Regularization

We estabilished Feynman Rules for QED and QCD

⇒ Able to make perturbative calculations of Xsec for arbitrary processes

* We have seen tree-level processes in QCD with quarks and gluons

⇒ lowest-order calculations reproduce the parton-model results, however, the dynamical effects of QCD do not appear at tree level.

⇒ It is essential to deal with higher order corrections in perturbation theory i.e. loops and more legs!

* loop contributions to amplitudes diverge Renormalization program needed.

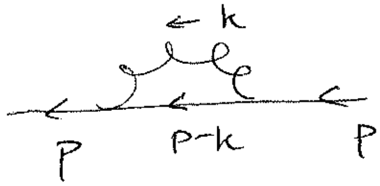
* Intermediate stage: Regularization

Renormalization: divergences are subtracted at in the final physical answer on the basis of renormalization.

We still require, at intermediate stage, that divergent integrals are mathematically manageable. The procedure that makes divergent integrals tentatively finite by introducing a suitable convergence device is generically called regularization.

Regularization: pure mathematical technique, not unique, has no physical consequences.

specific example of diverging diagrams (3)



quark self energy $\Sigma_{ij}(p)$

quark propagator at all orders $\tilde{S}_{ij}(p)$

→ full propagator which includes all the radiative corrections

$$\tilde{S}_{ij}(p) = \frac{\delta_{ij}}{m - \not{p} - \Sigma(p)} \qquad 1)$$

$$\Sigma_{ij}(p) = \delta_{ij}\, \Sigma(p)$$

$$\tilde{S}_{ij}(p) = i \int d^4x\, e^{-ip\cdot x} \langle 0 | T(\psi_i(x)\, \bar{\psi}_j(0)) | 0 \rangle_c$$

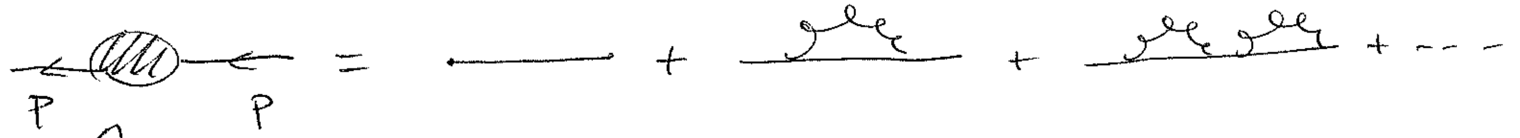
c = connected pieces.

$\Sigma(p)$ self energy part is one-particle irriducible, the propagator is not:

1) Can be expanded in powers of $\Sigma(p)$

$$\tilde{S}_{ij}(p) = \delta_{ij} \left\{ \tilde{S}_0(p) + \tilde{S}_0(p)\, \Sigma(p)\, \tilde{S}_0(p) + \tilde{S}_0(p)\, \Sigma(p)\, \tilde{S}_0(p)\, \Sigma(p)\, \tilde{S}_0(p) + \dots \right\}$$

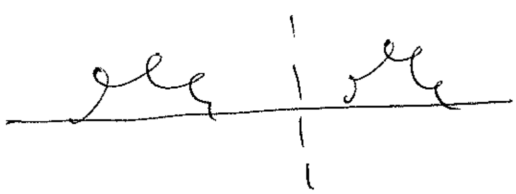
$$\tilde{S}_0(p) = \frac{1}{m - \not{p}} \qquad \text{tree-level propagator}$$



$\tilde{S}_{ij}(p)$

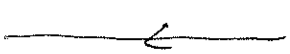
<u>one-particle irriducible</u> (1PI): it's a diagram that cannot become 2 nontrivial diagrams by cutting a single line.

 is 1PI

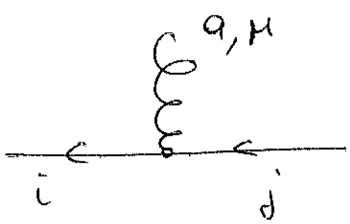
 is not 1PI

<u>Feynmal rules</u>

$\longrightarrow \displaystyle\int \frac{d^4k}{(2\pi)^4 i}$



$\rightarrow \dfrac{1}{m - \not{p}}$



$g_s \gamma_\mu T^a_{ij}$

Quark self energy to order g_s^2

$$\Sigma_{ij}(p) = \int \frac{d^4k}{(2\pi)^4 i} g_s \gamma_\mu T^a_{i\ell} \frac{\delta_{\ell n}}{m - \not{p} + \not{k}} g_s \gamma_\nu T^b_{nj} \frac{\delta_{ab}}{k^2} d^{\mu\nu}(k)$$

$$d_{\mu\nu}(k) = g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}$$

Color factors

$$T^a_{i\ell}\,\delta_{\ell n}\,T^b_{nj}\,\delta^{ab} = T^a_{in} T^a_{nj} = \left(T^a T^a\right)_{ij} = \delta_{ij} C_F$$

$C_F = \frac{N^2-1}{2N} \Rightarrow$ casimir of the fundamental representation of $SU(3)$.

$N = 3$ It labels all the irriducible representations of $SU(3)$.

$$\Sigma(p) = g_s^2 C_F \int \frac{d^4k}{(2\pi)^4 i} \gamma_\mu \frac{(m + \not{p} - \not{k})\gamma_\nu}{k^2\left[m^2 - (p-k)^2\right]} d^{\mu\nu}(k)$$

For simplicity, let's consider the Feynman gauge $\Rightarrow$ $\alpha = 1$ $\Rightarrow$ $d^{\mu\nu}(k) = g^{\mu\nu}$

$$\Sigma(p) = g_s^2 C_F \int \frac{d^4k}{(2\pi)^4 i} \frac{\gamma_\mu (m + \not{p} - \not{k})\gamma^\mu}{k^2(m^2 - (p-k)^2)}$$

The 4-dimensional integral in $\Sigma(p)$ is linearly divergent. By doing simple power counting we see that

$$\int \frac{d^4k}{k^2} \frac{\not{k}}{k^2} \sim \lim_{k \to \infty} k$$

The divergence comes from the high-momentum region $|k| \to \infty$. We need to regularize it, that is, we must write it as a suitable limit of a convergent integral.

Cut-off method

Simplest method: the high-momentum region is cut off in the divergent integrals.

Cons:
- it breaks translation invariance $\Rightarrow$ a shift in the momentum of the integral changes the result.
- breaks gauge invariance.

$\Rightarrow$ not good for gauge theories

Pauli - Villars

The integrand propagator is replaced by

$$\frac{1}{m^2-k^2} - \frac{1}{M^2-k^2} = \frac{M^2-m^2}{(m^2-k^2)(M^2-k^2)}$$

which reduces to the original propagator when $M \to \infty$

pros: translation and Lorentz invariance maintained. Gauge invariance in QED is preserved. Can be applied to massless QCD only

Cons: it does not maintain gauge invariance in massive Yang-Mills gauge theorres (like QCD with quark masses $\neq 0$)
Not good for the SM!

Analitical regulerization

$$\frac{1}{(m^2-k^2)} \longrightarrow \frac{1}{(m^2-k^2)^\alpha}$$

$\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 1$

In the limit $\alpha \to 1$ the original propagator is recovered.

Pros: extensively used for the proof of renormalizability of a theory.

Cons: Violates gauge invariance $\Rightarrow$ not good for QCD.

Lattice regularization

Here the space-time is discretized. That is, the Minkowski space is made of small cells of size a.

$\Rightarrow$ in the x-space or coordinate space the short-distance contribution to the space-time integration is eliminated.

In the momentum space, this means that we are cutting off the high-momentum region $\Rightarrow$ convergent momentum integral.

pros: good for non-perturbative calculations, e.g., configuration integrals in the functional integrals in QFT

Dimensional Regularization (DR)

A divergent multiple integral is made convergent by reducing the number of multiple integrals.

For example:

divergent 4-dim integral $\int d^4k \frac{k\!\!\!/}{k^2 k^2}$

would be finite if the space-time were 2-dim!

Therefore, in dimensional regularization

$$\int d^4k \rightarrow \int d^Dk \qquad D<4$$

$\Rightarrow$ we obtain the result of the integral in terms of analytic expressions as functions of D.

pros: in dimensional reg. or DR nothing is violated: gauge, Lorentz, unitarity invariant

cons: the space-time is not 4-dim.
Care must be given to the algebra in D-dim.

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Dimensional Regularization strategy:

$$\mu = 0, \ldots, 3 \quad \Rightarrow \quad \mu = 0, \ldots, D-1$$

$$p^\mu = (p^0, p^1, \ldots, p^{D-1})$$

$$g^\mu{}_\mu = g^{\mu\nu} g_{\mu\nu} = D$$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

$$\gamma^\mu \gamma_\mu = D$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2-D)\gamma^\nu$$

Ambiguities: 1) the measure $\frac{1}{(2\pi)^4} \to \frac{1}{(2\pi)^D}$

or it maybe the same as in 4-dim.

Requirement: the measure in D-dim must recover $1/(2\pi)^4$ when $D \Rightarrow 4$

2) Trace of the γ matrices.

Following the Clifford algebra one has

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 2^{D/2} g^{\mu\nu} \quad \text{for } D \text{ even}$$

which reduces to the 4-dim form as $D \to 4$.

As we are only interested in 4-dimensional space-time, the $\mathrm{Tr}[\gamma^\mu\gamma^\nu]$ expression above is in principle not needed.

To avoid this problem we fix our convention such that

$$\int \frac{d^D k}{(2\pi)^D}$$

and the trace of γ-matrices is normalized to

$$\mathrm{Tr}[\gamma_\mu \gamma_\nu] = 4 g_{\mu\nu}$$

* Always keep in mind this convention and be consistent in your calculations.

We can now evaluate our integral using DR. We set $m=0$ for simplicity, for now.

$$\Sigma_s(p) = g_s^2 C_F (2-D) \int \frac{d^D k}{(2\pi)^D} \frac{\not{k} - \not{p}}{k^2 (k-p)^2}$$

where we used $\gamma^\mu \gamma^\nu \gamma_\mu = (2-D)\gamma^\nu$.

We keep $D < 3$ to ensure convergence.

The next step is to introduce the Feynman parametrization:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\{xA + (1-x)B\}^2}$$

to reexpress the denominator of $\Sigma_s(p)$

$$\Sigma_s(p) = g_s^2 C_F (2-D) \int \frac{d^D k}{(2\pi)^D} (\not{k} - \not{p}) \int_0^1 \frac{dx}{\{x(k-p)^2 + (1-x)k^2\}^2}$$

As far as $D<3$ the k-integration is convergent and we can interchange $\int d^D k$ with $\int dx$

$$\Sigma_s(p) = g_s^2 C_F (2-D) \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D i} \frac{(\not{k} - \not{p})}{\{(k-xp)^2 + x(1-x)p^2\}^2}$$

$$\int_0^1 \frac{dx}{[xA + (1-x)B]^2} = \int_0^1 \frac{dx}{[x(A-B) + B]^2}$$

$$x(A-B) + B = y$$

$$dy = (A-B)\,dx \Rightarrow dx = \frac{dy}{A-B}$$

$$\begin{cases} x = 0 \Rightarrow y = B \\ x = 1 \Rightarrow y = A \end{cases}$$

$$\frac{1}{(B-A)}\int_A^B \frac{dy}{y^2} = \frac{1}{(B-A)}\frac{[y^{-1}]}{-1} = \frac{1}{A-B}\left[\frac{1}{y}\right]_A^B$$

$$= \frac{1}{(A-B)}\left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{\cancel{(A-B)}}\frac{\cancel{(A-B)}}{AB} = \frac{1}{AB}$$

This can be generalized to account for denominators with more terms.

where we rearranged the denominator. (13)

DR preserves translational invariance $\Rightarrow$ we can make a shift of the momentum variable:

$$k' = k - xp$$

$$\Sigma(p) = g_s^2 C_F (2-D) \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D i} \frac{\not{k}' - (1-x)\not{p}}{\{k'^2 + x(1-x)p^2\}^2}$$

DR preserves of symmetries of the space-time $\Rightarrow$ an integral of an odd function in k vanishes

$$\int d^D k \, k_\mu \, f(k^2) = 0$$

where $f(k^2)$ is an integrable function of k^2.

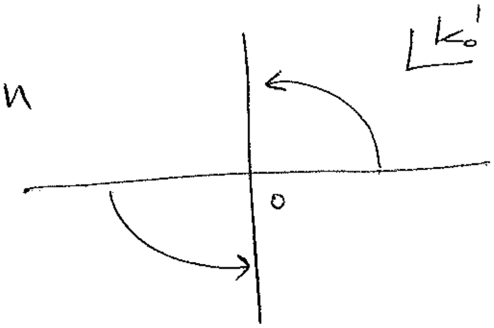
$\Rightarrow$ linearly divergent pieces disappear, leaving only logarithmically divergent contributions

$$\Sigma(p) = g_s^2 C_F (D-2) \not{p} \int_0^1 dx \, (1-x) \int \frac{d^D k'}{(2\pi)^D i} \frac{1}{\{k'^2 + x(1-x)p^2\}^2}$$

Now we want to perform the k' integral. In the Minkowski space, this is not easy.

To circumvent the problem we make a 90° rotation in the complex plane

Wick Rotation



k_0' - plane

$$k_0' = i K_0 \qquad K_0 = \text{real}, \in \mathbb{R}$$

$\vec{k}' = \vec{K} \Rightarrow$

$$d^D k' = i\, d^D K, \qquad k'^2 = -K^2 \qquad K^2 = K_0^2 + |\vec{K}|^2$$

this changes the integration from Minkowski to the Euclidean space.

$$\Sigma(p) = g_s^2\, C_F\, (D-2)\, \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D K}{(2\pi)^D} \frac{1}{(K^2 + L)^2}$$

$$L = -x(1-x)p^2$$

- Integral non singular for $L > 0 \Rightarrow$ we must keep $p \Rightarrow p \in$ space-like region $\Rightarrow p^2 < 0$

At this point, we use the polar coordinate system in D-dim

$$|K| = \sqrt{K_0^2 + |\vec{K}|^2}$$

$$K_0 = |\vec{K}| \cos\theta_1$$

$$K_1 = |K| \cos\theta_2 \sin\theta_1$$

$$\vdots$$

$$K_{D-1} = |K| \sin\theta_{D-1} \cdots \sin\theta_1$$

$$d^D K = K^{D-1} \, d|K| \, d\Omega_D$$

$$d\Omega_D = \prod_{l=1}^{D-1} \sin^{D-1-l}\theta_l \, d\theta_l$$

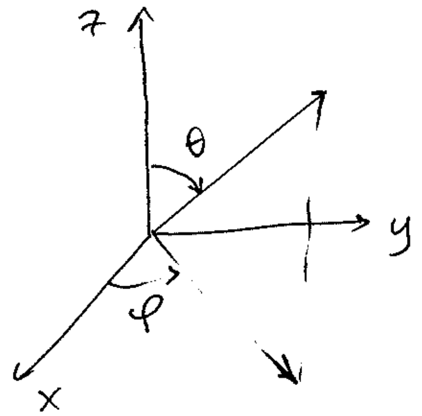
$$\int \frac{d^D K}{(2\pi)^D} \frac{1}{\left(K^2 + L\right)^2} = \frac{B\left(D/2, 2 - D/2\right)}{(4\pi)^{D/2}\,\Gamma(D/2)} L^{D/2-2}$$

where the Beta $B(x,y)$ and $\Gamma(x)$ are

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \; ; \quad B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$Re(z) > 0$

$$B(p,q) = \int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}} dt$$



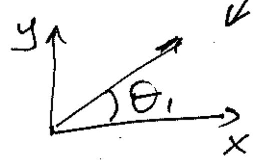
$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$k_0 \to z$

$z = r\cos\theta_1 = k_0$

$\underline{\underline{S^1 \to}}$ only θ_1 $\quad |\vec{r}|=1$

x-y plane



$$\begin{cases} x^1 = \cos\theta_1 = x \\ x^2 = \sin\theta_1 = y \end{cases}$$

$\underline{\underline{S^2 \to (\theta_1, \theta_2)}}$ $\quad |\vec{r}|=1$

$$\begin{cases} x^1 = \cos\theta_1 \\ x^2 = \sin\theta_1\cos\theta_2 \\ x^3 = \sin\theta_1\sin\theta_2 \end{cases}$$

$d\Omega = \sin\theta_1\, d\theta_1\, d\theta_2$

↳ (it would be $d\Omega_3$ in many text books)

$\underline{\underline{S^3 \to (\theta_1, \theta_2, \theta_3)}}$

$$\begin{cases} x^1 = \cos\theta_1 \\ x^2 = \sin\theta_1 \cos\theta_2 \\ x^3 = \sin\theta_1\sin\theta_2\cos\theta_3 \\ x^4 = \sin\theta_1\sin\theta_2\sin\theta_3 \end{cases}$$

$d\Omega = (\sin\theta_1)^2\sin\theta_2\, d\theta_1 d\theta_2 d\theta_3$

↳ (it would be $d\Omega_4$ in many text books)

$S^4 \Rightarrow (\theta_1, -, \theta_4)$

$$\begin{cases} X^1 = \cos\theta_1 \\ X^2 = \sin\theta_1 \cos\theta_2 \\ X^3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ X^4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4 \\ X^5 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \end{cases}$$

$d\Omega = (\sin\theta_1)^3 (\sin\theta_2)^2 \sin\theta_3 \, d\theta_1 \, d\theta_2 \, d\theta_3 \, d\theta_4$

$\hookrightarrow d\Omega_5$

$S^{D-1} \Rightarrow (\theta_1, \text{——}, \theta_{D-1})$

$$\begin{cases} X^1 = \cos\theta_1 \\ X^2 = - - \\ \vdots \\ X^{D-1} = \sin\theta_1 \cdots \cos\theta_{D-1} \\ X^D = \sin\theta_1 \cdots \sin\theta_{D-1} \end{cases}$$

$d\Omega = (\sin\theta_1)^{D-2} (\sin\theta_2)^{D-3} \cdots \sin\theta_{D-2} \, d\theta_1 \, d\theta_2 \cdots d\theta_{D-1}$

$\hookrightarrow d\Omega_D$

$$d\Omega_D = \prod_{\ell=1}^{D-1} (\sin\theta_\ell)^{D-1-\ell} \, d\theta_\ell$$

The integral of the solid angle in D-dim (15/c)

gives

$$\int d\Omega_D = \int_0^{\pi} d\theta_1 (\sin\theta_1)^{D-2} \cdots \int_0^{\pi} d\theta_{D-2} \sin\theta_{D-2} \int_0^{\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

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Unit sphere in D-dim

The Gauss integral is well known

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}} \Rightarrow (\sqrt{\pi})^D = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^D$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^D = \int d^Dx \, e^{-\sum_{i=1}^{D} x_i^2} = \int d\Omega_D \int_0^{+\infty} dx \, x^{D-1} e^{-x^2} =$$

$$= \int d\Omega_D \frac{1}{2} \int_0^{+\infty} d(x^2) (x^2)^{D/2-1} e^{-x^2} = \int d\Omega_D \frac{1}{2} \Gamma(D/2)$$

we used $\frac{1}{2} 2x \, x^D \frac{dx}{x^2} = dx \, x^{D-1}$ and

$$\boxed{\Gamma(D) = \int_0^{+\infty} dt \, t^{D-1} e^{-t}} \leftarrow \text{Euler Gamma function}$$

$$\boxed{\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}}$$

D	$\Gamma(D/2)$	$\int d\Omega_D$
1	$\sqrt{\pi}$	2
2	1	2π
3	$\sqrt{\pi}/2$	4π
4	1	$2\pi^2$

The integral of the solid angle gives

$$\int d\Omega_D = \int_0^\pi d\theta_1 (\sin\theta_1)^{D-2} \cdots \int_0^\pi d\theta_{D-2} \sin\theta_{D-2} \int_0^{2\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Therefore, we obtain the following expression for $\Sigma(\not p)$

$$\Sigma(p) = g_s^2 C_F (D-2) \not p \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} (-p^2)^{D/2-2} *$$

$$* \int_0^1 dx \, x^{D/2-2} (1-x)^{D/2-1}$$

The integral in this expression can be related to the $B(p,q)$ function (change of variable in B)

$$B(p,q) = \int_0^1 dx \, x^{p-1} (1-x)^{q-1}$$

and we finally obtain

$$\Sigma(p) = \frac{2 C_F g_s^2}{(4\pi)^{D/2}} \not p (-p^2)^{D/2-2} (D-1) B\left(\frac{D}{2}, \frac{D}{2}\right) \Gamma\left(2-\frac{D}{2}\right)$$

which is valid only for $D<3$ and $p^2<0$.

$\Sigma(p)$ is given as an explicit function of the space-time dimension and the momentum p $\Rightarrow$ analytical continuation to the region where D and p^2 are arbitrary complex numbers.

In fact, we observe that:

- $$\Gamma\left(2-D/2\right) = \int_0^{+\infty} t^{2-D/2-1}\, e^{-t}\, dt$$

$D = 4, 6, 8 \ldots$ are poles for this function.

ex.: $D=4 \Rightarrow \int_0^{+\infty} t^{-1} e^{-t}\, dt \rightarrow \infty$

as $\frac{e^{-t}}{t}$ does not converge at $t=0$.

- There is also a branch cut on the positive real axis in the p^2 plane:

$$\left(-p^2\right)^{D/2-2} \rightarrow \left(-p^2\right)^{\alpha} \quad \alpha\to 0 \approx 1 + \ln(-p^2)\alpha + \frac{1}{2!}\ln(-p^2)\alpha^2$$

For $D \sim 4$ we can write

$$\Sigma(p) \sim \frac{C_F g_s^2}{(4\pi)^2} \frac{2}{4-D} \not{p}$$

which diverges for $D \to 4$

Introducing a dimension for the g_s-coupling

The action in QFT is

$$S = \int d^D x \, \mathcal{L}$$

Is dimensionless (is $c = \hbar = 1$) in natural units

$\dim[\mathcal{L}] = D$ mass dimension of the lagrangian in natural units

Compton wave length: $\lambda = \frac{\hbar}{mc}$

$\hbar = c = 1 \Rightarrow [\lambda] = [m]^{-1}$

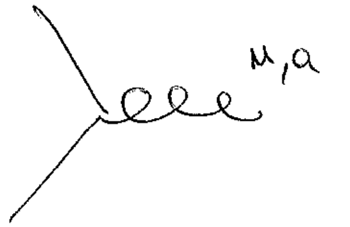
Partial derivative $\frac{\partial}{\partial x^\mu} \to$ inverse of length!

$[\partial] = 1$

$[d^4 x] = -4 \Rightarrow [d^D x] = -D$

$[\mathcal{L}] = D$ to have $[S] = 1$

Let's examin



$$g_s \bar{\psi} T^a \gamma^\mu A_\mu^a$$

?

$\downarrow$

$$\dim[g_s] + 2\dim[\psi] + \dim[A_\mu^a] = D$$

$\dim[A_\mu^a] = ?$ $\quad \dim[\psi] = ?$

Let's look at the kinetic terms in the lagrangian

$$\mathcal{L} = a_1 F^{\mu\nu} F_{\mu\nu} + a_2 \bar{\psi} \gamma^\mu D_\mu \psi + a_3 \bar{\psi}\psi$$

$$+ a_4 \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi + a_5 F^{\mu\nu} F_\nu{}^\lambda F_{\lambda\mu} + \cdots$$

In QED for example

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$$

$$F_{\mu\nu} F^{\mu\nu} \sim [\partial_\mu A_\nu \partial^\mu A^\nu]$$

$$\Rightarrow [A] = D-2$$

$$\dim\left[\int F^{\mu\nu} F_{\mu\nu}\, d^D x\right] = 0 \Leftrightarrow [\partial^2][A^2] - D = 0$$

$$[A^2] = D-2 \Rightarrow [A] = \frac{D-2}{2}$$

By the same token

$[\bar{\psi}\gamma^\mu \partial_\mu \psi] = D$ (kinetic term for fermions)

$[\psi^2][\partial] = D \Rightarrow [\psi] = \frac{D-1}{2}$

$\Rightarrow$

$$\dim[g_s] + 2\left(\frac{D-1}{2}\right) + \left(\frac{D-2}{2}\right) = D$$

We must introduce a mass scale μ and rewrite the gauge coupling constant as

$$\boxed{g_s = g_s^\circ \, \mu^{2-D/2}}$$

g_s° = dimensionless coupling

μ = DR scale parameter

We can rewrite $D = 4-2\varepsilon$ where ε is a small parameter

$$\varepsilon = \frac{4-D}{2}$$

$$\Sigma(p) = \frac{g_{0s}^2 C_F}{(4\pi)^2}\, \not{p} \left(\frac{-p^2}{4\pi\mu^2}\right)^{-\varepsilon} (1-\varepsilon)\, B(1-\varepsilon, 1-\varepsilon)\, \Gamma(\varepsilon)$$

A Laurent expansion around $\epsilon = 0$ gives

$$\Sigma(p) = \frac{g_0^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2} \right) \right) + \mathcal{O}(\epsilon)$$

where we used

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$$

$$(1-\epsilon)\, B(1-\epsilon, 1-\epsilon) = 1 + \epsilon + \mathcal{O}(\epsilon^2)$$

γ_E = Euler - Mascheroni constant $= 0.57721$

<u>Note</u>: In this calculation we haven't encountered γ^5. γ^5 in D-dim requires a special treatment as it cannot be defined explicitly for arbitrary dimensions.

DR conventions

1. D-dim space-time metric $g^{\mu\nu} = (+, -, \dots, -)$
2. $\mathrm{Tr}[1] = 4$ in the space of gamma matrices
3. $\int \frac{d^D k}{(2\pi)^D}$ defines the integral measure
4. γ_5 is an object that satisfies $\{\gamma_5, \gamma^\mu\} = 0$

A Laurent expansion around $\varepsilon = 0$ gives

$$\Sigma(p) = \frac{g_0^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\varepsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2} \right) \right) + \mathcal{O}(\varepsilon)$$

where we used

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)$$

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Self energy correction in an arbitrary covariant gauge

$$\Sigma(p) = g_s^2 C_F \int \frac{d^D k}{(2\pi)^D i} \frac{1}{k^2 (k-p)^2} \left\{ \gamma_\mu (\not{k} - \not{p}) \gamma^\mu - (1-\alpha) \frac{\not{k}(\not{k}-\not{p})\not{k}}{k^2} \right\}$$

we need to calculate the term prop to $(1-\alpha)$. We write

$$\Sigma(p) = \Sigma_1(p) - (1-\alpha)\Sigma_2(p)$$

$$\Sigma_1(p) = \frac{g_0^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\varepsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right) + O(\varepsilon)$$

calculated at pag 16.

$$\Sigma_2(p) = g_s^2 C_F \int \frac{d^D k}{(2\pi)^D} \frac{\not{k}(\not{k}-\not{p})\not{k}}{i (k^2)^2 (k-p)^2}$$

Now we use the following Feynman parametr.

$$\frac{1}{AB^2} = 2 \int_0^1 dx \frac{(1-x)}{\{xA + (1-x)B\}^3}$$

Again $L = -x(1-x)p^2$

At this point we need to "massage" the denominator in such a way that we can use a shift in the k momentum.

$$k \to k - xp + xp$$

The $\Sigma_2(p)$ expression then becomes

$$\Sigma_2(p) = -2g_s^2 C_F \int_0^1 dx\,(1-x) \int \frac{d^Dk}{(2\pi)^D i} \frac{\not{k}(\not{k}-\not{p})\not{k}}{\{-(k-xp)^2+L\}^3}$$

At this point we shift the k-momentum

$$k' = k - xp$$

and discard all the terms odd in k'

$$\Sigma_2(p) = 2g_s^2 C_F \int_0^1 dx\,(1-x) \int \frac{d^Dk'}{(2\pi)^D} \frac{(1-x)\not{k}'\not{p}\not{k}' - 2xk'^2\not{p} - xL\not{p}}{(-k'^2+L)^3}$$

For any integrable function $f(k'^2)$ we use the following formula

$$\int d^Dk'\, k'_\mu k'_\nu\, f(k'^2) = \frac{1}{D} g_{\mu\nu} \int d^Dk'\, k'^2 f(k'^2)$$

In fact, remember that $g^{\mu\nu} g_{\mu\nu} = D \Rightarrow$

$$\frac{D}{D} \int d^Dk'\, k'_\mu k'_\nu\, f(k'^2) = \frac{g_{\mu\nu}}{D} \int d^Dk'\, k'^2 f(k'^2)$$

We obtain

$$\Sigma_2(p) = 2g_s^2 C_F \not{p} \int_0^1 dx\,(1-x) \int \frac{d^Dk'}{(2\pi)^D i} \frac{1}{[-k'^2+L]^3} \left\{\left(\frac{2(1-x)}{D} - 1 - x\right)k'^2 - xL\right\}$$

which becomes, after a Wick rotation,

$$\Sigma_2(p) = 2 g_s^2 C_F \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{1}{(k^2+L)^3} \left[\left(\frac{2(1-x)}{D} - 1 - 2x \right) L \right. \right.$$

$$\left. \left. - \frac{1}{(k^2+L)^2} \left[\frac{2(1-x)}{D} - 1 - x \right] \right] \right\}$$

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We use the generalized result for the k integration

- $$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+L)^a} = \frac{\Gamma(a - D/2)}{(4\pi)^{D/2}\,\Gamma(a)} L^{D/2 - a} \qquad a \in \mathbb{C} \ni' \mathrm{Re}[a] > 0$$

We use the following relations for the x integration

- $B(p,q) = \int_0^1 dx\, x^{p-1} (1-x)^{q-1}$
- $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$
- $\Gamma(z+1) = z\,\Gamma(z)$

With these results the expression for $\Sigma_2(p)$ is given by

$$\Sigma_2(p) = 2 g_s^2 C_F \not{p} (-p^2)^{D/2-2} (D-1)\, B(D/2, D/2)\, \Gamma(2 - D/2)$$

which is exactly equal to $\Sigma(p)$ calculated at pag 16!

$$\Sigma_2(p) = 2 g_s^2 C_F \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D k}{(2\pi)^D} \; *$$

$$* \left\{ \frac{1}{(k^2+L)^3}\left(\frac{2(1-x)}{D} - 1 - 2x\right) L - \frac{1}{(k^2+L)^2}\left(\frac{2(1-x)}{D} - 1 - x\right) \right\}$$

At this point we use a generalization of a previous result

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+L)^a} = \frac{\Gamma(a - D/2)}{(4\pi)^{D/2}\,\Gamma(a)} L^{D/2-a} \qquad a \in \mathbb{C} \ni' \mathrm{Re}[a] > 0$$

$L = -x(1-x)p^2$. After the integration over k,

The first piece has 3 contributions

$$\int_0^1 dx\,(1-x) \left[\frac{2(1-x)}{D} - 1 - 2x\right] L^{D/2-2} =$$

$$= \int_0^1 dx\, \frac{2}{D}(1-x)^2 L^{D/2-2} - \int_0^1 dx\,(1-x) L^{D/2-2} - \int_0^1 dx\, 2x(1-x) L^{D/2-2}$$

$$= \int_0^1 dx\, \frac{2}{D} x^{D/2-2}(1-x)^{D/2}(-p^2)^{D/2-2} - \int_0^1 dx\, x^{D/2-2}(1-x)^{D/2-1}(-p^2)^{D/2-2} - 2\int_0^1 dx\, x^{D/2-1}(1-x)^{D/2-1}(-p^2)^{D/2-2}$$

$$B(p,q) = \int_0^1 dx\, x^{p-1}(1-x)^{q-1} \quad \Rightarrow$$

$$= (-p^2)^{D/2-2}\left\{ B(D/2-1, D/2+1)\frac{2}{D} - B(D/2-1, D/2) - 2\,B(D/2, D/2) \right\}$$

- $B(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$
- $\Gamma(z+1) = z\,\Gamma(z)$

We use these two properties to rewrite our result for the first piece

$$B(D/2-1, D/2+1) = \frac{\Gamma(D/2-1)\,\Gamma(D/2+1)}{\Gamma(D)} = \frac{(D/2-1)\,\Gamma(D/2-1)\,\Gamma(D/2+1)}{(D/2-1)\quad\Gamma(D)}$$

$$= \frac{\Gamma(D/2)\quad D/2\quad \Gamma(D/2)}{(D/2-1)\,\Gamma(D)} = \frac{\Gamma(D/2)}{\Gamma(D)}\,\frac{\Gamma(D/2)\,D}{(D-2)}$$

Then we have

$$B(D/2-1, D/2) = \frac{\Gamma(D/2-1)\,\Gamma(D/2)}{\Gamma(D-1)} = \frac{(D/2-1)\quad\Gamma(D/2-1)\,\Gamma(D/2)}{(D/2-1)\,\Gamma(D-1)} =$$

$$= \frac{\Gamma(D/2)\quad\Gamma(D/2)\quad(D-1)}{(D/2-1)\,(D-1)\,\Gamma(D-1)} = \frac{\Gamma(D/2)\,\Gamma(D/2)}{\Gamma(D)}\,\frac{(D-1)}{(D/2-1)}$$

$$B(D/2, D/2) = \frac{\Gamma(D/2)\quad\Gamma(D/2\;)}{\Gamma(D)}$$

$$B(D/2-1, D/2+1) = B(D/2, D/2)\ \frac{D}{D-2}$$

$$B(D/2-1, D/2) = B(D/2, D/2)\ \frac{D-1}{D/2-1} = B(D/2, D/2-1)$$

$$= \frac{2(D-1)}{D-2}$$

Therefore, the 1^{st} piece gives

$$\int_0^1 dx\,(1-x)\left[\frac{2(1-x)}{D} - 1 - 2x\right] L^{D/2-2} = (-p^2)^{D/2-2}\left\{ B(D/2, D/2)\frac{2}{D-2}\right.$$

$$\left. - B(D/2, D/2)\frac{2(D-1)}{D-2} - 2\,B(D/2, D/2)\right\} =$$

$$= (-p^2)^{D/2-2} B(D/2, D/2)\left\{\frac{2 - 2D + 2 - 2D + 4}{D-2}\right\} =$$

$$= (-p^2)^{D/2-2} B(D/2, D/2)\ \frac{(8-4D)}{D-2}$$

This must be multiplied by the rest of the result of the k-integration

$$= (-p^2)^{D/2-2} B(D/2; D/2)\left(\frac{8-4D}{D-2}\right)\frac{\Gamma(3-D/2)}{(4\pi)^{D/2}\,\Gamma(3)}$$

The second integral

$$\int_0^1 dx\,(1-x)\left[\frac{2(1-x)}{D} - 1 - x\right] L^{D/2-2}$$ because

the integral over k gives

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+L)^2} = \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}\,\Gamma(2)} L^{D/2-2}$$

Therefore we obtain

$$(-p^2)^{D/2-2} \int_0^1 dx\,(1-x)\left[\frac{2(1-x)}{D} - 1 - x\right]\left[x(1-x)\right]^{D/2-2} =$$

$$= (-p^2)^{D/2-2}\left\{\int_0^1 \frac{2x}{D}^{D/2-2}(1-x)^{D/2}\,dx - \int_0^1 dx\, x^{D/2-2}(1-x)^{D/2-1} - \int_0^1 dx\, x^{D/2-1}(1-x)^{D/2-1}\right\}$$

$$= (-p^2)^{D/2-2}\left\{\frac{2}{D} B(D/2-1, D/2+1) - B(D/2-1, D/2) - B(D/2, D/2)\right\}$$

The B-functions have been calculated before

$$= (-p^2)^{D/2-2}\left\{\frac{2}{\cancel{D}}\frac{\cancel{D}}{D-2} - 2\frac{(D-1)}{(D-2)} - 1\right\} B(D/2, D/2)$$

With the k integration result we get

$$= (-p^2)^{D/2-2}\,\frac{(6-3D)}{(D-2)}\,B(D/2, D/2)\,\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}\,\Gamma(2)}$$

Adding the two contributions together

$$\frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{(8-4D)}{D-2} \frac{\Gamma(3-D/2)}{\Gamma(3)} - \frac{(6-3D)}{D-2} \frac{\Gamma(2-D/2)}{\Gamma(2)} \right\}$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{(8-4D)(2-D/2)\,\Gamma(2-D/2)}{(D-2) \cdot 2\,\Gamma(2)} - \frac{(6-3D)}{(D-2)} \frac{\Gamma(2-D/2)}{\Gamma(2)} \right\}$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{(4-2D)(2-D/2)}{(D-2)\;\Gamma(2)} - \frac{(6-3D)}{(D-2)\,\Gamma(2)} \right\} \Gamma(2-D/2)$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ \frac{\cancel{(2-D)}(4-D)}{\cancel{(D-2)}} - \frac{(6-3D)}{(D-2)} \right\} \Gamma(2-D/2)$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2) \left\{ D-4 \quad +3 \right\} \Gamma(2-D/2)$$

$$= \frac{(-p^2)^{D/2-2}}{(4\pi)^{D/2}} B(D/2, D/2)\, (D-1)\, \Gamma(2-D/2) \Rightarrow$$

$$\boxed{\Sigma_2(p) = \frac{2 g_s^2 C_F}{(4\pi)^{D/2}} \not{p}\, (-p^2)^{D/2-2} (D-1)\, B(D/2, D/2)\, \Gamma(2-D/2)}$$

This is exactly equal to $\Sigma(p)$ we calculated at Pag 16!

We found that $\Sigma_2(\not{p}) = \Sigma_1(\not{p})$ where $\Sigma_1(\not{p})$ was obtained before at pag 16.

Therefore for a covariant gauge with arbitrary α

$$\Sigma(\not{p}) = \alpha \frac{2C_F g_s^2}{(4\pi)^{D/2}} \not{p} (-p^2)^{D/2-2} (D-1) B(D/2, D/2) \Gamma(2-D/2)$$

$$= \alpha \frac{g_{os}^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2} \right) \right) + O(\varepsilon)$$

Renormalization schemes (preliminaries)

Renormalization: redefinition of mass and coupling constant together with a re-adjustment of the normalization of Green functions by suitable multiplicative factors that may eliminate possible infinities in the Green functions.

Renormalization is not unique: divergent pieces in the Green functions are not uniquely defined.
$\Rightarrow$ ambiguity in the finite piece of the Green func.

How do we remove this ambiguity?

→ specify how the divergent piece is defined so that it can be consistently subtracted.

→ the subtraction prescription is called Renormalization scheme.

→ Different renormalization schemes are always connected by a finite renormalization.

Let's consider $\Sigma(p)$ which we have computed:

$$\Sigma(p) = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right) + O(\varepsilon)$$

if we substitute this into Eq (1) at pag 3 of these notes,

$$\tilde{S}_{ij}(p) = \frac{\delta_{ij}}{m - \not{p} - \Sigma(p)}$$

and we set $m = 0$ for simplicity, we obtain

$$\tilde{S}_{ij}(p) = -\frac{\delta_{ij}}{\not{p}} \frac{1}{1 + \sigma(p^2)}$$

$$\sigma(p^2) = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right) + O(g_{0s}^4)$$

where all terms of order ε have been set to zero.

- $\tilde{S}_{ij}(p)$ has a pole at $\not{p} = 0$
- massless quark stays massless after the inclusion of 1-loop corrections (this is generally true for massless quarks at all orders in perturbation theory)

We renormalize the quark propagator by a multiplicative factor Z_2

Z_2 = quark-field renormalization constant.

$$\tilde{S}_{R\,ij}(p) = Z_2^{-1}\,\tilde{S}_{ij}(p)$$

$\downarrow$

renormalized (finite) quark propagator

Z_2 can be expanded in g_{0s} powers

$$Z_2 = 1 - z_2 + \mathcal{O}(g_{0s}^4)$$

$\downarrow$

g_{0s}^2 - term (divergent)

substituting this into $\tilde{S}_{ij}(p)$ gives us

$$\tilde{S}_{R\,ij}(p) = -\frac{\delta_{ij}}{\not{p}}\,\frac{1}{1+\sigma(p^2)-z_2}$$

where we keep only the g_{0s}^2 terms. In fact

$$Z_2\,\sigma(p^2) = \sigma(p^2) + \mathcal{O}(g_{0s}^4)$$

Note that $\tilde{S}_{R\,ij}(p)$ should have the renormalized version of g_{0s}, but at this perturbative order there is no effect on g_{0s}. Therefore, we'll keep using g_{0s} for now.

- $\tilde{S}_{R\,ij}(p)$ should be free of divergences $\Rightarrow$

$\sigma(p^2) - Z_2$ must be finite, and the divergences in $\sigma(p^2)$ should be cancelled by Z_2

This requirement determines Z_2 up to a finite additive constant.

$\Rightarrow$ we need an extra requirement which sets up a renormalization scheme (prescription).

As discussed before, there are several renormalization schemes depending on this prescription.

Let's see a few examples.

1. On-shell subtraction

Z_2 is determined on the mass shell of quarks by imposing the condition

$$\tilde{S}_{R_{ij}}(p) \sim \frac{\delta_{ij}}{m - \not{p}} \quad \text{for } \not{p} \sim m$$

this is traditionally used in QED. In our case $m = 0$ and so $Z_2 = \sigma(0)$. $\sigma(0)$ is not well-defined in this example because for the massless quark the singularity is in $\sigma(p^2)$.

2. Off-shell subtraction

At an unphysical (off-shell) value of p^2, say $p^2=-\lambda^2$ with $-\lambda^2<0$, we require that $\tilde{S}_{Rij}(p)$ be of the form of the free (massless) propagator

$$\tilde{S}_{Rij}(p) \sim -\frac{\delta_{ij}}{\not{p}} \quad \text{for } p^2 \sim -\lambda^2$$

This condition determines z_2 such that

$$z_2 = \sigma(-\lambda^2) = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{\lambda^2}{4\pi\mu^2}\right)\right)$$

and the renormalized propagator reads

$$\tilde{S}_{Rij}(p) = -\frac{\delta_{ij}}{\not{p}} \left(1 - \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \ln\left(-\frac{p^2}{\lambda^2}\right)\right)^{-1}$$

This scheme is also called momentum-space subtraction scheme. (MOM)

3. Minimal subtraction (MS) ('t Hooft)

This is specific to DR. We only eliminate the $1/\varepsilon$ pole in the DR expression of the Green functions. This scheme is very economical and often used in QCD and other gauge theories. The requirement imposes that

$$Z_2 = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \frac{1}{\varepsilon}$$

Therefore, the renormalized propagator is

$$\tilde{S}_{R\,ij}(p) = -\frac{\delta_{ij}}{\not p}\left\{1 - \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F\left(\gamma_E - 1 + \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right)\right\}^{-1}$$

- renormalization constants $\rightarrow$ simple expression
- Green functions $\rightarrow$ complicated

Z_2 independent of mass parameters $\rightarrow$ easy to define renormalization group functions.

The $\tilde{S}_{R\,ij}(p)$ above can be converted in the off-shell subtraction (MOM) by setting

$$\Lambda^2 = 4\pi e^{1-\gamma_E} \mu^2$$

4. Modified Minimal subtraction ($\overline{MS}$)

In the expression for $\Gamma(p^2)$ the pole term is accompanied by γ_E and $\ln 4\pi$

$$\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi$$

It can be shown that this combination always appears in any calculation at 1-loop order.

$\Rightarrow$ more convenient to eliminate the whole factor in the renormalization process, instead of only eliminating $1/\varepsilon$. This procedure/prescription goes under the name of modified minimal subtr.

The renormalization constant in this ($\overline{MS}$) scheme is given by

$$Z_2 = \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right)$$

The renormalized propagator reads

$$\tilde{S}_{R_{ij}}(p) = -\frac{\delta_{ij}}{\not{p}} \left\{ 1 - \alpha \frac{g_{0s}^2}{(4\pi)^2} C_F \left(-1 + \ln\left(-\frac{p^2}{\mu^2}\right)\right)\right\}^{-1}$$

- $\overline{MS}$ $\rightarrow$ many advantages $\rightarrow$ compact expression for the renormalized propagator.

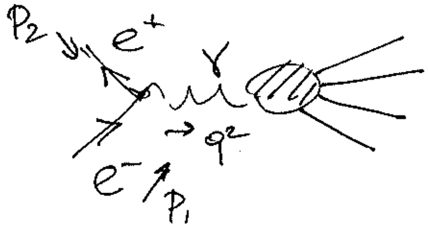
The Feynman parametrization: general formula

$$\prod_{i=1}^{n} \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{l=1}^{n} \Gamma(\alpha_i)} \int_0^1 \left(\prod_{i=1}^{n} dx_i \, x_i^{\alpha_i - 1}\right) \frac{\delta(1-x)}{\left(\sum_{i=1}^{n} x_i A_i\right)^{\alpha}}$$

Electron-positron annihilation

We'll review e^+e^- annihilation and the computation of the total X sec.

* e^+e^- annihilate through EM interactions producing hadrons in the final state
* Z_0 will not be considered for simplicity for now



X   X = system of hadrons

$$e^+ + e^- \to X$$

Feyman Amplitude

$$\langle X | T | e^+e^- \rangle = \bar{v}_{\lambda_2}(p_2)\, e \gamma^\mu\, u_{\lambda_1}(p_1) \, * $$
$$* \frac{1}{q^2} \langle X | (-e) J_\mu(0) | 0 \rangle$$

P_1, P_2 = incoming momenta of $e^+ e^-$

λ_1, λ_2 = spins of the incoming $e^+ e^-$.

$J_\mu(x)$ = quark part of the EM current.

$$\mathcal{L}_1 = (-e \bar{\psi}_e \gamma_\mu \psi_e + e J_\mu) A^\mu$$ } Little digression

$|X\rangle$ = state representing the final hadron system

Useful properties for current-current products

Completeness $\quad \sum_X |X\rangle\langle X| = 1$

translation invariance

$$J_\mu(x) = e^{iP\cdot x} J_\mu(0)\, e^{-iP\cdot x}$$

where P is the energy-momentum operator which satisfies the eigenvalue equation

$$P^\mu |X\rangle = p_X^\mu |X\rangle$$

We indicate with q the the total momentum

$$q = p_1 + p_2$$

The total Xsec for $e^+ + e^- \to X$ can be written as

$$\sigma = \frac{1}{2s}\frac{1}{4}\sum_{\lambda_1,\lambda_2}\sum_X (2\pi)^4 \delta^4(p_X - q)\,|\langle X|T|e^+e^-\rangle|^2$$

T is the operator that allows for the transition

* we are going to neglect the electron mass

$$s = q^2 = (p_1+p_2)^2 = 2m_e^2 + 2p_1\cdot p_2 \simeq 2p_1\cdot p_2$$

General formula for $\sigma \longrightarrow$

$$\sigma = \frac{1}{k(s)} \frac{1}{(2J_1+1)(2J_2+1)} \sum_{\substack{\lambda_1\lambda_2,\mu_1\ldots\mu_n \\ (\text{spin})}} \int \prod_{j=1}^{n} \frac{d^3k_j}{(2\pi)^3 2k_j^0} (2\pi)^4 \delta^{(4)}\left(\sum_{j=1}^{n} k_j - p_1 - p_2\right) *$$

$$* \left| \langle k_1\mu_1, k_2\mu_2, \ldots | T | p_1\lambda_1, p_2\lambda_2 \rangle \right|^2$$

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$$k(s) = \sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}$$

Inserting the expression for $\langle X | T | e^+e^- \rangle$ into that of the X sec σ we obtain

$$\sigma = \frac{e^4}{2s^3} \, l^{\mu\nu} w_{\mu\nu}$$

$l^{\mu\nu}$ = leptonic tensor

$$l^{\mu\nu} = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2} g^{\mu\nu}$$

$w^{\mu\nu}$ = hadronic tensor

$$w_{\mu\nu} = \sum_X (2\pi)^4 \delta^{(4)}(p_X - q) \langle 0 | j_\mu(0) | X \rangle \langle X | j_\nu(0) | 0 \rangle$$

This can be rewritten in a more compact form. using the completeness relation over $|X\rangle$, translation invariance and properties of the Fourier transf.

We observe that in general for a physical process

$$\int d^4x \, e^{iq\cdot x} \langle p | j_\nu(0) \, j_\mu(x) | p \rangle = 0$$

where E = initial energy (0-component of q^μ, or q^0)

E' = final-state energy

Physical process $\Rightarrow E > E' \Rightarrow q_0 > 0$

Using translation invariance we obtain

$$\int d^4x\, e^{iq\cdot x} \langle p | J_\nu(0)\, e^{i\hat{p}\cdot x} J_\mu(0)\, e^{-i\hat{p}\cdot x} | p \rangle$$

and using the completeness relation

$$\sum_X \int d^4x\, e^{iq\cdot x} \langle p | J_\nu(0)\, e^{i\hat{p}x} | X \rangle \langle X | J_\mu(0)\, e^{-i\hat{p}\cdot x} | p \rangle$$

Eigenvalue equs:

$$\begin{cases} e^{i\hat{p}\cdot x} | X \rangle = e^{i p_X\cdot x} | X \rangle \\ e^{-i\hat{p}\cdot x} | p \rangle = e^{-ip\cdot x} | p \rangle \end{cases}$$

Therefore we can write

$$\sum_X \int d^4x\, e^{iq\cdot x} e^{ip_X\cdot x} e^{-ipx} \langle p | J_\nu(0) | X \rangle \langle X | J_\mu(0) | p \rangle =$$

$$\sum_X (2\pi)^4 \delta^{(4)}(q - p + p_X) \langle p | J_\nu(0) | X \rangle \langle X | J_\mu(0) | p \rangle$$

For e^+e^- we found an expression with $p=0$

and with $\delta(p_X - q)$.

$$\sum_X (2\pi)^4 \delta^4(p_X - q) \langle 0 | J_\mu(0) | X \rangle \langle X | J_\nu(0) | 0 \rangle =$$

$$\sum_X \int e^{i(q - p_X)\cdot x} \langle 0 | J_\mu(0) | X \rangle \langle X | J_\nu(0) | 0 \rangle d^4x =$$

$$\sum_X \int e^{iq\cdot x} \langle 0 | J_\mu(0) e^{-ip_X \cdot x} | X \rangle \langle X | J_\nu(0) | 0 \rangle d^4x =$$

$$\langle 0 | J_\mu(x) | X \rangle = \langle 0 | e^{i0\cdot x} J_\mu(0) e^{-ip_X \cdot x} | X \rangle \Rightarrow$$

$$= \int e^{iqx} \langle 0 | J_\mu(x) J_\nu(0) | 0 \rangle d^4x$$

We can prove that

$$\int e^{iq\cdot x} \langle 0 | J_\nu(0) J_\mu(x) | 0 \rangle d^4x = 0$$

<u>Again</u>

$$\int e^{iq\cdot x} \langle 0 | J_\nu(0) J_\mu(x) | 0 \rangle d^4x =$$

$$\sum_X \int e^{iq\cdot x} \langle 0 | J_\nu(0) | X \rangle \langle X | e^{ip_X \cdot x} J_\mu(0) | 0 \rangle d^4x =$$

$$\sum_X (2\pi)^4 \delta^{(4)}(q + p_X) \langle 0 | J_\nu(0) | X \rangle \langle X | J_\mu(0) | 0 \rangle$$

But this cannot be satisfied:

$\delta^{(4)}(q+p_x) \Rightarrow \quad q = -p_x \quad$ with $q_0 < 0$

$q^\mu = (q_0, \vec{0}) \qquad q^2 = (p_1+p_2)^2 = 2p_1 \cdot p_2 = s > 0$

$P_1^\mu = (p_1^0, \vec{0}_\perp, p_z)$

$P_2^\mu = (p_2^0, \vec{0}_\perp, -p_z)$

$P_1^\mu + P_2^\mu = q^\mu = (p_1^0 + p_2^0, \vec{0})$ CM system

$q^2 \cong 2p_{10}p_{20} = s > 0 \quad \searrow \quad q_0 > 0 \Rightarrow$ contradiction

$\Rightarrow$

$$\omega_{\mu\nu} = \int d^4x \, e^{iq\cdot x} \langle 0 | [J_\mu(x), J_\nu(0)] | 0 \rangle$$

In high-energy annihilations, q_0 is large.

$\Rightarrow q_0 \to +\infty \Rightarrow e^{iq\cdot x}$: rapid oscillatory behavior with no bounds.

Therefore only the $x_0 \sim 0$ makes a major contribution to the integral.

The integrand on the other hand, has support only for $x^2 \geq 0$ due to causality requirement. $[J_\mu(x), J_\nu(0)] = 0$ for $x^2 < 0$

Therefore, $x_0 \sim 0 \Rightarrow x \sim 0$ and we conclude that the total Xsec for e^+e^- annihilation is governed by short-distance current commutators.

$W_{\mu\nu}$ can be written even more generally following Lorentz invariance and current conservation
→ only 1 invariant amplitude needed

$$W_{\mu\nu} = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\frac{1}{6\pi}\, w(q^2)$$

$1/(6\pi)$ is for convenience (we shall see this later)

Therefore σ is obtained as

$$\sigma = \frac{4\pi\alpha_{em}^2}{3s}\, w(s)$$

$$\alpha_{em} = \frac{e^2}{4\pi}$$

We have studied for example that $e^+e^- \to \mu^+\mu^-$

$$\sigma_{\mu\mu}^{(LO)} = \frac{4\pi\alpha_{em}^2}{3s}$$

$$R = \frac{\sigma}{\sigma_{\mu\mu}} = w(s) = \text{Drell ratio}$$

$$R = \frac{4\pi \alpha_{em}^2}{3s} w(s) \frac{3s}{4\pi\alpha_{em}^2} = w(s)$$

which can be written as

$$R = -\frac{2\pi}{q^2} \int d^4x \, e^{iq\cdot x} \langle 0 | J_\mu(x) J^\mu(0) | 0 \rangle$$

In a $e^+ + e^- \to X$ process at very high COM (i.e., large $\sqrt{q^2}$) massess of the quarks are negligible

$$R = R(s = q^2, g, \mu)$$

$\mu \to$ renormalization scale

$g \to$ renormalized coupling constant

$R = R(s/\mu^2, g)$: rearranged for dimensional reasons.

R : does not contain any hadronic state in its expression. It consists only of the short distance piece for large s

According to the renormalization group equation for R (which we will not discuss here for now) we can write

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] R\left(\frac{s}{\mu^2}, g\right) = 0$$

The general solution is given by

$$R\left(\frac{s}{\mu^2}, g\right) = R(1, \bar{g}(s))$$

where $\bar{g}(s)$ is defined in terms of the β-function $\ni'$

$$\frac{d\bar{g}}{dt} = \beta(\bar{g}) \qquad \bar{g}(\mu^2) = g$$

with $t = 1/2 \ln(s/\mu^2)$.

This means that the explicit s-dependence of R computed by using the coupling constant g can be completely absorbed in the s-dependence of the "running coupling" constant $\bar{g}(s)$

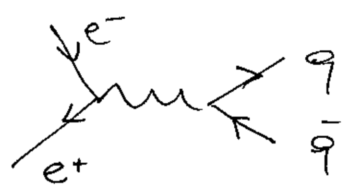
Asymptotically free field theories $\Rightarrow$ $\bar{g}(s) \to 0$ when $s \to \infty$

<u>Perturbative expansion for R</u>

$$R\left(\frac{s}{\mu^2}, g\right) = \sum_i Q_i^2 \left[1 + a(s/\mu^2)g^2 + b(s/\mu^2)g^4 + \cdots\right]$$

The first term in the expansion refers to

$e^+ + e^- \to q + \bar{q}$



$\sigma = Q^2 \frac{4\pi\alpha_{em}^2}{3s}$

$\sum_i Q_i$ $\to$ the index i runs over colors and flavors of quarks

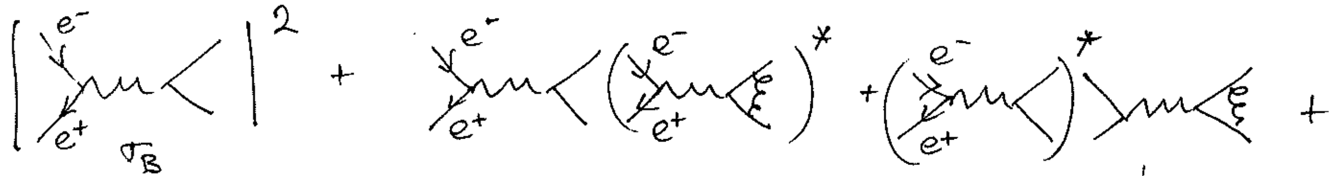
The other terms ~~___~~ represent radiative corrections

The coefficients a, b, etc, contain large logs $\ln(s/\mu^2)$

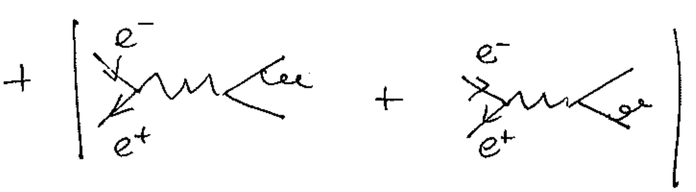
$$R\left(s/\mu^2, \bar{g}(s)\right) = R(1, \bar{g}(s)) \Rightarrow R\left(\frac{s}{\mu^2}, g\right) = \sum_i Q_i^2 \left[1 + a(1)\bar{g}(s)^2 + b(1)\bar{g}(s)^4 + \dots\right]$$

where $g \to \bar{g}(s)$ indicates that the large logs in the coeffic. disappear. The latter expression is more stable in perturbation theory because the expansion coefficients are smaller, and the expansion parameter $\bar{g}(s)$ is smaller than g for $s \gg \mu^2$ (according to asymptotic freedom)

Radiative corrections to e^+e^- annihilation

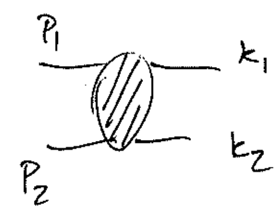


virtual contributions



→ real emission contributions

 $\sim O(g_s)$ $\qquad g_s = \sqrt{4\pi\alpha_s}$



$$\sigma_B = \frac{4\pi\alpha_{em}^2}{s} \sum_i Q_i$$

$$\sigma = Z_2^2\,\sigma_B + \sigma_V + \sigma_R = \sigma_B + \tilde{\sigma}_V + \sigma_R$$

where $\tilde{\sigma}_V = \sigma_V + (Z_2^2 - 1)\sigma_B$

Z_2 = field renormalization constant

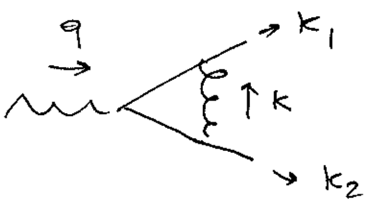
If we neglect the mass of the quarks

$$\sigma_V = \frac{1}{8s}\int \frac{d^3k_1}{(2\pi)^3 2k_{10}}\,\frac{d^3k_2}{(2\pi)^3 2k_{20}}\,(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2)\,F_V$$

$$F_V = \Big(\sum_i Q_i\Big)^2 \frac{e^4}{q^4}\,\mathrm{Tr}\big[\not{p}_2\,\gamma^\mu\,\not{p}_1\,\gamma^\nu\big]\,\mathrm{Tr}\big[\not{k}_1\,\Lambda_\mu\,\not{k}_2\,\gamma_\nu\big] + \text{c.c.}$$

$$\Lambda_\mu = g_s^2 C_F \int \frac{d^Dk}{(2\pi)^D i}\,\frac{1}{k^2}\,\gamma_\lambda\,\frac{1}{\not{k}-\not{k}_1}\,\gamma_\mu\,\frac{1}{\not{k}+\not{k}_2}\,\gamma^\lambda$$

↳ 1-loop vertex



* We can use the Feynman gauge for this calculation

$m_q \to 0 \Rightarrow$ mass singularity $\Rightarrow$ infrared divergences $\Rightarrow$ we need regularization!

Let's review the unrenormalized 1-loop self-energy contribution for $m_q = 0$ in the Feynman gauge

singular for $p^2=0$ (mass singularity)

$$\Sigma(p) = g_{0s}^2\, C_F\,(2-D)\int_0^1 dx \int \frac{d^Dk'}{(2\pi)^D i}\,\frac{\not{k}' - (1-x)\not{p}}{\{k'^2 + x(1-x)p^2\}^2}$$

$$\int \frac{d^Dk}{k^2} = 0$$ can be justified in the sense of DR.

More general case in DR

$$\int \frac{d^D k}{(-k^2)^\alpha} = 0 \qquad \alpha > 0$$

with a Wick Rotation we obtain

$$\int \frac{d^D k}{(-k^2)^\alpha} = i \, \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^{+\infty} (\bar{k}^2)^{D/2-\alpha-1} \, d\bar{k}^2$$

$\bar{k}^2 = -k^2$

$D > 2\alpha \rightarrow$ ultraviolet divergence UV - div

$D < 2\alpha \rightarrow$ infrared divergence IR - div

- No mathematical meaningful region in D.

We found in previous cases that the integral made sense for $D < 3$. We are going to use the same approach here and split the integration ~~part~~ in $\bar{k}$ in two parts:

$$\begin{cases} \bar{k}^2 > \Lambda^2 & \text{ultraviolet part} \\ \bar{k}^2 < \Lambda^2 & \text{infrared part} \end{cases}$$

$$\int \frac{d^D k}{(-k^2)^\alpha} = i \, \frac{\pi^{D/2}}{\Gamma(D/2)} \left[\int_0^{\Lambda^2} (\bar{k}^2)^{D/2-\alpha-1} d\bar{k}^2 + \int_{\Lambda^2}^{+\infty} (\bar{k}^2)^{D/2-\alpha-1} d\bar{k}^2 \right.$$

1^{st} integral convergent for $D > 2\alpha$

2^{nd} integral convergente for $D < 2\alpha$

D = regulator for IR and UV divergences.

We call $D_I = D$ in the 1^{st} integral and $D_U = D$ in the 2^{nd}.

Integrating, we obtain

$$\int \frac{d^D k}{(-k^2)^\alpha} = i \frac{\pi^{D/2}}{\Gamma(D/2)} \left[\frac{\Lambda^{D_I - 2\alpha}}{\frac{1}{2} D_I - \alpha} - \frac{\Lambda^{D_U - 2\alpha}}{\frac{1}{2} D_U - \alpha} \right]$$

The two terms have poles at $D_I = D_U = 2\alpha$

$i\varepsilon$

$\frac{\Lambda^{D_I - 2\alpha}}{\frac{1}{2} D_I - \alpha} - \frac{\Lambda^{D_U - 2\alpha}}{\frac{1}{2} D_U - \alpha}$ can be continued analitically and the constraints $D_I > 2\alpha$ and $D_U < 2\alpha$

$\frac{\Lambda^{D_I - 2\alpha}}{\frac{1}{2} D_I - \alpha + i\varepsilon'} - \frac{\Lambda^{D_U - 2\alpha}}{\frac{1}{2} D_U - \alpha + i\varepsilon}$ can be removed.

If D_I and D_U are identified after this analytic continuation, then when $D_I = D_U$ on the right hand side of the equation is zero. $\Rightarrow$

$$\int \frac{d^D k}{(-k)^\alpha} = 0$$

This result is justified within DR.

Therefore, if we look at $\Sigma(p)$ for $p^2=0$

$$\Sigma(p) = g_{s0}^2 C_F (D-2) \not{p} \int_0^1 dx(1-x) \int \frac{d^D k'}{(2\pi)^D i} \frac{1}{\{k'^2 + x(1-x)p^2\}^2}$$

after we have removed $\left(\int d^D k \, k_\mu f(k^2) = 0 \right)$ the linear terms in k_μ,

$$\Sigma(p)\Big|_{p^2=0} = g_{0s}^2 C_F (D-2) \not{p} \underbrace{\int_0^1 d(1-x)}_{1/2} \int \frac{d^D k}{(2\pi)^D i} \frac{1}{k^4}$$

This is like the case we analyzed before with $\alpha = 2$

$$\Sigma(p)\Big|_{p^2=0} = \frac{g_{0s}^2}{(4\pi)^2} C_F \not{p} \left[\frac{1}{\varepsilon'} - \frac{1}{\varepsilon} \right] \Rightarrow \boxed{Z_2 = 1 + \frac{g^2}{(4\pi)^2} C_F \left(\frac{1}{\varepsilon'} - \frac{1}{\varepsilon} \right)}$$

where ε' and ε are equal to the $(4-D)/2$ param.

* We do not set $\varepsilon' = \varepsilon$ here because we want to distinguish between the two divergences

* The integral for Λ_μ can be treated in the same way

$$\Lambda_\mu = \gamma_\mu \frac{g_{0s}^2 C_F}{8\pi^2} \left(\frac{4\pi\mu^2}{-q^2} \right)^{\varepsilon} \Gamma(1+\varepsilon) B(1-\varepsilon, 2-\varepsilon) \left(\frac{1}{\varepsilon'} - \frac{2}{\varepsilon^2} - 2 \right)$$

$\mu^2 \rightarrow$ mass scale to make g dimensionless.

$q = p_1 + p_2$

The knowledge of the singular structure of $\Sigma(p)$, allows us to determine the field renormalization const. Z_2

The remarkable task of renormalization is accomplished in this case:

$\tilde{\sigma}_V = \sigma_V + \left(Z_2^2 - 1\right)\sigma_B$ all UV div cancel out in $\tilde{\sigma}_V$!

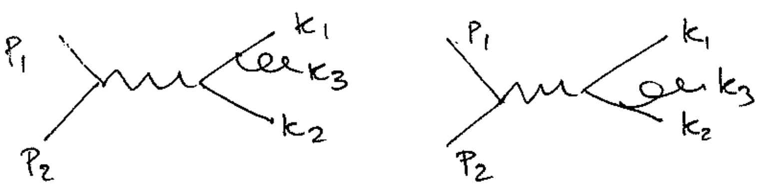
Putting all the ingredients together we obtain

$$\tilde{\sigma}_V = A_V\,\sigma_B$$

$$A_V = \frac{\alpha_s}{\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \frac{\cos(\pi\varepsilon)}{\Gamma(1-\varepsilon)} \left[-\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + O(\varepsilon)\right]$$

where $\alpha_s = \dfrac{g_s^2}{4\pi}$

Real emission corrections



$$\sigma_R = \frac{1}{8s}\int \prod_{i=1}^{3} \frac{d^{D-1}k_i}{(2\pi)^{D-1}\,2k_{i0}} (2\pi)^D \delta^{(D)}\left(\sum_{i=1}^{3} k_i - p_1 - p_2\right) F_R$$

$$F_R = -\left(\sum_i Q_i^2\right)\frac{e^4}{q^2} g_s^2 C_F \underbrace{\mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right]}_{L^{\mu\nu}} \underbrace{\mathrm{Tr}\left[\not{k}_1 S_{\lambda\mu} \not{k}_2 S^{\lambda}{}_{\nu}\right]}_{G_{\mu\nu}}$$

$$S_{\mu\nu} = \gamma_\mu \frac{-1}{\not{k}_1 + \not{k}_3}\gamma_\nu + \gamma_\nu \frac{1}{\not{k}_2 + \not{k}_3}\gamma_\mu$$

$$L^{\mu\nu} = \left[4\, p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2} g^{\mu\nu}\right]$$