Test 1

PHYS4500: Quantum Field Theory

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Problem 1.

The gamma matrices in the chiral representation are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{1.1}$$

So,

$$\begin{aligned}
\gamma^5 &= i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \\
&= i \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix} \begin{pmatrix} \sigma^2\sigma^3 & 0 \\ 0 & \sigma^2\sigma^3 \end{pmatrix}, \\
&= i \begin{pmatrix} -\sigma^1\sigma^2\sigma^3 & 0 \\ 0 & \sigma^1\sigma^2\sigma^3 \end{pmatrix}.
\end{aligned}$$

The Pauli spin matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2}$$

so

$$\begin{aligned}
\sigma^1\sigma^2\sigma^3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = i \ (2 \times 2 \text{ matrix})
\end{aligned}$$

therefore

$$\begin{aligned}
\gamma^5 &= i \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \\
\rightarrow & \boxed{\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.}
\end{aligned}$$

Problem 2.

We are to show that the spinor

$$v^{(1)} = \sqrt{\frac{E + mc^2}{c}} \begin{pmatrix} \frac{c(p_x - ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} \tag{2.1}$$

satisfies the momentum-space Dirac equation

$$\gamma^\mu p_\mu v^{(1)} = -mcv^{(1)}. \tag{2.2}$$

For notational simplicity, I will just write $v^{(1)} \rightarrow v$, and I will use natural units where $c = 1$. Additionally, since v appears on both sides in the Dirac equation, we can eliminate the normalization term (the square root) and rewrite

$$
v \rightarrow \begin{pmatrix} \frac{(p_x - ip_y)}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E+m \end{pmatrix}.
$$

Lastly, to keep with the simpler 2×2 convention (at least at first), I will let

$$
v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \quad \text{where,} \quad v_+ = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \quad \text{and} \quad v_- = \begin{pmatrix} 0 \\ E+m \end{pmatrix}.
$$

Now let's look at $\gamma^\mu p_\mu$:

$$
\begin{aligned}
\gamma^\mu p_\mu &= p_0\gamma^0 + p_1\gamma^1 + p_2\gamma^2 + p_3\gamma^3, \\
&= E\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - p_x \begin{pmatrix} 0 & \sigma^x \\ -\sigma^x & 0 \end{pmatrix} - p_y \begin{pmatrix} 0 & \sigma^y \\ -\sigma^y & 0 \end{pmatrix} - p_z \begin{pmatrix} 0 & \sigma^z \\ -\sigma^z & 0 \end{pmatrix}, \\
&= \begin{pmatrix} E & -\mathbf{p}\cdot\boldsymbol{\sigma} \\ \mathbf{p}\cdot\boldsymbol{\sigma} & -E \end{pmatrix}.
\end{aligned}
$$

So our momentum-space Dirac equation for v is

$$
\begin{aligned}
\begin{pmatrix} E & -\mathbf{p}\cdot\boldsymbol{\sigma} \\ \mathbf{p}\cdot\boldsymbol{\sigma} & -E \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix} &= -m \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \\
\begin{pmatrix} Ev_+ \\ \mathbf{p}\cdot\boldsymbol{\sigma}v_+ \end{pmatrix} - \begin{pmatrix} \mathbf{p}\cdot\boldsymbol{\sigma}v_- \\ Ev_- \end{pmatrix} &= \begin{pmatrix} -mv_+ \\ -mv_- \end{pmatrix}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathbf{p}\cdot\boldsymbol{\sigma} &= p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix},
\end{aligned}
$$

so,

$$
\mathbf{p}\cdot\boldsymbol{\sigma}v_- = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 0 \\ E+m \end{pmatrix} = \begin{pmatrix} (E+m)(p_x - ip_y) \\ -p_z(E+m) \end{pmatrix},
$$

and

$$
\begin{aligned}
\mathbf{p}\cdot\boldsymbol{\sigma}v_+ &= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}, \\
&= \begin{pmatrix} p_z(p_x - ip_y) - p_z(p_x - ip_y \\ (p_x + ip_y)(p_x - ip_y) + p_z^2) \end{pmatrix}, \\
&= \begin{pmatrix} 0 \\ \mathbf{p}^2 \end{pmatrix}.
\end{aligned}
$$

Returning back to the Dirac equation, we need to now expand fully to 4×4 matrices:

$$\rightarrow \begin{pmatrix} E(p_x - ip_y) \\ -Ep_z \\ 0 \\ \mathbf{p}^2 \end{pmatrix} - \begin{pmatrix} (E+m)(p_x - ip_y) \\ -p_z(E+m) \\ 0 \\ E(E+m) \end{pmatrix} = -m \begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E+m \end{pmatrix},$$

$$\begin{pmatrix} -m(p_x - ip_y) \\ -mp_z \\ 0 \\ \mathbf{p}^2 - E(E+m) \end{pmatrix} = \begin{pmatrix} -m(p_x - ip_y) \\ -mp_z \\ 0 \\ -Em - m^2 \end{pmatrix}.$$

The first three rows are obviously equal, so the last row's equality is all that's left to show:

$$\mathbf{p}^2 - E^2 - Em = -Em - m^2,$$
$$E^2 - \mathbf{p}^2 = m^2,$$

but the last line is just the mass-shell condition, so it must be true. At last, then, we have shown that the spinor $v^{(1)}$ satisfies the momentum space Dirac equation.

Problem 3.

The Mandelstam variables for a $a + b \rightarrow 1 + 2$ process are (this is my convention for u)

$$s = (p_a + p_b)^2, \quad t = (p_a - p_1)^2, \quad u = (p_a - p_2)^2. \tag{3.1}$$

Let's just expand (where all p's are understood to be 4-momenta):

$$\begin{aligned} s + t + u &= (p_a + p_b)^2 + (p_a - p_1)^2 + (p_a - p_2)^2, \\ &= p_a^2 + p_b^2 + 2p_a \cdot p_b + p_a^2 + p_b^2 - 2p_a \cdot p_1 + p_a^2 + p_2^2 - 2p_a \cdot p_2, \\ &= 3m_a^2 + m_b^2 + m_1^2 + m_2^2 + 2p_a \cdot p_b - 2p_a \cdot p_1 - 2p_a \cdot p_2. \end{aligned}$$

Looking at just the cross terms:

$$2p_a \cdot p_b - 2p_a \cdot p_1 - 2p_a \cdot p_2 = 2p_a(p_b - p_1 - p_2) = 2p_a(p_a + p_b - p_1 - p_2 - p_a).$$

But the first four terms in the parentheses are zero by momentum conservation, so the cross terms simplify to just $-2p_a^2 = -2m_a^2$. So,

$$s + t + u = 3m_a^2 + m_b^2 + m_1^2 + m_2^2 - 2m_a^2,$$
$$\rightarrow \boxed{s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2.}$$

Problem 4.

We are given a Lagrangian for a scalar field:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{4.1}$$

where λ is some constant.

The first term in the Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \left(\partial^\mu \phi \right) = \partial_\mu \partial^\mu \phi.$$

The second term is:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi - \frac{\lambda}{6}\phi^3.$$

So the solution to the Euler-Lagrange equation is:

$$\partial_\mu \partial^\mu \phi + m^2\phi + \frac{\lambda}{6}\phi^3 = 0.$$

The conjugate momentum can be found by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}. \tag{4.2}$$

We can rewrite the Lagrangian to make this a little more clear:

$$\mathcal{L} = \frac{1}{2}\partial_0\phi\partial^0\phi + \frac{1}{2}\partial_i\phi\partial^i\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4,$$

so now the conjugate momentum is simply:

$$\pi(x) = \partial^0\phi = \dot{\phi}.$$

The stress energy tensor is given by

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} && (4.3)\\ &= \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right). && (4.4) \end{aligned}$$

We have to use different indices for the 4-gradients inside the parentheses, since they are meant to be contracted only with each other, not the metric outside.

The Hamiltonian density $\mathcal{H}$ is given with T^{00}, so

$$\begin{aligned} \mathcal{H} = T^{00} &= \partial^0\phi\partial^0\phi - \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{2\lambda}{4!}\phi^4\right), \\ &= (\partial_0\phi)^2 - \frac{1}{2}\left(\partial_0\phi\partial^0\phi + \partial_i\phi\partial^i\phi - m^2\phi^2 - \frac{2\lambda}{4!}\phi^4\right), \\ &= (\partial_0\phi)^2 - \frac{1}{2}\left((\partial_0)^2 - (\boldsymbol{\nabla}\phi)^2 - m^2\phi^2 - \frac{2\lambda}{4!}\phi^4\right), \\ &= \frac{1}{2}\left[(\partial_0)^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 + \frac{2\lambda}{4!}\phi^4\right]. \end{aligned}$$

As expected, this is the same as the Hamiltonian energy density for the free real KG field plus an extra ϕ^4 term. Additionally, it is also positive-definite, which is a good sign.

Problem 5.

We are given the Fourier expansion for the creation and annihilation operators in terms of the fields for the real scalar (KG) field:

$$a^\dagger(\mathbf{p}) = i\int \mathrm{d}^3x\sqrt{2p^0}\left[\phi(\mathbf{x})\partial_0 e^{-ip\cdot x} - (\partial_0\phi(\mathbf{x}))e^{-ip\cdot x}\right]. \tag{5.1}$$

To simplify this, we know that the conjugate momentum is given by $\pi(\mathbf{x}) \equiv \partial_0\phi$ and

$$\partial_0 e^{-ip\cdot x} = e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{\mathrm{d}}{\mathrm{d}t}\left[-ip^0 t\right] e^{-ip^0 t} = -ip^0 e^{-ip^0 t} e^{-i\mathbf{p}\cdot\mathbf{x}},$$

Thus,

$$\begin{aligned} a^\dagger(\mathbf{p}) &= i\sqrt{2p^0} e^{-ip^0 t} \int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left[-ip^0\phi(\mathbf{x}) - \pi(\mathbf{x})\right], \\ &= \sqrt{2p^0} e^{-ip^0 t} \int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left[p^0\phi(\mathbf{x}) - i\pi(\mathbf{x})\right], \end{aligned}$$

or we can just let $t = 0$ to eliminate the first exponential outside the integral:

$$a^\dagger(\mathbf{p}) = \sqrt{2p^0} \int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left[p^0\phi(\mathbf{x}) - i\pi(\mathbf{x})\right]$$

The quantity $a^\dagger(\mathbf{p})a^\dagger(\mathbf{q})$ is found by integrating over the dummy variable y for the $a^\dagger(\mathbf{q})$ term:

$$a^\dagger(\mathbf{p})a^\dagger(\mathbf{q}) = 2\sqrt{p^0q^0} \int \mathrm{d}^3x\mathrm{d}^3y\, e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left[p^0\phi(\mathbf{x}) - i\pi(\mathbf{x})\right]\left[q^0\phi(\mathbf{y}) - i\pi(\mathbf{y})\right]$$

Looking just at the product of the two expressions in the brackets:

$$= p^0q^0\phi(\mathbf{x})\phi(\mathbf{y}) - ip^0\phi(\mathbf{x})\pi(\mathbf{y}) - iq^0\pi(\mathbf{x})\phi(\mathbf{y}) - \pi(\mathbf{x})\pi(\mathbf{y}).$$

When we do the commutator, the second term will have a term in brackets that looks identical with $\mathbf{x} \leftrightarrow \mathbf{y}$ and $\mathbf{p} \leftrightarrow \mathbf{q}$:

$$= p^0q^0\phi(\mathbf{y})\phi(\mathbf{x}) - iq^0\phi(\mathbf{y})\pi(\mathbf{z}) - ip^0\pi(\mathbf{y})\phi(\mathbf{x}) - \pi(\mathbf{y})\pi(\mathbf{x}).$$

Subtracting the two (as the definition of the commutator requires), these terms in brackets become:

$$p^0q^0[\phi(\mathbf{x}), \phi(\mathbf{y})] - ip^0[\phi(\mathbf{x}), \pi(\mathbf{y})] - iq^0[\pi(\mathbf{x}), \phi(\mathbf{y})] - [\pi(\mathbf{x}), \pi(\mathbf{y})].$$

From the equal time-commutation relations, we know the first and fourth terms are zero, so all we are left with is

$$-ip^0(i\delta^3(\mathbf{x}-\mathbf{y})) - iq^0(-i\delta^3(\mathbf{x}-\mathbf{y})) = (p^0 - q^0)\delta^3(\mathbf{x}-\mathbf{y}),$$

so the full commutator is

$$[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] = 2\sqrt{p^0q^0} \int \mathrm{d}^3x\mathrm{d}^3y\, e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})}(p^0 - q^0)\delta^3(\mathbf{x}-\mathbf{y}).$$

The delta function kills, say, the y integral, leaving:

$$[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] = 2\sqrt{p^0q^0}(p^0 - q^0) \int \mathrm{d}^3x\, e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}$$

The remaining integral leaves another delta function:

$$[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] = 2\sqrt{p^0q^0}(p^0 - q^0)(2\pi)^3\delta^3(\mathbf{p}+\mathbf{q}).$$

Now, when $\mathbf{p} \neq -\mathbf{q}$, the commutator is automatically zero via the delta function. When $\mathbf{p} = -\mathbf{q}$, we can take a closer look at p^0 and q^0; they are defined by:

$$p^0 = \sqrt{\mathbf{p}^2 + m^2} \quad \text{and} \quad q^0 = \sqrt{\mathbf{q}^2 + m^2}.$$

Since the 3-vectors only appear squared, when $\mathbf{q} = -\mathbf{p}$, $q^0 = \sqrt{\mathbf{p}^2 + m^2} = p^0$, so the quantity $(p^0 - q^0) \to (p^0 - p^0) = 0$. Hence,

$$\boxed{[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] = 0.}$$