Cross sections

Consider two colliding particle beams

Beam 1 has particles with mass m_1 and number density n_1 (no. per unit volume) with velocity $\vec{v}_1$

Beam 2 has $m_2, n_2, \vec{v}_2$ The relative velocity is $\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2$

If $\vec{v}_2 = 0$ then we have a fixed target.

The number of scattering events N per unit volume and per unit time is

$$\frac{dN}{dV\,dt} = \sigma v_{rel}\, n_1 n_2 \quad \text{or} \quad dN = \sigma v_{rel}\, n_1 n_2\, d^4x$$

where σ is the cross section which has units of area

In rest frame of beam 2 ($\vec{p}_2 = 0$) we have $p_1 \cdot p_2 = E_1 E_2 = \frac{m_1}{\sqrt{1-v_1^2}} m_2 = \frac{m_1 m_2}{\sqrt{1-v_{rel}^2}}$

Then $v_{rel} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2}$ Note that dN is Lorentz invariant

So $dN = \sigma \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} n_1 n_2\, dV\, dt$ then integrate over dV and dt

$$N = \sigma \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2\, V} N_1 N_2\, t \propto |\mathcal{M}|^2$$ where $\mathcal{M}$ is the amplitude

Final result for the process $p_1 + p_2 \to p_3 + p_4 + \cdots + p_n$ is

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_n) \prod_{i=3}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_i}$$

Cross section is Lorentz invariant

Cross section for $2 \to 2$ processes $\quad p_1 + p_2 \to p_3 + p_4$

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4) \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$\Rightarrow \sigma = \int \frac{|\mathcal{M}|^2 \, \delta^4(p_1+p_2-p_3-p_4)}{64\pi^2 E_3 E_4 \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} d^3p_3 \, d^3p_4 = \int \frac{|\mathcal{M}|^2 \, \delta(E_1+E_2-E_3-E_4)}{64\pi^2 E_3 E_4 \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}} d^3p_3$$

because $\delta^4(p_1+p_2-p_3-p_4) = \delta(E_1+E_2-E_3-E_4)\,\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$

Work in center-of-mass frame $\vec{p}_1+\vec{p}_2 = 0 = \vec{p}_3+\vec{p}_4$ so $\vec{p}_2 = -\vec{p}_1$ and $\vec{p}_4 = -\vec{p}_3$

Then $\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2} = (E_1+E_2)\,|\vec{p}_1|$

so $$\sigma = \frac{1}{64\pi^2 (E_1+E_2)|\vec{p}_1|} \int \frac{|\mathcal{M}|^2 \, d^3p_3}{\sqrt{\vec{p}_3^{\,2}+m_3^2}\sqrt{\vec{p}_3^{\,2}+m_4^2}} \delta\left(E_1+E_2-\sqrt{\vec{p}_3^{\,2}+m_3^2}-\sqrt{\vec{p}_3^{\,2}+m_4^2}\right)$$

But $d^3p_3 = |\vec{p}_3|^2 d|\vec{p}_3|\, d\Omega$ where $d\Omega = \sin\theta\, d\theta\, d\varphi$ $\quad$ Note that $\vec{p}_3^{\,2} = |\vec{p}_3|^2$

Then $$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2(E_1+E_2)|\vec{p}_1|} \int_0^\infty \frac{|\mathcal{M}|^2 \, |\vec{p}_3|^2 d|\vec{p}_3|}{\sqrt{\vec{p}_3^{\,2}+m_3^2}\sqrt{\vec{p}_3^{\,2}+m_4^2}} \delta\left(E_1+E_2-\sqrt{\vec{p}_3^{\,2}+m_3^2}-\sqrt{\vec{p}_3^{\,2}+m_4^2}\right)$$

Let $x = \sqrt{\vec{p}_3^{\,2}+m_3^2} + \sqrt{\vec{p}_3^{\,2}+m_4^2}$ $\Rightarrow$ $$\frac{dx}{d|\vec{p}_3|} = \frac{|\vec{p}_3|}{\sqrt{\vec{p}_3^{\,2}+m_3^2}} + \frac{|\vec{p}_3|}{\sqrt{\vec{p}_3^{\,2}+m_4^2}} = \frac{|\vec{p}_3|\, x}{\sqrt{\vec{p}_3^{\,2}+m_3^2}\sqrt{\vec{p}_3^{\,2}+m_4^2}}$$

Cross section $2\to 2$

Then $$\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2(E_1+E_2)|\vec{p}_1|}\int_{m_3+m_4}^{\infty}|\mathcal{M}|^2\,|\vec{p}_3|\,\frac{dx}{x}\,\delta(E_1+E_2-x)$$

$$\Rightarrow \frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2(E_1+E_2)|\vec{p}_1|}\,|\mathcal{M}|^2\,\frac{|\vec{p}_3|}{E_1+E_2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega}=\frac{|\mathcal{M}|^2}{64\pi^2(E_1+E_2)^2}\,\frac{|\vec{p}_3|}{|\vec{p}_1|}$$ But $s=(p_1+p_2)^2=(E_1+E_2)^2$ in c.m. frame

So $$\frac{d\sigma}{d\Omega}=\frac{|\mathcal{M}|^2}{64\pi^2 s}\,\frac{|\vec{p}_3|}{|\vec{p}_1|}$$ For elastic scattering ($m_1=m_3$ and $m_2=m_4$) we have $|\vec{p}_3|=|\vec{p}_1|$ so $\dfrac{d\sigma}{d\Omega}^{\text{elastic}}=\dfrac{|\mathcal{M}|^2}{64\pi^2 s}$

We can rewrite this in terms of $t=(p_1-p_3)^2=p_1^2+p_3^2-2p_1\cdot p_3$

$$\Rightarrow t=m_1^2+m_3^2-2E_1E_3+2|\vec{p}_1||\vec{p}_3|\cos\theta \Rightarrow dt=2|\vec{p}_1||\vec{p}_3|\,d\cos\theta$$

$$\Rightarrow dt\,d\varphi=2|\vec{p}_1||\vec{p}_3|\,d\cos\theta\,d\varphi \Rightarrow dt\,d\varphi=2|\vec{p}_1||\vec{p}_3|\,d\Omega$$

Then $$\frac{d\sigma}{dt}=\int_0^{2\pi}d\varphi\,\frac{1}{2|\vec{p}_1||\vec{p}_3|}\,\frac{d\sigma}{d\Omega}=\frac{2\pi}{2|\vec{p}_1||\vec{p}_3|}\,\frac{|\mathcal{M}|^2}{64\pi^2 s}\,\frac{|\vec{p}_3|}{|\vec{p}_1|}\Rightarrow \frac{d\sigma}{dt}=\frac{|\mathcal{M}|^2}{64\pi s|\vec{p}_1|^2}$$

or $$\frac{d\sigma}{dt}=\frac{|\mathcal{M}|^2}{16\pi\,\lambda(s,m_1^2,m_2^2)}$$ where $\lambda(s,m_1^2,m_2^2)=s^2+m_1^4+m_2^4-2s(m_1^2+m_2^2)-2m_1^2m_2^2$
$=(s-m_1^2-m_2^2)^2-4m_1^2m_2^2$

If $m_1=m_2=0$ then $$\left.\frac{d\sigma}{dt}\right|_{m_1=m_2=0}=\frac{|\mathcal{M}|^2}{16\pi s^2}$$

Decay rates

Heavier elementary particles can decay. Example: muon decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

We can only predict the probability of a given particle to decay (and this probability is independent of how long ago that particle was created)

The decay rate Γ is the probability of decay per unit time.

If $N(t)$ is the number of particles at time t, then

$$dN = -\Gamma N\, dt \quad \Rightarrow \quad N(t) = N(0)\, e^{-\Gamma t}$$

The average lifetime is $\tau = \frac{1}{\Gamma}$ and the half-life is $t_{1/2} = \tau \ln 2 = \frac{\ln 2}{\Gamma}$

For the decay $p \to p_1 + p_2 + \dots + p_n$ with amplitude $\mathcal{M}$

we have

$$d\Gamma = \frac{|\mathcal{M}|^2}{2E} (2\pi)^4 \delta^4(p - p_1 - p_2 - \dots - p_n) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

The n-body phase space $d\phi^{(n)}$ is defined by

$$d\phi^{(n)} = (2\pi)^4 \delta^4(p - p_1 - p_2 - \dots - p_n) \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

Thus

$$d\Gamma = \frac{|\mathcal{M}|^2}{2E}\, d\phi^{(n)}$$