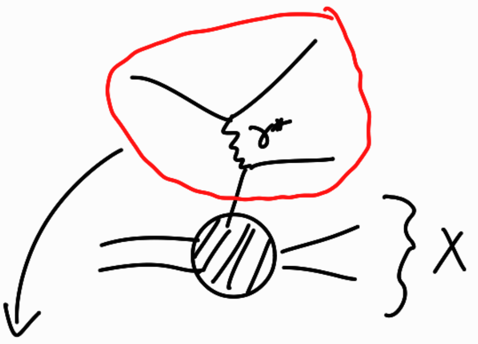
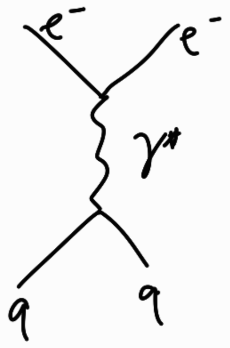
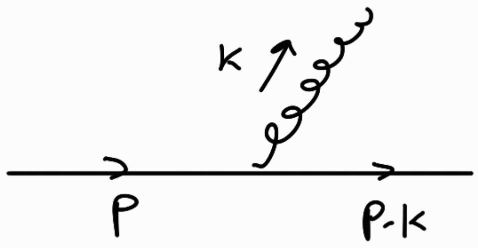
Renormalization

- Deep inelastic scattering at tree level:





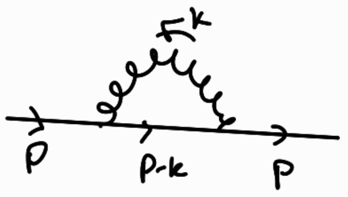
- no gluon, no QCD anything
- >30% off of experimental value

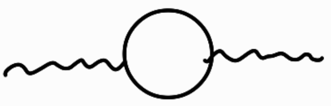


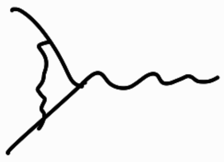
$k \to 0$ = soft gluon/photon

$\theta \to 0$ = collinear singularity

- these arise due to detector limitations and other things
- other "corrections":







"vertex correction"

- involves an $\int \frac{d^4k}{(2\pi)^4 i}$
- however, such integrals diverge for certain energies
- we can "manipulate, absorb, etc." these diagrams to, remarkably, avoid the infinites

· there is also a process called "regularization" that is a non-trivial mathematical magic

· involves turning integral into a function that we can do the limit of

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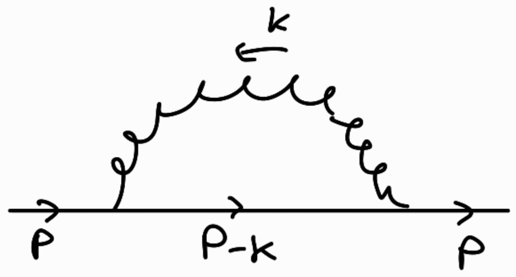
Self Energy Diagram for Quark or Electron:



-> $\Sigma_{ij}(p)$ is this self-energy

-> The full (including all self energies) $\equiv \tilde{S}_{ij}(p)$

· Can be shown that

$$\tilde{S}_{ij} = \frac{\delta_{ij}}{m - \not{p} - \Sigma(p)}$$

w/ $\Sigma_{ij}(p) = \delta_{ij}\,\Sigma(p)$

w/ $\tilde{S}_{ij} = i\int d^4_x \, e^{-ip\cdot x} \langle 0 | T\psi_i(x)\bar{\psi}_j(0)|0\rangle_c$

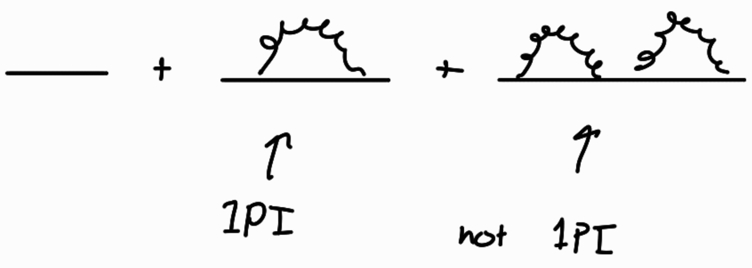
- fourier transform
- subscript c means "connected"

· series expand 1st $\tilde{S}_{ij}$ in terms of $\Sigma(p)$, the "1-particle irreducable" diagram (1PI)

$$\tilde{S}_{ij}(p) = \delta_{ij}\left\{ \tilde{S}_0(p) + \tilde{S}_0(p)\Sigma(p)\tilde{S}_0(p) + \left(\tilde{S}_0\Sigma(p)\tilde{S}_0\right)^2 + \dots \right\}$$

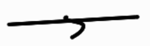
- we intepret it by original propagator. so

$\tilde{S}_{ij}(P) =$



· these 1PI cannot be cut to create 2 new diagram first step for renorm/regularization is finding these 1PI's.

Feynman Rules for QCD:



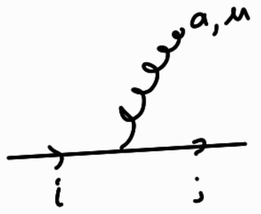
$\rightarrow \dfrac{1}{m - \not{p}}$,

$\rightarrow \dfrac{\delta_{ab}}{k^2} d^{\mu\nu}(k)$

$d^{\mu\nu}(k) = g^{\mu\nu}$ for Feynman gauge

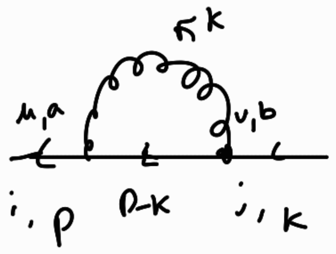
$= g^{\mu\nu} - (1-\alpha)\dfrac{k_\mu k_\nu}{k^2}$ for others

$\rightarrow \displaystyle\int \frac{d^4k}{(2\pi)^4 i}$

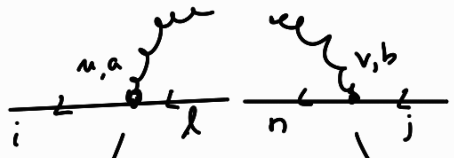


$\rightarrow g_s \gamma_\mu T^a_{ij}$ ← adjoint rep. 1-8 (the algebra) (for a); ← fund. rep 1-3 (for ij)

now:



start w/



, and a loop $\displaystyle\int \frac{d^4k}{(2\pi)^4 i}$

$$\boxed{\int \frac{d^4k}{(2\pi)^4 i}\, g_s T^a_{il} \gamma^\mu \left[\frac{\overset{*}{\delta_{ln}}}{m - (\not{p} - \not{k})} \right] g_s T^b_{nj} \gamma^\nu \cdot \frac{\delta_{ab}}{k^2} d^{\mu\nu}(k)}$$

fermion propagator

gluon propagator

* quark can't transform in between, l must be equal to n

- looking at SU(3) generators:

$$\rightarrow \Sigma_{ij}(p) : \delta_{ab}\delta_{ln} T^a_{il} T^b_{nj} = (T^a T^a)_{ij} = \delta_{ij} C_F ,$$

- $C_F = \dfrac{N_c^2 - 1}{N_c}$, N_c = num of colors; for $SU(3)_c$, $N_c = 3$
  - Casimir operator of the fundamental representation
- $C_F = \dfrac{8}{3}$

$$\Rightarrow \Sigma_{ij}(p) = g_s^2 \delta_{ij} C_F \int \frac{d^4k}{(2\pi)^4 i} \gamma^\mu \left[\frac{1}{m - \not{p} + \not{k}} \right] \gamma^\nu \cdot \frac{g_{\mu\nu}}{k^2} = \delta_{ij} \Sigma(p)$$

- in Feyman Gauge, $\alpha = 1$ so $d^{\mu\nu} = g^{\mu\nu}$

$$\Rightarrow \quad \Sigma(p) = g_s^2 C_F \int \frac{d^4k}{(2\pi)^4 i} \gamma^\mu \left[\frac{1}{m - \not{p} + \not{k}}\right] \gamma_\mu \cdot \frac{1}{k^2}$$

- we can do $\frac{1}{\not{p}} = \frac{\not{p}}{p^2}$ (mult top/bottom by $\not{p}$)

$$\rightarrow \quad = g_s^2 C_F \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{m - \not{p} + \not{k}}{m^2 - (p-k)^2} \gamma_\mu \cdot \frac{1}{k^2}$$

- For large k, and gammas irrelevant:

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{k^2 \cdot k^2}$$

$$\sim \int d^4k \, \frac{1}{k^3}$$

This is volume in 4D, but this is in large k behavior, so this diverges ("linearly divergent")

$$\lim_{k \to \infty} K$$

- In Cutoff regularization, we just cut the integral:

$$\int^\Lambda \frac{d^4k}{(2\pi)^4 i}$$

- In Pauli-Villars: replace propagator with

$$\frac{1}{m^2 - k^2} - \frac{1}{M^2 - k^2} = \frac{M^2 - m^2}{(m^2 - k^2)(M^2 - k^2)}$$ which reduces to

regular propagator in $M \to \infty$

- better for Lorentz invariance than Cutoff, but fails for gauge theories with massive gauge bosons

- Analytical Regularization

$$\frac{1}{m^2 - k^2} \rightarrow \frac{1}{(m^2 - k^2)^\alpha}, \quad \alpha \in \mathbb{C} \text{ w/ } Re(\alpha) > 1$$

- helpful to prove renormalizability
- but, violates gauge invariance

Lattice Regularization:

- discretize Minkowski spacetime
- Short-distance contribution in configuration space which corresponds to the large momentum contribution is eliminated

Dimensional Regularization (DR):

- spoils NO symmetries!
- maps integral into series of analytical terms interms of dimension:

$$\int \frac{d^4 k}{(2\pi)^4 i} \longrightarrow \int \frac{d^D k}{(2\pi)^D i}$$

- try for $D < 4$.

8/28: More Regularization

Green's Functions: quantum expect val of a time-ordered product of fields

$$\hat{\phi}(x) = \langle 0 | T[\phi(x_1) \dots \phi(x_2)] | 0 \rangle$$

- "correlator"

- Simplest ex: Dirac Propagator:

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$$

going from point x to y

- or

$$\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle$$

- We know solutions of Dirac EQ:

$$\psi_a(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(a^{(s)} u^{(s)}(p) e^{-ip\cdot x} + b^{(s)\dagger} v^{(s)}(p) e^{ip\cdot x} \Big)$$

a's / b's become operators in 2^{nd} quantization: satisfy anti-commutation relations

- we can now calc. the two amplitudes above; can group them together → "retarded" Green's function

$$S_R^{ab}(x-y) = \Theta(x^0-y^0)\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle$$

$$S_R(x-y) = (i\not{\partial}-m)D_R(x-y)$$

☆ look further at this ☆

- back to dimensional regularization

$$\int d^4k \rightarrow \int d^Dk$$

- we want $D \neq 4$ and to express our divergent integrals as an analytic function of D

- 1^{st} consequence: $\mu = 0,\dots,3$, $\mu = 0,\dots,D-1$

. now, $g^{\mu\nu}g_{\mu\nu} = D$,

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^\mu\gamma_\mu = D$$

$$\gamma^\mu\gamma_\nu\gamma_\mu = (2-D)\gamma_\nu$$

. in our integral, we had a $\frac{1}{(2\pi)^4}$, but this is a bit funny

. what we do is $\int \frac{1}{(2\pi)^4} \rightarrow \int \frac{1}{(2\pi)^D}$.

but the "measure" in D-dim must recover $\frac{1}{(2\pi)^4}$ for $D \to 4$

. 2^{nd} consequence: $Tr[\gamma^\mu\gamma^\nu] = 2^{D/2}g^{\mu\nu}$ for D even

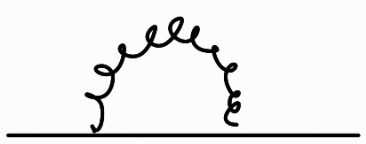
. correctly reduces to $4g^{\mu\nu}$ for $D \to 4$.

. this gets messy; what we do instead is keep D-dim measure, but use 4-dim traces

$$\rightarrow \int \frac{d^Dk}{(2\pi)^D} \quad \& \quad Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$$

· now we have conventions; can tackle self energy

· for simplicity, $m \to 0$, so



$$\rightarrow \quad \underset{\text{self energy}}{\Sigma(p)} = g_s^2 C_F (2-D) \int \frac{d^D k}{(2\pi)^K} \frac{\not{k} - \not{p}}{k^2 (k-p)^2}$$

- for $D < 3$, this converges for sure. but, what do we do? what is $d^D x$?

· We introduce "Feynman Parametrization":

$$\frac{1}{k^2 (k-p)} \rightarrow \frac{1}{AB} = \int_0^1 \frac{dx}{\{xA + (1-x)B\}^2}$$

$u = x(A-B) + B$

$du = (A-B)\,dx$

$\frac{du}{A-B} = dx$

$$= \frac{1}{B-A} \int_A^B \frac{du}{u^2} = \frac{1}{A-B} \left[\frac{1}{u}\right]_A^B$$

$$= \frac{1}{A-B} \left[\frac{1}{B} - \frac{1}{A}\right]$$

$$= \frac{1}{A-B} \left[\frac{A-B}{AB}\right] = \frac{1}{AB} \checkmark$$

- this can be generalized to any number of factors!

- Now:

$$\Sigma(p) = g_s^2 C_F (2-D) \int \frac{d^D k}{(2\pi)^D} (\not{k} - \not{p}) \int_0^1 \frac{dx}{[x(k-p)^2 + (1-x)k^2]^2}$$

- again, integral converges for $D < 3$

- apparently we can swap the two integral signs?

$$\Sigma(p) = g_s^2 C_F (2-D) \int \frac{d^D k}{(2\pi)^D} \int_0^1 \frac{\not{k} - \not{p}}{[(k-xp)^2 + x(1-x)p^2]^2}$$

- since this preserves translational symmetry, we can shift the momentum: $k' = k - xp$

$$\Sigma(p) = g_s^2 C_F (2-D) \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D} \frac{\cancel{\not{k}} - (1-x)\not{p}}{[k' + x(1-x)p^2]^2}$$

· now we have the sum of 2 functions. The first is a function of k', but we see it's odd: evaluates to zero w/ symmetric limits

$$\rightarrow \quad \Sigma(p) = g_s^2 C_F (D-2) \not{p} \int_0^1 dx\, (1-x) \int \frac{d^D k'}{(2\pi)^D} \frac{1}{[k'^2 + x(1-x)p^2]^2}$$

- now we do a "Wick Rotation" (again, DR preserves all symmetries)

- in Minkowski space, $k'^2 = k'^{0\,2} - |\vec{k}'|^2$

- go to Euclidean space where we want $-k^{0\,2} - |\vec{k}|^2$

· give it an i! basically a complex rotation

$$k_0' = i\underline{K}_0\,, \quad \underline{K} \in \mathbb{R}$$

$$\vec{k}' = \vec{\underline{K}}$$

now, $k'^2 = -\underline{K}_0^{\,2} - |\vec{\underline{K}}|^2$

and $d^D k \rightarrow i\, d^D \underline{K}$

so,

$$\Sigma(p) = g_s^2\, C_F (D-2)\, \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D \underline{K}}{(2\pi)^D} \frac{1}{(\underline{K}^2 + L)^2}$$

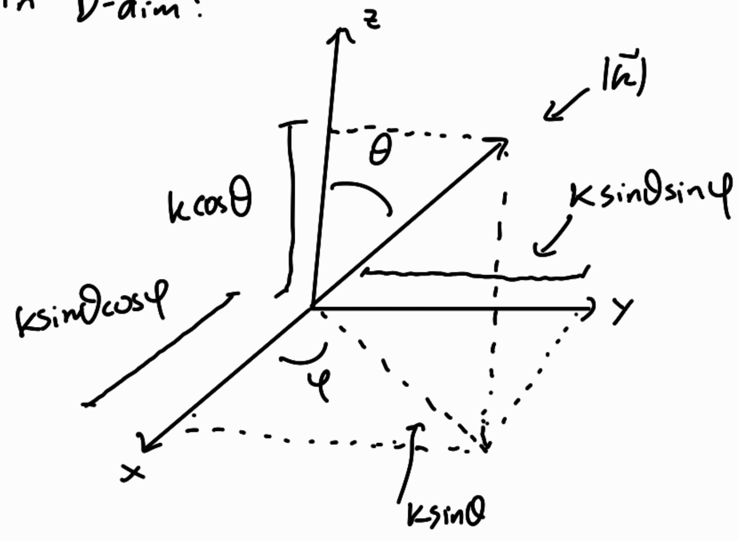
where $L = -x(1-x)p^2$

- Now it's Euclidean: we know it's non-singular for $L > 0$
- so, we want to keep $p^2 < 0$: "space-like"

in 3-D, we do spherical coords:

$$d^3\vec{k} = d\Omega_3 |\vec{k}|^2 d|\vec{k}|$$

- similar here! In D-dim:



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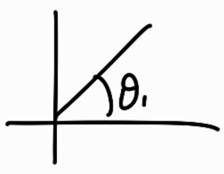
1) S_1

- one angle



$|\vec{k}| = 1$

$\rightarrow \begin{cases} x^1 = \cos\theta_1 \\ x^2 = \sin\theta_1 \end{cases}$

2) S_2

- two angles

$$\begin{cases} x^1 = \cos\theta_1 \\ x^2 = \sin\theta_1 \cos\theta_1 \\ x^3 = \sin\theta_1 \sin\theta_1 \end{cases}$$

after x_1, project to 2-D

here, project to 3-D

then to 2-D

3) S_3 (hypersphere!)

- three angles $\theta_1, \theta_2, \theta_3$

$$\begin{cases} x^1 = \cos\theta_1 \\ x^2 = \sin\theta_1 \cos\theta_1 \\ x^3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ x^4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \end{cases}$$

4) S^5

· four angles

$$\begin{cases} x_1 = \cos\theta_1 \\ x_2 = \sin\theta_1 \cos\theta_2 \\ x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ x_4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4 \\ x_5 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 \end{cases}$$

We can tell the pattern now:

S^D $D-1$ angles

$$\begin{cases} x_1 = \cos\theta_1 \\ x_2 = \sin\theta_1 \cos\theta_2 \\ \vdots \\ x_{D-1} = \sin\theta_1 \dots \sin\theta_{D-2} \cos\theta_{D-1} \\ x_D = \sin\theta_1 \dots \sin\theta_{D-1} \end{cases}$$

- now: $K_0 = |K| \cos\theta_1$

$$\vdots$$

$$K_{D-1} = |K| \sin\theta_1 \dots \sin\theta_{D-1}$$

and $d^4 K = d\Omega_D \, |K|^{D-1} d|K|$

$$d\Omega_D = \prod_{l=1}^{D-1} \sin^{D-1-l}\theta_l \, d\theta_l$$

$\rightarrow \int d\Omega_D =$ ☆notes☆

HW: Calc:

$$\boxed{\int_0^\pi d\theta \, (\sin\theta)^n}$$

a/4) Volume element in D dimension:

$$d^D K = |K|^{D-1} d|K| \, d\Omega_D$$

where

$$d\Omega_D = \prod_{l=1}^{D-1} \sin^{D-1-l}\theta_l \, d\theta_l$$

now, this is a relation in math, we can say

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+L)^2} = \frac{B\left(D/2,\, 2-D/2\right) L^{D/2-2}}{(4\pi)^{D-2}\,\Gamma(D/2)} \quad (\#1)$$

one representation of Γ is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \text{for } z \in \mathbb{C} \text{ w/ } \mathrm{Re}[z] > 0$$

the "beta" function $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

also has an integral representation:

$$B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt$$

to get (#1), we can get from the Jacobian:

$$\int_0^\pi d\Omega_D = \int_0^\pi d\theta_1 (\sin\theta)^{D-2} \cdots \int_0^\pi d\theta_{D-2} (\sin\theta)^{D-2} \int_0^\pi d\theta_{D-1}$$

$$= \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

To show this, let's examine unit sphere in D-dim

first, Gaussian integral

$$\sqrt{\pi} = \int_{-\infty}^\infty e^{-x^2}\, dx$$

$$\rightarrow (\sqrt{\pi})^D = \left(\int_{-\infty}^\infty e^{-x^2}\, dx \right)^D$$

in D-dim, we can take the integral over each independent coord:

$$= \int_{-\infty}^\infty e^{-\sum_{i=1}^D x_i^2}\, d^D x$$

• polar coords again! $d^D x = x^{D-1} e^{-x^2} d\Omega_D dx$,

where $x \equiv \sqrt{x_1^2 + x_2^2 + \cdots x_D^2}$

so $-\sum_{i=1}^D x_i^2 = -x^2$, so

$$\rightarrow \int d\Omega_D \int x^{D-1} e^{-x^2}\, dx$$

$$= \int d\Omega_D \; \underbrace{\frac{1}{2}\int_0^\infty (x^2)^{D/2-1} e^{-x^2} dx^2}_{\substack{\text{well-known} \\ = \frac{1}{2}\Gamma(D/2)}}$$

so $\quad (\sqrt{\pi})^D = \frac{1}{2}\Gamma(D/2)\int d\Omega_D$

or $\quad \int d\Omega_D = \dfrac{2\pi^{D/2}}{\Gamma(D/2)}$

$\rightarrow$

D	$\Gamma(D/2)$	$\int d\Omega_D$
1	$\sqrt{\pi}$	2
2	1	2π
3	$\pi/2$	4π
4	1	$2\pi^2$
$\vdots$	$\vdots$	$\vdots$

At last, we return to our quark self-energy.

$$\Sigma(p) = g_s^2 C_F (D-2)\cancel{p}\,\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}(-p^2)^{D/2-2} \times \int_0^1 dx\, x^{D/2-2}(1-x)^{D/2-1}$$

can express integral interms of Beta

$$\Sigma(p) = \frac{2C_F g_s^2}{(4\pi)^{D/2}}\cancel{p}(-p^2)^{D/2-2}(D-1)\,B\left(\frac{D}{2},\frac{D}{2}\right)\Gamma\left(2-\frac{D}{2}\right)$$

- analytic fn of D; valid for $D<3$ & $p^2<0$.
- want to "analytic continuation" in a region where D & p are arbitrary complex numbers.
- First: $\Gamma(2-\frac{D}{2}) = \int_0^\infty t^{2-D/2-1} e^{-t} dt \quad \left[\text{from } \int_0^\infty t^{z-1} e^{-t} dt\right]$
  - observe: $D = 4, 6, 8, \ldots$ are poles: lead to $\frac{\sim}{t}$, which can't do 0 lower limit
  - for $D=4$: $\Gamma(0) = \int_0^\infty \frac{e^{-t}}{t} dt \rightarrow \infty$; diverges...
- Second: there is a "branch cut" on the positive real axis in the p^2 plane
  - imagine $(-p^2)^\alpha$ for small α - taylor exp:

$$= 1 + \ln(-p^2)\alpha + \frac{1}{2!}\ln(-p^2)\alpha^2 + \ldots$$, we have a log!

- p can be $\pm$; approaching zero from either side gives diff infties.
- for $D=4$, we'd have $\Sigma(p) \sim \frac{C_F g_s}{(4\pi)^D} \frac{2}{4-D} \not{p}$!!

- Now, $S = \int L \, dt = \int \mathcal{L} \, d^4x \rightarrow L$ has [energy], so S has $[E \cdot t]$.

This is same unit as $\hbar$, which is angular momentum.

In natural units, $\hbar = c = 1$. so we need this to be satisfied.

In D-dim: $S = \int \mathcal{L} \, d^Dx$. for $\hbar = c = 1$, we'll have $\dim[\mathcal{L}] = D$

called "mass dimension"

- Compton wavelength: $\lambda = \underline{\lambda}^{\circ}(1-\cos\theta)$, where $\lambda = \frac{\hbar}{mc} \rightarrow \frac{1}{m}$ (mass),

so $[\lambda] = [m]^{-1}$

- Now we look inside the $\mathcal{L}$ and see the mass dimensions, because we must have that $[\mathcal{L}] = D$.

- let's examine

$i g_s \bar{\psi} T^a \gamma^\mu \psi A^a_\mu$

$\dim[g_s] = ?$

$\dim[\psi] = \dim[\bar{\psi}] =$

$\dim[A^a_\mu] =$

- first,

$$\mathcal{L} = a_1 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad a_2 \bar{\psi} \gamma^\mu D_\mu \psi + a_3 \bar{\psi} \psi$$

$$+ a_4 \bar{\psi} \sigma^{\mu\nu} F_{\mu\nu} \psi + a_5 F^{\mu\nu} F_\nu{}^\lambda F_{\lambda\mu} + \ldots\ldots$$

- these are all the bilinear covariants
- we restrict them by imposing Lorentz / Gauge invariance

- looking at field strength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

$\rightarrow F^{\mu\nu} F_{\mu\nu} \subset [\partial^\mu A^\nu \partial_\mu A_\nu] = D$, then $[A_\mu^2] = D-2$.

since $[\partial_\mu] = 1$, so

$$[A_\mu] = \frac{D-2}{2}.$$

- now we can look at $\psi / \bar{\psi}$ $\quad [\bar{\psi} \gamma^\mu \partial_\mu \psi] = D$

γ unitless, $[\partial_\mu] = 1$ $\rightarrow$ $[\psi^2][\partial] = D$

$$\rightarrow [\psi] = \frac{D-1}{2}$$

now, $[g_s] + 2[\psi] + [A_\mu] = D$

$$[g_s] + (D-1) + \frac{D-2}{2} = D$$

$$[g_s] = 2 - \frac{D}{2}$$

- our "constant" now dependent on dimension
- related to factorization / renormalization scale μ

$$\text{let } \quad g_s = \underset{\substack{\uparrow \\ \text{dimensionless, now}}}{g_0}\, \mu^{2-D/2}$$

Now, we can introduce a new constant epsilon, and let

$$D = 4 - 2\varepsilon,$$

$$\text{so } \varepsilon = \frac{4-D}{2} = 2 - D/2, \quad g_s = g_0 \mu^\varepsilon$$

In our self-energy:

$$\Sigma(p) = \frac{g_0^2 C_F}{(4\pi)^2} \not{p} \left(\frac{-p^2}{4\pi\mu^2}\right)^{-\varepsilon} (1-\varepsilon)\, B(1-\varepsilon, 1-\varepsilon)\, \Gamma(\varepsilon)$$

- now we perform a Laurent expansion (basically a Taylor expansion for complex)

$$\underline{\underline{\Sigma(p) = \frac{g_0^2}{(4\pi)^2} C_F \not{p} \left(\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right)}} + O(\varepsilon),$$

γ_E = Euler-Mascheroni constant = 0.57721

where we used $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)$,

and we can approximate $(1-\varepsilon)B(1-\varepsilon, 1-\varepsilon) \simeq$

- we have now mapped the singularities as poles $\left(\frac{1}{\varepsilon}\right)$

- One important note: we didn't use γ^5 ... in D-dim, γ^5 is weird and requires a special treatment (it cannot be defined uniquely explicitly)

- Another note: $\underline{\text{DR conventions}}$

1) D-dim spacetime metric is always mostly minus

2) $\mathrm{Tr}[\mathbb{1}] = 4$ (in space of γ's)

3) $\int \frac{d^Dk}{(2\pi)^D}$ defines the integral 'measure

4) only thing we can say now is that γ^5 is an object that satisfies $\{\gamma^5, \gamma^\mu\} = 0$

Fermion self-energy correction in an arbitrary covariant gauge

- we chose $\alpha = 1$ before; going back to before this step:

$$\Sigma(p) = g_s^2 C_F \int \frac{dK}{(2\pi)^D} \frac{1}{i} \frac{1}{k^2(k-p)^2}$$

$$\times \Big\{ \underbrace{\gamma_\mu(\not{k}-\not{p})\gamma^\mu}_{\text{calculated this already}} - (1-\alpha)\frac{\not{k}(\not{k}-\not{p})\not{k}}{k^2} \Big\}$$

let's call $\Sigma(p) = \Sigma_1(p) - (1-\alpha)\Sigma_2(p)$, where $\Sigma_1(p)$ is what we already found;

$$\Sigma_2(p) = g_s^2 C_F \int \frac{d^Dk}{(2\pi)^D} \frac{\not{k}(\not{k}-\not{p})\not{k}}{i(k^2)^2(k-p)^2}$$

- now, we need a more general case for Feynman Parameterization:

$$\frac{1}{AB^2} = 2\int_0^1 dx \frac{(1-x)}{\{xA + (1-x)B\}^3}, \quad B = k^2,\ A = (k-p)^2$$

- we choose $L = -x(1-x)p^2$ like before

- At this point we massage denominator to introduce a suitable shift in k:

$$k \to k' = k - xp$$

- doing this, we can show that

$$\frac{1}{[x(k-p)^2 + (1-x)k^2]^3} \to \frac{1}{[-\underbrace{(k-xp)^2}_{k} \underbrace{- x(1-x)p^2}_{L}]^2}$$

so, $\Sigma_2(p) = \cdot 2g_s^2 C_F \int_0^1 dx(1-x) \int \frac{d^Dk}{(2\pi)^D} \frac{1}{i} \frac{\not{k}(\not{k}-\not{p})\not{k}}{[-(k-xp)^2 + L]^3}$

then, doing $k \to k'$:

$$= 2g_s^2 C_F \int_0^1 dx\,(1-x) \int \frac{d^D k'}{(2\pi)^D} \frac{\overbrace{(1-x)k'\not{p}k' - 2xk'^2\not{p}}^{\sim k'^2} - xL\not{p}}{[-k'^2+L]^3}$$

- now, recall $g^{\mu\nu}g_{\mu\nu} = D$; we can isolate $\not{A}$, we have something like $k'\not{p}k' \sim p^\rho k'^\mu k'^\nu$. then our integral is of the form

$$\sim \int d^D k'\, k_\mu' k_\nu' \cdot f(k'^2)$$

$$\sim \frac{D}{D}\int \sim \;\to\; \frac{g^{\mu\nu}g_{\mu\nu}}{D}\int d^D k'\, k_\mu' k_\nu'$$

$$\to \frac{g_{\mu\nu}}{D}\int d^D k'\, k'^2 f(k'^2)$$

So,

$$\Sigma_2(p) = 2g_s^2 C_F \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D k'}{(2\pi)^D i} \frac{1}{[-k'^2+L]^3} \left\{ \left(\frac{2(1-x)}{D} - (1-x)\right) k'^2 - xL \right\}$$

want to again perform wick rotation to get to Euclidean space:

$$\Sigma_2(p) = 2g_s^2 C_F \not{p} \int_0^1 dx\,(1-x) \int \frac{d^D \underline{k}}{(2\pi)^D i}$$

$$\times \left\{ \frac{1}{[\underline{k}^2+L]^3}\left(\frac{2(1-x)}{D} - 1 - 2x\right) - \frac{1}{[\underline{k}^2+L]^2}\left(2(1-x) - 1 - x\right) \right\}$$

then we can use the generalization of a previous result:

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{[\underline{k}^2+L]^a} = \frac{\Gamma(a - D/2)}{(4\pi)^{D/2}\Gamma(a)} \cdot L^{D/2 - a},$$

where $a \in \mathbb{C}$, $\mathrm{Re}[a] > 0$

- we also need 3 properties:

- $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

- $\Gamma(z+1) = z\Gamma(z)$ for some $z \in \mathbb{C}$

- $B(p,q) = \int_0^1 dx\, x^{p-1}(1-x)^{q-1}$

- w/ this, we can show

$$\Sigma_2(p) = \frac{2 g_s^2 C_F \not{p}(-p^2)^{D/2-2}}{(4\pi)^{D/2}} (D-1)\, B\left(\tfrac{D}{2}, \tfrac{D}{2}\right)\Gamma\left(2-\tfrac{D}{2}\right)$$

- same as Σ_1!

- w/ this, we have calculated $\Sigma(p)$:

$$\Sigma(p) = \frac{2\alpha C_F g_s^2}{(4\pi)^{D/2}} \not{p}(-p^2)^{D/2-2}(D-1)\, B\left(\tfrac{D}{2}, \tfrac{D}{2}\right)\Gamma\left(2-\tfrac{D}{2}\right)$$

now, Laurent Expansion in ε:

$$\Sigma(p) = \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F \not{p}\left[\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(-\frac{p^2}{4\pi\mu^2}\right)\right] + O(\varepsilon)$$

- this was regularization; now to Renormalization

Renormalizational Schemes (preliminaries)

Def: redefinition of mass/coupling constant together w/ a readjustment of the normalization of the Green's function

- renormalization is not unique
- divergent pieces in the Green's fn's cannot be uniquely defined
- there is an ambiguity in the final piece of the Green's fn.

- Q: How to remove this ambiguity?
- specify how the divergent piece is defined so that we are able to systematically subtract it.
- this particular subtraction procedure is called a renormalization scheme.
- different schemes are related by a finite renormalization

Example:

- let's consider $\Sigma(p)$ we just calculated.

$$\Sigma(p) = \frac{\alpha g_s^2}{(4\pi)^2} C_F \not{p} \left[\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right] + O(\varepsilon)$$

- if we substitute this into

$$\tilde{S}_{ij}(p) = \frac{-\delta_{ij}}{\not{p} + \Sigma(p)}$$

we first define:

$$\sigma(p^2) = \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F \left[\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right] + O(g_{0s}^4) + O(\varepsilon)$$

$$\to \quad S_{ij}(p) = \frac{-\delta_{ij}}{\not{p}} \frac{1}{1 + \sigma(p^2)}$$

- now, $\tilde{S}_{ij}$ has a pole at $\not{p} = 0$
- also, our massless quark prescription will remain; it does not get changed by our renormalization
- this will be the case for all orders.

- we will use a multiplicative factor called Z_2 (historical reasons → it's the "quark field renormalization constant."
- Our renormalized propagator is:

$$\tilde{S}_{R,ij}(p) = Z_2^{-1} \tilde{S}_{ij}(p)$$

- Z_2 is a fn of g_0 → can expand:

$$Z_2 = 1 - z_2 + O(g_{0s}^4),$$

where (lowercase) z_2 contains a g_{0s}^2 and divergent piece.

- Substituting:

$$\tilde{S}_{R,ij}(p) = \frac{-\delta_{ij}}{\not{p}} \cdot \frac{1}{1 + \sigma(p^2) - z_2}$$

- now, we should also renormalize g_{0s}, but it turns out, it will have no effect at this perturbative order.

- now: $\tilde{S}_{R,ij}(p)$ should be free of divergences, so $\sigma(p^2) - z_2$ must be finite and the divergences in $\sigma(p^2)$ must be canceled by those in z_2.

- we can now define Z_2 to fulfill this. Of course, then, Z_2 can be determined up to a finite constant, hence why we said that it's not unique.

- so. we need some other procedure/requirement to set up the scheme.

- the first of which we will examine is called the:

On-shell Subtraction

- Z_2 is determined on the mass shell of quarks by imposing the condition

$$\tilde{S}_{R,ij}(p) \sim \frac{\delta_{ij}}{m-\not{p}} \quad \text{for } \not{p}\sim m$$

for our case w/ $m=0$, then $Z_2 = \sigma(0)$. but $\sigma(0)$ is not well defined, so we don't use it.

Off-shell Subtraction

- at some unphysical scale (hence off-shell) of p^2 say $p^2 = -\lambda^2$ with $-\lambda^2>0$ and we require that $\tilde{S}_{R,ij}(p)$ is of the form of the free massless propagator:

$$\tilde{S}_{R,ij}(p) \sim \frac{-\delta_{ij}}{\not{p}} \quad \text{for } p^2 \sim -\lambda^2$$

- this determines Z_2 such that

$$Z_2 = \sigma(-\lambda^2) = \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F \left[\frac{1}{\varepsilon} - \gamma_E + 1 - \ln\left(\frac{\lambda^2}{4\pi M^2}\right)\right]$$

so that

$$\tilde{S}_{R,ij}(p) = \frac{-\delta_{ij}}{\not{p}}\left[1 - \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F \ln\left(\frac{-p^2}{\lambda^2}\right)\right]^{-1}$$

- ε pole vanishes (lg e is reabsorbed)!

- This scheme is also called "momentum-space subtraction scheme"

Minimal Subtraction Scheme ('t Hooft)

- very specific to DR -> only eliminate the $\frac{1}{\varepsilon}$ pole that
- makes Z_2 very economical; often used in QCD (a modified ver.)
- Requirement imposes $Z_2 \propto \frac{g_{0s}^2}{(4\pi)^2} C_F \frac{1}{\varepsilon}$
- therefore,

$$\tilde{S}_{R,ij}(p) = \frac{-\delta_{ij}}{\not p}\left\{1 - \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F\left[\gamma_E - 1 + \ln\left(\frac{-p^2}{4\pi\mu^2}\right)\right]\right\}^{-1}$$

- advantage is a simpler expression for Z_2, but leads to more complicated Green's fn.

- Can be converted to off-shell subtraction by

$$\lambda^2 = 4\pi e^{1-\gamma_E}\mu^2$$

<u>Standard</u>: Modified Minimal Subtraction Scheme ($\overline{MS}$; "ms bar")

from normal MS

- the $\gamma_E - 1 + \ln(4\pi)$ shows up everywhere and is very annoying!
- we want to get rid of it.

$$\rightarrow Z_2 = \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right)$$

then,

$$\tilde{S}_{R\,ij}(p) = \frac{-\delta_{ij}}{\not p}\left\{1 - \frac{\alpha g_{0s}^2}{(4\pi)^2} C_F\left[-1 + \ln\left(\frac{-p^2}{\mu^2}\right)\right]\right\}$$

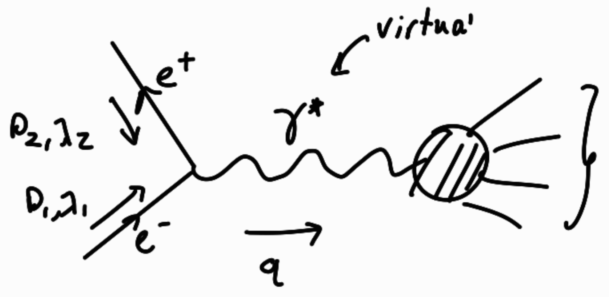
Real quick: general Feynman Parametrization

$$\prod_{i=1}^{n} \frac{1}{A_i^{\alpha_i}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_0^1 \left(\prod_{i=1}^{n} dx_i\, x_i^{\alpha_i - 1}\right) \frac{\delta(1-x)}{\left(\sum_{i=1}^{n} x_i A_i\right)^{\alpha}}$$

where $\alpha = \sum_{i=1}^{n} \alpha_i$

$x = \sum_{i=1}^{n} x_i$

<u>Electron/Positron Annihilation</u> : $e^- + e^+ \rightarrow X$



X = hadronic final state

The Feynman amplitude: $\langle X | T | e^+e^- \rangle$, T is similar to scattering matrix; it's a transition operator,

Don't know how to blob, but we can start w/ L.H.S. of diagram

$$-ie\ \bar{v}^{(\lambda_2)}(p_2)\gamma^\mu u^{(\lambda_1)}(p_1) \cdot \frac{-i}{q^2} \cdot \langle X | (-ie) J_\mu(0) | 0 \rangle,$$

$J_\mu(x)$ is some em current w/ quarks... but why a current?

Recall: $$\mathcal{L}_{QED} = (\underbrace{-e\bar{\psi}\gamma^\mu\psi}_{\text{well behaved: } e^+/e^-} + \underbrace{eJ^\mu}_{\text{this is the more "arbitrary" part}})A_\mu$$

- we have to square the amplitude. To do this, we need 2 properties:

- completeness: $\sum_x |x\rangle\langle x| = \mathbb{1}$
- translational invariance: $J_\mu(x) = e^{i\hat{P}\cdot x} J_\mu(0) e^{-i\hat{P}\cdot x}$

$\hat{P}$ has $\hat{P}^\mu |x\rangle = P_x^\mu |x\rangle$, and

$$e^{i\hat{P}\cdot x}|x\rangle = e^{iP_x\cdot x}|x\rangle$$

- now, generic cross section formula:

$$\sigma = \frac{1}{2s}\cdot\frac{1}{4}\sum_{\lambda_1,\lambda_2}\sum_x (2\pi)^4 \delta^4(P_x - q) \left| \langle X | \hat{T} | e^+e^- \rangle \right|^2$$

$$(\text{generally}) = \frac{1}{k(s)} \cdot \frac{1}{(2J_1+1)(2J_2+1)} \sum_{\lambda_1,\lambda_2,M_1,M_2\ldots}$$

$$\times \int \frac{d^3k_j}{(2\pi)^3 2k_j^0} (2\pi)^4 \delta^4\left(\sum_{j=1}^n k_j - P_1 - P_2\right) \times \left| \langle k_1 M_1, k_2 M_2 \ldots | \hat{T} | P_1\lambda_1, P_2\lambda_2, \ldots \rangle \right|^2,$$

where $k(s)$ is "flux factor": $k(s) = \sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}$

- we use the original one; substiting our exp for $\langle X | \hat{T} | e^+e^- \rangle$, we get

$$\sigma = \frac{e^4}{2s^3} l^{\mu\nu} W_{\mu\nu},$$

$$l^{\mu\nu} = p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2} g^{\mu\nu} \quad ; \text{ leptonic tensor}$$

$$W^{\mu\nu} = \sum_x (2\pi)^4 \delta^4(P_x - q) \langle 0 | J_\mu(0) | x \rangle \langle x | J_\nu(0) | 0 \rangle \quad ; \text{ hadronic tensor}$$

$$= \sum_x \int e^{-i(P_x - q)\cdot x} \langle 0 | J_\mu(0) | x \rangle \langle x | J_\nu(0) | 0 \rangle \, d^4x$$

exponential is just a number, so we can move around

$$= \sum_X \int e^{iq\cdot x} e^{-iP_X\cdot x} e^{i0x} \sim$$

$$= \sum_X \int e^{iq\cdot x} \langle 0| e^{i0x} J_\mu(0) e^{-iP_X\cdot x} |X\rangle\langle X| J_\nu(0)|0\rangle\, d^4x$$

☆

$$= \sum_X \int e^{iq\cdot x} \langle 0| J_\mu(x) |X\rangle\langle X| J_\nu(0)|0\rangle\, d^4x$$

completeness

$$\rightarrow \quad = \sum_X \int e^{iq\cdot x} \langle 0| J_\mu(x) J_\nu(0)|0\rangle\, d^4x$$

we can show that $\int e^{iq\cdot x} \langle 0| J_\nu(0) J_\mu(x)|0\rangle\, d^4x = 0$, then

$$W^{\mu\nu} = \sum_X \int e^{iq\cdot x} \langle 0| [J_\mu(x), J_\nu(0)] |0\rangle$$

10/9

☆ in general: we have something like $\langle p| \underbrace{e^{ip\cdot x} J_\mu(0) e^{-iP_x\cdot x}}_{J_\mu(x)} |X\rangle$,

but here $p=0$, so $\langle 0| \underbrace{J_\mu(0) e^{-iP_y\cdot y}}_{J_\mu(x)} |X\rangle$.

we will now show $\int e^{iq\cdot x} \langle 0| J_\nu(0) J_\mu(x)|0\rangle\, d^4x = 0$

- let's reinsert: $\sum_X \int e^{iq\cdot x} \langle 0| J_\nu(0) |X\rangle\langle X| J_\mu(x)|0\rangle\, d^4x$

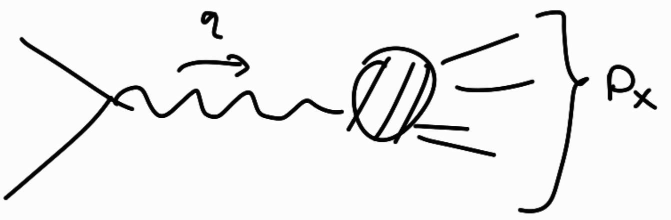
- now, $\langle X| J_\mu(x)|0\rangle = \langle X| e^{i\hat{P}_X\cdot x} J_\mu(0)|0\rangle$, and

$\langle X| e^{i\hat{P}_X\cdot x} = \langle X| e^{iP_X\cdot x}$ (a number, so we move)

$$\rightarrow \sum_X \int e^{i(q+P_X)\cdot x} \langle 0| J_\nu(0)|X\rangle\langle X| J_\mu(0)|0\rangle\, d^4x$$

$$= \sum_X (2\pi)^4 \delta^4(q+P_X) \langle 0| J_\nu(0) J_\mu(0)|0\rangle$$

Recall:



- we choose our reference frame such that $q^\mu = (q^0, \vec{0})$ (C.O.M. frame)
- now, $q^2 = (P_1{}^2 + P_2)^2 = 2P_1\cdot P_2 = s > 0$
- in this frame, $P_1^\mu = (P_1^0, \vec{0}_\perp, P_z)$, $P_2^\mu = (P_2^0, \vec{0}_\perp, -P_z)$,

so $q^\mu = (P_1^0 + P_2^0, \vec{0})$, so $q^2 \approx 2P_1^0 P_2^0 = s > 0$... so both q, P_x must be +, but delta function doesn't encompass this, so it's zero.

- so:

$$W_{\mu\nu} = \int e^{iq\cdot x} \langle 0 | [J_\mu(x), J_\nu(0)] | 0 \rangle d^4x$$

- in high-energy regime, $q^0 \to \infty$, but this leads to wild oscillations
- so only small values of x are allowed to keep this well behaved.
- since the commutator (operator $J_\mu(x)$) is a fn of x, we can now expand it perturbatively
- in order to satisfy causality, $[J_\mu(x), J_\nu(0)] = 0$ for $x^2 < 0$, so we must have $x^2 \geq 0$.
- now we use Lorentz-invariance/current-conservation to paramaterize $W_{\mu\nu}(q)$ into all second rank tensors involving q:

$$W_{\mu\nu} = A q_\mu q_\nu + B q^2 g_{\mu\nu}$$

applying these invariance/conservations:

$$W_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu}) \frac{1}{6\pi} W(q^2)$$

back to our cross sec: $\sigma = \frac{e^4}{2s^3} l^{\mu\nu} W_{\mu\nu}$:

$$= \frac{4\pi \alpha_{em}^2}{3s} W(s)$$

For $e^- + e^+ \to \mu^- + \mu^+$, we had

$$\sigma_{\mu\mu} = \frac{4\pi \alpha_{em}^2}{3s},$$

no $W(q^2)$, because no inner structure, just a vertex.

we can take ratios to cancel the prefactor:

$$R = \frac{\sigma}{\sigma_{\mu\mu}} = W(s),$$

$R \equiv$ Drell Ratio

- we can write R this way:

$$R = -\frac{2\pi}{q^2} \int e^{iq\cdot x} \langle 0 | [J_\mu(x), J_\nu(0)] | 0 \rangle$$

- R is a fn of s, the coupling g (generically), and some "renormalization scale" μ which is associated w/ where we cut our perturbative series off: $R(s, g, \mu^2) = R(\frac{s}{\mu^2}, g)$.

- according to renormalization group theory, R must satisfy

$$\left[\mu \frac{\partial}{\partial \mu} + B(g) \frac{\partial}{\partial g}\right] R\left(\frac{s}{\mu^2}, g\right) = 0$$

- this total derivative being zero indicates a desirable stability for R

- the general solution of this is

$$R\left(\frac{s}{\mu^2}, g\right) = R(1, \bar{g}(s)),$$

where we group energy/scale into coupling using B-fn;

$$\frac{d\bar{g}}{dt} = B(g); \quad \bar{g}(\mu^2) = g, \quad \text{where}$$

$$t = \frac{1}{2}\ln(s/\mu^2)$$

now,

$$R\left(\frac{s}{\mu^2}, g\right) = \sum_i Q_i^2 \left[1 + a(s/\mu^2) g^2 + b(s/\mu^2) g^4 + \dots\right]$$

all quark flavors/gens

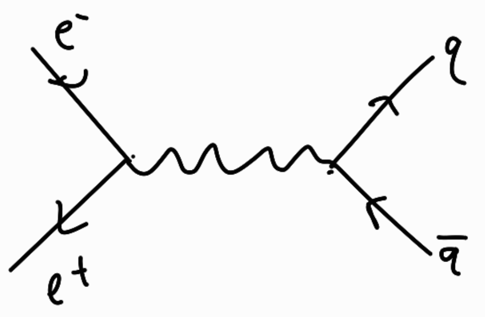
where the coefficients have $\ln(s/\mu^2)$ dependence... bad!

the new one:

$$R(1, \bar{g}(s)) = \sum_i Q_i^2 \left[1 + a(1)\bar{g}(s)^2 + b(1)\bar{g}(s)^4 + \dots\right]$$

now, coefficients are good, and pushing s up, $\bar{g}$ goes smaller!

- Born-level:



$$\sigma_B = \frac{4\pi\alpha_{em}^2}{s}$$

- calculating the virtual NLO diagrams gives a term like

$\sigma_B(\dots)$, which when added w/

$Z_2^2\sigma_B$ from our renormalization cancels the divergences

$$\sigma_v = \frac{1}{2s}\int \frac{d^3\vec{k}_1}{(2\pi)^3 2k_{10}} \frac{d^3\vec{k}_2}{(2\pi)^3 2k_{20}} (2\pi)^4 \delta^4(k_1+k_2-P_1-P_2)\, F_v,$$

where

$$F_v = \left(\sum_i Q_i\right)^2 \frac{e^2}{q^4} \mathrm{Tr}[\not{P}_2 \gamma^\mu \not{P}_1 \gamma^\nu]\,\mathrm{Tr}[\not{k}_1 \Lambda_\mu \not{k}_2 \gamma_\nu],$$

with Λ_μ being the vertex correction at 1-loop

- w/ our $m=0$ prescription, we get IR divergences with $\frac{1}{p^2}$ stuff.

· we go back to

$$\Sigma(p) = g_s^2 C_F (D-2) \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D i} \frac{-(1-x)\not{p}}{[k'^2 + x(1-x)p^2]^2}$$

for $p^2 \to 0$ $(m \to 0)$, $\not{p} \neq 0$:

$$= g_s^2 C_F (2-D) \underbrace{\int_0^1 dx\,(1-x)}_{\frac{1}{2}} \int \frac{d^D k'}{(2\pi)^D i} \frac{1}{(k')^4}$$

- turns out k bit is zero

- in context of DR: $\int \frac{d^D k}{(-k^2)^\alpha} = 0$ where $\alpha > 0$

· doing Wick rotation $\to -k^2 = \bar{K}^2$,

· then doing angular integration

$$\int \frac{d^D \bar{k}}{(-\bar{k}^2)^\alpha} = \frac{i\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty (\bar{k}^2)^{D/2-\alpha-1}\, d\bar{k}^2$$

- we now recognize if $D > 2\alpha$, we get a UV divergence,
  if $D < 2\alpha$, we get an IR divergence

- what we will do is split the integral into $[0,\Lambda]$, $[\Lambda,\infty]$, s.t. we can integrate in both ranges more nicely, then do some complex analysis stuff at Λ.

  · here, $\bar{k}^2 > \Lambda^2 \to$ UV ; $\bar{k}^2 < \Lambda^2 \to$ IR

so,
$$= \frac{i\pi^{D/2}}{\Gamma(D/2)}\left[\int_0^{\Lambda^2}(\bar{k}^2)^{D/2-\alpha-1}\,d\bar{k}^2 + \int_{\Lambda^2}^{\infty}(\bar{k})^{D/2-\alpha-1}\,d\bar{k}^2\right]$$

$\nearrow$ convergent for $D > 2\alpha$

con. for $D < 2\alpha$

- we will call first D D_i for IR, second D D_u for UV

$$= i\frac{\pi^{D/2}}{\Gamma(D/2)}\left[\frac{\Lambda^{D_I-2\alpha}}{\frac{1}{2}D_I-\alpha} - \frac{\Lambda^{D_u-2\alpha}}{\frac{1}{2}D_u-\alpha}\right]$$, which is divergent if $D_I = D_u = 2\alpha$

- now we use analytical continuation to move the domain into complex plane, such that the denominators are only divergent in real part $\rightarrow$ we just need a small imaginary part

$$\rightarrow \frac{\Lambda^{D_i/2-\alpha-1}}{\frac{1}{2}D_i-\alpha+i\varepsilon} - \frac{\Lambda^{D_u/2-\alpha-1}}{\frac{1}{2}D_u-\alpha+i\varepsilon}$$

- now we can do the $D_i = D_u \rightarrow 2\alpha$ so that they no longer diverge; instead they cancel

- thus, w/ analytical continuation, we have shown that in the context of DR,
$$\int \frac{d^Dk}{(-k^2)^\alpha} = 0.$$

Back to
$$\Sigma(p)\big|_{p^2=0} = g_s^2 C_F(D-2)\not{p}\cdot\frac{1}{2}\int\frac{d^Dk'}{(2\pi)^{D_i}}\frac{1}{(-k^2)^2}$$
$$= \frac{g_s^2}{(4\pi)^2}C_F(D-2)\not{p}\left[\frac{1}{\varepsilon'}-\frac{1}{\varepsilon}\right]$$

- we keep them separate for now to distinguish their origin
- they conspire later

- for vertex correction:

$$\Lambda_\mu = \gamma_\mu \frac{g_{0s}^2}{8\pi^2} C_F \left(\frac{4\pi\mu^2}{-q^2}\right)^{\varepsilon}$$

$$\times \Gamma(1+\varepsilon) B(1-\varepsilon, 2-\varepsilon)\left[\frac{1}{\varepsilon} - \frac{2}{\varepsilon^2} - 2\right],$$

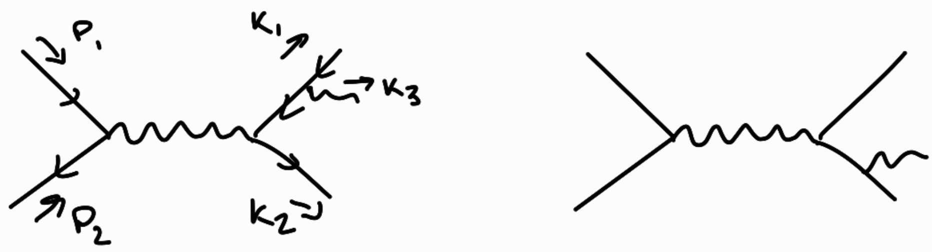
μ^2 is the scale to make g dimensionless

$$\tilde{\sigma}_v = \sigma_v + (Z_2^2 - 1)\sigma_B$$

$$= A_v \sigma_B, \quad \text{where}$$

$$A_v = \frac{\alpha_s}{\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \frac{\cos(\pi\varepsilon)}{\Gamma(1-\varepsilon)}\left[-\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + \mathcal{O}(\varepsilon)\right]$$

Real emission corrections



Fermi Golden Rule:

$$\sigma_R = \frac{1}{8s}\int \prod_{i=1}^{3} \frac{d^{D-1}k_i}{(2\pi)^{D-1} 2k_{i,0}} (2\pi)^D \delta^{(D)}\left(\sum_{i=1}^{3} k_i - p_1 - p_2\right) F_R,$$

$$F_R = -\sum_i Q_i^2 \frac{e^2}{q^2} g_s^2 C_F \,\mathrm{Tr}[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu] \leftarrow L^{\mu\nu}, \text{ leptonic part,}$$

$$\times \mathrm{Tr}[\not{k}_1 S_{\lambda\mu} \not{k}_2 S_\nu^{\lambda}] \leftarrow G_{\mu\nu}, \text{ "quark" part,}$$

$$S_{\mu\nu} = \gamma_\mu \cdot \frac{-1}{\not{k}_1 + \not{k}_3}\gamma_\nu + \gamma_\nu \cdot \frac{1}{\not{k}_2 + \not{k}_3}\gamma_\mu$$

- leptonic part:

$$L^{\mu\nu} = 4\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{q^2}{2} g^{\mu\nu}\right] \text{ (after trace)}$$

- can collect phase space parts together:

$$I_{\mu\nu} = \int \prod_{i=1}^{3} \frac{d^{D-1}k_i}{2k_{i,0}} \quad \delta^{(D)}\left(\sum_{i=1}^{3} k_i - q\right) G_{\mu\nu}$$

$$\sigma_R = \frac{-e^4 g_s^2}{8_s (2\pi)^{2D-3}} \cdot \frac{1}{q^4} \sum_i Q_i^2 \, C_F \, L^{\mu\nu} I_{\mu\nu}$$

- Now, $I_{\mu\nu} = I_{\mu\nu}(q)$, and it must satisfy current conservation
- We know $\partial_\mu J^\mu = 0$. Here, $J^\mu \to I_{\mu\nu}$, since that contains our current information
- The conjugate for ∂_μ is q^μ in position space, so we must have

$$q^\mu I_{\mu\nu} = 0$$

- We can parametrize $I_{\mu\nu}$ like so:

$$I_{\mu\nu}(q) = I(q)\left[\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right]$$

- Inversely, then, $\quad I(q) = \dfrac{-g^{\mu\nu} I_{\mu\nu}}{D-1}$

Now, $\quad L^{\mu\nu} I_{\mu\nu} = \dfrac{(D-2)}{(D-1)} q^2 g^{\mu\nu} I_{\mu\nu}$,

or, we can have

$$g^{\mu\nu} G_{\mu\nu} = -8(1-\varepsilon) \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)},$$

where $\varepsilon = \frac{4-D}{2}$ and $x_i = \dfrac{2k_i \cdot q}{q^2}$

- With all this,

$$g^{\mu\nu} I_{\mu\nu} = \frac{\pi (s\pi)^{1-2\varepsilon}}{4\Gamma(2-2\varepsilon)} \int_0^1 \prod_{i=1}^{3} (1-x_i)^{-\varepsilon} dx_i \; \delta\left(2 - \sum_{i=1}^{3} x_i\right) g^{\mu\nu} G_{\mu\nu}$$

side hint: $q^2 = q \cdot (k_1 + k_2 + k_3) \to$ can see x_i from here

Thus,

$$\sigma_R = \left(\sum_i Q_i\right) \alpha_{em}^2 \alpha_s^2 C_F \frac{2}{s} \left(\frac{4\pi\mu}{s}\right)^{2\varepsilon} \frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} K,$$

where $\quad K = \displaystyle\int_0^1 \prod_{i=1}^{3} (1-x_i)^{-\varepsilon} dx_i \; \delta\left(2 - \sum_{i=1}^{3} x_i\right) \frac{x_1^2 + x_2^2 - \varepsilon x_3^2}{(1-x_1)(1-x_2)}$

- We do the integral:

$$K = \left(\frac{4}{\varepsilon^2} - \frac{12}{\varepsilon} + 10 - 4\varepsilon\right) B(1-\varepsilon, 2-2\varepsilon)\, B(1-\varepsilon, 1-\varepsilon) + O(\varepsilon)$$

What we want to do is have $\sigma_R = A_R \sigma_B$

thus, we need σ_B in D-dim for consistency:

$$\rightarrow \sigma_B = \frac{4\pi\alpha_{em}^2}{3s}\left(\sum_i Q_i^2\right)\left(\frac{4\pi}{s}\right)^{\varepsilon} \frac{3(1-\varepsilon)\Gamma(2-\varepsilon)}{(3-2\varepsilon)\Gamma(2-2\varepsilon)}$$

- Now we can just do a ratio to pull out σ_B: $\frac{\sigma_R}{\sigma_B} = A_R$

$$A_R = \frac{\alpha_s}{\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \frac{\cos(\pi\varepsilon)}{\Gamma(1-\varepsilon)} \left[\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{19}{4} + O(\varepsilon)\right]$$

Now, $\sigma = (1 + A_V + A_R)\sigma_B$

Recalling A_V:

$$A_V = \frac{\alpha_s}{\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\varepsilon} \frac{\cos(\pi\varepsilon)}{\Gamma(1-\varepsilon)} \left[-\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - 4 + O(\varepsilon)\right]$$

- our poles cancel!

$$\sigma = \left[1 + \frac{3}{4} C_F \frac{\alpha_s}{\pi}\right]\sigma_B$$

!!!