Clebsch-Gordan Coefficients

They are numbers that arise in angular momentum addition in QM:

→ expansion coeff. of total angular momentum eigenstates

$[J_k, J_l] \equiv J_k J_l - J_l J_k = i\hbar \varepsilon_{klm} J_m \qquad k,l,m = \{1,2,3\} = \{x,y,z\}$

$\vec{J} = (J_x, J_y, J_z) \qquad \vec{J}^2 = J_x^2 + J_y^2 + J_z^2$

Raising and lowering operators

$J_\pm = J_x \pm i J_y$

$[J^2, J_k] = 0$

Spherical basis

$\vec{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle \qquad j = \{0, 1/2, 1, 3/2 \ldots\}$

$J_z |j, m\rangle = \hbar m |j, m\rangle \qquad m \in \{-j, -j+1, \ldots j\}$

$J_\pm |j, m\rangle = \hbar C_\pm(j, m) |j, m \pm 1\rangle$

Ladder coefficients

$C_\pm(j, m) = \sqrt{j(j+1) - m(m\pm 1)} = \sqrt{(j \mp m)(j \pm m + 1)}$

$\langle j\, m | j', m'\rangle = \delta_{jj'} \delta_{mm'}$

Tensor product space

Let us consider two physically different angular momenta j_1 and j_2, for example s and ℓ.

The angular momentum (AM) operators act on a space V_1 of dim $2j_1+1$ and also on V_2 of dim $2j_2+1$.

The "total AM" operators act on the tensor product space $V_1 \otimes V_2$ which has dim $(2j_1+1)(2j_2+1)$

tensor product definition:

V & W vector spaces

$V \otimes W$ is called tensor product of V and W

= vector space to which is associated a bilinear map $V \times W \to V \otimes W$ that maps a pair (v, w) with $v \in V$ and $w \in W$ to an element of $V \otimes W$ which is denoted as $v \otimes w$

Example V $\qquad$ W

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$ $\qquad$ $\vec{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ in $\mathbb{R}^2$

We can generate new vectors in two ways:

1. $\vec{v} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \vec{w} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\rightsquigarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow (\vec{v}, \vec{w}) \in \mathbb{R}^3 \oplus \mathbb{R}^2$

   stack!

   direct sum

2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1\cdot 4 \\ 1\cdot 5 \\ 2\cdot 4 \\ 2\cdot 5 \\ 3\cdot 4 \\ 3\cdot 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 10 \\ 12 \\ 15 \end{bmatrix}$

   multiply!

$$\vec{v} \otimes \vec{w} \in \mathbb{R}^3 \otimes \mathbb{R}^2$$

tensor product

A basis of $V \otimes W$ is a set of $\vec{v}_i \otimes \vec{w}_j$

$i = 1, 2, 3$ $\quad j = 1, 2$ $\quad$ such that

	$\vec{w}_1$	$\vec{w}_2$
$\vec{v}_1$	$\vec{v}_1 \otimes \vec{w}_1$	$\vec{v}_1 \otimes \vec{w}_2$
$\vec{v}_2$	$\vec{v}_2 \otimes \vec{w}_1$	$\vec{v}_2 \otimes \vec{w}_2$
$\vec{v}_3$	$\vec{v}_3 \otimes \vec{w}_1$	$\vec{v}_3 \otimes \vec{w}_2$

$\Rightarrow$ 6-dim space spanned by $\{\vec{v}_1 \otimes \vec{w}_1, \vec{v}_1 \otimes \vec{w}_2, \ldots\}$

Let V_1 be the $(2j_1+1)$-dimensional vector space

$|j_1 m_1\rangle \qquad m_1 \in \{-j_1, -j_1+1, \dots j_1\}$

and V_2 the $(2j_2+1)$-dim vector space

$|j_2, m_2\rangle \qquad m_2 \in \{-j_2, -j_2+1, \dots j_2\}$

Tensor product space $V_3 \equiv V_1 \otimes V_2$ has $(2j_1+1)(2j_2+1)$-dimensional basis

$$|j_1 m_1 j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

with $m_1 \in \{-j_1, -j_1+1, \dots j_1\}$ $\quad m_2 \in \{-j_2, j_2+1, \dots j_2\}$

AM operators act on states in V_3 as follows

$$(\vec{J} \otimes \mathbb{1})\, |j_1 m_1 j_2 m_2\rangle \equiv \vec{J}\, |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

and

$$\mathbb{1} \otimes \vec{J}\, |j_1 m_1 j_2 m_2\rangle \equiv |j_1 m_1\rangle \otimes \vec{J}\, |j_2 m_2\rangle$$

$\mathbb{1}$ = identity operator

Total AM operators

$$\vec{J} \equiv \vec{J}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{J}_2 \qquad [J_k, J_l] = i\hbar\, \varepsilon_{klm} J_m$$

A set of coupled eigenstates exists:

$$\hat{J}^2 \,|\, [j_1 j_2] J M \rangle = \hbar^2 J(J+1) \,|\, [j_1 j_2] J M \rangle$$

$$\hat{J}_z \,|\, [j_1 j_2] J M \rangle = \hbar M \,|\, [j_1 j_2] J M \rangle$$

$$M \in \{-J, -J+1, \dots J\}$$

$[j_1 j_2]$ is normally omitted.

(*) The total AM J must satisfy (triangular condition)

$$|j_1 - j_2| \leq J \leq j_1 + j_2$$

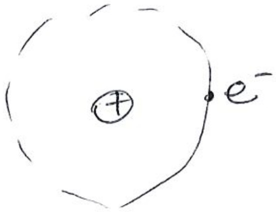
such that the three non-negative integer or half integer values could correspond to the three sides of a triangle.

(*) The total number of total AM states is necessary equal to the dimension of V_3

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1)$$

Essentially the tensor product decomposes as direct sum of $(2J+1)$-dim spaces where J ranges from $|j_1 - j_2|$ to $j_1 + j_2$ in increments of 1

Spin-orbit interaction



electron has spin 1/2

interaction between $\vec{S}$ and $\vec{L}$ where $\vec{L}$ is the orbital angular momentum

$H_{so} \sim \vec{S} \cdot \vec{L}$

$$\vec{J} = \vec{L} + \vec{S} \quad \Rightarrow \quad \vec{L} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}$$

↳ direct sum $\oplus$

The Hilbert state space of the particle is spanned by $\{|\vec{x}\rangle\}$ (position kets) and the 2-dim spin space spanned by $|\uparrow\rangle$ and $|\downarrow\rangle$ (or $|+\rangle$, $|-\rangle$)

In presence of weak spin-orbit coupling the Hilbert space of the wave functions is the product of the position space and the spin space:

$$|\vec{x}, \pm\rangle = |\vec{x}\rangle \otimes |\pm\rangle \qquad \langle \vec{x}, \pm | \alpha \rangle = \psi_{\pm}(\vec{x})$$

Rotation operator

$$U_R(\hat{n}, \theta) = e^{-i \vec{J} \cdot \hat{n} \theta / \hbar} = e^{-i \vec{L} \cdot \hat{n} \theta / \hbar} \, e^{-i \vec{S} \cdot \hat{n} \theta / \hbar}$$

Total wave function

spin part

$$\psi_\alpha(\vec{x}) = \psi(\vec{x}) \chi_\alpha \qquad \alpha = \uparrow \text{ or } \downarrow$$

space part

$$\psi_\alpha(\vec{x}) = \begin{pmatrix} \psi_\uparrow(\vec{x}) \\ \psi_\downarrow(\vec{x}) \end{pmatrix}$$ two-component tensor

Then we have $\hat{L}^2$ and $\hat{L}_z$ for the orbital AM and $\hat{S}^2$ and S_z for the spin AM.

Composite system $\{\hat{L}^2, \hat{L}_z, \hat{S}, \hat{S}_z\}$

state space $\rightarrow$ $|\ell, s, m_\ell, m_s\rangle = |\ell, m_\ell\rangle \otimes |s, m_s\rangle$

is a product of independent states where

$$L^2 |\ell\, m_\ell\rangle = \ell(\ell+1)\hbar^2 |\ell, m_\ell\rangle$$

$$L_z |\ell, m_\ell\rangle = m_\ell \hbar |\ell, m_\ell\rangle$$

$$S^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$

$$S_z |s, m_s\rangle = m_s \hbar |s, m_s\rangle$$

To be completely general, let us call

$$\hat{L} = \hat{J}_1 \quad \text{and} \quad \hat{S} = \hat{J}_2$$

$$\vec{J} = \vec{J}_1 \otimes \mathbb{1} \oplus \mathbb{1} \otimes \vec{J}_2 = \vec{J}_1 + \vec{J}_2$$

We can use the $\hat{\vec{J}}_1 + \vec{J}_2$ notation to stress the fact that we are dealing with vector operators

Commutation relations for $\hat{J}_1$ and $\hat{J}_2$

$$[\hat{J}_{1i}, \hat{J}_{1j}] = i\hbar \varepsilon_{ijk} \hat{J}_{1k}$$

$$[\hat{J}_{2i}, \hat{J}_{2j}] = i\hbar \varepsilon_{ijk} \hat{J}_{2k}$$

$$[\hat{J}_{1i}, \hat{J}_{2j}] = 0 \Rightarrow$$ $\hat{J}_1$ & $\hat{J}_2$ can have a common set of eigenstates

We can choose $\hat{J}_1^2$, $\hat{J}_{1z}$, $\hat{J}_2^2$, and $\hat{J}_{2z}$ to be a set of operators that have common eigenstates:

$$|j_1 j_2; m_1 m_2\rangle \equiv |j_1, m_1\rangle |j_2, m_2\rangle$$

↳ tensor product

Therefore we have

$$\hat{J}_1^2 |j_1, j_2; m_1, m_2\rangle = j_1(j_1+1)\hbar^2 |j_1 j_2; m_1, m_2\rangle$$

$$\hat{J}_{1z} |j_1, j_2; m_1, m_2\rangle = m_1 \hbar |j_1 j_2; m_1, m_2\rangle$$

$$\hat{J}_2^2 |j_1, j_2; m_1 m_2\rangle = j_2(j_2+1)\hbar^2 |j_1 j_2; m_1, m_2\rangle$$

$$\hat{J}_{2z} |j_1, j_2; m_1 m_2\rangle = \hbar m_2 |j_1 j_2; m_1, m_2\rangle$$

The dim of the space to which $\hat{J}_1$ and $\hat{J}_2$ belong is $(2j_1+1)(2j_2+1)$ and the set $|j_1 j_2; m_1 m_2\rangle$ states form a complete orthonormal set

$$(*) \quad \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |j_1 j_2; m_1 m_2\rangle\langle j_1, j_2; m_1, m_2| = 1 \leftarrow \text{(completeness)}$$

(orthonorm) → $\langle j_1, j_2; m_1, m_2 | j_1' j_2'; m_1', m_2'\rangle = \delta_{j_1 j_1'} \delta_{j_2 j_2'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}$