$$j^\mu = \bar{\psi}\gamma^\mu\psi \quad \text{so} \quad \rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = (\psi_1^* \ \psi_2^* \ \psi_3^* \ \psi_4^*)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

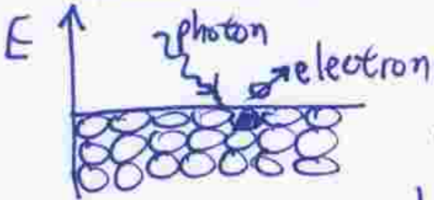
$$\Rightarrow \rho = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$$

So the problem with negative probabilities is resolved.

However, the problem with negative energies remains $E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}$

Two positive-energy solutions corresponding to the two states of a spin $\frac{1}{2}$ particle (electron) but also two negative-energy solutions

Hole theory - Dirac sea

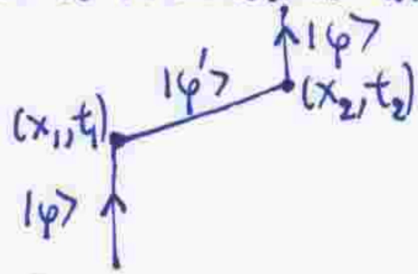


Dirac postulated that all negative-energy states are filled → Dirac sea, and by the Pauli exclusion principle, electrons or other spin-$\frac{1}{2}$ particles, cannot occupy them

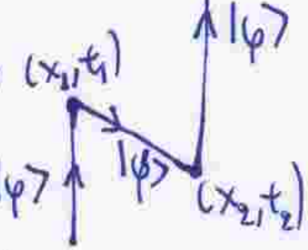
However, a photon could impart enough energy to a negative-energy electron to give it positive energy. Then the hole left in the Dirac sea would act like a positive-charge particle, a positron, i.e. an antiparticle. Prediction of antimatter. Positron was observed by Carl Anderson at Caltech in 1932.

The Dirac equation can be reinterpreted as a quantum field equation. The existence of antiparticles follows from combining quantum mechanics with relativity. Consider a state $|\varphi\rangle$ that changes to $|\varphi'\rangle$ at (x_1, t_1) and back to $|\varphi\rangle$ at (x_2, t_2) due to interactions with spacelike separation.

Frame S



Then in a different reference frame S' this looks as:



So in S' it looks as if a particle and an antiparticle (backwards in time) are created at (x_2, t_2) and later at (x_1, t_1) the antiparticle annihilates with a particle

Plane-wave solutions of the Dirac equation

positive-energy

$$\psi(x) = e^{-\frac{i}{\hbar}x^\mu p_\mu}\, u(p)$$

two independent normalized solutions (particles)

with $$u^{(1)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 1 \\ 0 \\ cp_z/(E+mc^2) \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

and $$u^{(2)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 0 \\ 1 \\ c(p_x-ip_y)/(E+mc^2) \\ \dfrac{-cp_z}{E+mc^2} \end{pmatrix}$$

Normalization $u^\dagger u = \dfrac{2E}{c}$

$$(\gamma^\mu p_\mu - mc)u = 0$$

negative-energy

$$\psi(x) = e^{+\frac{i}{\hbar}x^\mu p_\mu}\, v(p)$$

two independent normalized solutions (antiparticles)

with $$v^{(1)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} c(p_x-ip_y)/(E+mc^2) \\ -cp_z/(E+mc^2) \\ 0 \\ 1 \end{pmatrix}$$

and $$v^{(2)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} cp_z/(E+mc^2) \\ c(p_x+ip_y)/(E+mc^2) \\ 1 \\ 0 \end{pmatrix}$$

Normalization $v^\dagger v = \dfrac{2E}{c}$

$$(\gamma^\mu p_\mu + mc)v = 0$$

Weyl or chiral representation of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \text{for } i=1,2,3$$

Write $\psi = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}$ where φ_R, φ_L are two-component spinors

Under Lorentz transformations with θ for rotation and $\cosh\varphi = \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ for boost

$$\psi = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\varphi})} & 0 \\ 0 & e^{\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\varphi})} \end{pmatrix} \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}$$

For massless particles, the Dirac equation decouples into two Weyl equations:

for $m=0$, $\gamma^\mu p_\mu \psi = 0 \Rightarrow \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_0 + \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} p_i\right] \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} = 0 \Rightarrow (p_0 - \sigma^i p_i)\varphi_L = 0$ and $(p_0 + \sigma^i p_i)\varphi_R = 0$

$\Rightarrow$ $(p_0 + \vec{\sigma}\cdot\vec{p})\varphi_L = 0$ and $(p_0 - \vec{\sigma}\cdot\vec{p})\varphi_R = 0$ $\Rightarrow$ $\vec{\sigma}\cdot\vec{p}\,\varphi_L = -p_0\varphi_L$, $\vec{\sigma}\cdot\vec{p}\,\varphi_R = p_0\varphi_R$ $\Rightarrow$ $\vec{\sigma}\cdot\hat{p}\,\varphi_L = -\varphi_L$, $\vec{\sigma}\cdot\hat{p}\,\varphi_R = \varphi_R$ with $\hat{p} = \frac{\vec{p}}{|\vec{p}|} = \frac{\vec{p}}{p_0}$

But $\vec{\sigma}\cdot\hat{p}$ is the helicity $\rightarrow$ component of spin in direction of momentum

So φ_R has positive helicity $\rightarrow$ right-handed

and φ_L has negative helicity $\rightarrow$ left-handed

Dirac equation with electromagnetic field: $[\gamma^\mu(i\partial_\mu - qA_\mu) - m]\psi = 0$ with q the charge

Relativistic quantum mechanics

We have discussed two attempts to find a relativistic version of the Schrodinger equation $\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$ or $-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = i\hbar\frac{\partial\psi}{\partial t}$

The first was the Klein-Gordon eq. $\partial^\mu\partial_\mu\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0$ with φ a scalar (spin 0)

problems: negative probability density and negative-energy solutions

The second was the Dirac eq. $i\hbar\gamma^\mu\partial_\mu\psi = mc\psi$ with ψ a spinor (spin $\frac{1}{2}$)

problem: negative-energy solutions but turned into a triumph $\to$ prediction of antimatter

But single-particle interpretation of a wavefunction is untenable since special relativity + quantum mechanics implies particle-antiparticle creation and annihilation, i.e. a multi-particle theory.

Furthermore, electromagnetic field so far also treated classically.

The introduction of scalar, vector, and spinor fields and their quantization resolved all the above issues

In non-relativistic QM $\hat{x}$ and $\hat{p}_x$ are operators ($\hat{x} \to x$ and $\hat{p}_x \to -i\hbar\frac{\partial}{\partial x}$) and satisfy the commutation relation $[\hat{x}, \hat{p}_x] = i\hbar$ but t is a parameter

In Quantum Field Theory $\vec{x}$ and t are parameters, but fields φ and conjugate momenta π are operators (second quantization) and satisfy the commutation relation (equal-time)

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\hbar\,\delta^3(\vec{x}-\vec{y})$$