$$\underline{e^-(p_1)+\mu^-(p_2)\to e^-(p_3)+\mu^-(p_4)}$$

$$
\begin{aligned}
\text{Then } |\mathcal{M}|^2 = \frac{e^4}{4t^2}\,&\mathrm{tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu + m_e\not{p}_3\gamma^\mu\gamma^\nu + m_e\gamma^\mu\not{p}_1\gamma^\nu + m_e^2\gamma^\mu\gamma^\nu\right] \\
&\cdot\mathrm{tr}\left[\not{p}_4\gamma_\mu\not{p}_2\gamma_\nu + m_{\mu^-}\not{p}_4\gamma_\mu\gamma_\nu + m_{\mu^-}\gamma_\mu\not{p}_2\gamma_\nu + m_{\mu^-}^2\gamma_\mu\gamma_\nu\right]
\end{aligned}
$$

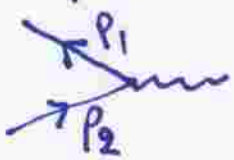
$$
\begin{aligned}
\Rightarrow |\mathcal{M}|^2 &= \frac{e^4}{4t^2}\mathrm{tr}\left[p_{3\rho}p_{1\sigma}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu + 0 + 0 + m_e^2\gamma^\mu\gamma^\nu\right]\mathrm{tr}\left[p_4^\lambda p_2^\tau\gamma_\lambda\gamma_\mu\gamma_\tau\gamma_\nu + 0 + 0 + m_{\mu^-}^2\gamma_\mu\gamma_\nu\right] \\
&= \frac{e^4}{4t^2}\left\{p_{3\rho}p_{1\sigma}4\left(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right) + m_e^2 4g^{\mu\nu}\right\}\left\{p_4^\lambda p_2^\tau 4\left(g_{\lambda\mu}g_{\tau\nu} - g_{\lambda\tau}g_{\mu\nu} + g_{\lambda\nu}g_{\mu\tau}\right) + m_{\mu^-}^2 4g_{\mu\nu}\right\} \\
&= \frac{e^4 16}{4t^2}\left\{p_3^\mu p_1^\nu - p_3^\sigma p_{1\sigma}g^{\mu\nu} + p_3^\nu p_1^\mu + m_e^2 g^{\mu\nu}\right\}\left\{p_{4\mu}p_{2\nu} - p_{4\tau}p_2^\tau g_{\mu\nu} + p_{4\nu}p_{2\mu} + m_{\mu^-}^2 g_{\mu\nu}\right\} \\
&= \frac{4e^4}{t^2}\Big[p_3\cdot p_4\, p_1\cdot p_2 - \cancel{p_3\cdot p_1\, p_4\cdot p_2} + p_3\cdot p_2\, p_1\cdot p_4 + m_{\mu^-}^2\, p_3\cdot p_1 \\
&\qquad - \cancel{p_3\cdot p_1\, p_4\cdot p_2} + \cancel{p_3\cdot p_1\, p_4\cdot p_2}\cdot 4 - \cancel{p_3\cdot p_1}\, p_4\cdot p_2 - m_{\mu^-}^2\, p_3\cdot p_1\cdot 4 \\
&\qquad + p_3\cdot p_2\, p_1\cdot p_4 - \cancel{p_3\cdot p_1\, p_4\cdot p_2} + p_3\cdot p_4\, p_1\cdot p_2 + m_{\mu^-}^2\, p_3\cdot p_1 \\
&\qquad + m_e^2\, p_4\cdot p_2 - m_e^2 4\, p_4\cdot p_2 + m_e^2\, p_4\cdot p_2 + m_e^2 m_{\mu^-}^2\cdot 4\Big] \\
&= \frac{4e^4}{t^2}\left[2\,p_1\cdot p_2\, p_3\cdot p_4 + 2\,p_1\cdot p_4\, p_2\cdot p_3 - 2m_e^2\, p_2\cdot p_4 - 2m_{\mu^-}^2\, p_1\cdot p_3 + 4m_e^2 m_{\mu^-}^2\right] \\
&= \frac{8e^4}{t^2}\left[p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3 - m_e^2\, p_2\cdot p_4 - m_{\mu^-}^2\, p_1\cdot p_3 + 2m_e^2 m_{\mu^-}^2\right]
\end{aligned}
$$

Useful relations

$$\sum_{\text{spins}} [\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma^\nu u(p_1)]^* = \mathrm{tr}[(\not{p}_3+m)\gamma^\mu(\not{p}_1+m)\gamma^\nu]$$

where m is the mass of the fermion line

$$\sum_{\text{spins}} [\bar{v}(p_1)\gamma^\mu u(p_2)][\bar{v}(p_1)\gamma^\nu u(p_2)]^* = \mathrm{tr}[(\not{p}_1-m)\gamma^\mu(\not{p}_2+m)\gamma^\nu]$$

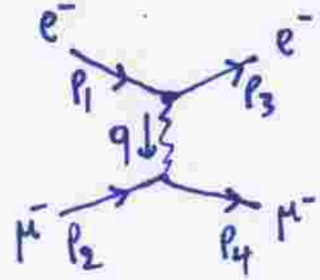


$$\sum_{\text{spins}} [\bar{u}(p_4)\gamma^\mu v(p_3)][\bar{u}(p_4)\gamma^\nu v(p_3)]^* = \mathrm{tr}[(\not{p}_4+m)\gamma^\mu(\not{p}_3-m)\gamma^\nu]$$

$$\sum_{\text{spins}} [\bar{v}(p_1)\gamma^\mu v(p_3)][\bar{v}(p_1)\gamma^\nu v(p_3)]^* = \mathrm{tr}[(\not{p}_1-m)\gamma^\mu(\not{p}_3-m)\gamma^\nu]$$

We can use these relations to quickly write the expressions for squared amplitudes (with sum over final spins, average over initial spins)

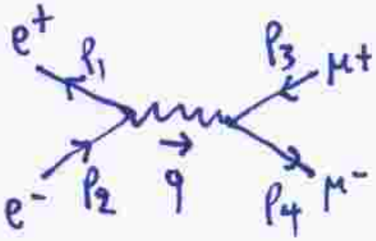
Examples



$$i\mathcal{M} = \bar{u}(p_3)(-ie\gamma^\mu)u(p_1)\,(-i)\frac{g_{\mu\nu}}{q^2}\,\bar{u}(p_4)(-ie\gamma^\nu)u(p_2)$$

$$\Rightarrow \mathcal{M} = e^2\bar{u}(p_3)\gamma^\mu u(p_1)\frac{1}{(p_1-p_3)^2}\bar{u}(p_4)\gamma_\mu u(p_2)$$

$$\Rightarrow |\mathcal{M}|^2 = \frac{e^4}{4(p_1-p_3)^4}\,tr[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu]\,tr[(\not{p}_4+m_{\mu^-})\gamma_\mu(\not{p}_2+m_{\mu^-})\gamma_\nu]$$



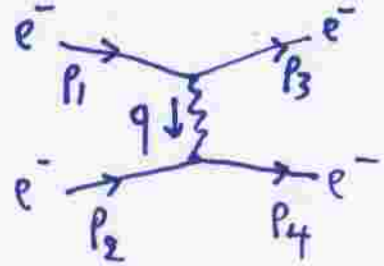
$$i\mathcal{M} = \bar{v}(p_1)(-ie\gamma^\mu)u(p_2)\,(-i)\frac{g_{\mu\nu}}{q^2}\,\bar{u}(p_4)(-ie\gamma^\nu)v(p_3)$$

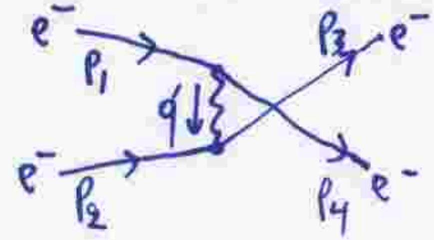
$$\Rightarrow \mathcal{M} = e^2\bar{v}(p_1)\gamma^\mu u(p_2)\frac{1}{(p_1+p_2)^2}\bar{u}(p_4)\gamma_\mu v(p_3)$$

$$\Rightarrow |\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4}\,tr[(\not{p}_1-m_e)\gamma^\mu(\not{p}_2+m_e)\gamma^\nu]\,tr[(\not{p}_4+m_{\mu^-})\gamma_\mu(\not{p}_3-m_{\mu^-})\gamma_\nu]$$

In the squared amplitude we have summed over final spins and averaged (factor of $\frac{1}{4}$) over initial spins

Electron-electron scattering





$e^-(p_1)+e^-(p_2)\to e^-(p_3)+e^-(p_4)$

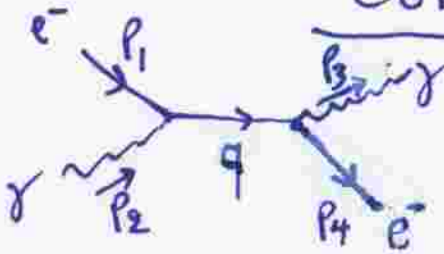
$$i\mathcal{M}=\bar{u}(p_3)(-ie\gamma^\mu)u(p_1)(-i)\frac{g_{\mu\nu}}{q^2}\bar{u}(p_4)(-ie\gamma^\nu)u(p_2)-\bar{u}(p_4)(-ie\gamma^\mu)u(p_1)(-i)\frac{g_{\mu\nu}}{q^2}\bar{u}(p_3)(-ie\gamma^\nu)u(p_2)$$

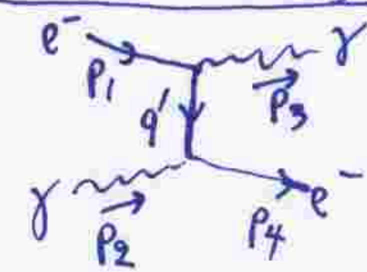
$$\Rightarrow \mathcal{M}=\frac{e^2}{(p_1-p_3)^2}\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)-\frac{e^2}{(p_1-p_4)^2}\bar{u}(p_4)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu u(p_2)$$

note the relative minus sign between the two terms

Antisymmetrization rule: insert a minus sign when combining amplitudes with interchange of two identical external fermions

Compton scattering





$e^-(p_1)+\gamma(p_2)\to\gamma(p_3)+e^-(p_4)$

$$i\mathcal{M}=\bar{u}(p_4)(-ie\gamma^\mu)\frac{i(\not{q}+m_e)}{q^2-m_e^2}(-ie\gamma^\nu)u(p_1)\varepsilon^*_\mu(p_3)\varepsilon_\nu(p_2)+\bar{u}(p_4)(-ie\gamma^\mu)\frac{i(\not{q}'+m_e)}{q'^2-m_e^2}(-ie\gamma^\nu)u(p_1)\varepsilon_\mu(p_2)\varepsilon^*_\nu(p_3)$$

$$\Rightarrow \mathcal{M}=\frac{-e^2}{(p_1+p_2)^2-m_e^2}\bar{u}(p_4)\not{\varepsilon}^*(p_3)(\not{p}_1+\not{p}_2+m_e)\not{\varepsilon}(p_2)u(p_1)-\frac{e^2}{(p_1-p_3)^2-m_e^2}\bar{u}(p_4)\not{\varepsilon}(p_2)(\not{p}_1-\not{p}_3+m_e)\not{\varepsilon}^*(p_3)u(p_1)$$