HW11

PHYS4500: Quantum Field Theory

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From the lecture notes, we know already that

$$\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)} = \left[-D(x_1-x_2) + \left(\int D(x-x_2)J(x)\ \mathrm{d}^4x\right)\left(\int D(x-x_1)J(x)\ \mathrm{d}^4x\right)\right] \times \exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\ \mathrm{d}^4x\mathrm{d}^4y\right). \quad (0.2)$$

If we take $x_2 \to x_3$ and $x_1 \to x_2$, then

$$\frac{\delta^3 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = \frac{\delta}{\delta J(x_1)}\left\{\left[-D(x_2-x_3) + \left(\int D(x-x_3)J(x)\ \mathrm{d}^4x\right)\left(\int D(x-x_2)J(x)\ \mathrm{d}^4x\right)\right] \times \exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\ \mathrm{d}^4x\mathrm{d}^4y\right)\right\}. \quad (0.3)$$

Our goal is to show that

$$G(x_1,x_2,x_3) = -i\left.\frac{\delta^3 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\right|_{J=0} = 0. \quad (0.4)$$

We have a product of two terms; the first is the expression in brackets, the second is the exponential. We will need to take derivatives of both. For notational simplicity, I will let

$$Y_1(x_1,x_2) \equiv -D(x_2-x_3) + \left(\int D(x-x_3)J(x)\ \mathrm{d}^4x\right)\left(\int D(x-x_2)J(x)\ \mathrm{d}^4x\right), \quad (0.5)$$

which is just the expression in brackets, and

$$Y_2 \equiv \exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\ \mathrm{d}^4x\mathrm{d}^4y\right). \quad (0.6)$$

Then

$$\frac{\delta^3 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} = Y_2\frac{\delta Y_1}{\delta J(x_1)} + Y_1\frac{\delta Y_2}{\delta J(x_1)}. \quad (0.7)$$

Looking at the first functional derivative:

$$\frac{\delta Y_1}{\delta J(x_1)} = \frac{\delta}{\delta J(x_1)}\left[-D(x_2-x_3) + \left(\int D(x-x_3)J(x)\ \mathrm{d}^4x\right)\left(\int D(x-x_2)J(x)\ \mathrm{d}^4x\right)\right] \quad (0.8)$$

$$= -D(x_1-x_3)\int D(x-x_2)J(x)\ \mathrm{d}^4x - D(x_1-x_2)\int D(x-x_3)J(x)\ \mathrm{d}^4x. \quad (0.9)$$

Our goal is to, at the end, evaluate this quantity at $J=0$. Since every term here contains a J, this will be zero. Further, the entire first term in Equation (0.7) is zero. Looking next at the second functional derivative:

$$\frac{\delta Y_2}{\delta J(x_1)} = \frac{\delta}{\delta J(x_1)}\exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\mathrm{d}^4x\right) \quad (0.10)$$

$$= \left(-\frac{1}{2}D(x-x_1)J(x)\ \mathrm{d}^4x\right)\exp\left(-\frac{1}{2}\int J(x)D(x-y)J(y)\mathrm{d}^4x\right). \quad (0.11)$$

There is really only one term here, and it contains a J, which, after evaluating it to zero, turns the entire expression zero. Therefore,

$$G(x_1,x_2,x_3) = -i\left.\frac{\delta^3 Z_0[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\right|_{J=0} = 0, \quad (0.12)$$

as we expect.