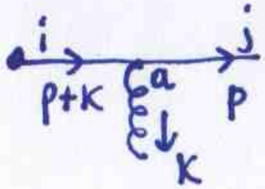
Soft gluons

In the limit when the gluon energy goes to zero (soft limit) the Feynman rules simplify.

Consider an outgoing quark emitting a soft gluon:



$$\bar{u}_j(p)(-ig_s)T^a_{ji}\gamma^\mu \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2} = \bar{u}_j(p)(-ig_s)T^a_{ji}\gamma^\mu \frac{i(\not{p}+\not{k}+m)}{k^2+2p\cdot k}$$

$$\xrightarrow[k\to 0]{} -ig_s T^a_{ji}\,\bar{u}_j(p)\gamma^\mu \frac{i(\not{p}+m)}{2p\cdot k} = g_s T^a_{ji}\,\bar{u}_j(p)\gamma^\mu \frac{(\gamma^\nu p_\nu + m)}{2p\cdot k}$$

$$= g_s T^a_{ji}\,\bar{u}_j(p)\,\frac{(2g^{\mu\nu}p_\nu - \gamma^\nu\gamma^\mu p_\nu + m\gamma^\mu)}{2p\cdot k}$$

$$= g_s T^a_{ji}\,\bar{u}_j(p)\,\frac{[2p^\mu - (\not{p}-m)\gamma^\mu]}{2p\cdot k}$$

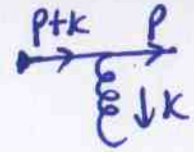
(but $\bar{u}_j(p)(\not{p}-m)=0$)

$$= g_s T^a_{ji}\,\bar{u}_j(p)\,\frac{p^\mu}{p\cdot k}$$

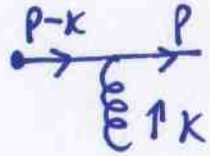
Thus the Feynman rule simplifies greatly → no Dirac γ matrix

If we write $p^\mu \propto v^\mu$ with v^μ a velocity vector, then this can be rewritten as $g_s T^a_{ji}\,\bar{u}_j(p)\,\dfrac{v^\mu}{v\cdot k}$ <u>eikonal approximation</u>

(similarly for QED – soft photons)

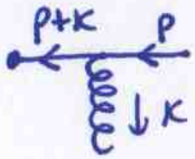
So, ignoring color factors, the vertex for 

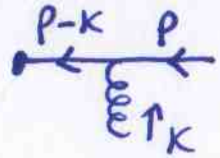
is simply $\dfrac{v^\mu}{v \cdot k}$ (times g_s)

Similarly 

$\dfrac{v^\mu}{-v \cdot k}$

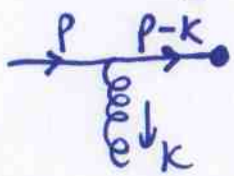
For outgoing antiquarks

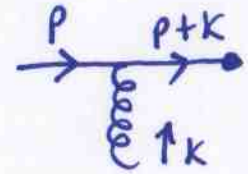


$\dfrac{-v^\mu}{v \cdot k}$ and 

$\dfrac{-v^\mu}{-v \cdot k}$

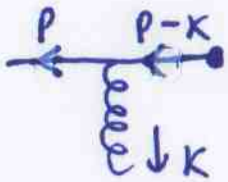
For incoming quarks

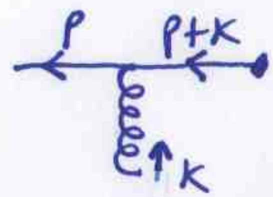


$\dfrac{v^\mu}{-v \cdot k}$ and 

$\dfrac{v^\mu}{v \cdot k}$

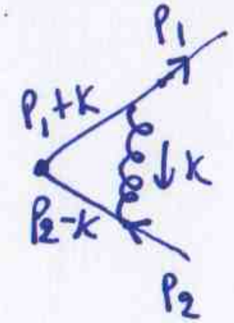
For incoming antiquarks



$\dfrac{-v^\mu}{-v \cdot k}$ and 

$\dfrac{-v^\mu}{v \cdot k}$

Cusp anomalous dimension at one loop



We consider a quark-antiquark pair with gluon exchange. The integral describing this cusp at one loop in the eikonal approximation is (ignoring color factors)

$$I = \int \frac{d^n k}{(2\pi)^n}\, g_s \frac{v_1^\mu}{v_1\cdot k}\,(-i)\frac{g_{\mu\nu}}{k^2}\, g_s \frac{(-v_2^\nu)}{(-v_2\cdot k)}$$

$$\Rightarrow I = -i g_s^2 \frac{v_1\cdot v_2}{(2\pi)^n}\int \frac{d^n k}{k^2\, v_1\cdot k\; v_2\cdot k} = -i g_s^2 \frac{v_1\cdot v_2}{(2\pi)^n}\, 2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^n k}{[x k^2 + y v_1\cdot k + (1-x-y) v_2\cdot k]^3}$$

$$= -2i g_s^2 \frac{v_1\cdot v_2}{(2\pi)^n}\int_0^1 dx\, x^{-3}\int_0^{1-x} dy \int d^n k \left[k^2 + 2\,\frac{(y v_1 + (1-x-y)v_2)}{2x}\cdot k\right]^{-3}$$

$$= -2i g_s^2 \frac{v_1\cdot v_2}{(2\pi)^n}\int_0^1 dx\, x^{-3}\int_0^{1-x} dy\; i\pi^{n/2}\frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)}\left[-\frac{(y v_1 + (1-x-y)v_2)^2}{4x^2}\right]^{\frac{n}{2}-3}$$

$$= 4\pi a_s\, v_1\cdot v_2\, 2^{6-2n}\,\pi^{-n/2}\,\Gamma(3-\tfrac{n}{2})\int_0^1 dx\, x^{-3} x^{6-n}\int_0^{1-x} dy\left[-y^2 v_1^2 - (1-x-y)^2 v_2^2 - 2y(1-x-y) v_1\cdot v_2\right]^{\frac{n}{2}-3}$$

Let $z = \frac{y}{1-x}$ Then $dy = (1-x)dz$ and $1-x-y = (1-x)(1-z)$ Also $n = 4-\varepsilon$
Then

$$I = \frac{a_s}{\pi}\, 2^{2\varepsilon}\,\pi^{\varepsilon/2}\, v_1\cdot v_2\,\Gamma(1+\tfrac{\varepsilon}{2})\int_0^1 dx\, x^{-1+\varepsilon}(1-x)\int_0^1 dz\left[\begin{matrix}-z^2(1-x)^2 v_1^2 - (1-x)^2(1-z)^2 v_2^2 \\ -2(1-x)z(1-x)(1-z)\, v_1\cdot v_2\end{matrix}\right]^{-1-\frac{\varepsilon}{2}}$$

$$\Rightarrow I = \frac{a_s}{\pi} 2^{2\varepsilon} \pi^{\varepsilon/2} v_1\cdot v_2 \,\Gamma(1+\tfrac{\varepsilon}{2}) \int_0^1 dx\, x^{-1+\varepsilon}(1-x)^{-1-\varepsilon} \int_0^1 dz \left[-z^2 v_1^2 - (1-z)^2 v_2^2 - 2z(1-z) v_1\cdot v_2\right]^{-1-\frac{\varepsilon}{2}}$$

If $p_1^\mu = \frac{\sqrt{s}}{2} v_1^\mu$ and $p_2^\mu = \frac{\sqrt{s}}{2} v_2^\mu$ with $s = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 = 2m^2 + 2p_1\cdot p_2$

Then $s = 2m^2 + 2\frac{s}{4} v_1\cdot v_2 \Rightarrow v_1\cdot v_2 = 2 - \frac{4m^2}{s} = 1+\beta^2$ with $\beta = \sqrt{1-\frac{4m^2}{s}}$ (the speed)

Also $p_1^2 = \frac{s}{4} v_1^2 \Rightarrow v_1^2 = \frac{4m^2}{s} = 1-\beta^2$ and also $v_2^2 = 1-\beta^2$

Then

$$I = \frac{a_s}{\pi} 2^{2\varepsilon} \pi^{\varepsilon/2} (-1)^{-1-\frac{\varepsilon}{2}} v_1\cdot v_2 \,\Gamma(1+\tfrac{\varepsilon}{2}) \int_0^1 dx\, x^{-1+\varepsilon}(1-x)^{-1-\varepsilon} \int_0^1 dz \left[(1-\beta^2) z^2 + (1-\beta^2)(1-z)^2 + 2(1+\beta^2) z(1-z)\right]^{-1-\frac{\varepsilon}{2}}$$

$$= \frac{a_s}{\pi} 2^{2\varepsilon} \pi^{\varepsilon/2} (-1)^{-1-\frac{\varepsilon}{2}} (1+\beta^2) \Gamma(1+\tfrac{\varepsilon}{2}) \int_0^1 dx\, x^{-1+\varepsilon}(1-x)^{-1-\varepsilon} \int_0^1 dz \left[4\beta^2 z(1-z) + 1-\beta^2\right]^{-1-\frac{\varepsilon}{2}}$$

Now $\int_0^1 dx\, x^{-1+\varepsilon}(1-x)^{-1-\varepsilon} = \int_0^1 dx\, x^{-1+\varepsilon} + \underbrace{\int_0^1 dx\, x^{-1+\varepsilon}\left[(1-x)^{-1-\varepsilon} - 1\right]}_{\text{infrared divergent integral } (= -\frac{1}{\varepsilon} \text{ if } \varepsilon<0)}$

$$= \left.\frac{x^\varepsilon}{\varepsilon}\right|_0^1 + IR = \frac{1}{\varepsilon} + IR$$

Also $\int_0^1 dz \left[4\beta^2 z(1-z) + 1-\beta^2\right]^{-1-\frac{\varepsilon}{2}} = \int_0^1 dz \left[4\beta^2 z(1-z)+1-\beta^2\right]^{-1} - \frac{\varepsilon}{2}\int_0^1 dz \frac{\ln(4\beta^2 z(1-z)+1-\beta^2)}{4\beta^2 z(1-z)+1-\beta^2} + O(\varepsilon^2)$

and $\int_0^1 dz \left[4\beta^2 z(1-z) + 1-\beta^2\right]^{-1} = -\frac{1}{2\beta}\ln\left(\frac{1-\beta}{1+\beta}\right)$

Thus keeping only the UV pole, we find

$$I_{UV} = \frac{a_s}{\pi}\frac{1}{\varepsilon}\frac{(1+\beta^2)}{2\beta}\ln\left(\frac{1-\beta}{1+\beta}\right)$$

One can define a <u>soft</u> function that describes noncollinear soft-gluon emission in scattering processes with quarks and gluons.

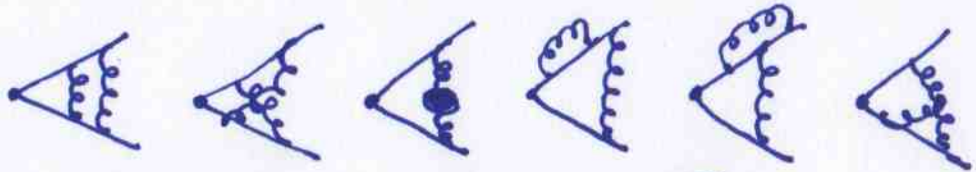
This soft function satisfies a renormalization-group equation

$\mu \frac{dS}{d\mu} = -\Gamma_S^{\dagger} S - S \Gamma_S$ where S is the soft function and Γ_S is the <u>soft anomalous dimension</u>

Evolution of $S \rightarrow$ resummation of soft-gluon corrections

For the <u>cusp</u> at <u>one loop</u>: $\Gamma^{(1)}_{cusp} = -C_F (L_\beta + 1)$ with $L_\beta = \frac{(1+\beta^2)}{2\beta} \ln\left(\frac{1-\beta}{1+\beta}\right)$

For the <u>cusp</u> at <u>two loops</u>



where ● denotes quark, gluon, and ghost loops

$$\Gamma^{(2)}_{cusp} = K_2 \Gamma^{(1)}_{cusp} + \frac{C_F C_A}{2} \left\{ 1 + \zeta_2 + \ln^2\left(\frac{1-\beta}{1+\beta}\right) + \frac{(1+\beta^2)}{2\beta}\left[\zeta_2 \ln\left(\frac{1-\beta}{1+\beta}\right) - \ln^2\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{3}\ln^3\left(\frac{1-\beta}{1+\beta}\right) - Li_2\left(4\beta/(1+\beta)^2\right)\right] \right.$$
$$\left. + \frac{(1+\beta^2)^2}{4\beta^2}\left[-\zeta_3 - \zeta_2 \ln\left(\frac{1-\beta}{1+\beta}\right) - \frac{1}{3}\ln^3\left(\frac{1-\beta}{1+\beta}\right) - \ln\left(\frac{1-\beta}{1+\beta}\right) Li_2\left(\frac{(1-\beta)^2}{(1+\beta)^2}\right) + Li_3\left(\frac{(1-\beta)^2}{(1+\beta)^2}\right)\right] \right\}$$

where $K_2 = C_A\left(\frac{67}{36} - \frac{\zeta_2}{2}\right) - \frac{5}{18} n_f$ and $\zeta_K = \sum_{n=1}^{\infty} \frac{1}{n^K}$, $\zeta_2 = \pi^2/6$, $\zeta_3 = 1.2020569\ldots$, $Li_K(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^K}$

For processes with complex color flow, such as $t\bar{t}$ production, Γ_S is a matrix in color space.