Noether's theorem

Consider a continuous transformation on field φ:

$$\varphi(x^\mu) \to \varphi'(x^\mu) = \varphi(x^\mu) + a\Delta\varphi(x)$$

with $\underline{a}$ an infinitesimal parameter

This is a symmetry if the equations of motion (the Euler-Lagrange eqs.) remain invariant. This is the case if $\mathcal{L}$ is invariant up to a divergence

$$\mathcal{L}(x^\mu) \to \mathcal{L}(x^\mu) + a\,\partial_\mu J^\mu(x^\mu) = \mathcal{L}(x^\mu) + a\Delta\mathcal{L}(x^\mu)$$

Then $$a\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}(a\Delta\varphi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(a\Delta\varphi) = a\frac{\partial\mathcal{L}}{\partial\varphi}\Delta\varphi + a\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) - a\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\Delta\varphi$$

$$= -a\left[\underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi}}_{=0}\right]\Delta\varphi + a\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right)$$

Thus $$\partial_\mu J^\mu(x^\mu) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) \Rightarrow \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - J^\mu\right] = 0$$

$$\Rightarrow \partial_\mu j^\mu = 0 \quad \text{where} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - J^\mu$$

So current j^μ is conserved

Charge $Q = \int j^0 d^3x$

Consider, next, spacetime transformations

Infinitesimal translation $x^\mu \to x^\mu + a^\mu$

Then $\varphi(x^\mu) \to \varphi(x^\mu + a^\mu) = \varphi(x^\mu) + a^\mu \partial_\mu \varphi(x)$

Also $\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + g^{\mu\nu} a_\nu \partial_\mu \mathcal{L} = \mathcal{L} + a_\nu \partial_\mu (g^{\mu\nu} \mathcal{L})$

($\mathcal{L}$ is scalar) Since also $\mathcal{L} \to \mathcal{L} + a^\nu \partial_\mu J^\mu_\nu$ then $J^\mu_\nu = g^\mu_\nu \mathcal{L}$ or $J^{\mu\nu} = g^{\mu\nu} \mathcal{L}$

Then we have four conserved currents $T^{\mu\nu} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$

$T^{\mu\nu}$ is the energy-momentum tensor or stress-energy tensor

The Hamiltonian (energy) H is $H = \int T^{00} d^3x = \int \mathcal{H} d^3x$ (with $\mathcal{H}$ the Hamiltonian density)

and it is the conserved quantity associated with time translations.

Also the physical momenta $p^i = \int T^{0i} d^3x$

are the conserved quantities associated with spatial translations

Note that stress-energy tensors appear in field equations of general relativity
$$\underset{\text{Ricci tensor}}{\underset{\downarrow}{R^{\mu\nu}}} - \frac{1}{2} g^{\mu\nu} \underset{\text{Ricci scalar}}{\underset{\downarrow}{R}} = -8\pi G T^{\mu\nu}$$

Real scalar field (Klein-Gordon field) $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\partial^0\varphi - g^{00}\mathcal{L} = \partial^0\varphi\partial^0\varphi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2$$

So the field energy is $H = \int T^{00}d^3x = \frac{1}{2}\int\left((\partial^0\varphi)^2 + \vec\nabla\varphi\cdot\vec\nabla\varphi + m^2\varphi^2\right)d^3x$

which is positive definite so not affected by negative-energy problem

Conjugate (or canonical) momentum $\pi(x^\mu) = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = \dot\varphi(x^\mu)$

Also Hamiltonian density $\mathcal{H}$ is given by

$\mathcal{H} = \pi\dot\varphi - \mathcal{L} = \dot\varphi\dot\varphi - \mathcal{L} = T^{00}$ consistent with above expressions

Fourier expansion for the field φ:

$$\varphi(x^\mu) = \int\frac{d^4p}{(2\pi)^4}\,2\pi\,\delta(p^2-m^2)\,\theta(p^0)\left[a(p^\mu)e^{-ip^\mu x_\mu} + a^\dagger(p^\mu)e^{ip^\mu x_\mu}\right](2p_0)^{1/2}$$

$$\Rightarrow \varphi(x) = \int\frac{d^3p}{(2\pi)^{3/2}(2p^0)^{1/2}}\left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right]$$

where $p^0 = (\vec p^{\,2} + m^2)^{1/2}$

Field φ is a Hermitian operator $\varphi^\dagger = \varphi$

Real scalar field φ

"second quantization"

Equal-time commutation relations

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$

$$[\varphi(\vec{x},t), \varphi(\vec{y},t)] = 0 \quad \text{and} \quad [\pi(\vec{x},t), \pi(\vec{y},t)] = 0$$

(these are the analogs of $[\hat{x}, \hat{p}_x] = i$, $[\hat{x}, \hat{x}] = 0$; $[\hat{p}_x, \hat{p}_x] = 0$)

Inverse Fourier transform to get operators $a(p)$ and $a^\dagger(p)$:

$$a(p) = \int d^3x\, (2p^0)^{-1/2}\, i\left[e^{ip\cdot x}\partial_0\varphi(x) - (\partial_0 e^{ip\cdot x})\varphi(x)\right]$$

and $$a^\dagger(p) = \int d^3x\, (2p^0)^{-1/2}\, i\left[\varphi(x)\,\partial_0 e^{-ip\cdot x} - (\partial_0\varphi(x))e^{-ip\cdot x}\right]$$

Then $$[a(p), a^\dagger(p')] = (2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')$$

$$[a(p), a(p')] = 0 \quad \text{and} \quad [a^\dagger(p), a^\dagger(p')] = 0$$

Construct operator $N(p) = a^\dagger(p)\,a(p)$ with eigenkets $|n(p)\rangle$ and eigenvalues $n(p)$: $\quad N(p)|n(p)\rangle = n(p)|n(p)\rangle$

We have $[N(p), a(p)] = -a(p)$ and $[N(p), a^\dagger(p)] = a^\dagger(p)$

(compare with simple harmonic oscillator in quantum mechanics)