HW1

PHYS4210: Quantum Mechanics

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Problem 1. (1.5)

We are looking at the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t} \tag{1.1}$$

a) To normalize, we can choose $t = 0$, though it really doesn't matter since the modulus square will eliminate the time-dependence anyways:

$$A^2 \int_{\infty}^{\infty} e^{-2\lambda|x|}\, \mathrm{d}x = 1.$$

The fact that there is an absolute value around the x in the exponential makes the integrand even, so we can change the limits, add a factor of 2, and remove the absolute value:

$$\begin{gathered} 2A^2 \int_0^{\infty} e^{-2\lambda x}\, \mathrm{d}x = 1, \\ 2A^2 \cdot -\frac{1}{2\lambda}\left[e^{-2\lambda x}\right]_0^{\infty} = 1, \\ A^2 \cdot \frac{1}{\lambda} = 1, \\ \boxed{A = \sqrt{\lambda}.} \end{gathered}$$

b) We can just use the definition of expectation value:

$$\langle x \rangle = \int_{\infty}^{\infty} \psi^* x \psi\, \mathrm{d}x = \lambda \int_{\infty}^{\infty} x e^{-2\lambda|x|}\, \mathrm{d}x.$$

Here, x is an odd function, and, as established in part (a), the exponential is an even function. The product of an odd and even function is an odd function, and since we are evaluating it over a symmetric interval, it is zero:

$$\boxed{\langle x \rangle = 0.}$$

Next,

$$\langle x^2 \rangle = \lambda \int_{\infty}^{\infty} x^2 e^{-2\lambda|x|}\, \mathrm{d}x = 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}\, \mathrm{d}x.$$

From the integral table, we have that

$$\int_0^{\infty} x^n e^{-x/a}\, \mathrm{d}x = n! a^{n+1}. \tag{1.2}$$

Here, we have that $n = 2$ and $a = 1/2\lambda$, so our integral becomes:

$$\langle x^2 \rangle = 2\lambda \cdot 2! \left(\frac{1}{2\lambda}\right)^3 = \boxed{\frac{1}{2\lambda^2}.}$$

c) The standard deviation is

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{2\lambda^2}} = \boxed{\frac{\sqrt{2}}{2\lambda}.} \tag{1.3}$$

A plot of the $|\Psi|^2$ distribution is given in Figure 1, where the vertical black lines are $x = \langle x \rangle - \sigma_x$ and $x = \langle x \rangle + \sigma_x$. To determine the probability that a particle lies outside this range, we integrate

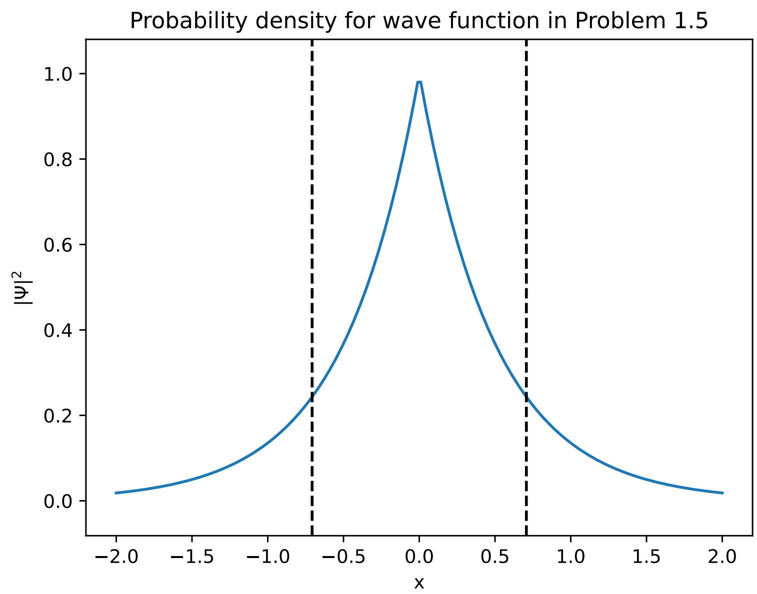

Figure 1: The $|\Psi|^2$ distribution with the standard deviations marked with black dotted lines.

over the whole space, excluding the bit inside. Again exploiting the fact that our integrand $|\Psi|^2$ is even, we can look at just the upper half, say, which is

$$\begin{aligned} P(x \notin [-\sigma_x, \sigma_x]) &= 2\lambda \int_{\sigma_x}^{\infty} e^{-2\lambda x}\,\mathrm{d}x, \\ &= -1\left[e^{-2\lambda x}\right]_{\sigma_x}^{\infty}, \\ &= e^{-2\lambda\cdot\sqrt{2}/2\lambda^2} = e^{-\sqrt{2}}, \end{aligned}$$

$$\boxed{P(x \notin [-\sigma_x, \sigma_x]) = e^{-\sqrt{2}} \approx 0.243.}$$

Problem 2. (1.9)

We are given the wavefunction:

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]}. \tag{2.1}$$

a) To find A can can just normalize:

$$\int_{\infty}^{\infty} |\Psi(x,0)|^2\,\mathrm{d}x = A^2\int_{\infty}^{\infty} e^{-2amx^2/\hbar}\,\mathrm{d}x = 2A^2\int_0^{\infty} e^{-2amx^2/\hbar}\,\mathrm{d}x = 1.$$

Using the integral table:

$$\int_0^{\infty} x^{2n}e^{-x^2/a^2}\,\mathrm{d}x = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}. \tag{2.2}$$

In our case, we have $n = 0$ and $a = \sqrt{\hbar/2am}$, so:

$$\int_{\infty}^{\infty} |\Psi(x,0)|^2\,\mathrm{d}x = 2A^2\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\frac{\hbar}{2am}} = A^2\sqrt{\frac{\hbar\pi}{2am}} = 1,$$

$$\boxed{A = \sqrt[4]{\frac{2am}{\hbar\pi}}.}$$

b) To find the potential, we can simply plug in to the Schrödinger Equation and solve for $V(x)$:

$$\begin{aligned}
i\hbar\frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = V(x)\Psi, \\
i\hbar(-ai)\Psi &= \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[-\frac{2amx}{\hbar}\Psi\right] + V(x)\Psi, \\
\hbar a\Psi &= -\frac{\hbar^2}{2m}\left[\frac{-2am}{\hbar}\Psi - \frac{2amx}{\hbar}\Psi\left(-\frac{2amx}{\hbar}\right)\right] + V(x)\Psi, \\
\hbar a &= \hbar a - \frac{\hbar^2}{2m}\left(-\frac{2amx}{\hbar}\right)^2 + V(x), \\
&\boxed{V(x) = 2ma^2x^2.}
\end{aligned}$$

c) Let's start with $\langle x\rangle$:

$$\langle x\rangle = \int_{\infty}^{\infty} x|\psi|^2\,\mathrm{d}x = \sqrt{\frac{2am}{\hbar\pi}}\int_{-\infty}^{\infty} xe^{-2amx^2/\hbar}\,\mathrm{d}x.$$

We can immediately stop here. x is an odd function, and the exponential is an even one. An odd function times an even function gives an odd function, and we are evaluating this function over symmetric intervals, meaning that its zero:

$$\boxed{\langle x\rangle = 0}.$$

Moving to $\langle x^2\rangle$:

$$\langle x^2\rangle = \sqrt{\frac{2am}{\hbar\pi}}\int_{-\infty}^{\infty} x^2e^{-2amx^2/\hbar}\,\mathrm{d}x = 2\sqrt{\frac{2am}{\hbar\pi}}\int_{0}^{\infty} xe^{-2amx^2/\hbar}\,\mathrm{d}x.$$

Using Eq (2.2), we have $n = 1$ and $a = \sqrt{\hbar/2am}$, so:

$$\langle x^2\rangle = \left(\frac{2am}{\hbar\pi}\right)^{1/2}\cdot 2\cdot\sqrt{\pi}\cdot 2\cdot\left(\frac{\hbar}{2am}\right)^{3/2} = \frac{1}{2}\left(\frac{\hbar}{2am}\right)^{3/2}\left(\frac{2am}{\hbar}\right)^{1/2} = \boxed{\frac{\hbar}{4am}}.$$

Since $\langle x\rangle = 0$, $\langle p\rangle = 0$ too. Lastly, we turn to $\langle p^2\rangle$:

$$\langle p^2\rangle = \int_{\infty}^{\infty}\psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)^2\psi\,\mathrm{d}x = -\hbar^2\int_{\infty}^{\infty}\psi^*\frac{\partial^2\psi}{\partial x^2}\,\mathrm{d}x.$$

Doing the derivatives:

$$\frac{\partial\psi}{\partial x} = -\frac{2amx}{\hbar}\psi \to \frac{\partial^2\psi}{\partial x^2} = -\frac{2am}{\hbar}\psi + \frac{4a^2m^2x^2}{\hbar^2}\psi.$$

So,

$$\begin{aligned}
\langle p^2\rangle &= -2\hbar^2\sqrt{\frac{2am}{\hbar\pi}}\int_0^{\infty}\left(-\frac{2am}{\hbar} + \frac{4a^2m^2x^2}{\hbar^2}\right)e^{-2amx^2/\hbar}\,\mathrm{d}x, \\
&= 4am\hbar\sqrt{\frac{2am}{\hbar\pi}}\int_0^{\infty}e^{-2amx^2/\hbar}\,\mathrm{d}x - 8a^2m^2\sqrt{\frac{2am}{\hbar\pi}}\int_0^{\infty}x^2e^{-2amx^2\hbar}\,\mathrm{d}x, \\
&= 4am\hbar\sqrt{\frac{2am}{\hbar\pi}}\cdot\frac{\sqrt{\pi}}{2}\sqrt{\frac{\hbar}{2am}} - 8a^2m^2\sqrt{\frac{2am}{\hbar\pi}}\cdot\frac{2\sqrt{\pi}}{8}\left(\frac{h}{2am}\right)^{3/2}, \\
&= 2am\hbar - 2a^2m^2\left(\frac{h}{2am}\right) = 2am\hbar - am\hbar,
\end{aligned}$$

$$\boxed{\langle p^2\rangle = am\hbar.}$$

d) The definition for the standard deviations are:

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}, \text{ and } \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}. \tag{2.3}$$

So, for x:

$$\sigma_x = \sqrt{\frac{\hbar}{4am}},$$

since $\langle x\rangle = 0$. For p:

$$\sigma_p = \sqrt{am\hbar}.$$

Plugging in:

$$\sqrt{\frac{\hbar}{4am}} \cdot \sqrt{am\hbar} = \frac{\hbar}{2} \geq \frac{\hbar}{2}.$$

And the Heisenberg Uncertainty Principle is satisfied!

Problem 3. (1.16)

We are given the wavefunction of a particle at $t = 0$:

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2), & -a \leq x \leq a, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

a) We just normalize:

$$\int_{\infty}^{\infty} |\Psi(x,0)| \,\mathrm{d}x = A^2 \int_{-a}^{a} (a^2 - x^2)^2 \,\mathrm{d}x = 2A^2 \left(\int_0^a a^4 \,\mathrm{d}x - 2\int_0^a a^2x^2 \,\mathrm{d}x + \int_0^a x^4 \,\mathrm{d}x \right),$$
$$= 2A^2 \left(a^5 - \frac{2}{3}a^5 + \frac{a^5}{5} \right) = 2A^2a^5 \cdot \frac{8}{15} = A^2 \frac{16a^5}{15} = 1,$$

$$\rightarrow \boxed{A = \sqrt{\frac{15}{16a^5}}.}$$

b) We have

$$\langle x\rangle = \int_{\infty}^{\infty} \Psi^* x \Psi \,\mathrm{d}x = \int_{-a}^{a} x|\Psi|^2 \,\mathrm{d}x.$$

Just as with the previous parts, we have, in the integrand, a product of an odd and an even function, which is an odd function, evaluated over a symmetric interval, meaning:

$$\boxed{\langle x\rangle = 0.}$$

To evaluate $\langle p\rangle$, we cannot use $\mathrm{d}\langle x\rangle/\mathrm{d}t$, because we are only given the wavefunction at $t = 0$; there is no way to assess its change with respect to t – we will have to do it manually:

$$\langle p\rangle = \int_{\infty}^{\infty} \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi \,\mathrm{d}x = -\frac{15i\hbar}{16a^5} \int_{-a}^{a} (a^2 - x^2) \frac{\mathrm{d}}{\mathrm{d}x} \left[a^2 - x^2\right] \,\mathrm{d}x,$$
$$\langle p\rangle = \frac{15i\hbar}{16} \int_{-a}^{a} 2x(a^2 - x^2) \,\mathrm{d}x.$$

$2x$ is an odd function, and $(a^2 - x^2)$ is an even function; hence, by the same reasoning as before, we have that

$$\boxed{\langle p\rangle = 0.}$$

c) For $\langle x^2 \rangle$ we have

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-a}^{a} x^2 |\Psi|^2 \, \mathrm{d}x = \frac{15}{16a^5} \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, \mathrm{d}x, \\
&= \frac{15}{8a^5} \int_0^a x^2 \left(a^4 - 2a^2 x^2 + x^4 \right) \, \mathrm{d}x, \\
&= \frac{15}{8a^5} \left[a^4 \int_0^a x^2 \, \mathrm{d}x - 2a^2 \int_0^a x^4 \, \mathrm{d}x + \int_0^a x^6 \, \mathrm{d}x \right], \\
&= \frac{15}{8a^5} \left[\frac{a^7}{3} - \frac{2a^7}{5} + \frac{a^7}{7} \right], \\
&= \frac{a^2}{8} \left[5 - 6 + \frac{15}{7} \right] = \frac{a^2}{8} \left[\frac{15}{7} - 1 \right], \\
&\boxed{\langle x^2 \rangle = \frac{a^2}{7}.}
\end{aligned}$$

d) For $\langle p^2 \rangle$ we have

$$\langle p^2 \rangle = \int_{-a}^{a} \Psi^* \left(-i\hbar \frac{\mathrm{d}^2}{\mathrm{d}x^2} \right) \Psi \, \mathrm{d}x = -\frac{15\hbar^2}{16a^5} \int_{-a}^{a} \Psi \frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} \, \mathrm{d}x,$$

where $\Psi^* = \Psi$. Doing the derivatives,

$$\frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x} [-2x] = -2,$$

so

$$\begin{aligned}
\langle p^2 \rangle &= \frac{15\hbar^2}{8a^5} \int_{-a}^{a} (a^2 - x^2) \, \mathrm{d}x, \\
&= \frac{15\hbar^2}{4a^5} \left[a^2 \int_0^a \mathrm{d}x - \int_0^a x^2 \, \mathrm{d}x \right], \\
&= \frac{15\hbar^2}{4a^5} \left[a^3 - \frac{a^3}{3} \right] = \frac{15\hbar}{4a^2} \cdot \frac{2}{3}, \\
&\boxed{\langle p^2 \rangle = \frac{5\hbar^2}{2a^2}.}
\end{aligned}$$

e) The standard deviation for x is

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{a^2}{7}} = \frac{a}{\sqrt{7}}.$$

f) The standard deviation for p is

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5\hbar^2}{2a^2}} = \frac{\hbar}{a} \sqrt{\frac{5}{2}}.$$

g) Plugging into the uncertainty relation:

$$\sigma_x \sigma_p = \frac{a}{\sqrt{7}} \cdot \frac{\hbar}{a} \sqrt{\frac{5}{2}} = \hbar \sqrt{\frac{5}{14}} = \frac{\hbar}{2} \sqrt{\frac{10}{7}} \geq \frac{\hbar}{2}.$$

The quantity $\sqrt{10/7}$ is definitely over 1, meaning the Heisenberg Uncertainty Principle is satisfied.

Problem 4.

We are given the wavefunction:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{i}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar}, \tag{4.1}$$

where $\psi_1(x)$ and $\psi_2(x)$ are two Hamiltonian eigenstates of the TISE for the infinite square well; the nth eigenstate is given by

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), \tag{4.2}$$

so

$$\psi_1(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right), \quad \text{and} \quad \psi_2(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right). \tag{4.3}$$

To determine whether this satisfies the Heisenberg Uncertainty Principle, we can start computing expectation values:

$$\langle x\rangle = \langle\Psi|\,\hat{x}\,|\Psi\rangle = \int x|\Psi|^2 \,\mathrm{d}x.$$

To make this a bit easier, I will define $\phi_1(t) = e^{-iE_1t/\hbar}$ and $\phi_2(t) = e^{-iE_2t/\hbar}$:

$$\begin{aligned}
|\Psi|^2 &= \frac{1}{2}|\psi_1|^2|\phi_1|^2 \,\mathrm{d}x + \frac{1}{2}|\psi_2|^2|\phi_2|^2 \,\mathrm{d}x + \frac{i}{2}\psi_1^*\psi_2\phi_1^*\phi_2 \,\mathrm{d}x - \frac{i}{2}\psi_1\psi_2^*\phi_1\phi_2^* \,\mathrm{d}x,\\
&= \frac{1}{2}\psi_1^2 + \frac{1}{2}\psi_1^2 + \frac{i}{2}\psi_1\psi_2(\phi_1^*\phi_2 - \phi_1\phi_2^*),
\end{aligned}$$

where I have dropped the "absolute value" bars (that signified modulus squared) for the ψ's since they are fully real, and $|\phi_1|^2 = |\phi_2|^2 = 1$. For the term in parentheses:

$$\begin{aligned}
\phi_1^*\phi_2 - \phi_1\phi_2^* &= \phi_1^*\phi_2 - (\phi_1^*\phi_2)^* = 2i\mathrm{Im}(\phi_1^*\phi_2),\\
&= 2i\mathrm{Im}\left(e^{iE_1t/\hbar}e^{-iE_2t/\hbar}\right) = 2i\mathrm{Im}\left(e^{-i(E_2-E_1)t/\hbar}\right),\\
&= 2i\sin\left(-\frac{E_2-E_1}{\hbar}t\right),\\
&= -2i\sin\left(\frac{E_2-E_1}{\hbar}t\right).
\end{aligned}$$

We know that

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},$$

meaning

$$E_2 - E_1 = \frac{3\pi^2\hbar^2}{2ma^2},$$

hence

$$\phi_1^*\phi_2 - \phi_1\phi_2^* = -2i\sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right).$$

The full result for the ϕ terms is

$$|\Psi|^2 = \frac{1}{2}\psi_1^2 + \frac{1}{2}\psi_2^2 + \psi_1\psi_2 \sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right).$$

Since our derivatives are always over x, I will stow away that sine term ϕ from now on for simplicity and only substitute back at the very end. Our expectation value for x is

$$\begin{aligned}
\langle x\rangle &= \frac{1}{2}\int x\psi_1^2 \,\mathrm{d}x + \frac{1}{2}\int x\psi_2^2 \,\mathrm{d}x + \phi\int x\psi_1\psi_2 \,\mathrm{d}x, \\
&= \frac{1}{a}\int_0^a x\sin^2\left(\frac{\pi x}{a}\right) \,\mathrm{d}x + \frac{1}{a}\int_0^a x\sin^2\left(\frac{2\pi x}{a}\right) \,\mathrm{d}x + \frac{2\phi}{a}\int x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \,\mathrm{d}x.
\end{aligned}$$

The first two integrals are almost identical; let's look at a general integral that encompasses both:

$$\begin{aligned}
\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right) \,\mathrm{d}x &= \frac{1}{2}\int_0^a x\left[1-\cos\left(\frac{2n\pi x}{a}\right)\right] \,\mathrm{d}x, \\
&= \frac{1}{2}\int_0^a x \,\mathrm{d}x - \frac{1}{2}\int_0^a x\cos\left(\frac{2n\pi x}{a}\right) \,\mathrm{d}x, \\
&= \frac{a^2}{4} - \left[\frac{xa}{4n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a - \frac{a}{4n\pi}\int_0^a \sin\left(\frac{2n\pi x}{a}\right) \,\mathrm{d}x, \\
&= \frac{a^2}{4} - \frac{a^2}{8n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{a}\right)\right]_0^a, \\
&= \frac{a^2}{4} - \frac{a^2}{8n^2\pi^2}[\cos(2n\pi) - \cos 0].
\end{aligned}$$

Now the term in brackets will always be zero, because any integer multiple of 2π always has a cosine of 1, and $\cos 0 = 1$. Hence,

$$\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right) \,\mathrm{d}x = \frac{a^2}{4},$$

which is independent of n, meaning that the first two terms of the expectation value for x both evaluate to this:

$$\langle x\rangle = \frac{a}{2} + \frac{2\phi}{a}\int x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \,\mathrm{d}x.$$

Looking now at the second integral:

$$\int x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \,\mathrm{d}x = 2\int x\sin^2\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) \,\mathrm{d}x,$$

using the double angle formula. Now, we can do integration by parts with

$$u = x \quad \text{and} \quad \mathrm{d}v = \sin^2\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right),$$

but the integral of $\mathrm{d}v$ needs another u-sub, but to avoid confusion with notation, I will use w as the substitution variable:

$$\begin{aligned}
\int \sin^2\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) &\rightarrow \left[\begin{aligned} w &= \sin(\pi x/a), \\ \mathrm{d}w &= \frac{\pi}{a}\cos(\pi x/a)\,\mathrm{d}x \end{aligned}\right] \rightarrow \frac{a}{\pi}\int w^2 \,\mathrm{d}w, \\
&= \frac{a}{3\pi}w^3 = \frac{a}{3\pi}\sin^3\left(\frac{\pi x}{a}\right).
\end{aligned}$$

Our integration by parts is now

$$= 2\left\{\left[\frac{xa}{3\pi}\sin^3\left(\frac{\pi x}{a}\right)\right]_0^a - \frac{a}{3\pi}\int \sin^3\left(\frac{\pi x}{a}\right)\,\mathrm{d}x\right\}.$$

It is easy to see that the sine term will be zero at both limits. Looking specifically at the new sine integral and using the power reduction formula:

$$\begin{aligned}\int \sin^3\left(\frac{\pi x}{a}\right)\,\mathrm{d}x &= \frac{1}{2}\int \sin\left(\frac{\pi x}{a}\right)\left[1-\cos\left(\frac{2\pi x}{a}\right)\right]\,\mathrm{d}x,\\ &= \frac{1}{2}\left[\int_0^a \sin\left(\frac{\pi x}{a}\right)\,\mathrm{d}x - \int \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x\right].\end{aligned}$$

Doing integration by parts on the second integral with

$$u = \sin\left(\frac{\pi x}{a}\right), \qquad \mathrm{d}v = \cos\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x,$$

$$\mathrm{d}u = \frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right), \quad \text{and} \quad v = \frac{a}{2\pi}\sin\left(\frac{2\pi x}{a}\right),$$

we get

$$\int \sin^3\left(\frac{\pi x}{a}\right)\,\mathrm{d}x =$$
$$\frac{1}{2}\left\{-\frac{a}{\pi}\left[\cos\left(\frac{\pi x}{a}\right)\right]_0^a - \left(\frac{a}{2\pi}\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\right]_0^a - \frac{1}{2}\int \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x.\right)\right\}.$$

The evaluated term with the sines will again be zero. Using similar methods as before on the final integral:

$$\begin{aligned}\int \sin^3\left(\frac{\pi x}{a}\right)\,\mathrm{d}x &= \frac{1}{2}\left[\frac{2a}{\pi} + \int_0^a \cos^2\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\,\mathrm{d}x\right],\\ &= \frac{1}{2}\left[\frac{2a}{\pi} + \frac{a}{\pi}\int_{-1}^{1} u^2\,\mathrm{d}u\right],\\ &= \frac{1}{2}\left(\frac{2a}{\pi} + \frac{2a}{3\pi}\right) = \frac{4a}{3\pi}.\end{aligned}$$

At last, our original integral is given by:

$$\int x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x = -\frac{2a}{3\pi}\left(\frac{4a}{3\pi}\right) = -\frac{8a^2}{9\pi^2},$$

so our expectation value is

$$\langle x\rangle = \frac{a}{2} + \frac{2\phi}{a}\left(-\frac{8a^2}{9\pi^2}\right) = \frac{a}{2} - \frac{16a}{9\pi^2}\sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right).$$

Since this was taken from the full wavefunction (not just the initial one), we can do

$$\langle p\rangle = m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = -\frac{8\hbar}{3a}\cos\left(\frac{3\pi^2\hbar}{2ma^2}t\right).$$

Now for $\langle x^2\rangle$:

$$\langle x^2\rangle = \frac{1}{a}\int_0^a x^2\sin^2\left(\frac{\pi x}{a}\right)\,\mathrm{d}x + \frac{1}{a}\int_0^a x^2\sin^2\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x + \frac{2\phi}{a}\int x^2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x.$$

Again looking at a general case for the first two integrals:

$$\int_0^a x^2\sin^2\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x,$$

$$\begin{aligned}
&= \frac{1}{2}\int_0^a x^2\,\mathrm{d}x - \frac{1}{2}\int_0^a x^2\cos\left(\frac{2n\pi x}{a}\right)\,\mathrm{d}x,\\
&= \frac{a^3}{6} - \frac{1}{2}\left[\frac{x^2 a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a - \frac{a}{2n\pi}\int_0^a x\sin\left(\frac{2n\pi x}{a}\right)\,\mathrm{d}x,\\
&= \frac{a^3}{6} + \frac{a}{2n\pi}\left\{\left[-\frac{xa}{2n\pi}\cos\left(\frac{2n\pi x}{a}\right)\right]_0^a + \frac{a}{4n\pi}\int_0^a \cos\left(\frac{2n\pi x}{a}\right)\,\mathrm{d}x\right\},
\end{aligned}$$

where in the second to last line, the sine would evaluate to zero at both limits. In this last line, the cosine integral will turn into a sine, which would evaluate to zero similarly. So,

$$\begin{aligned}
\int_0^a x^2\sin^2\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x &= \frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}\cos(2n\pi),\\
&= \frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}.
\end{aligned}$$

The first two terms in our equation for $\langle x^2\rangle$ are therefore:

$$\begin{aligned}
\frac{1}{a}\int_0^a x^2\sin^2\left(\frac{\pi x}{a}\right)\,\mathrm{d}x + \frac{1}{a}\int_0^a x^2\sin^2\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x &= \frac{1}{a}\left(\frac{a^3}{6} - \frac{a^3}{4\pi^2}\right) + \frac{1}{a}\left(\frac{a^3}{6} - \frac{a^3}{16\pi^2}\right),\\
&= a^2\left(\frac{1}{6} + \frac{1}{4\pi^2} + \frac{1}{6} + \frac{1}{16\pi^2}\right),\\
&= a^2\left(\frac{1}{3} + \frac{5}{16\pi^2}\right).
\end{aligned}$$

The final integral is super nasty, so I will just plug it into Mathematica:

$$\int x^2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\,\mathrm{d}x = -\frac{8a^3}{9\pi^2}.$$

So,

$$\langle x^2\rangle = a^2\left(\frac{1}{3} + \frac{5}{16\pi^2}\right) - \frac{16a^2}{9\pi^2}\sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right).$$

Lastly, we need to $\langle p^2\rangle$. Since we have no extra factors of x or anything, the time-dependent terms are going to cancel, and the cross terms will integrate to zero since the sines are orthogonal to each other, meaning all we have is:

$$\begin{aligned}
\langle p^2\rangle &= \langle\Psi|\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)^2|\Psi\rangle,\\
&= -\frac{2\hbar^2}{a}\left\{\int_0^a \sin\left(\frac{\pi x}{a}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\sin\left(\frac{\pi x}{a}\right)\right]\,\mathrm{d}x + \int_0^a \sin\left(\frac{2\pi x}{a}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\sin\left(\frac{2\pi x}{a}\right)\right]\,\mathrm{d}x\right.
\end{aligned}$$

Again looking at a general case first:

$$\begin{aligned}
\int_0^a \sin\left(\frac{n\pi x}{a}\right) \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\sin\left(\frac{n\pi x}{a}\right)\right] \, \mathrm{d}x &= -\frac{n^2\pi^2}{a^2}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \, \mathrm{d}x, \\
&= -\frac{n^2\pi^2}{2a^2}\left[\int_0^a \, \mathrm{d}x - \int_0^a \cos\left(\frac{2n\pi x}{a}\right) \, \mathrm{d}x\right], \\
&= -\frac{n^2\pi^2}{2a^2}\left\{a - \frac{a}{2n\pi}\left[\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a\right\}, \\
&= -\frac{n^2\pi^2}{2a}.
\end{aligned}$$

So,

$$\langle p^2 \rangle = -\frac{2\hbar^2}{a}\left(\frac{-\pi^2}{2a} - \frac{4\pi^2}{2a}\right) = \frac{\hbar^2\pi^2}{a^2}(1+4) = \frac{5\hbar^2\pi^2}{a^2}.$$

To recap:

$$\begin{aligned}
\langle x \rangle &= \frac{a}{2} - \frac{16a}{9\pi^2}\sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right), \\
\langle x^2 \rangle &= a^2\left(\frac{1}{3} + \frac{5}{16\pi^2}\right) - \frac{16a^2}{9\pi^2}\sin\left(\frac{3\pi^2\hbar}{2ma^2}t\right), \\
\langle p \rangle &= -\frac{8\hbar}{3a}\cos\left(\frac{3\pi^2\hbar}{2ma^2}t\right), \\
\langle p^2 \rangle &= \frac{5\hbar^2\pi^2}{a^2}.
\end{aligned}$$

The algebra will be insanely complex to find the standard deviations and then the uncertainty. So, I will just plug everything in Mathematica as the following function:

$$f(t) = \frac{2}{\hbar}\sigma_x(t)\sigma_p(t), \tag{4.4}$$

since both standard deviations will contain the time dependence. The factor of $2/\hbar$ is there so that we now are making sure that our function remains above 1. Then, I evaluate this over a few t values that give "angles" (meaning the quantity inside the sines and cosines) of 0, $\pi/2$, and $\pi/4$. These t values are $t = 0$, $t = (ma^2)/(3\pi\hbar)$, and $t = (ma^2)/(6\pi\hbar)$ respectively.

"θ"	t	$f(t)$
0	0	1.905
$\frac{\pi}{2}$	$\frac{ma^2}{3\pi\hbar}$	1.377
$\frac{\pi}{4}$	$\frac{ma^2}{6\pi\hbar}$	1.730

Table 1: Table of a few select values of t for Eq. (4.4).

It would appear that our wavefunction Eq. (4.1) satisfies the Heinsenberg Uncertainty Principle, as these values are a good bit over 1.

Problem 5.

We are given the wavefunction

$$\psi(x) = \begin{cases} Ae^{ikx}\cos\left(\dfrac{3\pi x}{L}\right), & -L/2 \leq x \leq L/2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

To find A, we just normalize:

$$\begin{aligned}
\int_{\infty}^{\infty} |\psi|^2 \, \mathrm{d}x &= A^2 \int_{-L/2}^{L/2} \cos^2\left(\frac{3\pi x}{L}\right) \, \mathrm{d}x, \\
&= 2A^2 \int_0^{L/2} \frac{1}{2}\left[1 + \cos\left(\frac{6\pi x}{L}\right)\right] \, \mathrm{d}x, \\
&= A^2 \left[\int_0^{L/2} \mathrm{d}x + \int_0^{L/2} \cos\left(\frac{6\pi x}{L}\right) \, \mathrm{d}x\right], \\
&= A^2 \left[\frac{L}{2} + \frac{L}{6\pi}\sin\left(\frac{6\pi x}{L}\right)\Big|_0^{L/2}\right], \\
&= A^2 L \left[\frac{1}{2} + \frac{1}{6\pi}\sin(3\pi)\right], \\
&= A^2 L \cdot \frac{1}{2}, \\
\rightarrow & \boxed{A = \sqrt{\frac{2}{L}}.}
\end{aligned}$$

Next, to determine the probability of finding the particle in between $x = 0$ and $x = L/4$, we just integrate the square of the wavefunction between that interval:

$$P\left(x \in [0,\ L/2]\right) = \int_0^{L/4} |\psi|^2 \, \mathrm{d}x.$$

Picking up from the third line in the previous step, changing the upper limit and dividing by 2 since we multiplied by 2 in that step due to the even integrand, we get:

$$\begin{aligned}
P\left(x \in [0,\ L/2]\right) &= \frac{A^2}{2}\left[\int_0^{L/4} \mathrm{d}x + \int_0^{L/4} \cos\left(\frac{6\pi x}{L}\right) \, \mathrm{d}x\right], \\
&= \frac{1}{L}\left[\frac{L}{4} + \frac{L}{6\pi}\sin\left(\frac{6\pi x}{L}\right)\Big|_0^{L/4}\right], \\
&= \frac{1}{4} + \frac{1}{6\pi}\sin(3\pi/2) = \frac{1}{4} - \frac{1}{6\pi}, \\
\rightarrow & \boxed{P\left(x \in [0,\ L/2]\right) = \frac{3\pi - 2}{12\pi} \approx 0.197.}
\end{aligned}$$