# HW6

# PHYS4500: Quantum Field Theory

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## Problem 1. (13.1)

We have the Lagrangian

$$\mathcal{L} = \frac{1}{2}|\phi\partial_\mu\phi|^2 - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}|\partial_\mu\eta|^2 - \frac{1}{2}m_\eta^2\eta^2 - \lambda n\phi. \tag{1.1}$$

We can define the rotated fields

$$\phi' = \cos(\theta)\phi + \sin(\theta)\eta, \quad \text{and} \tag{1.2}$$

$$\eta' = -\sin(\theta)\phi + \cos(\theta)\eta, \tag{1.3}$$

and recover another Lagrangian

or, since this is an element of $SO(2)$ we can get its inverse by taking the transpose, so we can also state that

$$\phi = \cos\theta\ \phi' - \sin\theta\ \eta, \quad \text{and} \tag{1.4}$$

$$\eta = \sin\theta\ \phi' + \cos\theta\ \eta. \tag{1.5}$$

We want to show that this Lagrangian, under a rotation of the fields like this, is physically equivalent to the Lagrangian:

$$\mathcal{L} = \frac{1}{2}|\phi\partial_\mu\phi'|^2 - \frac{1}{2}m_1^2\phi'^2 + \frac{1}{2}|\partial_\mu\eta'|^2 - \frac{1}{2}m_2^2\eta'^2 \tag{1.6}$$

for a suitable value of $\theta$. To look at this, let's plug Eqs. (1.4) and (1.5) into the original Lagrangian. First, we consider the kinetic terms; also, for notational simplicity, we will let $c \equiv \cos\theta$ and $s \equiv \sin\theta$:

$$|\partial_\mu\phi|^2 = (c\partial_\mu\phi' - s\partial_\mu\eta')\,(c\partial^\mu\phi' - s\partial^\mu\eta') = c^2|\partial_\mu\phi'|^2 + s^2|\partial_\mu\eta'|^2 - 2sc\partial_\mu\phi'\partial^\mu\eta',$$
$$|\partial_\mu\eta|^2 = s^2|\partial_\mu\phi'|^2 + c^2|\partial_\mu\eta'|^2 + 2sc\partial_\mu\phi'\partial^\mu\eta'.$$

Adding the two together removes the cross terms, and by using the all famous $c^2 + s^2 = 1$, any $\theta$ dependence cancels too, so it turns out the kinetic terms are invariant under this change. For the mass terms, we have

$$-\frac{1}{2}m_\phi^2\phi^2 = -\frac{1}{2}m_\phi^2(c^2\phi'^2 + s^2\eta'^2 - 2cs\phi'\eta'),$$
$$-\frac{1}{2}m_\eta^2\eta^2 = -\frac{1}{2}m_\eta^2(s^2\phi'^2 + c^2\eta'^2 + 2cs\phi'\eta'),$$

and for the cross term:

$$-\lambda\phi\eta = -\lambda(cs\phi'^2 - cs\eta'^2 + c^2\phi'\eta' - s^2\phi'\eta').$$

Grouping terms proportial to $\phi'^2$, $\eta'^2$, and $\phi'\eta'$:

$$-\frac{1}{2}\phi'^2(m_\phi^2c^2 + m_\eta^2s^2 + 2\lambda cs) - \frac{1}{2}\eta'^2(m_\phi^2s^2 + m_\eta^2c^2 - 2\lambda cs) - \phi'\eta'(-m_\phi^2cs + m_\eta^2cs + \lambda c^2 - \lambda s^2).$$

Now, in our new Lagrangian, we want to eliminate this last term, so we can set the expression in the parentheses equal to zero:

$$cs(m_\eta^2 - m_\phi^2) + \lambda(c^2 - s^2) = 0,$$
$$\lambda = \frac{cs(m_\phi^2 - m_\eta^2)}{(c^2 - s^2)} = \frac{1}{2}\frac{\sin 2\theta(m_\phi^2 - m_\eta^2)}{\cos 2\theta} = \frac{1}{2}\tan 2\theta(m_\phi^2 - m_\eta^2).$$

Now if we solve for $\theta$, we find:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\lambda}{m_\phi^2 - m_\eta^2}\right).$$

So, rotating our fields by this angle will eliminate the cross term. Now, we need to determine what the new mass terms are in relation to the previous mass terms and the coupling constant:

$$\begin{cases} m_1^2 = m_\phi^2 c^2 + m_\eta^2 s^2 + 2\lambda cs, \\ m_1^2 = m_\phi^2 s^2 + m_\eta^2 c^2 - 2\lambda cs. \end{cases}$$

I'm not sure if we have to simplify this any further; with our definition of $\theta$, it'll get a little ugly, so I'll leave it here since we have everything we need: we've defined $\theta$, and we have it in terms of $m_\phi^2$, $m_\eta^2$, and $\lambda$.

## Problem 2.

We need to show the relation

$$T\left\{\hat{\phi}_I(x^\mu)\hat{\phi}_I(y^\mu)\right\} \to T\left\{\phi(x)\phi(y)\right\} =: \phi(x)\phi(y) : + \left[\phi^+(y), \phi^-(x)\right] \tag{2.1}$$

for $x^0 < y^0$. First, with this condition, the time-ordered product switches the order

$$\begin{aligned} T\left\{\phi(x)\phi(y)\right\} &= \phi(y)\phi(x), \\ &= [\phi^+(y) + \phi^-(y)][\phi^+(x) + \phi^-(x)], \\ &= \phi^+(y)\phi^+(x) + \phi^+(y)\phi^-(x) + \phi^-(y)\phi^+(x) + \phi^-(y)\phi^-(x). \end{aligned}$$

What we can note is that $[\phi^\pm(y), \phi^\pm(x)] = 0$ (both plus or both minus), since, when we do this commutation, we will have either $[a(\mathbf{y}), a(\mathbf{x})]$ or $[a^\dagger(\mathbf{y}), a^\dagger(\mathbf{x})]$ which we know to be zero. Then, in the interest of normal-ordering, the only term that needs to change in the second term, which when we flip the order of the product, we pick up its commutator:

$$T\left\{\phi(x)\phi(y)\right\} = \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^-(y) + [\phi^+(y), \phi^-(x)].$$

But, the first four terms are just the normal-ordering of the product of the two original fields : $\phi(x)\phi(y)$ (this is why commuted the + and - terms), so what we have is

$$T\left\{\phi(x)\phi(y)\right\} =: \phi(x)\phi(y) : +[\phi^+(y), \phi^-(x)],$$

which is what we expect.

## Problem 3.

We are looking to compute

$$T\left\{\phi_1\phi_2\phi_3\phi_4\phi_5\right\}.$$

There are going to be a large number of terms here, so to better organize it for myself, I'll split it up into terms with only a single contraction and terms with a double contraction. For the single contractions terms, we have:

$$\begin{aligned} \text{single contractions} &= D_{12}\phi_3\phi_4\phi_5 + D_{13}\phi_2\phi_4\phi_5 + D_{14}\phi_2\phi_3\phi_5 + D_{15}\phi_2\phi_3\phi_4 \\ &+ D_{23}\phi_1\phi_4\phi_5 + D_{24}\phi_1\phi_3\phi_5 + D_{25}\phi_1\phi_3\phi_4 + D_{34}\phi_1\phi_2\phi_5 + D_{35}\phi_1\phi_2\phi_4 + D_{45}\phi_1\phi_2\phi_3. \end{aligned}$$

Then, for double contractions we can look at in terms of the leftover field. For any given left-over field, we will have three different ways to double contract the remaining four fields:

$$
\begin{aligned}
\text{double contractions } &= (D_{23}D_{45} + D_{24}D_{35} + D_{25}D_{34})\phi_1 \\
&+ (D_{13}D_{45} + D_{14}D_{35} + D_{15}D_{34})\phi_2 \\
&+ (D_{12}D_{45} + D_{14}D_{25} + D_{15}D_{24})\phi_3 \\
&+ (D_{12}D_{35} + D_{13}D_{25} + D_{15}D_{23})\phi_4 \\
&+ (D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23})\phi_5.
\end{aligned}
$$

So, joining everything together (not really sure how to best format this):

$$
\begin{aligned}
T\{\phi_1\phi_2\phi_3\phi_4\phi_5\} &= :\phi_1\phi_2\phi_3\phi_4\phi_5: + D_{12}\phi_3\phi_4\phi_5 + D_{13}\phi_2\phi_4\phi_5 + D_{14}\phi_2\phi_3\phi_5 + D_{15}\phi_2\phi_3\phi_4 && (3.1)\\
&+ D_{23}\phi_1\phi_4\phi_5 + D_{24}\phi_1\phi_3\phi_5 + D_{25}\phi_1\phi_3\phi_4 + D_{34}\phi_1\phi_2\phi_5 + D_{35}\phi_1\phi_2\phi_4 + D_{45}\phi_1\phi_2\phi_3 && (3.2)\\
&+ (D_{23}D_{45} + D_{24}D_{35} + D_{25}D_{34})\phi_1 + (D_{13}D_{45} + D_{14}D_{35} + D_{15}D_{34})\phi_2 && (3.3)\\
&+ (D_{12}D_{45} + D_{14}D_{25} + D_{15}D_{24})\phi_3 + (D_{12}D_{35} + D_{13}D_{25} + D_{15}D_{23})\phi_4 && (3.4)\\
&+ (D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23})\phi_5. && (3.5)
\end{aligned}
$$