The rotation operator (as before) (8)

$$U_{1R}(\hat{n},\theta)\, U_{2R}(\hat{n},\theta) = e^{-i\vec{J}_1\cdot\hat{n}\theta/\hbar}\, e^{-i\vec{J}_2\cdot\hat{n}\theta/\hbar}$$

$$[\hat{J}_i, \hat{J}_j] = i\hbar\,\varepsilon_{ijk}\hat{J}_k$$

$\hat{J}^2$ commutes with $\hat{J}_z$, $\hat{J}_1^2$ and $\hat{J}_2^2$ and we also have

$[\hat{J}^2, \hat{J}_{1z}] \neq 0 \quad [\hat{J}^2, \hat{J}_{2z}] \neq 0$

As a result, we can choose an alternative set of commuting operators which describes the same space exactly as did the old set $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}\}$.

We denote the new set as $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z^2\} \Rightarrow$

$$|j_1\, j_2;\, j\, m\rangle$$

and we therefore have

$$J^2 |j_1 j_2; j, m\rangle = j(j+1)\hbar^2\, |j_1 j_2; j, m\rangle$$

$$J_z\, |J_1, J_2; j, m\rangle = m\hbar\, |j_1, j_2; j, m\rangle$$

$|j_1, j_2; j, m\rangle$ are also eigenkets of J_1^2 end J_2^2

completeness $\rightarrow \quad \sum_{m=-j}^{+j} |j_1 j_2; j\, m\rangle\langle j_1 j_2; j\, m| = 1$

orthonormality $\rightarrow \quad \langle j_1, j_2; j\, m\,|\, j_1' j_2'; j', m'\rangle = \delta_{mm'}\,\delta_{jj'}$

We tacitly assume that j_1 and j_2 in a given problem are given fixed.

Using the completeness relation (*) at pag 7 for the old set $\{|j_1 j_2; m_1 m_2\rangle\}$ we may express a member of the new set $\{|j_1, j_2; j, m\rangle\}$ as

$$|j_1,j_2;j,m\rangle = \sum_{m_1=-j_1}^{+j_1}\sum_{m_2=-j_2}^{+j_2} |j_1,j_2;m_1,m_2\rangle \underbrace{\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle}$$

$$\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle = \text{Clebsch-Gordan coeff.}$$

Properties of C-G coeff.

1. $\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle = 0$ unless $m = m_1 + m_2$

In fact we can show that

$$\langle j_1,j_2;m_1,m_2|J_z - J_{1z} - J_{2z}|j_1,j_2;j,m\rangle = 0$$

$$(m - m_1 - m_2)\langle j_1,j_2;j,m|j_1,j_2;j,m\rangle = 0$$

and the C-G is zero if $m \neq m_1 + m_2$.

2. C-G are taken real by convention

$$\langle j_1,j_2,m_1,m_2|j_1,j_2,j\,m\rangle = \langle j_1\, j_2; j\,m|j_1\, j_2; m_1, m_2\rangle$$

3. Orthonormality

$$\langle j_1, j_2; j, m | j_1' j_2'; j', m' \rangle = \delta_{m,m'} \delta_{j,j'}$$

In fact, one can see it by considering

$$\sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \langle j_1, j_2; j, m | j_1, j_2; m_1, m_2 \rangle \langle j_1, j_2; m_1, m_2 | j_1', j_2'; j', m' \rangle = \delta_{mm'} \delta_{jj'}$$

substituting $j = j'$ and $m = m'$ above we get

$$\sum_{m_1, m_2} | \langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m \rangle |^2 = 1$$

In a similar fashion, if we use the orthonormality condition on the old basis set and insert the completeness relation we obtain

$$\sum_{j} \sum_{m=-j}^{+j} | \langle j_1 j_2; m_1 m_2 | j_1 j_2 j m \rangle |^2 = 1$$

4. The C-G coeff. vanish unless

$$| j_1 - j_2 | \leq j \leq j_1 + j_2$$

$j \leq j_1 + j_2$ is trivial because

$$m_1^{max} = j_1 \quad \text{and} \quad m_2^{max} = j_2 \Rightarrow$$

$$\Rightarrow m^{max} = j_1 + J_2 \qquad \text{but} \quad m^{max} = J$$

$$\Rightarrow \quad j \leq j_1 + j_2$$

$|j_1 - j_2| \leq j$ is more complicated to show

We have $(2j_1+1)(2j_2+1)$ number of states.

for each j we have $2j+1$ states.

$$\Rightarrow \sum_{j=j_{min}}^{j_{max}} (2j+1) = (2j_1+1)(2j_2+1)$$

LHS : $\sum_{j=j_{min}}^{j_{max}} (2j+1) \longrightarrow S_n = \frac{n(a_1+a_n)}{2}$

$$\Rightarrow \sum_{j=j_{min}}^{j_{max}} (2j+1) = \frac{(j_{max} - j_{min} + 1)}{2} (2j_{max} + 1 + 2j_{min} + 1)$$

$$= (j_{max} - j_{min} + 1)(j_{max} + j_{min} + 1)$$

$$= (j_{max} + 1)^2 - j_{min}^2$$

$$= (j_1 + j_2 + 1)^2 - j_{min}^2$$

If we equate this to the RHS

$$(j_1 + j_2 + 1)^2 - j_{min}^2 = (2j_1+1)(2j_2+1) \Rightarrow$$

$$j_{min}^2 = (j_1 - j_2)^2 \Rightarrow j_{min} = |j_1 - j_2|$$

$$\Rightarrow |j_1 - j_2| \leq j \leq j_1 + j_2$$

5. For a given j, the possible m values are

$$-j \leq m \leq j$$

Calculation of the C-G coeff.

This is normally done by using recursion relations

Note that

$$J_\pm |j_1 j_2; j m\rangle = (J_{1\pm} + J_{2,\pm})|j_1, j_2,; j, m\rangle =$$

$$= \sum_{m_1', m_2'} \langle j_1 j_2; m_1' m_2' | j_1 j_2; j, m\rangle (J_{1,\pm} + J_{2,\pm}) |j_1, j_2; m_1' m_2'\rangle$$

Using the eigenvalues of the ladder operators on both sides:

$$\sqrt{(j \mp m)(j \pm m+1)}\, |j_1, j_2; j, m\pm 1\rangle = \sum_{m_1' m_2'} \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle$$

$$\Big(\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)}\, |j_1, j_2; m_1' \pm 1, m_2'\rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)}\, |j_1, j_2; m_1', m_2' \pm 1\rangle\Big)$$

Now we multiply both sides of the above eqn by $\langle j_1, j_2; m_1, m_2|$ and use orthonormality.

On the LHS we obtain

$$\sqrt{(j \mp m)(j \pm m+1)}\, \underbrace{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\pm 1\rangle}_{\hookrightarrow\ C-G}$$

On the reight we have

$$\sum_{m_1' m_2'} \langle j_1 j_2; m_1' m_2' | j_1 j_2; j m \rangle \sqrt{(j_1 \mp m_1')(j_1 \pm m_1'+1)}\; \underbrace{\langle j_1, j_2; m_1 m_2 | j_1, j_2; m_1' \pm 1, m_2' \rangle}_{=1 \text{ if } m_1 = m_1' \pm 1 \,\wedge\, m_2 = m_2'}$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2'+1)}\; \underbrace{\langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1', m_2' \pm 1 \rangle}_{=1 \text{ if } m_1 = m_1' \,\wedge\, m_2 = m_2' \pm 1}$$

and using orthornormality, and the result obtained for the LHS, we obtain:

$$\sqrt{(j \mp m)(j \pm m+1)}\; \overset{\text{LHS}}{\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1 \rangle} =$$

$$\left.\begin{array}{l} \sqrt{(j_1 \mp m_1+1)(j_1 \pm m_1)}\; \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \rangle + \\ \sqrt{(j_2 \mp m_2+1)(j_2 \pm m_2)}\; \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \rangle \end{array}\right\} \text{RHS}$$

On the left-hand side, the C-G coeffs reequire that $m \pm 1 = m_1 + m_2$.

The recursion relation allows one to obtain the LHS coeff with $m \pm 1$ fraun a combination of coeff. on the RHS with m.

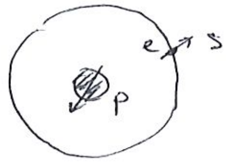
Example 1 — Sum of two spin 1/2 Angular momenta

$\hat{J}_1 \to \hat{S}_1$

$\hat{J}_2 \to \hat{S}_2$

$\hat{J} \to \hat{S}$

We can think about proton and electron spins in the Hydrogen atom



We clearly have $[\hat{S}_1, \hat{S}_2] = 0$ and S is such that

$$s_1 - s_2 \leq S \leq s_1 + s_2$$

For
$$\begin{cases} S = 0 \Rightarrow m = 0 & \text{only one state (singlet)} \\ S = 1 \Rightarrow m = -1, 0, 1 & \text{3 states (triplet)} \end{cases}$$

$s_1 = 1/2$ has 2 states

$s_2 = 1/2$ has 2 states

$\Rightarrow$ 4 states in total

Let us look at the $|s_1, s_2; S, m\rangle$ with $m = 1$:

$|1/2, 1/2; 1\,1\rangle$ is uniquely obtained from

$$|1/2, 1/2; 1/2\; 1/2\rangle \text{ of } |s_1\, s_2; m_1\, m_2\rangle$$

That is, $\quad |1/2, 1/2; 1, 1\rangle = |1/2, 1/2; 1/2, 1/2\rangle$

As we learn from the discussion at Pag 12,

We consider

$$\hat{S}_\pm |s_1 s_2; s\, m\rangle = (\hat{S}_{1\pm} + \hat{S}_{2\pm}) |s_1 s_2; s\, m\rangle$$

$$= \sum_{m_1' m_2'} \langle s_1 s_2; m_1' m_2' | s_1 s_2; s\, m\rangle (\hat{S}_{1\pm} + \hat{S}_{2\pm}) |s_1 s_2; m_1' m_2'\rangle$$

Let's start with S_- and recall the general result $\hat{J}_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$

$$\hat{S}_- |1/2, 1/2; 1, 1\rangle = (\hat{S}_{1-} + \hat{S}_{2-}) |1/2, 1/2; 1/2, 1/2\rangle \quad (**)$$

This is because $|1/2, 1/2; 1 1\rangle = |1/2\ 1/2; 1/2\ 1/2\rangle$ as we found before

But using $(**)$ we find that

$$\sqrt{2}\, |1/2, 1/2; 1, 0\rangle = |1/2, 1/2; -1/2, 1/2\rangle + |1/2, 1/2; 1/2, -1/2\rangle.$$

Sandwiching with $\langle s_1, s_2; m_1, m_2|$, we find the CG coefficients as follows:

$$\langle 1/2, 1/2; 1/2, 1/2 | 1/2\ 1/2; 1\ 1\rangle = 1$$

and

$$\langle 1/2, 1/2; 1/2, -1/2 | 1/2, 1/2; 1, 0\rangle = 1/\sqrt{2}$$

and

$$\langle 1/2, 1/2; -1/2, 1/2 | 1/2, 1/2; 1, 0\rangle = 1/\sqrt{2}$$

The case $|s_1 s_2; s\, m\rangle$ with $m=-1$ can be obtained by symmetry

$$\langle 1/2\ 1/2; -1/2, -1/2 | 1/2, 1/2; 1, -1\rangle = 1$$

For $s=0$ $m=0$ and we only have one state that is orthogonal to $|1/2, 1/2; 1, 0\rangle$.

We can find this state observing that the orthogonal state $|1/2, 1/2; 0, 0\rangle$ must satisfy

$$\langle 1/2\ 1/2; 0\,0 | 1/2, 1/2; 1, 0\rangle$$

$$\left[\langle 1/2\ 1/2; 1/2\ -1/2|\alpha + \langle 1/2, 1/2; -1/2, 1/2|\beta\right]\left[\frac{1}{\sqrt{2}}|1/2\ 1/2; 1/2\ -1/2\rangle + 1/\sqrt{2}\,|1/2, 1/2; -1/2, 1/2\rangle\right]$$

$= 0$ (orthogonality) $\Rightarrow$ $\dfrac{\alpha}{\sqrt{2}} + \dfrac{\beta}{\sqrt{2}} = 0 \Rightarrow \alpha = -\beta$

and imposing $\left|\,|1/2, 1/2; 0\,0\rangle\right|^2 = 1 \Rightarrow \alpha = \pm 1/\sqrt{2} \Rightarrow$

$$|1/2, 1/2; 0, 0\rangle = \frac{1}{\sqrt{2}}\left(|1/2, 1/2; 1/2, -1/2\rangle - |1/2, 1/2, -1/2, 1/2\rangle\right)$$

This implies that the C-G coeff. are

$$\langle 1/2, 1/2; 1/2, -1/2 \mid 1/2, 1/2; 0, 0 \rangle = 1/\sqrt{2}$$

$$\langle 1/2, 1/2; -1/2, 1/2 \mid 1/2, 1/2; 0, 0 \rangle = -1/\sqrt{2}$$

Convention $m = 1/2 \rightarrow |\uparrow\rangle$ or α

$m = -1/2 \rightarrow |\downarrow\rangle$ or β

Triplet

$m = +1 \quad |\uparrow\uparrow\rangle$

$m = 0 \quad \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right]$

$m = -1 \quad |\downarrow\downarrow\rangle$

Singlet $s = 0 \quad m = 0$

$$\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$$

this is antisymmetric under spin exchange.