Dimensional Regularization (DR)

A divergent multiple integral is made convergent by reducing the number of multiple integrals.

For example:

divergent 4-dim integral $\int d^4k \frac{k}{k^2 k^2}$

would be finite if the space-time were 2-dim!

Therefore, in dimensional regularization

$$\int d^4k \rightarrow \int d^Dk \qquad D<4$$

⇒ we obtain the result of the integral in terms of analytic expressions as functions of D.

pros: In dimensional reg. or DR nothing is violated: gauge, Lorentz, unitarity invariant

cons: the space-time is not 4-dim.
Care must be given to the algebra in D-dim.

Dimensional Regularization strategy:

$$\mu = 0, \dots, 3 \quad \Rightarrow \quad \mu = 0, \dots, D-1$$

$$p^\mu = (p^0, p^1, \dots, p^{D-1})$$

$$g^\mu{}_\mu = g^{\mu\nu} g_{\mu\nu} = D$$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

$$\gamma^\mu \gamma_\mu = D$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2-D)\gamma^\nu$$

Ambiguities: 1) the measure $\frac{1}{(2\pi)^4} \to \frac{1}{(2\pi)^D}$

or it maybe the same as in 4-dim.

Requirement: the measure in D-dim must recover $1/(2\pi)^4$ when $D \Rightarrow 4$

2) Trace of the γ matrices.

Following the Clifford algebra one has

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 2^{D/2} g^{\mu\nu} \quad \text{for } D \text{ even}$$

which reduces to the 4-dim form as $D \to 4$.

As we are only interested in 4-dimensional space-time, the $\mathrm{Tr}[\gamma^\mu\gamma^\nu]$ expression above is in principle not needed.

To avoid this problem we fix our convention such that

$$\int \frac{d^D k}{(2\pi)^D}$$

and the trace of γ-matrices is normalized to

$$\mathrm{Tr}[\gamma_\mu \gamma_\nu] = 4 g_{\mu\nu}$$

* Always keep in mind this convention and be consistent in your calculations.

We can now evaluate our integral using DR
We set $m=0$ for simplicity, for now.

$$\Sigma_s(p) = g_s^2\, C_F\,(2-D)\int \frac{d^Dk}{(2\pi)^D}\,\frac{\not{k}-\not{p}}{k^2(k-p)^2}$$

where we used $\gamma^\mu \gamma^\nu \gamma_\mu = (2-D)\gamma^\nu$.

We keep $D<3$ to ensure convergence.

The next step is to introduce the Feynman parametrization:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\{xA+(1-x)B\}^2}$$

to reexpress the denominator of $\Sigma_s(p)$

$$\Sigma_s(p) = g_s^2\, C_F\,(2-D)\int \frac{d^Dk}{(2\pi)^D}\,(\not{k}-\not{p})\int_0^1 \frac{dx}{\{x(k-p)^2+(1-x)k^2\}^2}$$

As far as $D<3$ the k-integration is convergent and we can interchange $\int d^Dk$ with $\int dx$

$$\Sigma_s(p) = g_s^2\, C_F\,(2-D)\int_0^1 dx\int \frac{d^Dk'}{(2\pi)^D i}\,\frac{(\not{k}-\not{p})}{\{(k-xp)^2+x(1-x)p^2\}^2}$$

$$\int_0^1 \frac{dx}{[xA + (1-x)B]^2} = \int_0^1 \frac{dx}{[x(A-B) + B]^2}$$

$$x(A-B) + B = y$$

$$dy = (A-B)\,dx \Rightarrow dx = \frac{dy}{A-B}$$

$$\begin{cases} x = 0 \Rightarrow y = B \\ x = 1 \Rightarrow y = A \end{cases}$$

$$\frac{1}{(B-A)}\int_A^B \frac{dy}{y^2} = \frac{1}{(B-A)}\frac{[y^{-1}]}{-1} = \frac{1}{A-B}\left[\frac{1}{y}\right]_A^B$$

$$= \frac{1}{(A-B)}\left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{\cancel{(A-B)}}\frac{\cancel{(A-B)}}{AB} = \frac{1}{AB}$$

This can be generalized to account for denominators with more terms.

where we rearranged the denominator.

DR preserves translational invariance $\Rightarrow$ we can make a shift of the momentum variable:

$$k' = k - xp$$

$$\Sigma(p) = g_s^2 C_F (2-D) \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D i} \frac{\not{k}' - (1-x)\not{p}}{\{k'^2 + x(1-x)p^2\}^2}$$

DR preserves of symmetries of the space-time $\Rightarrow$ an integral of an odd function in k vanishes

$$\int d^D k \, k_\mu f(k^2) = 0$$

where $f(k^2)$ is an integrable function of k^2.

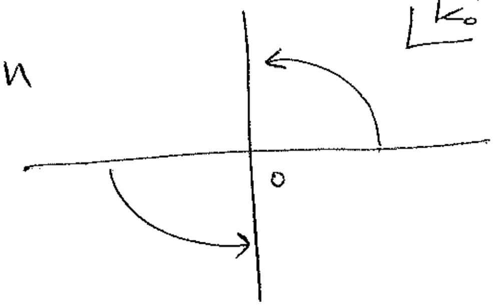
$\Rightarrow$ linearly divergent pieces disappear, leaving only logarithmically divergent contributions

$$\Sigma(p) = g_s^2 C_F (D-2) \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D k'}{(2\pi)^D i} \frac{1}{\{k'^2 + x(1-x)p^2\}^2}$$

Now we want to perform the k' integral. In the Minkowski space, this is not easy.

To circumvent the problem we make a 90° rotation in the complex plane

Wick Rotation



k_0' - plane

$$k_0' = i K_0 \qquad K_0 = \text{real}, \in \mathbb{R}$$

$\vec{k}' = \vec{K} \Rightarrow$

$$d^D k' = i\, d^D K, \qquad k'^2 = -K^2 \qquad K^2 = K_0^2 + |\vec{K}|^2$$

this changes the integration from Minkowski to the Euclidean space.

$$\Sigma(p) = g_s^2\, C_F\, (D-2)\, \not{p} \int_0^1 dx(1-x) \int \frac{d^D K}{(2\pi)^D} \frac{1}{(K^2+L)^2}$$

$$L = -x(1-x)p^2$$

- Integral non singular for $L > 0 \Rightarrow$ we must keep $p \Rightarrow p \in$ space-like region $\Rightarrow p^2 < 0$

At this point, we use the polar coordinate system in D-dim

$$|K| = \sqrt{K_0^2 + |\vec{K}|^2}$$

$$K_0 = |\vec{K}| \cos\theta_1$$

$$K_1 = |K| \cos\theta_2 \sin\theta_1$$

$$\vdots$$

$$K_{D-1} = |K| \sin\theta_{D-1} \cdots \sin\theta_1$$

$$d^D K = K^{D-1} d|K| \, d\Omega_D$$

$$d\Omega_D = \prod_{l=1}^{D-1} \sin^{D-1-l}\theta_l \, d\theta_l$$

$$\int \frac{d^D K}{(2\pi)^D} \frac{1}{(K^2 + L)^2} = \frac{B(D/2, 2 - D/2)}{(4\pi)^{D/2} \Gamma(D/2)} L^{D/2-2}$$

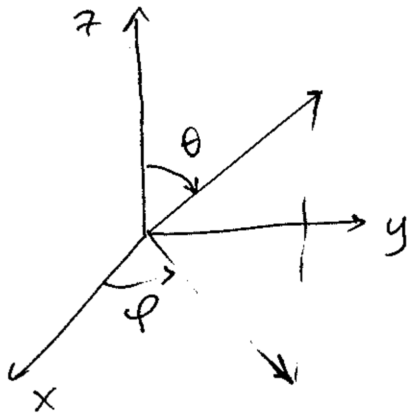
where the Beta $B(x,y)$ and $\Gamma(x)$ are

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \; ; \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$Re(z) > 0$

$$\Updownarrow$$

$$B(p,q) = \int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}} dt$$

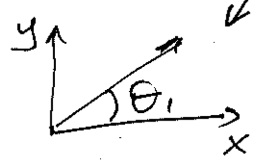


$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$K_0 \to z$

$z = r\cos\theta_1 = K_0$

x-y plane



$\underline{\underline{S^1 \to}}$ only θ_1 $\quad |\vec{r}|=1$

$$\begin{cases} X^1 = \cos\theta_1 = x \\ X^2 = \sin\theta_1 = y \end{cases}$$

$\underline{\underline{S^2 \to (\theta_1, \theta_2)}}$ $\quad |\vec{r}|=1$

$$\begin{cases} X^1 = \cos\theta_1 \\ X^2 = \sin\theta_1\cos\theta_2 \\ X^3 = \sin\theta_1\sin\theta_2 \end{cases}$$

$d\Omega = \sin\theta_1\, d\theta_1\, d\theta_2$

↳ (it would be $d\Omega_3$ in many text books)

$\underline{\underline{S^3 \to (\theta_1, \theta_2, \theta_3)}}$

$$\begin{cases} X^1 = \cos\theta_1 \\ X^2 = \sin\theta_1 \cos\theta_2 \\ X^3 = \sin\theta_1\sin\theta_2\cos\theta_3 \\ X^4 = \sin\theta_1\sin\theta_2\sin\theta_3 \end{cases}$$

$d\Omega = (\sin\theta_1)^2 \sin\theta_2\, d\theta_1 d\theta_2 d\theta_3$

↳ (it would be $d\Omega_4$ in many text books)

$S^4 \Rightarrow (\theta_1, -, \theta_4)$

$$\begin{cases} X^1 = \cos\theta_1 \\ X^2 = \sin\theta_1 \cos\theta_2 \\ X^3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ X^4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4 \\ X^5 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \end{cases}$$

$$d\Omega = (\sin\theta_1)^3 (\sin\theta_2)^2 \sin\theta_3 \, d\theta_1 \, d\theta_2 \, d\theta_3 \, d\theta_4$$

$\hookrightarrow d\Omega_5$

$S^{D-1} \Rightarrow (\theta_1, \text{——}, \theta_{D-1})$

$$\begin{cases} X^1 = \cos\theta_1 \\ X^2 = - - \\ \vdots \\ X^{D-1} = \sin\theta_1 \cdots \cos\theta_{D-1} \\ X^D = \sin\theta_1 \cdots \sin\theta_{D-1} \end{cases}$$

$$d\Omega = (\sin\theta_1)^{D-2} (\sin\theta_2)^{D-3} \cdots \sin\theta_{D-2} \, d\theta_1 \, d\theta_2 \cdots d\theta_{D-1}$$

$\hookrightarrow d\Omega_D$

$$d\Omega_D = \prod_{l=1}^{D-1} (\sin\theta_l)^{D-1-l} \, d\theta_l$$

The integral of the solid angle in D-dim (15/c)

gives

$$\int d\Omega_D = \int_0^{\pi} d\theta_1 (\sin\theta_1)^{D-2} \cdots \int_0^{\pi} d\theta_{D-2} \sin\theta_{D-2} \int_0^{\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$