HW14
PHYS4500: Quantum Field Theory

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Problem 1.

First, we have that

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

So,

$$\begin{aligned}
[\lambda^1, \lambda^2] &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (1.2) \\
&= \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} && (1.3) \\
&= 2i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (1.4) \\
&= 2i\lambda^3 = 2if^{123}\lambda^3, && (1.5)
\end{aligned}$$

where $f^{123} = 1$, as we expect. So, the relation is satisfied. Next, we have

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

So,

$$\begin{aligned}
[\lambda^3, \lambda^4] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && (1.7) \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} && (1.8) \\
&= i \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} && (1.9) \\
&= 2i \cdot \frac{1}{2} \cdot \lambda^5 = 2if^{345}\lambda^5, && (1.10)
\end{aligned}$$

where $f^{345} = 1/2$, as we expect. The relation is again satisfied.

Problem 2.

$$\begin{aligned}
\lambda^5\lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} && (2.1) \\
&= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & x & x \\ x & 0 & x \\ x & x & 0 \end{pmatrix}, && (2.2)
\end{aligned}$$

where, in the context of the trace, we don't care about the non-diagonal components, so I just labeled them with an x and didn't bother computing them. We can see quite clearly that since all the diagonal elements are zero that

$$\boxed{\mathrm{Tr}[\lambda^5 \lambda^8] = 0.} \tag{2.3}$$

We expect this, since

$$\mathrm{Tr}[\lambda^i, \lambda^j] = 2\delta_{ij} = 0 \tag{2.4}$$

for $i \neq j$.

Problem 3.

The only Gell-Mann matrix with non-zero elements in the $(1,1)$ and $(3,3)$ positions is λ^8, so

$$T^a_{11} T^a_{33} = \frac{1}{4} \lambda^8_{11} \lambda^8_{11} = \frac{1}{4} \cdot -\frac{2}{3} = -\frac{1}{6}. \tag{3.1}$$

We could also use the relation

$$T^a_{ij} T^a_{k\ell} = \frac{1}{2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{k\ell} \right). \tag{3.2}$$

In our case, have that $(i = j) \neq (k = \ell)$, so the first term in parentheses vanishes and we are only left with the second term with the deltas becoming 1, so

$$= \frac{1}{2} \cdot -\frac{1}{N_c} = -\frac{1}{6} \tag{3.3}$$

for 3 colors.

Problem 4.

For this, we basically matrix multiply each T matrix with itself and only grab the $(1,1)$ term. This can easily be done just by looking at the Gell-Mann matrices, so I won't write them out or anything:

$$T^a_{1j} T^a_{j1} = \frac{1}{4} \lambda^a_{1j} \lambda^a_{j1} = \frac{1}{4} \left(1 + 1 + 1 + 1 + 1 + 0 + 0 + \frac{1}{3} \right) \tag{4.1}$$

$$= \frac{1}{4} \left(5 + \frac{1}{3} \right) = \frac{1}{4} \cdot \frac{16}{3} = \frac{4}{3}. \tag{4.2}$$

We also have the general relation that

$$T^a_{ij} T^a_{j\ell} = C_F \delta_{i\ell} = C_F = \frac{4}{3} \tag{4.3}$$

if $i = \ell$, which we have, and also if we are in the fundamental representation then $C_F = 4/3$, which matches what we got explicitly.

Problem 5.

There are four non-zero structure constants with $f^{5bc} \neq 0$. They are

$$f^{516} = f^{572} = f^{534} = \frac{1}{2}, \quad \text{and} \tag{5.1}$$

$$f^{584} = \frac{\sqrt{3}}{2}. \tag{5.2}$$

So, for the relation

$$f^{abc}T^bT^c = \frac{i}{2}C_AT^a \quad \rightarrow \quad f^{abc}\lambda^b\lambda^c = 3i\lambda^a, \tag{5.3}$$

we will have a sum over all four possibilities:

$$f^{5bc}T^bT^c = f^{516}T^1T^6 + f^{572}T^7T^2 + f^{534}T^3T^4 + f^{584}T^8T^4 \tag{5.4}$$

Doing these calculations:

$$f^{516}T^1T^6 = \frac{1}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{5.5}$$

$$f^{572}T^7T^2 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix} \tag{5.6}$$

$$f^{534}T^3T^4 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{5.7}$$

$$f^{534}T^3T^4 = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -2 & 0 & 0\end{pmatrix} \tag{5.8}$$

So,

$$f^{5bc}T^bT^c = \frac{3}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix} = \frac{3i}{2}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix} = \frac{3i}{2}\lambda^5. \tag{5.9}$$

This matches with the relation

$$f^{abc}T^bT^c = \frac{i}{2}C_AT^a \tag{5.10}$$

where we have $a = 5$ and $C_A = N_c = 3$ for 3 colors.

Problem 6.

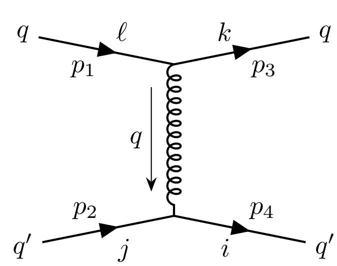

Figure 1: Feynman diagram for $q + q' \rightarrow q + q'$.

Figure 1 shows the one diagram for the process $qq' \rightarrow qq'$ (I give the bottom vertex a space-time index of μ and a gluon index of a and the top vertex gets a space-time index of ν and a gluon index of b). This is the only diagram since there is no s-channel as that would have a conversion from one quark to another and there is no u-channel since the two final state particles are distinct. We can use our Feynman rules to easily write down the amplitude:

$$i\mathcal{M} = \bar{u}(p_4)(-ig_s\gamma^\mu T^a_{ij})u(p_2)\left(\frac{-ig_{\mu\nu}\delta^{ab}}{q^2}\right)\bar{u}(p_3)(-ig_s\gamma^\nu T^b_{k\ell})u(p_1) \tag{6.1}$$

$$\mathcal{M} = \frac{g_s^2}{(p_1-p_3)^2}[\bar{u}(p_4)\gamma^\mu u(p_3)][\bar{u}(p_3)\gamma_\mu u(p_1)]\left(T^a_{ij}T^a_{k\ell}\right). \tag{6.2}$$

We cannot do much with the color factor now. We can use the relation

$$T^a_{ij}T^a_{k\ell} = \frac{1}{2}\left(\delta_{i\ell}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right), \tag{6.3}$$

but this makes things even more complicated since we have nothing to use the delta functions with. We'd be able to calculate the full color factor when squaring the amplitude, but for now when just considering the amplitude, I'll leave it as is.