So $j^\mu = (\rho, \vec{j}) = \frac{i\hbar}{2m}\left(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}, -\varphi^* \vec{\nabla}\varphi + \varphi \vec{\nabla}\varphi^*\right)$

$= \frac{i\hbar}{2m}\left(\varphi^* \frac{\partial\varphi}{\partial t}, -\varphi^*\vec{\nabla}\varphi\right) - \frac{i\hbar}{2m}\left(\varphi\frac{\partial\varphi^*}{\partial t}, -\varphi\vec{\nabla}\varphi^*\right)$

$= \frac{i\hbar}{2m}\varphi^*\left(\frac{\partial\varphi}{\partial t}, -\vec{\nabla}\varphi\right) - \frac{i\hbar}{2m}\varphi\left(\frac{\partial\varphi^*}{\partial t}, -\vec{\nabla}\varphi^*\right) = \frac{i\hbar}{2m}\varphi^*\partial^\mu\varphi - \frac{i\hbar}{2m}\varphi\partial^\mu\varphi^*$

So $j^\mu = \frac{i\hbar}{2m}\left(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*\right)$

Then $\partial_\mu j^\mu = \frac{i\hbar}{2m}\left(\cancel{\partial_\mu\varphi^*\partial^\mu\varphi} + \varphi^*\partial_\mu\partial^\mu\varphi - \cancel{\partial_\mu\varphi\partial^\mu\varphi^*} - \varphi\partial_\mu\partial^\mu\varphi^*\right) = \frac{i\hbar}{2m}\left(-\varphi^*\frac{m^2c^2}{\hbar^2}\varphi + \varphi\frac{m^2c^2}{\hbar^2}\varphi^*\right) = 0$

So continuity equation holds.

However, $\rho = \frac{i\hbar}{2m}\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right)$ is <u>not</u> positive definite [unlike $\rho = \varphi^*\varphi = |\varphi|^2$ for Schrodinger eq.]

ρ can take negative values so cannot be a probability density

So we cannot consider the Klein-Gordon eq. as a single-particle equation with wavefunction φ. We will reinterpret it as a field equation with field φ. ρ and $\vec{j}$ are then charge and current densities. Note that φ is complex in general

If φ is real then ρ and $\vec{j}$ vanish $\rightarrow$ electrically neutral particle

<u>Negative energy problem:</u> $E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}$ are both solutions to $p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2$

An interacting particle could keep losing energy to $-\infty$, emitting an infinite amount of energy

In quantum field theory these solutions correspond to positive-energy <u>antiparticles.</u>

Dirac equation

Paul Dirac derived a relativistic quantum equation that has first-order derivatives in space and time in order to avoid the negative probabilities of the second-order Klein-Gordon wave equation

Start with $p^\mu p_\mu - m^2c^2 = 0$ and try a factorization

$$(\gamma^\mu p_\mu - mc)(\gamma^\nu p_\nu + mc) = 0$$

This equals $\gamma^\mu p_\mu \gamma^\nu p_\nu + \cancel{mc\gamma^\mu p_\mu} - \cancel{mc\gamma^\nu p_\nu} - m^2c^2 = 0 \Rightarrow \gamma^\mu p_\mu \gamma^\nu p_\nu - m^2c^2 = 0$

So we need $p^\mu p_\mu = \gamma^\mu p_\mu \gamma^\nu p_\nu \Rightarrow (i\hbar\partial^\mu)(i\hbar\partial_\mu) = \gamma^\mu(i\hbar\partial_\mu)\gamma^\nu(i\hbar\partial_\nu)$

$\Rightarrow \partial^\mu\partial_\mu = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu \quad \Rightarrow \quad g^{\mu\nu}\partial_\nu\partial_\mu = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$

But since $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, we have $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \gamma^\mu\gamma^\nu\partial_\nu\partial_\mu)$

$$= \frac{1}{2}(\underbrace{\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu}_{\text{"dummy indices"}} + \gamma^\mu\gamma^\nu\partial_\nu\partial_\mu) = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\nu\partial_\mu$$

Thus $g^{\mu\nu}\partial_\nu\partial_\mu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\nu\partial_\mu \Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

or $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ where $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$ is the anticommutator

Thus $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$, and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ for $\mu \neq \nu$

The Dirac gamma matrices are 4x4 matrices

standard representation $\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}$ where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and for $i = 1, 2, 3$, $\gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}$ where $\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

and $\sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli spin matrices

In 1928, <u>Dirac equation</u>: $(\gamma^\mu p_\mu - mc)\psi = 0$ or $i\hbar\gamma^\mu \partial_\mu \psi = mc\psi$

or (with $\hbar = c = 1$) $i\gamma^\mu \partial_\mu \psi = m\psi$ or $i\gamma \cdot \partial \psi = m\psi$ or $\not{p}\psi = m\psi$ with $\not{p} = \gamma^\mu p_\mu$

where ψ is a four-component Dirac spinor (<u>not</u> a 4-vector)

$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ We also define the adjoint spinor $\bar{\psi} = \psi^\dagger \gamma^0$

and note that $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$

Then $i\gamma^\mu \partial_\mu \psi = m\psi \Rightarrow -i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} = m\psi^\dagger \Rightarrow -i\partial_\mu \psi^\dagger \gamma^{\mu\dagger}\gamma^0 = m\psi^\dagger \gamma^0$

$\Rightarrow -i\partial_0 \psi^\dagger \gamma^{0\dagger}\gamma^0 - i\partial_i \psi^\dagger \gamma^{i\dagger}\gamma^0 = m\bar{\psi} \Rightarrow -i\partial_0 \psi^\dagger \gamma^0 \gamma^0 + i\partial_i \psi^\dagger \gamma^i \gamma^0 = m\bar{\psi}$

$\Rightarrow -i\partial_0 \bar{\psi}\gamma^0 - i\partial_i \psi^\dagger \gamma^0 \gamma^i = m\bar{\psi} \Rightarrow -i\partial_0 \bar{\psi}\gamma^0 - i\partial_i \bar{\psi}\gamma^i = m\bar{\psi} \Rightarrow -i\partial_\mu \bar{\psi}\gamma^\mu = m\bar{\psi} \Rightarrow i\partial_\mu \bar{\psi}\gamma^\mu = -m\bar{\psi}$

Then the 4-current $j^\mu = \bar{\psi}\gamma^\mu \psi$ is conserved: $\partial_\mu j^\mu = \partial_\mu \bar{\psi}\gamma^\mu \psi + \bar{\psi}\gamma^\mu \partial_\mu \psi = im\bar{\psi}\psi - im\bar{\psi}\psi = 0$