Final Exam
PHYS4500: Quantum Field Theory

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Problem 1.

The QED Lagrangian is given by

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu \psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.1}$$

For the purposes of finding the Euler-Lagrange equation for the Dirac spinor $\overline{\psi}$, we don't really care about the last term with the field tensors. Also, we can easily tell that the term

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) = 0 \tag{1.2}$$

since there are no terms like $\partial_\mu \overline{\psi}$. So,

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 \quad \rightarrow \quad i\gamma^\mu \partial_\mu \psi - m\psi - q\gamma^\mu \psi A_\mu = 0 \quad \rightarrow \quad \boxed{\left(i\partial\!\!\!/ - m - qA\!\!\!/\right)\psi = 0.} \tag{1.3}$$

The EW (GWS) Lagrangian, before electroweak symmetry breaking (where the fermions are massless) is given by

$$\mathcal{L}_{EW} = i\bar{L}\gamma^\mu D_\mu L + \bar{e}_R\gamma^\mu D_\mu e_R - \frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{1.4}$$

where L is the left-handed lepton doublet and e_R is the right-handed lepton singlet. The covarient derivative is defined as

$$D_\mu L = \partial_\mu L + \frac{i}{2}g\left(\sigma^a W^a_\mu\right)L - \frac{i}{2}g' B_\mu L, \quad \text{and} \tag{1.5}$$

$$D_\mu e_R = \partial_\mu e_R - ig' B_\mu e_R. \tag{1.6}$$

Again, we don't care about the field-tensor terms for either the $SU(2)$ or $U(1)$ gauge fields, and there are no terms like $\partial_\mu \overline{\psi}$ for either the doublet or singlet. The terms for the individual spinors within the doublet will be identical, so I'll just leave it in the doublet. Starting with this doublet, I'll expand out the covarient derivative into one expression:

$$\mathcal{L}_{\text{EW,L}} = i\bar{L}\gamma^\mu \partial_\mu L - \frac{g}{2}\bar{L}\gamma^\mu\left(\sigma^a W^a_\mu\right)L + \frac{g'}{2}\bar{L}\gamma^\mu B_\mu L. \tag{1.7}$$

From here,

$$\frac{\partial \mathcal{L}}{\partial \bar{L}} = 0 \tag{1.8}$$

$$\rightarrow \quad i\partial\!\!\!/ L - \frac{g}{2}\gamma^\mu \sigma^a W^a_\mu L + \frac{g'}{2}B\!\!\!/ L = 0 \tag{1.9}$$

$$\rightarrow \quad \boxed{\left(i\partial\!\!\!/ - \frac{g}{2}\sigma^a W\!\!\!/^a + \frac{g'}{2}B\!\!\!/\right)L = 0.} \tag{1.10}$$

The Euler-Lagrange equation for the singlet is similar, but without any $SU(2)$ terms and a different weak hypercharge:

$$\boxed{\left(i\partial\!\!\!/ + g'B\!\!\!/\right)e_R = 0.} \tag{1.11}$$

The QCD Lagrangian is given by

$$\mathcal{L}_{\text{QCD}} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi - \frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} + \mathcal{L}_{\text{g-f}} + \mathcal{L}_{\text{ghost}}, \tag{1.12}$$

where $\mathcal{L}_{\text{g-f}}$ corresponds to gauge-fixing terms, and the covariant term is given by

$$D_\mu \psi = \partial_\mu \psi + i g_s T^a G^a_\mu. \tag{1.13}$$

Expanding this out and considering only the terms we care about for this calculation:

$$\mathcal{L}_{\text{QCD}} \to i\overline{\psi}\gamma^\mu \partial_\mu \psi - g_s \overline{\psi}\gamma^\mu \left(T^a G^a_\mu\right)\psi - m\overline{\psi}\psi. \tag{1.14}$$

In principle, there should be flavor and color indices on the spinors here, but that doesn't make any meaningful changes when it comes to the form of the Lagrangian and the Euler-Lagrange equations. Thus,

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 \tag{1.15}$$

$$\to \quad i\partial\!\!\!/\psi - g_s \gamma^\mu T^a G^a_\mu \psi - m\psi = 0 \tag{1.16}$$

$$\to \quad \boxed{\left(i\partial\!\!\!/ - g_s T^a G\!\!\!/^a - m\right) = 0.} \tag{1.17}$$

Problem 2.

The parts of the QCD Lagrangian containing only the spinors is:

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi, \tag{2.1}$$

where the covariant derivative is

$$D_\mu \psi = \partial_\mu \psi + i g_s T^a G^a_\mu \psi. \tag{2.2}$$

Under a local $SU(3)$ gauge transformation, we have that

$$\psi' \to e^{iT^a\theta^a}\psi, \tag{2.3}$$

$$\overline{\psi}' \to \overline{\psi}e^{-iT^a\theta^a}, \quad \text{and} \tag{2.4}$$

$$G^a_\mu \to G^a_\mu - \frac{1}{g_s}\partial_\mu \theta^a - f^{abc}\theta^b G^c_\mu. \tag{2.5}$$

If we can show how the covariant derivative transforms first, then it will be trivial to show how the Lagrangian as a whole will change, so I'll do that one first. We have that

$$(D_\mu\psi)' = \partial_\mu\left(e^{iT^a\theta^a}\psi\right) + i g_s T^a\left[G^a_\mu - \frac{1}{g_s}\left(\partial_\mu\theta^a\right) - f^{abc}\theta^b G^c_\mu\right]e^{iT^d\theta^d}\psi \tag{2.6}$$

where I have used different dummy indices in the exponentials in each term to differentiate them from the indices in the rest of the term. Working this out,

$$= iT^a\left(\partial_\mu\theta^a\right)e^{iT^b\theta^b}\psi + e^{iT^a\theta^a}\partial_\mu\psi + i g_s T^a G^a_\mu e^{iT^b\theta^b}\psi - i\left(\partial_\mu\theta^a\right)T^a e^{iT^b\theta^b}\psi - i g_s f^{abc}\theta^b T^a G^c_\mu e^{iT^d\theta^d}\psi \tag{2.7}$$

The first and fourth terms cancel:

$$= e^{iT^a\theta^a}\partial_\mu\psi + i g_s T^a G^a_\mu e^{iT^b\theta^b}\psi - i g_s f^{abc}\theta^b T^a G^c_\mu e^{iT^d\theta^d}\psi. \tag{2.8}$$

In the last term, we have that

$$i f^{abc} T^a = i f^{bca} T^a = [T^b, T^c]. \tag{2.9}$$

Additionally, since we are using the assumption that θ is small, we can expand the exponentials out in their power series expansion and eliminate terms of order $\mathcal{O}(\theta^2)$. With this,

$$= e^{iT^a\theta^a}\partial_\mu\psi + ig_sT^aG^a_\mu(1+iT^b\theta^b)\psi - g_s\theta^b[T^b,T^c]G^c_\mu(1+iT^d\theta^d)\psi. \tag{2.10}$$

No T^d terms remain in the final term since they would be of $\mathcal{O}(\theta^2)$. We can also rename some indices and flip the commutator at the cost of a minus:

$$= e^{iT^a\theta^a}\partial_\mu\psi + ig_sT^aG^a_\mu\psi - g_s\theta^bT^aT^bG^a_\mu\psi + g_s\theta^b[T^a,T^b]G^a_\mu\psi. \tag{2.11}$$

The third term cancels with the first term from the commutator:

$$\begin{aligned} &= e^{iT^a\theta^a}\partial_\mu\psi + ig_sT^aG^a_\mu\psi - g_s\theta^bT^bT^aG^a_\mu\psi && (2.12)\\ &= e^{iT^a\theta^a}\partial_\mu\psi + (1+iT^b\theta^b)\left(ig_sT^aG^a_\mu\psi\right). && (2.13)\end{aligned}$$

Undoing the power series expansion:

$$(D_\mu\psi)' = e^{iT^b\theta^b}\left(\partial_\mu\psi + ig_sT^aG^a_\mu\psi\right) = \boxed{e^{iT^b\theta^b}D_\mu\psi.} \tag{2.14}$$

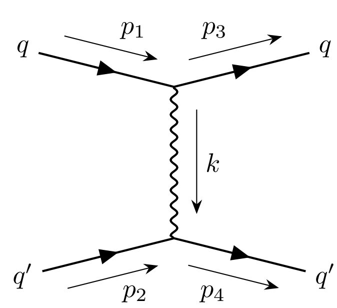
With this, then:

$$\mathcal{L}' = i\overline{\psi}\gamma^\mu e^{-iT^a\theta^a}e^{iT^a\theta^a}D_\mu\psi - m\overline{\psi}e^{-iT^a\theta^a}e^{iT^a\theta^a}\psi = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi = \mathcal{L}. \tag{2.15}$$

As expected, these terms are invariant under a local $SU(3)$ gauge transformation.

Problem 3.

We are considering the process in the Feynman diagram below, where the boson is either a photon, a Z-boson, or a gluon.



Considering first the case when the boson is a photon we get:

$$\begin{aligned} i\mathcal{M} &= \bar{u}(p_4)\left(-ie\gamma^\mu\right)u(p_2)\left(\frac{-ig_{\mu\nu}}{k^2}\right)\bar{u}(p_3)\left(-ie\gamma^\nu\right)u(p_1) && (3.1)\\ \mathcal{M} &= \frac{e^2}{(p_1-p_3)^2}[\bar{u}(p_4)\gamma^\mu u(p_2)][\bar{u}(p_3)\gamma_\mu u(p_1)]. && (3.2)\end{aligned}$$

If the boson is a Z-boson:

$$i\mathcal{M} = \bar{u}(p_4)\left[\frac{-ie}{\sin(2\theta_W)}\gamma^\mu\left(c_v^1 - c_A^1\gamma^5\right)\right]u(p_2)\left[\frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_z^2}\right)}{k^2-m_z^2}\right]\bar{u}(p_3)\left[\frac{-ie}{\sin(2\theta_W)}\gamma^\mu\left(c_v^2 - c_A^2\gamma^5\right)\right]u(p_1). \tag{3.3}$$

$$\mathcal{M} = \frac{e^2}{[(p_1-p_3)^2 - m_z^2]\sin^2(\theta_W)} \left[\bar{u}(p_4)\gamma^\mu \left(c_v^1 - c_A^1\gamma^5\right) u(p_2)\right]$$
$$\times \left(g_{\mu\nu} - \frac{(p_1-p_3)_\mu (p_1-p_3)_\nu}{m_z^2}\right) \left[\bar{u}(p_3)\gamma^\mu \left(c_v^2 - c_A^2\gamma^5\right) u(p_1)\right]. \quad (3.4)$$

where the superscripts on the c_v and c_A are meant to indicate the quark and primed quark.

Or, in the case that we are at very low energies where $k^2 << m_z^2$, we can disregard the second term in the numerator of the propagator and simplify its denominator to get

$$\mathcal{M} = \frac{-e^2}{m_z^2\sin^2(\theta_W)}[\bar{u}(p_4)\gamma^\mu(c_v^1 - c_A^1\gamma^5)u(p_2)][\bar{u}(p_3)\gamma_\nu(c_v^2 - c_A^2\gamma^5)u(p_1)]. \quad (3.5)$$

In the case that the boson is a gluon, we get

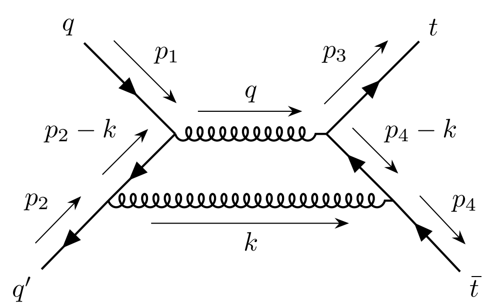
$$i\mathcal{M} = \bar{u}_i(p_4)\left(-ig_sT^a_{ij}\gamma^\mu\right)u_j(p_2)\left[\frac{-ig_{\mu\nu}\delta^{ab}}{k^2}\right]\bar{u}_k(p_3)\left(-ig_sT^b_{k\ell}\gamma^\nu\right)u_\ell(p_1) \quad (3.6)$$

$$\mathcal{M} = \frac{g_s^2}{(p_1-p_3)^2}[\bar{u}(p_4)\gamma^\mu u(p_2)][\bar{u}(p_3)\gamma_\mu u(p_1)]T^a_{ij}T^a_{k\ell}. \quad (3.7)$$

Just like with one of the previous homeworks, there is the possibility to use an identity for the color factor, but it wouldn't make any nice simplifications until we square it, so this is fine as it is for just the amplitude.

Problem 4.

First, we consider the process shown in this Feynman diagram:



Using our Feynman rules:

$$i\mathcal{M} = \int \frac{\mathrm{d}^n k}{(2\pi)^n}\bar{u}_i(p_3)\left(-ig_sT^a_{ij'}\gamma^\mu\right)\left(\frac{i(-\not{p}_4 + \not{k} + m)}{(k-p_4)^2 - m^2}\right)\left(-ig_sT^c_{j'j}\gamma^\sigma\right)v_j(p_4)$$
$$\times\left(\frac{-ig_{\mu\nu}\delta^{ab}}{q^2}\right)\left(\frac{-ig_{\rho\sigma}\delta^{cd}}{k^2}\right)\bar{v}_k(p_2)\left(-ig_sT^d_{kk'}\gamma^\sigma\right)\left(\frac{i(-\not{p}_2 + \not{k} + m)}{(k-p_2)^2 - m^2}\right)\left(-ig_sT^b_{k'\ell}\gamma^\nu\right)u_\ell(p_1). \quad (4.1)$$

$$i\mathcal{M} = g_s^4(T^aT^c)_{ij}(T^cT^a)_{k\ell}\int\frac{\mathrm{d}^4k}{(2\pi)^n}\frac{\bar{u}(p_3)\gamma^\mu(\not{k} - \not{p}_4 + m)\gamma_\sigma v(p_4)}{(k-p_4)^2 - m^2}$$
$$\times\frac{1}{(p_1+p_2-k)^2k^2}\cdot\frac{\bar{v}(p_2)\gamma^\sigma(\not{k} - \not{p}_2 + m)\gamma_\mu u(p_1)}{(k+p_2)^2 - m^2}. \quad (4.2)$$

The generic formula for the Feynman parameters is

$$
\prod_{i=1}^{n} \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \int_0^1 \left(\prod_{i=1}^{n} \mathrm{d}x_i \; x_i^{\alpha_i - 1} \right) \frac{\delta(1-x)}{\left(\sum_{i=1}^{n} x_i A_i\right)^{\alpha}}, \tag{4.3}
$$

where $x = \sum_i x_i$ and $\alpha = \sum_i \alpha_i$. With this, then,

$$
\begin{aligned}
\frac{1}{ABCD} &= \Gamma(4) \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}\alpha \; \frac{\delta(1-x-y-z-\alpha)}{(xA+yB+zC+\alpha D)^4} && (4.4) \\
&= 6 \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z \; \frac{1}{[xA+yB+zC+(1-x-y-z)D]^4}. && (4.5)
\end{aligned}
$$

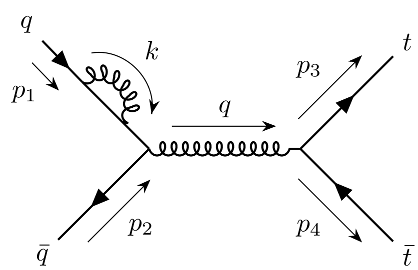
So,

$$
\begin{aligned}
\frac{1}{[(k-p_4)^2 - m^2](p_1+p_2-k)^2 k^2 [(k-p_2)^2 - m^2]} &= 6 \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
\times \frac{1}{\left\{ x\left[(k-p_4^2) - m^2\right] + y(p_1+p_2-k)^2 + zk^2 + (1-x-y-z)\left[(k-p_2)^2 - m^2\right] \right\}^4} & \qquad (4.6)
\end{aligned}
$$

Using Feynman parameters, our amplitude is

$$
\begin{aligned}
i\mathcal{M} = g_s^4 (T^a T^c)_{ij} (T^c T^a)_{k\ell} \int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z \int \frac{\mathrm{d}^n k}{(2\pi)^n} & \\
\times \frac{\bar{u}(p_3)\gamma^{\mu}(\not{k} - \not{p}_4 + m)\gamma_{\sigma} v(p_4) \bar{v}(p_2) \gamma^{\sigma} (\not{k} - \not{p}_2 + m)\gamma_{\mu} u(p_1)}{\left\{ x\left[(k-p_4)^2 - m^2\right] + y(p_1+p_2-k)^2 + zk^2 + (1-x-y-z)\left[(k-p_2)^2 - m^2\right] \right\}^4} & \qquad (4.7)
\end{aligned}
$$

This is as far as we are going, so I'll leave it here.
Next, we have the diagram



Following the Feynman rules:

$$
\begin{aligned}
i\mathcal{M} = \int \frac{\mathrm{d}^n k}{(2\pi)^n} \; \bar{u}_i(p_3)(-ig_s T^a_{ij} \gamma^{\mu}) v_j(p_4) \left(\frac{-i g_{\mu\nu} \delta^{ab}}{q^2} \right) \bar{v}_k(p_2)(-ig_s T^b_{k\ell'} \gamma^{\nu}) & \\
\times \frac{i(\not{p}_1 + m)}{p_1^2 - m^2 + i\varepsilon} (-ig_s T^c_{\ell' m} \gamma^{\sigma}) \frac{i(\not{p}_1 - \not{k} + m)}{(p_1-k)^2 - m^2 + i\varepsilon} (-ig_s T^d_{m'\ell} \gamma^{\rho}) u_{\ell}(p_1) \delta^{mm'} \delta^{\ell\ell'} \left(\frac{-i g_{\sigma\rho} \delta^{cd}}{k^2} \right). & \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M} = \frac{g_s^4}{(p_1+p_2)^2} T^a_{ij} T^b_{k\ell'} T^c_{\ell' m} T^d_{m'\ell} \, \delta^{mm'} \delta^{\ell\ell'} \delta^{ab} \delta^{cd} & \\
\times \int \frac{\mathrm{d}^n k}{(2\pi)^n} \; \frac{\bar{u}(p_3)\gamma^{\mu} v(p_4) \bar{v}(p_2) \gamma_{\mu} (\not{p}_1 + m) \gamma^{\sigma} (\not{p}_1 - \not{k} + m) \gamma_{\sigma} u(p_1)}{(p_1^2 - m^2 + i\varepsilon)[(p_1-k)^2 - m^2 + i\varepsilon] k^2}. & \qquad (4.9)
\end{aligned}
$$

$$i\mathcal{M} = \frac{g_s^4 T^a_{ij}(T^aT^cT^c)_{k\ell}}{(p_1+p_2)^2(p_1^2-m^2+i\varepsilon)} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{\bar{u}(p_3)\gamma^\mu v(p_4)\bar{v}(p_2)\gamma_\mu(\not{p}_1+m)\gamma^\sigma(\not{p}_1-\not{k}+m)\gamma_\sigma u(p_1)}{[(p_1-k)^2-m^2]k^2}. \quad (4.10)$$

We have two propagator denominators in the integral:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[xA+(1-x)B]^2}, \quad (4.11)$$

so in our case we have

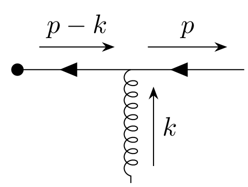
$$\frac{1}{[(p_1-k)^2-m^2]k^2} = \int_0^1 \mathrm{d}x \, \frac{1}{\{x[(p_1-k)^2-m^2]+(1-x)k^2\}^2}, \quad (4.12)$$

so our entire amplitude looks like:

$$\begin{aligned} i\mathcal{M} = \frac{g_s^4 T^a_{ij}(T^aT^cT^c)_{k\ell}}{(p_1+p_2)^2(p_1^2-m^2+i\varepsilon)} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n k}{(2\pi)^n} \\ \times \frac{\bar{u}(p_3)\gamma^\mu v(p_4)\bar{v}(p_2)\gamma_\mu(\not{p}_1+m)\gamma^\sigma(\not{p}_1-\not{k}+m)\gamma_\sigma u(p_1)}{\{x[(p_1-k)^2-m^2]+(1-x)k^2\}^2}. \quad (4.13) \end{aligned}$$

Problem 5.

We are considering the eikonal rule for the outgoing anti-quark, meaning we are looking at the following diagram:



Since the propagator has a "time" direction opposite that of its momentum, its momentum picks up a minus, meaning we have

$$\begin{aligned} \rightarrow \quad & \frac{i(-\not{p}+\not{k}+m)}{(p-k)^2-m^2}(-ig_s T^a\gamma^\mu)v(p) && (5.1) \\ & = g_s T^a \frac{(-\not{p}+m)}{-2p\cdot k}\gamma^\mu v(p), && (5.2) \end{aligned}$$

where I've taken $k \to 0$ here. Looking at the numerator and everything to the right of the fraction, we have

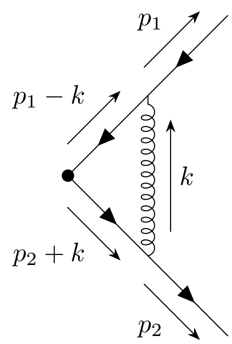
$$\begin{aligned} &= (-p_\nu\gamma^\nu\gamma^\mu + m\gamma^\mu)v(p) && (5.3) \\ &= [-p_\nu(2g^{\nu\mu}-\gamma^\mu\gamma^\nu)+m\gamma^\mu]v(p) && (5.4) \\ &= (-2p^\mu + \gamma^\mu\not{p} + m\gamma^\mu)v(p) && (5.5) \\ &= -2p^\mu v(p) + \gamma^\mu(\not{p}+m)v(p) && (5.6) \\ &= -2p^\mu, && (5.7) \end{aligned}$$

where the last term in the second-to-last expression vanishes by virtue of the Dirac equation for anti-particles. Our Feynman rules then read

$$\rightarrow \quad g_s T^a \frac{-2p^\mu}{-2p\cdot k} v(p) = g_s T^a \frac{v^\mu}{v\cdot k} v(p), \tag{5.8}$$

where the v in the fraction is the four-velocity.

We now consider the one-loop cusp diagram



Just as in the lecture notes, I will ignore the color factors, since they amount to, just like in other calculations, a product of $SU(3)$ generators that isn't really simplifiable unless we square the amplitude. What this diagram amounts to is the product of the eikonal rule for an outgoing quark and that for an outgoing anti-quark, then we contract the two four-velocities with the gluon propagator (minus color factor terms). We also need an integral over all possible k momenta, and we won't consider the external spinors since we just want to look at the integral itself:

$$I = \int \frac{\mathrm{d}^n k}{(2\pi)^n}\, g_s \frac{v_2^\mu}{v_2\cdot k}\left(\frac{-ig_{\mu\nu}}{k^2}\right) g_s \frac{v_1^\nu}{v_1\cdot k} = -ig_s^2(v_2\cdot v_1)\int \frac{\mathrm{d}^n k}{(2\pi)^n}\, \frac{1}{(v_2\cdot k)k^2(v_1\cdot k)}. \tag{5.9}$$

We can use Feynman parameters to simplify this. We have already done the case with $\frac{1}{ABC}$ before:

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{1}{[xA + yB + (1-x-y)C]^3}. \tag{5.10}$$

Therefore, in our case, we have that

$$\frac{1}{(v_2\cdot k)k^2(v_1\cdot k)} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{1}{[x(v_2\cdot k) + yk^2 + (1-x-y)(v_1\cdot k)]^3}, \tag{5.11}$$

so the total integral for the one-loop cusp diagram is

$$I = -2ig_s^2(v_2\cdot v_1)\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^n k}{(2\pi)^n}\, \frac{1}{[x(v_2\cdot k) + yk^2 + (1-x-y)(v_1\cdot k)]^3}. \tag{5.12}$$

We are asked not to calculate the integral, so this is as far as I'll go.