t_i\rangle = \int \langle q_f, t_f|q_n, t_n\rangle \langle q_n, t_n|q_{n-1}, t_{n-1}\rangle \dots \langle q_1, t_1|q_i, t_i\rangle \ \mathrm{d}q_1 \mathrm{d}q_2 \dots \mathrm{d}q_n. \tag{12.7}$$

- What we have now is an integral over not just every possible *single* intermediate *state*, but rather every possible *path*.

- Let's consider an arbitrary braket in this integral:

$$\begin{aligned}
\langle q_{m+1}, t_{m+1}, q_m, t_m\rangle &= \left\langle q_{m+1} \left| e^{iHt_{m+1}} e^{-iHt_m} \right| q_m \right\rangle && (12.8)\\
&= \left\langle q_{m+1} \left| e^{-iH\tau} \right| q_m \right\rangle && (12.9)\\
&= \langle q_{m+1}|1 - iH\tau + \mathcal{O}(\tau^2)|q_m\rangle && (12.10)\\
&= \langle q_{m+1}|q_m\rangle - i\tau \langle q_{m+1}|H|q_m\rangle + \mathcal{O}(\tau^2). && (12.11)
\end{aligned}$$

- In the first step, we brought out the exponentials which removed the time dependence then combined them with $\tau = t_{m+1} - t_m$. Then, we just expanded the exponential.

- Now, the first term is the delta function by the orthonormality of Hilbert space, so

$$\begin{aligned}
&= \delta(q_{m+1} - q_m) - i\tau \langle q_{m+1}|H|q_m\rangle && (12.12)\\
&= \frac{1}{2\pi} \int e^{ip\Delta q} \ \mathrm{d}p - i\tau \left\langle q_{m+1} \left| \frac{p^2}{2m} + U(q) \right| q_m \right\rangle . && (12.13)
\end{aligned}$$

- Let's consider just the second term, with the Hamiltonian:

$$\begin{aligned}
&\to \left\langle q_{m+1} \left| \frac{p^2}{2m} \right| q_m \right\rangle + \langle q_{m+1}|U(q)|q_m\rangle && (12.14)\\
&= \int \langle q_{m+1}|p'\rangle \left\langle p' \left| \frac{\hat{p}^2}{2m} \right| p \right\rangle \langle p|q_m\rangle \ \mathrm{d}p'\mathrm{d}p + U\left(\frac{q_{m+1} + q_m}{2}\right) \langle q_{m+1}|q_m\rangle . && (12.15)
\end{aligned}$$

- Now obviously $\hat{p}^2 \langle p| = p^2 \langle p|$. We also expressed the potential as a function of the average distance between the two positions assuming that they are close, which they are; we can define $\bar{q}_m$ as this quantity to make things a bit more simple. Again expressing the delta functions as exponentials:

$$\begin{aligned}
&= \frac{1}{2\pi} \int e^{ip'q_{m+1}} \frac{p^2}{2m} \delta(p' - p) e^{-ipq_m} \ \mathrm{d}p'\mathrm{d}p + U(\bar{q}_m)\Delta(q_{m+1} - q_m) && (12.16)\\
&= \frac{1}{2\pi} \int \mathrm{d}p_m \ e^{ip_m \Delta q_m} H(p_m, \bar{q}_m). && (12.17)
\end{aligned}$$

- If we plug this all back into our propagator K and do a bunch of tedious algebra, we find

$$K(q_f, t_f, q_i, t_i) = \int \prod_{j=0}^{n} \mathrm{d}q_m \prod_{k=0}^{n} \frac{\mathrm{d}p_m}{2\pi} \exp\left\{ \sum_{\ell=0}^{n} i[p_m \Delta q_m - \tau H(p_m, \bar{q}_m)] \right\} \tag{12.18}$$

- In the continuum limit (where $n \to \infty$), we can express the sum in the exponential as an integral after multiplying and dividing by a Δt and also rewrite the infinite product of differentials as:

$$= \int \mathcal{D}q\mathcal{D}p \ \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}t[p\dot{q} - H(q,p)] \right\}. \tag{12.19}$$

- But the quantity in the exponential is just the Lagrangian! Further, integrating it over time is just the action! After all this, then we have that

$$\boxed{K = \int \mathcal{D}q \ e^{iS}.} \tag{12.20}$$

- We will not use this itself, especially not for anything related to QM, but it is built off of in QFT, as we will see shortly.

Source Theory (Schwinger)

- Before looking at path integrals in QFT, we consider a "source" of a field $J(t)$. This source is analogous to the electromagnetic current in EM which is a "source" for the electromagnetic field.
- We then define the functional $Z[J]$ as the vacuum-to-vacuum transition amplitude of particle creation in the presence of the source J. With this, we have that

$$Z[J] \propto \langle 0, \infty | 0, -\infty \rangle . \tag{12.21}$$

- It turns out that this takes the Lagrangian $L \to L + J(t)q(t)$. And our functional $Z[J]$ is just the propagator K we found before but with this "modified" Lagrangian:

$$\boxed{Z[J] = \int \mathcal{D}q \ \exp\left\{ i \int_{t_i}^{t_f} \mathrm{d}t \ (L + Jq) \right\}.} \tag{12.22}$$

Path Integrals in QFT

- In QFT, we make some simple generalizations/substitutions:

$$Z[J] = \int \mathcal{D}\phi \ \exp\left\{ i \int \mathrm{d}^x \ [\mathcal{L} + J(x)\phi(x)] \right\}. \tag{12.23}$$

- For a free particle, we use $Z_0[J]$ and input the Lagrangian without any interaction term. For a scalar field, as we used above, this means that we have

$$Z_0[J] = \int \mathcal{D}\phi \ \exp\left\{ i \int \mathrm{d}^4x \ \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi \right] \right\}. \tag{12.24}$$

- Using integration by parts on the kinetic term in the Lagrangian gives us

$$\int \partial_\mu\phi\partial^\mu\phi \ \mathrm{d}^4x = \int \partial_\mu(\phi\partial^\mu\phi) \ \mathrm{d}^4x - \int \phi\partial_\mu\partial^\mu\phi \ \mathrm{d}^4x. \tag{12.25}$$

- Doing the integration on the first term just removes the 4-derivitave from the front, so all we are left with is what's in parentheses evaluated at $\pm\infty$, but we assume the fields vanish at infinity, so that term is zero, meaning

$$Z_0[J] = \int \mathcal{D}\phi \ \exp\left\{ i \int \mathrm{d}^4x \ \left[-\frac{1}{2}\phi\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi \right] \right\}. \tag{12.26}$$

- Now we can take $\phi \to \phi + \phi_0$, where

$$\partial_\mu\partial^\mu\phi_0 + m^2\phi_0 = J, \tag{12.27}$$

which nets us

$$Z_0[J] = \int \mathcal{D}\phi \ \exp\left\{ -i \int \mathrm{d}^4x \ \left[\frac{1}{2}\phi\partial_\mu\partial^\mu\phi\frac{1}{2}m^2\phi^2 + J\phi_0 \right] \right\}. \tag{12.28}$$

- Because ϕ_0 satisfies the EL equations for the KL Lagrangian, we can write it like so:

$$\phi_0(x) = i \int D(x-y)J(y) \ \mathrm{d}^4y, \tag{12.29}$$

where $D(x-y)$ is, as usual, the Feynman propagator.

- Therefore, we can write

$$Z_0[J] = \exp\left\{ -\frac{1}{2} \int J(x)D(x-y)J(y) \ \mathrm{d}^4x\mathrm{d}^4y \right\} \int \mathcal{D}\phi \ \exp\left\{ i \int \mathrm{d}^4x \ \left[-\frac{1}{2}\phi\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right] \right\}, \tag{12.30}$$

- We have effectively separated out the ϕ_0 and ϕ, meaning that the $\mathcal{D}\phi$ integral is no longer dependent on J, so in light of the functional being a function of J, that entire integral might as well be a constant, call it N, and we will take it so that the whole thing is normalized to 1.

- After this, we can Taylor expand the exponential to get

$$Z_0[J] = N\left[1 - \frac{1}{2} \int J(x)D(x-y)J(y) \ \mathrm{d}^4x\mathrm{d}^4y + \ldots \right]. \tag{12.31}$$

- Using complex integration, we can write the integrand as

$$-\frac{1}{2}J(x)D(x-y)J(x) \ \mathrm{d}^4x\mathrm{d}^4y = -\frac{i}{2(2\pi)^4} \int J(x)\frac{e^{-i(x-y)}}{p^2 - m^2 + i\epsilon}J(y) \ \mathrm{d}^4p\mathrm{d}^4x\mathrm{d}^4y. \tag{12.32}$$

- If we define the Fourier transform of $J(x)$ as

$$J(x) = \frac{1}{(2\pi)^4} \int J(p)e^{-ipx} \ \mathrm{d}^4p, \tag{12.33}$$

then

$$= -\frac{i}{2}(2\pi)^4 \int \frac{J(-p)J(p)}{p^2 - m^2 + i\epsilon} \ \mathrm{d}^4p. \tag{12.34}$$

Or, using the definition for the Feynman propagator in momentum space:

$$= \frac{1}{2}(2\pi)^4 \int J(-p)D(p)J(p) \ \mathrm{d}^4p. \tag{12.35}$$

- Now, if we make the new association that the term $i(2\pi)^4J(p)$ looks something like

$$J\times\!\!\!-\!\!\!-$$

then the term we found as a result of the integral is something like

$$J \times\!\!\longrightarrow\!\!\times J$$

- These are generating functions! In fact, this functional is a generator of Green's functions! Diagramatically,

$$Z_0[J] = 1 + \frac{1}{2} \times\!\!\longrightarrow\!\!\times + \frac{1}{2 \cdot 2!} \begin{matrix} \times\!\!\longrightarrow\!\!\times \\ \times\!\!\longrightarrow\!\!\times \end{matrix} + \dots$$

- Recall

$$G(x_1, x_2, \dots, x_n) = \langle 0 \,|\, T\left\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\right\} \,|\, 0\rangle\,, \tag{12.36}$$

and

$$D(x-y) = \langle 0 \,|\, T\left\{\phi(x_1)\phi(x_2)\right\} \,|\, 0\rangle\,. \tag{12.37}$$

- It turns out that

$$G(x_1, x_2, \dots, x_n) = (-i)^n \frac{\delta Z_0[J]}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)}\bigg|_{J=0}. \tag{12.38}$$

- To show this, let's first write down the only functional derivative identity we will need:

$$\frac{\delta J(y)}{\delta J(x)} = \delta^4(x-y), \tag{12.39}$$

where the deltas on the left-hand side are the functional derivative symbol and the delta on the right-hand side is the Dirac delta.

- From this, we can write that

$$\frac{\delta}{\delta J(x)} \int J(y)\phi(y)\, \mathrm{d}^4y = \phi(x). \tag{12.40}$$

- Now, from Equation (12.38), we know that

$$\langle 0 \,|\, T\left[\phi(x_1)\right] \,|\, 0\rangle = i\frac{\delta Z_0[J]}{\delta J(x_1)}\bigg|_{J=0}. \tag{12.41}$$

- Plugging in:

$$\frac{\delta Z_0[J]}{\delta J(x_1)} = \frac{\delta}{\delta J(x_1)} \exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\, \mathrm{d}^4x\mathrm{d}^4y\right), \tag{12.42}$$

$$= \left[-\frac{1}{2}\int D(x_1-y)J(y)\, \mathrm{d}^4y - \frac{1}{2}\int J(x)D(x-x_1)\, \mathrm{d}^4x\right] \times \exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\, \mathrm{d}^4x\mathrm{d}^4y\right). \tag{12.43}$$

- Since our goal is to evaluate this for $J = 0$, we can see that the entire thing vanishes, meaning

$$\langle 0|T\{\phi(x_1)\}|0\rangle = 0, \tag{12.44}$$

as we expect.

- Similarly, by continuing to do derivatives, we find that

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = D(x_1-x_2) \tag{12.45}$$

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\}|0\rangle = 0 \tag{12.46}$$

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = D(x_1-x_2)D(x_3-x_4) \tag{12.47}$$

$$+ D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3), \tag{12.48}$$

also as we expect.

QFT Path Integrals: Interacting Fields

- This has been using the generating functional $Z_0[J]$, which was defined using the Lagrangian for a free scalar field. Let's now include an interaction term. Continuing with our scalar field stuff, we have that

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4. \tag{12.49}$$

- Our total generating functional now is

$$Z[J] = \int \mathcal{D}\phi \ \exp\left[i \int \mathrm{d}^4x \ (\mathcal{L} + J(x)\phi(x))\right] \tag{12.50}$$

$$= \int \mathcal{D}\phi \ \exp\left(iS + i\int \mathrm{d}^4x \ J\phi\right). \tag{12.51}$$

- Similarly to before, we define a normalization factor N as the inverse of $\int \mathcal{D}\phi \ e^{iS}$. Thus, then $\mathcal{L}_{\text{int}} \to 0$, $Z[J] \to Z_0[J]$.

- Now, it can be shown that

$$Z[J] = N \exp\left[i \int \mathcal{L}_{\text{int}}\left(\frac{\delta}{\delta J} \ \mathrm{d}^4x\right)\right] Z_0[J]. \tag{12.52}$$

- For example, in ϕ^4 theory,

$$\mathcal{L}_{\text{int}}\left(\frac{\delta}{\delta J}\right) = -\frac{\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4. \tag{12.53}$$

- With this, we still have that the Green's functions are

$$G(x_1, x_2, \ldots, x_n) = (-i)^n \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\bigg|_{J=0}. \tag{12.54}$$

Path Integrals with Spinor Fields

- We will not prove anything relating to spinor fields, but rather just present some results.

- First, we know that spinors are different, by which the fields obey *anti*-commutation relations. Because of this, we introduct anti-commuting numbers, called the **Grassmann algebra**. For such numbers ζ, θ, we have

$$\{\zeta, \theta\} = \zeta\theta + \theta\zeta = 0, \quad \text{or} \quad \zeta\theta = -\theta\zeta. \tag{12.55}$$

- Interestingly, this implies that $\zeta^2 = \theta^2 = 0$, meaning that any polynominal we construct with these terms truncates after the linear term.

- Now, the free Dirac field (thus only containing a kinetic and mass term for a spinor field) is given by

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi, \tag{12.56}$$

- We now introduce two sources, η for $\overline{\psi}$ and $\bar{\eta}$ for ψ. With this,

$$Z_0[\eta, \bar{\eta}] = N \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \ \exp\left[i \int (i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi + \bar{\eta}\psi + \overline{\psi}\eta)\right]. \tag{12.57}$$

- With this, the Dirac propagator comes out from

$$S(x-y) = \left\langle 0 \,\middle|\, T\left\{\psi(x)\overline{\psi}(y)\right\} \,\middle|\, 0\right\rangle = -\frac{\delta^2 Z_0[\eta, \bar{\eta}]}{\delta\bar{\eta}(x)\delta\eta(y)}\bigg|_{\eta=\bar{\eta}=0}. \tag{12.58}$$

- For interacting spinor fields, we have a similar story where

$$Z[\eta, \bar{\eta}] = \exp\left[i \int \mathcal{L}_{\text{int}}\left(\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}\right) \ \mathrm{d}^4x\right] Z_0[\eta, \bar{\eta}]. \tag{12.59}$$

Final Note

- This path integral formalism was done very quickly, and with little derivation. Rather, it was just to present some results so that they are familiar, and so we could get a taste of another type of formalism as opposed to canonical quantization that we did before. We will continue no further with this subject.

13 The Standard Model

- Here we start to delve into some more stuff that is more related to the Standard Model. In class we started with describing particles and some basic conserved quantities as well as the CPT theorem, but I already know all this so I won't write it here. Instead, I will jump straight to the Yang-Mills theory stuff.

Yang-Mills Theories

- We now consider two spinor fields, ψ_a and ψ_b. With this, the Lagrangian (without gauge field stuff yet) is basically two identical copies for each spinor field:

$$\mathcal{L}_{\text{Dirac}} = i\overline{\psi}_a\gamma^\mu\partial_\mu\psi_a - m\overline{\psi}_a\psi_a + i\overline{\psi}_b\gamma^\mu\partial_\mu\psi_b - m\overline{\psi}_b\psi_b. \tag{13.1}$$

- The next thing we can do is combine the two fields into one new field:

$$\psi = \begin{pmatrix}\psi_a \\ \psi_b\end{pmatrix} \quad \rightarrow \quad \overline{\psi} = \begin{pmatrix}\overline{\psi}_a & \overline{\psi}_b\end{pmatrix}. \tag{13.2}$$

- With this, we recover (almost) the more familiar Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - \overline{\psi}M\psi, \tag{13.3}$$

  where now the capital M is a matrix:

$$M = \begin{pmatrix}m_a & 0 \\ 0 & m_b\end{pmatrix}. \tag{13.4}$$

- Let's now do a global gauge transformation $\psi \rightarrow \psi' = U\psi$, where U is some unitary 2×2 matrix (meaning $UU^\dagger = U^\dagger U = 1$; also $U^\dagger = U^{-1}$).

- We know from before that we can write any unitary matrix as the complex exponential of a Hermitian matrix: $U = e^{iH}$ (H is not the Hamiltonian here, but an arbitrary Hermitian matrix). Considering that we are working in the space of 2×2 matrices, this is the group $U(2)$.

- The last simplification we can make is to assume mass degeneracy, meaning $m_a = m_b = m$, where we actually do recover the familiar Dirac Lagrangian (so I won't type it).

- Just as before, doing a global transformation will leave this Lagrangian invariant. We can push the U past the derivatives, and it commutes with the gammas since it is a completely different space (U is a $2x2$, γ^μ is a 4-vector of 4×4 matrices), so everything cancels nicely.

- Now, in general, in $U(2)$, we can express the Hermitian matrix as $H = \phi + \sigma^i\theta^i$, where $i = 1, 2, 3$ (so I could've used normal 3-vector notation, but I choose to avoid that). Here, ϕ is just a phase factor. We associate this with $U(1)$ in EM, so we want to eliminate that here and just focus on the other part with the Pauli matrices. The way we can do this is to require that the determinant of U is 1, which has the effect of removing that phase factor.

- With this, we are now working in $SU(2)$, where $U = e^{i\sigma^i\theta^i}$.

- We now promote this symmetry to a local one, just like before, where the angles now depend on the space-time position: $\theta^i \to \theta^i(x)$ (x is assumed to be the 4-vector x^μ). Of course, we are no longer able to push the U past the derivative anymore, and we pick up an extra term.
- The way to alleviate this is to add another term in the Lagrangian involving gauge fields (3, to be exact, since we have 3 generators and three terms that come from the 4-derivative).
- First, let's re-express the angles by bringing out a scalar in front: $\theta(x) \to q\lambda(x)$. Then, we can add the term $-q(\overline{\psi}\gamma^\mu\sigma^i\psi)A^i_\mu$. There's a lot going on here! We have spinors with four terms, a 4-vector that is in a different space of the spinors containing 4×4 matrices that are back again in spinor space, a 3-vector of 2×2 matrices, etc.
- Anyway, these new scalar fields need to transform in a certain way to respect this local symmetry. Actually, we also need to consider the contraction between the Pauli matrices and the gauge fields:

$$\sigma^i A^i_\mu \to U\sigma^i A^i_\mu U^\dagger + \frac{i}{q}(\partial_\mu U)U^\dagger. \tag{13.5}$$

- Our Lagrangian is now

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - q(\overline{\psi}\gamma^\mu\sigma^i\psi)A^i_\mu. \tag{13.6}$$

- Testing out the local transformation:

$$\begin{aligned}\mathcal{L}' &= i\overline{\psi}U^\dagger\gamma^\mu\partial_\mu(U\psi) - m\overline{\psi}U^\dagger U\psi - q\overline{\psi}U^\dagger\gamma^\mu U\sigma^i A^i_\mu U^\dagger U\psi - q\overline{\psi}U^\dagger\gamma^\mu\frac{i}{q}(\partial_\mu U)U^\dagger U\psi &(13.7)\\ &= i\overline{\psi}U^\dagger\gamma^\mu(\partial_\mu U)\psi + i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu\sigma^i A^i_\mu\psi - i\overline{\psi}U^\dagger\gamma^\mu(\partial_\mu U)\psi &(13.8)\\ &= i\overline{\psi}\gamma^\mu\partial_\mu - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu\sigma^i A^i_\mu\psi. &(13.9)\end{aligned}$$

- This is exactly our original Lagrangian, so this local $SU(2)$ transformation is a symmetry!

- We can combine everything into the covariant derivative

$$D_\mu = \partial_\mu + iq\sigma^i A^i_\mu, \tag{13.10}$$

  wherein our Lagrangian becomes

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi, \tag{13.11}$$

  which is essentially just the normal Dirac Lagrangian.
- As before, when considering the gauge terms, mass terms like $\sim m_A^2 A^i_\mu A^{i,\mu}$ cannot be gauge invariant, so we must have that $m_A = 0$. We will still have a similar looking kinetic term, but with just a few extra components. We will have $-F^i_{\mu\nu}F^{i,\mu\nu}$, where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + iqf^{ijk}A^j_\mu A^k_\nu. \tag{13.12}$$

- Here, f^{ijk} are the structure constants for the group $SU(2)$, defined by

$$[\sigma_i, \sigma_j] = if_{ijk}\sigma_k = 2i\epsilon_{ijk}\sigma_k, \tag{13.13}$$

  where ϵ_{ijk} is the Levi-Civita. Thus, in 3-vector land, we can write this as

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + 2q(\mathbf{A}_\mu \times \mathbf{A}_\nu)^i. \tag{13.14}$$

- To make it a bit more familiar, we can consider a small λ where λ was defined by $\mathbf{Q} = q\lambda$, where an element of $SU(2)$ can now be found by $U = e^{iq\sigma^i\lambda^i}$. For small λ, then, we can expand the exponential in a power series and just keep the linear term:

$$U = 1 + iq\sigma^i\lambda^i + O(\lambda^2). \tag{13.15}$$

- Then, it can be shown that instead of having a transformation rule for the combined quantity $\sigma^i A^i_\mu$, we can just impose
$$A^i_\mu \to A^i_\mu - \partial_\mu \lambda^i - iqf^{ijk}\lambda^i A^k_\mu \tag{13.16}$$
and still have our Lagrangian be gauge invariant.

14 Electroweak Theory

- We will now apply this Yang-Mills theory stuff to some more concrete field theories.
- Recall: back when we were discussing the Dirac equation and stuff, we found that we were able to write a generic spinor in terms of its left and right handed components:
$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{14.1}$$
where
$$(\boldsymbol{\sigma}\cdot\mathbf{p})\psi_R = \psi_R \quad \text{and} \quad (\boldsymbol{\sigma}\cdot\mathbf{p})\psi_L = -\psi_L, \tag{14.2}$$
with $\boldsymbol{\sigma}\cdot\mathbf{p}$ being the helicity operator. This means that right-handed particles have positive helicity and left-handed particles have negative helicity.
- We also had the fifth gamma matrix, which is generally defined as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. In the chiral representation, we have
$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14.3}$$
- This matrix can be used to define the projection operator:
$$\psi_R = \frac{1-\gamma^5}{2}\psi \quad \text{and} \quad \psi_L = \frac{1+\gamma^5}{2}\psi. \tag{14.4}$$
- Now, with this, we can actually split up the Dirac equation into the sum of the left and right handed bits. Ignoring gauge and mass terms for now,
$$\mathcal{L} = i\overline{\psi}_R\gamma^\mu\partial_\mu\psi_R + i\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L. \tag{14.5}$$
- In light of this, we can consider the left and right handed components of the leptons (the same prescription will work for the quark sector). The left-handed component will actually be a doublet, which we will denote as capital L:
$$L = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L. \tag{14.6}$$
- Since (for this more simple, older theory) there are no right handed neutrinos, the right handed component is just a singlet consisting of the right handed electron e_R.
- Now, each of these handed components will get assigned an $SU(2)$ "charge", called **weak isospin**.[5] The doublet gets $I_w = \frac{1}{2}$, while the singlet gets $I_w = 0$, signifying that with a neutral charge, it does not interact with the force decscribed by this theory.
- Just as with ordinary spin in quantum mechanics, we only consider one of the three components, specifically the third, I^3_w, and the ν_e gets $I^3_w = \frac{1}{2}$ and the (left-handed) e gets $I^3_w = -\frac{1}{2}$.
- With this, we can write the Lagrangian (still without gauge or mass terms) as
$$\mathcal{L} = i\bar{e}_R\gamma^\mu\partial_\mu e_R + i\bar{L}\gamma^\mu\partial_\mu L. \tag{14.7}$$

[5] This naming comes from the earlier idea that the proton and neutron were really eigenstates of the same particle, differing only by their "isospin". This is honestly a bit of a unfortunate naming convention, but it's easier to stick with it than try and make up a bunch of new stuff.

- We can now consider the application of our Yang-Mills $SU(2)$ to this new handed theory. As I mentioned before, we know already that an element of this group will not transform the singlet (so $e_R \to e_R$.) The doublet will transform like

$$L \to e^{i\sigma^i \theta^i/2}, \tag{14.8}$$

where the factor of $\frac{1}{2}$ is because the generators of $SU(2)$ in the fundamental representation are different from the Pauli matrices by a factor of two.[6]

- However, there is nothing saying that we cannot also still have $U(1)$ phase rotations. This will not be exactly the EM symmetry we have already studied, but it will be close (and the machinery will of course be the same).

- With this, both of the handed components will transform this time:

$$L \to e^{i\beta_1/2} L \quad \text{and} \quad e_R \to e^{i\beta_2/2} e_R. \tag{14.9}$$

- As I just mentioned, this $U(1)$ is different from EM; the "charge" here is called **(weak) hypercharge** Y (or if we choose to add the "weak" to the name, we would write Y_w. I won't do this, though). There is a relation (that I have no idea how it was derived) that relates the hypercharge, ordinary EM charge, and the third component of weak isospin together called the **Gell-Mann-Nishijima relation**:

$$Q = I_w^3 + \frac{Y}{2}. \tag{14.10}$$

- Since we of course already know the charges of the particles and we just defined their weak isospins, we find that $Y(e_L) = Y(\nu_{e,L}) = -1$ and $Y(e_R) = -2$. We can then relate β_1 and β_2 to a singular β and write

$$L \to e^{i\beta/2} L \quad \text{and} \quad e_R \to e^{i\beta} e_R. \tag{14.11}$$

- Putting all this together, and bringing in interactions through the covariant derivative that I'll define in just a moment, our Lagrangian is

$$\mathcal{L} = i\bar{L}\gamma^\mu D_\mu L + i\bar{e}_R \gamma^\mu D_\mu e_R. \tag{14.12}$$

- As a quick note: L is a doublet consisting of two *spinors*, named e_L and $\nu_{e,L}$. Similarly, e_R is a *spinor*, we are just naming it this way to differentiate between the other spinors.

- This Lagrangian is now invariant under $SU(2)_L \otimes U(1)_Y$. We will have a single vector gauge field corresponding to the $U(1)_Y$ symmetry which we will denote as B_μ, and a similar case for our Yang-Mills formulation for $SU(2)_L$, where we will denote the three gauge bosons as $W^i_{\mu\nu}$.

- Therefore, we can define the action of the covariant derivatives per group. Generally, the total covariant derivative can be written as a sum of the contributions from each group:

$$D_\mu = \partial_\mu + \frac{i}{2} g \sigma^i W^i_\mu + \frac{i}{2} g' Y B_\mu, \tag{14.13}$$

where g and g' are the couplings to the gauge fields.

- Now, the action of this will be different on the different representations of the groups. For instance, considering the action of the covariant derivative on the left-handed doublet, we have both contributions:

$$D_\mu L = \left(\partial_\mu + g \frac{i}{2} \sigma^i W^i_\mu - \frac{i}{2} g' B_\mu \right) L, \tag{14.14}$$

where $Y = -1$ for the left-handed components. It is understood that the $SU(2)_L$ part acts on the doublet itself, while the $U(1)$ part acts on each Weyl spinor individually.

[6] My complete guess of why we choose to use something that is only such a factor off has to do with the fact that our eigenvalues and such are all 1/2, as the doublet has $I_w = 1/2$ and the two eigenstates (the electron and electron neutrino) have $I_w^3 = \pm 1/2$, so it is natural to define the generators in that way.

- Similarly,
$$D_\mu e_R = (\partial_\mu - ig' B_\mu)\, e_R, \tag{14.15}$$
where there is no action from $SU(2)_L$ since it is a singlet under that representation.

- We also distinguish between the kinetic terms for the B boson and the W bosons individually:
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad \text{and} \quad W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu \partial^i_\mu + ig f^{ijk} W^j_\mu W^k_\nu. \tag{14.16}$$

- Writing out everything, we have:
$$\mathcal{L} = i\bar{L}\gamma^\mu D_\mu L + i\bar{e}_R \gamma^\mu D_\mu e_R - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i,\mu\nu}. \tag{14.17}$$

- It is important to note that we have all fields as being massless currently. This is intended, as all this will eventually lead to the Higgs mechanism, which breaks this symmetry down to just a $U(1)_{\text{em}}$ symmetry and gives the fields masses.

15 The Higgs Mechanism

- Let's define a complex scalar field ϕ called the **Higgs** field. It'll carry a weak isospin $I_w = 1/2$, and thus we represent it as a doublet like
$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{15.1}$$

- It also carries hypercharge $Y = 1$, so we have that
$$D_\mu \Phi = \partial_\mu \Phi + \frac{i}{2} g \sigma^i W^i_{\mu\nu} \Phi + \frac{i}{2} g' B_\mu \Phi, \tag{15.2}$$
and it will carry additional stuff in the Lagrangian from the Klein-Gordon Lagrangian:
$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger D^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 - G(\bar{L}\Phi e_R + \bar{e}_R \Phi^\dagger L), \tag{15.3}$$
where G is some other coupling between the Higgs, the left-handed doublet, and the right-handed singlet. λ is a Yukawa coupling, a coupling we parametrize differently as it is a coupling between scalar particles.

15.1 Spontaneous Symmetry Breaking

- (The previous section was I guess just an intro). To delve into the Higgs stuff, we consider a simpler model consisting of a single complex scalar field ϕ and some gauge field A^μ (meaning $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$).

- The Klein-Gordon Lagrangian terms for this are
$$\mathcal{L}_\phi = (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2, \tag{15.4}$$
where, as always $D_\mu = \partial_\mu + iqA_\mu$.

- Now, where this deviates from normal complex scalar theory is we consider $m^2 < 0$, and define $\mu^2 = -m^2$ so that
$$\mathcal{L}_\phi = (D_\mu \phi)^* D^\mu \phi + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2. \tag{15.5}$$
At this point, $\phi = 0$ is no longer the ground state, it is actually a local maximum. Let's consider just the potential terms:
$$U = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2, \tag{15.6}$$

so, to find the value of ϕ for which the potential is a maximum we simply do the derivative and set it equal to zero:

$$\frac{\mathrm{d}U}{\mathrm{d}\phi} = 0 \to -\mu^2\phi^* + 2\lambda(\phi^*)^2\phi = 0 \tag{15.7}$$

$$\to \phi^*\phi = \frac{\mu^2}{2\lambda} \to \frac{1}{2}(\phi_1^2 + \phi_2^2) = \frac{\mu^2}{2\lambda} \to \phi_1^2 + \phi_2^2 = \frac{\mu^2}{\lambda}. \tag{15.8}$$

We can select, for instance, $\phi_2 = 0$ and $\phi_1 = \mu/\sqrt{2}$ which gives us, after plugging in, that $U_{\min} = -\mu^4/4\lambda$.

- Now, we can select two new fields $\eta = \phi_1 - \mu/\sqrt{\lambda}$ and $\xi = \phi_2$ so that

$$\phi = \frac{1}{\sqrt{2}}\left(\eta + \frac{\mu}{\sqrt{\lambda}} + i\chi\right). \tag{15.9}$$

At this point, U is now a minimum when both $\eta = \chi = 0$.

- With our new fields, we can do a bunch of algebra to rewrite the Lagrangian in terms of these new fields:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \mu^2\eta^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{q^2\mu^2}{2\lambda}A_\mu A^\mu + \frac{\mu^4}{4\lambda} + \frac{1}{2}q^2A_\mu A^\mu\left(\eta^2 + \frac{2\eta\mu}{\sqrt{\lambda}} + \chi^2\right) \tag{15.10}$$

$$+ qA^\mu\left(\eta\partial_\mu\chi - \chi\partial_\mu\eta + \frac{\mu}{\sqrt{2}}\partial_\mu\chi\right) - \frac{\lambda}{4}\left(\eta^4 + \chi^4 + 4\eta^3\frac{\mu}{\sqrt{\lambda}} + 2\eta^2\chi^2 + 4\eta\frac{\mu\chi^2}{\sqrt{\lambda}}\right). \tag{15.11}$$

- That's a lot. However, what is important to notice here is that there is not actually a mass term for the χ field! We call such a field a **Goldstone Boson**. It is given a special name because any spontaneous breaking of a symmetry entails the existsence of some massless particle.

- Interestingly, though, the gauge field A^μ now has a mass. We can entirely eliminate the χ field by performing a phase rotation (valid under local gauge invariance) where $\tan\theta = -\phi_2/\phi_1$. If we perform this rotation on a complex number(/field), the imaginary part vanishes, thus eliminating the Goldstone boson field. With this, our Lagrangian is now:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \mu^2\eta^2 + \frac{1}{2}m_A^2A_\mu A^\mu + \frac{q^2}{2}A_\mu A^\mu\left(\eta^2 + \frac{2\mu}{\sqrt{\lambda}}\eta\right) - \frac{\lambda}{4}\left(\eta^4 + \frac{4\mu}{\sqrt{\lambda}}\eta^3\right) + \frac{\mu^4}{4\lambda}. \tag{15.12}$$

- At this point, the interpretation of this is that we eliminated the Goldstone boson χ, and the gauge field "ate" it, giving it mass. The remaining η is now our massive scalar Higgs field.

15.2 Generalization to EW Theory

- This generalizes to (GWS) EW theory. The difference is that now we must consider a doublet

$$\varphi(x) = \begin{pmatrix} 0 \\ \rho + \frac{1}{\sqrt{2}}\phi(x) \end{pmatrix} \tag{15.13}$$

where $\rho = \mu/\sqrt{2\lambda}$.

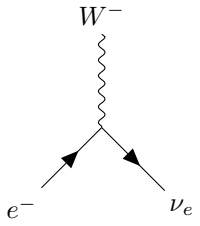
- I won't go into any of the funny business with what the Lagrangian looks like, especially because there is nothing really new involved. This time, though, we end up with four total gauge bosons, three W's and one B. By doing rotations and eating up of Goldstone bosons, the B remains massless and becomes the photon, the three W's pick up mass, and the rotations recombine them in particular ways to give the $W^\pm$ and the Z (sometimes written Z^0 boson).

15.3 Feynman Rules for EQ Theory

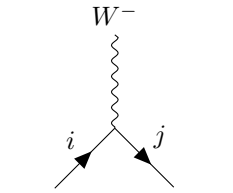
- With (precious little) formalism, we now present the Feynman rules for EW theory. The propagator for massive gauge bosons is given by

$$\rightarrow \quad \frac{-i\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right]}{p^2 - m^2 + i\epsilon}.$$

- We can also write the possible verices. There can be a flavor changing reaction mediated by either W which converts a lepton into its neutrino, or converts a quark to its corresponding other quark (for instance, from a u to a d):



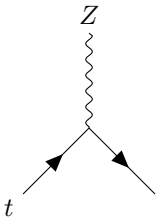
$$\rightarrow \quad \frac{-i\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right]}{p^2 - m^2 + i\epsilon}.$$



$$\rightarrow \quad \frac{-ie}{2\sqrt{2}\,\sin\theta_W}\gamma^\mu(1-\gamma^5)V_{ij},$$

where V_{ij} is an element of the CKM matrix, which defines how quark mixing occurs.

- There are also neutral current processes, but these cannot be flavor changing (in the current SM, at least):



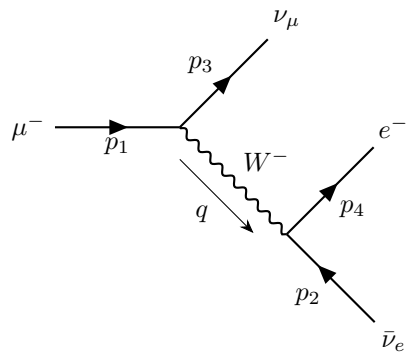
$$\rightarrow \quad \frac{-ie}{\sin(2\theta_W)}\gamma^\mu(c_v^t - c_A^t),$$

where c_v and c_A are defined differently for the different quarks and leptons (hence the superscript t).

- This theory also admits self-interactions, so there are tons of different diagrams with three and four bosons interacting, including the Higgs. I won't write any of them thouhg, because I don't think we really look at them.

15.4 Muon Decay

- We are now in a position to write the amplitude for the decay of the muon into an electron, a muon neutrino, and an anti electron neutrino.



- First, we note that $p_1 = p_2 + p_3 + p_4$ and $q = p_1 - p_3 = p_2 + p_4$. Now, we can write down that

$$i\mathcal{M} = \bar{u}(p_3)\frac{(-ie)\gamma^\mu(1-\gamma^5)}{2\sqrt{2}\,\sin\theta_W}u(p_1)\cdot\frac{(-i)\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{m_W^2}\right)}{q^2-m_W^2}\cdot\bar{u}(p_4)\frac{(-ie)\gamma^\nu(1-\gamma^5)}{2\sqrt{2}\,\sin\theta_W}v(p_2). \tag{15.14}$$

  We can assume that $q^2 << m_W^2$, in which the term in parentheses in the propagator simplifies to just the metric and the denominator turns into $q^2 - m_W^2 \to -m_W^2$. So,

$$\mathcal{M} = -\frac{e^2}{8\sin^2\theta_W\, m_W^2}\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)\bar{u}(p_4)\gamma_\mu(1-\gamma^5)v(p_2). \tag{15.15}$$

  After some more work we find that

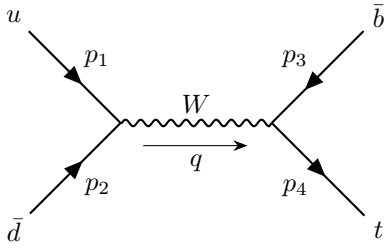
$$|\mathcal{M}|^2 = \frac{2e^4}{\sin^4\theta_W\, m_W^4}(p_1\cdot p_2)(p_3\cdot p_4). \tag{15.16}$$

- From here, we can find the decay rate via Fermi's Golden Rule:

$$\mathrm{d}\Gamma = \frac{|\mathcal{M}|^2}{2E_1}(2\pi)^4\delta^4(p_1-p_2-p_3-p_4)\prod_{i=2}^{4}\frac{\mathrm{d}^3p_i}{(2\pi)^3 2E_i}. \tag{15.17}$$

15.5 Single Top-Quark Production

- We also have the ability to singly produce top quarks via some flavor-changing process. As an example, we have the diagram of



- Again, we note that $p_1 + p_2 = p_3 + p_4$ and $q = p_1 + p_2 + p_3 + p_4$. The amplitude is

$$i\mathcal{M} = \bar{u}(p_4)\frac{(-ie)\gamma^\mu(1-\gamma^5)V_{tb}}{2\sqrt{2}\sin\theta_W}v(p_3)\frac{(-i)\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{m_W^2}\right)}{q^2-m_W^2}\bar{v}(p_2)\frac{(-ie)\gamma^\nu(1-\gamma^5)V_{ud}}{2\sqrt{2}\sin\theta_W}u(p_1) \tag{15.18}$$

$$\mathcal{M} = \frac{e^2V_{tb}V_{ud}}{7\sin^2\theta_W[(p_1+p_2)^2-m_W^2]}\bar{u}(p_4)\gamma^\mu(1-\gamma^5)v(p_3) \times\left(g_{\mu\nu}-\frac{(p_1+p_2)_\mu(p_1+p_2)_\nu}{m_W^2}\right)\bar{v}(p_2)\gamma^\nu(1-\gamma^5)u(p_1). \tag{15.19}$$

Again, after quite a bit more work, we find

$$|\mathcal{M}|^2 = \frac{4\pi^2\alpha^2 V_{tb}^2 V_{ud}^2}{\sin^4\theta_W}\frac{t(t-m_t^2)}{(s-m_W^2)^2}, \tag{15.20}$$

and we can plug this into our formula for the differential cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|}{16\pi s^2}. \tag{15.21}$$

16 QCD

- QCD is the consideration of local $SU(3)$ gauge transformations which come in the form

$$\psi(x) \to \psi'(x) = e^{iT^a\theta^a}, \tag{16.1}$$

where T^a is a vector of the 8 generators of $SU(3)$, which are the analog of the Pauli matrices (times two).

- In this "dimension", we admit three types of **color charge**, along with 8 gluons.
- The generators follow the same generalized commutation relations:

$$[T^a, T^b] = if^{abc}T^c. \tag{16.2}$$

- We define the covariant derivative as

$$D_\mu = \partial_\mu + ig_s T^a G_\mu^a, \tag{16.3}$$

where G_μ^a denote the 8 gluon fields. Under a local $SU(3)$ gauge transformation, these fields must transform like

$$G_\mu^a \to G_\mu^a - \frac{1}{g_s}\partial_\mu\theta^a - f^{abc}\theta^b G_\mu^c. \tag{16.4}$$

- The QCD Lagrangian, at this point, is given by

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi - \frac{1}{4}G_{\mu\nu}^a G^{a,\mu\nu}, \tag{16.5}$$

where there are implicit color (and flavor) indices on the quarks.

- The generators, just like with the $SU(2)$ generators, have a more common matrix representation which is $T^a = \frac{1}{2}\lambda^a$, where λ^a are the **Gell-Mann matrices**, and it is these that are analogous to the Pauli matrices. I'm not going to write any of them out here.
- There is a more complicated case when it comes to the structure constants, though. Fortunately, most of them are zero. I also will not write those here, nor will I write any of the identies or anything else that I can just look up in the back of a QFT book or something.

16.1 Ghosts

- Now that gluons are able to couple to themselves, we admit spurious degrees of freedom that we parametrize via what are called "ghost" diagrams. These only appear in close loops and take into account these extra degrees of freedom. We either need to include diagrams containing these ghosts, or use a different (and more often more complex) gauge.

17 QCD Renormalization

- Let's first consider the analog of the lepton self-energy, in which we have a quark and gluon (and color factors). The kinematics are actually identical, so we can say that

$$\Sigma_{\text{QCD}}(p) = C_F \Sigma_{\text{QED}}(p) = \frac{g_s^2}{6\pi^2\epsilon}(\not{p} - 4m) + \mathcal{O}(\epsilon^0). \tag{17.1}$$

  Note that this is not just arbitrary, i.e. we are not just adding this color factor because there *should* be one; if we do the actual calculation we find that this just happens to be the case.

- Next, we do a similar move by redefining the field like so:

$$\psi_b = \sqrt{Z_\psi}\psi \quad \text{with} \quad Z_\psi = 1 - \frac{g_s^2}{6\pi^2\epsilon}. \tag{17.2}$$

- The vacuum polarization diagrams are a little more complex, since the gluon can self couple. Additionally, as it is a loop, there is also the contribution from ghosts (assuming a Feynman gauge). We won't do any of these calculations, but it turns out, the 4-gluon vertex is zero as it is proportional to $\int \mathrm{d}^n k/k^2$, which we have found to be zero.

- The quark loop one is $n_f/2$ times the QED result, where n_f is proportional to the number of flavors under consideration. In general this is of course 6, but when doing calculations we often neglect top quarks or even bottom quarks due to their high mass, in which case we'd only consider the other four quarks and set $n_f = 4$. So,

$$\Pi^{\mu\nu}_{ab,\text{quark-loop}}(k) = \frac{n_f}{2}\delta_{ab}\frac{g_s^2}{6\pi^2\epsilon}(k^\mu k^\nu - g^{\mu\nu}k^2) + \mathcal{O}(\epsilon^0). \tag{17.3}$$

- The gluon loop is a little more complex, but still follows the same structure:

$$\Pi^{\mu\nu}_{ab,\text{gluon-loop}}(k) = \frac{-g_s^2}{16\pi^2\epsilon}f^{acd}f^{bcd}(\frac{11}{3}k^\mu k^\nu - \frac{19}{6}g^{\mu\nu}k^2) + \mathcal{O}(\epsilon^0). \tag{17.4}$$

- Lastly, the ghost loop looks like

$$\Pi^{\mu\nu}_{ab,\text{ghost-loop}}(k) = \frac{g_s^2}{16\pi^2\epsilon}f^{acd}f^{bcd}(\frac{1}{3}k^\mu k^\nu + \frac{1}{6}g^{\mu\nu}k^2) + \mathcal{O}(\epsilon^0). \tag{17.5}$$

- Summing over all contributions:

$$\Pi^{\mu\nu}_{ab}(k) = \frac{g_s^2}{24\pi^2\epsilon}\delta_{ab}(5C_A - 2n_f)(g^{\mu\nu}k^2 - k^\mu k^\nu) + \mathcal{O}(\epsilon^0). \tag{17.6}$$

- With all of this, we redefine the gluon field like so:

$$(G_b)^a_\mu = \sqrt{Z_G}G^a_\mu, \quad \text{with} \quad Z_G = 1 + \frac{g_s^2}{24\pi^2\epsilon}(5C_A - 2n_f). \tag{17.7}$$

- Vertex diagrams follow similarly. There is the QED analog, then one with a 3-gluon vertex, and end up with

$$\Lambda^a_\mu = -\frac{g_s^3}{8\pi^2\epsilon}(C_F + C_A)\gamma_\mu T^a. \tag{17.8}$$

  With some dimensional analysis, we redefine the bare coupling like so

$$g_{b,s} = \frac{Z_L}{Z_\psi \sqrt{Z_G}} g_s \mu^{\epsilon/2}, \tag{17.9}$$

where μ is a renormalization scale that keeps the actual coupling dimensionless, and Z_L is defined like $\mathcal{L}_{b,\text{int}} = Z_L \mathcal{L}_{\text{int}}$.

- We find that

$$g_{b,s} = \left[1 - \frac{g_s^2}{48\pi^2\epsilon}\left(11C_A - 2n_f\right)\right] g_s \mu^{\epsilon/2}. \tag{17.10}$$

- Now, the *bare* strong coupling should be entirely independent of the renormlization scale, as it is the coupling we are supposed to be able to see at an infinitely large scale:

$$\frac{\partial g_{b,s}}{\partial \mu} = 0 \quad \rightarrow \quad \beta(g_s) = \mu \frac{\partial g_s}{\partial \mu} = -\frac{g_s^3}{48\pi^2}(11C_A - 2n_f) = -\frac{g_s^3}{16\pi^2}\beta_0. \tag{17.11}$$

- This time, we find that $\beta(g_s) < 0$, meaning that with increaing energy, we get a decreasing coupling. This phenomena is called **Asymptotic Freedom**, and was an absolutely massive discovery.

- We can make the alternative definition (after a little bit of work)

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \alpha_s(\mu_0)\frac{\beta_0}{4\pi}\log\frac{\mu^2}{\mu_0^2}}, \tag{17.12}$$

where we must know the coupling at some initial energy scale μ_0. If we do, then we can calculate the new energy scale at any μ.

- What we can also do is define a "QCD Scale" Λ that is characteristic of typical QCD energies (usually it is around $\sim 200\,\text{MeV}$ like

$$\log\Lambda^2 = \log\mu_0^2 - \frac{4\pi}{\beta_0 \alpha_s(\mu_0)}. \tag{17.13}$$

With this,

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \log\frac{\mu^2}{\Lambda^2}}. \tag{17.14}$$

18 Final Words

We covered some stuff specific to soft gluons on the final day. I won't be including any of that here, mostly because I am lazy, but also because it is less related to a more general QFT course, and is something more specific to what Dr. Kidonakis does for his research. I doubt it is what I'm going to go into specifically; this is the other reason why I won't be typing it all out.