<u>QED Lagrangian</u> $\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - q\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

or $\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $D_\mu = \partial_\mu + iqA_\mu$ is the covariant derivative

<u>Euler-Lagrange Equations</u> for $\psi, \bar{\psi}, A^\mu$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi} \Rightarrow \partial_\mu(i\bar{\psi}\gamma^\mu) = -m\bar{\psi} - q\bar{\psi}\gamma^\mu A_\mu \Rightarrow i\partial_\mu\bar{\psi}\gamma^\mu = -m\bar{\psi} - q\bar{\psi}\gamma^\mu A_\mu$$
$$\Rightarrow i(\partial_\mu - iqA_\mu)\bar{\psi}\gamma^\mu = -m\bar{\psi} \Rightarrow iD_\mu\bar{\psi}\gamma^\mu = -m\bar{\psi}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} \Rightarrow \partial_\mu(0) = i\gamma^\mu\partial_\mu\psi - m\psi - q\gamma^\mu\psi A_\mu \Rightarrow i\gamma^\mu\partial_\mu\psi - q\gamma^\mu\psi A_\mu = m\psi$$
$$\Rightarrow i\gamma^\mu(\partial_\mu + iqA_\mu)\psi = m\psi \Rightarrow i\gamma^\mu D_\mu\psi = m\psi$$

This is simply the Dirac equation with $\partial_\mu \to D_\mu$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial\mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu\left(-(\partial^\mu A^\nu - \partial^\nu A^\mu)\right) = -q\bar{\psi}\gamma^\nu\psi \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu$$

where the current $j^\mu = q\bar{\psi}\gamma^\mu\psi$ acts as the source of gauge field A^μ

These contain the inhomogeneous Maxwell equations (Gauss' law for electric fields and the Ampere-Maxwell law).

We can also rewrite $\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - j^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

Also, $\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \partial_\nu j^\nu = \partial_\nu\partial_\mu F^{\mu\nu} = 0$ continuity equation $\rightarrow$ conservation of charge

Electromagnetic field

The 4-potential $A^\mu = (V, \vec{A})$ where V is the scalar (electric) potential and $\vec{A}$ is the vector potential

The electric and magnetic fields are given by

$$\vec{E} = -\vec{\nabla} V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

Then $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

Also $j^\mu = (\rho, \vec{j})$

Then $\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \partial_\mu F^{\mu 0} = j^0 \Rightarrow \vec{\nabla} \cdot \vec{E} = \rho$ Gauss' law for $\vec{E}$

with $\rho = j^0 = q\bar{\psi}\gamma^0\psi = q\psi^\dagger\psi$

and $\partial_\mu F^{\mu i} = j^i$ (with $i = 1,2,3$) $\Rightarrow \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$ Ampere-Maxwell law

Also note that $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ (div curl = 0) which is Gauss' law for $\vec{B}$

and $\vec{\nabla} \times \vec{E} = -\underbrace{\vec{\nabla} \times \vec{\nabla} V}_{=0 \text{ (curl grad} = 0)} - \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\vec{\nabla} \times \vec{A} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's law of induction

If we define $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ with $\varepsilon^{\mu\nu\rho\sigma}$ antisymmetric then

$\partial_\mu \tilde{F}^{\mu\nu} = 0$ which contains the homogeneous Maxwell equations

Gauge freedom $A_\mu \to A_\mu - \partial_\mu \lambda$

We can fix the gauge by imposing additional constraints

Lorenz gauge: $\partial_\mu A^\mu = 0$ reduces independent components of A^μ from four to three

still remaining freedom in this gauge if $\partial_\mu \partial^\mu \lambda = 0$

Additional constraints $V=0$ and $\vec{\nabla}\cdot\vec{A}=0$ Coulomb or radiation gauge

For $j^\mu = 0$ (free photon) $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$

$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 0$ In Lorenz gauge this becomes $\partial_\mu \partial^\mu A^\nu = 0$

which is the Klein-Gordon equation for a massless field – the photon field A^μ

Quantization of the electromagnetic field in Coulomb gauge

$$A(x) = \int \frac{d^3p}{(2\pi)^3 (2p^0)^{1/2}} \sum_{\lambda=1,2} \left[\varepsilon^{(\lambda)}(p)\, a^{(\lambda)}(p)\, e^{-ip\cdot x} + \varepsilon^{(\lambda)*}(p)\, a^{(\lambda)\dagger}(p)\, e^{ip\cdot x} \right]$$

where $\varepsilon^{(\lambda)}(p)$ are polarization vectors $\vec{\nabla}\cdot\vec{A}=0 \Rightarrow \vec{\varepsilon}\cdot\vec{p}=0$ and $\varepsilon^0 = 0$

So $\vec{\varepsilon}$ is perpendicular to direction of travel $\to$ transversely polarized

Two independent polarization vectors

Commutation relations $[a^{(\lambda)}(p), a^{(\lambda')\dagger}(p')] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}')\, \delta_{\lambda\lambda'}$

and $[a^{(\lambda)}(p), a^{(\lambda')}(p')] = 0$, $[a^{(\lambda)\dagger}(p), a^{(\lambda')\dagger}(p')] = 0$

Coulomb gauge quantization

Conjugate momenta $\pi^\mu = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$ with $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

Then $\pi^0 = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$ and $\pi^i = \dfrac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i$ for $i = 1, 2, 3$

So both A^0 and π^0 vanish – not dynamical variables

Commutation relations (equal-time)

$$[A^i(\vec{x},t), \pi^j(\vec{y},t)] = -i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(\delta^{ij} - \frac{p^i p^j}{\vec{p}^2} \right) = -i\, \delta^{(3)ij}_{\text{transverse}}(\vec{x}-\vec{y})$$

$$[A^i(\vec{x},t), A^j(\vec{y},t)] = 0 \;, \quad [\pi^i(\vec{x},t), \pi^j(\vec{y},t)] = 0$$

Hamiltonian $H = \frac{1}{2} \int d^3x\, (\vec{E}^2 + \vec{B}^2) = \int \frac{d^3p}{(2\pi)^3} \frac{p^0}{2} \sum_{\lambda=1,2} [a^{(\lambda)}(p) a^{(\lambda)\dagger}(p) + a^{(\lambda)\dagger}(p) a(p)]$

after normal ordering $H = \int \frac{d^3p}{(2\pi)^3} p^0 \sum_{\lambda=1,2} a^{(\lambda)\dagger}(p) a^{(\lambda)}(p)$ which is positive definite

So the operators $a(p)$ and $a^\dagger(p)$ are annihilation and creation operators for photons

Also note that $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$

Field quantization in Coulomb gauge is not manifestly Lorentz invariant because the conditions $V = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$ are not Lorentz invariant

However, the theory still has Lorentz invariance.