The integral of the <u>solid angle in D-dim</u> gives

$$\int d\Omega_D = \int_0^{\pi} d\theta_1 (\sin\theta_1)^{D-2} \cdots \int_0^{\pi} d\theta_{D-2} \sin\theta_{D-2} \int_0^{\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Unit sphere in D-dim

The Gauss integral is well known

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}} \Rightarrow (\sqrt{\pi})^D = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^D$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^D = \int d^D x \, e^{-\sum_{i=1}^{D} x_i^2} = \int d\Omega_D \int_0^{+\infty} dx \, x^{D-1} e^{-x^2} =$$

$$= \int d\Omega_D \frac{1}{2} \int_0^{+\infty} \underbrace{d(x^2)(x^2)^{D/2-1}}_{} e^{-x^2} = \int d\Omega_D \frac{1}{2} \Gamma(D/2)$$

we used $\frac{1}{2} 2x \, x^D \frac{dx}{x^2} = dx \, x^{D-1}$ and

$$\boxed{\Gamma(D) = \int_0^{+\infty} dt \, t^{D-1} e^{-t}} \leftarrow \text{Euler Gamma function}$$

$$\boxed{\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}}$$

D	$\Gamma(D/2)$	$\int d\Omega_D$
1	$\sqrt{\pi}$	2
2	1	2π
3	$\sqrt{\pi}/2$	4π
4	1	$2\pi^2$

The integral of the solid angle gives

$$\int d\Omega_D = \int_0^{\pi} d\theta_1 (\sin\theta_1)^{D-2} \cdots \int_0^{\pi} d\theta_{D-2} \sin\theta_{D-2} \int_0^{2\pi} d\theta_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Therefore, we obtain the following expression for $\Sigma(\not{p})$

$$\Sigma(p) = g_s^2 C_F (D-2) \not{p} \frac{\Gamma(2-D/2)}{(4\pi)^{D/2}} (-p^2)^{D/2-2} *$$

$$* \int_0^1 dx\, x^{D/2-2} (1-x)^{D/2-1}$$

The integral in this expression can be related to the $B(p,q)$ function (change of variable in B)

$$B(p,q) = \int_0^1 dx\, x^{p-1} (1-x)^{q-1}$$

and we finally obtain

$$\Sigma(p) = \frac{2 C_F g_s^2}{(4\pi)^{D/2}} \not{p} (-p^2)^{D/2-2} (D-1) B\left(\frac{D}{2}, \frac{D}{2}\right) \Gamma\left(2-\frac{D}{2}\right)$$

which is valid only for $D<3$ and $p^2<0$.

$\Sigma(p)$ is given as an explicit function of the space-time dimension and the momentum p $\Rightarrow$ analytical continuation to the region where D and p^2 are arbitrary complex numbers.

In fact, we observe that:

- $$\Gamma(2-D/2) = \int_0^{+\infty} t^{2-D/2-1} e^{-t}\, dt$$

$D = 4, 6, 8 \ldots$ are poles for this function.

ex.: $D=4 \Rightarrow \int_0^{+\infty} t^{-1} e^{-t}\, dt \to \infty$

as $\frac{e^{-t}}{t}$ does not converge at $t=0$.

- There is also a branch cut on the positive real axis in the p^2 plane:

$$(-p^2)^{D/2-2} \quad \to \quad (-p^2)^{\alpha} \underset{\alpha \to 0}{\approx} 1 + \ln(-p^2)\alpha + \frac{1}{2!}\ln^2(-p^2)\alpha^2$$

For $D \sim 4$ we can write

$$\Sigma(p) \sim \frac{C_F g_s^2}{(4\pi)^2} \frac{2}{4-D} \not{p}$$