Lorentz transformation $\quad x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$

If frame S' moves with speed v along the x-axis relative to frame S, then

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad x'^0 = \gamma\left(x^0 - \frac{v}{c}x^1\right)$$

$$x' = \gamma(x - vt) \quad \text{or} \quad x'^1 = \gamma\left(x^1 - \frac{v}{c}x^0\right) \quad \text{and} \quad \Lambda = \begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$y' = y \qquad x'^2 = x^2$$

$$z' = z \qquad x'^3 = x^3$$

A 4-vector A^{μ} is a four-component object that transforms in the same way as x^{μ} under Lorentz transformations, i.e. $A'^{\mu} = \Lambda^{\mu}_{\ \nu} A^{\nu}$

A second-rank tensor $T^{\mu\nu}$ (two indices) transforms with two Λ's:

$$T'^{\mu\nu} = \Lambda^{\mu}_{\ \rho} \Lambda^{\nu}_{\ \sigma} T^{\rho\sigma}$$

An n-rank tensor $T^{a_1 a_2 \cdots a_n}$ (n indices) transforms with n Λ's

$$T'^{a_1 a_2 \cdots a_n} = \Lambda^{a_1}_{\ b_1} \Lambda^{a_2}_{\ b_2} \cdots \Lambda^{a_n}_{\ b_n} T^{b_1 b_2 \cdots b_n}$$

For a 4-vector A^{μ}, $A^2 = A \cdot A = (A^0)^2 - \vec{A}^2$

If $A^2 > 0$ then A^{μ} is timelike

If $A^2 = 0$ then A^{μ} is lightlike

If $A^2 < 0$ then A^{μ} is spacelike

Energy-momentum conservation

Consider collisions $a+b \to 1+2$

particles a and b with 4-momenta p_a^μ and p_b^μ collide and produce particles 1 and 2 with p_1^μ and p_2^μ

Then $p_a^\mu + p_b^\mu = p_1^\mu + p_2^\mu$ or $p_a + p_b = p_1 + p_2$

Since $p_a^\mu = \left(\frac{E_a}{c}, \vec{p}_a\right)$, $p_b^\mu = \left(\frac{E_b}{c}, \vec{p}_b\right)$, $p_1^\mu = \left(\frac{E_1}{c}, \vec{p}_1\right)$, $p_2^\mu = \left(\frac{E_2}{c}, \vec{p}_2\right)$

this is $\left(\frac{E_a+E_b}{c}, \vec{p}_a + \vec{p}_b\right) = \left(\frac{E_1+E_2}{c}, \vec{p}_1 + \vec{p}_2\right)$

so $\underbrace{E_a + E_b = E_1 + E_2}_{\text{energy conservation}}$ and $\underbrace{\vec{p}_a + \vec{p}_b = \vec{p}_1 + \vec{p}_2}_{\text{momentum conservation}}$

$\to$ 4-momentum conservation

We also have $p_a^2 = m_a^2 c^2$ $p_b^2 = m_b^2 c^2$ $p_1^2 = m_1^2 c^2$ $p_2^2 = m_2^2 c^2$

$E_a = \sqrt{\vec{p}_a^{\,2} c^2 + m_a^2 c^4}$ $E_b = \sqrt{\vec{p}_b^{\,2} c^2 + m_b^2 c^4}$ $E_1 = \sqrt{\vec{p}_1^{\,2} c^2 + m_1^2 c^4}$ $E_2 = \sqrt{\vec{p}_2^{\,2} c^2 + m_2^2 c^4}$

Note that $p_a^2, p_b^2, p_1^2, p_2^2, (p_a+p_b)^2, (p_1+p_2)^2, (p_a-p_1)^2, (p_a-p_2)^2, (p_b-p_1)^2, (p_b-p_2)^2$ are relativistic invariants

Mandelstam variables

For the process $a + b \to 1 + 2$

we define $s = (p_a + p_b)^2$ $\quad t = (p_a - p_1)^2$ $\quad u = (p_a - p_2)^2$

They are kinematical variables but also relativistic invariantes (i.e. have the same value in all reference frames)

Since $p_a + p_b = p_1 + p_2 \Rightarrow p_a - p_1 = p_2 - p_b$ and $p_a - p_2 = p_1 - p_b$

Also $s = (p_a + p_b)^2 = p_a^2 + p_b^2 + 2\, p_a \cdot p_b = m_a^2 + m_b^2 + 2\,(E_a E_b - \vec{p}_a \cdot \vec{p}_b)$

and $s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2$ (used $c = 1$)

We can choose any frame of reference we like to calculate s, t, u, since they are invariant.

In center of mass frame (C.M.) $\vec{p}_a + \vec{p}_b = 0 = \vec{p}_1 + \vec{p}_2$ so $\vec{p}_b = -\vec{p}_a$ and $\vec{p}_2 = -\vec{p}_1$

In C.M. frame $p_a + p_b = \left(\frac{E_a^{cm} + E_b^{cm}}{c}, \vec{0} \right)$ Thus $s = (E_a^{cm} + E_b^{cm})^2 / c^2$ (in any frame)

So $\sqrt{s} = E_a^{cm} + E_b^{cm}$ (with $c=1$)

is the total energy in the C.M. frame

Schrodinger equation $\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$ where ψ is the wavefunction and $\hat{H}$ is the Hamiltonian operator (energy)

In quantum mechanics $\vec{p} \to -i\hbar \vec{\nabla}$ and $E \to i\hbar \frac{\partial}{\partial t}$

Non-relativistic: $\hat{H} = \frac{\hat{p}^2}{2m} + U = -\frac{\hbar^2}{2m}\nabla^2 + U$

Thus we get the non-relativistic Schrodinger equation (time-dependent)

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = i\hbar \frac{\partial \psi}{\partial t}$$

or $$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + U\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This equation has second-order derivatives in space coordinates x, y, z but first-order derivative in time.

Clearly unacceptable for a relativistic theory where space and time must be on equal footing.

Search for a relativistic quantum-mechanical wave equation

We can start from $p^2 = p \cdot p = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2$ or $p^\mu p_\mu = m^2c^2$

Now $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$ and $\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

and since $E \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to -i\hbar\vec{\nabla}$ we have

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \to \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, -i\hbar\vec{\nabla}\right) = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) = i\hbar\partial^\mu$$

So $p^\mu \to i\hbar\partial^\mu$ Similarly, $p_\mu = \left(\frac{E}{c}, -\vec{p}\right) \to \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, i\hbar\vec{\nabla}\right) = i\hbar\partial_\mu$

Then $p^\mu p_\mu = m^2c^2 \to i\hbar\partial^\mu i\hbar\partial_\mu = m^2c^2 \Rightarrow -\hbar^2\partial^\mu\partial_\mu = m^2c^2$

<u>Klein-Gordon equation</u> $\partial^\mu\partial_\mu\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0$

or $\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0$ second-order in space & time

In natural units $\hbar = c = 1$ this becomes $\partial^\mu\partial_\mu\varphi + m^2\varphi = 0$

Klein-Gordon equation $\quad \partial^\mu \partial_\mu \varphi + m^2 \varphi = 0$ (natural units)

$$\text{or} \quad \partial^\mu \partial_\mu \varphi + \frac{m^2 c^2}{\hbar^2} \varphi = 0$$

Plane-wave solutions: $\varphi(x^\mu) = \varphi(\vec{x}, t) = A\, e^{-\frac{i}{\hbar} p_\mu x^\mu}$ with A a constant

Check: $\partial_\mu \varphi = \dfrac{\partial \varphi}{\partial x^\mu} = -\dfrac{i}{\hbar} p_\mu A\, e^{-\frac{i}{\hbar} p_\mu x^\mu} = -\dfrac{i}{\hbar} p_\mu A\, e^{-\frac{i}{\hbar} p^\mu x_\mu}$

so $\partial^\mu \partial_\mu \varphi = \dfrac{\partial}{\partial x_\mu}(\partial_\mu \varphi) = -\dfrac{i}{\hbar} p_\mu \left(-\dfrac{i}{\hbar} p^\mu\right) A\, e^{-\frac{i}{\hbar} p^\mu x_\mu} = -\dfrac{p_\mu p^\mu}{\hbar^2} \varphi = -\dfrac{m^2 c^2}{\hbar^2} \varphi$

The free-particle Schrodinger eq. $-\dfrac{\hbar^2}{2m} \nabla^2 \varphi = i\hbar \dfrac{\partial \varphi}{\partial t}$ is the non-relativistic approximation to Klein-Gordon

For Schrodinger eq. $\underbrace{\rho = \varphi^* \varphi = |\varphi|^2}_{\text{probability density}}$ and $\underbrace{\vec{j} = -\dfrac{i\hbar}{2m}\left(\varphi^* \vec{\nabla} \varphi - \varphi \vec{\nabla} \varphi^*\right)}_{\text{probability current}}$

which obey the continuity equation $\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \dfrac{\partial}{\partial t}(\varphi^* \varphi) - \dfrac{i\hbar}{2m}\left(\varphi^* \nabla^2 \varphi - \varphi \nabla^2 \varphi^*\right)$

$$= \varphi^* \left(\frac{\partial \varphi}{\partial t} - \frac{i\hbar}{2m} \nabla^2 \varphi \right) + \varphi \left(\frac{\partial \varphi^*}{\partial t} + \frac{i\hbar}{2m} \nabla^2 \varphi^* \right) = 0$$

For Klein-Gordon eq. ρ should be the time component of a 4-vector

$\rho = \dfrac{i\hbar}{2m}\left(\varphi^* \dfrac{\partial \varphi}{\partial t} - \varphi \dfrac{\partial \varphi^*}{\partial t}\right)$ So $j^\mu = (\rho, \vec{j})$ with $\vec{j}$ as above