PHYS 4210/ Fall 2024
Midterm Exam 2

KENNESAW STATE UNIVERSITY

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100/100

This exam consists of six questions, each worth 25 points. Answer them on the provided sheets. You have 70 minutes to complete the exam. You may use a calculator and your own integration formula sheet. All other work must be your own, without assistance from peers, notes, books, or online resources.

After grading your answers to all six questions, your two lowest scores will be dropped, i.e. the maximum possible score for this exam is 100.

1. The solutions to time-independent Schrodinger equation for a particle in infinite square well potential of width a are given as:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\, sin\left(\frac{n\pi}{a}x\right)$$

Where n =1, 2, 3, ...
Show that these solutions are mutually orthogonal.

We can choose two states ψ_n and ψ_m where $n \neq m$. For two functions to be mutually orthogonal, their inner product is non-zero only if $n = m$.

−5

So, here: $\int_0^a \psi_n^* \psi_m \, dx = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) dx$

But we know $\boxed{\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) dx = \frac{2}{a}\delta_{nm}}$, show work

which vanishes if $n \neq m$, and the total inner product is 1 if $n = m$. Thus, the Hamiltonian eigenstates are mutually orthogonal (technically they are orthonormal!)

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2. The ground state of quantum harmonic oscillator of mass m is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Assume that the ladder operators have the form

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x)$$

Find the first excited state and normalize it.

We can find the next state by $\psi_1 = A\hat{a}_+\psi_0$. I'll let $\alpha_0 \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

First,

$$A\hat{a}_+\psi_0 = \frac{A}{\sqrt{2\hbar m\omega}}\left[-i\hat{p}\psi_0 + m\omega x\psi_0\right]$$

$$= \frac{A\alpha_0}{\sqrt{2\hbar m\omega}}\left[-\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$= \frac{A\alpha_0}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\hbar} + m\omega\right) x e^{-\frac{m\omega}{2\hbar}x^2}$$

- I'll just let the constant absorb all the other constant stuff

$$\psi_1 = A x e^{-\frac{m\omega}{2\hbar}x^2}$$

Now, $\langle\psi_1|\psi_1\rangle = |A|^2\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx$, or w/ $a = \sqrt{\frac{\hbar}{m\omega}}$,

$$= |A|^2\int_{-\infty}^{\infty} x^2 e^{-x^2/a^2}dx = |A|^2 \cdot \frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} = 1$$

$$= |A|^2 \cdot \left(\frac{\pi}{4}\right)^{1/2}\left(\frac{\hbar^3}{m^3\omega^3}\right)^{1/2} = |A|^2\left(\frac{\pi\hbar^3}{4m^3\omega^3}\right)^{1/2} = 0$$

$$\rightarrow \underline{A = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}},$$

so $\boxed{\psi_1(x) = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x e^{-\frac{m\omega}{2\hbar}x^2}}$

Scratch!

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega x\right)$$

$$a_+\psi_0 = \frac{1}{\sqrt{2\hbar m\omega}}\left[-\hbar\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x\, e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left[\frac{m\omega}{\hbar}x + m\omega x\right] x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_1(x) = A x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\langle\psi_1(x)|\psi_1(x)\rangle = |A|^2\int x^2 e^{-\frac{m\omega}{\hbar}x^2}\,dx \quad \text{, or w/ } a = \sqrt{\frac{\hbar}{m\omega}}\text{ ,}$$

$$= |A|^2\int_{-\infty}^{\infty} x^2 e^{-x^2/a^2}\,dx = |A|^2\cdot\frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2} = 1$$

$$\rightarrow \quad A = \sqrt{\frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2}}$$

$$= \left(\frac{4}{\pi}\right)^{1/4}\left(\frac{m\omega}{\hbar}\right)^{3/4} = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}$$

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3. Using the definition of ladder operator from question 2, show that $\hat{a}_-$ is the Hermitian conjugate of $\hat{a}_+$.

The Hermitian conjugate of an operator $\hat{Q}$ is $\hat{Q}^\dagger$, where $\hat{Q}$ acts on kets and $\hat{Q}$ acts on bras:

$$\hat{Q}|\psi\rangle \ , \ \langle\psi|\hat{Q}^\dagger .$$

To find the Hermitian conjugate of $\hat{a}_+$, we can find how it acts on bras First:

$$\langle f|\hat{a}_+|f\rangle = \frac{1}{\sqrt{2\hbar m\omega}}\langle f|-i\hat{p}+m\omega\hat{x}|f\rangle$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left[\langle f|-\frac{d}{dx}|f\rangle + \langle f|m\omega x|f\rangle\right]$$

1) $\langle f|-\frac{d}{dx}|f\rangle = \int f^*\left(-\frac{df}{dx}\right)dx = -f^*f\Big|_{-\infty}^{\infty} + \int \frac{df^*}{dx} f\,dx$

f lives in Hilbert-space, so it's square-integrable and vanisles at $\pm\infty$. So,

$$\langle f|-\frac{d}{dx}|f\rangle = \int\left(\frac{df}{dx}\right)^* f\,dx, \quad \text{so}$$

$$(-i\hat{p})^\dagger = i\hat{p}$$

2) $\langle f|m\omega x|f\rangle = \int f^*(m\omega x f)\,dx = \int (m\omega x f)^* f\,dx$, so $(m\omega x)^\dagger = m\omega x$.

Thus, $\hat{a}_+^\dagger = \left(\frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p}+m\omega x)\right)^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{p}+m\omega x) = \hat{a}_-$,

by definition.

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4. The state of a quantum particle is written as

$$|\psi\rangle = \sqrt{\frac{7}{15}}\,|\phi_1\rangle + \sqrt{\frac{1}{3}}\,|\phi_2\rangle + \sqrt{\frac{1}{5}}\,|\phi_3\rangle$$

Where $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ are eigenstates to a Hermitian operator $\hat{B}$ such that $\hat{B}|\phi_n\rangle = (3n^3 - 1)|\phi_n\rangle$

1) For operator $\hat{B}$, what is the probability of getting an eigen value of 80? (10 credits)
2) Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$. (15 credits)

1) An eigenvalue of 80 corresponds to

$$3n^3 - 1 = 80$$
$$n^3 = 27$$
$$n = 3.$$

The probability to obtain the n^{th} eigenval is $|C_n|^2$, so

$$P(B=3) = |C_3|^2 = \left(\sqrt{\tfrac{1}{5}}\right)^2 = \underline{\tfrac{1}{5}.}$$

2) $$\langle\psi|\hat{B}|\psi\rangle = \int \frac{7}{15}\langle\phi_1|\hat{B}|\phi_1\rangle dx + \int \frac{1}{3}\langle\phi_2|\hat{B}|\phi_2\rangle dx + \int \frac{1}{5}\langle\phi_3|\hat{B}|\phi_3\rangle dx$$

$$= \frac{7}{15}(3(1)^3-1)\int\langle\phi_1|\phi_1\rangle dx + \frac{1}{3}(3(2)^3-1)\int\langle\phi_2|\phi_2\rangle dx + \frac{1}{5}80\int\langle\phi_3|\phi_3\rangle dx$$

These are all normalized, so $\int\langle\phi_n|\phi_n\rangle dx = 1$, so

$$\langle\psi|\hat{B}|\psi\rangle = \frac{7}{15}(2) + \frac{1}{3}(23) + \frac{81}{5}$$

$$= \frac{14}{15} + \frac{115}{15} + \frac{273}{15} \quad \frac{402}{15} \approx \underline{26.8} = \langle\hat{B}\rangle$$

23 × 5 = 115
81 × 3 = 273
273 + 115 + 14 = 402

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5. Evaluate

(a) $\int_0^{2\pi} cosx\,\delta(x^2 - \pi^2)dx$ (12.5 credits)

(b) $\int_{-3}^{1} f(x)\,\delta(x+2)\,dx$,

where $f(x) = 10x - 1$ for $0 \le x < \infty$ and $f(x) = x^3 - 3x^2 + 2x - 1$ for $-\infty < x \le 0$

(12.5 credits)

a) $\int_0^{2\pi} \cos(x)\cdot\delta(x^2-\pi^2)\,dx = \int_0^{2\pi} \cos(x)\ \delta\big((x+\pi)(x-\pi)\big)\,dx.$

The delta function is nonzero only when $x = \pm\pi$. Since $+\pi$ is the only one in the integration range, we pick out $x=\pi$ so

$$\int_0^{2\pi} \cos(x)\,\delta(x^2-\pi^2)\,dx = \cos(\pi) = \underline{-1}.$$

b) $\int_{-3}^{1} f(x)\,\delta(x+2) = \int_{-3}^{0} (x^3-3x^2+2x-1)\,\delta(x+2)\,dx + \int_0^1 (10x-1)\,\delta(x+2)\,dx$

The delta function is non-zero for $x=-2$, which is outside this integral's range, so it's zero. We pick out $x=-2$ in the first so

$$\int_{-3}^{1} f(x)\,\delta(x+2) = (-2)^3 - 3(-2)^2 + 2(-2) - 1$$

$$= -8 - 12 - 4 - 1$$

$$= \underline{-25}.$$

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6. In this question, I am gauzing your understanding of basic ideas in this course. Short answers explaining basic math will get you full credit. Show your work.

(a) Show that $[Ae^{ikx} + Be^{-ikx}]$ and $[C\cos kx + D\sin kx]$ are equivalent ways of writing the same function of x, and determine the constants C and D in terms of A and B, and vice versa. (10 points)

(b) Why is $\psi(x) = Ae^{-kx} + Be^{kx}$ a general solution to the equation to the equation $\frac{d^2\psi}{dx^2} = k^2\psi$? Assume k is real and positive. (10 points)

(c) Explain the fundamental difference between the solutions of type (a) and type (b)? (5 points)

a) From the all-famous Euler's formula, we have that

$$e^{ikx} = \cos(kx) + i\sin(kx),$$

so

$$Ae^{ikx} + Be^{-ikx} = A\cos(kx) + iA\sin(kx) + B\cos(-kx) + iB\sin(-kx)$$
$$= A\cos(kx) + iA\sin(kx) + B\cos(kx) - iB\sin(kx)$$
$$= (A+B)\cos(kx) + i(A-B)\sin(kx).$$

Then, we can define $C \equiv A+B$ and $D \equiv i(A-B)$ so that

$$Ae^{ikx} + Be^{-ikx} = C\cos(kx) + D\sin(kx).$$

(and vice versa?)

-1

b) For both terms, a double differentiation brings down two powers of K w/ the same sign (the minus cancels in the A term). So, we end up w/ k^2 times the function we started with, which is exactly what this EQ represents.

c) The sols. in a) are oscillatory and stay finite forever, whereas the sols. in b) are true exponentials that go to zero at one end and blow up at the other.

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Potentially Useful Identities

$$cos(A+B) = cosA\ cosB - sinA\ sinB$$
$$cos(A-B) = cosA\ cosB + sinA\ sinB$$

$$sin^2x = \frac{1-cos2x}{2}$$

$$\int dx = x + C$$

$$\int coskx\ dx = \frac{1}{k}\ sinkx + C$$

Odd functions are those which satisfy f(-x) = -f(x). Integral of an odd function in symmetric limit is zero.

$$\int_{-\infty}^{\infty} e^{-x^2/k^2} dx = k\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/k^2} dx = \frac{k^3\sqrt{\pi}}{2}$$

$$\frac{d(e^{kx})}{dx} = ke^{kx}$$

Chain rule of derivatives: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

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HW7

PHYS4210: Quantum Mechanics

Casey Hampson

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Problem 1. (5.4)

a) Equation (5.17) is

$$\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = A[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)] \tag{1.1}$$

By inspection, it's probably going to be $1/\sqrt{2}$. But, of course, let's check. Imposing the normalization condition,

$$\int |\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2)|^2 \, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 = 1 \tag{1.2}$$

The square of the wavefunction is

$$|\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2)|^2 = |A|^2 \left[\psi_a^*(\mathbf{r}_1)\psi_b^*(\mathbf{r}_2) \pm \psi_b^*(\mathbf{r}_1)\psi_a^*(\mathbf{r}_2)\right] \times \left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right]. \tag{1.3}$$

Now, when we do the multiplication, the cross terms will look like $\psi_b^*(\mathbf{r}_1)\psi_a(\mathbf{r}_2)$, which, since ψ_a and ψ_b are orthogonal, will integrate to zero when we normalize. Therefore, the square of the wavefuntion is *effectively*

$$|\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2)|^2 = |A|^2 \left[|\psi_a(\mathbf{r}_1)|^2|\psi_b(\mathbf{r}_2)|^2 + |\psi_b(\mathbf{r}_1)|^2|\psi_a(\mathbf{r}_2)|^2\right] \tag{1.4}$$

Doing the integration:

$$\int |\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2)|^2 \, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 = |A|^2 \Bigg[\left(\int \mathrm{d}^3\mathbf{r}_1 \, |\psi_a(\mathbf{r}_1)|^2\right)\left(\int \mathrm{d}^3\mathbf{r}_2 \, |\psi_b(\mathbf{r}_2)|^2\right) + \left(\int \mathrm{d}^3\mathbf{r}_1 \, |\psi_b(\mathbf{r}_1)|^2\right)\left(\int \mathrm{d}^3\mathbf{r}_2 \, |\psi_a(\mathbf{r}_2)|^2\right)\Bigg] \tag{1.5}$$

$$\int |\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2)|^2 \, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 = 2|A|^2. \tag{1.6}$$

So, $A = 1/\sqrt{2}$, as expected.

b) If the two wavefunctions are the same, then

$$\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = 2\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2), \tag{1.7}$$

meaning

$$\int |\psi_\pm(\mathbf{r}_1, \mathbf{r}_2)|^2 \, \mathrm{d}^3\mathbf{r}_1 \mathrm{d}^3\mathbf{r}_2 = 4|A|^2 \left(\int |\psi_a(\mathbf{r}_1)|^2 \, \mathrm{d}^3\mathbf{r}_1 \times \int |\psi_b(\mathbf{r}_2)|^2 \, \mathrm{d}^3\mathbf{r}_2 \right) = 4|A|^2 = 1, \tag{1.8}$$

so this time, $A = 1/2$.

Problem 2. (5.6)

We will need the expectation values which we already solved for in a previous HW:

$$\langle x \rangle = \frac{a}{2} \tag{2.1}$$

$$\langle x^2 \rangle = a^2 \left(\frac{1}{3} - \frac{1}{2(n\pi)^2} \right) \tag{2.2}$$

a) For the case of distinguishable particles, we can use Equation (5.23):

$$\left\langle (x_1 - x_2)^2 \right\rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b \tag{2.3}$$

$$= a^2 \left[\frac{2}{3} - \frac{1}{2} \left(\frac{1}{(n\pi)^2} + \frac{1}{(\ell\pi)^2} \right) \right] \tag{2.4}$$

$$\boxed{= a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \left(\frac{1}{n^2} + \frac{1}{\ell^2} \right) \right].} \tag{2.5}$$

b) In the case of identical bosons, we need a term $-2|\langle x \rangle_{ab}|^2$ where

$$\langle x \rangle_{ab} = \int x \psi_a^*(x) \psi_b(x) \, \mathrm{d}x. \tag{2.6}$$

In our case then

$$\langle x \rangle_{n\ell} = \frac{2}{a} \int_0^a \int x \sin\left(\frac{n\pi x}{a} \right) \sin\left(\frac{\ell \pi x}{a} \right) \, \mathrm{d}x. \tag{2.7}$$

I'll just use Mathematica:

$$|\langle x \rangle_{n\ell}|^2 = \frac{16 a^2 n^2 \ell^2 ((-1)^{n+\ell} - 1)^2}{\pi^4 (n^2 - \ell^2)^4}. \tag{2.8}$$

So,

$$\boxed{\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \left(\frac{1}{n^2} + \frac{1}{\ell^2} \right) \right] - \frac{32 a^2 n^2 \ell^2 ((-1)^{n+\ell} - 1)^2}{\pi^4 (n^2 - \ell^2)^4}.} \tag{2.9}$$

c) For identical fermions, the last term just picks up a minus:

$$\boxed{\langle (x_1 - x_2)^2 \rangle = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \left(\frac{1}{n^2} + \frac{1}{\ell^2} \right) \right] - \frac{32 a^2 n^2 \ell^2 ((-1)^{n+\ell} - 1)^2}{\pi^4 (n^2 - \ell^2)^4}.} \tag{2.10}$$

Problem 3. (5.17)

a) This part is super straightforward, so I am guessing that the "Explain your answers for each element" is for the second part. We fill up the first s orbital, then the s orbital in the next energy level, then since the next energy level admits another value of l, we get a p orbital. That's it.

$$
\begin{aligned}
\text{H: } & (1s) \\
\text{He: } & (1s)^2 \\
\text{Li: } & (1s)^2(2s) \\
\text{Be: } & (1s)^2(2s)^2 \\
\text{B: } & (1s)^2(2s)^2(2p) \\
\text{C: } & (1s)^2(2s)^2(2p)^2 \\
\text{N: } & (1s)^2(2s)^2(2p)^3 \\
\text{O: } & (1s)^2(2s)^2(2p)^4 \\
\text{F: } & (1s)^2(2s)^2(2p)^5 \\
\text{Ne: } & (1s)^2(2s)^2(2p)^6
\end{aligned}
$$

b) The first four atoms in their ground states have $\ell = 0$, meaning the letter will be S for all of them. For hydrogren, the single electron can only have spin $s = 1/2$, so $2S + 1 = 2$. Therefore the grand total $J = 1/2$, so Hydrogen has ${}^2S_{1/2}$.

Helium fills the $1s$ orbital, so they now occupy a singlet configuration with spin 0: thus Helium has ${}^1S_0$.

Lithium has a new electron in the $2s$ orbital. There is still no orbital angular momentum and the spin of a single filled s orbital is zero so really this is the same as Hydrogen: ${}^2S_{1/2}$.

In a similar vein, Beryllium will be the same as Helium: ${}^1S_0$.

Boron fills both $(1s)$ and $(2s)$ orbitals. Again, these have 0 angular momentum, so all we really care about is the electron in the $(2p)$ orbital. It is a single electron, so it has spin 1/2, and with orbital angular momentum $\ell = 1$, the total angular momentum is either 3/2 or 1/2. The letter is now P since $\ell = 0$. So the two possibilities for Boron are: ${}^2P_{3/2}$ and ${}^2P_{1/2}$.

For carbon, the two electrons can have total spin 1 or 0, and the total orbital angular momentum can now be 2, 1, or 0, so it's a bit more complicated. For $L = 0$, it is simple, but when $L = 1$, S can be 0 or 1. In the latter, we therefore have $J = 2, 1, 0$. Similarly, when $L = 2$ and $S = 1$, we will have $J = 3, 2, 1$. So: ${}^1S_0$, ${}^3S_1$, ${}^1P_1$, ${}^3P_2$, ${}^3P_1$, ${}^3P_0$, ${}^1D_2$, ${}^3D_3$, ${}^3D_2$, ${}^3D_1$.

Lastly, for nitrogen, $L = 3, 2, 1$ or 0, and $S = 3/2$ or $1/2$. Following a similar process as before we get: ${}^2S_{1/2}$, ${}^4S_{3/2}$, ${}^2P_{3/2}$, ${}^2P_{1/2}$, ${}^4P_{5/2}$, ${}^4P_{3/2}$, ${}^4P_{1/2}$, ${}^2D_{5/2}$, ${}^2D_{3/2}$, ${}^4D_{7/2}$, ${}^4D_{5/2}$, ${}^4D_{3/2}$, ${}^4D_{1/2}$, ${}^2F_{7/2}$, ${}^2F_{5/2}$, ${}^4F_{9/2}$, ${}^4F_{7/2}$, ${}^4F_{5/2}$, ${}^4F_{3/2}$.

Problem 4. (6.1c)

Parity only affects the angular part like $\hat{\Pi}\, Y_\ell^m(\theta, \phi) = Y_\ell^m(\pi - \theta, \phi + \pi)$. Recall,

$$
Y_\ell^m(\theta, \phi) = \sqrt{\frac{(2\ell + 1)}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} e^{im\phi} P_\ell^m(\cos\theta). \tag{4.1}
$$

The coefficient obviously doesn't change. The exponential turns into:

$$
e^{im(\phi+\pi)} = e^{im\pi} e^{im\phi} = (-1)^m e^{im\phi}. \tag{4.2}
$$

For the associated Legendre polynomials, we are taking $\cos\theta \to \cos(\pi-\theta) = -\cos\theta$ and $\sin\theta \to \sin(\pi-\theta) = \sin\theta$. Using the definition of the associated Legendre functions and the ℓth Legendre function, we can see that taking $x \to -x$ results in a factor of -1 only if the quantity $\ell + m$ is odd, because we only see x^2, but the derivatives will pick up a minus. Therefore, we can have a general factor of $(-1)^{\ell+m}$, which, combining with the $(-1)^m$ from before:

$$(-1)^m(-1)^{\ell+m} = (-1)^{2m}(-1)^{\ell}. \tag{4.3}$$

-1 raised to any even number is always 1, so all we have leftover is $(-1)^{\ell}$. Therefore:

$$\boxed{\hat{\Pi}\,\psi_{n\ell m}(r,\theta,\phi) = (-1)^{\ell}\psi_{n\ell m}(r,\theta,\phi).} \tag{4.4}$$

Problem 5. (6.8)

We did this one in class!

a) First,

$$\left\langle f \middle| \hat{\Pi} \middle| g \right\rangle = \int_{-\infty}^{\infty} f^*(x)g(-x)\,\mathrm{d}x. \tag{5.1}$$

Taking $x \to -x$:

$$= \int_{\infty}^{-\infty} f^*(-x)g(x)\,(-\mathrm{d}x) = \int_{-\infty}^{\infty} f^*(-x)g(x)\,\mathrm{d}x = \left\langle f \middle| \hat{\Pi}^{\dagger} \middle| g \right\rangle, \tag{5.2}$$

so $\boxed{\hat{\Pi}^{\dagger} = \hat{\Pi}.}$

b) As a unitary operator, we have that

$$\hat{\Pi}^{\dagger}\hat{\Pi}\,\psi = |\lambda|^2\psi = \psi, \tag{5.3}$$

where the second expression follows from applying the operators to the wavefunction and getting the eigenvalues and the third follows from the unitarity of the parity operator rendering $\hat{\Pi}^{\dagger}\hat{\Pi} = 1$. Now, since the parity operator is also Hermitian, it means that its eigenvalues must also be real, so it must be that $\lambda^2 = 1$, meaning $\lambda = 1$ or -1.

Problem 6. (6.10)

a) For $\hat{x}$:

$$\left\langle f \middle| \hat{\Pi}^{\dagger}\hat{x}\hat{\Pi} \middle| g \right\rangle = \int_{-\infty}^{\infty} f^*(-x)xg(-x)\,\mathrm{d}x \tag{6.1}$$

$$= \int_{\infty}^{-\infty} f^*(x)(-x)g(x)\,(-\mathrm{d}x) \tag{6.2}$$

$$= \int_{-\infty}^{\infty} f^*(x)(-x)g(x)\,\mathrm{d}x = \langle f\,|\,(-\hat{x})\,|\,g\rangle, \tag{6.3}$$

so $\boxed{\hat{x}' = -\hat{x}.}$

b) For $\hat{p}$, we have

$$\left\langle f \middle| \hat{\Pi}^{\dagger}\hat{p}\hat{\Pi} \middle| g \right\rangle = \int_{-\infty}^{\infty} f^*(-x)(-i\hbar)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\mathrm{d}x. \tag{6.4}$$

When we do integration by parts, the term that we evaluate at the limits will go to zero like always so we have

$$\left\langle f \middle| \hat{\Pi}^\dagger \hat{p} \hat{\Pi} \middle| g \right\rangle = -\int_{-\infty}^{\infty} (-i\hbar) \frac{\mathrm{d}f^*(-x)}{\mathrm{d}x} g(-x)\,\mathrm{d}x \tag{6.5}$$

$$= -\int_{\infty}^{-\infty} (-i\hbar) - \frac{\mathrm{d}f^*(x)}{\mathrm{d}x} g(x)\,(-\mathrm{d}x) \tag{6.6}$$

$$= -\int_{-\infty}^{\infty} i\hbar \frac{\mathrm{d}f^*(x)}{\mathrm{d}x} g(x)\,\mathrm{d}x \tag{6.7}$$

$$= -\int_{-\infty}^{\infty} \left(-i\hbar \frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^* g(x)\,\mathrm{d}x = \langle f \,|\, (-\hat{p}) \,|\, g \rangle\,, \tag{6.8}$$

so $\boxed{\hat{p}' = -\hat{p}.}$

Problem 7. (6.13)

a) For a single electron in the ground state of the hydrogren atom, there is perfect spherical symmetry, meaning that $\langle \mathbf{r} \rangle = 0$, so $\langle \mathbf{p}_e \rangle = q \langle \mathbf{r} \rangle = 0$.

b) For $n = 2$, we need to make use of Equation (6.26), and we can tell that there is possibility for two different values of ℓ and ℓ' such that $\ell + \ell'$ is not even. We need a single state/wavefunction, so we need a linear combination of two states with different ℓ values, say

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|210\rangle + |200\rangle), \tag{7.1}$$

where

$$\psi_{200} = \frac{1}{4\sqrt{a^3\pi}} \left(2 - \frac{r}{a}\right) e^{-r/2a}, \quad \text{and} \tag{7.2}$$

$$\psi_{210} = \frac{1}{4\sqrt{a^3\pi}} \left(\frac{r}{a}\right) e^{-r/2a} \cos\theta. \tag{7.3}$$

Thus,

$$\langle \psi \,|\, \hat{p}_e \,|\, \psi \rangle = \frac{1}{2} \left(\langle 200|\hat{p}_e|200\rangle + \langle 210|\hat{p}_e|210\rangle + \langle 210|\hat{p}_e|200\rangle + \langle 200|\hat{p}_e|210\rangle \right). \tag{7.4}$$

The first two terms in parentheses are zero since $\ell = \ell'$, so we are only left with

$$\langle \psi \,|\, \hat{p}_e \,|\, \psi \rangle = \frac{1}{2} \left(\langle 210|\hat{p}_e|200\rangle + \langle 200|\hat{p}_e|210\rangle \right). \tag{7.5}$$

For a generic complex number z, we have that $(z + z^*)/2 = \mathrm{Re}[z]$, and since the second term in parentheses is the complex conjugate of the first (because the position operator is Hermitian) then we can write this as

$$\langle \psi \,|\, \hat{p}_e \,|\, \psi \rangle = \mathrm{Re}[\langle 210|\hat{p}_e|200\rangle]. \tag{7.6}$$

Doing the actual calculation, $\mathbf{r}$ is a vector so: $\hat{\mathbf{r}} = \begin{pmatrix}\hat{x} & \hat{y} & \hat{z}\end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\phi \end{pmatrix}$. The x component is

$$\langle \hat{p}_e \rangle_x = -e \cdot \mathrm{Re}\left[\int \psi^*_{210}(r \sin\theta \cos\phi)\psi_{200} \, \mathrm{d}^3\mathbf{r} \right] \hat{\imath} \tag{7.7}$$

The ϕ integration will be super easy, since neither wavefunctions contribute a ϕ component and $\mathrm{d}^3 r = r^2 \sin\theta \mathrm{d}r$ also doesn't contribute a ϕ component. Fortunately, however, we have that

$$\int_0^{2\pi} \cos\phi \, \mathrm{d}\phi = \int_0^{2\pi} \sin\phi \, \mathrm{d}\phi = 0, \tag{7.8}$$

so both the x and y components will be zero. The z component is (and since we know everything will be real now, we can drop the real specifier)

$$\langle 210|\hat{p}_e|200\rangle_x = -\frac{e}{16a^5\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \cos^2\theta \sin\phi \, \mathrm{d}\theta \int_0^{\infty} r^4(2a - r)e^{-r/a} \, \mathrm{d}r. \tag{7.9}$$

The ϕ integration is obviously 2π, Mathematica tells me the θ integral is $2/3$, and it also tells me the r integration is $-72/a^6$, so

$$\boxed{\langle \hat{p}_e \rangle = 6ea\,\hat{k}.} \tag{7.10}$$

99/100

HW6

PHYS4210: Quantum Mechanics

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Problem 1. (4.1a)

a) We are to work out all of the anti-commutators for each component of the position and momentum operators. We can first qualitatively consider the commutators of the position operators amongst themselves. They are just numbers, there are no derivatives or anything involved with them, and we know that numbers commute, so we can immediately say that

$$[r_i, r_j] = 0. \tag{1.1}$$

Next, let's consider the commutators of the momentum operators among each other:

$$[p_i, p_j]f = -\hbar^2 \left(\frac{\partial^2 f}{\partial r_i \partial r_j} - \frac{\partial^2 f}{\partial r_j \partial r_i} \right). \tag{1.2}$$

By construction, f lives in Hilbert space (rather, we only care about such functions), so it is well behaved, and we are able to switch the order of the derivatives. Hence, the two quantities in parentheses are just identical, so we can say

$$[p_i, p_j] = 0. \tag{1.3}$$

Next, we have the commutator of the position components with the momentum components:

$$[r_i, p_j]f = -i\hbar \left(r_i \frac{\partial f}{\partial r_j} - \frac{\partial}{\partial r_j}[r_i * f] \right). \tag{1.4}$$

If $i \neq j$, we can just pull r_j out of the second derivative:

$$[r_i, p_j]f = i\hbar r_i \left(\frac{\partial f}{\partial r_j} - \frac{\partial f}{\partial r_j} \right). \tag{1.5}$$

But this is zero. The other case is if $i = j$. This is just $[x, p_x] = i\hbar$ that we have done in class before. To combine the two, then, we can use a Dirac delta and say that

$$[r_i, p_j] = i\hbar \delta_{ij}. \tag{1.6}$$

Of course, $[p_i, r_j] = -[r_i, p_j] = -i\hbar\delta_{ij}$.

b) The "generalized" Ehrenfest theorem, given in Eq (3.73) in Griffiths is

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{i}{\hbar} \left\langle [\hat{H}, \hat{Q}] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle. \tag{1.7}$$

However, the operators never depend on time, so really we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q \rangle = \frac{i}{\hbar} \left\langle [\hat{H}, \hat{Q}] \right\rangle. \tag{1.8}$$

Here we have $Q = r_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle r_i \rangle = \frac{i}{\hbar} \left\langle [\hat{H}, r_i] \right\rangle. \tag{1.9}$$

Looking at the commutator:

$$[\hat{H}, r_i] = \left[\frac{\hat{\mathrm{p}}^2}{2m} + V, r_i\right]. \tag{1.10}$$

The potential is just a number, and r_i is just a number, so we can get rid of V. Further, from the previous part we know that only like components of momentum don't commute with like components of position, so the only non-zero terms are p_i^2:

$$= \frac{1}{2m}[\hat{p}_i^2, r_i] \tag{1.11}$$

$$= \frac{1}{2m}\left(\hat{p}_i\hat{p}_i r_i - r_i\hat{p}_i\hat{p}_i + \hat{p}_i r_i\hat{p}_i - \hat{p}_i r_i\hat{p}_i\right), \tag{1.12}$$

where I added and subtracted the same term for the third and fourth terms. Now,

$$= \frac{1}{2m}\left(\hat{p}_i[\hat{p}_i, r_i] + [\hat{p}_i, r_i]\hat{p}_i\right) \tag{1.13}$$

$$= \frac{1}{2m}\left(-i\hbar\hat{p}_i - i\hbar\hat{p}_i\right) \tag{1.14}$$

$$= -\frac{i\hbar}{m}\hat{p}_i. \tag{1.15}$$

Plugging back in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle r_i\rangle = \frac{i}{\hbar}\left\langle -\frac{i\hbar}{m}\hat{p}_i\right\rangle = \frac{1}{m}\langle\hat{p}_i\rangle. \tag{1.16}$$

Since there were no cross-terms between position or momentum components, this will be the same for all three components, meaning we can generally express it in vector form:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathrm{r}\rangle = \frac{1}{m}\langle\mathrm{p}\rangle.} \tag{1.17}$$

Next, we'll let $\hat{Q} = \hat{p}_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}_i\rangle = \left\langle[\hat{H}, \hat{p}_i]\right\rangle. \tag{1.18}$$

Looking at the commutator:

$$[\hat{H}, \hat{p}_i] = \left[\frac{\hat{\mathrm{p}}^2}{2m} + V, \hat{p}_i\right]. \tag{1.19}$$

We know from before that all momentum components commute among each other, so all we have is

$$= [V, \hat{p}_i]. \tag{1.20}$$

Using a test function f:

$$[V, \hat{p}_i]f = -i\hbar\left(V\frac{\partial f}{\partial r_i} - \frac{\partial}{\partial r_i}[Vf]\right) \tag{1.21}$$

$$= i\hbar f\frac{\partial V}{\partial r_i}. \tag{1.22}$$

$$\rightarrow [V, \hat{p}_i] = i\hbar\frac{\partial V}{\partial r_i}. \tag{1.23}$$

Plugging this back in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{p}_i\rangle = \frac{i}{\hbar}\left\langle i\hbar\frac{\partial V}{\partial r_i}\right\rangle = \left\langle -\frac{\partial V}{\partial r_i}\right\rangle. \tag{1.24}$$

Again, there are no cross-terms among components, so we can express this in vector form:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{\mathbf{p}}\rangle = \langle -\nabla V\rangle .} \tag{1.25}$$

c) This is easy. We know only like components of position and momentum don't compute, and since there are no cross terms, they are each equal to $\hbar/2$, as we very well know, so long as $i = j$:

$$\boxed{\sigma_{r_i}^2\sigma_{\hat{p}_j}^2 \geq \frac{\hbar}{2}\delta_{ij}.} \tag{1.26}$$

Problem 2. (4.12)

The radial wavefunction is

$$R_{n\ell} = \frac{1}{r}\rho^{\ell+1}e^{-\rho}v(\rho), \tag{2.1}$$

where ρ is an nth order polynominal whose coefficients are given by

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)}c_j, \tag{2.2}$$

and $\rho = kr$ where $k = 1/an = 1/3a$ for us with $n = 3$.

Starting with R_{30}, let's start with computing the coefficients for the polynomial $v(\rho)$. Since we are told to not bother normalizing, I'll just set $c_0 = 1$ for simplicity. Therefore,

$$c_1 = \frac{2(1-3)}{(1)(2)} = -2 \tag{2.3}$$

$$c_2 = (-2)\frac{2(1+1-3)}{(2)(3)} = \frac{-2}{6}(-2) = \frac{2}{3}, \tag{2.4}$$

$$c_3 = \left(\frac{2}{3}\right)\frac{2(2+1-3)}{\cdots} = 0, \tag{2.5}$$

where we don't care about the denominator in the last term since the numerator is zero. Therefore, our polynominal

$$1 - 2\rho + \frac{2}{3}\rho^2 \to 1 - 2\frac{r}{3a} + \frac{2}{3}\left(\frac{r}{3a}\right)^2 = 1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2. \tag{2.6}$$

Thus,

$$R_{30} = \frac{1}{r}\left(\frac{r}{3a}\right)e^{-r/3a}\left[1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2\right] \tag{2.7}$$

$$\boxed{R_{30} = \frac{1}{3a}e^{-r/3a}\left[1 - \frac{2}{3}\left(\frac{r}{a}\right) + \frac{2}{27}\left(\frac{r}{a}\right)^2\right].} \tag{2.8}$$

Next, we consider R_{31}. The coefficients are, with $c_0 = 1$:

$$c_1 = \frac{2(1+1-3)}{(1)(2+2)} = \frac{-1}{2}, \tag{2.9}$$

$$c_2 = \left(-\frac{1}{2}\right)\frac{2(1+1+1-3)}{\dots} = 0, \tag{2.10}$$

where again we can stop before doing the denominator since the numerator is zero. Thus,

$$R_{31} = \frac{1}{r}\left(\frac{r}{3a}\right)^2 e^{-r/3a}\left[1 - \frac{1}{2}\left(\frac{r}{3a}\right)\right] \tag{2.11}$$

$$\boxed{R_{31} = \frac{r}{9a^2}e^{-r/3a}\left[1 - \frac{1}{6}\left(\frac{r}{a}\right)\right].} \tag{2.12}$$

Lastly, for R_{32}, I'll take $c_0 = 1$ so

$$c_1 = \frac{2(2+1-3)}{\dots} = 0, \tag{2.13}$$

so we only have the constant term. Thus,

$$R_{32} = \frac{1}{r}\left(\frac{r}{3a}\right)^3 e^{-r/3a} \tag{2.14}$$

$$\boxed{R_{32} = \frac{r^2}{27a^3}e^{-r/3a}.} \tag{2.15}$$

Problem 3. (4.15)

a) The ground state of an electron in a hydrogen atom is

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}. \tag{3.1}$$

So,

$$\langle r\rangle = \langle \psi_{100} \,|\, r \,|\, \psi_{100}\rangle = \frac{1}{\pi a^3}\int r e^{-2r/a} \mathrm{d}^3 r. \tag{3.2}$$

Converting to spherical coordinates:

$$\langle r\rangle = \frac{1}{\pi a^3}\int \mathrm{d}\Omega \int_0^\infty r^3 e^{-2r/a}\,\mathrm{d}r. \tag{3.3}$$

The solid angle integral is 4π, and we can use the formula from the back of the book

$$\int_0^\infty x^n e^{-x/a}\,\mathrm{d}x = n!a^{n+1} \tag{3.4}$$

to say that

$$\langle r\rangle = \frac{4}{a^3}3!\left(\frac{a}{2}\right)^4 4a\left(\frac{6}{16}\right) = \boxed{\frac{3}{2}a^2.} \tag{3.5}$$

$\frac{3}{2}a$

Similarly,

$$\langle r^2\rangle = \frac{4}{a^3}\int r^4 e^{-2r/a}\,\mathrm{d}r = \frac{4}{a^3}4!\left(\frac{a}{2}\right)^5 = 4a^2\left(\frac{24}{32}\right) = \boxed{3a^2.} \tag{3.6}$$

b) The ground state only depends on r, the distance from the center; there is no dependence on anything else, so there is perfect spherical symmetry. Therefore, it easily follows that the expectation value of x must be zero: $\boxed{\langle x \rangle = 0x}$

x^2 is a little different, but still doesn't require integration. Since $r^2 = x^2 + y^2 + z^2$, then $\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle$. But again, by spherical symmetry, all three terms on the right should be equal, so

$$\langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = \boxed{a^2.} \tag{3.7}$$

c) First we must figure out ψ_{211}. We know $\psi_{211} = R_{21} Y_1^1$, and R_{21} is given in Equation (4.83):

$$R_{21} = \frac{c_0}{4a^2} r e^{-r/2a}. \tag{3.8}$$

We must normalize it an find c_0. We know that we can normalize the radial equation separately by

$$\int_0^\infty |R|^2 r^2 \, dr = 1, \tag{3.9}$$

so for us we have

$$\int_0^\infty |R_{21}|^2 r^2 \, dr = \frac{c_0^2}{16a^4} \int_0^\infty r^4 e^{-r/a} \, dr \tag{3.10}$$

$$= \frac{c_0^2}{16a^4} 4! a^5 = c_0^2 \frac{3}{2} a = 1, \tag{3.11}$$

so

$$c_0 = \sqrt{\frac{2}{3a}}. \tag{3.12}$$

Next, we can use one of the tables in the book to find that

$$Y_1^1 = -\left(\frac{3}{8\pi}\right) \sin\theta e^{i\phi}. \tag{3.13}$$

So, the total wavefunction is

$$\psi_{211} = \sqrt{\frac{2}{3a}} \frac{1}{4a^2} r e^{-r/2a} \cdot -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \tag{3.14}$$

$$= -\frac{1}{\sqrt{\pi a}} \frac{1}{8a^2} r e^{-r/2a} \sin\theta e^{i\phi}. \tag{3.15}$$

This carries an angular dependence, so it isn't perfectly spherically symmetric, so we must use the fact that $x = r \sin\theta \cos\phi$ to find that

$$\langle x^2 \rangle = \langle \psi_{211} \,|\, r^2 \sin^2\theta \cos^2\phi \,|\, \psi_{211} \rangle = \frac{1}{64\pi a^5} \int_0^\pi \sin^5\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi \int r^6 e^{-r/a} \, dr. \tag{3.16}$$

I'll just use Mathematica for the angular integrals:

$$\int_0^{\pi} \sin^5\theta \, \mathrm{d}\theta = \frac{16}{15} \quad \text{and} \quad \int_0^{2\pi} \cos^2\phi \, \mathrm{d}\phi = \pi, \tag{3.17}$$

so

$$\langle x^2 \rangle = \frac{1}{64\pi a^5}\left(\frac{16}{15}\right) \cdot \pi \cdot 6! a^7 = \frac{a^2}{60} 720 = \boxed{12a^2.} \tag{3.18}$$

Problem 4. (4.21)

We are considering the raising and lowering operators for angular momentum, where

$$L_+ f_\ell^m = A_\ell^m f_\ell^{m+1} \quad \text{and} \quad L_- f_\ell^m = B_\ell^m f_\ell^{m-1}, \tag{4.1}$$

where A_ℓ^m and B_ℓ^m are undetermined functions of ℓ and m and

$$L_\pm \equiv L_x \pm iL_y. \tag{4.2}$$

First, we are to show that $L_+ = (L_-)^\dagger$ (and vice versa). This is straightforward:

$$\langle f|L_+|g\rangle = \int f^*[(L_x + iL_y)g] \, \mathrm{d}x = \int [(L_x - iL_y)f]^* g \, \mathrm{d}x = \left\langle f \,\middle|\, (L_-)^\dagger \,\middle|\, g\right\rangle, \tag{4.3}$$

so, evidently, $L_+ = (L_-)^\dagger$.

Next, what we can do is use this fact so that

$$\langle f_\ell^m|L_- L_+|f_\ell^m\rangle = \left\langle f_\ell^m \,\middle|\, (L_+)^\dagger L_+ \,\middle|\, f_\ell^m\right\rangle = |A_\ell^m|^2 \langle f_\ell^{m+1}|f_\ell^{m+1}\rangle = |A_\ell^m|^2. \tag{4.4}$$

We can use Equation (4.112) from the book to get that

$$L_- L_+ = L^2 - L_z^2 - \hbar L_z \tag{4.5}$$

so that

$$\begin{aligned} |A_\ell^m|^2 &= \langle f_\ell^m|L_- L_+|f_\ell^m\rangle = \left\langle f_\ell^m \,\middle|\, L^2 - L_z^2 - \hbar L_z \,\middle|\, f_\ell\right\rangle && (4.6) \\ &= [\hbar^2 \ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m] \langle f_\ell^m|f_\ell^m\rangle && (4.7) \\ |A_\ell^m|^2 &= [\hbar^2\ell(\ell+1) - \hbar^2 m^2 - \hbar^2 m]. && (4.8) \end{aligned}$$

So,

$$\boxed{A_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}.} \tag{4.9}$$

For the other case, we have

$$\langle f_\ell^m|L_+ L_-|f_\ell^m\rangle = |B_\ell^m|^2, \tag{4.10}$$

and the only difference between $L_- L_+$ (from before) and $L_+ L_-$ (now) is a relative minus on the $\hbar L_z$ term in Equation (4.112) in Griffiths. Therefore, we have a plus instead in the third term in brackets in Equation (4.8) (in this document) meaning

$$\boxed{B_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m-1)}.} \tag{4.11}$$

More succinctly, we could say

$$L_\pm f_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)} f_\ell^{m\pm 1}. \tag{4.12}$$

top and bottom of ladder? -1

Problem 5. (4.22a)

We are considering the commutators of the z angular momentum operator with the position and linear momentum operators. $\hat{L}_z$ is defined like

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{5.1}$$

Looking first at the x operator:

$$\begin{aligned}
[\hat{L}_z, x] &= (x\hat{p}_y x - y\hat{p}_x x) - (xx\hat{p}_y - xy\hat{p}_x) && (5.2)\\
&= xy\hat{p}_x - y\hat{p}_x x && (5.3)\\
&= xy\hat{p}_x - y\hat{p}_x x - yx\hat{p}_x + yx\hat{p}_x && (5.4)\\
&= y[x, \hat{p}_x] = \boxed{i\hbar y.} && (5.5)
\end{aligned}$$

In the second line, we canceled the first term in the first parentheses with the first term in the second parentheses since $[\hat{p}_y, x] = 0$. Then in the third line I added and subtracted the same term, so that the first and third terms cancel and we are left with a commutator we know.

Next,

$$\begin{aligned}
[\hat{L}_z, y] &= (x\hat{p}_y y - y\hat{p}_x y) - (yx\hat{p}_y - yy\hat{p}_x) && (5.6)\\
&= x\hat{p}_y y - yx\hat{p}_y && (5.7)\\
&= x\hat{p}_y y - yx\hat{p}_y - xy\hat{p}_y + xy\hat{p}_y && (5.8)\\
&= x[\hat{p}_y, y] = \boxed{-i\hbar x.} && (5.9)
\end{aligned}$$

When we do $[\hat{L}_z, z]$, we notice that there are no $\hat{p}_z$ operators, so z will commute with all them, and of course it'll commute with the other position operators, so since it commutes with everything, then we can say $\boxed{[\hat{L}_z, z] = 0.}$

Looking next at the linear momentum operators:

$$\begin{aligned}
[\hat{L}_x, \hat{p}_x] &= (x\hat{p}_y\hat{p}_x - y\hat{p}_x\hat{p}_x) - (\hat{p}_x x\hat{p}_y - \hat{p}_x y\hat{p}_x) && (5.10)\\
&= x\hat{p}_y\hat{p}_x - \hat{p}_x x\hat{p}_y - x\hat{p}_x\hat{p}_y + x\hat{p}_x\hat{p}_y && (5.11)\\
&= [x, \hat{p}_x]\hat{p}_y = \boxed{i\hbar\hat{p}_y.} && (5.12)
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\hat{L}_x, \hat{p}_y] &= (x\hat{p}_y\hat{p}_y - y\hat{p}_x\hat{p}_y) - (\hat{p}_y x\hat{p}_y - \hat{p}_y y\hat{p}_x) && (5.13)\\
&= \hat{p}_y y\hat{p}_x - y\hat{p}_x\hat{p}_y - y\hat{p}_y\hat{p}_x + y\hat{p}_y\hat{p}_x && (5.14)\\
&= [\hat{p}_y, y]\hat{p}_x = \boxed{-i\hbar\hat{p}_x}. && (5.15)
\end{aligned}$$

No z's at all appear in $\hat{L}_z$, and we know that different components of position and linear momentum commute, so like before, $\hat{p}_z$ commutes with everything in $\hat{L}_z$ so $\boxed{[\hat{L}_z, \hat{p}_z] = 0.}$

Problem 6. (4.30)

a) Spinors are states in Hilbert space, so they must be normalized:

$$\chi^\dagger\chi = |A|^2[|3Ai|^2 + (4A)^2] = 25|A|^2 = 1, \tag{6.1}$$

so

$$\boxed{A = \frac{1}{5}.} \tag{6.2}$$

b) We first measure the $\langle \hat{S}_x \rangle$:

$$\langle \hat{S}_x \rangle = \left\langle \chi \middle| \hat{S}_x \middle| \chi \right\rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.3}$$

$$= \frac{\hbar}{50}\begin{pmatrix} 4 & -3i \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.4}$$

$$= \frac{\hbar}{50}(12i - 12i) = \boxed{0.} \tag{6.5}$$

Next,

$$\langle \hat{S}_y \rangle = \left\langle \chi \middle| \hat{S}_y \middle| \chi \right\rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.6}$$

$$= \frac{\hbar}{50}\begin{pmatrix} 4i & -3 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.7}$$

$$= \frac{\hbar}{50}(-12 - 12) = \boxed{-\frac{12\hbar}{25}.} \tag{6.8}$$

Lastly,

$$\langle \hat{S}_z \rangle = \left\langle \chi \middle| \hat{S}_z \middle| \chi \right\rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.9}$$

$$= \frac{\hbar}{50}\begin{pmatrix} -3i & -4 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} \tag{6.10}$$

$$= \frac{\hbar}{50}(9 - 16) = \boxed{-\frac{7\hbar}{50}.} \tag{6.11}$$

c) To get the standard deviations, we need $\langle \hat{S}_i^2 \rangle$. First, we need

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{6.12}$$

$$\sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{6.13}$$

$$\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{6.14}$$

$$\tag{6.15}$$

So,

$$\langle \hat{S}_i^2 \rangle = \frac{\hbar^2}{4}\langle \chi | \chi \rangle = \frac{\hbar^2}{4} \tag{6.16}$$

no matter what the spinor χ happens to be and for all three components. So,

$$\sigma_{S_x} = \sqrt{\frac{\hbar^2}{4} - 0} = \boxed{\frac{\hbar}{2}}. \tag{6.17}$$

$$\sigma_{S_y} = \sqrt{\frac{\hbar^2}{4} - \hbar^2\frac{144}{625}} = \hbar\sqrt{\frac{625-576}{2500}} = \hbar\sqrt{\frac{49}{2500}} = \boxed{\frac{7\hbar}{50}}. \tag{6.18}$$

$$\sigma_{S_z} = \sqrt{\frac{\hbar^2}{4} - \hbar^2\frac{49}{2500}} = \hbar\sqrt{\frac{625-49}{2500}} = \hbar\sqrt{\frac{576}{2500}} = \boxed{\frac{12\hbar}{25}}. \tag{6.19}$$

d) We now check if this satisfies the uncertainty principle with Equation (4.100) in Griffiths:

$$\sigma_{S_x}\sigma_{S_y} = \frac{\hbar}{2}\frac{7\hbar}{50} \geq \frac{\hbar}{2}\frac{7\hbar}{50} \checkmark \tag{6.20}$$

$$\sigma_{S_y}\sigma_{S_z} = \frac{7\hbar}{50}\frac{12\hbar}{25} \geq 0 \checkmark \tag{6.21}$$

$$\sigma_{S_z}\sigma_{S_x} = \frac{12\hbar}{25}\frac{\hbar}{2} \geq \frac{\hbar}{2}\frac{12\hbar}{25} \checkmark \tag{6.22}$$

For the non-zero ones, we are right at the limit!

30+70=100

100/100

HW5

PHYS4210: Quantum Mechanics

Casey Hampson

November 10, 2024

Problem 1.

The normalized spherical harmonics are given by

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\phi}P_\ell^m(\cos\phi), \tag{1.1}$$

where

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_\ell(x), \tag{1.2}$$

and

$$P_\ell(x) = \frac{1}{2^\ell l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell (x^2-1)^\ell. \tag{1.3}$$

Starting with $m=\ell=0$, we can tell pretty easily that $P_0(\cos\theta)=1$, and so too will $P_0^0(\cos\theta)$. Therefore,

$$\boxed{Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}}.} \tag{1.4}$$

For $\ell=1$ and $m=0$,

$$P_1(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) = x \to \cos\theta, \tag{1.5}$$

$$\to P_1^0 = \cos\theta, \tag{1.6}$$

$$\boxed{Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta.}. \tag{1.7}$$

For $\ell=m=1$:

$$P_1(x) = x \to \cos\theta \tag{1.8}$$

$$P_1^1(x) = (-1)\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}(x) = -\sqrt{1-x^2} \to -\sqrt{1-\cos^2\theta} = -\sin\theta, \tag{1.9}$$

$$\boxed{Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta.} \tag{1.10}$$

For $\ell=2$ and $m=0$:

$$P_2(x) = \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 \tag{1.11}$$

$$= \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}x}[4x(x^2-1)] \tag{1.12}$$

$$= \frac{1}{8}[4(x^2-1)+8x^2] \tag{1.13}$$

$$= \frac{1}{8}(12x^2-4) \tag{1.14}$$

$$\to \frac{1}{2}(\cos^2\theta - 1). \tag{1.15}$$

$$P_2^0(\cos\theta) = \frac{1}{2}(\cos^2\theta-1). \tag{1.16}$$

$$\boxed{Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(\cos^2\theta-1),} \tag{1.17}$$

where I brought the 1/2 from $P_2(x)$ inside the square root. For $\ell = 2$ and $m = 1$,

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \to \frac{1}{2}(3\cos^2\theta - 1), \tag{1.18}$$

$$P_2^1(x) = -\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(3x^2 - 1)\right] \tag{1.19}$$

$$\to -\sin\theta \cdot 3x \to -3\sin\theta\cos\theta, \tag{1.20}$$

$$Y_2^1(\theta, \phi) = -3\sqrt{\frac{5}{4\pi}\frac{1}{3!}}e^{i\phi}\sin\theta\cos\theta, \tag{1.21}$$

$$\boxed{Y_2^1(\theta, \phi) = -\sqrt{\frac{15}{8\pi}}e^{i\phi}\sin\theta\cos\theta.} \tag{1.22}$$

Lastly, for $\ell = 2$ and $m = -2$, we need to use the footnote 5 on page 135 for the definition of the associated Legendre functions for a negative m:

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \to \frac{1}{2})c\cos^2\theta - 1) \tag{1.23}$$

$$P_2^2(x) = (-1)^2(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{1}{2}(3x^2 - 1)\right] \tag{1.24}$$

$$= 3x^2 \to 3\sin^2\theta, \tag{1.25}$$

$$P_2^{-2}(x) = (-1)^2\frac{1}{4!} \cdot 3x^2 \to \frac{1}{8}\sin^2\theta, \tag{1.26}$$

$$Y_2^{-2}(\theta, \phi) = \frac{1}{8}\sqrt{\frac{5}{4\pi} \cdot 4!}e^{-2i\phi}\sin^2\theta \tag{1.27}$$

$$\boxed{Y_2^{-2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}}e^{-2i\phi}\sin^2\theta.} \tag{1.28}$$

100/100

HW4

PHYS4210: Quantum Mechanics

Casey Hampson

October 12, 2024

Problem 1.

The eigenvalue equation for the position operator is

$$\hat{x}f(x) = x_0 f(x) \quad \rightarrow \quad xf(x) = x_0 f(x). \tag{1.1}$$

where x_0 is the eigenvalue. The only function that is the same when multiplied by *any* x and the singular x_0 is the Dirac delta:

$$f(x) = \delta(x - x_0). \tag{1.2}$$

The eigenfunctions of the position operator being the Dirac delta sort of make sense - such a function (roughly speaking) is a localization entirely at $x = x_0$, which is exactly what the position is!

With this in mind, it is pretty straightforward to see in this case that the eigenvalues $\{x_n\}$ are all the possible positions the particle can take. Since this spectrum is continuous and since the particle can be anywhere in space, then the eigenvalues must be the set of all real numbers.

Problem 2.

Considering two observables A and B, we can define

$$|f\rangle = (\hat{A} - \langle A\rangle)|\psi\rangle, \quad \text{and} \quad |g\rangle = (\hat{B} - \langle B\rangle)|\psi\rangle, \tag{2.1}$$

just as the book did. The product of the variances is given, by the Schwartz inequality, as

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2. \tag{2.2}$$

To achieve minimum uncertainty, this means that we have an equality rather than a inequality in the above equation:

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle = |\langle f|g\rangle|^2. \tag{2.3}$$

However, the only way for this to be the case in the vector space of square-integrable functions (Hilbert space) is for the two functions f and g to be equal, up to a constant which can in general be complex:

$$f = ag. \tag{2.4}$$

But, Griffiths states that for a complex number z:

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(x)]^2 \geq [\mathrm{Im}(z)]^2. \tag{2.5}$$

Again, in the minimum-uncertainty regime, this is now an equality, meaning

$$|\langle f|g\rangle|^2 = |a\langle f|f\rangle|^2 = [\mathrm{Im}(z)]^2, \tag{2.6}$$

or

$$[\mathrm{Re}(a\langle f|f\rangle)]^2 = 0. \tag{2.7}$$

We could have taken it to be equal to the

However, since f lies in Hilbert space, it is square-integrable, meaning its inner product with itself must be real. For the entire quantity to be zero, a must be purely imaginary:

$$a = i\alpha. \tag{2.8}$$

We now have that

$$f = i\alpha g \quad \rightarrow \quad (\hat{A} - \langle A\rangle)\psi = i\alpha(\hat{B} - \langle B\rangle)\psi, \tag{2.9}$$

or replacing $\hat{A} \rightarrow \hat{p} \rightarrow -i\hbar\mathrm{d}/\mathrm{d}x$ and $\hat{B} \rightarrow \hat{x} = x$, we get

$$\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x} - \langle p\rangle\right)\psi = i\alpha(x - \langle x\rangle)\psi, \tag{2.10}$$

$$-i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}x} - \langle p\rangle\,\psi = i\alpha(x - \langle x\rangle)\psi, \tag{2.11}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{i}{\hbar}[i\alpha(x - \langle x\rangle) + \langle p\rangle]\psi, \tag{2.12}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \left(-\frac{\alpha}{\hbar}(x - \langle x\rangle) + \frac{i}{\hbar}\langle p\rangle\right)\psi. \tag{2.13}$$

This will be an exponential. For the first term, since we are differentiating with respect to x, we need a $\frac{1}{2}(x - \langle x\rangle)^2$ in the exponential (with the other constants as well, of course), and the second term is easy since $\langle p\rangle$ is not a function of x - it's just a number. With this:

$$\boxed{\psi = Ae^{-\alpha(x-\langle x\rangle)^2/2\hbar + i\langle p\rangle x/\hbar} = Ae^{-\alpha(x-\langle x\rangle)^2/2\hbar}e^{i\langle p\rangle x/\hbar}.} \tag{2.14}$$

The first exponential has a square of x, which is Gaussian. The second exponential is a "wiggle" factor, but the point is that we still get a Gaussian-looking function.

Problem 3. (3.3)

First we assume that $\langle h|\hat{Q}|h\rangle = \langle h|\hat{Q}^\dagger|h\rangle$ (i.e. $\hat{Q}$ is Hermitian). This doesn't actually say much about the function $h(x)$, since are are already assuming it is in Hilbert space. But the stronger condition that operators must satisfy is that $\langle f|\hat{Q}|g\rangle = \langle f|\hat{Q}^\dagger|g\rangle$ for any *two* functions $f(x)$ and $g(x)$. To show this, let's first consider the case where $h(x) = f(x) + g(x)$:

$$\begin{aligned}\langle f+g|\hat{Q}|f+g\rangle &= \int (f^* + g^*)[\hat{Q}(f+g)]\,\mathrm{d}x \\ &= \int [f^*(\hat{Q}g) + g^*(\hat{Q}g) + f^*(\hat{Q}g) + g^*(\hat{Q}f)]\,\mathrm{d}x.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle f+g|\hat{Q}^\dagger|f+g\rangle &= \int [\hat{Q}(f^* + g^*)](f+g)\,\mathrm{d}x \\ &= \int [(\hat{Q}f^*)f + (\hat{Q}g^*)g + (\hat{Q}f^*)g + (\hat{Q}g^*)f]\,\mathrm{d}x.\end{aligned}$$

Since we started with $\langle h|\hat{Q}|h\rangle = \langle h|\hat{Q}^\dagger|h\rangle$, these two equations must be equal, or better, their integrands must be equal

$$f^*(\hat{Q}f) + g^*(\hat{Q}g) + f^*(\hat{Q}g) + g^*(\hat{Q}f) = (\hat{Q}f^*)f + (\hat{Q}g^*)g + (\hat{Q}f^*)g + (\hat{Q}g^*)f.$$

But we can eliminate the first two terms on each side since they are equal by our assumption, so we have

$$f^*(\hat{Q}g) + g^*(\hat{Q}f) = (\hat{Q}f^*)g + (\hat{Q}g^*)f.$$

If we now let $h(x) = f(x) + ig(x)$,

$$\begin{aligned}\langle f+ig|\hat{Q}|f+ig\rangle &= \int (f^* - ig^*)[\hat{Q}(f+ig)]\,\mathrm{d}x \\ &= \int [f^*(\hat{Q}g) + g^*(\hat{Q}g) + if^*(\hat{Q}g) - ig^*(\hat{Q}f)]\,\mathrm{d}x,\end{aligned}$$

and

$$\langle f+ig|\hat{Q}^\dagger|f+ig\rangle = \int [\hat{Q}(f^*-ig^*)](f+ig)\,\mathrm{d}x$$
$$= \int [(\hat{Q}f^*)f + (\hat{Q}g^*)g + i(\hat{Q}f^*)g - i(\hat{Q}g^*)f]\,\mathrm{d}x.$$

Again, these must be equal:

$$f^*(\hat{Q}g) + g^*(\hat{Q}g) + if^*(\hat{Q}g) - ig^*(\hat{Q}f) = (\hat{Q}f^*)f + (\hat{Q}g^*)g + i(\hat{Q}f^*)g - i(\hat{Q}g^*).$$

The first two terms on each side cancel again, so we have (canceling the i's):

$$f^*(\hat{Q}g) - g^*(\hat{Q}f) = (\hat{Q}f^*)g - (\hat{Q}g^*).$$

If we now add to this the result from the previous $h(x) = f(x) + g(x)$, then we get

$$2f^*(\hat{Q}g) = 2(\hat{Q}f^*)g, \tag{3.1}$$

or

$$\boxed{\langle f|\hat{Q}|g\rangle = \langle f|\hat{Q}^\dagger|g\rangle\,.}$$

Problem 4. (3.5)

a) First, for a real x,

$$\langle f|x|f\rangle = \int f^*(xf)\,\mathrm{d}x = \int (xf)^* f\,\mathrm{d}x = \langle f|x^\dagger|f\rangle\,, \tag{4.1}$$

so $x^\dagger = x$, meaning it's Hermitian.

Looking next at i:

$$\langle f|i|f\rangle = \int f^*(if)\,\mathrm{d}x = \int (-if)^* f\,\mathrm{d}x = \langle f|i^\dagger|f\rangle\,, \tag{4.2}$$

so $i^\dagger = -i$.

Finally for $\mathrm{d}/\mathrm{d}x$:

$$\left\langle f\left|\frac{\mathrm{d}}{\mathrm{d}x}\right|f\right\rangle = \int f^*\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x = f^*f\Big|_{\infty}^{\infty} - \int \frac{\mathrm{d}f^*}{\mathrm{d}x}f\,\mathrm{d}x = \int\left(-\frac{\mathrm{d}f}{\mathrm{d}x}\right)^* f\,\mathrm{d}x = \left\langle f\left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\dagger\right|f\right\rangle, \tag{4.3}$$

so $(\mathrm{d}/\mathrm{d}x)^\dagger = -\mathrm{d}/\mathrm{d}x$.

b) Using Griffiths' notation will be a little more illuminating than the notation I've been using (which is because I've been doing QFT stuff, and they tend to use that notation there more). Starting with $\langle f|(\hat{Q}\hat{R})f\rangle$, this means that $\hat{R}$ acts on f first, then $\hat{Q}$, so to move the two operators to act on the bra instead, we have to start with the outer $\hat{Q}$ the move the inner $\hat{R}$, meaning

$$\langle f|(\hat{Q}\hat{R})f\rangle = \langle(\hat{R}^\dagger\hat{Q}^\dagger)f|f\rangle\,. \tag{4.4}$$

But as Hermitian operators, this means that we must have

$$\langle(\hat{R}^\dagger\hat{Q}^\dagger)f|f\rangle = \langle(\hat{Q}\hat{R})^\dagger f|f\rangle\,, \tag{4.5}$$

meaning that

$$(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger \hat{Q}^\dagger. \tag{4.6}$$

c) With the normalization scheme we used in class, we have that

$$\hat{a}_+ = \frac{1}{\sqrt{2m}} \left(\hat{p} + im\omega\hat{x} \right). \tag{4.7}$$

Taking the Hermitian conjugate, the constant out front doesn't change since it's real, and we know that $\hat{x}$ and $\hat{p}$ are Hermitian, so the only thing that changes is that the i picks up a negative sign:

$$(\hat{a}_+)^\dagger = \frac{1}{\sqrt{2m}} \left(\hat{p} - im\omega x \right). \tag{4.8}$$

Incidentally, this says that

$$(\hat{a}_+)^\dagger = \hat{a}_-. \tag{4.9}$$

Problem 5. (3.18)

For all of these, the last term involving the rate of change of the operator will be zero since none of the operators depend on time.

a) Starting with $Q = 1$, the entire right-hand side is zero since $[\hat{H}, 1] = \hat{H} - \hat{H} = 0$ and 1 obviously has no time dependence, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle 1\rangle\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\psi|\psi\rangle\right) = 0. \tag{5.1}$$

This is something we proved before - the normalization of the wave-function is independent of time, so we can normalize it at the most convenient time (usually $t = 0$), and we are set forever.

b) For $\hat{Q} = \hat{H}$, it obviously commutes with itself and since it almost never has time dependence, this means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle H\rangle = 0, \tag{5.2}$$

which is just conservation of energy! Technically, it's that measurements of the energy on any given system are guaranteed to give back the same energy; it cannot change with time so it cannot somehow gain a different energy sometime later.

c) For $\hat{Q} = \hat{x}$, we first need to look at the commutator

$$[\hat{H}, x] = \left[\left(\frac{p^2}{2m} + V\right), x\right] = \frac{1}{2m}[p^2, x], \tag{5.3}$$

since x obviously commutes with $V(x)$. Now,

$$[p^2, x] = p^2x - xp^2 = p\cdot px - xp\cdot p = p[p,x] - pxp - [x,p]p + pxp = -p[x,p] - [x,p]p = -2i\hbar p, \tag{5.4}$$

so

$$[\hat{H}, x] = -\frac{i\hbar p}{m}. \tag{5.5}$$

Plugging this in:

$$\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\left\langle\left(-\frac{i\hbar p}{m}\right)\right\rangle = \frac{\langle p\rangle}{m}, \tag{5.6}$$

or

$$m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \langle p\rangle. \tag{5.7}$$

This is just what we had before when doing all the problems with determining the expectation values for various wavefunctions!

d) For $\hat{Q} = \hat{p}$, $[p^2, p] = p^3 - p^3 = 0$ and (using a test function f)

$$\left[V, -i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right] f = -i\hbar V\frac{\mathrm{d}f}{\mathrm{d}x} + i\hbar\frac{\mathrm{d}}{\mathrm{d}x}(Vf) = i\hbar\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{5.8}$$

So, plugging everything in:

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\left(-\hbar\frac{\mathrm{d}V}{\mathrm{d}x}\right) = -\left\langle\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle. \tag{5.9}$$

This is the same as Equation (1.28) in Griffiths, which is **Ehrenfest's theorem**.