HW3

PHYS4210: Quantum Mechanics

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Problem 1. (2.22)

a)

$$\int_{-3}^{1} (x^3 - 3x^2 + 2x - 1)\delta(x+2)\ \mathrm{d}x = (-2)^3 - 3(-2)^2 + 2(-2) - 1 = -8 - 12 - 4 - 1 = \boxed{-25}.$$

b)

$$\int_{0}^{\infty} [\cos(3x) + 2]\delta(x - \pi)\ \mathrm{d}x = \cos(3\pi) + 2 = -1 + 2 = \boxed{1}.$$

c)

$$\int_{-1}^{1} e^{|x|+3}\delta(x-2)\ \mathrm{d}x = 0,$$

since the delta function delivers its spike at $x = 2$, which is outside the domain of the integral.

Problem 2. (2.26)

The definition of the Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx}\ \mathrm{d}k,$$

and the inverse Fourier transform is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\ \mathrm{d}x.$$

Here we have $f(x) \equiv \delta(x)$, so its Fourier transform is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x)e^{-ikx}\ \mathrm{d}x = \frac{1}{\sqrt{2\pi}}.$$

Therefore

$$f(x) = \delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{ikx}\ \mathrm{d}k = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\ \mathrm{d}x.$$

Problem 3. (2.29)

The odd bound states involve a choice of the odd function in the $0 < x < a$ region, which in this case is the sine, so our total wavefunction is

$$\psi(x) = \begin{cases} Fe^{-kx} & x > a, \\ C\sin(\ell x) & 0 < x < a, \\ -\psi(-x) & x < 0. \end{cases} \tag{3.1}$$

Continuity of $\psi(x)$ at $x = a$ means that

$$Fe^{-ka} = C\sin(\ell a).$$

Continuity of $\mathrm{d}\psi(x)/\mathrm{d}x$ at $x = a$ means that

$$-kFe^{-ka} = \ell C\cos(\ell a).$$

Dividing the second by the first gives

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$$-k = \ell \cot(\ell a) \quad \rightarrow \quad \frac{ka}{\ell a} = -\cot(\ell a).$$

Using the definition $z = \ell a$ and $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$, as well as the fact from the book that $ka = \sqrt{z_0^2 - z^2}$, we arrive at a very similar (but not exact) transcendental equation for z:

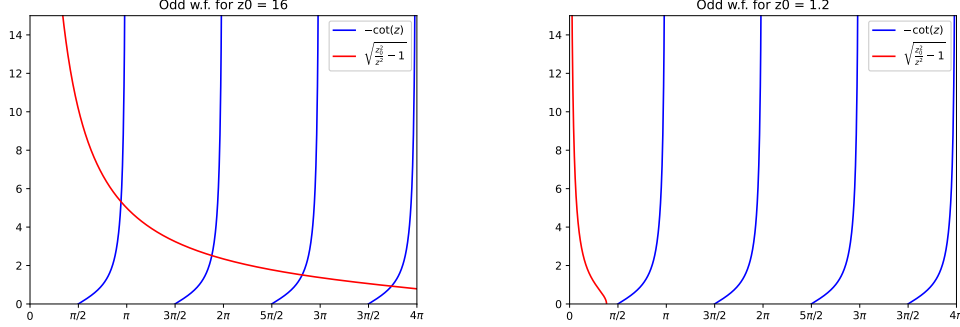$$-\cot(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}. \tag{3.2}$$

Figure 1: Plots of both sides of the transcendental equation in Equation (3.2) for $z_0 = 16$ (left) and $z_0 = 1.2$ (right).

In Figure 1, I plotted the left and right-hand sides of Equation (3.2). On the left I set $z_0 = 16$, in which we can see a similar looking plot for that of the even solutions, where we have a number of intersections. However, in the right plot, I let z_0 go way down to 1.2, where we can see that there are actually no intersections, meaning that for some suitably small cutoff value z_0, there are no longer any bound states. This corresponds to a particularly shallow or small well.

In the other limiting case with $z_0 \to \infty$ (a wide, deeper well), we get a similar limiting case to before where the intersections occur at integer multiples of π. This time, though, they occur at the even multiples, which complement the odd multiples from the even wavefunction.

Problem 4. (2.34)

a) In the region $x \leq 0$, the Schrödinger EQ reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \rightarrow \quad \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi.$$

Since we must have $E > V_{\min}$, $E > 0$, so we have to let $k = \sqrt{2mE}/\hbar$:

$$\rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi.$$

This has solutions

$$\psi(x) = Ae^{ikx} + Be^{-ikx}. \tag{4.1}$$

For the other region ($x > 0$), the Schrödinger EQ reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \quad \rightarrow \quad \frac{d^2\psi}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2}\psi.$$

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If $0 < E < V_0$, then we define $\ell = \sqrt{2m(V_0 - E)}/\hbar$ (a different variable) to get

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = ell^2\psi,$$

which has solutions

$$\psi(x) = C^{\ell x} + D^{-\ell x}. \tag{4.2}$$

In this case, we can eliminate the first term since it blows up as $x \to \infty$, so we have

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x \le 0, \\ D^{-\ell x} & x > 0. \end{cases} \tag{4.3}$$

However, regardless if it actually blew up, we could set it zero under the assumption that the experiement we are conducting only involves shooting in, say, electrons from the left. Regardless, continuity of $\psi(x)$ at $x = 0$ means that

$$A + B = D, \tag{4.4}$$

and continuity of $\mathrm{d}\psi(x)/\mathrm{d}x$ means that

$$ik(A - B) = -\ell D \to D = -\frac{ik}{\ell}(A - B).$$

Using these two equations, we have that

$$A + B = -\frac{ik}{\ell}(A - B) \to A\left(1 + \frac{ik}{\ell}\right) = B\left(\frac{ik}{\ell} - 1\right).$$

The reflection coefficient is given by

$$R = \frac{|B|^2}{|A|^2} = \frac{|1 + ik/\ell|^2}{|ik/\ell - 1|^2} = \frac{k^2/\ell^2 + 1}{k^2/\ell^2 + 1} = 1. \tag{4.5}$$

Interestingly, we have a non-zero transmitted wave, but eventually all of the wave gets reflected back. This is because the energy is less than the potential on the right.

b) In the opposite case, where $E > V_0$, the left-side stays the same, but for the right side, the Schrödinger EQ reads

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\frac{2m(E - V_0)}{\hbar^2}\psi.$$

Defining $\ell = \sqrt{2m(E - V_0)}/\hbar$, we get

$$\psi(x) = Ce^{i\ell x} + De^{-i\ell x}.$$

But, as mentioned before, in experiments under our consideration, the D term vanishes, so we have

$$\psi(x) = \begin{cases} Ae^{ikx} + De^{-ikx} & x \le 0, \\ Ce^{i\ell x} & x > 0. \end{cases} \tag{4.6}$$

Continuity of $\psi(x)$ at $x = 0$ means that

$$A + B = C, \tag{4.7}$$

and continuity of $\mathrm{d}\psi(x)/\mathrm{d}x$ means

$$ik(A - B) = i\ell D \rightarrow D = \frac{k}{\ell}(A - B),$$

so

$$A + B = \frac{k}{\ell}(A - B) \rightarrow A\left(1 - \frac{k}{\ell}\right) = B\left(-\frac{k}{\ell} - 1\right).$$

The reflection coefficient is therefore

$$R = \frac{|B|^2}{|A|^2} = \frac{(1 - k/\ell)^2}{(1 + k/l)^2} = \frac{(\ell - k)^2}{(\ell + k)^2} = \frac{(\ell - k)^4}{(\ell + k)^2(\ell - k)^2} = \frac{(\ell - k)^4}{(\ell^2 - k^2)^2}.$$

Now that we have the denominator in a super nice form:

$$\ell^2 - k^2 = \left(\frac{2m}{\hbar^2}\right)(E - V_0 - E) = \left(\frac{2m}{\hbar^2}\right)V_0.$$

The numerator won't be so nice, but normally we are more interested in a cleaner looking denominator:

$$\ell - k = \left(\frac{\sqrt{2m}}{\hbar}\right)(\sqrt{E - V_0} - \sqrt{E}),$$

so

$$R = \frac{(\sqrt{E - V_0} - \sqrt{E})^4}{V_0^2}.$$

c) As the book explains, the transmission coefficient is a little harder, since the speeds of the two waves are different. To account for this, we can use the ratio of the two speeds from Equation 2.98 in the book:

$$v_{\text{quantum}} = \frac{\hbar|k|}{2m}. \tag{4.8}$$

For the wave on the left, this is identical to the case in the book to

$$v_i = \frac{E}{2m}.$$

For the wave on the right, we use ℓ instead (for the case when $E > V_0$):

$$v_f = \frac{\hbar}{2m}\frac{\sqrt{2m(E - V_0)}}{\hbar} = \sqrt{\frac{E - V_0}{2m}}.$$

Thus the ratio of the two speeds is

$$\frac{v_f}{v_i} = \sqrt{\frac{E - V_0}{E}}. \tag{4.9}$$

By taking this factor into account for the calculation of the transmission coefficient, we arrive at

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|F|^2}{|A|^2}. \tag{4.10}$$

This part of the problems asks for the transmission coefficient for the case of $E < V_0$, which since we found the reflection coefficient to be 1, then the transmission coefficient must be zero.

d) For the other case of $E > V_0$, we need to use our new formula for the transmission coefficient. First, we use our two equations for A, B, and D (where D is my F):

$$A + B = D \rightarrow B = D - A,$$

and

$$ik(A - B) = i\ell D \rightarrow A - \frac{\ell}{k}D = B,$$

so

$$D - A = A - \frac{\ell}{l}D,$$

$$D\left(1 + \frac{\ell}{k}\right) = 2A,$$

$$2kA = (k + l)D.$$

Thus,

$$\frac{|D|^2}{|A|^2} = \frac{4k^2}{(k+l)^2} = \frac{4k^2(k-l)^2}{(k^2 - l^2)^2} = \frac{4E(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2}.$$

The transmission coefficient therefore is

$$T = \sqrt{\frac{E - V_0}{E}} \frac{4E(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2} = \frac{4\sqrt{E(E - V_0)}(\sqrt{E} - \sqrt{E - V_0})^2}{V_0^2}.$$

To check,

$$T + R = \sqrt{\frac{E - V_0}{E}} \frac{4k^2}{(k+l)^2} + \frac{(\ell - k)^2}{(\ell + k)^2}.$$

To make things simpler, the square root quantity at the beginning is just ℓ/k:

$$T + R = \frac{4k\ell}{(k+l)^2} + \frac{(\ell - k)^2}{(\ell + k)^2} = \frac{4k\ell + k^2 - 2k\ell + \ell^2}{(\ell + k)^2} = \frac{k^2 + 2k\ell + \ell^2}{(\ell + k)^2} = \frac{(k + \ell)^2}{(\ell + k)^2} = 1,$$

as expected.