Test 1
PHYS4500: Quantum Field Theory

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Problem 1.

We are to show that the first component of the conjugate momentum for the free Lagrangian for a gauge field with the Feynman gauge fixing term given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{1.1}$$

is

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu. \tag{1.2}$$

Firstly, it is easy to note that

$$\frac{\partial}{\partial \dot{A}_0}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = 0,$$

since the only components containing $\partial_0 A_0$ at all are the F^{00} terms, which are zero. Turning to the other term, we have:

$$-\frac{1}{2}\left(\partial_\mu A^\mu\right)^2 = -\frac{1}{2}\left(\partial_0 A^0 + \partial_i A^i\right)^2 = -\frac{1}{2}\left(\dot{A}_0 + \partial_i A^i\right)^2.$$

With this,

$$\frac{\partial}{\partial \dot{A}_0}\left(-\frac{1}{2}(\partial_\mu A^\mu)^2\right) = -\left(\dot{A}_0 + \partial_i A^i\right) = -\partial_\mu A^\mu,$$

as expected.

Problem 2.

We are to show that the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi - q\overline{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.1}$$

is invariant under a local phase transformation, where

$$\begin{aligned}
\psi \to \psi' &= e^{i\theta(x)}\psi, \\
\overline{\psi} \to \overline{\psi}' &= \overline{\psi}e^{-i\theta(x)}, \quad \text{and} \\
A_\mu \to A'_\mu &= A_\mu - \frac{1}{q}\partial_\mu\theta(x),
\end{aligned}$$

where x is the four-vector x^μ.

We have seen a number of times already that the mass term will be invariant; there are no derivatives to push the exponentials past, so we are good there. For the first term:

$$\begin{aligned}
i\overline{\psi}'\gamma^\mu\partial_\mu\psi' &= i\overline{\psi}e^{-i\theta(x)}\gamma^\mu\partial_\mu\left(e^{i\theta(x)}\psi\right) \\
&= i\overline{\psi}e^{-i\theta(x)}\gamma^\mu\left(\partial_\mu\psi\right)e^{i\theta(x)} + i\overline{\psi}e^{-i\theta(x)}\gamma^\mu\left(\partial_\mu e^{i\theta(x)}\right)\psi \\
&= i\overline{\psi}\gamma^\mu\partial_\mu\psi - \overline{\psi}e^{-i\theta(x)}\gamma^\mu\left(\partial_\mu\theta\right)e^{i\theta(x)\psi} \\
&= i\overline{\psi}\gamma^\mu\partial_\mu\psi - \overline{\psi}\gamma^\mu\left(\partial_\mu\theta\right)\psi.
\end{aligned}$$

For the third term:

$$
\begin{aligned}
-q\overline{\psi}'\gamma^\mu\psi' A'_\mu &= -q\overline{\psi}e^{-i\theta(x)}\gamma^\mu e^{i\theta(x)}\psi\left(A_\mu - \frac{1}{q}\partial_\mu\theta\right)\\
&= -q\overline{\psi}\gamma^\mu\psi A_\mu + \overline{\psi}\gamma^\mu\psi\left(\partial_\mu\theta\right).
\end{aligned}
$$

The last term in this expression exactly cancels the last term in the previous expression, leaving us with only terms from the original Lagrangian. Now, turning to the kinetic term for the gauge field, we can first examine just one of the stress-energy tensors:

$$
\begin{aligned}
F'_{\mu\nu} = (\partial_\mu A'_\nu - \partial_\nu A'_\mu) &= \left[\partial_\mu\left(A_\nu - \frac{1}{q}\partial\theta\right) - \partial_\nu\left(A_\mu - \frac{1}{q}\partial_\mu\theta\right)\right]\\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{q}\partial_\mu\partial_\nu\theta + \frac{1}{q}\partial_\nu\partial_\mu\theta).
\end{aligned}
$$

But, the second and third terms are equal since we can exchange the four-gradients without incurring any extra terms, meaning that they cancel, and the field-strength tensor is invariant under this transformation, and so too is its contraction with itself.

Therefore, the QED Lagrangian is invariant under a local $U(1)$ transformation.

Problem 3.

We want to prove the completeness relation for the spinors $u^{(s)}(p)$:

$$
\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p} + m. \tag{3.1}
$$

We are going to use the standard representation for the gamma matrices:

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{3.2}
$$

The spinors in question are (in natural units):

$$
u^{(1)}(p) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad \text{and} \quad u^{(2)}(p) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}. \tag{3.3}
$$

Considering the $s = 1$ case first:

$$
u^{(1)}(p)\bar{u}^{(1)}(p) = u^{(1)}(p)u^{\dagger(1)}(p)\gamma^0 = (E+m)\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m} \end{pmatrix}\gamma^0.
$$

We can easily see how the gamma interacts with this expression by considering

$$
\begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \end{pmatrix}
$$

So it makes the second half of the row vector to its left negative. Now,

$$
= \begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x + ip_y \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{-p_z}{E+m} & \frac{-p_x+ip_y}{E+m} \end{pmatrix},
$$

$$
= \begin{pmatrix} E+m & 0 & -p_z & -p_x+ip_y \\ 0 & 0 & 0 & 0 \\ p_z & 0 & \frac{-p_z^2}{E+m} & \frac{-p_z(p_x-ip_y)}{E+m} \\ p_x+ip_y & 0 & -\frac{p_z(p_x+ip_y)}{E+m} & -\frac{|p_x+ip_y|^2}{E+m} \end{pmatrix}.
$$

Now for the $s = 2$ case:

$$
u^{(2)}(p)\bar{u}^{(2)}(p) = (E+m) \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{p_x+ip_y}{E+m} & \frac{-p_z}{E+m} \end{pmatrix} \gamma^0
$$

$$
= \begin{pmatrix} 0 \\ E+m \\ p_x - ip_y \\ -p_z \end{pmatrix} \begin{pmatrix} 0 & 1 & -\frac{(p_x+ip_y)}{E+m} & \frac{p_z}{E+m} \end{pmatrix},
$$

$$
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & E+m & -(p_x+ip_y) & p_z \\ 0 & p_x-ip_y & -\frac{|p_x-ip_y|^2}{E+m} & \frac{p_z(p_x-ip_y)}{E+m} \\ 0 & -p_z & \frac{p_z(p_x+ip_y)}{E+m} & -\frac{p_z^2}{E+m} \end{pmatrix}.
$$

Summing the two together (which is the completeness relation) gives us:

$$
\sum_{s=1,2} u^{(s)}\bar{u}^{(s)}(p) = \begin{pmatrix} p^0+m & 0 & -p_z & -p_x+ip_y \\ 0 & p^0+m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -\frac{\mathbf{p}^2}{p^0+m} & 0 \\ p_x+ip_y & -p_z & 0 & -\frac{\mathbf{p}^2}{p^0+m} \end{pmatrix}.
$$

Simplifying the two terms in the bottom right "quadrant":

$$
\frac{-\mathbf{p}^2}{E+m} = \frac{E^2 - \mathbf{p}^2 - E^2}{E+m} = \frac{m^2 - E^2}{E+m} = \frac{(m-E)(m+E)}{E+m} = -p^0 + m,
$$

so we have:

$$
\sum_{s=1,2} u^{(s)}\bar{u}^{(s)}(p) = \begin{pmatrix} p^0+m & 0 & -p_z & -p_x+ip_y \\ 0 & p^0+m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -p^0+m & 0 \\ p_x+ip_y & -p_z & 0 & -p^0+m \end{pmatrix}.
$$

We can easily note the additive factor of $+m$ down the diagonal corresponding to the $+m\mathbf{1}$ in the completeness relation. Additionally, we can note the positive p^0's in the diagonal in the top left and the negiatve p^0's in the diagonal in the bottom right, which is the $\gamma^0 p^0$ factor. Let's now consider

$$
\mathbf{p} \cdot \boldsymbol{\sigma} = p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}.
$$

This is exactly what are in the off-diagonal quadrants; what we have then is

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)}(p) = \begin{pmatrix} p_0 & -\mathbf{p}\cdot\boldsymbol{\sigma} \\ \mathbf{p}\cdot\boldsymbol{\sigma} & -p_0 \end{pmatrix} + m = \gamma^0 p_0 + \begin{pmatrix} 0 & -\mathbf{p}\cdot\boldsymbol{\sigma} \\ \mathbf{p}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} + m.$$

Now, $\gamma^\mu p_\mu = \gamma^0 p_0 + \gamma^i p_i = \gamma^0 p_0 - \boldsymbol{\gamma}\cdot\mathbf{p}$, so if we bring back Lorentz indices in the middle term, we undo a minus sign, so

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)}(p) = \gamma^0 p_0 + p_i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + m = \gamma^0 p_0 + \gamma^i p_i + m = \boxed{\not{p} + m}.$$

Problem 4.

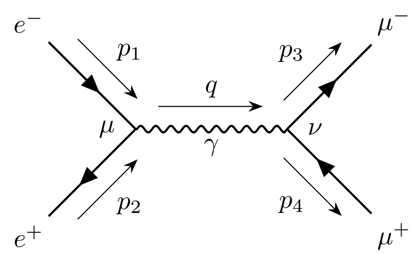

Figure 1: Single Feynman diagram contributing to $e^-(p_1) + e^+(p_2) \to \mu^-(p_3) + \mu^+(p_4)$.

For the process $e^-(p_1) + e^+(p_2) \to \mu^-(p_3) + \mu^+(p_4)$, there is only one Feynman diagram: the s-channel, as shown in Figure 1. We can pretty easily use the Feynman rules to write down the amplitude for this diagram:

$$\mathcal{M} = -i\,\bar{v}^{(s_2)}(p_2)(-ie\gamma^\mu)u^{(s_1)}(p_1)\left[\frac{-ig_{\mu\nu}}{q^2}\right]\bar{u}^{(s_3)}(p_3)(-ie\gamma^\nu)v^{(s_4)}(p_4).$$

For notational simplicity, I will let $u_i \equiv u^{(s_i)}(p_i)$. Additionally, by virtue of momentum conservation, the virtual photon momentum is contrained to $q = p_1 + p_2$. So,

$$\mathcal{M} = \frac{e^2}{(p_1+p_2)^2}[\bar{v}_2\gamma^\mu u_1][\bar{u}_3\gamma_\mu v_4].$$

Squaring this,

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^2}[\bar{v}_2\gamma^\mu u_1][\bar{u}_3\gamma_\mu v_4][\bar{v}_2\gamma^\nu u_1]^*[\bar{u}_3\gamma_\nu v_4]^*,$$

where we have used another dummy index on the second set of terms. We know that

$$[\bar{v}_i\gamma^\mu u_j]^* = [\bar{u}_j\gamma^\mu v_i],$$

so

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4}[\bar{v}_2\gamma^\mu u_1][\bar{u}_1\gamma^\nu v_2][\bar{u}_3\gamma_\mu v_4][\bar{v}_4\gamma_\nu u_3].$$

Summing over s_1 and s_4, we get

$$\sum_{s_1,s_4}|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4}[\bar{v}_2\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu v_2][\bar{u}_3\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu u_3].$$

We also worked out in class that a term like

$$[\bar{v}_2 \Gamma v_2] = \mathrm{Tr}[v_2 \bar{v}_2 \Gamma],$$

so, also doing the sum over s_2 and s_3 spins and average over initial spins (which incurs a factor of 1/4)

$$\frac{1}{4}\sum_{\text{allspins}} |\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4}\mathrm{Tr}[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu] \times \mathrm{Tr}[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_v].$$

Looking the first trace, we note that in our expansion of the terms we can eliminate any term linear in the masses because those carry an odd number of gamma matrices, the trace of which is zero. So all we are left with is:

$$\begin{aligned}
\mathrm{Tr}[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu] &= \mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu - m_e^2\gamma^\mu\gamma^\nu], \\
&= p_{2,\sigma}p_{1,\rho}\mathrm{Tr}[\gamma^\sigma\gamma^\mu\gamma^\rho\gamma^\nu] - m_e^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] \\
&= 4p_{2,\sigma}p_{1,\rho}(g^{\sigma\mu}g^{\rho\nu} - g^{\sigma\rho}g^{\mu\nu} + g^{\sigma\nu}g^{\mu\rho}) - 4m_e^2 g^{\mu\nu} \\
&= 4[p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2 + m_e^2) + p_2^\nu p_1^\mu].
\end{aligned}$$

The other trace will be identical, with $p_2 \to p_3$, $p_1 \to p_4$, and $m_e \to m_\mu$, as well as lowered indices:

$$\mathrm{Tr}[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_v] = 4[p_{3,\mu}p_{4,\nu} - g_{\mu\nu}(p_3 \cdot p_4 + m_\mu^2) + p_{3,\nu}p_{4,\mu}].$$

To multiply these out, I will make a notational simplification of $p_i \cdot p_j = p_{ij}$. Doing the product:

$$\begin{aligned}
= 16[p_{14}p_{23} - p_{12}(p_{34} + m_\mu^2) + p_{13}p_{24} - p_{34}(p_{12} + m_e^2) + 4(p_{12} + m_e^2)(p_{34} + m_\mu^2) \\
- p_{34}(p_{12} + m_e^2) + p_{13}p_{24} - p_{12}(p_{34} + m_\mu^2) + p_{14}p_{23}].
\end{aligned}$$

$$\begin{aligned}
= 32[p_{14}p_{23} + p_{13}p_{24} - p_{12}(p_{34} + m_\mu^2) - p_{34}(p_{12} + m_e^2) + 2(p_{12} + m_e^2)(p_{34}m_\mu^2)] \\
= 32[p_{14}p_{23} + p_{13}p_{24} + p_{12}m_\mu^2 + p_{34}m_e^2 + 2m_e^2 m_\mu^2].
\end{aligned}$$

or, putting back the original dot product notation,

$$= 32[p_1 \cdot p_4 p_2 \cdot p_3 + p_1 \cdot p_3 p_2 \cdot p_4 + p_1 \cdot p_2 m_\mu^2 + p_3 \cdot p_4 m_e^2 + 2m_e^2 m_\mu^2].$$

Thus, our amplitude squared is

$$|\mathcal{M}|^2 = \frac{8e^4}{(p_1+p_2)^4}[p_1 \cdot p_4 p_2 \cdot p_3 + p_1 \cdot p_3 p_2 \cdot p_4 + p_1 \cdot p_2 m_\mu^2 + p_3 \cdot p_4 m_e^2 + 2m_e^2 m_\mu^2].$$

To get this in terms of the Mandelstam variables, we first need

$$s = (p_1+p_2)^2 = 2m_e^2 + 2p_1 \cdot p_2 = (p_3+p_4)^2 = 2m_\mu^2 + 2p_3 \cdot p_4, \quad \text{so}$$
$$p_1 \cdot p_2 = \frac{s - 2m_e^2}{2} \quad \text{and} \quad p_3 \cdot p_4 = \frac{s - 2m_\mu^2}{2}.$$

For t:

$$t = (p_1 - p_3)^2 = m_e^2 + m_\mu^2 - 2p_1 \cdot p_3 = (p_4 - p_2)^2 = m_e^2 + m_\mu^2 - 2p_2 \cdot p_4, \quad \text{so}$$
$$p_1 \cdot p_3 = p_2 \cdot p_4 = \frac{m_e^2 + m_\mu^2 - t}{2}.$$

Lastly, for u:

$$u=(p_1-p_4)^2=m_e^2+m_\mu^2-2p_1\cdot p_4=(p_3-p_2)^2=m_e^2+m_\mu^2-2p_2\cdot p_3, \quad \text{so}$$
$$p_1\cdot p_4=p_2\cdot p_3=\frac{m_e^2+m_\mu^2-u}{2}.$$

With these, the term in brackets in the amplitude squared becomes

$$\frac{1}{4}(m_e^2+m_\mu^2-u)^2+\frac{1}{4}(m_e^2+m_\mu^2-t)^2+\frac{1}{2}(s-2m_e^2)m_\mu^2+\frac{1}{2}(s-2m_\mu^2)m_e^2+2m_e^2m_\mu^2$$

$$\begin{aligned}=\frac{1}{4}[&m_e^4+m_\mu^4+u^2+2m_e^2m_\mu^2-2m_e^2u-2m_\mu^2u\\ &m_e^4+m_\mu^4+t^2+2m_e^2m_\mu^2-2m_e^2t-2m_\mu^2t\\ &2sm_\mu^2-4m_e^2m_\mu^4+2sm_e^2-4m_e^2m_\mu^2+8m_e^2m_\mu^2].\end{aligned}$$

$$\begin{aligned}&=\frac{1}{4}[u^2+t^2+2(s-t-u)(m_e^2+m_\mu^2)+4m_e^2m_\mu^2+2m_e^4+2m_\mu^4]\\ &=\frac{1}{4}[u^2+t^2+2(s-t-u)(m_e^2+m_\mu^2)+2(m_e^2+m_\mu^2)^2].\end{aligned}$$

We could keep going with this simplification, however, we have everything in terms of Lorentz-invariant quantities and it looks decently clean, so we can stop here. Our total amplitude squared is therefore

$$|\mathcal{M}|^2=\frac{2e^4}{s^2}[u^2+t^2+2(s-t-u)(m_e^2+m_\mu^2)+2(m_e^2+m_\mu^2)^2]. \tag{4.1}$$

To get the differential cross section, we turn to the formula we developed in class:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\frac{|\mathcal{M}|^2}{16\pi\lambda(s,m_1^2,m_2^2)},$$

where we will replace $m_1^2\to m_e^2$ and $m_2^2\to m_\mu^2$ to have that

$$\lambda(s,m_e^2,m_\mu^2)=(s-m_e^2-m_\mu^2)^2-4m_e^2m_\mu^2.$$

There isn't much simplification that can be done here, so

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\frac{e^4}{8\pi s^2}\frac{u^2+t^2+2(s-t-u)(m_e^2+m_\mu^2)+2(m_e^2+m_\mu^2)^2}{(s-m_e^2-m_\mu^2)^2-4m_e^2m_\mu^2}.}$$

Again, I am not sure how really to simplify this without ending up splitting the fraction into multiple other fractions, so I think this is as good as it is going to get (hopefully).