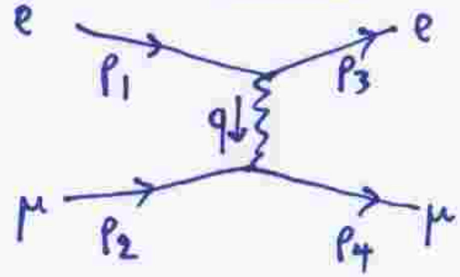
Electron-muon scattering $e^-(p_1)+\mu^-(p_2)\to e^-(p_3)+\mu^-(p_4)$



with $p_1 = q + p_3$ and $p_2 + q = p_4$

so $q = p_1 - p_3$ or $q = p_4 - p_2$ Also $p_1 + p_2 = p_3 + p_4$

$$i\mathcal{M} = \bar{u}(p_3)(-ie\gamma^\mu)u(p_1)\,\frac{(-i)g_{\mu\nu}}{q^2}\,\bar{u}(p_4)(-ie\gamma^\nu)u(p_2)$$

where $\mathcal{M}$ is the amplitude

$$\Rightarrow \mathcal{M} = e^2\,\bar{u}(p_3)\gamma^\mu u(p_1)\,\frac{1}{(p_1-p_3)^2}\,\bar{u}(p_4)\gamma_\mu u(p_2)$$

To calculate cross sections we need $|\mathcal{M}|^2$

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1-p_3)^4}[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma_\mu u(p_2)][\bar{u}(p_3)\gamma^\nu u(p_1)]^*[\bar{u}(p_4)\gamma_\nu u(p_2)]^*$$

We have $[\bar{u}(p_3)\gamma^\nu u(p_1)]^* = [\bar{u}(p_3)\gamma^\nu u(p_1)]^\dagger = [u^\dagger(p_3)\gamma^0\gamma^\nu u(p_1)]^\dagger$

$$= u^\dagger(p_1)\gamma^{\nu\dagger}\gamma^{0\dagger}u(p_3) = u^\dagger(p_1)\gamma^{\nu\dagger}\gamma^0 u(p_3) = u^\dagger(p_1)\gamma^0\gamma^0\gamma^{\nu\dagger}\gamma^0 u(p_3)$$

$= \bar{u}(p_1)\gamma^0\gamma^{\nu\dagger}\gamma^0 u(p_3)$ We note that $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$, $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$

For $\nu = 0 \to \bar{u}(p_1)\gamma^0\gamma^{0\dagger}\gamma^0 u(p_3) = \bar{u}(p_1)\gamma^0\gamma^0\gamma^0 u(p_3) = \bar{u}(p_1)\gamma^0 u(p_3)$

For $\nu = i$ with $i = 1,2,3 \to \bar{u}(p_1)\gamma^0\gamma^{i\dagger}\gamma^0 u(p_3) = -\bar{u}(p_1)\gamma^0\gamma^i\gamma^0 u(p_3) = \bar{u}(p_1)\gamma^0\gamma^0\gamma^i u(p_3) = \bar{u}(p_1)\gamma^i u(p_3)$

So for any $\nu \to [\bar{u}(p_3)\gamma^\nu u(p_1)]^* = \bar{u}(p_1)\gamma^\nu u(p_3)$ Also $[\bar{u}(p_4)\gamma_\nu u(p_2)]^* = \bar{u}(p_2)\gamma_\nu u(p_4)$

$$e^-(p_1) + \mu^-(p_2) \to e^-(p_3) + \mu^-(p_4)$$

Then $|\mathcal{M}|^2 = \dfrac{e^4}{(p_1-p_3)^4} [\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_1)\gamma^\nu u(p_3)][\bar{u}(p_4)\gamma_\mu u(p_2)][\bar{u}(p_2)\gamma_\nu u(p_4)]$

We sum over spins and note that $\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p} + m$

$\left(\text{and } \sum_{s=1,2} v^{(s)}(p)\bar{v}^{(s)}(p) = \not{p} - m\right)$

Then

$$\sum_{s,s'} \bar{u}^{(s')}(p_3)\gamma^\mu u^{(s)}(p_1)\bar{u}^{(s)}(p_1)\gamma^\nu u^{(s')}(p_3) = \sum_{s'} \bar{u}^{(s')}(p_3)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu u^{(s')}(p_3)$$

(where m_e is the electron mass)

$= \sum_{s'} \bar{u}^{(s')}_m(p_3)\left(\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right)_{mn} u^{(s')}_n(p_3)$ where we wrote explicitly the matrix elements m, n

$= \left(\sum_{s'} u^{(s')}_n(p_3)\bar{u}^{(s')}_m(p_3)\right)\left(\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right)_{mn} = (\not{p}_3 + m_e)_{nm}\left(\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right)_{mn}$

$= \mathrm{tr}\left[(\not{p}_3 + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]$

trace

Similarly $\sum_{s,s'} \bar{u}^{(s')}(p_4)\gamma_\mu u^{(s)}(p_2)\bar{u}^{(s)}(p_2)\gamma_\nu u^{(s')}(p_4) = \mathrm{tr}\left[(\not{p}_4 + m_{\mu^-})\gamma_\mu(\not{p}_2 + m_{\mu^-})\gamma_\nu\right]$

For unpolarized beams we average over initial spins $\to \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

Then $|\mathcal{M}|^2 = \dfrac{e^4}{4t^2}\,\mathrm{tr}\left[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right]\mathrm{tr}\left[(\not{p}_4+m_{\mu^-})\gamma_\mu(\not{p}_2+m_{\mu^-})\gamma_\nu\right]$

where t is the Mandelstam variable $t = (p_1-p_3)^2$

Trace identities for matrices A, B, C

we have $tr(A+B) = trA + trB$, $tr(aA) = a\, trA$

$tr(AB) = tr(BA)$, $tr(ABC) = tr(BCA) = tr(CAB)$ cyclicity

for an $n \times n$ unit matrix $tr(1) = n$ so $tr(1) = 4$ for 4×4 unit matrix

$tr(\gamma^\mu) = 0$ as can be seen explicitly in both Dirac and chiral reps. but this is a general property in any representation

$tr(\gamma^5) = 0$ where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$tr(\gamma^\mu\gamma^\nu) = tr(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2g^{\mu\nu} tr(1) - tr(\gamma^\nu\gamma^\mu) = 2g^{\mu\nu}\cdot 4 - tr(\gamma^\mu\gamma^\nu)$$

$$\Rightarrow 2\, tr(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu} \Rightarrow tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$tr(\gamma^\mu\gamma^\nu\gamma^5) = 0$$

The trace of a product of an <u>odd</u> number of γ matrices is <u>zero</u>

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)$$

$$= tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma)$$

$$= tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\rho 2g^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)$$

$$\Rightarrow 2\, tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho)$$

since $tr(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu) = tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$

$$\Rightarrow tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

Other useful relations

$$g_{\mu\nu} g^{\mu\nu} = 4$$

$$\gamma^\mu \gamma_\mu = 4$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu \quad \text{and} \quad \gamma_\mu \not{p} \gamma^\mu = -2\not{p}$$

$$\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4 g^{\nu\lambda} \quad \text{and} \quad \gamma_\mu \not{p} \not{q} \gamma^\mu = 4\, p\cdot q$$

$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \quad \text{and} \quad \gamma_\mu \not{p} \not{q} \not{p}' \gamma^\mu = -2 \not{p}' \not{q} \not{p}$$

$$\mathrm{tr}(\not{p}\not{q}) = 4\, p\cdot q$$

$$\mathrm{tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4(p_1\cdot p_2\ p_3\cdot p_4 - p_1\cdot p_3\ p_2\cdot p_4 + p_1\cdot p_4\ p_2\cdot p_3)$$

$$\mathrm{tr}(\gamma^5 \not{p}\not{q}) = 0$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) = -4i\,\varepsilon^{\mu\nu\rho\sigma}$$

where $\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ +1 & \text{if } \mu\nu\rho\sigma \text{ is an even } \quad \text{"} \quad \text{"} \\ 0 & \text{if any two indices are the same} \end{cases}$

$$\mathrm{tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4 \gamma^5) = -4i\,\varepsilon^{\mu\nu\rho\sigma} p_{1\mu} p_{2\nu} p_{3\rho} p_{4\sigma}$$

Also the trace of a product of an odd number of terms with γ matrices and slashed momenta is zero: $\mathrm{tr}(\gamma^{\mu_1} \dots \gamma^{\mu_n} \not{p}_1 \cdots \not{p}_m) = 0$ if $n+m$ is odd