Homework 11

Problem 1

$$\langle q_f t_f | q_i t_i \rangle = \lim_{n\to\infty} \int \prod_{j=1}^{n} dq_i \int \prod_{j=1}^{n} \frac{dp_j}{(2\pi)\hbar}$$
$$\times \exp\left\{ \frac{i\tau}{\hbar} \sum_{j=0}^{n} \left[p_j \cdot \frac{q_{j+1} - q_j}{\tau} - \frac{p_j^2}{2m} - V(\bar{q}_j) \right] \right\}$$

· Using

$$\int_{-\infty}^{\infty} dx \; e^{-ax^2 + bx + c} = \sqrt{\frac{\pi}{a}} \exp\left\{ \frac{b^2}{4a} + c \right\}$$

- we have $n+1$ of these, w/ $a = \frac{i\tau}{\hbar} \cdot \frac{1}{2m}$, $b = \frac{i\tau}{\hbar} \cdot \frac{q_{j+1} - q_j}{\tau}$, $c = -\frac{i\tau}{\hbar} V(\bar{q}_j)$, so

$$\langle q_f t_f | q_i t_i \rangle = \lim_{n\to\infty} \left(\pi \cdot \frac{2\hbar m}{i\tau} \right)^{(n+1)/2}$$
$$\times \int \prod_{j=1}^{n} dq_j \exp\left\{ \frac{i\tau}{\hbar} \sum_{j=0}^{n} \left[\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\tau} \right)^2 - V \right] \right\}$$

- The sum turns to integral for $n \to \infty$:

$$\langle q_f t_f | q_i t_i \rangle = N \int Dq \; \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} \left(\frac{m\dot{q}^2}{2} - V \right) \right]$$

- but $\frac{m\dot{q}^2}{2} = \frac{p^2}{2m} \rightarrow \;= L$, so

$$\boxed{\langle q_f t_f | q_i t_i \rangle = N \int Dq \; \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} L \, dt \right]}$$

Problem 2

$$K_0 = \lim_{n\to\infty} \left(\frac{m}{i\hbar\tau}\right)^{(n+1)/2} \int_{-\infty}^{\infty} \prod_{i=1}^{n}$$

$$\cdot \exp\left[\frac{im}{2\hbar\tau} \sum_{j=0}^{n} (x_{j+1}-x_j)^2\right]$$

- with

$$\int_{-\infty}^{\infty} \exp\left\{i\lambda\left[(x_1-a)^2 + (x_2-x_1)^2 + \cdots + (b-x_n)^2\right] dx_1 \cdots dx_n\right]\Big\}$$

$$= \left[\frac{(i\pi)^n}{(n+1)\lambda^n}\right]^{1/2} \exp\left\{\frac{i\lambda}{n+1}(b-a)^2\right\}$$

- here $\lambda = \frac{m}{2\hbar\tau}$, so

$$K_0 = \cancel{\left(\frac{m}{i\hbar\tau}\right)^{n/2}} \left(\frac{m}{i\hbar\tau}\right)^{1/2} \cdot \cancel{\left(\frac{i\pi \cdot 2\hbar\tau}{m}\right)^{n/2}} \cdot \sqrt{\frac{1}{n+1}}$$

$$\times \exp\left\{\frac{im}{2\hbar(n+1)\tau}(x_f - x_i)^2\right\}$$

- since $(n+1)\tau = t_f - t_i$,

$$K_0(x_f t_f; x_i t_i) = \left(\frac{m}{i\hbar(t_f-t_i)}\right)^{1/2} \exp\left\{\frac{im(x_f-x_i)^2}{2\hbar(t_f-t_i)}\right\}$$

- since we must have $t_f > t_i$, we can insert a $\theta(t_f - t_i)$ to get the final result:

$$\boxed{K_0(x_f t_f; x_i t_i) = \theta(t_f - t_i)\left(\frac{m}{i\hbar\Delta t}\right)^{1/2} \exp\left[\frac{im(\Delta x)^2}{2\hbar\Delta t}\right]}$$