

Solutions to Exercises in Modern Condensed Matter Physics

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Note to Instructors

For a few of the more difficult problems, we include notes to the instructor suggesting simplifications, specializations and hints that the instructor may wish to give the students when assigning those problems.

Note also that there are some useful exercises within the appendices of the textbook.

Chapter 2

Ex. 2.1

(i)

The radiated electric field is

$$\vec{\epsilon}_a \approx r_e \frac{e^{ikR_D}}{R_D} \left[\hat{n} \times (\hat{n} \times \vec{E}_{\text{in}}) \right] e^{-i\omega t} e^{-i\vec{q} \cdot \vec{r}}. \quad (1)$$

Replace $e^{-i\vec{q} \cdot \vec{r}}$ by $\langle e^{-i\vec{q} \cdot \vec{r}} \rangle = f(\vec{q})$. The vector $(\hat{n} \times \vec{E}_{\text{in}})$ is perpendicular to both \hat{n} and to \vec{E}_{in} and has length $|E_{\text{in}} \sin \theta|$. Hence $|\hat{n} \times (\hat{n} \times \vec{E}_{\text{in}})|^2 = E_{\text{in}}^2 \sin^2 \theta$. Thus

$$|\epsilon_a|^2 = E_{\text{in}}^2 \frac{r_e^2}{R_D^2} \sin^2 \theta |f(\vec{q})|^2. \quad (2)$$

The total radiated power passing through a sphere of radius R_D is

$$P = c R_D^2 \int d\Omega \left(\frac{\epsilon_a^2}{8\pi} \times 2 \right) \quad (3)$$

$$= \frac{c}{4\pi} r_e^2 E_{\text{in}}^2 \int d\Omega \sin^2 \theta |f(\vec{q})|^2. \quad (4)$$

Let us normalize the incident electric field to that associated with a single photon in the normalization volume L^3

$$\frac{E_{\text{in}}^2}{4\pi} = \frac{\hbar\omega}{L^3} = \frac{\hbar ck}{L^3} \quad (5)$$

which yields

$$P = \hbar c^2 k \frac{r_e^2}{L^3} \int d\Omega \sin^2 \theta |f(\vec{q})|^2. \quad (6)$$

(ii)

Now compare this to the quantum result using the photon scattering matrix element in Eq. (2.28)

$$M = r_e f(\vec{q}) \wedge_k^2 \hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{\epsilon}_{\vec{k}'\lambda}. \quad (7)$$

Fermi's Golden Rule for the transition rate is

$$\Gamma = \frac{2\pi}{\hbar} \sum_{\lambda'} \frac{L^3}{(2\pi)^3} \int d^3 k' r_e^2 \wedge_k^4 [\hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{\epsilon}_{\vec{k}'\lambda'}]^2 \delta(\hbar\omega - \hbar c k') |f(\vec{q})|^2. \quad (8)$$

Noting that the two polarization vectors $\hat{\epsilon}_{\vec{k}\lambda}$ and the vector \hat{k}' are all mutually perpendicular, we find $\sum_{\lambda'} [\hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{\epsilon}_{\vec{k}'\lambda'}]^2 = 1 - [\hat{\epsilon}_{\vec{k}\lambda} \cdot \hat{k}']^2 = 1 - \cos^2 \theta = \sin^2 \theta$. The radiated power is

$$P = \hbar c^2 k \frac{r_e^2}{L^3} \int d\Omega \sin^2 \theta |f(\vec{q})|^2, \quad (9)$$

in agreement with the result from the semiclassical calculation.

Ex. 2.2

$$S(\vec{q}) = \frac{1}{N} < |W(\vec{q})|^2 > = \frac{1}{N} < \sum_{i=1}^N e^{i\vec{q} \cdot \vec{r}_i} \sum_{j=1}^N e^{-i\vec{q} \cdot \vec{r}_j} > \quad (10)$$

$$= \frac{1}{N} < \sum_{i=j}^N e^{i\vec{q} \cdot \vec{r}_i - i\vec{q} \cdot \vec{r}_j} > + \frac{1}{N} < \sum_{i \neq j}^N \int d^3 \vec{r} d^3 \vec{r}' e^{i\vec{q} \cdot \vec{r}_i - i\vec{q} \cdot \vec{r}_j} \delta(\vec{r} - \vec{r}_i) \delta(\vec{r}' - \vec{r}_j) > \quad (11)$$

$$= \frac{1}{N} N + \frac{1}{N} \int d^3 \vec{r} d^3 \vec{r}' e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} < \sum_{i \neq j}^N \delta(\vec{r} - \vec{r}_i) \delta(\vec{r}' - \vec{r}_j) > \quad (12)$$

Remembering

$$< \sum_{i \neq j}^N \delta(\vec{r} - \vec{r}_i) \delta(\vec{r}' - \vec{r}_j) > = n^{(2)}(\vec{r}_i - \vec{r}_j),$$

then obviously

$$S(\vec{q}) = 1 + \frac{1}{N} \int d^3 \vec{r} d^3 \vec{r}' e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} n^{(2)}(\vec{r}' - \vec{r}) = 1 + n \int d^3 \vec{r} e^{i\vec{q} \cdot \vec{r}} g(\vec{r})$$

where we used $N/V = n$ and $n^{(2)}(\vec{R}) = n^2 g(\vec{R})$.

P.S. " <>" indicates thermal average in liquid or amorphous materials. It is unnecessary only for perfect lattices. Generally " <>" must be in the formula.

Chapter 3

Ex. 3.1

•	SC	FCC	BCC
radius	$a/2$	$\frac{\sqrt{2}}{4}a$	$\frac{\sqrt{3}}{4}a$
volume of one sphere	$\frac{\pi}{6}a^3$	$\frac{\sqrt{2}\pi}{24}a^3$	$\frac{\sqrt{3}\pi}{16}a^3$
number of sites in unit cell	1	4	2
volume fraction	$\frac{\pi}{6}$	$\frac{\sqrt{2}}{6}\pi$	$\frac{\sqrt{3}}{8}\pi$

Ex. 3.2

$$\begin{aligned} FCC \ 8 \times \frac{1}{8} + 6 \times \frac{1}{2} &= 4 \\ BCC \ 8 \times \frac{1}{8} + 1 &= 2 \end{aligned}$$

Ex. 3.3

(i)

Suppose the lattice spacing is a . The three primitive vectors are

$$\begin{aligned} \vec{a}_1 &= a(0, \frac{1}{2}, \frac{1}{2}) \\ \vec{a}_2 &= a(\frac{1}{2}, 0, \frac{1}{2}) \\ \vec{a}_3 &= a(\frac{1}{2}, \frac{1}{2}, 0) \end{aligned}$$

Thus, the coordinates of the four points of the tetrahedron spanned by the three vectors are

$$\begin{aligned} O &= (0, 0, 0) \\ A &= a(0, \frac{1}{2}, \frac{1}{2}) \\ B &= a(\frac{1}{2}, 0, \frac{1}{2}) \\ C &= a(\frac{1}{2}, \frac{1}{2}, 0) \end{aligned}$$

By calculating the distance of any two points, we can prove that the edges of the tetrahedron are equal. So it is a regular tetrahedron.

(ii)

Another lattice site (on the opposite sublattice) locates at $P = \frac{a}{4}(1, 1, 1)$

The distance of P to each corner of the tetrahedron is

$$\overline{PO} = a\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

$$\overline{PA} = a\left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$$

$$\overline{PB} = a\left(-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right)$$

$$\overline{PC} = a\left(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right)$$

We get $\overline{PO} = \overline{PA} = \overline{PB} = \overline{PC} = \frac{\sqrt{3}a}{4}$. Therefore, P is at the geometrical center of this tetrahedron.

Ex. 3.4

(i) $\vec{R}_{\vec{m} \pm \vec{n}} = (m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3) \pm (n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3) = (m_1 \pm n_1)\vec{a}_1 + (m_2 \pm n_2)\vec{a}_2 + (m_3 \pm n_3)\vec{a}_3$

Indeed have the form of (3.15) and are lattice vectors characterized by $\vec{m} \pm \vec{n}$.

(ii) Start in 1D. Pick the lattice site closest to the origin, whose distance from the origin is a . Claim: all sites satisfying 1D version of (3.16) can be written as $R_m = ma$ with m being an integer.

Proof: Assume $R' = \alpha a$ is a lattice site with α being a non-integer. $[\alpha]$ represents the integer part of α , and $\Delta\alpha = \alpha - [\alpha]$ is its fractional part. Thus $0 < \Delta\alpha < 1$.

From (3.16), we know $[\alpha]a$ and thus $R' - [\alpha]a = \Delta\alpha a$ is also a lattice site, but its distance to the origin is less than a , leading to a contradiction.

For 2D, let us look for the closest site to the origin, located at \vec{a}_1 . This immediately gives us a lattice line, $m\vec{a}_1$, with m being an integer. Any lattice site $\vec{R} \neq m\vec{a}_1$, gives us a parallel lattice line, $\vec{R} + m\vec{a}_1$. Look for $\vec{R} = \vec{a}_2$ such that $\vec{a}_2 + m\vec{a}_1$ is the lattice line closest to the line $m\vec{a}_1$. Then $m\vec{a}_1 + m_2\vec{a}_2$ are all lattice sites. Now assume $\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2$ is also a lattice site, with $\Delta\alpha_1 > 0$ and $\Delta\alpha_2 > 0$. Then we know $(m_1 + \Delta\alpha_1)\vec{a}_1 + [\alpha_2]\vec{a}_2$ forms a lattice line, which is closer to the $m\vec{a}_1$ line than the $\vec{a}_2 + m\vec{a}_1$ line! Contradiction again. It is now obvious how to generalize this to 3D.

Ex. 3.5

For a Bravais lattice, it has a set of primitive vectors, \vec{a}_i . The locations of all lattice points could be expressed as

$$\vec{R}_{\vec{c}} = \sum_i c_i \vec{a}_i, \quad c_i \in Z(\text{integer}).$$

The mid-point of any two lattice sites, say $\vec{R}_{\vec{m}}$ and $\vec{R}_{\vec{n}}$ is

$$\begin{aligned} \vec{R}_{mid} &= \frac{1}{2} (\vec{R}_{\vec{m}} + \vec{R}_{\vec{n}}) \\ &= \sum_i \frac{1}{2} (m_i + n_i) \vec{a}_i \end{aligned}$$

where m_i and n_i are integers.

We can always shift the origin of the coordinate of the lattice to this mid-point so that all lattice points have new coordinates as

$$\vec{R}'_{\vec{c}} = \vec{R}_{\vec{c}} - \vec{R}_{mid} = \sum_i \left[c_i - \frac{1}{2} (m_i + n_i) \right] \vec{a}_i \quad (13)$$

If \vec{R}_{mid} is an inversion center, given a lattice site $\vec{R}'_{\vec{c}}$, $-\vec{R}'_{\vec{c}}$ must be a lattice point as well. Namely, there exists a set of p_i which are integers such that

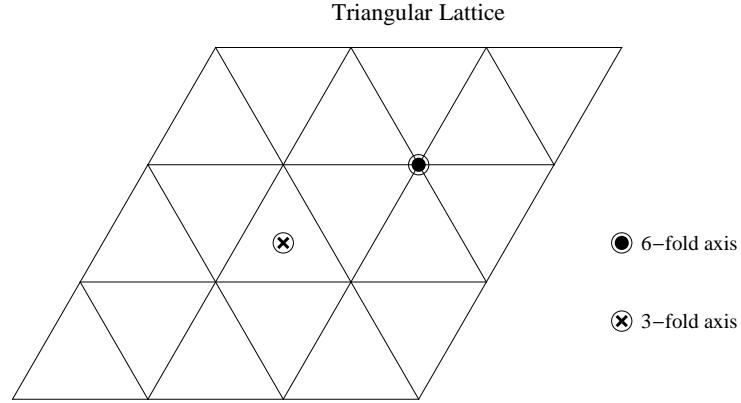
$$-\vec{R}'_{\vec{c}} = \vec{R}'_{\vec{p}} = \sum_i \left[p_i - \frac{1}{2} (m_i + n_i) \right] \vec{a}_i \quad (14)$$

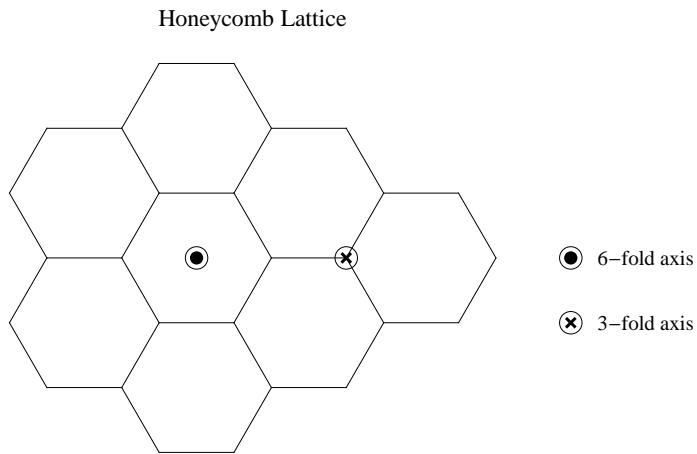
Combining Eq. (13) and Eq. (14), we get

$$p_i = (m_i + n_i) - c_i \in Z.$$

This tells us that \vec{R}_{mid} is indeed an inversion center.

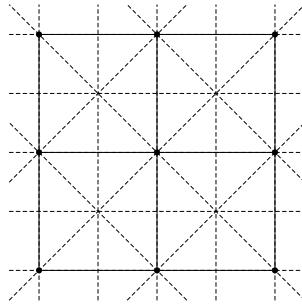
Ex. 3.6





If the atoms on A and B sublattice are different, C_6 symmetry will be broken. The original 6-fold axis will become a 3-fold axis. And the original 3-fold axis is still 3-fold.

Ex. 3.7



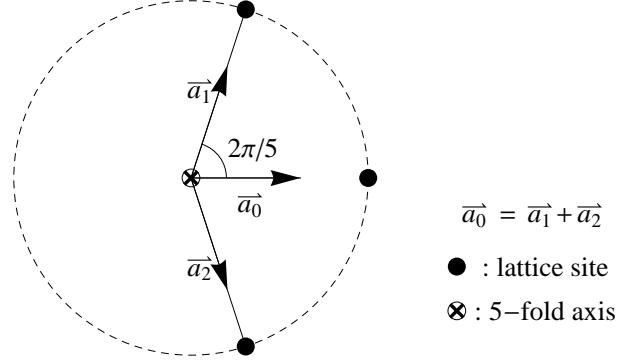
Ex. 3.8

A diamond structure could be viewed as a FCC lattice with a basis containing two atoms called A and B. A and B have a $\frac{a}{4}(1, 1, 1)$ shift where a is the lattice constant. If we take the mid-point of A and B as an inversion center, the positions of the sublattices will exchange applying to all the lattice sites. As a result, the lattice is unchanged if the atoms on different sublattice sites are the same; otherwise, the lattice is not centrosymmetric. Therefore, diamond structures are centrosymmetric, but Zincblende lattices are not.

Ex. 3.9

Suppose the lattice has n -fold symmetry. Then rotating by $\alpha = \frac{2\pi}{n}$ about the origin (assumed to be a lattice site) should leave the lattice invariant. Assume \vec{a}_0 is the shortest lattice vector connecting

the origin to one of its neighbors. After a rotation of $\alpha = \pm \frac{2\pi}{5}$, it becomes \vec{a}_1 and \vec{a}_2 respectively, which should be lattice vectors themselves (see figure). Then $\vec{a}_1 + \vec{a}_2$ should also be a lattice vector. But simple trigonometry finds it is shorter than \vec{a}_0 (see figure), which leads to contradiction. Thus 5-fold symmetry is not allowed in 2D. Since 3D lattices are made of parallel 2D planes, this implies such symmetry is impossible in 3D as well.



Ex. 3.10

The construction of 1D reciprocal lattice $\{\vec{b}_j\}$ with $\vec{b}_j \cdot \vec{a}_m = 2\pi\delta_{mj}$:

$$\vec{b} \cdot \vec{a} = 2\pi \Rightarrow \vec{b} = \frac{2\pi}{a} \hat{x}.$$

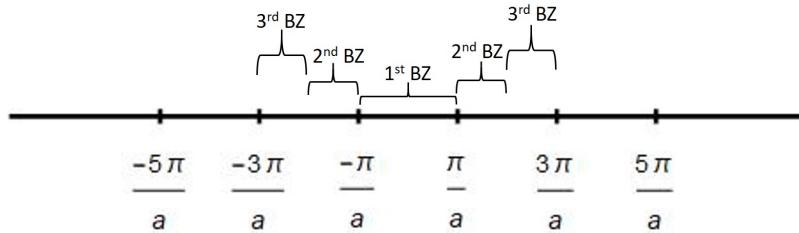
The reciprocal lattice vectors are

$$\vec{G} = m\vec{b} = m\frac{2\pi}{a} \hat{x}. \quad (m=\text{integer})$$

1st BZ: $[-\pi/a, \pi/a]$

2nd BZ: $[-3\pi/a, -\pi/a], [\pi/a, 3\pi/a]$

n th BZ: $[(1-2n)\pi/a, (3-2n)\pi/a], [(2n-3)\pi/a, (2n-1)\pi/a]$



Ex. 3.11

1) FCC

$$\begin{aligned}
 \vec{a}_1 &= \frac{a}{2}(\hat{y} + \hat{z}) = (0, a/2, a/2) \\
 \vec{a}_2 &= \frac{a}{2}(\hat{x} + \hat{z}) = (a/2, 0, a/2) \\
 \vec{a}_3 &= \frac{a}{2}(\hat{x} + \hat{y}) = (a/2, a/2, 0) \\
 w &= \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{8} + \frac{a^3}{8} = \frac{a^3}{4} \\
 \vec{b}_1 &= \frac{2\pi}{w}(\vec{a}_2 \times \vec{a}_3) = \frac{2\pi}{a}(-1, 1, 1) \\
 \vec{b}_2 &= \frac{2\pi}{w}(\vec{a}_3 \times \vec{a}_1) = \frac{2\pi}{a}(1, -1, 1) \\
 \vec{b}_3 &= \frac{2\pi}{w}(\vec{a}_1 \times \vec{a}_2) = \frac{2\pi}{a}(1, 1, -1)
 \end{aligned}$$

This is indicative that $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ forms a BCC lattice with lattice constant $\frac{4\pi}{a}$.

2) BCC

$$\begin{aligned}
 \vec{a}_1 &= \frac{a}{2}(\hat{y} + \hat{z} - \hat{x}) = (-a/2, a/2, a/2) \\
 \vec{a}_2 &= \frac{a}{2}(\hat{x} + \hat{z} - \hat{y}) = (a/2, -a/2, a/2) \\
 \vec{a}_3 &= \frac{a}{2}(\hat{x} + \hat{y} - \hat{z}) = (a/2, a/2, -a/2) \\
 w &= \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \frac{a^3}{2} \\
 \vec{b}_1 &= \frac{2\pi}{w}(\vec{a}_2 \times \vec{a}_3) = \frac{2\pi}{a}(0, 1, 1) \\
 \vec{b}_2 &= \frac{2\pi}{w}(\vec{a}_3 \times \vec{a}_1) = \frac{2\pi}{a}(1, 0, 1) \\
 \vec{b}_3 &= \frac{2\pi}{w}(\vec{a}_1 \times \vec{a}_2) = \frac{2\pi}{a}(1, 1, 0)
 \end{aligned}$$

$(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ constructs a FCC lattice with lattice constant $\frac{4\pi}{a}$.

Ex. 3.12

For a direct lattice $\vec{R} = \sum_i n_i \vec{a}_i$ whose reciprocal lattice $\vec{K} = \sum_i k_i \vec{b}_i$, we have

$$e^{i\vec{R} \cdot \vec{K}} = e^{\sum n_i \vec{a}_i \cdot \sum k_j \vec{b}_j} = 1.$$

Let us call the reciprocal lattice of \vec{K} \vec{R}' , then we have $e^{i\vec{K} \cdot \vec{R}'} = 1$, thus $\vec{R}' = \vec{R}$.

Ex. 3.13

(a)

Begin with a 1D array of disks of radius r_0 . If the centers of the disks are on the 1D lattice $\{\vec{R}_j = jd(1, 0, 0); j \in \mathbb{Z}\}$ where $d = 2r_0$, then the disks are just touching as shown in Fig. 1a.

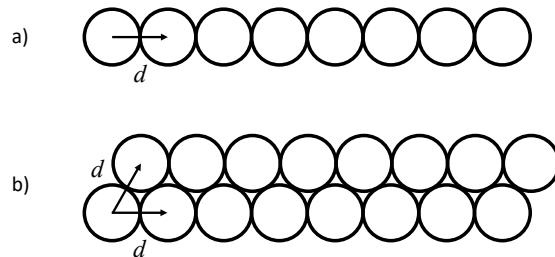


Figure 1

Now consider adding a second line of atoms as shown in Fig. 1b with lattice positions $\{\vec{R}_j = d(j + \delta, y, 0); j \in \mathbb{Z}\}$. The lowest possible allowed value of y (and hence the densest lattice) occurs for horizontal displacement $\delta = 1/2$. At $y_{\min} = \frac{\sqrt{3}}{2}$ each disk in the second row touches two disks in the first row. Extending this to an arbitrary number of rows yields the triangular lattice with lattice vectors $\vec{a}_1 = d(1, 0, 0)$ and $\vec{a}_2 = d(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$.

(b)

Now consider the triangular lattice A defined by points $\vec{R}_{jk} = j\vec{a}_1 + k\vec{a}_2$ as shown in the left panel of Fig. 2.

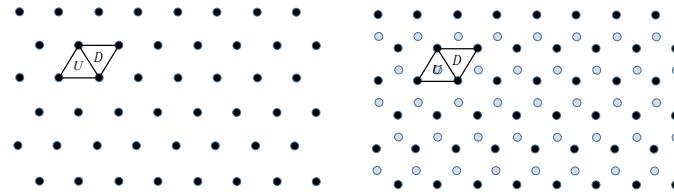


Figure 2

We see that each unit cell contains an upward facing triangle (labelled U) and a downward facing triangle (labelled D). We can choose to place a second identical layer B on top of the first positioned either with the lattice sites located in the center of the U triangles or the D triangles of the first layer. Both choices result in a honeycomb lattice as shown in the right panel of Fig. 2 (where we have arbitrarily chosen to center on the U triangles). If the lower left corner of the triangle labelled U is taken as the origin, the center of the triangle is located at the ‘center of mass’ position $\vec{r}_c = \frac{1}{3}[\vec{0} + \vec{a}_1 + \vec{a}_2] = d(\frac{1}{2}, \frac{1}{2\sqrt{3}}, 0)$.

If each lattice point is the center of a hard sphere of radius $r_0 = d/2$, the spheres in a given layer are close-packed. If the second layer is raised a distance λd above the first, we can define a third vector $\vec{b} = \vec{r}_c + \lambda d(0, 0, 1)$ connecting this point to the origin. The close-packed hard-sphere constraint requires $|\vec{b} - \vec{0}| = |\vec{b} - \vec{a}_1| = |\vec{b} - \vec{a}_2| = d$ which implies $\lambda = \sqrt{\frac{2}{3}}$. This minimum vertical displacement required to satisfy the hard-sphere constraint maximizes the packing density.

(c)

Calling the lower lattice A and the upper lattice B, we can repeat the stacking in the pattern ABABAB...AB to form a hexagonal close-packed (HCP) lattice. The vectors \vec{a}_1, \vec{a}_2 defined above constitute two of the lattice vectors for the hcp lattice. The vector \vec{b} defined above is *not* a lattice vector because (for example) there is no lattice site at $2\vec{b}$. \vec{b} simply defines the location of a second atom in the unit cell of this non-Bravais lattice. Under the ABABAB...AB repetition, the third plane matches the A plane and hence the lattice vector lies in the z direction and is elevated twice as high as the B plane. It thus has length $2\vec{b} \cdot \hat{z}$ in order to satisfy the hard-sphere constraint: $\vec{a}_3 = d(0, 0, 2\sqrt{\frac{2}{3}})$.

(d)

If the third layer is centered on the other choice (downward facing triangles) we can repeat this ABCABCABC...ABCC stacking to obtain the face-centered-cubic (FCC) lattice. In this case the third lattice vector is $\vec{a}_3 = \vec{b}$. In this case \vec{b} , defined above, *is* a lattice vector because the ABC stacking places an atom at both \vec{b} and $2\vec{b}$, unlike the case of AB stacking. To see that this is indeed the FCC lattice, notice that $\vec{a}_1 \cdot \vec{a}_2 = \vec{a}_1 \cdot \vec{a}_3 = \vec{a}_2 \cdot \vec{a}_3 = \frac{1}{2}d^2$ indicating that there is a 60 degree angle between each of the three pairs of vectors just as there is for the standard basis vectors for the FCC lattice:

$$\begin{aligned}\vec{A}_1 &= \Delta(0, \frac{1}{2}, \frac{1}{2}) \\ \vec{A}_2 &= \Delta(\frac{1}{2}, 0, \frac{1}{2}) \\ \vec{A}_3 &= \Delta(\frac{1}{2}, \frac{1}{2}, 0) \\ \vec{A}_1 \cdot \vec{A}_2 &= \vec{A}_1 \cdot \vec{A}_3 = \vec{A}_2 \cdot \vec{A}_3 = \frac{1}{2}d^2,\end{aligned}$$

where $\Delta \equiv \sqrt{2}d$ is the length of the side of the cube defining the conventional unit cell (i.e., the second-neighbor distance). There exists an orthogonal rotation matrix that preserves the angles between the vectors while effecting the coordinate change needed to map $\vec{a}_j \rightarrow A_j$ for $j = 1, 2, 3$. Hence the lattice is FCC.

(e)

There are an infinite number of ways to do the stacking by randomly choosing each layer to be A, B, or C, subject to the constraint that two adjacent layers cannot be the same. For example: ABACBABCABACABCABABABCABCBC.

Ex. 3.14

For diamond lattice the two point basis can be written as $\vec{\tau}_1 = 0$, $\vec{\tau}_2 = \frac{a}{4}(\hat{x} + \hat{y} + \hat{z})$.

Scattering is possible for momentum transfer $\vec{q} = \vec{G} = K\vec{b}_1 + L\vec{b}_2 + M\vec{b}_3$ where

$$\begin{aligned}\vec{b}_1 &= \frac{2\pi}{a}(-\hat{x} + \hat{y} + \hat{z}) \\ \vec{b}_2 &= \frac{2\pi}{a}(\hat{x} - \hat{y} + \hat{z}) \\ \vec{b}_3 &= \frac{2\pi}{a}(\hat{x} + \hat{y} - \hat{z})\end{aligned}$$

are primitive reciprocal lattice vectors of FCC. The scattering amplitude is proportional to

$$f(\vec{q}) = \sum_{s=1}^2 e^{-i\vec{q} \cdot \vec{\tau}_s} f_s(\vec{q}) \quad (15)$$

$$= e^{-i\vec{q} \cdot \vec{\tau}_1} f_1(\vec{q}) + e^{-i\vec{q} \cdot \vec{\tau}_2} f_2(\vec{q}). \quad (16)$$

For identical atoms on the two sublattices $f_1(\vec{q}) = f_2(\vec{q}) = f_a(\vec{q})$, where f_a is the atomic form factor. Then

$$f(\vec{G}) = (1 + e^{-i\vec{G} \cdot \vec{\tau}_2}) f_a(\vec{G}) = (1 + e^{-i\frac{\pi}{2}(K+L+M)}) f_a(\vec{G}) = 0$$

when $K + L + M = 2(2n + 1)$ (twice of odd number).

Ex. 3.15

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

(i)

$$\begin{aligned}\int_{-\infty}^{\infty} dx P(x) &= \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \\ (\text{Gaussian integral}) &= \frac{1}{\sqrt{2\pi}\sigma} \cdot \sqrt{2\pi\sigma^2} \\ &= 1\end{aligned}$$

$\therefore P(x)$ is normalized.

(ii)

Consider the integrals

$$\begin{aligned}
<<x^{2n}>> &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x^{2n} \cdot e^{-\alpha x^2} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(\frac{-\partial}{\partial\alpha} \right)^n \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \\
&= \frac{1}{\sqrt{2\pi}\sigma} (-1)^n \left(\frac{\partial}{\partial\alpha} \right)^n \sqrt{\pi} \alpha^{-\frac{1}{2}} \\
&= \frac{(2\sigma^2)^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \\
&= \sigma^{2n} \frac{(2n)!}{2^n n!}
\end{aligned}$$

where we set $\alpha = \frac{1}{2\sigma^2}$ at the last steps. We thus have $<<x^{2n}>> = \int_{-\infty}^{\infty} dx x^{2n} P(x)$ as desired.

(iii)

$$\begin{aligned}
<<e^{-i\alpha x}>> &= \int_{-\infty}^{\infty} dx e^{-i\alpha x} P(x) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx e^{-i\alpha x} \cdot e^{\frac{-x^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2\sigma^2}(x+i\alpha\sigma^2)^2} \cdot e^{-\frac{\alpha^2\sigma^2}{2}} \\
&= e^{\frac{-\alpha^2\sigma^2}{2}}.
\end{aligned}$$

According to (ii), $<<x^2>> = \sigma^2$. Hence, we get $<<e^{-i\alpha x}>> = e^{\frac{-\alpha^2 <<x^2>>}{2}}$. Note here α is a fixed (but arbitrary) parameter unrelated to the auxiliary variable we introduced in (ii).

(iv)

$$\begin{aligned}
<<e^{-i\alpha x}>> &= \int_{-\infty}^{\infty} dx e^{-i\alpha x} P(x) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx e^{-i\alpha x} \cdot e^{\frac{-x^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!} (\alpha x)^{2n} + \sum_{n=0}^{\infty} \frac{(-i)^{2n+1}}{(2n+1)!} (\alpha x)^{2n+1} \right) e^{\frac{-x^2}{2\sigma^2}}
\end{aligned}$$

The second sum vanishes because odd functions have no contribution to this integral.

$$\begin{aligned}
\therefore \langle\langle e^{-i\alpha x} \rangle\rangle &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\alpha x)^{2n} \right) e^{\frac{-x^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \alpha^{2n} \int_{-\infty}^{\infty} dx x^{2n} e^{\frac{-x^2}{2\sigma^2}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \alpha^{2n} \langle\langle x^{2n} \rangle\rangle \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\alpha^2}{2}\right)^n \sigma^{2n} \\
&= e^{-\frac{\alpha^2\sigma^2}{2}},
\end{aligned}$$

in agreement with (iii).

Ex. 3.16

Let us introduce

$$M(\alpha) = \langle\langle e^{\alpha A} \rangle\rangle = 1 + \sum_{n=1}^{\infty} \frac{\langle\langle A^n \rangle\rangle}{n!} \alpha^n = e^{\sum_{n=1}^{\infty} \frac{c_n}{n!} \cdot \alpha^n} = e^{C(\alpha)}$$

where $M(\alpha)$ is the moment generating function and $C(\alpha) = \sum_{n=1}^{\infty} \frac{c_n}{n!} \alpha^n$ is cumulant generating function. We thus have

$$\begin{aligned}
c_n &= \frac{d^n}{d\alpha^n} \ln M(\alpha) |_{\alpha=0}, \\
\langle\langle A^n \rangle\rangle &= \frac{d^n}{d\alpha^n} e^{C(\alpha)} |_{\alpha=0}.
\end{aligned}$$

By applying the relations above, we can get cumulant coefficients c_n .

$$\begin{aligned}
<< A >> &= \frac{d}{d\alpha} \left(e^{\sum_{n=1}^{\infty} \frac{c_n}{n!} \alpha^n} \right) \\
&= \sum_{n=1}^{\infty} c_n \cdot \frac{1}{(n-1)!} \cdot \alpha^{n-1} e^{C(\alpha)} |_{\alpha=0} \\
&= c_1 \\
\therefore c_1 &= << A >> = 0
\end{aligned}$$

$$\begin{aligned}
<< A^2 >> &= \frac{d^2}{d\alpha^2} \left(e^{\sum_{n=1}^{\infty} \frac{c_n}{n!} \alpha^n} \right) |_{\alpha=0} \\
&= \sum_{n=2}^{\infty} c_n \cdot \frac{1}{(n-2)!} \alpha^{n-2} e^{C(\alpha)} |_{\alpha=0} + \left(\sum_{n=1}^{\infty} c_n \cdot \frac{1}{(n-1)!} \alpha^{n-1} \right)^2 e^{C(\alpha)} |_{\alpha=0} \\
&= c_2 + c_1^2 \\
\therefore c_2 &= << A^2 >> - << A >>^2 = << A^2 >>
\end{aligned}$$

$$\begin{aligned}
<< A^3 >> &= \frac{d^3}{d\alpha^3} \left(e^{\sum_{n=1}^{\infty} \frac{c_n}{n!} \alpha^n} \right) |_{\alpha=0} \\
&= \sum_{n=3}^{\infty} c_n \cdot \frac{1}{(n-3)!} \alpha^{n-3} e^{C(\alpha)} |_{\alpha=0} \\
&\quad + 3 \cdot \left(\sum_{n=2}^{\infty} c_n \cdot \frac{1}{(n-2)!} \alpha^{n-2} e^{C(\alpha)} |_{\alpha=0} \right) \cdot \left(\sum_{n=1}^{\infty} c_n \cdot \frac{1}{(n-1)!} \alpha^{n-1} e^{C(\alpha)} |_{\alpha=0} \right) \\
&\quad + \left(\sum_{n=1}^{\infty} c_n \cdot \frac{1}{(n-1)!} \alpha^{n-1} \right)^3 e^{C(\alpha)} \\
&= c_3 + 3 \cdot c_2 c_1 + c_1^3 \\
\therefore c_3 &= << A^3 >>
\end{aligned}$$

$$\begin{aligned}
<< A^4 >> &= \frac{d^4}{d\alpha^4} \left(e^{\sum_{n=1}^{\infty} \frac{c_n}{n!} \alpha^n} \right) |_{\alpha=0} \\
&= e^{C(\alpha)} \left[C^{(4)} + 4 \cdot C^{(1)} C^{(3)} + 3 \cdot \left(C^{(2)} \right)^2 + 6 \cdot \left(C^{(1)} \right)^2 C^{(2)} \right] |_{\alpha=0} \\
&= c_4 + 4 \cdot c_1 c_3 + 3 \cdot c_2^2 + 6 \cdot c_1^2 c_2 \\
&= c_4 + 3 \cdot c_2^2 \\
\therefore c_4 &= << A^4 >> - 3 << A^2 >>^2
\end{aligned}$$

where $C^{(n)} \equiv \frac{d^n}{d\alpha^n} C(\alpha)$.

Chapter 4

Ex. 4.1

Since X-ray scattering involves interactions with electrons, atomic form factor in X-ray scattering originates from the electron distribution near an atom. In neutron scattering, it only interacts with nuclei. In principle there is a form factor in this case related to the inner structure of nuclei. However, since the wavelength of neutrons are much larger than the length scale of the inner structure, we can in practice view a nucleus as a point particle and ignore its inner structure. As a result this form factor is nothing but a delta function which gives a constant in momentum (or Fourier) space.

Ex. 4.2

$$H_{int} = \alpha' \sum_{j=1}^N \delta^3(\vec{R} - \vec{r}_j) \vec{S}_n \vec{S}_j$$

Considering neutron spin, the wave functions become:

The initial state:

$$\Psi_i = \Psi_{\vec{p},\uparrow}(\vec{R}) \Phi_i.$$

The final state:

$$\Psi_f = \Psi_{\vec{p},\downarrow}(\vec{R}) \Phi_f.$$

Note Φ_i and Φ_f are the states of the system that include both spatial and spin dependence.

Also we can write

$$\vec{S}_n \cdot \vec{S}_j = \frac{1}{2} S_n^+ S_j^- + \frac{1}{2} S_n^- S_j^+ + S_n^z S_j^z.$$

It is clear the 2nd term flips the neutron spin from up to down, which is the process of interest here.

$$\Gamma_{if} = \frac{2\pi}{\hbar} \delta(E_f - E_i - \hbar\omega) |M_{if}|^2$$

$$\begin{aligned}
M_{if} &= \langle \Psi_f | H_{int} | \Psi_i \rangle \\
&= \frac{1}{L^3} \int d^3 R e^{-i\vec{p}' \cdot \vec{R}} \langle \downarrow, \Phi_f | \alpha' \sum_j^N \delta^3(\vec{R} - \vec{r}_j) \vec{S}_n \cdot \vec{S}_j | \uparrow, \Phi_i \rangle e^{i\vec{p} \cdot \vec{R}} \\
&= \frac{\alpha'}{L^3} \sum_j^N \langle \Phi_f | e^{-i\vec{q} \cdot \vec{r}_j} S_j^+ | \Phi_i \rangle.
\end{aligned}$$

This is very similar to Eq. (4.14), only having an additional S_j^+ operator in the expectation value. Following the same manipulations that lead to Eq. (4.23) we find the spin-flip scattering rate measures

$$S^{+-}(\vec{q}, \omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} \sum_{jk} \langle \langle e^{i\vec{q} \cdot \vec{r}_j(t)} S_j^- (t) e^{-i\vec{q} \cdot \vec{r}_k(0)} S_k^+ \rangle \rangle.$$

Ex. 4.3

Recall (4.23):

$$\begin{aligned}
S(\vec{q}, \omega) &\equiv \frac{1}{N} \int_{-\infty}^{\infty} dt e^{i\omega t} \sum_{j,k=1}^N \langle \langle e^{i\vec{q} \cdot \vec{r}_j(t)} e^{-i\vec{q} \cdot \vec{r}_k(0)} \rangle \rangle \\
\Pi(q, t) &\equiv \langle \langle e^{iqX(t)} e^{-iqX(0)} \rangle \rangle \\
(eq \ 4.80) : \quad &= e^{-q^2 \langle \langle X_0^2 \rangle \rangle (1 - \cos(\Omega t))}
\end{aligned}$$

In the 1D classical case with one particle, the analog of Eq. (4.23) is obtained by using the following relations.

$$\begin{aligned}
\vec{r} &\rightarrow X \\
\vec{q} &\rightarrow q \\
N &\rightarrow 1
\end{aligned}$$

and get rid of $\sum_{j,k}$ for summing different particles. We thus obtain

$$\begin{aligned}
S_{classical}(q, \omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \cdot \langle\langle e^{iqX(t)} e^{-iqX(0)} \rangle\rangle \\
&= \int_{-\infty}^{\infty} dt e^{i\omega t} \cdot \Pi(q, t) \\
(\text{let } \alpha \equiv -q^2 \langle\langle X_0^2 \rangle\rangle) &= e^{-q^2 \langle\langle X_0^2 \rangle\rangle} \int_{-\infty}^{\infty} dt e^{i\omega t} \cdot e^{\alpha \cos(\Omega t)} \\
&= e^{-q^2 \langle\langle X_0^2 \rangle\rangle} \int_{-\infty}^{\infty} dt e^{i\omega t} \cdot \left[I_0(\alpha) + 2 \sum_{n=1}^{\infty} I_n(\alpha) \cdot \cos(n\Omega t) \right] \\
&= e^{-q^2 \langle\langle X_0^2 \rangle\rangle} \left[I_0(\alpha) \delta(\omega) + 2 \sum_{n=1}^{\infty} I_n(\alpha) \cdot \int_{-\infty}^{\infty} dt e^{i\omega t} \cdot \frac{(e^{in\Omega t} + e^{-in\Omega t})}{2} \right] \\
&= \boxed{e^{-q^2 \langle\langle X_0^2 \rangle\rangle} \left[I_0(\alpha) \delta(\omega) + \sum_{n=1}^{\infty} I_n(\alpha) (\delta(\omega + n\Omega) + \delta(\omega - n\Omega)) \right]}
\end{aligned}$$

where I_i is modified Bessel function. In the above we used (a la. Eq. 4.81)

$$\Pi(q, t) \equiv \langle\langle e^{iqX(t)} e^{-iqX(0)} \rangle\rangle = e^{-q^2 \langle\langle X_0^2 \rangle\rangle (1 - \cos(\Omega t))}.$$

Ex. 4.4

Classically if the initial momentum is p_0 , the final momentum is $p_0 - q$, and energy change

$$E = \frac{(p_0 - q)^2}{2m} - \frac{p_0^2}{2m} = \frac{q^2}{2m} - \frac{qp_0}{m}$$

We neglect change in potential energy as classically there is no change in position. So $P(E) dE = P(p_0) dp_0$

$$\begin{aligned}
P(E) &= \frac{P(p_0)}{|dE/dp_0|} = \frac{m}{|q|} P(p_0) \\
&= \frac{m}{|q|} \frac{1}{\sqrt{2\pi m^2 \Omega^2 X_{ZPF}^2}} e^{-\frac{p_0^2}{2m^2 \Omega^2 X_{ZPF}^2}} \\
&= \frac{m}{|q|} \frac{1}{\sqrt{2\pi m \Omega X_{ZPF}}} e^{-\frac{(E - q^2/2m)^2}{2\lambda^2 \Omega^2}} \\
&= \frac{1}{\sqrt{2\pi |\lambda| \Omega}} e^{-\frac{(E - q^2/2m)^2}{2\lambda^2 \Omega^2}}
\end{aligned}$$

Comparing with (4.102), for large λ

$$P_n \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{(n-\bar{n})^2}{2\bar{n}}}$$

with $\bar{n} = \lambda = q^2 X_{ZPF}^2 = \frac{q^2}{2m\Omega^2}$. Upon identifying $E = n\Omega$, $P(E)$ is consistent with P_n .

Ex. 4.5

$$\begin{aligned} S(\vec{q}, \omega) &= \frac{1}{Z} \sum_i e^{-\beta E_i} S_i(\vec{q}, \omega) \\ S_i(\vec{q}, \omega) &= \frac{1}{N} \sum_f | \langle i | \rho_{\vec{q}} | f \rangle |^2 \cdot 2\pi\delta(\omega - \frac{E_f - E_i}{\hbar}) \\ S(\vec{q}, -\omega) &= \frac{1}{Z} \sum_i e^{-\beta E_f} \frac{1}{N} | \langle i | \rho_{\vec{q}} | f \rangle |^2 \cdot 2\pi\delta(\omega - \frac{E_f - E_i}{\hbar}) \end{aligned}$$

Using

$$E_f = \hbar\omega + E_i$$

we find

$$S(\vec{q}, -\omega) = e^{-\beta\hbar\omega} S(\vec{q}, \omega).$$

Ex. 4.6

$$\begin{aligned} \text{RHS} &= \frac{-i}{\hbar} \int_0^\infty \langle \langle \rho_{-\vec{q}}(t) \rho_{\vec{q}}(0) - \rho_{\vec{q}}(0) \rho_{-\vec{q}}(t) \rangle \rangle e^{i\omega t} e^{-\eta t} dt \\ &= \frac{-i}{\hbar} \frac{1}{Z} \sum_{i,j} e^{-\beta E_i} \left[\int_0^\infty \langle i | \rho_{-\vec{q}} | j \rangle \langle j | \rho_{\vec{q}} | i \rangle e^{i(\omega + \omega_i - \omega_j)t - \eta t} dt - \int_0^\infty \langle i | \rho_{\vec{q}} | j \rangle \langle j | \rho_{-\vec{q}} | i \rangle e^{i(\omega - \omega_i + \omega_j)t - \eta t} dt \right] \\ &= \frac{1}{\hbar} \sum_{i,j} e^{-\beta E_i} \left[\frac{\langle i | \rho_{\vec{q}} | j \rangle \langle j | \rho_{-\vec{q}} | i \rangle}{\omega + \omega_i - \omega_j + i\eta} - \frac{\langle i | \rho_{-\vec{q}} | j \rangle \langle j | \rho_{\vec{q}} | i \rangle}{\omega - \omega_i + \omega_j + i\eta} \right] \\ &= \chi(\vec{q}, \omega) \end{aligned}$$

Ex. 4.7

(i) The single particle eigen wave functions are $\psi_{\vec{k}}(\vec{r}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}}$, where V is the system volume. The N -particle ground state has all particles in the $\vec{k} = 0$ state, so

$$\psi_0(\vec{r}_1, \dots, \vec{r}_N) = V^{-\frac{N}{2}}$$

$$\rho_{\vec{q}} \psi_0 = V^{-\frac{N}{2}} \sum_{j=1}^N e^{-i\vec{q}\cdot\vec{r}_j}$$

has $N-1$ particles in $\vec{k} = 0$ and one particle in $\vec{k} = -\vec{q}$ state, and is an exact eigenstate. $E_{\vec{q}} = \frac{\hbar^2 q^2}{2M}$, where M is particle mass.

(ii)

$$\begin{aligned} S(\vec{q}) &= \frac{1}{N} \langle \psi_0 | \rho_{-\vec{q}} \rho_{\vec{q}} | \psi_0 \rangle \\ &= \frac{1}{N} \sum_{j,k=1}^N \langle \psi_0 | e^{i\vec{q} \cdot \vec{r}_j} e^{-i\vec{q} \cdot \vec{r}_k} | \psi_0 \rangle \\ &= \frac{1}{N} \sum_{j=1}^N \langle \psi_0 | e^{i\vec{q} \cdot \vec{r}_j} e^{-i\vec{q} \cdot \vec{r}_j} | \psi_0 \rangle = 1 \end{aligned}$$

Thus, $\epsilon(\vec{q}) = \frac{f(\vec{q})}{S(\vec{q})} = \frac{\hbar^2 q^2}{2M} = E(\vec{q})$

Ex. 4.8

(i) In this case, $\rho_{\vec{q}}$ changes one of the fermion's momentum from \vec{k} to $\vec{k} - \vec{q}$, if $|\vec{k}| < k_F$ and $|\vec{k} - \vec{q}| > k_F$. This results in a change of energy

$$\frac{\hbar^2}{2m} \left[(\vec{k} - \vec{q})^2 - \vec{k}^2 \right] = \frac{\hbar^2}{2m} (\vec{q}^2 - 2\vec{k} \cdot \vec{q}).$$

Since \vec{k} is not fixed, neither is the energy, as a result $\rho_{\vec{q}}|\psi_0\rangle$ is a superposition of states with different energies, but not an energy eigenstate itself.

(ii) The largest energy change is obviously $\frac{\hbar^2}{2m} (q^2 + 2k_F q)$.

The minimum energy is zero for $q \leq 2k_F$, and $\frac{\hbar^2}{2m} (q^2 - 4k_F q)$ for $q > 2k_F$.

Chapter 5

Ex. 5.1

Discrete system:

$$T = \frac{1}{2} \sum_i m_i |\dot{\vec{u}}(\vec{R}_i)|^2.$$

Taking the continuum limit: $\dot{\vec{u}}(\vec{R}_i) \rightarrow \dot{\vec{u}}(\vec{r})$, $\sum_i m_i \rightarrow \int dm = \rho_0 \int d^3r$, thus

$$T = \frac{\rho_0}{2} \int d^3r \dot{\vec{u}}(\vec{r})^2.$$

Ex. 5.2

Consider a distorted volume

$$V \rightarrow V + \Delta V$$

, in which

$$\rho = \frac{\rho_0 V}{V + \Delta V} = \rho_0 \left(1 - \frac{\Delta V}{V}\right).$$

Since

$$\frac{\Delta V}{V} = \frac{du_x}{dx} + \frac{du_y}{dy} + \frac{du_z}{dz} = \vec{\nabla} \cdot \vec{u},$$

we have

$$\rho = \rho_0 [1 - \vec{\nabla} \cdot \vec{u}(\vec{r})].$$

Ex. 5.3

With periodic boundary condition, when can define the Fourier components of $\vec{u}(\vec{r})$:

$$\vec{u}(\vec{q}) = \int d^3r e^{-i\vec{q} \cdot \vec{r}} \vec{u}(\vec{r}),$$

in terms of which we have

$$u_{\mu\nu}(\vec{r}) = \frac{1}{2L^3} \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} [q_\mu u_\nu(\vec{q}) + q_\nu u_\mu(\vec{q})],$$

$$\nabla \cdot \vec{u}(\vec{r}) = \frac{1}{L^3} \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} q_\mu u_\mu(\vec{q}),$$

$$\nabla \times \vec{u}(\vec{r}) = \frac{1}{L^3} \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} \epsilon_{\mu\nu\delta} q_\mu u_\nu(\vec{q}) \hat{e}_\delta.$$

Plugging these into Eqs. (5.2) and (5.7), (5.8) and (5.9) follow.

Ex. 5.4

For a shear (transverse) distortion, $u_{\alpha\beta}$ is traceless. So its two independent components are (a) $u_{11} = -u_{22} = \delta$, and (b) $u_{12} = u_{21} = \delta$, corresponding to the two shapes illustrated in the figure.

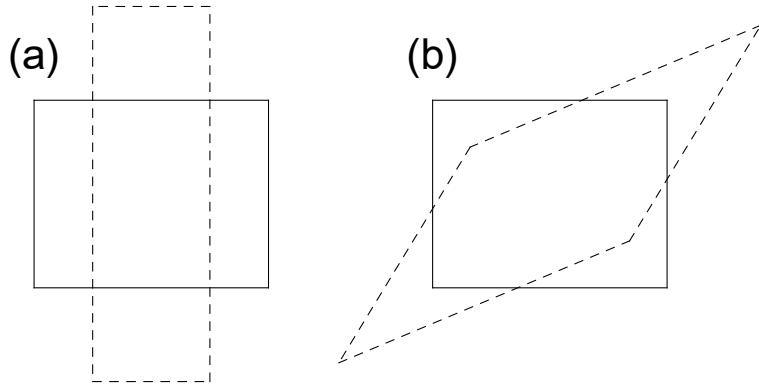


Figure 3: A square that gets distorted by the shear distortion corresponding to (a) and (b) above. Here we assume $\delta > 0$.

Ex. 5.5

$$V = \frac{1}{2} \int d^3r C_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta}$$

First consider the independent number of components in $u_{\alpha\beta}$: Since $u_{\alpha\beta} = u_{\beta\alpha}$ (symmetric tensor). in d -dimension the number of independent components is $N = \frac{d(d+1)}{2}$. Now consider the symmetry: $C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}$, we find the number of independent constants in C to be $\frac{N}{2}(N+1)$.

Ex. 5.6

- i) In the adiabatic approximation, we first treat X_1 and X_2 as classical variables, and solve the ground state energy of the electron Hamiltonian:

$$H_e = \frac{p^2}{2m} + \frac{K_2}{2} [(x - X_1)^2 + (x - X_2)^2] = \frac{p^2}{2m} + K_2 [x - (X_1 + X_2)/2]^2 + \frac{K_2}{4} (X_1 - X_2)^2,$$

so

$$E_e = \hbar\omega_e/2 + \frac{K_2}{4}(X_1 - X_2)^2,$$

where $\omega_e = \sqrt{2K_2/m}$. As a result

$$U(X_1, X_2) = \left(\frac{K_2}{4} - \frac{K_1}{2} \right) (X_1 - X_2)^2,$$

and the atom is stable for $K_2 > 2K_1$.

ii) Now the effective Hamiltonian for the two atoms is

$$H_a = \frac{P_1^2}{2M_1} + \frac{P_2^2}{2M_2} + \left(\frac{K_2}{4} - \frac{K_1}{2} \right) (X_1 - X_2)^2.$$

Separating into center-of-mass and relative coordinates, we find the latter has oscillating frequency of

$$\omega_a = \sqrt{\left(\frac{K_2}{2} - K_1 \right) \frac{M_1 + M_2}{M_1 M_2}}.$$

iii) For such a harmonic problem, one obtains the exact frequencies from solving the classical equations of motion which result in

$$\begin{vmatrix} -m\omega^2 + 2K_2 & -K_2 & -K_2 \\ -K_2 & -M_1\omega^2 + K_2 - K_1 & K_1 \\ -K_2 & K_1 & -M_2\omega^2 + K_2 - K_1 \end{vmatrix} = 0.$$

This reduces to a quadratic equation for ω^2 upon recognizing $\omega = 0$ is one of the three solutions corresponding to the free motion of the center of mass.

If we take the limit $m \rightarrow 0$, the $m\omega^2$ term drops out and we obtain a single solution identical to what we obtained in part (ii).

For the special case of $M_1 = M_2 = M$, the equation above simplifies and it is easy to show what we obtained in part (ii) is an exact frequency. The reason can be understood as follows. We can re-write V as

$$V(X_1, X_2, x) = \left(\frac{1}{4}K_2 - \frac{1}{2}K_1 \right) (X_1 - X_2)^2 + K_2[x - (X_1 + X_2)/2]^2.$$

For $M_1 = M_2 = M$, $(X_1 + X_2)/2$ is the center of mass of the two nuclei. Thus in this case the electron is coupled to the center-of-mass coordinate of the two nuclei, but *not* to their relative coordinate $X_1 - X_2$, as a result the latter is an eigen mode whose oscillation frequency is independent of the electron mass m , and the electron state.

Ex. 5.7

Let us first construct the D tensor cf. Eq. (5.28). Obviously D is non-zero only for nearest (n) and next nearest (nn) neighbors:

$$D_n^{11} = \lambda_1$$

for horizontal neighbors and zero otherwise, and

$$D_n^{22} = \lambda_1$$

for vertical neighbors and zero otherwise. Fourier transforming cf. Eq. (5.35) yields

$$D_n(\vec{q}) = 2\lambda_1 \begin{pmatrix} 1 - \cos q_x a & 0 \\ 0 & 1 - \cos q_y a \end{pmatrix}.$$

A slightly more complicated calculation yields

$$D_{nn}(\vec{q}) = 4\lambda_2 \begin{pmatrix} 1 - \cos(q_x a) \cos(q_y a) & \sin(q_x a) \sin(q_y a) \\ \sin(q_x a) \sin(q_y a) & 1 - \cos(q_x a) \cos(q_y a) \end{pmatrix}.$$

A quick way to obtain the above is to recognize that if we rotate our coordinate system by 45 degrees, $D_{nn}(\vec{q})$ would have essentially the same form as $D_n(\vec{q})$ in the original coordinate system.

Using the fact $D(\vec{q}) = D_n(\vec{q}) + D_{nn}(\vec{q})$ and taking $\vec{q} = q\hat{x}$ we obtain two modes with frequencies

$$\omega_l(q) = \sqrt{\frac{(2\lambda_1 + 4\lambda_2)(1 - \cos qa)}{M}}$$

with polarization along x-direction (longitudinal mode, and

$$\omega_t(q) = \sqrt{\frac{4\lambda_2(1 - \cos qa)}{M}}$$

with polarization along y-direction (transverse mode).

Ex. 5.8

The corresponding equations are similar to (5.53) and (5.54).

$$\begin{aligned} & \left\{ \begin{array}{l} [-M_A \omega^2 + 2\lambda] \epsilon_\mu - \lambda (1 + e^{-2iq a}) \epsilon_v = 0 \\ [-M_B \omega^2 + 2\lambda] \epsilon_v - \lambda (1 + e^{2iq a}) \epsilon_u = 0 \end{array} \right. \\ & \Rightarrow \begin{vmatrix} -M_A \omega^2 + 2\lambda & -\lambda (1 + e^{-2iq a}) \\ -\lambda (1 + e^{2iq a}) & -M_B \omega^2 + 2\lambda \end{vmatrix} = 0 \\ & \Rightarrow \omega_I^2 = \frac{\lambda}{M_A M_B} \left(M_A + M_B \pm \sqrt{M_A^2 + M_B^2 + 2M_A M_B \cos(2qa)} \right) \end{aligned}$$

Chapter 6

Ex. 6.1

For $\epsilon < k_B\Theta_D$, the number of modes per unit volume up to energy ϵ is

$$n(\epsilon) = \frac{d\Omega_d}{(2\pi)^d} \left(\frac{\epsilon}{v_s} \right)^d,$$

where Ω_d is the total solid angle in d -dimension, and $n(\epsilon > k_B\Theta_D) = d/\omega$ where ω is the unit cell volume. As a result

$$\rho(\epsilon) = \frac{dn(\epsilon)}{d\epsilon} = \frac{d^2\Omega_d}{(2\pi v_s)^d} \epsilon^{d-1} \theta(k_B\Theta_D - \epsilon) \propto \epsilon^{d-1} \theta(k_B\Theta_D - \epsilon).$$

Using arguments similar to those around Eq. (6.13), we find at low T we have $C_V(T) \propto T^d$.

Ex. 6.2

Since $a(k)$ and $a(k')$ are both made of annihilation operators, we have $[a(k), a(k')] = 0$.

$$\begin{aligned} [a(k), a^\dagger(k')] &= \frac{1}{N} \sum_{j,j'=1}^N \left[e^{-ik \cdot R_j} a_j, e^{ik' \cdot R_{j'}} a_{j'}^\dagger \right] \\ &= \frac{1}{N} \sum_{j=1}^N e^{i(k' - k) \cdot R_j} = \delta_{k,k'}. \end{aligned}$$

Ex. 6.3

$$\begin{aligned} [u(k), P(k')] &= \frac{1}{N} \sum_{j,j'} \left[e^{ikR_j} u_j, e^{ik'R_{j'}} P_{j'} \right] \\ &= \frac{1}{N} \sum_j e^{i(k+k')R_j} i\hbar = i\hbar \delta_{k,-k'}. \end{aligned}$$

Ex. 6.4

Since $\{\hat{e}_\alpha\}$ form an orthogonal set of vectors,

$$\left[\hat{e}_\alpha \cdot \vec{u}_j, \hat{e}_\beta \cdot \vec{P}_{j'} \right] = i\hbar \delta_{\alpha,\beta} \delta_{j,j'}.$$

$$\text{Thus } \left[a_\alpha(\vec{k}), a_\beta^\dagger(\vec{q}) \right] = \frac{\delta_{\alpha\beta}}{N} \sum_{j=1}^N e^{i(\vec{q}-\vec{k}) \cdot \vec{R}} = \delta_{\alpha\beta} \delta_{\vec{k},\vec{q}}.$$

$$\text{Similarly, } \left[a_\alpha(\vec{k}), a_\beta(\vec{q}) \right] = 0.$$

Ex. 6.5

i) With the change of phonon dispersion, Eq. (6.88) (for finite T) should be modified to

$$\Gamma \propto \int_0^{k_D} dk \frac{k^{d-1}}{\beta k^{2\alpha}},$$

which is convergent for $d = 2$ and $0 < \alpha < 1$. Similarly for $T = 0$, Eq. (6.81) needs to be modified to

$$\Gamma \propto \int_0^{k_D} dk \frac{k^{d-1}}{k^\alpha},$$

which is convergent for $d = 1$ and $0 < \alpha < 1$. As a result in both cases the Debye-Waller factor is finite.

ii)

In the case of 1D at finite T the relevant integral is

$$\Gamma \propto \int_0^{k_D} dk \frac{1}{\beta k^{2\alpha}},$$

which is convergent for $0 < \alpha < 1/2$.

iii)

For 2D at finite T the relevant integral is

$$\Gamma \propto \int_0^{k_D} dk \frac{k}{\beta k^2 |\ln k|^{2\gamma}} \propto \int_{|\ln k_D|}^\infty \frac{dt}{t^{2\gamma}},$$

which is convergent for $\gamma > 1/2$.

In the case of 1D at $T = 0$ the relevant integral is

$$\Gamma \propto \int_0^{k_D} \frac{dk}{k |\ln k|^\gamma} \propto \int_{|\ln k_D|}^\infty \frac{dt}{t^\gamma},$$

which is convergent for $\gamma > 1$.

Ex. 6.6

From Eq. (6.97)

$$P(\vec{q}, \omega) = e^{-2\Gamma(\vec{q})} \int_{-\infty}^{+\infty} e^{i\omega t} e^{f(t)}.$$

The zero-phonon contribution is

$$\begin{aligned} P_0(\vec{q}, \omega) &= e^{-2\Gamma(\vec{q})} \int_{-\infty}^{+\infty} e^{i\omega t} \times 1 \\ &= e^{-2\Gamma(\vec{q})} 2\pi\delta(\omega). \end{aligned}$$

The one-phonon contribution is

$$P_1(\vec{q}, \omega) = e^{-2\Gamma(\vec{q})} \int_{-\infty}^{+\infty} e^{i\omega t} f(t)$$

with

$$\begin{aligned} f(t) &= \langle\langle [\vec{q} \cdot \vec{u}_j(t)] [\vec{q} \cdot \vec{u}_j(0)] \rangle\rangle \\ &= \frac{1}{N} \sum_{\vec{k}, \alpha} \left(\frac{\hbar}{2Mv_s k} \right) (\vec{q} \cdot \epsilon_\alpha)^2 \{ e^{-i\omega_k t} [n_B(\hbar\omega_k) + 1] + [e^{+i\omega_k t} n_B(\hbar\omega_k)] \}, \end{aligned}$$

which gives

$$\begin{aligned} P_1(\vec{q}, \omega) &= e^{-2\Gamma(\vec{q})} \frac{1}{N} \frac{L^3}{(2\pi)^3} \int_0^\infty dk 4\pi k^2 e^{-k/k_D} \frac{\hbar q^2}{2Mv_s k} \{ 2\pi\delta(\omega - \omega_k) [n_B(\hbar\omega_k) + 1] + 2\pi\delta(\omega + \omega_k) [n_B(\hbar\omega_k)] \} \\ &= e^{-2\Gamma(\vec{q})} \frac{L^3}{N} \frac{1}{2\pi} e^{-|\omega|/\omega_D} \frac{\hbar q^2}{2Mv_s^3} \{ \omega [n_B(\hbar\omega) + 1] \theta(\omega) + |\omega| [n_B(|\omega|)] \theta(-\omega) \}, \end{aligned}$$

where θ is the step function and $\omega_D = v_s k_D$. Using $n_B(\hbar\omega) + 1 = -n_B(-\omega)$ gives

$$P_1(\vec{q}, \omega) = e^{-2\Gamma(\vec{q})} \frac{L^3}{N} \frac{1}{2\pi} e^{-|\omega|/\omega_D} \frac{\hbar q^2}{2Mv_s^3} \omega [n_B(\hbar\omega) + 1],$$

At zero temperature, $I(\omega) \equiv \omega [n_B(\hbar\omega) + 1] = \omega\theta(\omega)$. At finite temperature the discontinuity in slope at $\omega = 0$ is smoothed out over a frequency interval $\hbar\omega \sim k_B T$.

The contribution from the m -phonon term involves a Fourier transform of $f(t)^m$ and is thus an m -fold convolution of the one-phonon term that takes the form

$$I_m(\omega) = \int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 \dots \int_{-\infty}^{+\infty} d\omega_m I(\omega_1) I(\omega_2) \dots I(\omega_m) \delta(\omega_1 + \omega_2 + \dots + \omega_m - \omega)$$

At zero temperature (or for $\hbar\omega \gg k_B T$), simple dimensional analysis shows that $I_m(\omega) \sim \omega^{(2m-1)}\theta(\omega)$. In the opposite limit, $\hbar\omega \ll k_B T$, the result is simply a smooth approximately constant function.

Ex. 6.7

The first-order Doppler shift of the frequency is $\delta\omega = \frac{v}{c}\omega$. Hence the velocity required for an energy shift δE is $v = c\frac{\delta E}{E}$. For the $\approx 14.4\text{keV}$ gamma ray from ^{57}Co decay this yields $v \approx 3 \times 10^8 \frac{1\text{meV}}{14.4\text{keV}} \approx 20.8\text{m/s}$. In order to have the energy to create the phonon during absorption, the absorber must be moving *towards* the emitter so that the Doppler shift raises the energy when viewed in the frame of the absorber.

Ex. 6.8

$\frac{v}{c} = \frac{\delta\nu}{\nu_0} = \frac{gh}{c^2} \Rightarrow v = \frac{gh}{c}$. At $h = 20\text{m}$, this corresponds to a downward speed of $v \approx 0.65$ microns/sec. The relative frequency shift is $\frac{\delta\nu}{\nu_0} = \frac{0.65 \times 10^{-6}\text{m/s}}{3 \times 10^8\text{m/s}} \approx 2.2 \times 10^{-15}$. For a gamma ray energy of 14.4keV this corresponds to an energy shift of $\delta E \approx 3 \times 10^{-11}\text{eV}$ or about 2×10^{-3} of the linewidth. No wonder this was a hard experiment!

Chapter 7

Ex. 7.1

$$\langle \dot{V} \rangle = -\frac{e\vec{E}_0 e^{j\omega t}}{m} - \frac{1}{\tau} \langle V(t) \rangle$$

Making the ansatz $\langle V(t) \rangle \propto e^{j\omega t}$ yields the solution

$$\langle V(t) \rangle = -\frac{e\vec{E}_0 \tau e^{j\omega t}}{m(1 + j\omega\tau)},$$

leading to

$$\vec{J} = \frac{ne^2 \vec{E} \tau}{m(1 + j\omega\tau)}.$$

Thus

$$\sigma = \frac{\vec{J}}{\vec{E}} = \frac{ne^2 \tau}{m(1 + j\omega\tau)}.$$

i)

At high frequency the electric field oscillation period is much smaller than the scattering time τ , as a result the electron changes its direction before it gets a chance to be scattered by impurity, thus τ drops out. Also in this limit the electron motion is out of phase from that of electric field, and thus non-dissipative; this is encoded by the imaginary conductivity.

ii)

Impedance of inductor L is $X_L = j\omega L$, and the impedance of resistor R is $X_R = R$. Thus the total impedance is $X = j\omega L + R$, so $1/X$ has the same frequency dependence as Drude conductivity. The electron mass plays a role similar to inductance because it is its inertia against change of its velocity, thus the current.

Ex. 7.2

For a 2×2 matrix, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus,

$$\vec{\sigma} = \vec{\rho}^{-1} = \begin{pmatrix} \frac{\sigma}{1+(\frac{\sigma_B}{nec})^2} & \frac{-\sigma^2 \frac{B}{nec}}{1+(\frac{\sigma_B}{nec})^2} & 0 \\ \frac{\sigma^2 \frac{B}{nec}}{1+(\frac{\sigma_B}{nec})^2} & \frac{\sigma}{1+(\frac{\sigma_B}{nec})^2} & 0 \\ 0 & 0 & \sigma \end{pmatrix}$$

Thus σ_{xx} and σ_{yy} both depend on B ; only σ_{zz} is independent of B .

Ex. 7.3

i)

The probability that the electron doesn't collide is for a small interval dt is $1 - dt/\tau$. As a result the probability of no collision between $t = 0$ and $t = t_c$ is $\lim_{N \rightarrow \infty} [1 - t_c/(N\tau)]^N = e^{-t_c/\tau}$. So the probability the electron doesn't collide in $[0, t]$ but collides in $[t, t + dt]$ then the probability is $\frac{dt}{\tau} e^{-t_c/\tau}$, so $P(t_c) = \frac{1}{\tau} e^{-t_c/\tau}$.

ii)

$$\langle t_c \rangle = \int_0^\infty t_c P(t_c) dt_c = \tau$$

$$\langle t_c^2 \rangle = \int_0^\infty t_c^2 P(t_c) dt_c = 2\tau^2$$

iii)

Similar to part (i), we divide $[0, t]$ to N intervals and take the limit $N \rightarrow \infty$:

$$P(n) = \lim_{N \rightarrow \infty} \frac{N!}{(N-n)!n!} [1 - t/(N\tau)]^{N-n} [t/(N\tau)]^n = \frac{(t/\tau)^n}{n!} e^{-t/\tau},$$

so $\bar{n} = t/\tau$.

Ex. 7.4

$$T_{\vec{l}} = e^{i\vec{p} \cdot \vec{l}/\hbar} = \sum_{n=0}^{\infty} \frac{(i\vec{p} \cdot \vec{l}/\hbar)^n}{n!}, \quad \vec{p} = -i\hbar \frac{\partial}{\partial \vec{r}}$$

$$T_{\vec{l}} \psi(\vec{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\vec{l} \cdot \frac{\partial}{\partial \vec{r}})^n \psi(\vec{r})$$

$$= \psi(\vec{r}) + \frac{\partial \psi}{\partial \vec{r}} \cdot \vec{l} + \sum_{ij} \frac{1}{2!} \frac{\partial^2 \psi}{\partial r_i \partial r_j} l_i l_j + \dots$$

$$= \psi(\vec{r} + \vec{l})$$

Ex. 7.5

The highest energy of the 1BZ is at $\vec{k} = (\pi/a, \pi/a)$, with energy $\frac{\hbar^2}{m_e}(\pi/a)^2$. The lowest energy of the 2BZ is at $\vec{k} = (\pi/a, 0)$, with energy $\frac{\hbar^2}{2m_e}(\pi/a)^2$. So the overlap is for $\frac{\hbar^2}{2m_e}(\pi/a)^2 < \epsilon < \frac{\hbar^2}{m_e}(\pi/a)^2$.

Ex. 7.6

$$E(\lambda) = \langle n(\lambda) | H(\lambda) | n(\lambda) \rangle$$

$$\frac{dE(\lambda)}{d\lambda} = \langle n(\lambda) | \frac{\partial}{\partial \lambda} H(\lambda) | n(\lambda) \rangle + \langle n(\lambda) | H(\lambda) | \frac{\partial}{\partial \lambda} n(\lambda) \rangle.$$

Since $H(\lambda)|n(\lambda)\rangle = E(\lambda)|n(\lambda)\rangle$ and $\langle n(\lambda)|H(\lambda) = E(\lambda)\langle n(\lambda)|$, we have

$$\frac{dE(\lambda)}{d\lambda} = \langle n(\lambda) | \frac{\partial}{\partial \lambda} H(\lambda) | n(\lambda) \rangle + E(\lambda) \frac{\partial}{\partial \lambda} \langle n | n \rangle = \langle n(\lambda) | \frac{\partial}{\partial \lambda} H(\lambda) | n(\lambda) \rangle$$

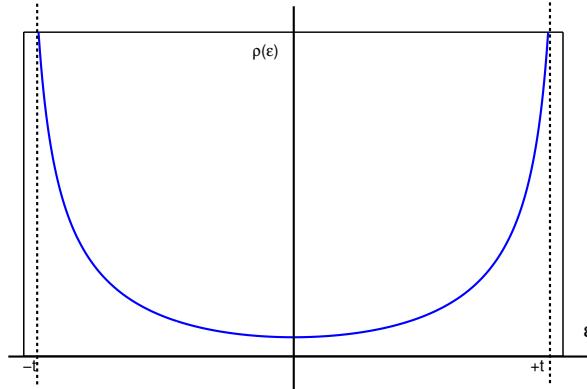
Ex. 7.7

$$\epsilon_k = -t \cos ka$$

$$\rho(\epsilon) = \frac{1}{2\pi} \int_{-\pi/a}^{\pi/a} dk \delta(\epsilon + t \cos ka) = \frac{1}{a\pi t} \frac{1}{\sqrt{1 - \epsilon^2/t^2}}, \text{ if } -1 < \epsilon/t < 1,$$

and zero otherwise (see plot).

The van Hove singularities are at $\epsilon = \pm t$, the minimum and maximum of energy.



Ex. 7.8

The dispersion $\epsilon_k = -t[\cos k_x a + \cos k_y a]$, $\nabla_k \epsilon_k = (at \sin k_x, at \sin k_y a)$. The van Hove singularity correspond to $\nabla_k \epsilon_k = 0$:

$$k_x = 0, \pm\pi/a, \quad k_y = 0, \pm\pi/a;$$

$$\epsilon = -2t, 0, +2t$$

The density of states can be written as a complete elliptical integral,

$$\rho(\epsilon) = \int \frac{d^2k}{(2\pi)^2} \delta(\epsilon - \epsilon_k) \propto \frac{1}{2t + |\epsilon|} K\left(\frac{2t - |\epsilon|}{2t + |\epsilon|}\right)$$

The complete elliptical integral $K(z)$ can be expanded as:

$$K(z) \propto -\frac{1}{2} \log(1-z)(1 - \frac{z-1}{4} + \frac{9}{64}(z-1)^2 + \dots)$$

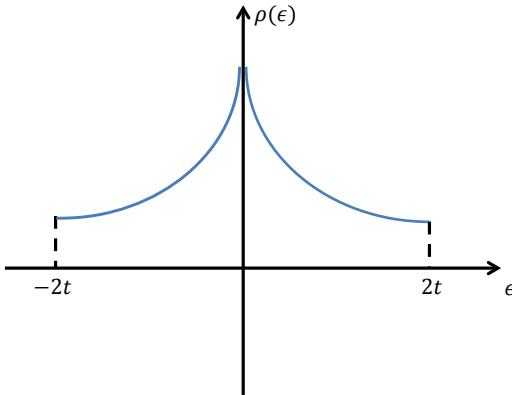
$K(z)$ diverges at $z = 1$

$$\epsilon = 0, \quad \rho(\epsilon) \rightarrow K(1) \quad \text{diverges} \propto \log \frac{2|\epsilon|}{2t + |\epsilon|}$$

$$\epsilon = \pm 2t, \quad \rho(\epsilon) \rightarrow K(0) \quad \text{converges}$$

The van Hove singularity locates at $\epsilon = 0$.

$$\begin{aligned} \vec{k} &= (0, \pm\pi/a) \\ \vec{k} &= (\pm\pi/a, 0) \end{aligned}$$



Ex. 7.9

$$v_k = \frac{1}{\hbar} \frac{\partial \epsilon(k)}{\partial k} = ta \sin ka.$$

It is clear v_k and k are parallel for $|ka| < \pi/2$ and antiparallel for $\pi/2 < |ka| < \pi$.

Ex. 7.10

Expand the band dispersion in Fourier series:

$$\epsilon_k = k^2 = \sum_{n=0}^{\infty} t_n \cos(nk),$$

so for $n \neq 0$ (which is what matters)

$$t_n = \frac{1}{\pi} \int_{-\pi}^{\pi} k^2 \cos(nk) dk = (-1)^n \frac{4}{n^2}.$$

Thus one can obtain a free-electron like dispersion (but only within 1BZ and then repeated periodically) with the carefully chosen hoppings above .

Ex. 7.11

(i)

$$H = \sum_j (t_1 |j\rangle_{AB} \langle j| + t_2 |j\rangle_{BA} \langle j+1| + h.c.)$$

$$|j\rangle_A = \frac{1}{\sqrt{N}} \sum_q e^{iqR_j} |q\rangle_A$$

$$|j\rangle_B = \frac{1}{\sqrt{N}} \sum_q e^{iqR_j} |q\rangle_B$$

Substitute the above into Hamiltonian, we obtain

$$H = \sum_q [|q\rangle_A \quad |q\rangle_B] \begin{bmatrix} 0 & t_1 e^{iqa_1} + t_2 e^{-iqa_2} \\ t_1 e^{-iqa_1} + t_2 e^{iqa_2} & 0 \end{bmatrix} \begin{bmatrix} {}_A\langle q| \\ {}_B\langle q| \end{bmatrix}$$

Eigenvalues are:

$$\epsilon_{\pm} = \pm \sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos(q(a_1 + a_2))}.$$

So we have two bands with a band gap separating them.

(ii)

As $a = a_1 = a_2$, $t = t_1 = t_2$

$$\epsilon_{\pm} = \pm \sqrt{2t^2 + 2t^2 \cos(2qa)} = \pm \sqrt{4t^2 \cos^2(qa)} = \pm 2t |\cos(qa)|.$$

with

$$q \in [-\pi/2a, \pi/2a].$$

So the band gaps vanishes at the zone boundary. This is, of course, because in this case the two bands are actually parts of the same band of Eq. (7.131) (see figure 7.9), with the segments for

$q \in [-\pi/a, -\pi/2a]$ shifted by π/a to $q \in [\pi/2a, \pi/a]$ and similarly for $q \in [\pi/2a, \pi/a]$. Such a shift is legitimate once we choose the unit cell size to be $2a$, even though in this special case the real unit cell size is a .

Ex. 7.12

(i)

$$\begin{aligned}
 |\chi_{nj}\rangle &= \frac{1}{\sqrt{N}} \sum_{\vec{q}} e^{-i\vec{q} \cdot \vec{R}_j} |\Psi_{n\vec{q}}\rangle \\
 &= \frac{1}{\sqrt{N}} \sum_{\vec{q}} e^{-i\vec{q} \cdot \vec{R}_j} \frac{1}{\sqrt{\eta(\vec{q})}} \sum_l e^{i\vec{q} \cdot \vec{R}_l} |nl\rangle \\
 &= \frac{1}{\sqrt{N}} \sum_{\vec{q}, l} \frac{1}{\sqrt{\eta(\vec{q})}} e^{-i\vec{q} \cdot (\vec{R}_j - \vec{R}_l)} |nl\rangle \\
 \langle \chi_{n'j'} | \chi_{nj} \rangle &= \frac{1}{N^2} \sum_{\vec{q}, l} \sum_{\vec{q}', l'} \frac{1}{\sqrt{\eta(\vec{q})} \sqrt{\eta(\vec{q}')}} e^{i\vec{q}' \cdot (\vec{R}_{j'} - \vec{R}_{l'}) - i\vec{q} \cdot (\vec{R}_j - \vec{R}_l)} \langle n'l' | nl \rangle \\
 &= \frac{1}{N^2} \sum_{\vec{q}\vec{q}'} \frac{1}{\sqrt{\eta(\vec{q})} \sqrt{\eta(\vec{q}')}} e^{i\vec{q}' \cdot \vec{R}_{j'} - i\vec{q} \cdot \vec{R}_j} \sum_{ll'} e^{i\vec{q} \cdot \vec{R}_l - i\vec{q}' \cdot \vec{R}_{l'}} \langle n'l' | nl \rangle \\
 &= \frac{1}{N} \sum_{\vec{q}\vec{q}'} \frac{1}{\sqrt{\eta(\vec{q})} \sqrt{\eta(\vec{q}')}} e^{i\vec{q}' \cdot \vec{R}'_j - i\vec{q} \cdot \vec{R}_j} \eta(\vec{q}) \delta_{\vec{q}\vec{q}'} \delta_{n,n'} \\
 &= \frac{1}{N} \sum_{\vec{q}} e^{i\vec{q} \cdot (\vec{R}'_j - \vec{R}_j)} \delta_{n,n'} \\
 &= \delta_{\vec{q}\vec{q}'} \delta_{n,n'}.
 \end{aligned}$$

(ii) This is simply the counter-Fourier transformation of (7.137).

Ex. 7.13

$$\begin{aligned}
 H &= \sum_n H_n \\
 H_n &= \sum_{\vec{q}} \epsilon_{n\vec{q}} |\Psi_{n\vec{q}}\rangle \langle \Psi_{n\vec{q}}|
 \end{aligned}$$

Using

$$|\Psi_{n\vec{q}}\rangle = \frac{1}{\sqrt{N}} \sum_l e^{i\vec{q} \cdot \vec{R}_l} |\chi_{nl}\rangle$$

we have

$$\begin{aligned}
H_n &= \sum_{\vec{q}} \epsilon_{n\vec{q}} \frac{1}{N} \sum_{ll'} e^{i\vec{q} \cdot \vec{R}_l} e^{-i\vec{q} \cdot \vec{R}'_l} |\chi_{nl'}\rangle \langle \chi_{nl}| \\
&= \sum_{ll'} |\chi_{nl}\rangle \langle \chi_{nl}| \frac{1}{N} \sum_{\vec{q}} \epsilon_{n\vec{q}} \sum_{ll'} e^{i\vec{q} \cdot (\vec{R}_l - \vec{R}'_l)} \\
&= \sum_{ll'} t_n(\vec{R}_l - \vec{R}'_l) |\chi_{nl}\rangle \langle \chi_{nl}|
\end{aligned}$$

where:

$$t_n(\vec{R}) = \frac{1}{N} \sum_{\vec{q}} \epsilon_{n\vec{q}} e^{i\vec{q} \cdot \vec{R}}$$

Ex. 7.14

This follows from $\langle j|H|\Psi\rangle = \epsilon \langle j|\Psi\rangle$.

Ex. 7.15

For $\vec{q} \approx \vec{k}$ we have Bloch state

$$\Phi_j = \frac{1}{\sqrt{N}} e^{i(\vec{k} + \delta\vec{q}) \cdot \vec{R}_j},$$

thus $\Psi_j = e^{-i\vec{k} \cdot \vec{R}_j} \Phi_j$ is slow varying in j . For $K = \pi/a$ we simply have $\Psi_j = (-1)^j \Phi_j$. It is then straightforward to follow steps in the text to obtain $\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} \Psi(x) = \epsilon' \Phi(x)$. Note the sign change for the mass.

Ex. 7.16

Let Π be the parity operator.

$$\Pi |\Phi_{p\vec{k}}\rangle = -|\Phi_{p(-\vec{k})}\rangle$$

$$\Pi |\Phi_{s\vec{k}}\rangle = +|\Phi_{s(-\vec{k})}\rangle$$

$$\Pi H \Pi = H$$

. Combining these yields

$$I_{\vec{k}} = \langle \Phi_{p\vec{k}} | H | \Phi_{s\vec{k}} \rangle = -I_{-\vec{k}}$$

which implies $I_{\vec{0}} = -I_{\vec{0}} = 0$.

Similarly in 1D, $I_{\frac{\pi}{a}} = -I_{-\frac{\pi}{a}}$. But since $\pm \frac{\pi}{a}$ are on the zone boundary, they are equivalent points. Hence $I_{\pm \frac{\pi}{a}} = 0$.

Ex. 7.17

a) Examination of Fig. (4) shows that if the transfer integrals from S to P on the left are J_x, J_y, J_z , then the corresponding transfer integrals to the right are $-J_x, J_y, J_z$. This is because the S orbital connects to the negative lobe of the P_x orbital on the right but connects to the positive lobe of the P_x orbital to the left. Hence the Hamiltonian has the following action on the basis states

$$H|S\rangle_j = \epsilon_S|S\rangle_j + J_x[-|P_x\rangle_j + |P_x\rangle_{j-1}] \quad (17)$$

$$H|P_x\rangle_j = \epsilon_P|P_x\rangle_j + J_x[|S\rangle_{j+1} - |S\rangle_j]. \quad (18)$$

b) The S orbital is symmetric under $y \rightarrow -y$ and under $z \rightarrow -z$, whereas P_y and P_z are odd under the corresponding reflection. Hence $J_y = J_z = 0$.

c,d) There are four atomic orbitals and hence there must be four bands. Since $J_y = J_z = 0$ by symmetry, there must be two flat bands with no dispersion, each with constant energy ϵ_p . We therefore concentrate only on the S and P_x orbitals which hybridize to form two bands. Assume a plane wave basis of the form

$$|\psi_k\rangle = \sum_j e^{ika_j} \begin{pmatrix} \phi_S(k)|S\rangle_j \\ \phi_{Px}(k)|P_x\rangle_j \end{pmatrix}$$

where a is the lattice constant and k is the wave vector in the first BZ. In this basis the tight-binding Hamiltonian has the form

$$H_k = \begin{pmatrix} \epsilon_S & -J_x(1 - e^{-ika}) \\ -J_x(1 - e^{+ika}) & \epsilon_P \end{pmatrix}.$$

We can simplify this by making a ‘gauge change’ that inserts a relative phase $e^{ika/2}$ between the S and P orbitals which yields

$$H_k = \begin{pmatrix} \epsilon_S & -J_x(e^{+ika/2} - e^{-ika/2}) \\ +J_x(e^{+ika/2} - e^{-ika/2}) & \epsilon_P \end{pmatrix}.$$

This can be expressed in terms of the Pauli matrices as

$$H_k = \frac{\epsilon_S + \epsilon_P}{2} + \frac{\epsilon_S - \epsilon_P}{2}\sigma^z + 2J_x \sin\left(\frac{ka}{2}\right)\sigma^y,$$

and the eigenvalues are thus

$$E_{\pm}(k) = \frac{\epsilon_S + \epsilon_P}{2} \pm \sqrt{\left(\frac{\epsilon_S - \epsilon_P}{2}\right)^2 + \left(2J_x \sin\left(\frac{ka}{2}\right)\right)^2}.$$

An example band dispersion is shown in Fig. (5), computed for the case $\epsilon_S = 2, \epsilon_P = 1$. Note that the bands derived from the P_y and P_z orbitals are flat (and degenerate) because $J_y = J_z = 0$. Also notice that at $k = 0$ there is no hybridization between the S and P_x bands because of parity symmetry. One can see in the spectrum that the level repulsion indicating hybridization starts at zero but grows larger as $|k|$ increases.

e) To prevent confusion with the J notation for the transfer integrals let us call the total orbital plus spin angular momentum on the P atom

$$\vec{Q} = \vec{L} + \vec{S}. \quad (19)$$

The spin-orbit Hamiltonian is then given by

$$\Delta H = \Gamma \vec{L} \cdot \vec{S} = \frac{\Gamma}{2} [Q^2 - L^2 - S^2] = \frac{\Gamma}{2} \left[q(q+1) - \frac{11}{4} \right]. \quad (20)$$

It will simplify matters if we choose coordinates so that the Q_z, L_z, S_z components are measured along the line of the 1D lattice (the direction previously called x above). The spin and orbital angular momentum on the P atom can couple to $q = 3/2$ with $m_q = 3/2, 1/2, -1/2, -3/2$ and to $q = 1/2$ with $m_q = \pm 1/2$. The eigenvalues of ΔH are therefore $\pm \Gamma$.

The corresponding eigenstates are given by

$$|q = 3/2, m_q = 3/2\rangle = |m_\ell = +1, m_s = +\frac{1}{2}\rangle \quad (21)$$

$$|q = 3/2, m_q = 1/2\rangle = \sqrt{2/3}|m_\ell = 0, m_s = +\frac{1}{2}\rangle + \sqrt{1/3}|m_\ell = +1, m_s = -\frac{1}{2}\rangle \quad (22)$$

$$|q = 3/2, m_q = -1/2\rangle = \sqrt{2/3}|m_\ell = 0, m_s = -\frac{1}{2}\rangle + \sqrt{1/3}|m_\ell = -1, m_s = +\frac{1}{2}\rangle \quad (23)$$

$$|q = 3/2, m_q = -3/2\rangle = |m_\ell = -1, m_s = -\frac{1}{2}\rangle \quad (24)$$

$$|q = 1/2, m_q = +1/2\rangle = \sqrt{1/3}|m_\ell = 0, m_s = +\frac{1}{2}\rangle - \sqrt{2/3}|m_\ell = +1, m_s = -\frac{1}{2}\rangle \quad (25)$$

$$|q = 1/2, m_q = -1/2\rangle = \sqrt{1/3}|m_\ell = 0, m_s = -\frac{1}{2}\rangle - \sqrt{2/3}|m_\ell = -1, m_s = +\frac{1}{2}\rangle \quad (26)$$

On the S atom the orbital angular momentum is zero and the spin is $m_s = \pm 1/2$. By rotational symmetry, the transfer matrix element from the S atom to the P atom can only take the electron into the P orbital with $m_\ell = 0$. The transfer matrix element from $|L = 0, m_s = \pm 1/2\rangle$ into $|q = 3/2, m_q = \pm 3/2\rangle$ therefore vanishes, producing two degenerate flat bands with energy $\epsilon_P + \Gamma$. The matrix element for transfer from $|L = 0, m_s = \pm 1/2\rangle$ into $|q = 3/2, m_q = \pm 1/2\rangle$ is renormalized from J_x to $\sqrt{2/3}J_x$. The matrix element for transfer from $|L = 0, m_s = \pm 1/2\rangle$ into $|q = 1/2, m_q = \pm 1/2\rangle$ is renormalized from J_x to $\sqrt{1/3}J_x$. [Here we continue to use the J_x notation as previously, despite the change of coordinates.]

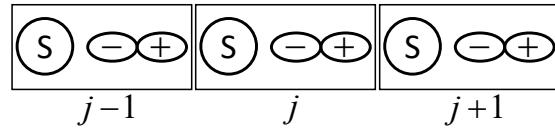


Figure 4: 1D lattice with two sites per unit cell. The first is the S orbital atom and the second is the P orbital atom. Only the P_x orbital is shown. The P_y and P_z orbitals are not shown because symmetry prevents them from hybridizing with their neighboring S orbitals.

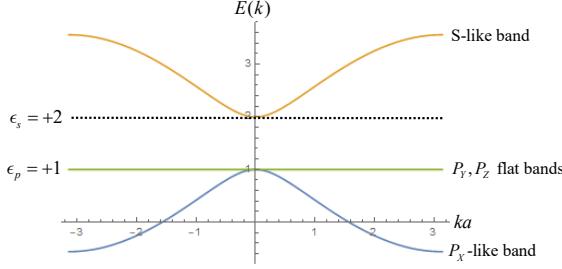


Figure 5: Band structure showing the two unhybridized (and hence flat) $P_{y,z}$ bands and the hybridized S and P_x bands.

Ex. 7.18

We use the basis states of Eqs. (7.186 – 7.190), whose angular parts are properly normalized already. For symmetry reasons it is clear ΔV only has diagonal matrix elements in this basis. Also the difference can only come from the angular part, as all $\ell = 2$ wave functions share the same radial part.

$$\langle \Phi_{xy} | \Delta V | \Phi_{xy} \rangle \propto \int d\Omega \sin^4 \theta \sin^2 \phi \cos^2 \phi [\sin^4 \theta (\sin^4 \phi + \cos^4 \phi) + \cos^4 \theta] = \frac{44\pi}{315},$$

where $\int d\Omega = \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi$. For symmetry reasons we obtain exactly the same for $\langle \Phi_{yz} | \Delta V | \Phi_{yz} \rangle$ and $\langle \Phi_{zx} | \Delta V | \Phi_{zx} \rangle$.

$$\langle \Phi_{x^2-y^2} | \Delta V | \Phi_{x^2-y^2} \rangle \propto \frac{1}{4} \int d\Omega \sin^4 \theta (\sin^2 \phi - \cos^2 \phi)^2 [\sin^4 \theta (\sin^4 \phi + \cos^4 \phi) + \cos^4 \theta] = \frac{4\pi}{21},$$

and

$$\langle \Phi_{z^2} | \Delta V | \Phi_{z^2} \rangle \propto \frac{1}{12} \int d\Omega (3 \cos^2 \phi - 1)^2 [\sin^4 \theta (\sin^4 \phi + \cos^4 \phi) + \cos^4 \theta] = \frac{4\pi}{21}.$$

As a result the 5 d-orbitals are split into 2 levels, made of the 3 t_{2g} and 2 e_g orbitals respectively.

Ex. 7.19

i)

$$H = -t \sum_{\vec{q}} \begin{bmatrix} |\vec{q}\rangle_A \\ |\vec{q}\rangle_B \end{bmatrix} \begin{bmatrix} V/t & f(\vec{q}) \\ f^*(\vec{q}) & -V/t \end{bmatrix} \begin{bmatrix} {}_A\langle \vec{q}| \\ {}_B\langle \vec{q}| \end{bmatrix}$$

$$f(\vec{q}) = i \sum_{j=1}^3 e^{i\vec{q} \cdot \vec{\delta} j} = i(1 + e^{i\vec{q} \cdot \vec{a}_1} + e^{i\vec{q} \cdot \vec{a}_2})$$

$$h_{\vec{q}} = -t \begin{bmatrix} V/t & f(\vec{q}) \\ f^*(\vec{q}) & -V/t \end{bmatrix}$$

Diagonalizing $h_{\vec{q}}$:

$$\begin{vmatrix} V - E & -tf(\vec{q}) \\ -tf^*(\vec{q}) & -t - E \end{vmatrix} = 0$$

$$\Rightarrow E = \pm \sqrt{V^2 + t^2|f(\vec{q})|^2}$$

The band gap:

$$\Delta = |E_+(\vec{q}) - E_-(\vec{q})|_{\vec{q}+\vec{k}=0} = |2\sqrt{V^2 + t^2|f(\vec{q})|^2}|_{\vec{q}+\vec{k}=0} = 2|V|$$

ii)

When $V \neq 0$

At the bandgap $\vec{q} \sim \vec{k}$

$$h_{\vec{q} \sim \vec{k}} \simeq v_F \begin{bmatrix} V/v_F & k_x - ik_y \\ k_x + ik_y & -V/v_F \end{bmatrix} = v_F(\sigma_x k_x + \sigma_y k_y) + V\sigma_z$$

Diagonalizing $h_{\vec{q}}$:

$$\begin{vmatrix} V - E & v_F(k_x - ik_y) \\ v_F(k_x + ik_y) & -V - E \end{vmatrix} = 0$$

$$\Rightarrow E = \pm \sqrt{V^2 + v_F^2(k_x^2 + k_y^2)} = \pm \sqrt{V^2 + v_F^2 k^2}$$

Obviously the mass term $m = V$,

$$\Delta = |E_+ - E_-|_{\vec{k}=0} = 2\sqrt{V^2 + v_F^2 k^2}|_{\vec{k}=0} = 2m$$

Ex. 7.20

Near \vec{K} the slowly-varying, two-component wave function (with component labels A and B) are

$$\psi_A(\vec{R}_i) = e^{-i\vec{K}\cdot\vec{R}_i} \Phi_A(\vec{R}_i),$$

$$\psi_B(\vec{R}_j) = e^{-i\vec{K}\cdot\vec{R}_j} \Phi_B(\vec{R}_j),$$

where i, j belong to A and B sublattices respectively. Expressing the eigen equations in terms of ψ_A and ψ_B and replacing the differences by appropriate (1st order) derivatives lead to the Dirac equation.

Ex. 7.21

$$\{H, C\} = 0 \quad & H|\psi\rangle = \epsilon|\psi\rangle$$

$$HC + CH = 0 \Rightarrow HC = -CH.$$

$$|\psi'\rangle = C|\psi\rangle \Rightarrow H|\psi'\rangle = HC|\psi\rangle = -CH|\psi\rangle = -\epsilon|\psi'\rangle$$

Then $\langle\psi|\psi'\rangle = 0$, since $\epsilon \neq \epsilon' = -\epsilon$ (eigenstates of H with different eigenvalues).

Ex. 7.22

We can write the hopping Hamiltonian

Consider a generic term in such a hopping Hamiltonian, $h = t|l\rangle\langle m|$, and without loss of generality assume l is even and m is odd. Then

$$h\Sigma_z = t|l\rangle\langle m|(|l\rangle\langle l| - |m\rangle\langle m|) = -t|l\rangle\langle m| = -(|l\rangle\langle l| - |m\rangle\langle m|)t|l\rangle\langle m| = -\Sigma_z h.$$

So h anti-commutes with Σ_z , and so is the case for a hopping Hamiltonian made of such terms.

Ex. 7.23

(i)

Obviously Σ_z commutes instead of anti-commutes with itself, thus once H' is included, the Hamiltonian no longer anti-commutes with Σ_z , and particle-hole symmetry is lost.

(ii)

The zero mode wave functions have non-zero amplitudes at either even sites or odd sites, which are eigenmodes of Σ_z with eigenvalues ± 1 respectively. As a result their energies are shifted by $\pm m_z$ without changing their wave functions. We thus find the zero energy of these "zero" modes require protection from particle-hole symmetry.

(iii)

the two cases of (ii) correspond to solitons and anti-solitons respectively, thus the conclusion follow.

(iv)

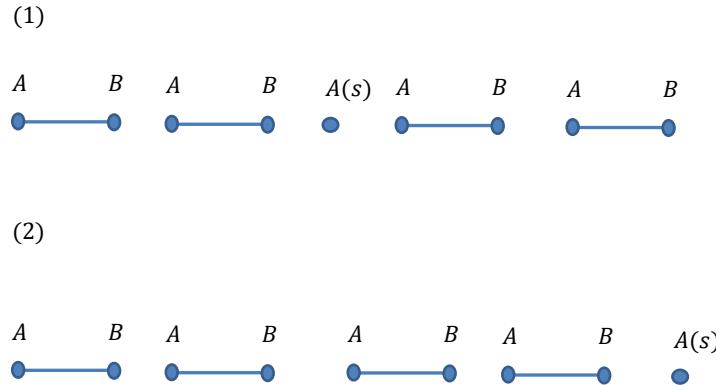
Once $|m_z|$ reaches half the band gap, these soliton/anti-soliton bound states energies enter the continuum formed by conduction and valence band states.

(v) In this case the even or oddness of a site cannot be properly defined, as a result the operator Σ_z is not well-defined, thus a term of the form (7.250) does not exist.

Ex. 7.24

Let us illustrate this with a specific case, with odd number of sites, $j = 1, 2, \dots, 2l, 2l+1$, and the bond $J_{12} = J > 0$ along with all odd bonds, while even bonds are all 0. Clearly there is a zero mode on site $2l+1$, because $J_{2l,2l+1} = 0$ (see figure below, case (2)). With one soliton (or odd number of solitons) we have $J_{2l,2l+1} = J$, thus the zero mode on site $2l+1$ disappears (see figure below, case

(1)). While for even number of solitons we have $J_{2l,2l+1} = 0$, and the zero mode reappears. Thus each time the bulk soliton number changes by one, the boundary zero mode changes accordingly.



Ex. 7.25

(i)

Yes in this case there is a (similar) trimerization instability, in which groups of 3 neighboring atoms (A, B, and C, with A and C move in opposite directions toward B) move together to triple the unit cell. As a result the BZ size becomes 1/3 of the original BZ, opening up a gap at the Fermi energy and lowering electronic energy. Essentially the same calculation as in text shows the electronic energy gain overwhelms the elastic energy cost by a logarithmic factor.

(ii)

In this case we have 3 types of domains corresponding to patterns starting with the 1st atom: ABCABC..., BCABCA..., and CABCAB.... There are thus 6 types of solitons. The charge quantum numbers of the soliton bound states are integer multiples of 1/3, while spin quantum number is 0 or 1/2. These are most easily seen in the extreme trimerization limit, in which hopping only exists among groups of 3 atoms that move together.

Ex. 7.26

(i) Define $\Delta\epsilon = \epsilon - \mu$. Then

$$\Delta f^\circ(\Delta\epsilon) = \frac{1}{e^{\beta\Delta\epsilon} + 1} - \theta(-\Delta\epsilon) = \begin{cases} \frac{1}{e^{\beta\Delta\epsilon} + 1}, & \Delta\epsilon > 0; \\ \frac{1}{e^{\beta\Delta\epsilon} + 1} - 1 = -\frac{1}{e^{-\beta\Delta\epsilon} + 1}, & \Delta\epsilon < 0. \end{cases}$$

Clearly $\Delta f^\circ(\Delta\epsilon) = -\Delta f^\circ(-\Delta\epsilon)$.

(ii)

$$H(\epsilon) = H(\mu) + \sum_{m=1}^{\infty} \frac{1}{m!} H^{(m)}(\mu) \Delta\epsilon^m.$$

Putting the above in the integral, we obtain zero for even m , and for odd m

$$\int_{-\infty}^{\infty} \Delta\epsilon^m \Delta f(\delta\epsilon) d\epsilon = 2 \int_0^{\infty} \frac{(\Delta\epsilon)^m}{e^{\beta\Delta\epsilon} + 1} d\Delta\epsilon = 2(k_B T)^{m+1} \int_0^{\infty} \frac{x^m}{e^x + 1} dx = 2(k_B T)^{m+1} I_m.$$

Eq. (7.278) follows.

Ex. 7.27

$\frac{\partial}{\partial\epsilon} f^o(\epsilon) = \frac{\beta e^{\beta\Delta\epsilon}}{(e^{\beta\Delta\epsilon} + 1)^2}$ is an even function of $\Delta\epsilon$, peaking at $\Delta\epsilon = 0$. Expanding

$$H(\epsilon) = H(\mu) + \sum_{m=1}^{\infty} \frac{1}{m!} H^{(m)}(\mu) \Delta\epsilon^m,$$

we find

$$I = -H(\mu) + \sum_{n=1}^{\infty} \frac{(k_B T)^{2n-1}}{(2n)!} H^{2n}(\mu) I'_{2n},$$

where

$$I'_{2n} = \int_0^{\infty} \frac{x^{2n} e^x}{(e^x + 1)^2} dx.$$

Ex. 7.28

$$\begin{aligned} n &= \int_{-\infty}^{\mu} \rho_B(\epsilon) d\epsilon = n(B=0, \mu) + \int_{\mu}^{\mu_B B} \rho(\epsilon) d\epsilon - \int_{\mu - \mu_B B}^{\mu} \rho(\epsilon) d\epsilon \\ &= n(B=0, \mu) + \rho'(\mu) \left[\int_0^{\mu_B B} \Delta\epsilon d\Delta\epsilon - \int_{-\mu_B B}^0 \Delta\epsilon d\Delta\epsilon \right] + O((\mu_B B)^4) \\ &= n(B=0, \mu) + \frac{\partial n}{\partial \mu}|_{\mu_0} (\mu - \mu_0) + \rho'(\mu_0)(\mu_B B)^2 + O((\mu - \mu_0)^2) + O((\mu - \mu_0)(\mu_B B)^2) + O((\mu_B B)^4) \\ &= n(B=0, \mu) + 2\rho(\mu_0)(\mu - \mu_0) + \rho'(\mu_0)(\mu_B B)^2 + \text{corrections}. \end{aligned}$$

Since the above yields $\mu - \mu_0 = O(\mu_B B)^2$, all corrections are $O(\mu_B B)^4$. Thus from Eq. (7.270), to $O(\mu_B B)^2$, we have

$$\begin{aligned} \mathcal{E} &= \mathcal{E}(B=0, \mu_0) + \frac{\partial \mathcal{E}(B=0, \mu)}{\partial \mu}|_{\mu_0} (\mu - \mu_0) + \mu \rho'(\mu)(\mu_B B)^2 - \rho(\mu_0)(\mu_B B)^2 \\ &= \mathcal{E}(B=0, \mu_0) - \rho(\mu_0)(\mu_B B)^2. \end{aligned}$$

Ex. 7.29

Obviously the magnetization energy density is

$$\mathcal{E}_B = -MB.$$

The magnetization induces a Fermi energy shift of $\Delta\epsilon = M/(2\mu_B B)$ in the opposite direction for up- and down-spin electrons. This results in a change of kinetic energy density

$$\Delta\mathcal{E}_K = \rho(\mu)(\Delta\epsilon)^2 = \rho(\mu)[M/(2\mu_B B)]^2.$$

Minimizing $\mathcal{E}_B + \Delta\mathcal{E}_K$ w.r.t. M yields $M = 2\mu_B^2\rho(\mu)$, in agreement with (7.275).

Ex. 7.30

i)

$$\rho(\epsilon) = \frac{1}{V} \sum_j \delta(\epsilon - \epsilon_j).$$

With p-h symmetry, $\epsilon_{j,j'}$ come in $\pm\epsilon_j$ pairs, as a result $\rho(\epsilon) = \rho(-\epsilon)$.

ii)

We know when $\mu = 0$, $\langle n(\epsilon) \rangle + \langle n(-\epsilon) \rangle = 1$ is fixed for any T . With (i) this implies the density is also fixed. This implies $\mu(T) = 0$ if $\mu(T=0) = 0$.

Ex. 7.31

It is clear from the spectrum (7.204) that graphene band structure is particle-hole symmetric. More formally, one can define an operator a la Eq. (7.247), where j is even/odd on A/B sublattices, which anti-commutes with the Hamiltonian, resulting in particle-hole symmetry and $\rho(\epsilon) = \rho(-\epsilon)$.

At low T we can use the linear spectrum $\epsilon_k = v_F k$, resulting in $\rho(\epsilon) = \frac{|\epsilon|}{\pi(\hbar v_F)^2}$. Internal energy density is thus

$$\mathcal{E}(T) = 2 \int_{-\infty}^{\infty} d\epsilon \rho(\epsilon) \epsilon f(\epsilon) - \mathcal{E}_0 = 4 \int_0^{\infty} d\epsilon \rho(\epsilon) \epsilon f(\epsilon) = \frac{4(k_B T)^3}{\pi(\hbar v_F)^2} \frac{3}{2} \xi(3),$$

resulting in

$$c_v = \frac{18k_B^3 \xi(3)}{\pi(\hbar v_F)^2} T^2.$$

Magnetization can be calculated using the method of Ex. 7.29; in this case we have to first determine $\Delta\epsilon$ in terms of M :

$$M = 2\mu_B \int_0^{\Delta\epsilon} \rho(\epsilon) d\epsilon = 2\alpha\mu_B \int_0^{\Delta\epsilon} \epsilon d\epsilon = \alpha\mu_B (\Delta\epsilon)^2,$$

where $\alpha = 1/[\pi(\hbar v_F)^2]$. Thus

$$\Delta\mathcal{E}_K = 2 \int_0^{\Delta\epsilon} \epsilon \rho(\epsilon) d\epsilon = \frac{2}{3} \alpha (\Delta\epsilon)^3 = \frac{2}{3} \alpha \left(\frac{M}{\alpha\mu_B}\right)^{3/2}.$$

Minimizing $\Delta\mathcal{E}_K - MB$ yields

$$M = \alpha\mu_B^3 B^2.$$

Ex. 7.32

1)

Time reversal symmetry requires $[\Theta, H] = 0$, or $\Theta H \Theta^{-1} = H$. For spin-less particles hit a Bloch state [namely $H\psi_{n,\vec{k}}(\vec{r}) = E_n(\vec{k})\psi_{n,\vec{k}}(\vec{r})$] with Θ :

$$\Theta\psi_{n,\vec{k}}(\vec{r}) = \psi_{n,-\vec{k}}(\vec{r}) \quad \text{change } \vec{k} \text{ to } -\vec{k}.$$

Thus

$$H\Theta\psi_{n,\vec{k}}(\vec{r}) = \Theta H\psi_{n,\vec{k}}(\vec{r}) = E_n(\vec{k})\Theta\psi_{n,\vec{k}}(\vec{r}),$$

or equivalently

$$\begin{aligned} H\psi_{n,-\vec{k}}(\vec{r}) &= E_n(\vec{k})\psi_{n,-\vec{k}}(\vec{r}) \\ \Rightarrow E_n(\vec{k}) &= E_n(-\vec{k}) \end{aligned}$$

2)

For spin-1/2 electrons $\Theta = K\sigma_y$. Without spin-orbit coupling S_z is a good quantum number, the Bloch states came in degenerate spin-up and -down pairs represented by

$$\psi_{n,\vec{k}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \psi_{n,\vec{k}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $\psi_{n,\vec{k}} = e^{i\vec{k}\cdot\vec{r}} u_{n\vec{k}}(\vec{r})$.

$$\begin{aligned} \Theta\psi_{n,\vec{k}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= ie^{-i\vec{k}\cdot\vec{r}} u_{n\vec{k}}^*(\vec{r}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i\psi_{n,\vec{k}}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i\psi_{n,-\vec{k}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \Theta\psi_{n,\vec{k}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= -ie^{-i\vec{k}\cdot\vec{r}} u_{n\vec{k}}^*(\vec{r}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i\psi_{n,\vec{k}}^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i\psi_{n,-\vec{k}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ H\psi_{n,\vec{k}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \epsilon_{n,\vec{k}} \psi_{n,\vec{k}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Theta H = H\Theta, \Theta H\Theta^{-1} = H$$

$$\Rightarrow E_n(\vec{k}) = E_n(-\vec{k})$$

Since E is an even function of \vec{k} , its gradient at $\vec{k} = 0$ vanishes.

Ex. 7.33

Under time-reversal transformation (Θ), $\frac{\vec{G}}{2} + \vec{k} \rightarrow -\frac{\vec{G}}{2} - \vec{k} \sim \frac{\vec{G}}{2} - \vec{k}$, thus $\epsilon_n(\frac{\vec{G}}{2} + \vec{k}) = \epsilon_n(\frac{\vec{G}}{2} - \vec{k})$ if Θ is a symmetry. Thus $\nabla \epsilon_n(\vec{k})|_{\vec{k}=\frac{\vec{G}}{2}}$ must be zero, as otherwise

$$\epsilon_n(\frac{\vec{G}}{2} + \delta\vec{k}) \approx \epsilon_n(\frac{\vec{G}}{2}) + \nabla \epsilon_n(\vec{k})|_{\vec{k}=\frac{\vec{G}}{2}} \cdot \delta\vec{k} \neq \epsilon_n(\frac{\vec{G}}{2} - \delta\vec{k})$$

if $\frac{\vec{G}}{2} \cdot \delta\vec{k} \neq 0$. Thus $\vec{v}(\vec{k} = \vec{G}/2) = 0$.

Ex. 7.34

To show the desired relation we need to be able to integrate by parts an expression of the form

$$\int_{\Omega} d^3r \vec{C} \cdot (\vec{\nabla} \times \vec{D})$$

where Ω is the integration volume. We can make use of the vector identity

$$\vec{C} \cdot (\vec{\nabla} \times \vec{D}) = \vec{D} \cdot (\vec{\nabla} \times \vec{C}) - \vec{\nabla} \cdot (\vec{C} \times \vec{D})$$

to obtain

$$\int_{\Omega} d^3r \vec{C} \cdot (\vec{\nabla} \times \vec{D}) = \int d^3r \vec{D} \cdot (\vec{\nabla} \times \vec{C}) - \int_{\partial\Omega} d\vec{A} \cdot (\vec{C} \times \vec{D}),$$

where the last integral is over the surface $\partial\Omega$ of the region Ω . We must assume that the symmetry of the problem is such that the surface integral (the 'boundary term' in the integration by parts) vanishes, or else that magnetic field vanishes sufficiently rapidly and the boundary is sufficiently far away that the integrand vanishes everywhere on the boundary.

Applying this integration by parts to the inner product

$$\begin{aligned} (\vec{B}_1, \Xi \vec{B}_2) &= \int_{\Omega} d^3r \vec{B}_1^* \cdot \left\{ \vec{\nabla} \times \left[\frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times \vec{B}_2 \right] \right\} \\ &= \int_{\Omega} d^3r (\vec{\nabla} \times \vec{B}_2) \cdot \left[\frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times \vec{B}_1^* \right] \end{aligned}$$

Integrating by parts a second time yields

$$\begin{aligned} (\vec{B}_1, \Xi \vec{B}_2) &= \int_{\Omega} d^3r \vec{B}_2 \cdot \left\{ \vec{\nabla} \times \left[\frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times \vec{B}_1^* \right] \right\} \\ &= (\Xi \vec{B}_1, \vec{B}_2), \end{aligned}$$

which proves the result.

Ex. 7.35

Eliminating \vec{B} in favor of \vec{E} yields

$$\frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = \frac{\omega^2}{c^2} \vec{E}.$$

This is not in Sturm-Liouville form. That is, the factor of $\frac{1}{\epsilon(\vec{r})}$ is not sitting symmetrically between the two derivative operators. Hence the operator is not symmetric (Hermitian). Defining

$$\Xi = \frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times \vec{\nabla},$$

we have

$$(\vec{E}_1, \Xi \vec{E}_2) = \int_{\Omega} d^3 r \vec{E}_1^* \cdot \left\{ \frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times [\vec{\nabla} \times \vec{E}_2] \right\}.$$

Integration by parts twice (assuming boundary terms can be neglected) yields

$$\begin{aligned} (\vec{E}_1, \Xi \vec{E}_2) &= \int_{\Omega} d^3 r \vec{E}_1^* \cdot \left\{ \frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times [\vec{\nabla} \times \vec{E}_2] \right\} \\ &= \int_{\Omega} d^3 r \cdot \left\{ \vec{\nabla} \times \vec{\nabla} \times \left[\frac{1}{\epsilon(\vec{r})} \vec{E}_1^* \right] \right\} \cdot \vec{E}_2, \\ &\neq (\Xi \vec{E}_1, \vec{E}_2). \end{aligned}$$

QED.

Ex. 7.36

The reflection coefficient r and transmission coefficient t for a wave traveling to the right in medium 1 with index of refraction n_1 when it reaches the interface with medium 2 with index of refraction n_2 is

$$r = \frac{n_1 - n_2}{n_1 + n_2} \quad (27)$$

$$t = 2 \frac{\sqrt{n_1 n_2}}{n_1 + n_2}. \quad (28)$$

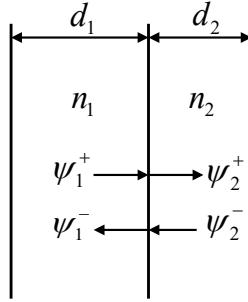
Notice that $r^2 + t^2 = 1$ as expected and that r is odd under interchange of the two media. Hence the reflection coefficient for a wave traveling to the left in medium 2 and reaching the interface is $-r$. The transmission amplitude t however is the same in both cases.

The outgoing amplitudes associated with incoming waves in both media is given by the scattering matrix S

$$\begin{pmatrix} \psi_1^- \\ \psi_2^+ \end{pmatrix} = S \begin{pmatrix} \psi_1^+ \\ \psi_2^- \end{pmatrix}, \quad (29)$$

$$S = \begin{pmatrix} r & t \\ t & -r \end{pmatrix}, \quad (30)$$

where the superscript \pm refers to right and left moving waves respectively (as illustrated in the figure below).



It is useful to convert the scattering matrix S into the transfer matrix M which gives the amplitudes in the two waves in medium 2 in terms of the amplitudes of the wave in medium 1

$$\begin{pmatrix} \psi_2^+ \\ \psi_2^- \end{pmatrix} = M \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}, \quad (31)$$

$$M = \begin{pmatrix} 1/t & -r/t \\ -r/t & 1/t \end{pmatrix}. \quad (32)$$

From one edge of medium 2 to the far edge, the phase change of the waves is described by the propagation matrix

$$P = \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{pmatrix}, \quad (33)$$

where $\theta_2 = k_0 n_2 d_2$, with k_0 being the free space wave vector at the corresponding frequency of the wave and d_2 is the thickness of layer 2.

At this point the wave has reached the interface with the next layer of medium 1. The transfer matrix into medium 1 is $M'(r) = M(-r)$

$$M' = \begin{pmatrix} 1/t & +r/t \\ +r/t & 1/t \end{pmatrix}. \quad (34)$$

Notice the fact that

$$M' = M^{-1}. \quad (35)$$

The importance of this will become clear shortly.

Finally the waves must be propagated forward in medium 1 a distance d_1 using the matrix $P(\theta_1)$ where $\theta_1 = k_0 n_1 d_1$. Having propagated through both medium 2 and medium 1, the cycle is ready to repeat. The overall transfer matrix for this step of the cycle is

$$\bar{M} = P(\theta_1) M' P(\theta_2) M. \quad (36)$$

The fact that $M' = M^{-1}$ guarantees that if we take the thickness of medium 2 to zero (so that $P(\theta_2)$ approaches the identity matrix), there are no reflections and the overall transfer matrix becomes

$$\bar{M} = P(\theta_1), \quad (37)$$

as it should.

The parameters of this particular problem have been chosen so that both layers have the same optical thickness

$$n_1 d_1 = n_2 d_2 \quad (38)$$

and hence $\theta_1 = \theta_2 \equiv \theta$. This simplification yields

$$\bar{M} = \begin{pmatrix} \frac{e^{2i\theta}}{t^2} - \frac{r^2}{t^2} & \frac{r}{t^2} - \frac{e^{2i\theta}r}{t^2} \\ \frac{r}{t^2} - \frac{e^{-2i\theta}r}{t^2} & \frac{e^{-2i\theta}}{t^2} - \frac{r^2}{t^2} \end{pmatrix} \quad (39)$$

Since the medium is periodic (with period $d = d_1 + d_2$), Bloch's theorem applies and propagating solutions will be eigenvectors of \bar{M} obeying

$$\bar{M}\Psi = e^{i\xi}\Psi, \quad (40)$$

where $\xi = kd$. It is straightforward to show that $\text{Det}\bar{M} = 1$ and hence the two eigenvalues of \bar{M} are $e^{\pm i\xi}$. It is convenient to relate this to the trace of \bar{M} via

$$\cos \xi = \frac{\cos(2\theta) - r^2}{t^2}. \quad (41)$$

Propagating solutions exist only if ξ is real which means that $|\cos \xi| \leq 1$. From the above equation we see that no propagating solutions exist in the regime

$$\cos(2\theta) < r^2 - t^2 = 1 - 2 \frac{n_1 n_2}{\left[\frac{n_1 + n_2}{2}\right]^2} \approx -0.959184. \quad (42)$$

where the last equality is for the particular parameters given in the problem, $n_1 = 1.5$ and $n_2 = 2.0$. If each material has an optical thickness of one-quarter wavelength, $\theta = \pi/2$ and $\cos(2\theta) = -1$ producing the greatest damping factor. This is the center of the (first) stop band. The band edges occur at

$$\theta = \frac{1}{2} \arccos(r^2 - t^2) \approx -0.959184. \quad (43)$$

This equation has solutions $\theta_{\pm} \approx \frac{\pi}{2} \pm 0.143348$. Since $\theta = k_0 n_1 d_1 = \frac{2\pi}{\lambda_0} 0.3 \mu\text{m}$, the corresponding wavelengths of the light (in vacuum) are

$$\lambda_0^{\pm} = \frac{2\pi}{\theta_{\pm}} 0.3 \mu\text{m} \approx 1.09965 \mu\text{m}; 1.32051 \mu\text{m}, \quad (44)$$

so that light with vacuum wavelengths in the range

$$1.09965 \mu\text{m} < \lambda_0 < 1.32051 \mu\text{m} \quad (45)$$

cannot propagate in the material.

Chapter 8

Ex. 8.1

$$H = \epsilon_n(\frac{1}{\hbar}[\vec{p} + \frac{e}{c}\vec{A}(\vec{R})]) + U(\vec{R}),$$

from which follows the Heisenberg equation of motion for \vec{R} :

$$\frac{d\vec{R}}{dt} = \frac{1}{i\hbar}[\vec{R}, H] = \frac{1}{i\hbar}[\vec{R}, \epsilon_n(\frac{1}{\hbar}[\vec{p} + \frac{e}{c}\vec{A}(\vec{R})])] = \frac{1}{\hbar} \left[\nabla \epsilon_n(\frac{1}{\hbar}[\vec{p} + \frac{e}{c}\vec{A}(\vec{R})]) \right] \left[\vec{p} + \frac{e}{c}\vec{A}(\vec{R}) \right].$$

We obtain Eq. (8.9) upon replacing the operators by their expectation values in the equation above.

Eq. (8.15) can be obtained similarly by inspecting the Heisenberg equation of motion for $\vec{p} + \frac{e}{c}\vec{A}(\vec{R})$. Note the 1st term in brackets follow trivially from the commutator between \vec{p} and $U(\vec{R})$. To obtain the 2nd term one needs to invoke the fact that \vec{B} is slowly varying and thus can be treated as a constant (with local value at \vec{R}) in the lowest order approximation.

Ex. 8.2

$$\begin{aligned} \vec{v}(\vec{k}) &= \frac{1}{\hbar} \nabla_{\vec{k}} \epsilon_n(\vec{k}) = -\frac{\hbar}{m^*} (\vec{k} - \vec{k}_0) \\ \frac{d^2}{dt^2} \vec{R} &= \frac{d}{dt} \vec{v}(\vec{k}) = -\frac{\hbar}{m^*} \frac{d}{dt} \vec{k} = \frac{e}{m^*} [\vec{E}(\vec{R}, t) + \frac{1}{c} \vec{v} \times \vec{B}(\vec{R}, t)] \end{aligned}$$

The reason \vec{k}_0 doesn't enter is that it shows up as an additional constant in $\vec{v}(\vec{k})$ and disappears when taking derivative w.r.t. t .

Ex. 8.3

We consider a single band and suppress band index n . The number of states for $\epsilon' < \epsilon$ is $N(\epsilon) = \frac{L^2}{(2\pi)^2} A(\epsilon)$.

$$\text{Thus } \rho(\epsilon) = \frac{1}{L^2} \frac{\partial N(\epsilon)}{\partial \epsilon} = \frac{1}{(2\pi)^2} \frac{\partial}{\partial \epsilon} A(\epsilon) = \frac{1}{(2\pi\hbar)^2} \frac{eB}{c} T(\epsilon).$$

Ex. 8.4

i)

$$\epsilon(\vec{k}) = 2t_x \cos(k_x) - 2t_y \cos(k_y)$$

with $t_x > t_y > 0$.

Electron-like closed orbits:

$$-2t_x - 2t_y < \epsilon < -2t_x + 2t_y.$$

Open orbits:

$$-2t_x + 2t_y < \epsilon < 2t_x - 2t_y.$$

Hole-like closed orbits:

$$2t_x - 2t_y < \epsilon < 2t_x + 2t_y.$$

ii) van Hove singularities correspond to

$$\nabla_{\vec{k}} \epsilon_{\vec{k}} = -2t_x \sin k_x \hat{x} + 2t_y \sin k_y \hat{y} = 0.$$

This equation has solutions at $\epsilon = 2t_x - 2t_y$ and $\epsilon = -2t_x + 2t_y$ where one has extended equal-energy lines (as opposed to single points) in 1BZ, which are the energies separating the closed and open orbits.

Ex. 8.5

$$\left(\frac{\partial f}{\partial t} \right)_{coll} = -\frac{\Omega}{(2\pi)^3} \int d^3 k' \{ W_{kk'} [1 - f_{\vec{k}'}] f_{\vec{k}} - W_{k'k} [1 - f_{\vec{k}}] f_{\vec{k}'} \}.$$

Under equilibrium,

$$f = f^0 = \frac{1}{1 + e^{\beta(\epsilon - \mu)}}.$$

We thus find the integrand vanished if

$$\frac{W_{kk'}}{W_{k'k}} = \frac{1 - f_{\vec{k}}}{1 - f_{\vec{k}'}} \frac{f_{\vec{k}'}}{f_{\vec{k}}} = e^{-\beta[\epsilon(\vec{k}') - \epsilon(\vec{k})]}.$$

Ex. 8.6

(i)

According to the Fermi Golden rule: $\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} | < f | H' | i > |^2 \delta(\epsilon_{\vec{k}} - \epsilon_{\vec{k}'} - \hbar\omega_{ph})$. In the present case H' describes electron-phonon interaction:

$$H_{e-ph} = \sum_{k,k',\sigma,\lambda} g_{\vec{k},\vec{k}'} C_{\vec{k}'\sigma}^\dagger C_{\vec{k}\sigma} (a_{\vec{q}\lambda}^\dagger + a_{-\vec{q}}, \lambda)$$

where $\vec{q} = \vec{k} - \vec{k}'$, λ is the polarization of the phonon. The detailed form is not that important. What matters is there is a phonon factor $\langle n+1 | a^\dagger | n \rangle = \sqrt{n+1}$ in M for emission, where n is the phonon number in the relevant mode before the electron emits an *additional* phonon, resulting in the factor $n_B(\hbar\omega_{ph}) + 1$ once squared.

ii)

For absorption, the corresponding phonon factor is $\langle n-1 | a | n \rangle = \sqrt{n+1}$ in M , resulting in the factor $n_B(\hbar\omega_{ph})$ once squared.

iii)

Under the conditions stated we have

$$\frac{W_{kk'}}{W_{k'k}} = \frac{n_B(\hbar\omega_{ph}) + 1}{n_B(\hbar\omega_{ph})} = e^{\beta\hbar\omega_{ph}} = e^{-\beta(\epsilon_{\vec{k}'} - \epsilon_{\vec{k}})}.$$

Ex. 8.7

Integration by parts:

$$\int_0^\infty d\epsilon \rho(\epsilon) \epsilon \left(-\frac{\partial f^0}{\partial \epsilon} \right) = \int_0^\infty d\epsilon \rho(\epsilon) f^0(\epsilon) + \int_0^\infty f^0(\epsilon) \epsilon \left(\frac{\partial \rho}{\partial \epsilon} \right) d\epsilon = \frac{n}{2} + \frac{n}{4} = \frac{3}{4}n,$$

where we used the fact $\epsilon \left(\frac{\partial \rho}{\partial \epsilon} \right) = \rho(\epsilon)/2$ for $\rho(\epsilon) \propto \sqrt{\epsilon}$. Eq. (8.137) follows.

Ex. 8.8

We have the Mott formula $Q = (L^{12}/L^{11}) = -\frac{\pi^2}{3} \left(\frac{k_B}{e} \right) k_B T \left(\frac{1}{\sigma} \frac{\partial \sigma}{\partial \epsilon} \right)$. Here $\sigma = ne^2\tau/m$, and we are assuming that τ is not energy dependent. We then have that $\frac{\partial \sigma}{\partial \epsilon} = \frac{e^2\tau}{m} \frac{\partial n}{\partial \epsilon}$. Placing this into the Mott formula we have

$$Q = -\frac{\pi^2}{3} \left(\frac{k_B}{e} \right) k_B T \frac{1}{n} \frac{\partial n}{\partial \epsilon} = -\frac{\pi^2}{3} \left(\frac{k_B}{e} \right) k_B T \frac{1}{N_e} \frac{\partial N_e}{\partial \epsilon}.$$

where $N_e = nV$ = (number of electrons), and the derivative is with respect to the Fermi energy $\epsilon = \epsilon_F$. Thus

$$\frac{\partial N_e}{\partial \epsilon} = 2V\rho(\epsilon_F).$$

In the meantime we have specific heat

$$C_v = 2 \frac{\pi^2}{3} k_B^2 T \rho(\epsilon_F),$$

thus the entropy is

$$S = V \int_0^T \frac{C_V}{T} dT = \frac{\pi^2}{3} V T k_B^2 2 \rho(\epsilon_F).$$

We thus have

$$Q = -\frac{S}{N_e e}.$$

Ex 8.9

Part i

Mechanical equilibrium requires that the forces on a small area volume element (cross-sectional area A and width dx) cancel. Without loss of generality, say that the E field in this location is parallel to the x direction: $\vec{E} = E\hat{x}$. Orient the A face of the volume element to be perpendicular to the electric field. For forces to cancel, we then have that

$$-E enA dx - P(x)A + P(x+dx)A = 0 \implies E en = \frac{\partial P}{\partial x}.$$

On the other hand, $\frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$, because there is no electric field in that direction, and hence no force to cancel.

Since the above reasoning holds at all points in the volume, this means that

$$\vec{E} en = \nabla P \implies \nabla \Phi en = \nabla P.$$

Part ii

Starting with the energy U as a thermodynamic potential as a function of extensive parameters, we can perform Legendre transforms on the N and S parameters to obtain the grand potential $\Omega(V, T, \mu)$. The thermodynamic identity here is $d\Omega = -SdT - PdV - Nd\mu$.

Differentiating, we find $P = -\frac{\partial \Omega}{\partial V}$ and $N = -\frac{\partial \Omega}{\partial \mu}$. Differentiating again, we find

$$\left(\frac{\partial P}{\partial \mu}\right)_{T,V} = -\frac{\partial \Omega}{\partial \mu \partial V} \quad \text{and} \quad \left(\frac{\partial N}{\partial V}\right)_{T,\mu} = -\frac{\partial \Omega}{\partial V \partial \mu},$$

so $\left(\frac{\partial P}{\partial \mu}\right)_{T,V} = \left(\frac{\partial N}{\partial V}\right)_{T,\mu}$. We similarly find that $\left(\frac{\partial P}{\partial T}\right)_{\mu,V} = \left(\frac{\partial S}{\partial V}\right)_{T,\mu}$ since

$$S = -\frac{\partial \Omega}{\partial T} \quad \text{and} \quad \left(\frac{\partial P}{\partial T}\right)_{\mu,V} = -\frac{\partial \Omega}{\partial T \partial V} \quad \text{and} \quad \left(\frac{\partial N}{\partial V}\right)_{T,\mu} = -\frac{\partial \Omega}{\partial V \partial \mu}.$$

When all intensive parameters are held fixed, extensive parameters must be proportional to each other. Hence, $\left(\frac{\partial P}{\partial \mu}\right)_{T,V} = \left(\frac{\partial N}{\partial V}\right)_{T,\mu} = \frac{N}{V}$ and $\left(\frac{\partial P}{\partial T}\right)_{\mu,V} = \left(\frac{\partial S}{\partial V}\right)_{T,\mu} = \frac{S}{V}$.

Part iii

Combining results of (i), (ii) and Eq. (8.181), we have

$$n\nabla\mu + (S/V)\nabla T = n\nabla\Phi,$$

from which we obtain

$$Q = \frac{\nabla(\Phi - \mu/e)}{\nabla T} = -\frac{S}{N_e e},$$

where we used the fact the difference in electrochemical potential $\mu - e\Phi$ is what is measured in a voltage measurement.

Thermopower Q is defined by the equation

$$\nabla\phi + \frac{1}{e}\nabla\mu = Q\nabla T \quad (\text{mechanical equilibrium thus no current flow})$$

In the clean limit there are no impurities and hence the scattering time τ is large, meaning that the conductivity σ is large as well. Hence, $\vec{J} = 0 \implies \vec{E} = \nabla\phi = 0$. Using the results from part 1, the above reduces to $\nabla\mu = eQ\nabla T$.

Because P is an intensive quantity, it must be a function of other intensive quantities alone, hence $\nabla P = \left(\frac{\partial P}{\partial \mu}\right)_T \nabla\mu + \left(\frac{\partial P}{\partial T}\right)_\mu \nabla T$. However, from part (i), we know that $\nabla P = \nabla\Phi(en) = 0$, so

$$0 = \left(\frac{\partial P}{\partial \mu}\right)_T \nabla\mu + \left(\frac{\partial P}{\partial T}\right)_\mu \nabla T.$$

Using the results of the previous part, this reduces to $\frac{N}{V}\nabla\mu = -\frac{S}{V}\nabla T \implies \nabla\mu = -\frac{S}{N}\nabla T$, and because $\nabla\mu = eQ\nabla T$, we find $\implies Q = -\frac{S}{Ne}$.

Chapter 9

Ex. 9.1

(i) For $\epsilon - \mu \gg k_B T$, $e^{\beta(\epsilon - \mu)} \gg 1$, thus

$$f^0(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \approx \frac{1}{e^{\beta(\epsilon - \mu)}} = e^{-\frac{(\epsilon - \mu)}{k_B T}} \ll 1.$$

$$(ii) 1 - f^0(\epsilon) = 1 - \frac{1}{e^{\beta(\epsilon - \mu)} + 1} = \frac{e^{\beta(\epsilon - \mu)}}{e^{\beta(\epsilon - \mu)} + 1} = \frac{1}{e^{\beta(\mu - \epsilon)} + 1}$$

For $\mu - \epsilon \gg k_B T$, the above becomes $\frac{1}{e^{\beta(\mu - \epsilon)}} = e^{\frac{(\epsilon - \mu)}{k_B T}} \ll 1$.

Ex. 9.2

Part i

We will use the Boltzmann distribution to find the number density n_c of electrons in the conduction band at low temperature, which are mostly near the bottom of the conduction band:

$$Vn_c = \int_{\epsilon_c}^{\infty} f(\epsilon)g(\epsilon)d\epsilon,$$

where ϵ_c is the bottom of the conduction band. Near ϵ_c ,

$$\epsilon(\vec{k}) = \epsilon_c + \frac{\hbar^2}{2m_c}k^2.$$

A similar equation holds for holes near the top of the valence band ϵ_v :

$$\epsilon(\vec{k}) = \epsilon_v - \frac{\hbar^2}{2m_v}k^2.$$

These dispersion relations lead to the following density of states g :

$$g_{c,v}(\epsilon) = \sqrt{2|\epsilon - \epsilon_{c,v}|} m_{c,v}^{3/2} / (\hbar^3 \pi^2).$$

(See part (ii) for derivation. Also note this is the density of states per unit volume.).
Thus

$$\begin{aligned}
n_c &= \frac{m_c^{3/2}}{\hbar^3 \pi^2} \int_{\epsilon_c}^{\infty} f(\epsilon) \sqrt{2(\epsilon - \epsilon_c)} d\epsilon & [f = \text{Fermi-Dirac function}] \\
&= \frac{m_c^{3/2}}{\hbar^3 \pi^2} \sqrt{2} \int_{\epsilon_c}^{\infty} e^{-\beta(\epsilon - \mu)} \sqrt{(\epsilon - \epsilon_c)} d\epsilon & \left[f \xrightarrow{\text{low } T} e^{-\beta(\epsilon - \mu)} \right] \\
&= \frac{m_c^{3/2}}{\hbar^3 \pi^2} \sqrt{2} e^{-\beta(\epsilon_c - \mu)} \int_{\epsilon_c}^{\infty} e^{-\beta(\epsilon - \epsilon_c)} \sqrt{(\epsilon - \epsilon_c)} d\epsilon \\
&= \frac{m_c^{3/2}}{\hbar^3 \pi^2} \sqrt{2} e^{-\beta(\epsilon_c - \mu)} \int_0^{\infty} e^{-\beta\epsilon} \epsilon^{1/2} d\epsilon \\
&= \left\{ \frac{1}{\hbar^3 \pi^2} \sqrt{\frac{\pi m_c^3}{2\beta^3}} \right\} e^{-\beta(\epsilon_c - \mu)}
\end{aligned}$$

The calculation is similar for holes in the valence band (density p_v). However, now the probability of a hole is $(1 - f)$. At low temperature this becomes

$$\frac{e^{\beta(\epsilon - \mu)}}{e^{\beta(\epsilon - \mu)} + 1} = \frac{1}{1 + e^{\beta(\mu - \epsilon)}} \approx e^{-\beta(\mu - \epsilon)},$$

because in a semiconductor hole energies are always below μ . Therefore:

$$\begin{aligned}
p_v &= \frac{m_v^{3/2}}{\hbar^3 \pi^2} \int_0^{\epsilon_v} e^{-\beta(\mu - \epsilon)} \sqrt{2(\epsilon_v - \epsilon)} d\epsilon \\
&= \frac{m_v^{3/2}}{\hbar^3 \pi^2} \sqrt{2} e^{-\beta(\mu - \epsilon_v)} \int_0^{\epsilon_v} e^{-\beta(\epsilon_v - \epsilon)} \sqrt{(\epsilon_v - \epsilon)} d\epsilon \\
&= \frac{m_v^{3/2}}{\hbar^3 \pi^2} \sqrt{2} e^{-\beta(\mu - \epsilon_v)} \int_0^{\epsilon_v} e^{-\beta\epsilon} \sqrt{\epsilon} d\epsilon \\
&\approx \frac{m_v^{3/2}}{\hbar^3 \pi^2} \sqrt{2} e^{-\beta(\mu - \epsilon_v)} \int_0^{\infty} e^{-\beta\epsilon} \sqrt{\epsilon} d\epsilon & [\text{since } e^{-\beta\epsilon} \text{ is small}] \\
&= \left\{ \frac{1}{\hbar^3 \pi^2} \sqrt{\frac{\pi m_v^3}{2\beta^3}} \right\} e^{-\beta(\mu - \epsilon_v)}
\end{aligned}$$

Since each electron in the conduction band leads to a hole in the valence band, we have that $n_c = p_v$, so

$$\left(\frac{m_v}{m_c} \right)^{3/2} = e^{-\beta(2\mu - \epsilon_c - \epsilon_v)}$$

Taking logs of both sides:

$$\frac{1}{2} \left\{ -k_B T \frac{3}{2} \ln \left(\frac{m_v}{m_c} \right) + (\epsilon_c + \epsilon_v) \right\} = \mu.$$

At very low temperatures we recover $\frac{1}{2}(\epsilon_c + \epsilon_v) = \mu$.

Part ii

In the case of anisotropy at the band extrema, we have

$$\begin{aligned}
\vec{\epsilon}(k) &= \epsilon_c + \frac{\hbar^2}{2} \sum_{\alpha\beta} k_\alpha (M_c^{-1})_{\alpha\beta} k_\beta & [\text{conduction band}] \\
\vec{\epsilon}(k) &= \epsilon_v - \frac{\hbar^2}{2} \sum_{\alpha\beta} k_\alpha (M_v^{-1})_{\alpha\beta} k_\beta & [\text{valence band}]
\end{aligned}$$

Choosing a coordinate systems lying along the principal axes of the M matrices above, this reduces to

$$\begin{aligned}\epsilon(\vec{k}) &= \epsilon_c + \frac{\hbar^2}{2} \sum_{\alpha} k_{\alpha}^2 / m_{c,\alpha} & [\text{conduction band}] \\ \epsilon(\vec{k}) &= \epsilon_v - \frac{\hbar^2}{2} \sum_{\alpha} k_{\alpha}^2 / m_{v,\alpha} & [\text{valence band}]\end{aligned}$$

From here we can calculate the density of states per unit volume. We do the calculation for the conduction band:

$$g(\epsilon) = 2 \times \left(\frac{1}{2\pi} \right)^3 \int d^3 k \delta \left([\epsilon_c - \epsilon] - \frac{\hbar^2}{2} \sum_{\alpha} k_{\alpha}^2 / m_{c,\alpha} \right),$$

where the factor of “2” comes from two spin orientations.

We change variables to $(k')_{\alpha} = k_{\alpha} / \sqrt{m_{c,\alpha}}$, resulting in

$$\begin{aligned}g(\epsilon) &= 2 \left(\frac{1}{2\pi} \right)^3 \sqrt{\Pi_a m_{c,\alpha}} \int d^3 k' \delta \left([\epsilon - \epsilon_c] - \frac{\hbar^2}{2} k'^2 \right) & [\Pi = \text{product}] \\ &= 2 \left(\frac{1}{2\pi} \right)^3 \sqrt{\Pi_a m_{c,\alpha}} 4\pi \int dk' k'^2 \delta \left([\epsilon - \epsilon_c] - \frac{\hbar^2}{2} k'^2 \right)\end{aligned}$$

Changing variables again:

$$\begin{aligned}&= 2 \left(\frac{1}{2\pi} \right)^3 \sqrt{\Pi_a m_{c,\alpha}} 4\frac{\pi}{2} \int dq \sqrt{q} \delta \left([\epsilon - \epsilon_c] - \frac{\hbar^2}{2} q \right) \\ &= 2 \left(\frac{1}{2\pi} \right)^3 \sqrt{\Pi_a m_{c,\alpha}} \frac{4\pi}{\hbar^2} \int dq' \sqrt{2q'/\hbar^2} \delta \left([\epsilon - \epsilon_c] - q' \right) \\ &= \frac{2}{(2\pi)^3} \sqrt{\Pi_a m_{c,\alpha}} \frac{4\pi}{\hbar^3} \sqrt{2[\epsilon - \epsilon_c]}\end{aligned}$$

Finally:

$$g_c(\epsilon) = \sqrt{2|\epsilon - \epsilon_c|} \sqrt{\Pi_a m_{c,\alpha}} / (\hbar^3 \pi^2).$$

The calculation for g_v is similar, resulting in

$$g_v(\epsilon) = \sqrt{2|\epsilon - \epsilon_v|} \sqrt{\Pi_a m_{v,\alpha}} / (\hbar^3 \pi^2).$$

Thus, only the mass term changes in the calculation in part (i), so the $T \rightarrow 0$ limit for μ remains the same.

Now, in the case of valley degeneracy, there are multiple minima and maxima, labeled with the index γ , and the dispersion relations become:

$$\begin{aligned}\epsilon(\vec{k}) &= \epsilon_c + \frac{\hbar^2}{2} \sum_{\gamma\alpha\beta} k_{\alpha} (M(\gamma)_c^{-1})_{\alpha\beta} k_{\beta} & [\text{conduction band}] \\ \epsilon(\vec{k}) &= \epsilon_v - \frac{\hbar^2}{2} \sum_{\gamma\alpha\beta} k_{\alpha} (M(\gamma)_v^{-1})_{\alpha\beta} k_{\beta} & [\text{valence band}]\end{aligned}$$

Choosing a coordinate systems lying along the principal axes of the $M(\gamma)$ matrices above, this reduces to

$$\begin{aligned}\epsilon(\vec{k}) &= \epsilon_c + \frac{\hbar^2}{2} \sum_{\gamma\alpha} k_{\alpha}(\gamma)^2 / m(\gamma)_{c,\alpha} & [\text{conduction band}] \\ \epsilon(\vec{k}) &= \epsilon_v - \frac{\hbar^2}{2} \sum_{\gamma\alpha} k_{\alpha}(\gamma)^2 / m(\gamma)_{v,\alpha} & [\text{valence band}]\end{aligned}$$

The above derivation for g now leads to

$$g_c(\epsilon) = \sum_{\gamma} \sqrt{2|\epsilon - \epsilon_c|} \sqrt{\Pi_a m(\gamma)_{c,\alpha}} / (\hbar^3 \pi^2)$$

and

$$g_v(\epsilon) = \sum_{\gamma} \sqrt{2|\epsilon - \epsilon_v|} \sqrt{\Pi_a m(\gamma)_{v,\alpha}} / (\hbar^3 \pi^2)$$

. Again, the $T \rightarrow 0$ limit for μ remains the same.

Ex. 9.3

(i)

In the zero temperature limit the chemical potential μ is exactly equal to the donor level ϵ_d , as in this limit each donor level has one electron, corresponding to half filling due to the spin degeneracy. At very low temperature we expect the chemical potential μ to saty very close the donor level ϵ_d , the electrons in the conduction band whose density is n_c come almost exclusively from the ionized donors, while the hole density in the valence band p_v is essentially zero. This will change once we have $n_c \sim n_d$, at which point we the donors get all ionized. Using the conduction band DOS

$$g_c(\epsilon) = A_c(\epsilon - \epsilon_c)^{1/2},$$

where A_c is a constant determined from the electron effective mass (see Ex. 9.2), we obtain at $T < T^*$

$$n_c(T) \approx \int_{\epsilon_c}^{\infty} A_c(\epsilon - \epsilon_c)^{1/2} e^{-\beta(\epsilon - \mu)} d\epsilon = A_c \frac{\sqrt{\pi}}{2} \beta^{-3/2} e^{-\beta\epsilon_b},$$

where $\epsilon_b = \epsilon_c - \epsilon_d$ is the binding energy of the donor level. The condition $n_c(T^*) \sim n_d$ thus yields

$$T^* \approx \frac{\epsilon_b}{k_B \ln \frac{A_c}{k_B \epsilon_b n_d}}.$$

(ii)

Using the valence band DOS

$$g_v(\epsilon) = A_v(\epsilon_v - \epsilon)^{1/2},$$

where A_v is a constant determined from the hole effective mass (see Ex. 9.2), we obtain at $T < T^*$ where $\mu \approx \epsilon_d$

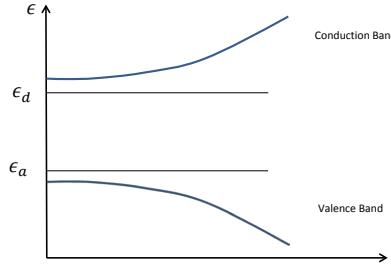
$$p_v(T) \approx \int_{\epsilon_v}^{\infty} A_v(\epsilon' - \epsilon_c)^{1/2} e^{-\beta(\epsilon' - \mu)} d\epsilon' = A_v \frac{\sqrt{\pi}}{2} \beta^{-3/2} e^{-\beta(\Delta - \epsilon_b)} \ll n_d, n_c(T),$$

but

$$n_c(T)p_v(T) = \frac{\pi}{4} A_v A_c \beta^{-3} e^{-\beta\Delta}$$

is the same as that of the intrinsic case, so the law of mass action is satisfied.

Ex. 9.4



The effective mass reflects the inverse of the band curvature:

$$m^* = \frac{1}{\hbar^2 \frac{\partial^2 \epsilon}{\partial k^2}} \begin{cases} > 0, & \text{conduction band} \\ < 0, & \text{valence band} \end{cases}$$

If an extra electron were living in a vacuum and bound to a unit nuclear charge, then we simply have a hydrogen atom problem: $\epsilon = -\frac{m_e e^4}{2\hbar^2} = -13.6 \text{eV}$, $a_B = \frac{\hbar^2}{me^4} = 0.5 \text{\AA}$.

The host semiconductor has two important effects:

$$m \rightarrow m^*, \quad e^2 \rightarrow e^2/\epsilon.$$

For a donor impurity, electron effective mass is calculated from the bottom of the conduction band, while the Coulomb impurity is attractive just like in the hydrogen atom problem, resulting in a bound state whose energy is slightly below the conduction band edge (ϵ_c) as discussed in the text.

For an acceptor impurity the situation is very different. First of all since the impurity ion has one fewer (positive) charge compared to that of the atom it replaces, it results in a *repulsive* Coulomb potential to the electron. In the meantime (as will be justified *a posteriori*), the resultant bound state is mostly formed from the valence band using its states near the band top, the appropriate effective mass is calculated from the top of the valence band, which is negative! So what we have is an effective Hamiltonian that has the opposite sign of that of the hydrogen atom problem, which result in a bound state whose energy is positive, while all the scattering states have zero or negative energy; the latter are nothing but the valence band states. As illustrated in the figure this acceptor bound state (energy ϵ_a) is thus right above the valence band edge (ϵ_v), with a binding energy

$$\epsilon_b = \epsilon_a - \epsilon_v = \frac{|m^*|e^4}{\epsilon^2 2\hbar^2} = \frac{|m^*|}{m\epsilon^2} (\text{Ryd}) > 0$$

$$a_b = \frac{\epsilon \hbar^2}{|m^*|e^2} = \frac{m}{m^*} \epsilon (0.5 \text{\AA})$$

In the above $|m^*|$ is the valence band (or hole) effective mass.

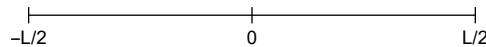
Ex. 9.5

Bound state

$$\begin{aligned}
 H &= \frac{-1}{2} \frac{d^2}{dx^2} - \lambda \delta(x) \\
 -\frac{1}{2} \psi'' - \lambda \delta(x) \psi(x) &= \epsilon \psi(x) \\
 -\frac{1}{2} (\psi'_+ - \psi'_-) - \lambda \psi(0) &= 0 \\
 \psi'_+ - \psi'_- &= -2\lambda \psi(0) \\
 \psi &= e^{-k|x|} \quad (\text{not normalized}) \\
 \psi(0) &= 1 \\
 \psi'_+ - \psi'_- &= -2k \\
 \Rightarrow k &= \lambda \quad (\text{require } \lambda > 0) \\
 \int_{-\infty}^{\infty} dx |\psi|^2 &= 2 \frac{1}{2k} = \frac{1}{k} \\
 \psi_0 &= \sqrt{k} e^{-k|x|} \\
 \underline{\psi_0(x) \psi_0(x') = k e^{-k(|x| + |x'|)}} \quad (k = \lambda)
 \end{aligned}$$

Scattering states

$$\begin{aligned}
 \text{odd parity: } \phi_k &= \sqrt{\frac{2}{L}} \sin kx \\
 \text{even parity: } \Psi_k &= \sqrt{\frac{2}{L}} \sin(k|x| + \theta_k)
 \end{aligned}$$



$$\begin{aligned}
 \text{n.b. } k > 0 \quad k_n \frac{L}{2} &= \pi n \\
 \delta k = \frac{2\pi}{L} \quad \text{DOS} = \frac{L}{2\pi} \quad \text{but } k > 0
 \end{aligned}$$

Completeness integrals

odd parity

$$\begin{aligned}
I_0 &\equiv L \int_0^\infty \frac{dk}{2\pi} \phi_k(x) \phi_k(x') \\
&= L \int_0^\infty \frac{dk}{2\pi} \frac{2}{L} \sin kx \cdot \sin kx' \\
&= \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{[e^{ikx} - e^{-ikx}]}{2i} \frac{[e^{ikx'} - e^{-ikx'}]}{2i} \\
&= \int_{-\infty}^\infty \frac{dk}{2\pi} \left(-\frac{1}{4}\right) \{ (e^{ik(x+x')} + \text{c.c.}) - (e^{ik(x+x')} + \text{c.c.}) \} \\
I_0 &\equiv -\frac{1}{2} \delta(x+x') + \frac{1}{2} \delta(x+x')
\end{aligned}$$

even parity

$$\begin{aligned}
I_E &= \int_{-\infty}^\infty \frac{dk}{2\pi} \sin(k|x| + \theta_k) \cdot \sin(k|x'| + \theta_k) \\
&= \int_{-\infty}^\infty \frac{dk}{2\pi} \left(-\frac{1}{4}\right) [e^{i\theta_k} e^{ik|x|} - e^{-i\theta_k} e^{-ik|x|}] \times [e^{i\theta_k} e^{ik|x'|} - e^{-i\theta_k} e^{-ik|x'|}]
\end{aligned}$$

$$I_E = I_1 + I_2$$

$$I_1 \equiv -\frac{1}{4} \int_{-\infty}^\infty \frac{dk}{2\pi} \{ e^{2i\theta_k} e^{ik(|x|+|x'|)} + \text{c.c.} \}$$

$$I_2 \equiv +\frac{1}{4} \int_{-\infty}^\infty \frac{dk}{2\pi} \{ e^{ik(|x|-|x'|)} + \text{c.c.} \}$$

$$I_2 = \frac{1}{2} \delta(|x| - |x'|) = \frac{1}{2} \delta(x-x') + \frac{1}{2} \delta(x+x')$$

$$I_0 + I_E = I_1 + \delta(x-x')$$

I_1 must cancel the contribution from the bound state if completeness is to work.

Now we need θ_k

$$\psi'_{k+} - \psi'_{k-} = \sqrt{\frac{2}{L}} 2k \cos \theta_k = -2\lambda \sqrt{\frac{2}{L}} \sin \theta_k$$

$$\tan \theta_k = \frac{-k}{\lambda}$$

$$\cos \theta_k = \frac{\lambda}{\sqrt{\lambda^2 + k^2}} \quad \sin \theta_k = \frac{-k}{\sqrt{k^2 + \lambda^2}}$$

$$e^{i\theta_k} = \cos \theta_k + i \sin \theta_k$$

$$= \frac{\lambda - ik}{\sqrt{\lambda^2 + k^2}}$$

$$e^{2i\theta_k} = \frac{(\lambda - ik)^2}{k^2 + \lambda^2}$$

$$I_1 = \frac{-1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(\lambda - ik)^2}{k^2 + \lambda^2} e^{ik(|x| + |x'|)}$$

pole at $k = +i|\lambda|$ where the absolute value of λ is because need $\operatorname{Im} k > 0$.

$$I_1 = \frac{-1}{2} \operatorname{Re} \frac{2\pi i}{2\pi} \frac{(\lambda + |\lambda|)^2}{2i\lambda} e^{-|\lambda|(|x| + |x'|)}$$

If $\lambda < 0$ (repulsive potential), $I_1 = 0 \rightarrow$ NO BOUND STATE

If $\lambda > 0$ (attractive potential), $I_1 = -\lambda e^{-\lambda(|x| + |x'|)}$.

$$\Rightarrow \underline{\psi_0(x) = \sqrt{k} e^{-k|x|}}$$

Completeness relation

$$\underline{\psi_0(x)\psi_0(x') + I_0 + I_E = \delta(x - x')}$$

N.B. I ignored the small corrections to the DOS from $\frac{d\theta_k}{dk}$ referred to in the exercise. We should reword the excise because these terms only have an exponentially small effect on the eigenstates and eigenvalues for large kL .

Ex. 9.6

$$\begin{aligned} \epsilon &= \frac{\hbar^2}{2} \left(\frac{k_x^2}{m_x} + \frac{k_y^2}{m_y} + \frac{k_z^2}{m_z} \right) \\ &\rightarrow \frac{k_x^2}{m_x} + \frac{k_y^2}{m_y} = \frac{2\epsilon}{\hbar^2} - \frac{k_z^2}{m_z} \end{aligned}$$

Assuming B to be in z direction, k_z is constant of the motion and this is the the equation of an ellipse (ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area $A = \pi ab$). Thus the area is

$$\begin{aligned} A &= \pi \sqrt{m_x \left(\frac{2\epsilon}{\hbar^2} - \frac{k_z^2}{m_z} \right)} \sqrt{m_y \left(\frac{2\epsilon}{\hbar^2} - \frac{k_z^2}{m_z} \right)} = \pi \sqrt{m_x m_y} \left(\frac{2\epsilon}{\hbar^2} - \frac{k_z^2}{m_z} \right) \\ T &= \frac{\hbar^2 c \partial A}{eB \partial \epsilon} = \frac{\hbar^2 c}{eB} \pi \sqrt{m_x m_y} \frac{2}{\hbar^2} \\ \omega &= \frac{2\pi}{T} = \frac{eB}{c \sqrt{m_x m_y}} \end{aligned}$$

Solution for other directions of the magnetic field is similar.

Ex. 9.7

NOTE TO INSTRUCTOR: Unfortunately, this exercise fails to mention the fact that, in order to obtain a sensible result, one needs to include Thomas-Fermi screening, a concept which is not introduced until later. We therefore suggest that the students be given the expression for the form of the screening and the value of q_{TF} as described further below.

i) The electron interaction with an impurity lying a distance d above the origin in the 2D plane is

$$V(r) = -\frac{1}{\epsilon} \frac{e^2}{\sqrt{r^2 + d^2}}$$

where ϵ is the dielectric constant of the host semiconductor.

ii) The 2D Fourier transform of the interaction is

$$\begin{aligned} V(q) &= \int d^2 r e^{-i\vec{q}\cdot\vec{r}} V(r) \\ &= - \int_0^\infty dr r \int_0^{2\pi} d\theta e^{-iqr \cos \theta} \frac{e^2}{\epsilon} \frac{1}{\sqrt{r^2 + d^2}} \end{aligned}$$

We know that $\int_0^{2\pi} d\theta e^{-iqr \cos \theta} = 2\pi J_0(rq)$, so:

$$V(q) = -\frac{2\pi e^2}{\epsilon} \int_0^\infty J_0(rq) \frac{r}{\sqrt{r^2 + d^2}} = -\frac{2\pi e^2}{\epsilon} \frac{e^{-qd}}{q}.$$

We can include Thomas-Fermi screening by modifying this expression to:

$$V(q) = -\frac{2\pi e^2}{\epsilon} \frac{e^{-qd}}{q + q_{\text{TF}}}.$$

The parameter q_{TF} for the case of three dimensions is defined in Eq. (15.85) in the textbook. The analogous expression for two dimensions in the presence of background dielectric constant ϵ is

$$q_{\text{TF}} \equiv \frac{2\pi e^2}{\epsilon} \frac{dn}{d\mu} = \frac{4\pi e^2}{\epsilon} D(0). \quad (46)$$

Below we will make use of the dimensionless quantity

$$\frac{q_{\text{TF}}}{k_F} = \frac{4\pi}{\epsilon} \frac{m^*}{m} \frac{\alpha}{k_F \lambda_e}, \quad (47)$$

where α is the fine-structure constant

$$\alpha \equiv \frac{e^2}{\hbar c} \approx 1/(137.04), \quad (48)$$

and the Compton wavelength of the electron is

$$\lambda_e \equiv \frac{\hbar}{m_e c} \approx 2.43 \times 10^{-10} \text{ cm}. \quad (49)$$

For the particular parameters given in the problem, we find $\frac{q_{\text{TF}}}{k_F} \approx 2.56$.

If the impurity is not at the origin, but rather a distance d above the point \vec{R}_i in the 2D plane then we have

$$V_i(\vec{q}) = e^{i\vec{q}\cdot\vec{R}_i} V(q).$$

Ensemble averaging overall all possible positions of a pair of impurities yields the useful result

$$\langle \langle V_i(\vec{q}) V_j^*(\vec{q}) \rangle \rangle = \delta_{ij} V(q)^2.$$

Hence the ensemble averaged power spectrum of the total scattering potential from N_I randomly located impurities is

$$\langle \langle |V_{\text{Tot}}(q)|^2 \rangle \rangle = N_I V(q)^2.$$

This will be useful in the Fermi Golden Rule calculation of the scattering rates.

iii) Assuming (from charge neutrality) that the areal density of electrons n equal to the density of singly-charged donors $n_I = N_I L^{-2}$, the Fermi Golden Rule expression for the scattering rate for an electron at the Fermi energy with initial wave vector \vec{k} is

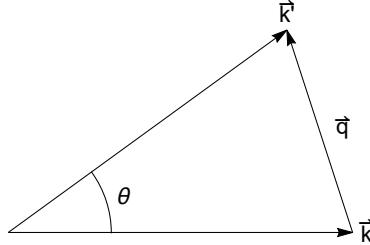
$$\frac{1}{\tau} = \frac{2\pi}{\hbar} n_I \int d^2 p \left(\frac{2\pi e^2}{\epsilon} \right)^2 \frac{e^{-2qd}}{(q + q_{\text{TF}})^2} \delta(\epsilon_p - \epsilon_F) f(\theta), \quad (50)$$

where $\vec{q} = \vec{p} - \vec{k}$, θ is the angle between the final (\vec{p}) and initial (\vec{k}) wave vectors, and $f(\theta) = 1$ for the case of the ordinary scattering rate and $f(\theta) = 1 - \cos(\theta) = 2 \sin^2(\theta/2)$ for the case of the transport scattering rate.

Defining the single-spin density of states (which is energy independent in 2D) $\rho_0 = \frac{n}{2\epsilon_F} = \frac{m^*}{2\pi\hbar^2}$ and doing the p integral in polar coordinates yields

$$\begin{aligned} \frac{1}{\tau} &= \frac{n_I}{\hbar} \rho_0 \left(\frac{2\pi e^2}{\epsilon} \right)^2 \int_0^{2\pi} d\theta f(\theta) \frac{e^{-4k_F d |\sin(\theta/2)|}}{(2k_F |\sin(\theta/2)| + q_{\text{TF}})^2}, \\ &= 2 \frac{\hbar}{m^*} \left(\rho_0 \frac{2\pi e^2}{\epsilon} \right)^2 \int_0^\pi d\theta f(\theta) \frac{e^{-4k_F d \sin(\theta/2)}}{(2 \sin(\theta/2) + q_{\text{TF}}/k_F)^2}, \end{aligned}$$

where we have used the fact that for elastic scattering around the Fermi surface, $q = 2k_F |\sin(\theta/2)|$ (see figure below).



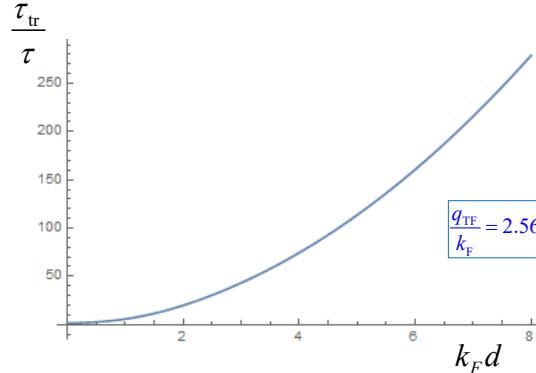
Defining

$$\begin{aligned} S_1 &= \int_0^\pi d\theta \frac{e^{-4k_F d \sin(\theta/2)}}{(2 \sin(\theta/2) + q_{\text{TF}}/k_F)^2}, \\ S_2 &= \int_0^\pi d\theta 2 \sin^2(\theta/2) \frac{e^{-4k_F d \sin(\theta/2)}}{(2 \sin(\theta/2) + q_{\text{TF}}/k_F)^2}, \end{aligned}$$

we have for the ratio of the transport lifetime to the ordinary lifetime

$$\frac{\tau_{\text{tr}}}{\tau} = \frac{S_1}{S_2}. \quad (51)$$

This quantity is plotted below as a function of $k_F d$. For the parameters given in the problem we find $k_F d \approx 7.9$.



iv) The conductivity from the Drude formula is

$$\sigma = \frac{ne^2\tau_{\text{tr}}}{m^*}. \quad (52)$$

The conductivity relates the current density to the electric field and the mobility relates the mean carrier drift velocity to the electric field. Hence the mobility is related to the conductivity by

$$\mu = \frac{\sigma}{ne} = \frac{e\tau_{\text{tr}}}{m^*} = \frac{F}{S_2}, \quad (53)$$

where

$$F \equiv \frac{2e}{h} \frac{\epsilon^2}{8\pi} \alpha^{-2} \left(\frac{m_e}{m^*} \right)^2 \lambda_e^2 \quad (54)$$

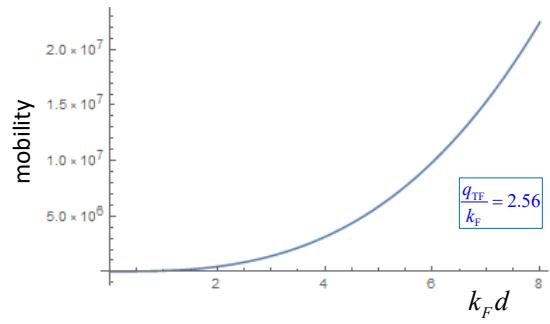
where the Josephson constant is

$$\frac{2e}{h} \approx 4.84 \times 10^{14} \frac{\text{Hz}}{\text{Volt}}. \quad (55)$$

Substituting the numerical values yields

$$F \approx 734 \frac{\text{cm}^2}{\text{Vs}}. \quad (56)$$

The mobility (in units of cm^2/Vs) is plotted as a function of $k_F d$ in the figure below for various values of q_{TF} . The integral S_2 was computed numerically. We see that for the experimentally relevant value $k_F d \approx 7.9$ that the mobility is $\mu \approx 2.5 \times 10^7 \frac{\text{cm}^2}{\text{Vs}}$. This is in agreement with the typical values observed for high-quality GaAs samples with these parameters.



Chapter 10

Ex. 10.1

Part i

For a loop in a horizontal plane perpendicular to \hat{z} , of radius R, we get

$$\oint \vec{A} \cdot d\vec{l} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \int B \cdot d\vec{a} = \Phi = \Phi_0 \beta,$$

where Φ is the flux through the loop. Assuming that the azimuthal component is constant, this turns into

$$\frac{A_\theta 2\pi R}{\Phi_0} = \beta = \frac{A_\theta 2\pi e R}{hc} = \frac{A_\theta e R}{\hbar c}.$$

Part ii

Now we have $H = \frac{\hbar^2}{2I} (-i \frac{\partial}{\partial \theta} + \beta)^2$. The eigenstate is simply $e^{i\alpha\theta}$, which is the eigenstate of angular momentum. α here should be an integer due to single-value principle

$$e^{i\alpha\theta} = e^{i\alpha(\theta+2\pi)} \Rightarrow \alpha \in \text{integer} \quad (57)$$

Therefore, the eigenfunctions are independent of the flux.

Part iii

Hitting $e^{i\alpha\theta}$ into H , we get

$$E_\alpha = \frac{\hbar^2}{2I} (\alpha + \beta)^2 \quad (58)$$

It is clear that the eigenvalues E do change with the flux β . However, since α can take any integer value, the integer part of β could be absorbed into α . Thus, the spectrum is periodic and the periodicity in β is 1.

Part iv

current = $\langle \psi_0 | \frac{-e\hbar}{mR} (-i \frac{\partial}{\partial \theta} + \beta) | \psi_0 \rangle = \frac{-e\hbar}{mR} (\alpha_0 + \beta)$, where α_0 is the α corresponding to the ground state energy ($|\alpha_0 + \beta|$ is minimum).

From Eq. 58 in part ii, $\frac{dE_0}{d\beta} = \frac{\hbar^2}{T} (\alpha_0 + \beta)$.

Hence the current $\propto \frac{dE_0}{d\beta}$.

Part v

If there exists a small flux Φ , based on the result of part iv, the current is in the opposite direction of the vector potential (if $\vec{A} = A_\theta \hat{\theta}$.) Then the magnetic field generated by the current will be anti-parallel to the external field so that the corresponding flux will reduce Φ . Therefore, the response is diamagnetic.

Ex. 10.2

We start with the general equation for current in a quantum wire:

$$I = -\frac{2e}{h} \int_0^\infty d\bar{\epsilon} \sum_{n=1}^\infty [f^\circ(\bar{\epsilon} + \gamma n^2 - \mu_L) - f^\circ(\bar{\epsilon} + \gamma n^2 - \mu_R)]$$

where f° is the fermi function at temperature T . We assume that at most one channel can be open at any temperature (so $n = 1$). Say that $\mu_R = \mu$, so that $\mu_L = \mu + \delta\mu$. Furthermore, the dispersion relation in the problem statement is equivalent to setting $\gamma = 0$ (and $\bar{\epsilon} = \epsilon$). Our current becomes

$$I = -\frac{2e}{h} \int_0^\infty d\epsilon [f^\circ(\epsilon - \mu - \delta\mu) - f^\circ(\epsilon - \mu)].$$

At zero temperature, we get that

$$\begin{aligned} I &= -\frac{2e}{h} \int_0^\infty d\epsilon [-\theta(\epsilon - \mu - \delta\mu) + \theta(\epsilon - \mu)] \\ &= -\frac{2e}{h} \delta\mu = \frac{2e^2}{h} V \end{aligned}$$

because $\delta\mu = -eV$. Thus we find that $G(\mu) = I/V = 2e^2/h$. In the case that $\mu < 0$ (which implies $\mu + \delta\mu < 0$ for small $\delta\mu$), the integral evaluates to zero, so we write $G(\mu) = \theta(\mu) 2e^2/h$.

Part i

For non-zero T and small $\delta\mu$, we can Taylor expand the Fermi-Dirac function about $\bar{\epsilon} - \mu$:

$$\begin{aligned} I &= -\frac{2e}{h} \int_0^\infty d\epsilon [f^\circ(\epsilon - \mu - \delta\mu) - f^\circ(\epsilon - \mu)] \\ &= -\frac{2e}{h} \int_0^\infty d\epsilon \left[\frac{d}{d\epsilon} f^\circ(\epsilon - \mu) (-\delta\mu) \right], \\ &= \delta\mu \frac{2e}{h} \left[\frac{1}{1+e^{\beta(\epsilon-\mu)}} \right]_0^\infty = \frac{2e^2}{h} V \frac{1}{1+e^{-\beta\mu}} \end{aligned}$$

where the last line comes from $\delta\mu = -eV$. Therefore, $G(\mu, T) = \frac{2e^2}{h} \frac{1}{e^{-\beta\mu} + 1}$.

Part ii

From $G(\mu, T) = \frac{2e^2}{h} \frac{1}{e^{-\beta\mu} + 1}$, we clearly have that $G(\mu, T) = G(\mu / (k_B T))$. When $\mu=0$, this reduces to $G(\mu = 0, T) = \frac{e^2}{h}$, independent of T .

Ex. 10.3

In this situation we need to recognize that both left and right movers in the energy range $[-e\Delta\Phi, 0)$ come from the right lead (we set Φ to be zero at the left lead without loss of generality). Since all the left movers coming from the right lead in this energy range are bounced and become right movers, their contribution to current cancel. We are then allowed to focus only on the states with positive energy, and follow the derivations from Eq. (10.5) to Eq. (10.13), with the substitution $\mu_R \rightarrow \mu_R + e\Delta\phi$, and set the transmission coefficient to be 1 which is appropriate for a smooth potential.

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk \left(\frac{L}{2\pi} \right) \left(\frac{-ev_{nk}}{L} \right) \\ &\quad \times [f^\circ(\epsilon_{nk} - \mu_L)\theta(k) + f^\circ(\epsilon_{nk} - \mu_R - e\Delta\phi)\theta(-k)] \\ &= -\frac{2e}{h} \int_0^{\infty} d\bar{\epsilon} \sum_{n=1}^{\infty} [f^\circ(\bar{\epsilon} + \gamma n^2 - \mu_L) + f^\circ(\bar{\epsilon} + \gamma n^2 - \mu_R - e\Delta\phi)], \end{aligned} \quad (59)$$

where the notations $\bar{\epsilon}$, γ , etc., are defined the same way as the textbook.

In the limit $T \rightarrow 0$ and $\mu_L \approx \mu_R + e\Delta\phi$, only channels with $\gamma n^2 < \mu_R + e\Delta\phi$ will be 'open' and contribute to the current

$$\begin{aligned} I &= -\frac{2e}{h} \sum_{n=1}^{\infty} \int_{\gamma n^2}^{\infty} d\epsilon [\theta(\mu_L - \epsilon) - \theta(\mu_R + e\Delta\phi - \epsilon)] \\ &= -\frac{2e}{h} (\mu_L - \mu_R - e\Delta\phi) N \\ &= 2N \frac{e^2}{h} V, \end{aligned} \quad (60)$$

where

$$N = \sum_{n=1}^{\infty} \theta(\mu_R + e\Delta\phi - \gamma n^2), \quad (61)$$

and

$$-eV = \mu_L - \mu_R - e\Delta\phi. \quad (62)$$

Ex. 10.4

(i) We have

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m^*} \geq \frac{\pi^2 \hbar^2}{2m^* W^2} N^2,$$

thus $N \approx \frac{k_F W}{\pi}$.

(ii) In d-dimension, the phase space DOS is $\frac{1}{(2\pi\hbar)^d}$. We thus expect the mode number to be $\sim \frac{Ap_F^2}{(2\pi\hbar)^2} = \frac{k_F^2 A}{(2\pi)^2}$

where $p_F^2 \hbar^2 A$ is the phase space volume in the transverse direction.

Ex. 10.5

$$\langle T \rangle = T_1 T_2 \frac{1}{2\pi} \int \frac{d\theta}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}} = \frac{T_1 T_2}{1 - R_1 R_2}.$$

Thus

$$\langle G \rangle = \frac{2e^2}{h} \frac{T_1 T_2}{1 - R_1 R_2}.$$

On the other hand according to Eq. (10.42) the average resistance of the wire is

$$\frac{h}{2e^2} (1 + \langle Z \rangle) = \frac{h}{2e^2} \frac{1 + R_1 R_2}{T_1 T_2} > \frac{1}{\langle G \rangle}.$$

Ex. 10.6

The system under consideration has N modes on the left and M modes on the right. In Section 10.4.1 the problem of transmission from left to right is discussed. Here we are asked to consider the reverse problem of transmission from the right to the left.

Altogether, there are $K = N + M$ modes and the S matrix is a $K \times K$ square (and unitary) matrix describing the scattering from the K input modes to the K output modes. The S matrix has the block form shown in Fig. 6. r_L is an $N \times N$ matrix representing reflections on the left, r_R is an $M \times M$ matrix representing reflections on the right. t is the $M \times N$ matrix representing transmission from the left to the right and q is the $N \times M$ matrix representing transmission from the right to the left.

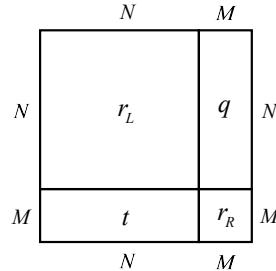


Figure 6: $(M + N) \times (M + N)$ -dimension S matrix in block form.

If the scattering is reciprocal then $S = S^T$ and hence $q = t^T$. In general however we must write $q(+\vec{B}) = t(-\vec{B})^T$. From Eq. (10.53) in the text it follows that

$$q = t(-\vec{B})^T = U^T(-\vec{B})\tau^T(-\vec{B})V^T(-\vec{B}) \quad (63)$$

To keep the notation simple, we will drop the $(-\vec{B})$ arguments henceforth. The $M \times M$ Hermitian transmission matrix is

$$\mathcal{T} = q^\dagger q = (V^T)^\dagger \tau (U^T)^\dagger U^T \tau V^T = V^* \tau^2 (V^*)^\dagger \quad (64)$$

Ex. 10.7

(i)

Shrödinger equation for the single delta potential is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0 \delta(x)\psi = E\psi. \quad (65)$$

At $x = 0$ we have boundary conditions

$$\psi(x = 0^-) = \psi(x = 0^+), \quad (66)$$

$$\frac{\hbar^2}{2m} [\psi'(x = 0^+) - \psi'(x = 0^-)] = V_0 \psi(0). \quad (67)$$

The solution takes the form of a piecewise function

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0 \\ te^{ikx} & x > 0 \end{cases}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}. \quad (68)$$

Solving for r and t based on the boundary conditions we have

$$t = \frac{1}{1 + i \frac{mV_0}{\hbar^2 k}}, \quad (69)$$

$$r = \frac{-i \frac{mV_0}{\hbar^2 k}}{1 + i \frac{mV_0}{\hbar^2 k}}. \quad (70)$$

For a single channel, the two-terminal conductance

$$G = \frac{2e^2}{h} |t|^2 = \frac{2e^2}{h} \frac{1}{1 + \frac{m^2 V_0^2}{\hbar^4 k^2}}. \quad (71)$$

(ii)

For the double-delta potential $V(x) = V_0[\delta(x - \frac{a}{2}) + \delta(x + \frac{a}{2})]$, we have Shrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0[\delta(x - \frac{a}{2}) + \delta(x + \frac{a}{2})]\psi = E\psi. \quad (72)$$

The solution takes the form

$$\psi(x) = \begin{cases} e^{ikx} + r_1 e^{-ikx} & x < -\frac{a}{2} \\ t_2 e^{ikx} + r_2 e^{-ikx} & -\frac{a}{2} < x < \frac{a}{2}, \quad k = \sqrt{\frac{2mE}{\hbar^2}} \\ t_3 e^{ikx} & x > \frac{a}{2} \end{cases} \quad (73)$$

Similar to the single delta potential case we have boundary conditions

$$e^{-ika/2} + r_1 e^{ika/2} = t_2 e^{-ika/2} + r_2 e^{ika/2}, \quad (74)$$

$$ik \frac{\hbar^2}{2m} [(t_2 - 1)e^{-ika/2} - (r_2 - r_1)e^{ika/2}] = V_0(e^{-ika/2} + r_1 e^{ika/2}); \quad (75)$$

$$t_2 e^{ika/2} + r_2 e^{-ika/2} = t_3 e^{ika/2}, \quad (76)$$

$$ik \frac{\hbar^2}{2m} [(t_3 - t_2)e^{ika/2} + r_2 e^{-ika/2}] = V_0 t_3 e^{ika/2}. \quad (77)$$

Let $\beta = \frac{\hbar^2 k}{mV_0} = \tan \phi$, and then we can express t_3 as

$$t_3 = \frac{-\beta^2}{(1 - \beta)^2 - e^{i2ka}} = \frac{-\beta^2}{(1 - i\beta - e^{ika})(1 - i\beta + e^{ika})}, \quad (78)$$

and

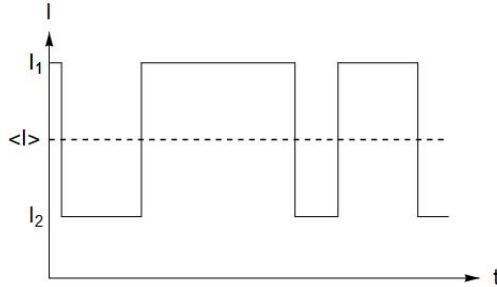
$$|t_3|^2 = \frac{\beta^4}{(2 + \beta^2)^2 - 4(1 + \beta^2)^2 \cos^2(ka + \phi)}. \quad (79)$$

So the conductance is

$$G = \frac{2e^2}{h} |t_3|^2 = \frac{2e^2}{h} \frac{\beta^4}{(2 + \beta^2)^2 - 4(1 + \beta^2)^2 \cos^2(ka + \phi)}, \quad (80)$$

and it oscillates with a .

Ex. 10.8



$$(ii) \langle I \rangle = \frac{I_1 + I_2}{2}$$

$$(iii) \bar{n}(t) = \frac{|t|}{\tau}$$

(iv) Define $\Delta I = \frac{I_1 - I_2}{2}$. Then

$$\Delta I(t') \cdot \Delta I(t' + t) = \begin{cases} (\Delta I)^2 & \text{for even number of jumps} \\ -(\Delta I)^2 & \text{for odd number of jumps} \end{cases}$$

Assume there are n jumps between t' and $t' + t$, and $P(n)$ is its probability. Then $\langle \Delta I(t') \Delta I(t' + t) \rangle = (\Delta I)^2 \cdot P(n \text{ even}) + (-\Delta I^2) \cdot P(n \text{ odd})$

$$\begin{aligned} &= (\Delta I)^2 \cdot e^{-\frac{|t|}{\tau}} \left\{ 1 + \frac{1}{2!} \left(\frac{|t|}{\tau} \right)^2 + \frac{1}{4!} \left(\frac{|t|}{\tau} \right)^4 + \cdots - \left[\frac{|t|}{\tau} + \frac{1}{3!} \left(\frac{|t|}{\tau} \right)^3 + \frac{1}{5!} \left(\frac{|t|}{\tau} \right)^5 + \cdots \right] \right\} \\ &= \frac{(I_1 - I_2)^2}{4} e^{-\frac{|t|}{\tau}} \cdot e^{-\frac{|t|}{\tau}} \\ &= \frac{(I_1 - I_2)^2}{4} \cdot e^{-\frac{2|t|}{\tau}}. \end{aligned}$$

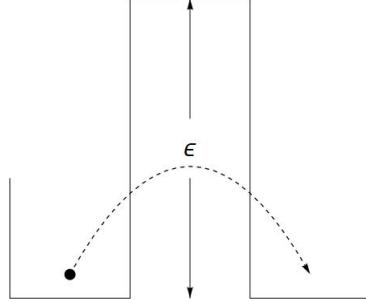
(v)

$$\begin{aligned} S_I(w) &= \int_{-\infty}^{\infty} dt \langle \Delta I(t') \Delta I(t' + t) \rangle \cdot e^{iwt} \\ &= \frac{(I_1 - I_2)^2}{4} \int_{-\infty}^{\infty} dt e^{-\frac{2|t|}{\tau}} e^{iwt} \\ &= \frac{(I_1 - I_2)^2}{4} \left[\int_0^{\infty} dt e^{-\frac{2t}{\tau}} e^{iwt} + \int_{-\infty}^0 dt e^{\frac{2t}{\tau}} e^{iwt} \right] \\ &= 2 \times \frac{(I_1 - I_2)^2}{4} \int_0^{\infty} dt \cos wt e^{-\frac{2t}{\tau}} \\ &= 2 \times \frac{(I_1 - I_2)^2}{4} \cdot \frac{2}{\tau} \frac{1}{\tau \left[w^2 + \left(\frac{2}{\tau} \right)^2 \right]} \\ &= \frac{(I_1 - I_2)^2}{\tau \left(w^2 + \left(\frac{2}{\tau} \right)^2 \right)} \end{aligned}$$

(vi)

$$\begin{aligned}
S_I(w) &\propto \int_0^\infty \frac{1}{\tau \left(w^2 + \left(\frac{2}{\tau}\right)^2 \right)} \frac{1}{\tau} d\tau \\
&\propto \int_0^\infty \frac{1}{w^2 + \left(\frac{2}{\tau}\right)^2} \frac{1}{\tau^2} d\tau \\
&\propto \int_0^\infty \frac{1}{w^2 + \left(\frac{2}{\tau}\right)^2} d\frac{1}{\tau} \\
&\propto \frac{1}{w} \int_0^\infty \frac{dx}{1+x^2} \quad x = \frac{1}{\tau w} \\
&\propto \frac{1}{w}
\end{aligned}$$

(vii) Thermal activation



Assume ϵ is the height of the barrier (see figure), then the activated tunneling rate takes the form (Ω is a normalization factor)

$$\frac{1}{\tau} = \Omega e^{-\beta\epsilon}$$

$$\tau = \frac{1}{\Omega} e^{\beta\epsilon}$$

$$d\tau = \frac{\beta}{\Omega} e^{\beta\epsilon} d\epsilon$$

$$P_\tau(\tau) d\tau = P_\epsilon d\epsilon = P_\epsilon d\tau \frac{\Omega}{\beta e^{\beta\epsilon}} = \frac{P_\epsilon}{\beta} \frac{1}{\tau} d\tau$$

$$\therefore P_\tau(\tau) = \frac{P_\epsilon}{\beta} \frac{1}{\tau} \propto \frac{1}{\tau}$$

if P_ϵ is a constant over a wide range of ϵ .

Ex. 10.9

(i) cf Eq. (10.8), we have

$$\begin{aligned} I &= \frac{-2e}{h} \int_0^\infty d\epsilon [f^0(\epsilon - \mu_L) - f^0(\epsilon - \mu_R)] |t(\epsilon)|^2 \\ &= \frac{2e^2}{h} V \int_0^\infty d\epsilon \frac{\partial f^0(\epsilon)}{\partial \mu} |t(\epsilon)|^2 \end{aligned}$$

$$\text{where } \frac{\partial f^0(\epsilon)}{\partial \mu} = \frac{1}{k_B T} \frac{e^{\beta(\epsilon - \mu)}}{(e^{\beta(\epsilon - \mu)} + 1)^2} = \frac{1}{k_B T} f^0(\epsilon) [1 - f^0(\epsilon)].$$

$$\text{Thus } G(T) = \frac{I}{V} = \frac{e^2}{h k_B T} \int_0^\infty |t(\epsilon)|^2 f^0(\epsilon) [1 - f^0(\epsilon)] d\epsilon.$$

(ii)

$$\langle n^2 \rangle = f^0(\epsilon) \cdot 1^2 + (1 - f^0(\epsilon)) \cdot 0^2 = f^0(\epsilon).$$

$$\langle n \rangle^2 = [f^0(\epsilon)]^2$$

$$\text{Thus } \langle n^2 \rangle - \langle n \rangle^2 = f^0(\epsilon) - [f^0(\epsilon)]^2 = f^0(\epsilon) [1 - f^0(\epsilon)].$$

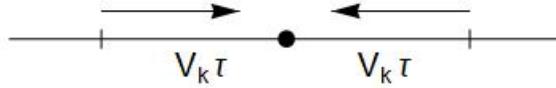
Ex. 10.10

For equilibrium we have $\langle N_T \rangle = 0$. For each wave packet that passes through the current detector during a period τ , which must be within $|v_k| \tau$ to the detector (see figure), its contribution to $\Delta N_T^2 = N_T^2$ is $f^0(\epsilon) (1 - f^0(\epsilon))$. Thus

$$N_T^2 = 2 \int_{-\infty}^\infty \frac{|v_k| \tau}{2\pi\hbar} dk f^0(\epsilon) (1 - f^0(\epsilon)) = \frac{4\tau}{2\pi\hbar} \int_0^\infty f^0(\epsilon) (1 - f^0(\epsilon)) d\epsilon = \frac{2\tau}{e^2} G k_B T.$$

[see Eq. (10.75) and set $|t(\epsilon)|^2 = 1$]

Thus $S_I(\omega \rightarrow 0) = 2k_B T G$, using (10.70).



Ex. 10.11

First check the normalization of (10.56):

$$\int_\delta^1 \frac{1}{T\sqrt{1-T}} dT = 2 \int_0^{\sqrt{1-\delta}} \frac{dx}{1-x^2} = \ln \left. \frac{1+x}{1-x} \right|_0^{\sqrt{1-\delta}} \approx \ln \frac{1}{\delta} = \frac{2L}{l}.$$

Thus $\int_\delta^1 P(T) dT = 1$. In the above we used the transformation $x^2 = 1 - T$.

$$\langle T(1-T) \rangle = \int_\delta^1 T(1-T) P(T) \approx \frac{l}{2L} \int_0^1 \sqrt{1-T} dT = \frac{l}{3L}.$$

Using (10.57), we have

$$F = \langle T(1-T) \rangle / \langle T \rangle = \frac{1}{3}.$$

An alternative solution that does not involve normalization of distribution of T , and the divergence of the distribution at small T is irrelevant: For a multichannel wire, the quantum shot noise spectrum is (see Eq. (10.74))

$$S_I(w) = \frac{2e^2}{h} eV \sum_n T_n (1 - T_n).$$

For continuous T_n ,

$$\begin{aligned} S_I(w) &= \frac{2e^2}{h} eV \alpha \int_{\delta}^1 \left(\frac{dT}{T(1-T)^{\frac{1}{2}}} \right) T (1 - T) \\ &= \frac{2e^2}{h} eV \langle T(1-T) \rangle \\ &= \frac{2e^2}{h} \langle T \rangle eV \frac{\langle T(1-T) \rangle}{\langle T \rangle} \\ &= GeVF \\ &= e\langle I \rangle F \end{aligned}$$

where α is a normalization factor that does not show up in the end, $G = \frac{2e^2}{h} \langle T \rangle$ is conductance, and $F = \frac{\langle T(1-T) \rangle}{\langle T \rangle}$, Fano factor

$$F = \frac{\int_{\delta}^1 \sqrt{1-T} dT}{\int_{\delta}^1 \frac{dT}{\sqrt{1-T}}} = \frac{1}{3};$$

$\langle I \rangle = GV$: average current.

Ex. 10.12

The entanglement entropy is:

$$\begin{aligned} -\text{Tr}(\rho_e \log \rho_e) &= \sum_s \langle s | \rho_e \log \rho_e | s \rangle = \\ \sum_s \langle s | &\left(\left(\alpha |\psi_1\rangle\langle\psi_2| + \alpha^* |\psi_2\rangle\langle\psi_1| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \right) \text{Log} N \left(\alpha |\psi_1\rangle\langle\psi_2| + \alpha^* |\psi_2\rangle\langle\psi_1| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \right) \right) | s \rangle \end{aligned}$$

where N is a normalizing constant such that the trace of ρ_e is 1. In terms of matrices, this can be found as

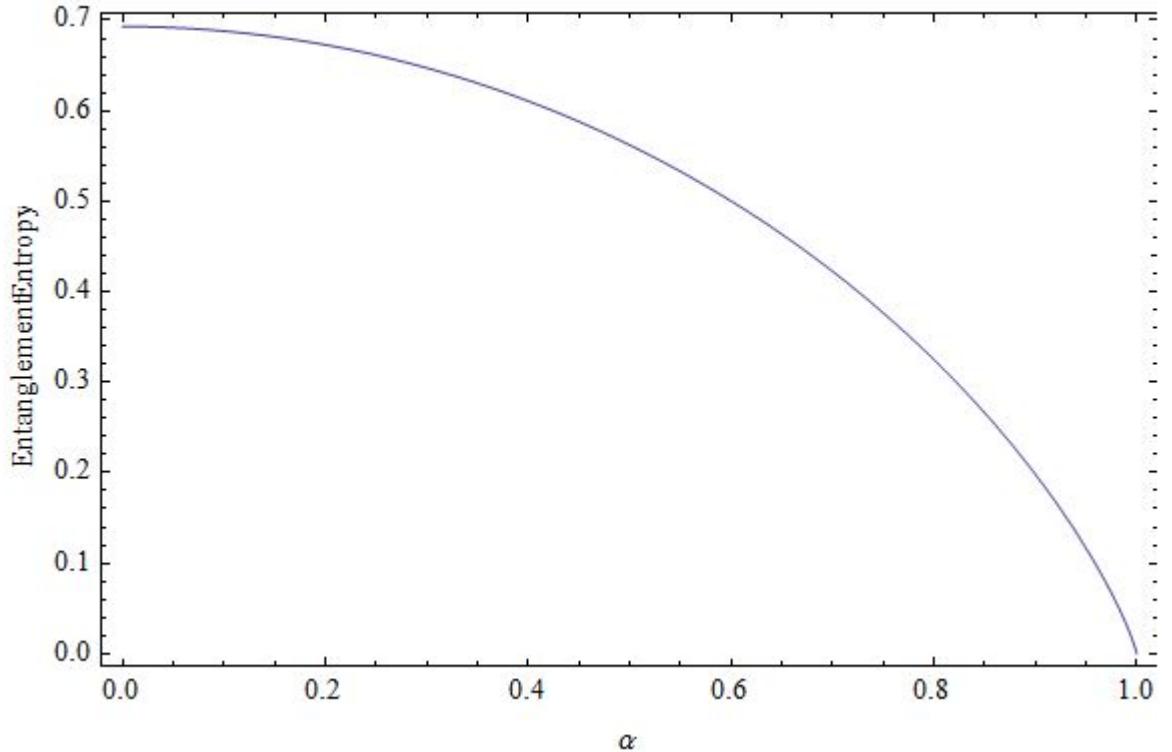
$$\text{Tr} \left\{ N \begin{pmatrix} 1 & \alpha \\ \alpha^* & 1 \end{pmatrix} \text{Log} \left(N \begin{pmatrix} 1 & \alpha \\ \alpha^* & 1 \end{pmatrix} \right) \right\}.$$

This trace is easier to evaluate in the diagonal basis, where we get the entanglement entropy to be

$$-\text{Tr} \left\{ N \begin{pmatrix} 1 - |\alpha| & 0 \\ 0 & 1 + |\alpha| \end{pmatrix} \text{Log} \left(N \begin{pmatrix} 1 - |\alpha| & 0 \\ 0 & 1 + |\alpha| \end{pmatrix} \right) \right\}.$$

Clearly, the normalizing term is $1/2$. The entropy is

$$-\left(\frac{1}{2}(1 - |\alpha|)\text{Log} \left(\frac{1}{2}(1 - |\alpha|) \right) + \frac{1}{2}(1 + |\alpha|)\text{Log} \left(\frac{1}{2}(1 + |\alpha|) \right) \right).$$



Thus, the entropy (i.e. the degree of entanglement) goes down to 0 as α approaches its limiting value of 1, as we would expect.

Chapter 11

Ex. 11.1

In d-dimension, we have:

$$\int d^d x = \Omega_d \int dx x^{d-1}$$

Where $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the total solid angle of a d-dimensional unit sphere. The density of particles is:

$$\begin{aligned} n &= \frac{2}{(2\pi)^d} \int d^d k = \frac{2}{(2\pi)^d} \Omega_d \int_0^{k_F} dk k^{d-1} \\ &= \frac{2}{(2\pi)^d} \frac{\Omega_d}{d} k_F^d \propto \epsilon_F^{d/2} \end{aligned}$$

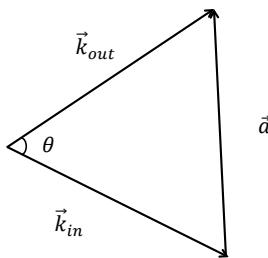
because $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$. As a result $\frac{dn}{d\mu} = \frac{d}{2} \frac{n}{\epsilon_F}$, where we used the fact $\epsilon_F = \mu$ at $T = 0$.

Einstein relation thus gives

$$\sigma = e^2 \frac{dn}{d\mu} D = e^2 \frac{d}{2} \frac{n}{\epsilon_F} \frac{1}{d} v_F^2 \tau_{tr} = e^2 \frac{n}{\epsilon_F} \frac{2}{d} v_F^2 \tau_{tr} = \frac{ne^2 \tau_{tr}}{m}$$

where we used the fact $\epsilon_F = \frac{1}{2}mv_F^2$.

Ex. 11.2



Elastic scattering $\Rightarrow |\vec{k}_{in}| = |\vec{k}_{out}| = k$, as a result

$$|\vec{k}_{out} - \vec{k}_{in}|^2 = \vec{k}_{out}^2 - 2\vec{k}_{out} \cdot \vec{k}_{in} + \vec{k}_{in}^2 = 2k^2(1 - \cos \theta) = \vec{q}^2 \Rightarrow 1 - \cos \theta = \frac{\vec{q}^2}{2k^2}$$

Ex. 11.3

$$\vec{E}(\omega) = \int dt e^{i\omega t} \vec{E}(t) = \vec{E}_0$$

$$\vec{J}(\omega) = \sigma(\omega) \vec{E}(\omega)$$

$$\begin{aligned} \vec{J}(t) &= \frac{1}{2\pi} \int d\omega e^{-i\omega t} \vec{J}(\omega) \\ &= \frac{1}{2\pi} \int d\omega e^{-i\omega t} \frac{ne^2 \tau_{tr}}{m} \frac{\vec{E}_0}{-i\tau_{tr}} \frac{1}{\omega + i\frac{1}{\tau_{tr}}} \end{aligned}$$

For $t > 0$, we close the integration contour over the lower half of the complex ω -plane and picks up the residue of the pole:

$$\vec{J}(t > 0) = \frac{ne^2}{m} \vec{E}_0 e^{-t/\tau_{tr}}.$$

For $t < 0$, we must integrate over the upper half ω -plane. Because there is no pole in the lower half plane, $\vec{J}(t < 0) = 0$. Thus

$$\vec{J}(t) = \frac{ne^2}{m} \vec{E}_0 e^{-t/\tau_{tr}} \theta(t).$$

Ex. 11.4

In 2D we have $n = \frac{k_F^2}{2\pi}$. Thus

$$\sigma_0 = \frac{ne^2 \tau_{tr}}{m} = \frac{k_F^2}{2\pi} \frac{e^2 \tau_{tr}}{m} = e^2 \frac{k_F}{2\pi} \frac{mv_F}{\hbar} \frac{\tau_{tr}}{m} = \frac{e^2}{h} k_F \ell$$

where we used $\ell = v_F \tau_{tr}$, $\hbar k_F = mv_F$.

Ex. 11.5

The physical difference between these two situations is as follows. Let's first look at Figure 10.2, and its idealization Fig. 10.1. The interference results from the phase difference between the two arms G_1 and G_2 , which is given by the flux threading through **the single ring** multiplied by e/\hbar . But the oscillation in Figure 11.7 results from **a pair of loops** with opposite directions, and their the phase difference due to the flux is actually twice that going through the single loop. So the phase difference due to the same flux is twice large as that of Figure 10.2. As a result, the period is half of Figure 10.2.

Ex. 11.6

Our localization analysis would only work within length scale l_ϕ . So for a piece of conductor with size l_ϕ

$$G \sim \frac{2e^2}{h} e^{-l_\phi/l}, \quad (81)$$

and the corresponding resistance is

$$R \sim \frac{h}{2e^2} e^{l_\phi/l}. \quad (82)$$

Then a large piece of conductor with size L can be viewed as L/l_ϕ pieces of conductors connected in series in a classical way. According to classical Ohm's law,

$$R_{\text{total}} \sim \frac{L}{l_\phi} R = \frac{h}{2e^2} \frac{L}{l_\phi} e^{l_\phi/l}, \quad (83)$$

and

$$G_{\text{total}} \sim \frac{h}{2e^2} \frac{l_\phi}{L} e^{-l_\phi/l}. \quad (84)$$

Ex. 11.7

Bloch's theorem says that for a Hamiltonian $H = \frac{p^2}{2m} + V(x)$ with periodic potential $V(x+a) = V(x)$, the eigenfunction is $\psi_{nk}(x) = e^{ikx} u_{nk}(x)$. The periodic function $u_{nk}(x)$ satisfies the following eigenvalue equation:

$$\left[\frac{(p + \hbar k)^2}{2m} + V(x) \right] u_{nk}(x) = \epsilon_n(k) u_{nk}(x)$$

For the electron moving around the ring, the eigenvalue equation is:

$$\left[\frac{(p + \frac{e}{c}A)^2}{2m} + V(x) \right] \psi_n(x) = E_n(\Phi) \psi_n(x)$$

with $A = \frac{\Phi}{L}$. Comparing it with the eigenvalue equation of the $u_{nk}(x)$ we have:

$$\begin{aligned} E_n(\Phi) &\rightarrow \epsilon_n(k), \quad \hbar k = \frac{e}{c} A = \frac{e}{c} \frac{\Phi}{L} \\ &\Rightarrow k = \frac{e}{\hbar c} \frac{\Phi}{L} \stackrel{\Phi_0 = hc/e}{\Rightarrow} k = \frac{2\pi}{L} \frac{\Phi}{\Phi_0}. \end{aligned}$$

Ex. 11.8

Part (i)

$$\frac{d \ln g}{d \ln L} = d - 2 - a/g \Rightarrow \frac{dg}{(d-2)g - a} = dL/L$$

- $d = 1$: $-\frac{dg}{g+a} = \frac{dL}{L} \Rightarrow -\ln \frac{g+a}{g_0+a} = \ln \frac{L}{l} \Rightarrow g = (g_0 + a)e^{-L/l} - a.$
- $d = 2$: $-\frac{dg}{a} = \frac{dL}{L} \Rightarrow \frac{1}{a}(g - g_0) = \ln \frac{L}{l} \Rightarrow g = g_0 - a \ln \frac{L}{l}.$
- $d = 3$, $\frac{dg}{g-a} = \frac{dL}{L} \Rightarrow \ln \frac{g-a}{g_0-a} = \ln \frac{L}{l} \Rightarrow g = (g_0 - a)e^{L/l} + a.$

Part (ii)

We expect $g \rightarrow 0$ as $L \rightarrow \xi$. Thus

$$\xi \sim \ell e^{g_0/a} = \ell e^{\pi k_F \ell / 2},$$

where we used the fact $a = 1/\pi$ and Eq. (11.89) which leads to $g_0 = k_F \ell / 2$ at scale ℓ .

Ex. 11.9

(i)

$$l_V \sim \tau_V^{1/z}. \quad (85)$$

At zero temperature but finite voltage

$$\frac{h}{\tau_V} \sim eV. \quad (86)$$

So we have

$$l_V \sim V^{-1/z}. \quad (87)$$

(ii)

$$\frac{dI}{dV} \sim V^\theta \sim \rho \sim L_V^{-d} \tau_L \sim L_V^{z-d} \sim V^{d/z-1}. \quad (88)$$

So we have

$$\theta = d/z - 1. \quad (89)$$

Ex. 11.10

$$(i) \int_{-\infty}^{\infty} \rho(E) dE = \frac{1}{N} \sum_{\alpha=1}^N 1 = 1.$$

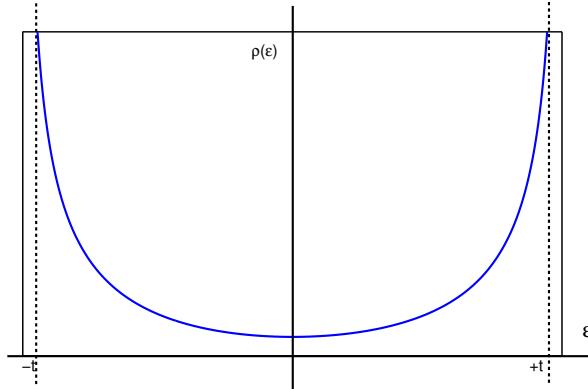
$$(ii) \int_{-\infty}^{\infty} \rho_i(E) dE = \sum_{\alpha=1}^N |\langle i | \psi_\alpha \rangle|^2 = \sum_{\alpha=1}^N \langle i | \psi_\alpha \rangle \langle \psi_\alpha | i \rangle = \langle i | i \rangle = 1.$$

(iii)

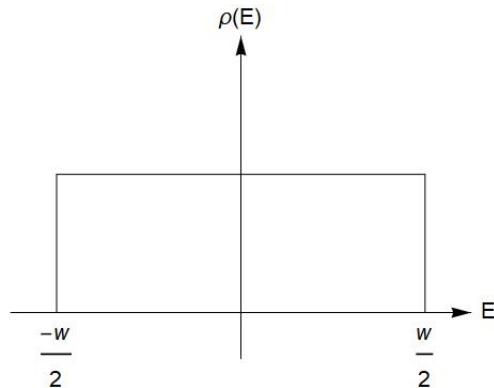
$$\begin{aligned}
\frac{1}{N} \sum_i \rho_i(E) &= \frac{1}{N} \sum_{\alpha=1}^N \sum_i \langle \psi_\alpha | i \rangle \langle i | \psi_\alpha \rangle \delta(E - E_\alpha) \\
&= \frac{1}{N} \sum_{\alpha=1}^N \langle \psi_\alpha | \psi_\alpha \rangle \delta(E - E_\alpha) \\
&= \frac{1}{N} \sum_{\alpha=1}^N \delta(E - E_\alpha) = \rho(E).
\end{aligned}$$

Ex. 11.11

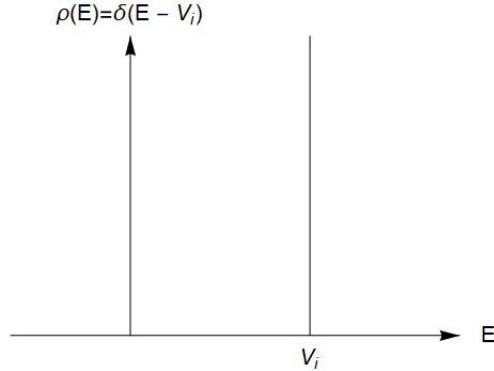
(i) From translation invariance we expect $\rho_i(E)$ to be independent of i . Thus $\rho_i(E) = \rho(E)$ cf (11.162). The 1D tight-binding DOS has been calculated in Ex. 7.7 already.



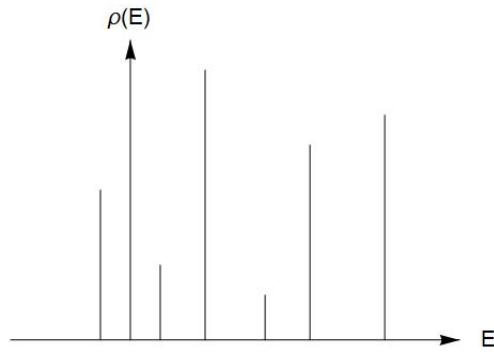
(ii) $\rho(E)$ takes the same form as distribution of V_i , because we have $|\psi_\alpha\rangle = |i\rangle$, $E_\alpha = V_i$.



(iii) In this case $\rho_i(E)$ is a δ -function, resulting in the plot below



(iv) In this case $\rho_i(E)$ is a sum of a dense but discrete set of delta-functions, different from a continuous function of (i).



11.12

For small but non-zero t , we have the Hamiltonian from Eq. (11.163)

$$H = V_1 S_1^z + V_2 S_2^z - t(S_1^+ S_2^- + S_2^+ S_1^-) \quad (90)$$

$$= H_0 - t(T + T^\dagger), \quad (91)$$

where the unperturbed Hamiltonian $H_0 = V_1 S_1^z + V_2 S_2^z$. We consider the unitary transformation

$$U = e^{\lambda(T - T^\dagger)}, \quad (92)$$

where λ is a real parameter, to be determined.

$$\begin{aligned} UH_0U^\dagger &= e^{\lambda(T-T^\dagger)}(V_1S_1^z + V_2S_2^z)e^{-\lambda(T-T^\dagger)} \\ &= V_1S_1^z + V_2S_2^z + [\lambda(T-T^\dagger), V_1S_1^z + V_2S_2^z] + \frac{\lambda^2}{2}[T-T^\dagger, [T-T^\dagger, V_1S_1^z + V_2S_2^z]] + \dots \\ &= V_1S_1^z + V_2S_2^z - \lambda(V_1-V_2)(T+T^\dagger) - \frac{\lambda^2}{2}(V_1-V_2)[T-T^\dagger, T+T^\dagger] + \dots. \end{aligned} \quad (93)$$

We also apply this to the perturbation term:

$$U[-t(T+T^\dagger)]U^\dagger = -t(T+T^\dagger) - \lambda t[T-T^\dagger, T+T^\dagger] + \dots \quad (94)$$

To cancel out the first-order term in perturbation we simply let

$$\lambda = -\frac{t}{V_1 - V_2}. \quad (95)$$

Stopping the expansion at second order, we have

$$\begin{aligned} UHU^\dagger &\approx V_1S_1^z + V_2S_2^z + \frac{t^2}{V_1 - V_2}[T, T^\dagger] \\ &\approx V_1S_1^z + V_2S_2^z + \frac{t^2}{V_1 - V_2}(S_1^z - S_2^z). \end{aligned} \quad (96)$$

We can also transform the S^z operators as

$$\begin{aligned} L_1 &= US_1^zU^\dagger = S_1^z + \lambda[T-T^\dagger, S_1^z] + \frac{\lambda^2}{2!}[T-T^\dagger, [T-T^\dagger, S_1^z]] + \dots \\ &= S_1^z + \frac{t}{V_1 - V_2}(T+T^\dagger) - \frac{t^2}{(V_1 - V_2)^2}(S_1^z - S_2^z) + \dots \end{aligned} \quad (97)$$

$$\begin{aligned} L_2 &= US_2^zU^\dagger = S_2^z + \lambda[T-T^\dagger, S_2^z] + \frac{\lambda^2}{2!}[T-T^\dagger, [T-T^\dagger, S_2^z]] + \dots \\ &= S_2^z - \frac{t}{V_1 - V_2}(T+T^\dagger) + \frac{t^2}{V_1 - V_2}(S_1^z - S_2^z) + \dots \end{aligned} \quad (98)$$

which has the form of Eq. (11.169) in the textbook. Notice that

$$L_1 + L_2 = S_1^z + S_2^z, \quad (99)$$

which is consistent with the fact that $S_1^z + S_2^z$ is an exact constant of the motion, and L_1 and L_2 (when computed exactly to all orders) are each separately constants of the motion.

Chapter 12

Ex. 12.1

The canonical momentum $\vec{p} = m\vec{v} - \frac{e\vec{A}}{c}$, thus

$$\begin{aligned} \oint \vec{p}.d\vec{x} &= 2\pi R_n m v - \frac{e}{c} \oint \vec{A}.d\vec{x} \\ &= 2\pi R_n m R_n \frac{eB}{mc} - \frac{e}{c} \int B d(\text{area}) \\ &= 2\pi R_n^2 \frac{eB}{c} - \frac{e}{c} B \cdot \pi R_n^2 \\ &= \frac{eB}{c} \pi R_n^2 = nh \\ \Rightarrow \pi R_n^2 &= m \frac{hc}{eB} \Rightarrow (\pi R_{n+1}^2 - \pi R_n^2)B = \Phi_0. \end{aligned}$$

Ex. 12.2

The Hamiltonian is $H = \frac{1}{2m}(\vec{p} + \frac{e}{c}\vec{A}(\vec{r}))^2$. In the Landau gauge $H = \frac{1}{2m}(p_x^2 + (p_y - \frac{eB}{c}x)^2)$.

Under a translation in x direction: $x \rightarrow x + \Delta x$

$$\vec{A}(\vec{r}) = -xB\hat{y} \rightarrow -xB\hat{y} - \Delta xB\hat{y}.$$

So $H(x) \neq H(x + \Delta x)$, thus the Hamiltonian is not invariant. However, after the translation, $\vec{A}(x)$ can be written as $A(x + \Delta x) = A(x) - \nabla f(\Delta x)$ where $f(\Delta x) = \Delta xB\hat{y}$. Thus translation in the x direction is equivalent to a gauge transformation.

For arbitrary gauge $\nabla \times \vec{A}(\vec{r}) = \vec{B}(\vec{r})$, after translation we have $\nabla \times \vec{A}(\vec{r} + \Delta \vec{r}) = \vec{B}(\vec{r} + \Delta \vec{r})$. For a uniform magnetic field $\vec{B}(\vec{r}) = \vec{B}(\vec{r} + \Delta \vec{r})$, we thus have $\vec{A}(\vec{r} + \Delta \vec{r}) = \vec{A}(\vec{r}) - \nabla f$, namely for translation the change in Hamiltonian is equivalent to a gauge transformation under uniform magnetic field.

Ex. 12.3

In order to verify Eq. (12.48) is a solution of the Schrödinger equation, it is sufficient to prove $H_n(x/l - kl)e^{-\frac{1}{2l^2}(x-kl)^2}$ obeys Eq. (12.43). By changing variables $\epsilon' = \frac{2\epsilon_k}{\hbar\omega}$ and $x' = \frac{x-kl^2}{l}$, the Schrödinger equation is simplified to

$$\frac{d^2 f_k(x')}{dx'^2} + (\epsilon' - x'^2) f_k(x') = 0. \quad (100)$$

It is easy to see that $f_k(x') = H_n(x')e^{-\frac{1}{2}x'^2}$ is a solution of the above equation with $\epsilon' = 2n + 1$ by utilizing the following important identity

$$\frac{d^2 H_n(x')}{dx'^2} - 2x' \frac{dH_n(x')}{dx'} + 2nH_n(x') = 0. \quad (101)$$

Ex. 12.4

$$\epsilon_n = \frac{1}{2}m_e v_n^2 = \frac{1}{2}m_e(\omega_c R_n)^2 = (n + 1/2)\hbar\omega_x \Rightarrow R_n \approx \sqrt{\frac{n\hbar}{m_e\omega_c}} = \sqrt{n\ell}$$

for large n , where semi-classical arguments are valid.

Ex. 12.5

Let us assume the x- and y-coordinates of \vec{R} are X and Y . In order for Ψ to peak at X we need the magnitude of a_k to peak at X/ℓ^2 . Its phase factor should be of the form e^{-ikY} , cf. Eq. (8.2). We thus have

$$a_k \propto e^{-ikY - k'^2 \lambda^2 / 2},$$

where $k' = k - X/\ell^2$, resulting in

$$\Psi \propto \int dk' e^{ik'y' - \frac{k'^2 \lambda^2}{2} - \frac{(x' - k'\ell^2)^2}{2\ell^2}} \propto \exp \left[-\frac{x'^2}{2\ell^2} + \frac{x'^2}{2(\ell^2 + \lambda^2)} - \frac{y'^2}{2(\ell^2 + \lambda^2)} - \frac{ix'y'}{\ell^2 + \lambda^2} \right],$$

where $x' = x - X$, $y' = y - Y$. If we make the choice $\lambda = \ell$ we find

$$|\Psi|^2 \propto \exp \left[-\frac{x'^2 + y'^2}{2\ell^2} \right],$$

which is a desired isotropic gaussian with variance ℓ in both directions. This is the smallest isotropic wave packet we can construct using lowest Landau level wave functions.

Ex. 12.6

Note to instructor: This is a challenging problem to solve for the general case. We recommend that the instructor suggest the students study the particular minimum-uncertainty wave packet example we provide in the solution below.

From Eq. (12.52) in the textbook, we have the general wave function

$$|\Psi(t)\rangle = e^{-i\frac{1}{2}\omega_c t} \sum_{k,n} a_{kn}|k,n\rangle e^{-in\omega_c t}, \quad (102)$$

where henceforth we will drop the irrelevant overall phase factor in front, and where

$$\langle x, y | \Psi(t) \rangle = \sum_{k,n} a_n(k) \frac{e^{iky}}{\sqrt{L_y Z_n}} H_n(x/\ell - k\ell) e^{-\frac{1}{2\ell^2}(x-k\ell^2)^2} \quad (103)$$

is a plane wave in the y direction and the n th harmonic oscillator wave function in the x direction. Here $Z_n \equiv 2^n n! \pi^{\frac{1}{2}} \ell$ is a normalization factor.

It is possible to compute the explicit values of the $a_n(k)$ that would yield an arbitrary Gaussian wave packet with specified initial position and momentum. We will provide an particular example of this below. However, this step is not essential to see that the packet evolution corresponds to circular motion. To see this, we will for simplicity look only at the evolution of the center of mass position of the wave packet. $\langle \Psi(t) | x | \Psi(t) \rangle$ has a time-independent part

$$X_0 = \sum_{k,n} k\ell^2 |a_n(k)|^2, \quad (104)$$

and a time-dependent part

$$\delta X(t) = \sum_{k,n,m} a_m^*(k) a_n(k) x_{\text{ZPF}} \langle m | b(t) + b^\dagger(t) | n \rangle, \quad (105)$$

where the ‘zero-point’ uncertainty of the position (in the lowest Landau level) is $x_{\text{ZPF}} = \frac{\ell}{\sqrt{2}}$, and we are using the second-quantized harmonic oscillator operators $b(t) = e^{-i\omega_c t} b$, $b^\dagger(t) = e^{+i\omega_c t} b^\dagger$ obeying $[b, b^\dagger] = +1$ to represent $\hat{x}(t) = x_{\text{ZPF}}(b(t) + b^\dagger(t))$ to represent the position operator of the oscillator (relative to $k\ell^2$).

Similarly for the y coordinate, we have

$$\begin{aligned} \langle \Psi(t) | y | \Psi(t) \rangle &= \frac{1}{L_y} \sum_{k,k',n,m} \int dx \int dy e^{-i\omega_c(n-m)t} \frac{1}{\sqrt{Z_n Z_m}} a_m^*(k') H_m(x/\ell - k'\ell) e^{-\frac{1}{2\ell^2}(x-k\ell^2)^2} \\ &\quad [(-i\partial_k) e^{iky}] a_n(k) H_n(x/\ell - k\ell) e^{-\frac{1}{2\ell^2}(x-k\ell^2)^2}. \end{aligned}$$

Converting the sum on k to an integral and then integrating by parts yields time-independent piece

$$Y_0 = \sum_{k,m} a_m^*(k) (i\partial_k) a_m(k), \quad (106)$$

and a time-dependent piece

$$\delta Y(t) = - \int \frac{dk}{2\pi} \sum_{k',n,m} 2\pi \delta(k - k') a_m^*(k) a_n(k) [i\ell^2 \langle m | \partial_x | n \rangle] e^{i(m-n)\omega_c t}, \quad (107)$$

where we have used the relation between the dependence of the harmonic oscillator function on $(x - k\ell^2)$ to replace $i\partial_k$ by $-i\ell^2 \partial_x$. Since $[\partial_x, x] = +1$ we can represent the derivative operator and the exponential time-dependence via

$$\partial_x = \frac{1}{2x_{\text{ZPF}}} (b(t) - b^\dagger(t)).$$

Thus we finally arrive at the pair of equations

$$\delta X(t) = \sum_{k,n,m} a_m^*(k) a_n(k) \frac{\ell}{\sqrt{2}} \langle m | (b(t) + b^\dagger(t)) | n \rangle \quad (108)$$

$$\delta Y(t) = - \sum_{k,n,m} a_m^*(k) a_n(k) \frac{i\ell}{\sqrt{2}} \langle m | (b(t) - b^\dagger(t)) | n \rangle \quad (109)$$

from which we have the equations of motion

$$\delta \dot{X} = +\omega_c \delta Y, \quad (110)$$

$$\delta \dot{Y} = -\omega_c \delta X, \quad (111)$$

$$(112)$$

corresponding to clockwise circular motion centered on (X_0, Y_0) . Note this is the correct sense of rotation for a charge of $-e$ in a magnetic field $\vec{B} = -B\hat{z}$.

In principle, it is possible to compute $a_n(k)$ analytically for an arbitrary Gaussian wave packet, but it is fairly complex. As a simple example where the algebra is not too bad, we specialize to the case of a minimum-uncertainty wave packet in the lowest Landau level which we boost in momentum in the y direction (so that it is no longer just in the lowest Landau level)

$$\begin{aligned} \Psi(x, y) &= \frac{1}{\sqrt{\pi\ell^2}} e^{iq_0 y} e^{-\frac{1}{2\ell^2}(x^2+y^2)} \\ a_n(k) &= \langle k, n | \Psi \rangle = \frac{1}{\sqrt{\pi\ell^2 Z_n L_y}} I_y I_x, \end{aligned}$$

where

$$\begin{aligned} I_y &= \int dy e^{-\frac{1}{2\ell^2}y^2} e^{i(q_0-k)y} = \sqrt{2\pi\ell^2} e^{-\frac{\ell^2}{2}(q_0-k)^2} \\ I_x &= \int dx e^{-\frac{1}{2\ell^2}x^2} H_n(x/\ell - k\ell) e^{-\frac{1}{2\ell^2}(x-k\ell^2)^2} \\ &= \sqrt{Z_n Z_0} \langle 0 | e^{-k\ell^2 \partial_x} | n \rangle = \sqrt{Z_n Z_0} \langle 0 | e^{-\frac{k\ell}{\sqrt{2}}(b-b^\dagger)} | n \rangle \\ &= \sqrt{Z_n Z_0} \langle 0 | e^{-\frac{k\ell}{\sqrt{2}}b} e^{+\frac{k\ell}{\sqrt{2}}b^\dagger} | n \rangle e^{-\frac{1}{4}k^2\ell^2} = \sqrt{Z_n Z_0} \frac{(-\frac{k\ell}{\sqrt{2}})^n}{\sqrt{n!}} e^{-\frac{1}{4}k^2\ell^2}, \end{aligned}$$

from which it follows that

$$a_n(k) = \sqrt{\frac{2\sqrt{\pi}\ell}{L_y}} \frac{(-\frac{k\ell}{\sqrt{2}})^n}{\sqrt{n!}} e^{-\frac{1}{4}k^2\ell^2} e^{-(k-q_0)^2\ell^2/2}$$

and

$$\begin{aligned} \sum_n |a_n(k)|^2 &= \frac{2\sqrt{\pi}\ell}{L_y} e^{-(k-q_0)^2\ell^2} \\ X_0 &= \frac{L_y}{2\pi} \int dk k\ell^2 \sum_n |a_n(k)|^2 = q_0\ell^2 \\ Y_0 &= \frac{L_y}{2\pi} \int dk \sum_n a_n^*(k) (i\partial_k) a_n(k) = 0, \end{aligned}$$

as expected.

Let us turn now to an evaluation of $\delta X(t)$

$$\begin{aligned}\delta X(t) &= L_y \int \frac{dk}{2\pi} \sqrt{2\ell} \cos(\omega_c t) \sum_n a_{n+1}(k) a_n(k) \sqrt{n+1} \\ &= \frac{L_y \ell \sqrt{2}}{2\pi} \int dk \cos(\omega_c t) \sum_n \frac{2\sqrt{\pi}\ell}{L_y} \left(\frac{-k\ell}{\sqrt{2}} \right) \frac{(k^2\ell^2/2)^n}{n!} e^{-\frac{1}{2}k^2\ell^2} e^{-(k-q_0)^2\ell^2} \\ &= -\cos(\omega_c t) \int dk \frac{k\ell^3}{\sqrt{\pi}} e^{-(k-q_0)^2\ell^2} \\ &= -q_0\ell^2 \cos(\omega_c t).\end{aligned}$$

Hence we finally arrive at the full solution for $X(t)$

$$X(t) = q_0\ell^2[1 - \cos(\omega_c t)]. \quad (113)$$

From the equations of motion for the clockwise circular motion, it then follows that

$$Y(t) = q_0\ell^2 \sin(\omega_c t). \quad (114)$$

To solve the more general case where the position variance of the wave packet is different than $\ell^2/2$, one has to include (in the language of quantum optics) squeezing terms (bb and $b^\dagger b^\dagger$) terms in the operator exponentials in the equations for I_x and I_y . We have ignored this complication here.

Ex. 12.7

We focus on the x -dependence of the wave function, as the plane wave along y -direction has no bearing on the following.

The unperturbed wave functions are

$$\begin{aligned}\psi_0^{(0)} &= \left(\frac{1}{\pi l^2} \right)^{1/4} e^{-\frac{1}{2l^2}(x-kl^2)^2}, \\ \psi_1^{(0)} &= \left(\frac{1}{\pi l^2} \right)^{1/4} \frac{\sqrt{2}}{l} (x-kl^2) e^{-\frac{1}{2l^2}(x-kl^2)^2}.\end{aligned}$$

Now adding perturbations $V = eEx$.

$$\begin{aligned}<\psi_1^{(0)}|V|\psi_0^{(0)}> &= \sqrt{2/\pi} \frac{eE}{l^2} \int dx e^{-\frac{(x-kl^2)^2}{l^2}} x(x-kl^2) \\ &= \sqrt{2/\pi} \frac{eE}{l^2} \int dx' e^{-\frac{x'^2}{l^2}} (x'^2 + kl^2 x') \\ &= \sqrt{2/\pi} \frac{eE}{l^2} \frac{\sqrt{\pi}}{2} l^3 = \frac{eEl}{\sqrt{2}}.\end{aligned}$$

As a result to 1st order in E ,

$$<\psi_1^{(0)}|\psi_0^{(1)}> = \frac{<\psi_1^{(0)}|V|\psi_0^{(0)}>}{E_0^{(0)} - E_1^{(0)}} = \frac{eEl}{\sqrt{2}\hbar\omega_c}$$

The exact new ground state wave function is:

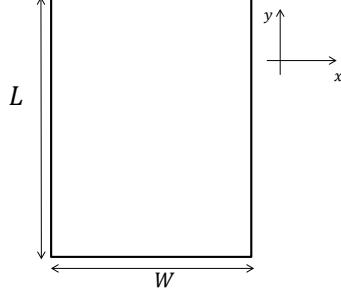
$$\begin{aligned}
\psi_0 &= \left(\frac{1}{\pi l^2}\right)^{1/4} e^{\frac{1}{2l^2}(x-kl^2+\frac{eE}{m\omega_c^2})^2} \\
&= \left(\frac{1}{\pi l^2}\right)^{1/4} e^{\frac{1}{2l^2}((x-kl^2)+2(x+kl^2)\frac{eE}{m\omega_c^2})^2} + \mathcal{O}(E^2) \\
&= \left(\frac{1}{\pi l^2}\right)^{1/4} e^{\frac{1}{2l^2}(x-kl^2)^2} \left(1 - \frac{1}{l^2}(x-kl^2)\frac{eE}{m\omega_c^2}\right) + \mathcal{O}(E^2) \\
&= \psi_0^{(0)} - \frac{1}{\sqrt{2}} \frac{eE}{m\omega_c^2 l} \psi_1^{(0)} + \mathcal{O}(E^2),
\end{aligned}$$

consistent with the perturbative result above.

The amount of mixing, which determines the Hall current J , is inversely proportional to the energy denominator $\hbar\omega_c \propto B$, as a result $J \propto 1/B$. The dependence of J on mass m_e is trickier: while $\hbar\omega_c \propto 1/m_e$, the current operator has the same dependence: $\hat{J} \propto 1/m_e$, as a result m_e -dependence is cancelled in J .

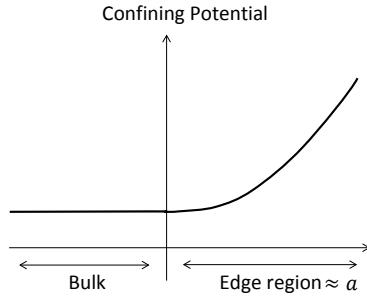
Ex. 12.8

Our sample geometry:



Let's use Landau gauge, and impose periodic boundary condition along y-direction. In x direction, the distance of neighboring guiding center: $\Delta x = \Delta_k l_B^2$, where magnetic length $l_B = \sqrt{\frac{\hbar c}{eB}}$, $\Delta k = \frac{2\pi}{L}$.

For a certain confining potential $V(x)$ the edge region's has a width (the distance over which the potential rises from constant in bulk to that of "vacuum", essentially the work function) $a \ll W$ that has no dependence on L .



The number of edge states $N_{edge} \approx a/\Delta x \approx La/(2\pi l_B^2)$.

The number of bulk states $N_{bulk} \approx W/\Delta x \approx LW/(2\pi l_B^2)$.

The ratio: $N_{edge}/N_{bulk} \approx a/W \rightarrow 0$ in the thermodynamic limit where L and W go to infinity.

When B is changing, so does N_{bulk} and N_{edge} , and the chemical potential moves from one Landau level to another when $\Delta N_{bulk} \approx N_{edge}$. From the above we find the corresponding change of B (or plateau width) scales as $\frac{\Delta B}{B} \sim \frac{N_{edge}}{N_{bulk}} \approx a/W \rightarrow 0$ without disorder.

Ex. 12.9

i)

The incident wave function is of the form:

$$\psi_i(x) = \frac{1}{\sqrt{L}} e^{ikx}$$

and the reflected:

$$\psi_r(x) = \frac{1}{\sqrt{L}} e^{-ikx}$$

So that the back scattering matrix element is:

$$\langle \psi_r | V | \psi_i \rangle = \frac{1}{L} \int dx e^{-ikx} V_0 \delta(x) e^{-ikx} = \frac{V_0}{L}$$

Which is proportional to $1/L$.

ii)

In the nth Landau gauge the wave function is:

$$\psi_k(x, y) = \frac{1}{\sqrt{L_y}} e^{iky} H_n\left(\frac{x - kl^2}{l}\right) e^{-\frac{(x + kl^2)^2}{2l^2}}.$$

The two edge states are localized at the points with wave vectors $k = \pm \frac{L_x}{2l^2}$.

So the matrix element is now:

$$\langle \psi_{-k}(x, y) | V | \psi_k(x, y) \rangle \propto \frac{e^{2iky_0}}{L_y} \int dx H_n\left(\frac{x - kl^2}{l}\right) H_{n'}\left(\frac{x + kl^2}{l}\right) e^{-\frac{(x + kl^2)^2}{2l^2}} e^{-\frac{(x - kl^2)^2}{2l^2}} \delta(x - x_0) \propto e^{-\frac{L_x^2}{l^2}},$$

where we used $k = \frac{L_x}{2l^2}$. So the matrix element is exponentially small in terms of L_x/l .

Ex. 12.10

Near K point of the 1st BA.

$$H_K = v_F \vec{\sigma} \cdot \vec{\Pi} = v_F \begin{bmatrix} 0 & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & 0 \end{bmatrix} = \epsilon_D \begin{bmatrix} 0 & a^\dagger \\ a & 0 \end{bmatrix}, \epsilon_D = \sqrt{2\hbar v_F/L}$$

It is easy to check for eigenenergy $\epsilon_n = \pm \epsilon_D \sqrt{|n|}$, the eigenstates is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} |n\rangle \\ \pm|n-1\rangle \end{bmatrix}$$

where the $|n\rangle$ is normalized eigenstate of $a^\dagger a$ for $n > 0$. The $n = 0$ case is special, and the eigenstate is

$$\begin{bmatrix} |0\rangle \\ 0 \end{bmatrix}.$$

To obtain the Landau gauge wave function with momentum k_y , simply replace $|n\rangle$ in the above by

$$\psi_{k_y, n}(x, y) = (1/\pi)^{1/4} \left(\frac{1}{2^n n! \ell} \right)^{1/2} \exp \left[-\frac{(x-X)^2}{2\ell^2} \right] H_n \left(\frac{x-X}{\ell} \right) e^{ik_y y}$$

where guiding center $X = k_y \ell^2$, H_n is the Hermit polynomial, and magnetic length $\ell = \sqrt{\frac{\hbar c}{eB}}$

Near K' point of the 1st BZ:

$$H_{K'} = -v_F \begin{bmatrix} 0 & \Pi_x + i\Pi_y \\ \Pi_x - i\Pi_y & 0 \end{bmatrix} = -\epsilon_D \begin{bmatrix} 0 & a \\ a^\dagger & 0 \end{bmatrix},$$

so the eigenstates are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} |n-1\rangle \\ \mp|n\rangle \end{bmatrix}$$

for eigenenergy $\epsilon_n = \pm \epsilon_D \sqrt{|n|}$ with $n > 0$, while the $n = 0$ eigenstate is

$$\begin{bmatrix} 0 \\ |0\rangle \end{bmatrix}.$$

The wave functions can be obtained the same way as the K point case.

Ex. 12.11

The 2D wave vector \vec{k} ranges from $k_x \in \left[\frac{-G_x}{2}, \frac{G_x}{2} \right]$, $k_y \in \left[\frac{-G_y}{2}, \frac{G_y}{2} \right]$, where $G_x = \frac{2\pi}{a_x}$, $G_y = \frac{2\pi}{a_y}$. Then according to periodic boundary condition, we will have

$$e^{ik_x \cdot x} = e^{ik_x \cdot (x + N_x a_x)}, e^{ik_y \cdot y} = e^{ik_y \cdot (y + N_y a_y)}$$

Thus $k_x N_x a_x = 2\pi n$, $k_y N_y a_y = 2\pi n'$, when $n = n' = 1$ we have the spacing between the neighboring k 's:

$$\delta k_x = \frac{2\pi}{N_x a_x}, \delta k_y = \frac{2\pi}{N_y a_y}$$

Then,

$$N = \frac{G_x}{\delta k_x} \times \frac{G_y}{\delta k_y} = \frac{\frac{2\pi}{a_x}}{\frac{2\pi}{N_x a_x}} \times \frac{\frac{2\pi}{a_y}}{\frac{2\pi}{N_y a_y}} = N_x N_y = N_\Phi.$$

Ex. 12.12

- According to the argument below Eq. (12.106), we need to repeat the unit cell q times in order to obtain a magnetic unit cell which contains an integer number of flux quanta. Thus the size of the magnetic unit cell is qA where A is the area of the unit cell. A possible choice for the primitive lattice vectors are \vec{a}_1 and $qveca_2$.

- To form them magnetic translation operators, we have

$$\begin{aligned} \because \vec{B}(\vec{r}) &= \vec{B}(\vec{r} + q\vec{a}_1) = \vec{B}(\vec{r} + q\vec{a}_2) \\ \therefore \nabla \times \vec{A}(\vec{r}) &= \nabla \times \vec{A}(\vec{r} + \vec{a}_1) = \nabla \times \vec{A}(\vec{r} + \vec{a}_2) \\ \Rightarrow \vec{A}(\vec{r} + \vec{a}_1) &= \vec{A}(\vec{r}) + \nabla f_{\vec{a}_1}(\vec{r}) \quad \vec{A}(\vec{r} + \vec{a}_2) = \vec{A}(\vec{r}) + \nabla f_{q\vec{a}_2}(\vec{r}) \\ \text{let } \phi_{\vec{a}_1}(\vec{r}) &= \frac{e f_{\vec{a}_1}(\vec{r})}{\hbar c} \quad \phi_{q\vec{a}_2}(r) = \frac{e f_{q\vec{a}_2}(r)}{\hbar c} \end{aligned}$$

Thus, we can define two magnetic translation operators:

$$\tilde{T}_{\vec{a}_1} = e^{i\phi_{\vec{a}_1}(\vec{r})} T_{\vec{a}_1} \quad \tilde{T}_{q\vec{a}_2} = e^{i\phi_{q\vec{a}_2}(\vec{r})} T_{q\vec{a}_2}$$

Chapter 13

Ex. 13.1

$$\begin{aligned}
\gamma_n(t) &= \int_0^t dt' \langle n(R(\tau(t'))) | \frac{\partial}{\partial t'} | n(R(\tau(t'))) \rangle \\
&= \int_0^t dt' \frac{\partial \tau}{\partial t'} \langle n(R(\tau)) | \frac{\partial}{\partial \tau} | n(R(\tau)) \rangle \\
&= \int_0^t d\tau \langle n(R(\tau)) | \frac{\partial}{\partial \tau} | n(R(\tau)) \rangle,
\end{aligned}$$

independent of the functional form of $\tau(t')$.

Ex. 13.2

$$\nabla_{\vec{R}} \langle \psi_n | \psi_n \rangle = \langle \psi_n | \nabla_{\vec{R}} \psi_n \rangle + \langle \nabla_{\vec{R}} \psi_n | \psi_n \rangle = 0.$$

Because the two terms are complex conjugates of each other, we immediately conclude $\langle \psi_n | \nabla_{\vec{R}} \psi_n \rangle$ is pure imaginary. As a result $A^n(\vec{R}) = i \langle \psi_n | \nabla_{\vec{R}} \psi_n \rangle$ is real.

Ex. 13.3

From Eq. (13.17) and use ∂_μ as a short hand for ∂_{R_μ} to simplify notation, we have

$$\omega_{\mu\nu}^n = i[\partial_\mu \langle n | \partial_\nu n \rangle - \partial_\nu \langle n | \partial_\mu n \rangle] = i[\langle \partial_\mu n | \partial_\nu n \rangle + \langle n | \partial_\mu \partial_\nu n \rangle - \langle \partial_\nu n | \partial_\mu n \rangle - \langle n | \partial_\nu \partial_\mu n \rangle] = i[\langle \partial_\mu n | \partial_\nu n \rangle - \langle \partial_\nu n | \partial_\mu n \rangle].$$

Ex. 13.4

This is elementary.

Ex. 13.5

Take

$$|+> = \begin{bmatrix} \cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2) \end{bmatrix} e^{i\Theta_+(\theta, \phi)}$$

$$|-> = \begin{bmatrix} \sin(\theta/2)e^{-i\phi} \\ -\cos(\theta/2) \end{bmatrix} e^{i\Theta_-(\theta, \phi)}$$

Then

$$A_\theta^- = -\frac{\partial \Theta_-}{\partial \theta}$$

$$A_\phi^- = \sin^2(\theta/2) - \frac{\partial \Theta_-}{\partial \theta}$$

$$\omega_{\theta\phi}^- = \partial_\theta A_\phi^- - \partial_\phi A_\theta^- = \frac{1}{2} \sin \theta$$

which is independent of Θ_- . Calculation for $|+>$ is similar.

Ex. 13.6

We set $\hbar = 1$.

$$H = \vec{h} \cdot \vec{S} \quad (115)$$

Redefine coordinates so that $\vec{h} = (0, 0, h) = \hat{z}h$, then the ground state has $S_z| -s \rangle = -s|s\rangle$.

$$\begin{aligned} \omega_{h_x h_y}^- &= i \sum_{m \neq -s} \frac{\langle -s | \frac{\partial H}{\partial h_x} | m \rangle \langle m | \frac{\partial H}{\partial h_y} | -s \rangle - c.c.}{(E_n - E_m)^2} \\ &= i \sum_m \frac{\langle -s | S_x | m \rangle \langle m | S_y | -s \rangle - c.c.}{h^2} \\ &= i \frac{\langle -s | [S_x, S_y] | -s \rangle}{h^2} \\ &= -\frac{\langle -s | S_z | -s \rangle}{h^2} \\ &= \frac{s}{h^2}. \end{aligned} \quad (116)$$

Note in the 2nd line only $m = -s + 1$ contributes to the sum. Eq. (13.55) follows based on rotation invariance, and Eq. (13.54) follows from that.

Ex. 13.7

$$\begin{aligned}
 |+> &= \begin{bmatrix} \cos(\theta/2)e^{-i\phi} \\ \sin(\theta/2) \end{bmatrix} \\
 A_\theta^+ &= <+|i\partial_\theta|+> = 0 \\
 A_\phi^+ &= <+|i\partial_\phi|+> = \cos^2(\theta/2) \\
 \omega_{\theta\phi}^+ &= \partial_\theta A_\phi^+ - \partial_\phi A_\theta^+ = -(1/2) \sin \theta.
 \end{aligned}$$

Thus

$$\omega_{\theta\phi}^+ + \omega_{\theta\phi}^- = (1/2) \sin \theta - (1/2) \sin \theta = 0$$

Ex. 13.8

(i) For $k_y = 0$, we can set the $\sigma_y p_y$ term in H to zero, and it reduces to

$$H_{1D} = v_F \sigma_x p_x + m(x) \sigma_z.$$

This is unitarily equivalent to

$$H' = e^{i\sigma_x \pi/4} H_{1D} e^{-i\sigma_x \pi/4} = v_F \sigma_x p_x + m(x) \sigma_y,$$

which is the same as the domain wall Hamiltonian of Eq. (7.238), where it was found there is precisely one zero mode.

(ii) Performing the same unitary transformation, we have (when $k_y \neq 0$)

$$H' = e^{i\sigma_x \pi/4} H_{1D} e^{-i\sigma_x \pi/4} = v_F \sigma_x p_x + m(x) \sigma_y - \hbar v_F k_y \sigma_z.$$

As we learned in Sec. 7.6.3, the “zero” mode is an eigen state of σ_z , as a result its wave function remains the same when $k_y \neq 0$, only its energy is shifted to

$$\epsilon(k_y) = \hbar v_F k_y$$

for $m_0 > 0$ (thus $\sigma_z = -1$ for the zero mode, see Eq. (7.242)), with linear dispersion and positive velocity (forward propagating).

(iii) For $m_0 < 0$ we have $\sigma_z = -1$ for the zero mode, so its energy is shifted to

$$\epsilon(k_y) = -\hbar v_F k_y$$

with linear dispersion and negative velocity (backward propagating).

Ex. 13.9

The regular mass Hamiltonian near K is given by

$$H_k = \vec{h} \cdot \vec{\sigma} = v_f(\sigma_x p_x + \sigma_y p_y) + m\sigma_z = \begin{bmatrix} m & v_f p e^{-i\theta_p} \\ v_f p e^{i\theta_p} & -m \end{bmatrix} \quad (117)$$

with $p = \sqrt{p_x^2 + p_y^2}$, $\theta_p = \arctan(p_x/p_y)$, and $h = \sqrt{m^2 + v_f^2 p^2}$. The eigenvalues of H_k are $\epsilon = \pm h$.

First find the non-normalized eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} m & v_f p e^{-i\theta_p} \\ v_f p e^{i\theta_p} & -m \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = m + a v_f p e^{-i\theta_p} = \pm h \quad (118)$$

so $a = \frac{\pm h - m}{v_f p} e^{i\theta_p}$. Normalizing the eigenvectors and multiplying by a phase:

$$\begin{aligned} \frac{e^{-i\theta_p}}{\sqrt{1+|a|^2}} \begin{bmatrix} 1 \\ a \end{bmatrix} &= \frac{e^{-i\theta_p} v_f p}{v_f p \sqrt{1+|a|^2}} \begin{bmatrix} 1 \\ a \end{bmatrix} \\ &= \frac{1}{\sqrt{v_f^2 p^2 + (\pm h - m)^2}} \begin{bmatrix} e^{-i\theta_p} v_f p \\ \pm h - m \end{bmatrix} \\ &= \frac{1}{\sqrt{2h(h \mp m)}} \begin{bmatrix} e^{-i\theta_p} v_f p \\ \pm h - m \end{bmatrix} = |\mathcal{U}^\pm\rangle \end{aligned} \quad (119)$$

The Berry curvature is then calculated by

$$\omega_{p\theta_p}^\pm = i \frac{\langle \mathcal{U}^\pm | \frac{\partial H_k}{\partial p} | \mathcal{U}^\mp \rangle \langle \mathcal{U}^\mp | \frac{\partial H_k}{\partial \theta_p} | \mathcal{U}^\pm \rangle - c.c.}{(\epsilon^+ - \epsilon^-)^2} \quad (120)$$

Define $A_p = \langle \mathcal{U}^+ | \frac{\partial H_k}{\partial p} | \mathcal{U}^- \rangle$ and $A_{\theta_p} = \langle \mathcal{U}^+ | \frac{\partial H_k}{\partial \theta_p} | \mathcal{U}^- \rangle$, then

$$A_p = \frac{1}{2h\sqrt{(h+m)(h-m)}} \begin{bmatrix} e^{i\theta_p} v_f p \\ h-m \end{bmatrix} \begin{bmatrix} 0 & v_f e^{-i\theta_p} \\ v_f e^{i\theta_p} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta_p} v_f p \\ -h-m \end{bmatrix} = \frac{-v_f^2 p m}{h\sqrt{h^2 - m^2}} \quad (121)$$

$$A_{\theta_p} = \frac{1}{2h\sqrt{(h+m)(h-m)}} \begin{bmatrix} e^{i\theta_p} v_f p \\ h-m \end{bmatrix} \begin{bmatrix} 0 & -iv_f p e^{-i\theta_p} \\ iv_f p e^{i\theta_p} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta_p} v_f p \\ -h-m \end{bmatrix} = \frac{iv_f^2 p^2 m}{\sqrt{h^2 - m^2}} \quad (122)$$

$$\begin{aligned} \omega_{p\theta_p}^+ &= i \frac{A_p A_{\theta_p}^* - c.c.}{(\epsilon^+ - \epsilon^-)^2} \\ &= i \frac{2A_p A_{\theta_p}^*}{4h^2} \\ &= \frac{v_f^4 p^3 m}{2h^3(h^2 - m^2)} \\ &= \frac{v_f^2 p m (h^2 - m^2)}{2h^3(h^2 - m^2)} \\ &= \frac{v_f^2 p m}{2(m^2 + v_f^2 p^2)^{3/2}} \end{aligned} \quad (123)$$

To get the Chern number for $|\mathcal{U}^+\rangle$ we integrate $\omega_{p\theta_p}^+$ over 1BZ. Since $m \ll t$, $\omega_{p\theta_p}^+$ approaches zero before the 1BZ boundary and only the integration close to points k and k' contribute to the Chern number. So the integral can be given infinite bounds in momentum space. Around point k we have

$$\begin{aligned} & \frac{1}{2\pi} \iint_{1BZ} \omega_{p\theta_p}^+ dp d\theta_p \\ &= \int_0^\infty \frac{v_f^2 pm}{2(m^2 + v_f^2 p^2)^{3/2}} dp \\ &= \frac{m}{2} \int_0^\infty \frac{x}{(m^2 + x^2)^{3/2}} dx \\ &= \frac{1}{2} \int_0^\infty \frac{y}{(1+y^2)^{3/2}} dy \\ &= \frac{1}{2} = C^k \end{aligned} \tag{124}$$

Near k'

$$H'_k = v_f(-\sigma_x p_x + \sigma_y p_y) + m\sigma_z = \begin{bmatrix} m & v_f p e^{i\theta_p} \\ v_f p e^{-i\theta_p} & -m \end{bmatrix} \tag{125}$$

The the only difference when calculating the contribution to the Chern number form k and k' is a minus sign

$$\left. \frac{\partial H_k}{\partial \theta_p} \right|_{k'} = - \left. \frac{\partial H_k}{\partial \theta_p} \right|_k \implies C_+^{k'} = -C_+^k \tag{126}$$

$$C_+ = C_+^{k'} + C_+^k = \frac{1}{2} - \frac{1}{2} = 0 \tag{127}$$

From exercise 13.7 we know that $\omega_{p\theta_p}^- = -\omega_{p\theta_p}^+$, so

$$C_- = -C_+ = 0 \tag{128}$$

For Haldane mass, $H_k = v_f(\tau_z \sigma_x p_x + \sigma_y p_y) + m\tau_z \sigma_z$, $\tau_z = \pm 1$, for k and k' . Near k , H_k is the same as with the regular mass, so $C_+^k = \frac{1}{2}$. Near k' , H_k is the same as with the regular mass except that m becomes $-m$, so the m in equation 124 gains a negative sign making $C_+^{k'} = \frac{1}{2}$, thus

$$C_\pm = \pm 1 \tag{129}$$

Ex. 13.10

- (i) $A'_y = A_y + \frac{d\phi(y)}{dy} = 0 \Rightarrow \phi(y) = -A_y y$.
- (ii) $\psi'(x, y) = e^{e\phi(y)/\hbar c} \psi(x, y) \Rightarrow \psi'(x, y + L_y) = e^{eA_y L_y / \hbar c} \psi'(x, y) \neq \psi'(x, y)$ for generic A_y .

(iii) In order to have $\psi'(x, y + L_y) = \psi'(x, y)$ we need $\frac{eA_y L_y}{\hbar c} = 2\pi n$, or

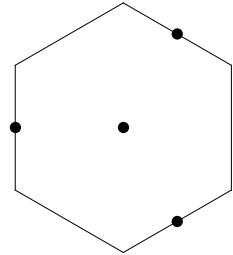
$$A_y L_y = \Phi = n \frac{2\pi\hbar c}{e} = n\Phi_0.$$

(iv) This is consistent with what we learned in Ex. 10.1 because having an integer number of flux quanta through the hole of the system does not change the physics (including energy spectrum).

Chapter 14

Ex. 14.1

- (i) They are $(0,0)$, $(0, \pi)$, $(\pi, 0)$ and (π, π) . Lattice constant is set to be 1.
- (ii) They are $(0,0)$ and the three inequivalent midpoints of the 1BZ boundary:



Ex. 14.2

Since $\vec{\alpha}$ is block diagonal and each block is $\pm \vec{\sigma}$, these anticommutators are identical to those of $\vec{\sigma}$, thus $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$.

$$\vec{\alpha}\beta + \beta\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} = 0$$

$$\beta^2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = 1$$

Ex. 14.3

(a)

$$H = \begin{pmatrix} 0 & v_F p_x & m & 0 \\ v_F p_x & 0 & 0 & m \\ m & 0 & 0 & -v_F p_x \\ 0 & m & -v_F p_x & 0 \end{pmatrix}$$

The zero mode equation is

$$H \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = 0$$

We find it decomposes into two decoupled equations:

$$H \begin{pmatrix} 0 \\ \psi_2 \\ \psi_3 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad H \begin{pmatrix} \psi_1 \\ 0 \\ 0 \\ \psi_4 \end{pmatrix} = 0$$

Each of these are unitarily equivalent to the zero mode equation corresponding to that of eqns. (7.239-7.240) with 2×2 Hamiltonian $v_F \sigma_z p_x + m(x) \sigma_x$. We thus have two zero modes. More specifically, the first equation takes the form:

$$\begin{aligned} v_F p_x \psi_2 + m(x) \psi_3 &= 0 \\ -v_F p_x \psi_3 + m(x) \psi_2 &= 0 \end{aligned}$$

This makes the proper combinations of them as follows:

$$\begin{aligned} v_F p_x (\psi_2 - i\psi_3) + im(x)(\psi_2 - i\psi_3) &= 0 \\ v_F p_x (\psi_2 + i\psi_3) - im(x)(\psi_2 + i\psi_3) &= 0 \end{aligned}$$

These are identical to (7.239-7.240).

(b)

Without losing generality (due to rotational invariance in yz plane), we may choose the momentum to be in the z-direction, contributing

$$H' = \begin{pmatrix} \sigma_z p_z & 0 \\ 0 & -\sigma_z p_z \end{pmatrix} \quad (130)$$

to the Hamiltonian. This is a diagonal term, which does not mix the two bound state solutions. We can thus consider the two subspaces separately. In these subspaces, H' is diagonal and equal to $-v_F k_z$ and $+v_F k_z$ respectively. We thus find the dispersion to be $\mp v_F |\vec{k}|$, corresponding to the valence and conduction bands of a 2D massless Dirac fermion!

Ex. 14.4

It is clear that Σ_x mixes the two bound state solutions. Thus in the subspace of the two $\epsilon = 0$ bound states for $\mu B = 0$, it shows up as a purely off-diagonal matrix, with matrix element μB . As a result, the bound state energies are $\pm \mu B$.

Ex. 14.5

(a) When static magnetic field is applied without considering Zeeman effect we have

$$h^\pm = \pm v \vec{\sigma} \cdot (\vec{p} + \frac{e}{c} \vec{A})$$

where $\vec{B} = \nabla \times \vec{A}$; $\vec{B} = -B\hat{z}$. One possible choice of \vec{A} would be $A = (\frac{By}{2}, -\frac{Bx}{2}, 0)$ thus, We take the positive case for example,

$$\begin{aligned} h^+ &= v \begin{bmatrix} p_z & (p_x + \frac{eBy}{c^2}) - i(p_y - \frac{eBx}{c^2}) \\ (p_x + \frac{eBy}{c^2}) + i(p_y - \frac{eBx}{c^2}) & -p_z \end{bmatrix} \\ &= v \begin{bmatrix} p_z & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & -p_z \end{bmatrix}, \end{aligned}$$

with

$$[\Pi_x, \Pi_y] = [p_x + \frac{eBy}{2c}, p_y - \frac{eBx}{2c}] = i \frac{\hbar e B}{c} = i \frac{\hbar^2}{l^2}$$

where $l = \sqrt{\frac{\hbar c}{eB}}$

$$\Rightarrow h^+ = vp_z \sigma^z + (v\sigma_x \Pi_x + v\sigma_y \Pi_y)$$

since

$$[p_z, \Pi_x] = [p_z, p_x + \frac{By}{2}] = 0,$$

$$[p_z, \Pi_y] = [p_z, p_y - \frac{Bx}{2}] = 0,$$

we can replace p_z by its eigen value, and h^+ reduces to the Hamiltonian of a 2D massive Dirac particle in a uniform magnetic field with mass vp_z , whose spectrum can be solved in a way analogous to that of Sec. 12.8, namely first solve the spectrum of $(h^+)^2$, and then take the square root of the resultant eigen values are (compare with Eq. (12.87)):

$$\epsilon_n^+(p_z) = \sqrt{v^2 p_z^2 + |n| \epsilon_D^2} \text{sgn}(n),$$

where $\epsilon_D = \sqrt{2}\hbar v_F / \ell$. The $n = 0$ case is special: $\epsilon_0^+(p_z) = vp_z$. The spectrum for h^- is the same except $\epsilon_0^-(p_z) = -vp_z$.

(b) When Zeeman effect is taken into consideration, we need to add $h_Z = -\mu_B \vec{\sigma} \cdot \vec{B} = \mu_B B \sigma^z$ to the Hamiltonian, then

$$H^+ = h^+ + h_Z, H^- = h^- + h_Z,$$

$$\Rightarrow H^+ = \begin{bmatrix} vp_z + \mu_B B & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & -vp_z - \mu_B B \end{bmatrix}, H^- = \begin{bmatrix} -vp_z + \mu_B B & -\Pi_x + i\Pi_y \\ -\Pi_x - i\Pi_y & vp_z - \mu_B B \end{bmatrix}.$$

As a result we simply replace vp_z by $vp_z \pm \mu_B B$ in the spectrum of h^\pm .

Chapter 15

Ex. 15.1

The standard electron Hamiltonian is:

$$\hat{H} = \hat{T} + \hat{V}_{\text{ext}} + \hat{U} = \sum_{i=1}^N \left(-\frac{\hbar^2 \nabla_i^2}{2m} + V_{\text{ext}}(\vec{r}_i) \right) + \frac{1}{2} \sum_{i \neq j}^N U(\vec{r}_i, \vec{r}_j)$$

We will take the expectation value with respect to the distinguishable many-body wavefunction:
 $\Psi = \prod_i^N \phi_i(\vec{r}_i, \sigma_i)$:

$$\langle \hat{H} \rangle = \sum_{i=1}^N \int dr_i \phi_i^*(\vec{r}_i, S_i) \left(-\frac{\hbar^2 \nabla_i^2}{2m} + V_{\text{ext}}(\vec{r}_i) \right) \phi_i(\vec{r}_i, S_i) + \frac{1}{2} \sum_{i \neq j}^N \int dr_i dr_j \phi_i^* \phi_j^* U(\vec{r}_i, \vec{r}_j) \phi_i \phi_j,$$

where we have suppressed the parameters of ϕ 's in the second term.

We use the variational principle to find the ϕ 's that minimize the energy:

$$\begin{aligned} & (\delta \langle \hat{H} \rangle) / (\delta \phi_i^*(\vec{r}_i, S_i)) = 0 \\ \implies & \left[-\left(\frac{\hbar^2 \nabla^2}{2m} \right) + V_{\text{ext}}(\vec{r}_i) \right] \phi_i(\vec{r}_i, S_i) + \left[\sum_{j \neq i, S_j}^N \int dr_j U(\vec{r}_i, \vec{r}_j) |\phi_j(\vec{r}_j, S_j)|^2 \right] \phi_i(\vec{r}_i, S_i) = \varepsilon_i \phi_i(\vec{r}_i, S_i). \end{aligned}$$

The term on the right hand side includes a Lagrange multiplier found from including the equation constraint $(\delta \langle \Psi | \Psi \rangle) / (\delta \phi_i^*(\vec{r}_i, \sigma)) = 0$.

The main difference between the last term of LHS above and Eq. (15.10) is the former does not include self-interaction, while the latter does. On the other hand the Hartree Hamiltonian of (15.11) is the same for all orbitals, while the equation above depends on the particle index i .

Ex. 15.2

The exchange part is

$$E_X = -\frac{1}{2} \sum_{i,j}^N \langle \phi_i \phi_j | u(r_i, r_j) | \phi_j \phi_i \rangle.$$

Since we have $|\phi_i\rangle = |\phi_i(r_i)\rangle|S_i\rangle$, where S_i is the S_z quantum number of $\phi_i\rangle$,

$$\begin{aligned} E_X &= -\frac{1}{2} \sum_{i,j}^N \langle \phi_i(r_i) \phi_j(r_j) | u(r_i, r_j) | \phi_j(r_i) \phi_i(r_j) \rangle \times \langle S_i S_j | S_j S_i \rangle \\ &= -\frac{1}{2} \sum_{i,j}^N \langle \phi_i(r_i) \phi_j(r_j) | u(r_i, r_j) | \phi_j(r_i) \phi_i(r_j) \rangle \times \delta_{S_i, S_j} \end{aligned}$$

Thus we know that the exchange contribution from the pair (i, j) vanishes unless they have the same spin.

Ex. 15.3

$$\begin{aligned} &\frac{1}{2} \int d\vec{r} d\vec{r}' u(\vec{r}, \vec{r}') \langle \Psi | [\delta \hat{n}(\vec{r}) \delta \hat{n}(\vec{r}')] | \Psi \rangle \\ &= \frac{1}{2} \int d\vec{r} d\vec{r}' u(\vec{r}, \vec{r}') \langle \Psi | [\hat{n}(\vec{r}) \hat{n}(\vec{r}')] | \Psi \rangle - \frac{1}{2} \int d\vec{r} d\vec{r}' u(\vec{r}, \vec{r}') \langle \Psi | \hat{n}(\vec{r}) | \Psi \rangle \langle \Psi | \hat{n}(\vec{r}') | \Psi \rangle \\ &= \langle \Psi | \hat{U} | \Psi \rangle - E_H, \end{aligned}$$

therefore the expression above must be equal to E_X .

Ex. 15.4

$$\begin{aligned} &\langle \Psi' | \hat{H} | \Psi' \rangle - \langle \Psi | \hat{H} | \Psi \rangle = \\ &\sum_{i=1}^N \langle \psi_i | \hat{T} + \hat{V}_{\text{ext}} | \psi_i \rangle + \langle \psi_l | \hat{T} + \hat{V}_{\text{ext}} | \psi_l \rangle + \langle \Psi' | \hat{U} | \Psi' \rangle - \sum_{j=1}^N \langle \psi_i | \hat{T} + \hat{V}_{\text{ext}} | \psi_i \rangle - \langle \Psi | \hat{U} | \Psi \rangle \\ &= \langle \psi_l | \hat{T} + \hat{V}_{\text{ext}} | \psi_l \rangle + \frac{1}{2} \sum_{i,j}^{N+1} [\langle \psi_i \psi_j | \hat{U} | \psi_i \psi_j \rangle - \langle \psi_i \psi_j | \hat{U} | \psi_j \psi_i \rangle] - \frac{1}{2} \sum_{i,j}^N [\langle \psi_i \psi_j | \hat{U} | \psi_i \psi_j \rangle - \langle \psi_i \psi_j | \hat{U} | \psi_j \psi_i \rangle]. \end{aligned}$$

In terms of position space orbitals $\psi_i(\vec{r})$, this becomes

$$\begin{aligned} &\int d\vec{r} \psi_l^*(\vec{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\vec{r}) \right] \psi_l(\vec{r}) + \sum_{j \neq l}^{N+1} \int d\vec{r} d\vec{r}' u(\vec{r}, \vec{r}') \psi_l^*(\vec{r}) \\ &\times \left([\psi_j^*(\vec{r}') \psi_j(\vec{r}')] \psi_l(\vec{r}) - [\psi_j^*(\vec{r}) \psi_j(\vec{r}')] \psi_l(\vec{r}') \right). \end{aligned}$$

Here we noticed that only terms that include l survive. We have also re-indexed $(r \longleftrightarrow r')$ terms to make sure that ψ_l^* always has the primed index.

The equation now has the form

$$\begin{aligned}\left\langle \Psi' \left| \hat{H} \right| \Psi' \right\rangle - \left\langle \Psi \left| \hat{H} \right| \Psi \right\rangle &= \int d\vec{r} \psi^* (\vec{r}) \times \{\text{Hartree Fock Equation for } \psi_l\} \\ &= \int d\vec{r} \psi^* (\vec{r}) \epsilon_l \psi_l (\vec{r}) = \epsilon_l\end{aligned}$$

where the last comes from the normalization of the ψ 's.

Ex. 15.5

(i) spinless case

The total kinetic energy : $T = \frac{L}{2\pi} \cdot 2 \int_0^{k_F} dk \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k_F^3 L}{6\pi m}$

Since : $N = \frac{L}{2\pi} \cdot 2 \int_0^{k_F} dk = \frac{L}{\pi} k_F$

$$\frac{T}{N} = \frac{n^2 \hbar^2 \pi^2}{6m}$$

This interaction : $V(x_1, x_2) = V_0 \delta(x_1 - x_2)$

For the spinless case, two fermions can not occupy the same position. Therefore, the interaction energy is zero.

(ii) spin-1/2 case

The total kinetic energy : $T = \frac{L}{2\pi} \cdot 4 \int_0^{k_F} dk \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k_F^3 L}{3\pi m}$

$$N = \frac{L}{\pi} \cdot 2 \cdot \int_0^{k_F} dk = \frac{L}{\pi} \cdot 2 k_F \Rightarrow k_F = \frac{1}{2} n \pi$$

$$\frac{T}{N} = \frac{n^2 \hbar^2 \pi^2}{24m} \text{ where } n = \frac{N}{L}$$

Total interaction energy :

$$\begin{aligned}\langle \Psi | U | \Psi \rangle &= \frac{1}{2} \sum_{i \neq j, s, s'} (\langle \phi_{i,s}, \phi_{j,s'} | u(x_1, x_2) | \phi_{i,s}, \phi_{j,s'} \rangle - \langle \phi_{i,s}, \phi_{j,s'} | u(x_1, x_2) | \phi_{j,s'}, \phi_{i,s} \rangle) \\ &= \frac{1}{2} \sum_{i \neq j, s, s'} (\langle \phi_i, \phi_j | u(x_1, x_2) | \phi_i, \phi_j \rangle - \langle \phi_i, \phi_j | u(x_1, x_2) | \phi_j, \phi_i \rangle \times (1 - \delta_{s,s'})) \\ &= \frac{1}{2} \sum_{i \neq j}^{N/2} \left(\frac{4V_0}{L} - \frac{2V_0}{L} \right);\end{aligned}$$

$$\frac{\langle \Psi | U | \Psi \rangle}{N} = nV_0/4.$$

Overall : Spinless fermion: $\epsilon = \frac{\hbar^2 n^2 \pi^2}{6m}$

Spin-1/2 fermion : $\epsilon = \frac{\hbar^2 n^2 \pi^2}{24m} + nV_0/4$

For a rough criterion of ferromagnetism:

$$\frac{\hbar^2 n^2 \pi^2}{6m} < \frac{\hbar^2 n^2 \pi^2}{24m} + nV_0/4 \Rightarrow V_0 > \frac{\hbar^2 n \pi^2}{2m}$$

Note: The purpose of this exercise is to show that exchange energy favors ferromagnetism. It is important to keep in mind though Hartree-Fock is an approximation. It turns out for delta-function in 1D the Hamiltonian is actually exactly solvable, and the ground state is actually non-magnetic (or a spin singlet) for any V_0 .

Ex. 15.6

Let $\frac{1}{k_F}$ be the unit of length and $\epsilon_F = \frac{\hbar^2 k_F^2}{2m_e}$ be the unit of energy and then the dimensionless Hamiltonian is

$$\begin{aligned} H &= \frac{1}{\epsilon_F} \frac{\hbar^2}{2m_e} \sum_i k_F^2 \left(-\tilde{\nabla}_i^2 \right) + \frac{1}{2} \sum_{j \neq i} \frac{e^2 k_F}{\epsilon_F} \frac{1}{|\vec{r}_i - \vec{r}_j|} - \frac{e^2 k_F}{\epsilon_F} \sum_i \int \frac{d^3 \vec{r}}{r} (\bar{n} k_F^{-3}), \\ \frac{e^2 k_F}{\epsilon_F} &= e^2 k_F \frac{2m_e}{\hbar^2 k_F^2} = \frac{2e^2 m_e}{\hbar^2 k_F}, \\ a_B &\equiv \frac{\hbar^2}{m_e e^2} \quad \text{so} \quad \frac{e^2 k_F}{\epsilon_F} = \frac{2}{k_F a_B}, \\ \frac{4\pi}{3} (r_s a_B)^3 &= \left[\left(\frac{1}{2\pi} \right)^3 \frac{4\pi}{3} k_F^3 2 \right]^{-1} \\ \Rightarrow (k_F r_s a_B)^3 &= \frac{8\pi^3}{2} \left(\frac{3}{4\pi} \right)^2 = \frac{9\pi}{4} \\ \Rightarrow \frac{1}{k_F a_B r_s} &= \left(\frac{4}{9\pi} \right)^{1/3}, \quad \frac{1}{k_F a_B} = r_s \left(\frac{4}{9\pi} \right)^{1/3}, \quad \frac{1}{2} \frac{e^2 k_F}{\epsilon_F} = r_s \left(\frac{4}{9\pi} \right)^{1/3}. \end{aligned}$$

We thus have

$$\begin{aligned} H &= \sum_i (-i \nabla_i^2) + r_s \left(\frac{4}{9\pi} \right)^{1/3} \sum_{j \neq i} \frac{1}{|\vec{r}_i - \vec{r}_j|} - 2r_s \left(\frac{4}{9\pi} \right)^{1/3} (\bar{n} k_F^{-3}) \int d^3 r \frac{1}{r}, \\ \bar{n} &= \left(\frac{1}{2\pi} \right)^3 \frac{4\pi}{3} k_F^3 \cdot 2 \Rightarrow \bar{n} k_F^{-3} = \frac{1}{3\pi^2} \\ \Rightarrow H &= \sum_i \left\{ (-\nabla_i^2) + r_s \left(\frac{4}{9\pi} \right)^{1/3} \left[\sum_{j \neq i} \frac{1}{|\vec{r}_i - \vec{r}_j|} - \frac{2}{3\pi^2} \int d^3 r \frac{1}{r} \right] \right\} \end{aligned}$$

Ex. 15.7

We consider an external potential of the form

$$H' = \lambda (\mathcal{O}_a + \mathcal{O}_a^\dagger) = \lambda (e^{-i\vec{q} \cdot \vec{r}} + e^{i\vec{q} \cdot \vec{r}}) = 2\lambda \cos(\vec{q} \cdot \vec{r}).$$

Following Appendix A, for a particle in state \vec{k}_i ,

$$\chi_{i,a} = \frac{1}{\lambda} \langle \mathcal{O}_a + \mathcal{O}_a^\dagger \rangle_i \stackrel{\text{firstorder}}{=} \sum_{j \neq i} \left(\frac{|\langle \psi_i | O_a | \psi_j \rangle|^2 + |\langle \psi_i | O_a^\dagger | \psi_j \rangle|^2}{E_i - E_j} \right).$$

Because $O_a = e^{-i\vec{q} \cdot \vec{r}}$, we have that

$$\begin{aligned} \langle \psi_i | O_a | \psi_j \rangle &= \frac{1}{V} \delta_{\vec{k}_j, \vec{k}_i + \vec{q}} \\ \langle \psi_i | O_a^\dagger | \psi_j \rangle &= \frac{1}{V} \delta_{\vec{k}_j, \vec{k}_i - \vec{q}} \end{aligned}.$$

Thus,

$$\chi_{i,a}(q) = \frac{1}{V} \left(\frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}}} + \frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}}} \right).$$

This is all for a single particle at a single energy state \vec{k}_i . Including Boltzman weights for multiple fermions, we get

$$\chi(q) = \sum_i \frac{1}{V} f_{\vec{k}_i} \left(\frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}}} + \frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}}} \right).$$

Changing variables to symmetrize the two terms, we get

$$= \sum_i \frac{1}{V} \left(+ \frac{f_{\vec{k}_i - \vec{q}/2}}{E_{\vec{k}_i - \vec{q}/2} - E_{\vec{k}_i + \vec{q}/2}} + \frac{f_{\vec{k}_i + \vec{q}/2}}{E_{\vec{k}_i + \vec{q}/2} - E_{\vec{k}_i - \vec{q}/2}} \right) = \sum_i \frac{1}{V} \left(\frac{f_{\vec{k}_i + \vec{q}/2} - f_{\vec{k}_i - \vec{q}/2}}{\frac{\hbar^2}{m} \vec{k} \cdot \vec{q}} \right).$$

Taking the continuous limit we get

$$\chi(q) = -2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{f_{\vec{k} - \vec{q}/2} - f_{\vec{k} + \vec{q}/2}}{\frac{\hbar^2}{m} \vec{k} \cdot \vec{q}} \right),$$

where we have added a factor of 2 to account for spin.

Ex. 15.8

1D

We assume without loss of generality $q > 0$, and first consider $0 < q < 2k_F$. In this case we have (using Eq. (15.92))

$$\chi_0(q) = -\frac{m}{\pi\hbar^2 q} \left[\int_{k_F - q/2}^{k_F + q/2} \frac{dk}{k} - \int_{-k_F - q/2}^{-k_F + q/2} \frac{dk}{k} \right] = -\frac{m}{\pi\hbar^2 k_F x} \ln \frac{1+x}{1-x},$$

where $x = q/(2k_F)$. Using the fact in 1D $\frac{\partial n_0}{\partial \mu} = \frac{2m}{\pi \hbar^2 k_F}$, we have for $0 < x < 1$

$$L_{1D}(x) = \frac{1}{2x} \ln \frac{1+x}{1-x},$$

where $L_{1D}(x \rightarrow 0) = 1$ as expected, while $L_{1D}(x \rightarrow 1)$ diverges logarithmically.

A similar calculation for $x > 1$ yields for general x

$$L_{1D}(x) = \frac{1}{2|x|} \ln \left| \frac{1+x}{1-x} \right|.$$

2D

Using 1st line of Eq. (15.92):

$$\chi_0(q) = -\frac{2m}{\pi^2 \hbar^2} \int_0^{k_F} k dk \int_0^{2\pi} \frac{d\theta}{q^2 + 2kq \cos \theta} = -\frac{m}{2\pi^2 \hbar^2} \int_0^1 t dt \int_0^{2\pi} \frac{d\theta}{x^2 + 2xt \cos \theta},$$

where $x = q/(2k_F)$, $t = k/k_F$, and θ is the angle between \vec{k} and \vec{q} . For $x > 1$, using

$$\int_0^{2\pi} \frac{d\theta}{x^2 + 2xt \cos \theta} = \frac{2\pi}{x^2 \sqrt{1-t^2/x^2}},$$

we find

$$\chi_0(q > 2k_F) = \frac{\partial n_0}{\partial \mu} L_{2D}(x)$$

with

$$L_{2D}(x > 1) = \frac{1}{x} \int_0^1 \frac{tdt}{\sqrt{x^2 - t^2}} = 1 - \sqrt{1 - \frac{1}{x^2}}.$$

A similar calculation yields

$$L_{2D}(x < 1) = \frac{1}{x} \int_0^x \frac{tdt}{\sqrt{x^2 - t^2}} = 1.$$

There is a cusp at $x = 1$, which is a stronger singularity than $L(x)$ of Eq. (15.94).

Ex. 15.9

Part a

We do this very similarly to the static linear response theory case.

$$\chi_{i,a}(\omega, q) = \sum_{j \neq i} \left(\frac{|\langle \psi_i | O_a | \psi_j \rangle|^2}{E_i - E_j + \hbar\omega + i\eta} + \frac{|\langle \psi_i | O_a^\dagger | \psi_j \rangle|^2}{E_i - E_j - \hbar\omega - i\eta} \right).$$

Because $O_a = e^{-i\vec{q} \cdot \vec{r}}$, we have

$$\begin{aligned}\langle \psi_i | O_a | \psi_j \rangle &= \delta_{\vec{k}_j, \vec{k}_i + \vec{q}} \\ \langle \psi_i | O_a^\dagger | \psi_j \rangle &= \delta_{\vec{k}_j, \vec{k}_i - \vec{q}}\end{aligned}$$

Thus

$$\chi_{i,a}(\omega, q) = \frac{1}{V} \left(\frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}} + \hbar\omega + i\eta} + \frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}} - \hbar\omega - i\eta} \right).$$

This is all for a single particle at a single energy state \vec{k}_i . Including Boltzman weights for multiple fermions, we get

$$\begin{aligned}& \sum_i \frac{1}{V} f_{\vec{k}_i} \left(\frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}} + \hbar\omega + i\eta} + \frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}} - \hbar\omega - i\eta} \right) \\&= \sum_i \frac{1}{V} \left(\frac{f_{\vec{k}_i - \vec{q}/2}}{E_{\vec{k}_i - \vec{q}/2} - E_{\vec{k}_i + \vec{q}/2} + \hbar\omega + i\eta} + \frac{f_{\vec{k}_i + \vec{q}/2}}{E_{\vec{k}_i + \vec{q}/2} - E_{\vec{k}_i - \vec{q}/2} - \hbar\omega - i\eta} \right) \\&= \sum_i \frac{1}{V} \left(\frac{f_{\vec{k}_i - \vec{q}/2}}{-\frac{\hbar^2}{m} \vec{k} \cdot \vec{q} + \hbar\omega + i\eta} + \frac{f_{\vec{k}_i + \vec{q}/2}}{\frac{\hbar^2}{m} \vec{k} \cdot \vec{q} - \hbar\omega - i\eta} \right) \\&= 2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{f_{\vec{k} - \vec{q}/2} - f_{\vec{k} + \vec{q}/2}}{-\frac{\hbar^2}{m} \vec{k} \cdot \vec{q} + \hbar\omega + i\eta} \right).\end{aligned}$$

Note: minus sign in the first term of the denominator of RHS of Eq. (15.99) is missing in the first printing of the book.

Part b

$$\begin{aligned}\chi(\omega) &= \frac{1}{V} \sum_{i,\sigma} f_{\vec{k}_i} \left(\frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}} + \hbar\omega + i\eta} + \frac{1}{E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}} - \hbar\omega - i\eta} \right) \\&= \frac{E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}} - \hbar\omega - i\eta + E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}} + \hbar\omega + i\eta}{\left(E_{\vec{k}_i} - E_{\vec{k}_i - \vec{q}} + \hbar\omega + i\eta \right) \left(E_{\vec{k}_i} - E_{\vec{k}_i + \vec{q}} - \hbar\omega - i\eta \right)} \\&\approx \frac{1}{V} \sum_{i,\sigma} f_{\vec{k}_i} \frac{2\hbar^2 q^2 / 2m}{\hbar^2 \omega^2} = \frac{q^2}{m \omega^2} \frac{1}{V} \sum_{i,\sigma} f_{\vec{k}_i} = \frac{q^2}{m \omega^2} n.\end{aligned}$$

The above becomes exact in the limit $q \rightarrow 0$.

Ex. 15.10

(a) The Maxwell equations are

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J}\end{aligned}$$

then

$$\begin{aligned}\nabla \times (\nabla \times \vec{E}) &= -\nabla \times \left(\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right) = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \vec{B}) \\ &= -\frac{1}{c} \frac{\partial}{\partial t} \left[\frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{J} \right] \\ &= -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi}{c^2} \frac{\partial \vec{J}}{\partial t}\end{aligned}$$

According to Equation (15.107), we have

$$\frac{\partial J}{\partial t} = -n_0 e \frac{\partial \vec{v}}{\partial t} = \frac{n_0 e^2}{m} \vec{E}.$$

Then,

$$\begin{aligned}\nabla \times (\nabla \times \vec{E}) &= -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} - \frac{4\pi}{c^2} \frac{n_0 e^2}{m} \vec{E} \\ &= -\frac{\omega^2}{c^2} \vec{E} - \frac{\Omega_{pl}^2}{c^2} \vec{E} \quad [\text{here, we assume } \vec{E} \propto e^{-i\omega t}]\end{aligned}$$

Since $\omega \ll \Omega$, we have

$$\nabla \times (\nabla \times \vec{E}) \approx -\lambda^{-2} \vec{E}.$$

Using

$$\nabla(\times \nabla \times \vec{E}) = \nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) = \nabla^2 \vec{E}$$

we obtain

$$\nabla^2 \vec{E} = -\lambda^{-2} \vec{E}.$$

we can solve this equation to get

$$\vec{E} = \vec{E}_0 e^{-i\omega t - \frac{x}{\lambda}}$$

where

$$\lambda = \frac{c}{\Omega_{pl}}.$$

This is an exponentially damped solution with damping length λ .

(b)

$$\begin{aligned}\lambda &= \frac{c}{\Omega} \\ \Rightarrow 50\text{nm} &= \frac{c}{\Omega} \\ \Rightarrow \Omega &= \frac{3.0 \times 10^8 \text{m/s}}{50 \times 10^{-9} \text{m}} = 0.6 \times 10^{16} \text{1/s}\end{aligned}$$

at this frequency

$$\hbar\Omega_{pl} = \frac{6.62607 \times 10^{-34} \times 0.6 \times 10^{16}}{1.602 \times 10^{-19}} = 24.82 \text{eV}$$

Ex. 15.11

Consider a simple harmonic oscillator

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}kx^2, \quad (131)$$

subject to the perturbation

$$V = \lambda x. \quad (132)$$

This can be solved exactly by ‘completing the square’

$$H_0 + V = \frac{p^2}{2m} + \frac{1}{2}k \left[x + \frac{\lambda}{k} \right]^2 - \frac{\lambda^2}{2k}, \quad (133)$$

from which it is clear that

$$\frac{d\langle x \rangle}{d\lambda} = -\frac{1}{k}. \quad (134)$$

Thus the susceptibility is $\chi = -1/k$ and the oscillation frequency is

$$\Omega^2 = -\frac{1}{m\chi}. \quad (135)$$

Now consider the electron gas and consider the density $\delta n_{\vec{q}}$ to be the coordinate of a harmonic oscillator

$$\delta n_{\vec{q}} = \chi(\vec{q})v_{\text{ext}}(\vec{q}). \quad (136)$$

From the continuity equation, the time rate of change of the coordinate is related to the current density via

$$\vec{\nabla} \cdot \vec{J} + (-e) \frac{\partial \delta n}{\partial t} = 0. \quad (137)$$

Writing the current density in terms of the local particle velocity and the mean density n

$$\vec{J} \approx -en\vec{v}, \quad (138)$$

we can write the kinetic energy as

$$\frac{1}{2} \frac{m}{nq^2} |\delta\dot{n}_q|^2. \quad (139)$$

This yields an effective mass for the mode of

$$m_q^* = \frac{m}{nq^2} \quad (140)$$

and a mode frequency of

$$\Omega_q^2 = -\frac{nq^2}{m\chi(q)}, \quad (141)$$

in agreement with Eq. (15.117) of the textbook.

Ex. 15.12

RPA

The random phase approximation (RPA) for $\chi(q)$ gives us

$$\begin{aligned} \chi_{\text{RPA}}(q, \omega) &= \frac{\chi_0(q, \omega)}{1 - \frac{2\pi}{q} e^2 \chi_0(q, \omega)} \\ \epsilon_{\text{RPA}}(q, \omega) &= 1 - \frac{2\pi}{q} e^2 \chi_0(q, \omega) \end{aligned}.$$

The plasma frequency is determined by the condition $\epsilon[q, \omega = \Omega_{\text{pl}}(q)] = 0$. Using the fact

$$\chi_0(q \rightarrow 0, \omega) = \frac{nq^2}{m\omega}$$

we have

$$\begin{aligned} 1 - \frac{2\pi e^2}{q} \frac{nq^2}{m\Omega_{\text{pl}}^2} &= 1 - 2\pi e^2 \frac{nq}{m\Omega_{\text{pl}}^2} = 0. \\ \implies \Omega_{\text{pl}}(q) &= \sqrt{2\pi e^2 n q / m}. \end{aligned}$$

SMA

The single mode approximation (SMA) [see Eq. (15.117)] gives

$$\Omega^2(q) \approx -\frac{nq^2}{m\chi(q)}.$$

Using the static RPA result for $q \rightarrow 0$ (which is exact in this limit): $\chi_{\text{RPA}}(q \rightarrow 0, \omega = 0) = -q/(2\pi e^2)$, we find

$$\Omega_{\text{pl}}(q) = \sqrt{2\pi e^2 n q / m}.$$

Ex. 15.13

Part a

To simplify notation we suppress spin and momentum indices for the moment. Since the symmetry is generated by \hat{n} , we expect

$$\hat{U} = e^{-i\theta\hat{n}} = e^{-i\theta c^\dagger c}.$$

Now let us verify:

$$\hat{U}c\hat{U}^\dagger = e^{-i\theta\hat{n}}ce^{i\theta\hat{n}} = e^{-i\theta\hat{n}}e^{i\theta(\hat{n}+1)}c = e^{i\theta}c,$$

where we used the fact $[c, \hat{n}] = c$ or $c\hat{n} = (\hat{n} + 1)c$.

Part b

$$\hat{T} = \sum_{\vec{k}, \sigma} \frac{\hbar^2 k^2}{2m} \hat{n}_{\vec{k}, \sigma}$$

We have already proven that $\hat{U}_{\vec{k}, \sigma}(\theta)c_{\vec{k}, \sigma}\hat{U}_{\vec{k}, \sigma}^\dagger = e^{i\theta}c_{\vec{k}, \sigma}$, and thus $\hat{U}_{\vec{k}, \sigma}(\theta)c_{\vec{k}, \sigma}^\dagger\hat{U}_{\vec{k}, \sigma}^\dagger = e^{-i\theta}c_{\vec{k}, \sigma}^\dagger$, therefore

$$\begin{aligned} \hat{U}_{\vec{k}, \sigma} \hat{n}_{\vec{k}, \sigma} \hat{U}_{\vec{k}, \sigma}^\dagger &= \hat{U}_{\vec{k}, \sigma} c_{\vec{k}, \sigma}^\dagger \hat{U}_{\vec{k}, \sigma}^\dagger \hat{U}_{\vec{k}, \sigma} \hat{n}_{\vec{k}, \sigma} \hat{U}_{\vec{k}, \sigma}^\dagger \\ &= (e^{-i\theta}c_{\vec{k}, \sigma}^\dagger)(e^{i\theta}c_{\vec{k}, \sigma}) = \hat{n}_{\vec{k}, \sigma}. \end{aligned}$$

Thus, \hat{T} is invariant.

Ex. 15.14

From the differential of Gibbs energy

$$dG = VdP - SdT + \mu dN$$

we obtain

$$\left(\frac{\partial \mu}{\partial P}\right)_{N,T} = \left(\frac{\partial V}{\partial N}\right)_{P,T} = \frac{V}{N} = \frac{1}{n},$$

where we used the fact that when intensive quantities P and T are fixed, extensive quantities V and N are proportional to each other. Now let us fix T and N as in Eq. (15.161). Then

$$\frac{1}{n} = \left(\frac{\partial \mu}{\partial P}\right) = \left(\frac{\partial \mu}{\partial n}\right) \left(\frac{\partial n}{\partial P}\right) = \left(\frac{\partial \mu}{\partial n}\right) \left(-\frac{N}{V^2} \frac{\partial V}{\partial P}\right),$$

resulting in Eq. (15.162).

Ex. 15.15

The derivation closely mimics that of the compressibility in the text; the main change is from Eq. (15.164) to

$$\delta\mu_\sigma = \sigma\hbar v_F^* \delta k_F + \sum_{k_F - \delta k_F < k' < k_F + \delta k_F, \sigma'} f_{\vec{k}, \sigma, \vec{k}', \sigma'} \delta n_{\vec{k}', \sigma'} = \sigma\hbar v_F^* \delta k_F (1 + F_0^a),$$

where we define $\delta k_F = \delta k_{F\uparrow} = -\delta k_{F\downarrow}$. The rest is identical to the manipulation for compressibility, leading to Eq. (15.167).

Ex. 15.16

From Galilean invariance we know the energy shift is

$$\Delta E = \frac{Nm}{2}v^2 = \frac{N\hbar^2(\Delta k)^2}{2m},$$

where $\Delta k = mv/\hbar$. Now we calculate ΔE using the energy functional (15.147). Obviously the single particle term (2nd term of RHS) yields

$$\Delta E_1 = \frac{N\hbar^2(\Delta k)^2}{2m^*}.$$

Now let us focus on the two particle term (3rd term of RHS). The shift yields an angular dependent change of Fermi wave vector:

$$\delta k(\theta) = (\Delta k) \cos \theta,$$

where θ is measured from the direction of \vec{v} . We find

$$\Delta E_2 = \frac{1}{2} \frac{V^2}{(2\pi)^{2d}} \sum_{\sigma\sigma'} \int d^d k \int d^d k' f_{\vec{k}, \sigma, \vec{k}', \sigma'} \delta n_{\vec{k}, \sigma} \delta n_{\vec{k}', \sigma'} = 2 \frac{V^2}{(2\pi)^{2d}} k_F^{2d-2} (\Delta k)^2 \int d\omega \int d\omega' f^s(\Delta\theta) \cos \theta \cos \theta',$$

where $d\omega$ and $d\omega'$ are solid angle elements in the directions of \vec{k} and \vec{k}' , and $\Delta\theta$ is the angle between them.

In 2D we have $\Delta\theta = \theta - \theta'$, and the integrals can be carried out trivially after using (15.154), yielding

$$\Delta E_2 = \frac{N\hbar^2(\Delta k)^2}{2m^*} F_1^s.$$

Using $\Delta E = \Delta E_1 + \Delta E_2$, we find

$$\frac{m^*}{m} = 1 + F_1^s.$$

The 3D case proceeds similarly, and we need to deal with a slightly more complicated integral of the form

$$\int d\omega \int d\omega' P_l(\Delta\theta) \cos \theta \cos \theta' = \int \sin \theta d\theta d\phi \int \sin \theta' d\theta' d\phi' P_l(\Delta\theta) \cos \theta \cos \theta',$$

where $\Delta\theta$ is the angle between (θ, ϕ) and (θ', ϕ') . To this end it is very useful to use the expansion

$$P_l(\Delta\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) = P_l(\theta)P_l(\theta') + 2 \sum_{m=1}^l \frac{(l-m)!}{(l+m)!} P_l^m(\theta)P_l^m(\theta') \cos[m(\phi-\phi')]. \quad (142)$$

Clearly only the first term above contributes to the integral, as the integration over ϕ vanishes for the 2nd term. Using the orthonormality of Legendre polynomials we again find

$$\Delta E_2 = \frac{N\hbar^2(\Delta k)^2}{2m^*} F_1^s.$$

and

$$\frac{m^*}{m} = 1 + F_1^s.$$

Unfortunately, the 1/3 factor in Eq. (15.168) should not be there!

Ex. 15.17

Consider a generic Fermi surface distortion in 3D:

$$\delta k_F(\sigma, \theta, \phi) = \sum_{lm} a_{\sigma lm} Y_{lm}(\theta, \phi).$$

Plugging into (15.147), we obtain

$$\Delta E_1 = \frac{1}{2} \frac{V}{(2\pi)^3} v_F^* k_F^2 \sum_{\sigma} \int d\omega |\delta k_F(\sigma, \theta, \phi)|^2 = \frac{1}{2} \frac{V}{(2\pi)^3} v_F^* k_F^2 \sum_{\sigma lm} |a_{\sigma lm}|^2 = \frac{V}{(2\pi)^3} v_F^* k_F^2 \sum_{lm} (|a_{lm}^s|^2 + |a_{lm}^a|^2),$$

where $a^s = (a_{\uparrow} + a_{\downarrow})/2$ and $a^a = (a_{\uparrow} - a_{\downarrow})/2$. In order to avoid notation cluttering, in the following we assume $a^a = 0$, and suppress spin or symmetry/antisymmetry indices. Then

$$\Delta E_2 = \frac{2V^2}{(2\pi)^6} k_F^4 \sum_{lm l' m'} a_{lm} a_{l' m'} \int d\omega \int d\omega' f(\Delta\theta) Y_{lm}(\theta, \phi) Y_{l' m'}(\theta', \phi').$$

Using the expansions of Eq. (15.154) for $f(\Delta\theta)$ and then Eq. (142) above, we obtain

$$\Delta E_2 = \frac{V}{(2\pi)^3} v_F^* k_F^2 \sum_{lm} a_{lm}^s |F_l|^2.$$

Thus

$$\Delta E = \Delta E_1 + \Delta E_2 = \frac{V}{(2\pi)^3} v_F^* k_F^2 \sum_{lm} a_{lm}^s (1 + F_l),$$

namely $\Delta E < 0$ for $a_{lm} \neq 0$ if $1 + F_l < 0$. In the above we only considered symmetry Fermi surface distortion in 3D, and F_l is understood as F_l^s . Antisymmetric distortion and/or 2D case proceed similarly with identical conclusion.

Ex. 15.18

The pairing Hamiltonian with cutoff Λ has the form

$$\begin{aligned} H &= H_0 + H_1 \\ H_0 &= \sum_{\vec{k}, \{0 \leq \epsilon_k \leq \Lambda\}} 2\epsilon_k c_{-\vec{k}, \downarrow}^\dagger c_{+\vec{k}, \uparrow}^\dagger c_{+\vec{k}, \uparrow} c_{-\vec{k}, \downarrow} \\ H_1 &= V_0(\Lambda) \sum_{\vec{k}, \vec{k}', \{0 \leq \epsilon_k, \epsilon_{k'} \leq \Lambda\}} c_{-\vec{k}', \downarrow}^\dagger c_{+\vec{k}', \uparrow}^\dagger c_{+\vec{k}, \uparrow} c_{-\vec{k}, \downarrow} \end{aligned}$$

Here we are considering only Cooper pairs of particles above the Fermi surface and ignoring pairing of holes below the Fermi surface. Now divide the energy region $0 \leq \epsilon \leq \Lambda$ into two parts $0 \leq \epsilon \leq \Lambda - \delta\Lambda$ and $\Lambda - \delta\Lambda \leq \epsilon \leq \Lambda$ and write

$$H_1(\Lambda) = H_1(\Lambda - \delta\Lambda) + T^+ + T^- + H_2, \quad (143)$$

where T^+ is the part of H_1 that scatters pairs from below $\Lambda - \delta\Lambda$ to above, T^- does the reverse, and $H_2 = H_1(\Lambda) - H_1(\Lambda - \delta\Lambda)$ scatters pairs only within the narrow energy slice above the new cutoff. In our perturbative RG calculation, we will neglect H_2 .

We want to perform a unitary transformation which eliminates $T^+ + T^-$ to first order in V_0 . This is accomplished via

$$U \approx e^{\lambda(T^+ - T^-)}, \quad (144)$$

where λ is approximated as a constant independent of \vec{k} . To second order in V_0 we have

$$\begin{aligned} UHU^\dagger &\approx H_0 + H_1(\Lambda - \delta\Lambda) + T^+ + T^- + \lambda[T^+ - T^-, H_0 + H_1(\Lambda - \delta\Lambda)] \\ &\quad + \lambda[T^+ - T^-, T^+ + T^-] + \frac{1}{2}\lambda^2[T^+ - T^-, [T^+ - T^-, H_0]]. \end{aligned} \quad (145)$$

In order to cancel the first-order term, we want to have

$$\lambda[T^+ - T^-, H_0] = -(T^+ + T^-). \quad (146)$$

Let us see if this is true for an appropriately chosen value of λ .

$$\begin{aligned} [T^+, H_0] &= V_0(\Lambda) \sum_{\vec{k}, \vec{k}', \{\epsilon_k \leq \Lambda - \delta\Lambda, \Lambda - \delta\Lambda \leq \epsilon_{k'} \leq \Lambda\}} 2(\epsilon_k - \epsilon_{k'}) c_{-\vec{k}', \downarrow}^\dagger c_{+\vec{k}', \uparrow}^\dagger c_{+\vec{k}, \uparrow} c_{-\vec{k}, \downarrow} \\ &\approx -2\Lambda T^+. \end{aligned}$$

Here, since we are interested in the very low energy effective Hamiltonian, we have neglected ϵ_k relative to $\epsilon_{k'} \approx \Lambda$ (consistent with our approximation of neglecting the k dependence of λ). Similarly, we have

$$[T^-, H_0] \approx +2\Lambda T^-. \quad (147)$$

Hence

$$\lambda[T^+ - T^-, H_0] = -2\lambda\Lambda(T^+ + T^-), \quad (148)$$

from which we obtain the desired cancellation, provided that $\lambda = \frac{1}{2\Lambda}$.

Using Eq. (146) in Eq. (145), the λ and λ^2 terms combine nicely and we obtain

$$\begin{aligned} UHU^\dagger &= H_0 + H_1(\Lambda - \delta\Lambda) \\ &+ \lambda[T^+ - T^-, H_1(\Lambda - \delta\Lambda)] \\ &+ \frac{\lambda}{2}[T^+ - T^-, T^+ + T^-]. \end{aligned} \quad (149)$$

The term in the second line above is a higher-order renormalization of T^\pm which we will ignore in our perturbative RG treatment. The term in the third line above is gives the renormalization of the coupling V_0

$$\frac{\lambda}{2}[T^+ - T^-, T^+ + T^-] = \lambda[T^+, T^-]. \quad (150)$$

Projecting this onto the space of low-energy initial states and carrying out the summation over the intermediate virtual states near the cutoff, we obtain

$$\lambda[T^+ - T^-] \approx -\lambda T^- T^+ \approx -\lambda V_0(\Lambda) D(0) \delta\Lambda H_1(\Lambda - \delta\Lambda), \quad (151)$$

where $D(0)$ is the density of states (assumed constant). H_1 enters because the initial low-energy state goes through a virtual high-energy state into a new low-energy state with a different momentum label. Hence we obtain the desired flow equation for the renormalization of the low-energy coupling

$$V_0(\Lambda - \delta\Lambda) \approx V_0(\Lambda) - \frac{1}{2} \frac{\delta\Lambda}{\Lambda} D(0) V_0^2. \quad (152)$$

Ex. 15.19

Eq. (15.181) yields

$$2E_c - E = -E e^{\frac{2}{D(0)|V_0|}} \Rightarrow E = -2E_c / (e^{\frac{2}{D(0)|V_0|}} - 1) \approx -E_c D(0) |V_0|$$

in the limit $D(0)|V_0| \gg 1$.

Ex. 15.20

In this case the 1st equality of Eq. (15.181) is replaced by

$$1 = - \sum_{\vec{k}} \frac{V_0}{\epsilon_{\vec{k}+\vec{q}/2} + \epsilon_{\vec{k}-\vec{q}/2} - E(q)},$$

where $E(q)$ is the bound state energy of a pair with total momentum $\hbar\vec{q}$, and the summation over \vec{k} is restricted to region with $|\vec{k} + \vec{q}/2| > k_F$ and $|\vec{k} - \vec{q}/2| > k_F$. As a result when turning the sum into an integral over energy, the lower end is no longer zero as in Eq. (15.181), but should be replaced by $\hbar v_F q |\cos \theta|$, where θ is the angle between \vec{k} and \vec{q} . Replacing $|\cos \theta|$ by its average 3D value 1/2, and evaluate the integral of Eq. (15.181) with this lower end, we find

$$E(q) = E + \hbar v_F q / 2,$$

where $E = E(0)$ is the bound state energy of zero momentum pair given by Eq. (15.182).

Source for this problem: page 33 of Schrieffer's book: Theory of Superconductivity, page 33, Eq. (2-15).

Ex. 15.21

From Eq. (15.185) we obtain for channel ℓ and a fixed magnetic quantum number m ,

$$\frac{dn}{dk} = \frac{1}{\pi} \left(R + \frac{d\delta_l}{dk} \right).$$

Thus the corresponding change to DOS due to the impurity is

$$\Delta D_l(k) = \frac{1}{\pi} \frac{d\delta_l}{dk}.$$

Thus the total energy shift of the system from adding up contributions from all channels and spins is

$$\begin{aligned} \Delta E &= 2 \sum_{\ell} (2\ell + 1) \int_0^{k_F} \frac{\hbar^2 k^2}{2m} \frac{1}{\pi} \frac{d\delta_l}{dk} dk = 2 \sum_{\ell} (2\ell + 1) \frac{1}{\pi} \left[\frac{\hbar^2 k_F^2}{2m} \delta_l(k_F) - \int_0^{k_F} \frac{\hbar^2 k}{m} \delta_l(k) dk \right] \\ &= Z\epsilon_F - 2 \sum_{\ell} (2\ell + 1) \frac{1}{\pi} \frac{\hbar^2 k_F^2}{2m} \delta_l(k_F) - \int_0^{k_F} \frac{\hbar^2 k}{m} \delta_l(k) dk \end{aligned}$$

where we used (15.183) at the last step.

Ex. 15.22

The interaction between the impurity and each of the conduction electrons is spin-dependent

$$V = J\Sigma^z\sigma^z, \quad (153)$$

where J is a coupling constant, Σ^z refers to the impurity spin and σ^z is an electron spin. The energy of the single-particle electron state $|k\sigma\rangle$, ϵ_k is modified by the Zeeman energy

$$\epsilon_{k\sigma} = \epsilon_k + \mu_B B \sigma. \quad (154)$$

This polarizes the electron gas which then modifies the energy of the impurity through the interaction V . This modification of the system energy is what renormalizes the impurity susceptibility. This is the essential physical picture. Now let us derive the appropriate equations.

Both spin states have the same Fermi level ϵ_F ,

$$\epsilon_{k_F\uparrow} + \mu_B B = \epsilon_F \quad (155)$$

$$\epsilon_{k_F\downarrow} - \mu_B B = \epsilon_F \quad (156)$$

and therefore different Fermi surface wave vectors, $k_{F\downarrow} > k_{F\uparrow}$. For a short-range scatterer, we need only consider the S-wave scattering phase shift $\delta_{0\Sigma^z\sigma^z}$ which depends on whether the impurity and electron spins are parallel or antiparallel

$$\delta_+ \equiv \delta_{0\uparrow\uparrow} = \delta_{0\downarrow\downarrow} \quad (157)$$

$$\delta_- \equiv \delta_{0\uparrow\downarrow} = \delta_{0\downarrow\uparrow}. \quad (158)$$

In the Born approximation, the phase shift is linear in V and thus $\delta_+ = -\delta_-$, but in general this is not true. However from the Friedel sum rule charge neutrality gives $\frac{\delta_+ + \delta_-}{\pi} = Z$, where Z is the impurity charge. For simplicity, let us take $Z = 0$. Then we have $\delta_+ = -\delta_-$, even beyond the Born approximation.

From Fumi's theorem (Ex. 15.21 in the textbook), the energy of the electron gas when the impurity spin is $\Sigma^z = +1$ is

$$\Delta E_\uparrow = \int_0^{k_{F\uparrow}} dk \epsilon_{k\uparrow} \frac{1}{\pi} \frac{d\delta_{0\uparrow\uparrow}}{dk} + \int_0^{k_{F\downarrow}} dk \epsilon_{k\downarrow} \frac{1}{\pi} \frac{d\delta_{0\uparrow\downarrow}}{dk}, \quad (159)$$

and the energy of the electron gas when the impurity spin is $\Sigma^z = -1$ is

$$\Delta E_\uparrow = \int_0^{k_{F\uparrow}} dk \epsilon_{k\uparrow} \frac{1}{\pi} \frac{d\delta_{0\downarrow\uparrow}}{dk} + \int_0^{k_{F\downarrow}} dk \epsilon_{k\downarrow} \frac{1}{\pi} \frac{d\delta_{0\downarrow\downarrow}}{dk}. \quad (160)$$

The energy difference can be written

$$\begin{aligned} \Delta E_\uparrow - \Delta E_\downarrow &= \int_0^{k_{F\uparrow}} dk \epsilon_{k\uparrow} \frac{1}{\pi} \frac{d(\delta_+ - \delta_-)}{dk} + \int_0^{k_{F\downarrow}} dk \epsilon_{k\downarrow} \frac{1}{\pi} \frac{d(\delta_- - \delta_+)}{dk}, \\ &= \frac{2}{\pi} \int_0^{k_{F\uparrow}} dk \epsilon_{k\uparrow} \frac{d\delta_+}{dk} - \frac{2}{\pi} \int_0^{k_{F\downarrow}} dk \epsilon_{k\downarrow} \frac{d\delta_+}{dk}. \end{aligned} \quad (161)$$

Assuming particle-hole symmetry so that the Fermi energy does not shift with B , we have

$$\left. \frac{dk_F \sigma}{dB} \right|_{B=0} = -\sigma \frac{\mu_B k_F}{2\epsilon_F}. \quad (162)$$

Using this we can expand the energy shift to first-order in B to obtain

$$\Delta E_\uparrow - \Delta E_\downarrow \approx -\mu_B B \frac{4}{\pi} k_F \left. \frac{d\delta_+}{dk} \right|_{k_F}. \quad (163)$$

This energy shift modifies the g-factor of the impurity from its bare value g_I to

$$g_I^{\text{eff}} \approx g_I - \frac{2}{\pi} k_F \left. \frac{d\delta_+}{dk} \right|_{k_F}. \quad (164)$$

The polarization of the impurity spin at inverse temperature β is

$$\langle \Sigma^z \rangle = \tanh(\beta g_I^{\text{eff}} \mu_B B) \approx \beta g_I^{\text{eff}} \mu_B B. \quad (165)$$

Assuming the impurity has magnetic moment μ_I the susceptibility is therefore

$$\chi = \mu_I \frac{\partial \langle \Sigma^z \rangle}{\partial B} \approx g_I^{\text{eff}} \frac{\mu_I \mu_B}{k_B T}. \quad (166)$$

Ex. 15.23

(i)

First process: the electron with initial momentum \vec{k} gets scattered to a state with $\Lambda - d\Lambda < \epsilon < \Lambda$, and then gets scattered again to a state with momentum \vec{k}' . This is identical to Fig. 15.9 (b).

Second the electron with initial momentum \vec{k} gets scattered to a state with $-\Lambda < \epsilon < -\Lambda + d\Lambda$, and then gets scattered again to a state with momentum \vec{k}' . This is somewhat similar to Fig. 15.9 (c), but different in a crucial way: there the two scattering events occur to two electrons and there is an exchange between them, while here it is the *same* electron that gets scattered twice.

(ii)

The first process above gives rise to a renormalization identical to Eq. (15.197). Their calculation for the contribution from the second process is almost identical, except the energy denominator changes from Λ to $-\Lambda$, because the intermediate state has a *lower* energy of $-\Lambda$ compared to the initial states energy $\epsilon_{\vec{k}} \approx 0$. As a result the contributions from the two processes cancel, resulting in Eq. (15.199).

(iii)

Unlike here, in the process illustrated in Fig. 15.9 (c) the intermediate state also has higher energy as in Fig. 15.9 (b). But due to the exchange of two fermions the resultant renormalization of the scattering matrix element picks up a minus sign. This minus sign has the same effect as the sign change of the energy denominator here, leading to identical results for the two cases that are physically different.

Ex. 15.24

(i)

The other class of 2nd-order processes that result in the same final state as those illustrated in Fig. 15.6 is the following: two electrons deep in the Fermi sea with momenta \vec{k}'' and $-\vec{k}''$, energy (measured from Fermi energy) $-\Lambda < \epsilon_{\vec{k}''} = \epsilon_{-\vec{k}''} < -\Lambda + d\Lambda$ get scattered to \vec{k}' and $-\vec{k}'$, and then the original pair of electron get scattered from \vec{k} and $-\vec{k}$ to \vec{k}'' and $-\vec{k}''$.

(ii)

The processes of part (i) give rise to the same contribution as those of Fig. 15.6, Eq. (15.173). This is because (a) The intermediate state has the same energy denominator 2Λ , and (b) These processes involve exchange of two pairs of fermions, as a result $(-1)^F = 1$ resulting in exactly the same matrix element.

(iii)

With contributions from both sets of processes included the flow equation becomes

$$\frac{dV(\Lambda)}{d \ln \frac{E_c}{\Lambda}} = -D(0)V^2(\Lambda).$$

Integrating it yields

$$V(\Lambda) = \frac{V_0}{1 + D(0)V_0 \ln \frac{E_c}{\Lambda}},$$

and we find $V(\Lambda)$ *diverges* when Λ reaches

$$\Lambda_c = E_c e^{-\frac{1}{D(0)|V_0|}},$$

which is the scale at which our perturbative analysis breaks down. This turns out to be the correct scale for superconducting gap, which is much larger than the binding energy of the two-body Cooper pair problem (15.152) and the scale found in (15.175). The source of the big difference is the electrons forming the Fermi sea also contribute to the RG flow, resulting in a much faster flow.

Chapter 16

Ex. 16.1

$$A(\text{area}) = 10^{-6} \text{ m}^2$$

$$l = \frac{257}{\sqrt{10}} \text{ \AA} = \frac{257}{\sqrt{10}} \times 10^{-10} \text{ m}$$

$$N_\phi \text{ (Landau degeneracy or orbital number)} = \frac{A}{2\pi l^2} = \frac{10^{15}}{2\pi(257)^2} \approx 2.41 \times 10^9$$

* Non-interacting degeneracy (= deg) = number of configurations = $C_{N_\phi}^{N_e} = \frac{N_\phi!}{N_e!(N_\phi - N_e)!}$ with N_e the particle number.

$$*\nu \text{ (the filling factor)} = \frac{1}{3} = \frac{N_\phi}{N_e} \rightarrow N_e = \frac{N_\phi}{3}$$

$$\therefore \ln \text{deg} = \ln \left(\frac{N_\phi!}{N_e!(N_\phi - N_e)!} \right)$$

$$= \ln \frac{N_\phi!}{\left(\frac{1}{3}N_\phi\right)!\left(\frac{2}{3}N_\phi\right)!}$$

$$\approx N_\phi \ln N_\phi - N_\phi - \frac{N_\phi}{3} \ln N_\phi + \frac{N_\phi}{3} \ln 3 + \frac{N_\phi}{3} - \frac{2N_\phi}{3} \ln N_\phi - \frac{2N_\phi}{3} \ln \frac{2}{3} + \frac{2N_\phi}{3}$$

$$= \frac{N_\phi}{3} \ln 3 - \frac{2N_\phi}{3} \ln 2 + \frac{2N_\phi}{3} \ln 3$$

$$= \ln \left(\frac{3}{\sqrt[3]{4}} \right)^{N_\phi}$$

$$\therefore \text{deg} \approx \left(\frac{3}{\sqrt[3]{4}} \right)^{N_\phi}$$

$$N_\phi \approx 2.41 \times 10^9$$

$$\therefore \text{deg} \approx (4.64)^{10^9} \#$$

Ex. 16.2

Verify Eqs.(16.9) and (16.11).

1. Eq.(16.9)

$$\begin{aligned}
 \vec{R} &= \vec{r} - \frac{l^2 \cdot (\hat{z} \times \vec{\Pi})}{\hbar} \\
 \Rightarrow \begin{cases} R_x = r_x + \frac{l^2 \Pi_y}{\hbar} \\ R_y = r_y - \frac{l^2 \Pi_x}{\hbar} \end{cases} \\
 [R_x, R_y] &= \left[r_x + \frac{l^2 \Pi_y}{\hbar}, r_y - \frac{l^2 \Pi_x}{\hbar} \right] \\
 &= \left[r_x, -\frac{l^2 \Pi_x}{\hbar} \right] + \left[\frac{l^2 \Pi_y}{\hbar}, r_y \right] + \frac{l^4}{\hbar^2} [\Pi_x, \Pi_y] \\
 \because \Pi_x &= p_x + \frac{e}{c} A_x, \Pi_y = p_y + \frac{e}{c} A_y \\
 [R_x, R_y] &= \left[r_x, -\frac{l^2}{\hbar} \left(p_x + \frac{e}{c} A_x \right) \right] + \left[\frac{l^2}{\hbar} \left(p_y + \frac{e}{c} A_y \right), r_y \right] + \frac{l^4}{\hbar^2} [\Pi_x, \Pi_y] \\
 &= \frac{-l^2}{\hbar} [r_x, p_x] + \frac{l^2}{\hbar} [p_y, r_y] + \frac{l^4}{\hbar^2} [\Pi_x, \Pi_y] \\
 &= \frac{-l^2}{\hbar} \cdot i\hbar + \frac{-l^2}{\hbar} \cdot i\hbar + \frac{l^4}{\hbar^2} \frac{i\hbar^2}{l^2} \\
 &= \frac{-il^2}{\hbar} \quad QED
 \end{aligned}$$

2. Eq.(16.11)

$$\begin{aligned}
 * [R_i, \Pi_\alpha] &= \left[r_i + \epsilon_{ijz} \frac{l^2}{\hbar} \Pi_j, \Pi_\alpha \right] \quad (j \neq i) \\
 &= [r_i, \Pi_\alpha] + \epsilon_{ijz} \frac{l^2}{\hbar} [\Pi_j, \Pi_\alpha] \\
 &= i\hbar \delta_{i\alpha} + \epsilon_{ijz} (1 - \delta_{j\alpha}) \epsilon_{j\alpha z} \frac{l^2}{\hbar} \frac{i\hbar^2}{l^2} \\
 &= i\hbar \delta_{i\alpha} + (\epsilon_{ijz} \epsilon_{j\alpha z} - \epsilon_{ijz} \epsilon_{j\alpha z}) \frac{l^2}{\hbar} \frac{i\hbar^2}{l^2} \\
 &= i\hbar \delta_{i\alpha} - i\hbar \delta_{i\alpha} \\
 &= \frac{0}{\hbar} \quad QED
 \end{aligned}$$

where $\epsilon_{ijk} = \begin{cases} 0 & \text{if any two of } \{i, j, k\} \text{ are the same} \\ 1 & \text{even permutation} \\ -1 & \text{odd permutation} \end{cases}$

$$\begin{aligned}
* [R_i, H] &= \left[R_i, \frac{1}{2m} (\Pi_x^2 + \Pi_y^2) \right] \\
&= \left[R_i, \frac{1}{2m} \Pi_x^2 \right] + \left[R_i, \frac{1}{2m} \Pi_y^2 \right] \\
&= \underline{\underline{0}} \quad (\because [R_i, \Pi_j] = 0)
\end{aligned}$$

$$\begin{aligned}
* [a, b] &= \left[\frac{l}{\sqrt{2}\hbar} (\Pi_x + i\Pi_y), \frac{l}{\sqrt{2}\hbar} (R_x - iR_y) \right] \\
&= \underline{\underline{0}} \quad (\because [R_i, \Pi_j] = 0) \\
* [a, b^\dagger] &= \underline{\underline{0}} \quad (\text{the same})
\end{aligned}$$

Ex. 16.3

$$\vec{A} = B (0, -x, 0) \quad a = \frac{l}{\sqrt{2}\hbar} (\Pi_x + i\Pi_y) \quad , \text{ where } \Pi_x = p_x \quad \text{and} \quad \Pi_y = p_y - \frac{e}{c} B x$$

$$a = \frac{l}{\sqrt{2}\hbar} [(p_x + i p_y) - i \frac{e}{c} B x]$$

Wave functions are eigenstates of p_y and can be written as

$$\psi_{n,k}(x, y) = e^{iky} \psi_n(x - kl^2)$$

where n is the Landau level index and k is the momentum in x direction. Hence,

$$\langle \psi_k | a | \psi_{k'} \rangle = C \times \delta_{k,k'}$$

because a is an operator which conserves the translational invariant in x direction so that $a | \psi_{k'} \rangle$ is still an eigenstate of p_x with eigenvalue $\hbar k'$. And the inner product of two eigenstates of p_x will be zero if k and k' are different.

Ex. 16.4

$$\begin{aligned}
\vec{R} &= \vec{r} - \frac{l^2(\hat{z} \times \vec{\Pi})}{\hbar} \\
\vec{r} &= \vec{R} + \frac{l^2(\hat{z} \times \vec{\Pi})}{\hbar} = \vec{R} + \frac{l^2}{\hbar} (-\Pi_y, \Pi_x) \\
a &= \frac{l}{\sqrt{2}\hbar} (\Pi_x + i\Pi_y) \quad a^\dagger = \frac{l}{\sqrt{2}\hbar} (\Pi_x - i\Pi_y)
\end{aligned}$$

a and a^\dagger are the lowering and raising operators from one Landau level to another.

$$\Pi_x = \frac{\hbar}{\sqrt{2}l} (a + a^\dagger) \quad \Pi_y = \frac{-i\hbar}{\sqrt{2}l} (a - a^\dagger)$$

$$\begin{aligned}
P_n \vec{r} P_n &= P_n \left(\vec{R} + \frac{l^2}{\hbar} (-\Pi_y, \Pi_x) \right) P_n \\
&= P_n \vec{R} P_n + \frac{l^2}{\hbar} P_n (-\Pi_y, \Pi_x) P_n
\end{aligned}$$

Assume a wave function $|\psi\rangle$, which has a linear combination of various single-particle states in

different Landau levels

$$\langle \psi | P_n \frac{\hbar}{\sqrt{2}l} [(a + a^\dagger) \hat{x} - i(a - a^\dagger) \hat{y}] P_n | \psi \rangle = \frac{\hbar}{\sqrt{2}l} \langle n | (a + a^\dagger) \hat{x} - i(a - a^\dagger) \hat{y} | n \rangle = 0$$

where $|n\rangle = P_n |\psi\rangle$ project $|\psi\rangle$ onto the n th Landau level. Obviously, this term is zero. Therefore, $P_n \vec{r} P_n = P_n \vec{R} P_n = \vec{R}$ because \vec{R} is only related to b and b^\dagger which are the lowering and raising operators of particles' angular momentum within the same Landau level.

Ex. 16.5

$$R_x = x + \frac{l^2}{\hbar} \Pi_y \quad R_y = y - \frac{l^2}{\hbar} \Pi_x$$

$$\text{Thus, } x = R_x - \frac{l^2}{\hbar} \Pi_y \quad y = R_y + \frac{l^2}{\hbar} \Pi_x$$

$$\text{And also, } R_x = \frac{l}{\sqrt{2}} (b + b^\dagger) \quad , \quad R_y = \frac{il}{\sqrt{2}} (b - b^\dagger)$$

$$\Pi_x = \frac{\hbar}{\sqrt{2}l} (a + a^\dagger) \quad , \quad \Pi_y = \frac{i\hbar}{\sqrt{2}l} (a^\dagger - a)$$

$$x = \frac{l}{\sqrt{2}} (b + b^\dagger + i(a^\dagger - a))$$

$$y = \frac{l}{\sqrt{2}} (i(b - b^\dagger) - (a + a^\dagger))$$

Assume without loss of generality $\langle x \rangle = \langle y \rangle = 0$.

$$\langle x^2 \rangle = \frac{l^2}{2} (bb^\dagger + b^\dagger b + a^\dagger a + aa^\dagger)$$

$$\langle y^2 \rangle = \frac{l^2}{2} (bb^\dagger + b^\dagger b + a^\dagger a + aa^\dagger)$$

$$\Rightarrow \langle x^2 \rangle = \langle y^2 \rangle = l^2 (m + n + 1) \geq l^2 (n + 1) \text{ where } m \geq 0 \text{ is the angular momentum}$$

$$\therefore \Delta x \Delta y \geq l^2 (n + 1) \quad QED$$

with $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ and $A \in x$ or y .

Ex. 16.6

Useful relations:

$$\begin{aligned}
\vec{R} &= \vec{r} - \frac{l^2}{\hbar} \left(\hat{z} \times \vec{\Pi} \right), & \Pi_i &= p_i + \frac{e}{c} A_i, & z &= \frac{(x + iy)}{l}, & z^* &= \frac{(x - iy)}{l} \\
a &= \frac{l}{\sqrt{2}\hbar} (\Pi_x + i\Pi_y), & a^\dagger &= \frac{l}{\sqrt{2}\hbar} (\Pi_x - i\Pi_y), & \vec{A} &= \left(\frac{By}{2}, \frac{-Bx}{2} \right) \\
b &= \frac{1}{\sqrt{2}l} (R_x - iR_y), & b^\dagger &= \frac{1}{\sqrt{2}l} (R_x + iR_y)
\end{aligned}$$

- $$\begin{aligned} a &= \frac{l}{\sqrt{2}\hbar} (\Pi_x + i\Pi_y) \\ &= \frac{l}{\sqrt{2}\hbar} \left(p_x + ip_y + (-i) \frac{eB}{2c} (x + iy) \right) \\ &= \frac{-il}{\sqrt{2}} \left[\partial_x + i\partial_y + \frac{z}{2l} \right] \\ &= \frac{-il}{\sqrt{2}} \left[\frac{\partial z}{\partial x} \partial_z + \frac{\partial z^*}{\partial x} \partial_{z^*} + i \frac{\partial z}{\partial y} \partial_z + i \frac{\partial z^*}{\partial y} \partial_{z^*} + \frac{z}{2l} \right] \\ &= \frac{-il}{\sqrt{2}} \left[\frac{1}{l} \partial_z + \frac{1}{l} \partial_{z^*} - \frac{1}{l} \partial_z + \frac{1}{l} \partial_{z^*} + \frac{z}{2l} \right] \\ &= \frac{-i}{\sqrt{2}} \left[\frac{z}{2} + 2\partial_{z^*} \right] \# \end{aligned}$$
- $$a^\dagger = (a)^\dagger = \frac{i}{\sqrt{2}} \left[\frac{z^*}{2} - 2\partial_z \right] \#$$
- $$\begin{aligned} b &= \frac{1}{\sqrt{2}l} (R_x - iR_y) \\ &= \frac{1}{\sqrt{2}l} \left(x + \frac{l^2}{\hbar} \Pi_y - iy + i \frac{l^2}{\hbar} \Pi_x \right) \\ &= \frac{1}{\sqrt{2}l} \left[x - iy + \frac{l^2}{\hbar} \left(p_y - \frac{eB}{2c} x + ip_x + i \frac{eB}{2c} y \right) \right] \\ &= \frac{1}{\sqrt{2}} \left(z^* + \partial_z - \partial_{z^*} + \partial_z + \partial_{z^*} - \frac{z^*}{2} \right) \\ &= \frac{1}{\sqrt{2}} \left(\frac{z^*}{2} + 2\partial_z \right) \# \end{aligned}$$
- $$b^\dagger = (b)^\dagger = \frac{1}{\sqrt{2}} \left(\frac{z}{2} - 2\partial_{z^*} \right) \#$$

Ex. 16.7

According to Eq.(16.14)

$$\psi_m(x, y) = \psi_m(z, z^*) = \frac{1}{\sqrt{2\pi \cdot 2^m \cdot m! \cdot l^2}} z^m e^{\frac{-1}{4}|z|^2}$$

$$\psi_0(z, z^*) = \frac{1}{\sqrt{2\pi}l} e^{\frac{-1}{4}|z|^2}$$

$$\begin{aligned} a\psi_0(z, z^*) &= \frac{-i}{\sqrt{2}} \left(\frac{z}{2} + 2 \frac{\partial}{\partial z^*} \right) \frac{1}{\sqrt{2\pi}l} e^{\frac{-1}{4}|z|^2} \\ &= \frac{-i}{\sqrt{2}} \left(\frac{z}{2} + 2 \times \left(\frac{-1}{4} \right) z \right) \frac{1}{\sqrt{2\pi}l} e^{\frac{-1}{4}z \cdot z^*} \\ &= 0 \quad (\text{means } \psi_0(z, z^*) \text{ is in the LLL } \because a \text{ changes the label of LL, namely, n.}) \end{aligned}$$

$$\begin{aligned} b\psi_0(z, z^*) &= \frac{1}{\sqrt{2}} \left(\frac{z^*}{2} + 2 \frac{\partial}{\partial z} \right) \frac{1}{\sqrt{2\pi}l} e^{\frac{-1}{4}|z|^2} \\ &= \frac{1}{2\sqrt{\pi}l} \left(\frac{z^*}{2} + 2 \times \left(\frac{-1}{4} \right) z^* \right) e^{\frac{-1}{4}|z|^2} \\ &= 0 \quad (\text{means } \psi_0(z, z^*) \text{ has the angular momentum } m=0) \\ &\quad \because b \text{ changes the angular momentum in the same LL} \end{aligned}$$

Thus, the wave function in the LLL with angular momentum m could be generated from $\psi_0(z, z^*)$ by applying b^\dagger operator.

$$\begin{aligned} \frac{(b^\dagger)^m}{\sqrt{m!}} \psi_0(z, z^*) &= \frac{1}{\sqrt{2 \cdot m!}} \left(\frac{z}{2} - 2 \frac{\partial}{\partial z^*} \right)^m \frac{1}{\sqrt{2\pi}l} e^{\frac{-1}{4}z \cdot z^*} \\ &= \frac{1}{\sqrt{2\pi l^2 2^m \cdot m!}} z^m e^{\frac{-1}{4}z \cdot z^*} \\ &= \psi_m(z, z^*) \end{aligned}$$

So $\psi_m(z, z^*)$ is the solution of Schrodinger equation and lies in the LLL with angular momentum m.

Ex. 16.8

Define

$$\begin{aligned} X_{Mm} &\equiv \langle m M | V | m M \rangle \\ &= N \int d^2\vec{r}_1 d^2\vec{r}_2 (z_1 - z_2)^{*m} (z_1 + z_2)^{*M} V(|\vec{r}_1 - \vec{r}_2|) (z_1 + z_2)^M (z_1 - z_2)^m e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \\ &= N' \int d^2\vec{r} d^2\vec{R} R^{2M} r^{2m} V(r) e^{-R^2/\ell^2 - r^2/(4\ell^2)} = N'' \int d^2\vec{r} r^{2m} V(r) e^{-r^2/(4\ell^2)} \end{aligned}$$

where N , N' and N'' are normalization factors, $\vec{r} = \vec{r}_1 - \vec{r}_2$, and $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$. Obviously X_{Mm} is independent of M , which we identify as v_m .

Ex. 16.9

Haldane pseudopotential V_c for Coulomb potential $\frac{e^2}{\epsilon r}$

$$V_c = \frac{\langle m M | \frac{l}{r} | m M \rangle}{\langle m M | m M \rangle} \quad (\text{in } \frac{e^2}{\epsilon l} \text{ unit})$$

\therefore Coulomb potential is only related to the relative coordinate, namely, $\vec{r}_1 - \vec{r}_2$

$$\therefore V_c = \frac{\langle m M | \frac{l}{r} | m M \rangle}{\langle m M | m M \rangle} = \frac{\langle m | \frac{l}{r} | m \rangle}{\langle m | m \rangle}$$

The wave function for the lowest Landau level

$$\psi_{mM}(z_1, z_2) = (z_1 - z_2)^m (z_1 + z_2)^M \times e^{\frac{-1}{4}(|z_1|^2 + |z_2|^2)}$$

$$\text{let } r_m \equiv z_1 - z_2 = r_m e^{i\theta} = r e^{i\theta} \quad (\because |r_m| = |\vec{r}|)$$

$$\text{Since } \psi_{mM}(z_1, z_2) = \psi_m(z_1, z_2) \bar{\psi}_M(z_1, z_2) = \langle \vec{r}_m | m \rangle \langle \vec{R} | M \rangle$$

$$\therefore \langle \vec{r} | m \rangle = r^m e^{im\theta} e^{\frac{-r^2}{8l^2}}$$

(i) First, calculate the normalization of the wave function $\psi_m(z_1, z_2)$

$$\langle m | m \rangle = \int d^2 \vec{r} r^{2m} e^{\frac{-r^2}{4l^2}}$$

$$= 2\pi \int_0^\infty dr r^{2m+1} e^{\frac{-r^2}{4l^2}}$$

$$= 2\pi (-1)^m \left(\frac{\partial}{\partial \alpha} \right)^m \int_0^\infty dr r e^{-\alpha r^2} \Big|_{\alpha=\frac{1}{4l^2}}$$

$$= 2\pi(-1)^m \left(\frac{\partial}{\partial \alpha} \right)^m \frac{1}{2\alpha} \Big|_{\alpha=\frac{1}{4l^2}}$$

$$= \pi m! \cdot \left(\frac{1}{4l^2} \right)^{-(m+1)}$$

$$= \underline{\pi \times m! \times 2^{2m+2} \times l^{2m+2}}$$

(ii)

$$\begin{aligned}
\langle m | \frac{1}{r} | m \rangle &= \int d^2 \vec{r} r^{2m-1} e^{\frac{-r^2}{4l^2}} \\
&= 2\pi \int_0^\infty dr r^{2m} e^{\frac{-r^2}{4l^2}} \\
&= 2\pi (-1)^m \left(\frac{\partial}{\partial \alpha} \right)^m \int_0^\infty dr e^{-\alpha r^2} \Big|_{\alpha=\frac{1}{4l^2}} \\
&= 2\pi (-1)^m \left(\frac{\partial}{\partial \alpha} \right)^m \frac{1}{2} \sqrt{\frac{\pi}{\alpha}} \Big|_{\alpha=\frac{1}{4l^2}} \\
&= \pi \sqrt{\pi} (-1)^m \left(\frac{-1}{2} \right) \left(\frac{-3}{2} \right) \cdots \left(\frac{-(2m-1)}{2} \right) \times \left(\frac{1}{4l^2} \right)^{-\left(\frac{2m+1}{2}\right)} \\
&= \underline{2\pi \sqrt{\pi} (2m-1)!! \times 2^m \times l^{2m+1}}
\end{aligned}$$

$$\therefore v_m = \frac{\sqrt{\pi} (2m-1)!!}{m! \cdot 2^{m+1}} \text{ (in unit of } \frac{e^2}{\epsilon l})$$

(iii) convert the unit to Kelvin

$$\epsilon = 10 \times 4\pi \times \epsilon_0$$

$$\text{According to Eq. (12.38), } l = \frac{257}{\sqrt{\frac{B}{Tesla}}} \text{ Å} = \frac{257 \times 10^{-10}}{\sqrt{10}} \text{ (m)}$$

$$[\text{energy E}] = [k_B T] = \left[\frac{e^2}{4\pi\epsilon_0 r} \right]$$

$$\Rightarrow \frac{e^2}{\epsilon l} = \frac{(1.6 \times 10^{-19})^2 (C)}{\frac{257 \times 10^{-10}}{\sqrt{10}} (m) \times 10 \times (4\pi\epsilon_0) \times k_B} \approx 206(K) \text{ where } k_B \text{ is Boltzmann constant.}$$

$$\therefore v_m = \frac{\sqrt{\pi} (2m-1)!! m! \cdot 2^{m+1}}{m! \cdot 2^{m+1}} \times 206(K).$$

Ex. 16.10

Assume $\varphi_m(z_1 - z_2)$ and $\Phi(s_1 - s_2)$ are normalized, we have the Haldane pseudopotential

$$v_m = \int dx_1 dy_1 dx_2 dy_2 ds_1 ds_2 |\varphi_m(z_1 - z_2)|^2 |\Phi(s_1 - s_2)|^2 V \left(\frac{1}{\sqrt{|z_1 - z_2|^2 + (s_1 - s_2)^2}} \right). \quad (167)$$

Note that when $s_1 \neq s_2$, V will no longer diverge even if $z_1 = z_2$. Also if $|s_1 - s_2|$ increases, V and $|\Phi(s_1 - s_2)|^2$ will decrease. Therefore the finite thickness will soften the Haldane pseudopotential in general, and this softening decreases with increasing m .

Ex. 16.11

If we exchange two particles z_k and z_l with $k < l$, the parts that will change sign are $(z_k - z_l)$, $\prod_{k < m < l} (z_k - z_m)(z_m - z_l)$. The other parts such as $\prod_{n < k} (z_n - z_k)(z_n - z_l)$ and $\prod_{l < p} (z_k - z_p)(z_l - z_p)$ will not change. Thus the change in the sign upon exchanging two particles z_k and z_l will be $(-1)^{1+2\#}$, where $\#$ is the number of particles between k and l . Thus the Vandemonde polynomial is totally anti-symmetric.

Ex. 16.12

$$\begin{aligned}\phi(z_1, z_2) &= (z_1 - z_2)^3 e^{\frac{-1}{4}(|z_1|^2 + |z_2|^2)} \\ &= (z_1^3 - z_2^3 - 3z_1^2 z_2 + 3z_1 z_2^2) e^{\frac{-1}{4}(|z_1|^2 + |z_2|^2)} \\ &= \underbrace{\left(-\begin{vmatrix} 1 & 1 \\ z_1^3 & z_2^3 \end{vmatrix} + 3 \begin{vmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \end{vmatrix} \right) e^{\frac{-1}{4}(|z_1|^2 + |z_2|^2)}}_{\#}\end{aligned}$$

Ex. 16.13

In anticipation of Ex. 16.14, in the following we first use notation appropriate for 2D electrons in the lowest Landau level, and then switch to 1D electrons.

N body wave function $\psi(z_1, \dots, z_N)$ as a single Slater determinant can be expanded as

$$\psi(z_1, \dots, z_N) = \sum_{\sigma_1 \sigma_2 \dots \sigma_N} \epsilon_{\sigma_1 \sigma_2 \dots \sigma_N} \varphi_0(z_{\sigma_1}) \varphi_1(z_{\sigma_2}) \dots \varphi_{N-1}(z_{\sigma_N}), \quad (168)$$

where $\sigma_1 \sigma_2 \dots \sigma_N$ is a permutation of $12 \dots N$, $\epsilon_{\sigma_1 \sigma_2 \dots \sigma_N}$ is the Levi-Civita symbol and $\varphi_i(z)$ is the single particle wave function. Then we have

$$\begin{aligned}\int d^2 z_1 \dots d^2 z_N |\psi(z_1, \dots, z_N)|^2 &= \sum_{\sigma_1 \sigma_2 \dots \sigma_N} \sum_{\sigma'_1 \sigma'_2 \dots \sigma'_N} \epsilon_{\sigma_1 \sigma_2 \dots \sigma_N} \epsilon_{\sigma'_1 \sigma'_2 \dots \sigma'_N} \\ &\times \int d^2 z_1 \dots d^2 z_N \varphi_0(z_{\sigma_1}) \varphi_1(z_{\sigma_2}) \dots \varphi_{N-1}(z_{\sigma_N}) \varphi_0^*(z_{\sigma'_1}) \varphi_1^*(z_{\sigma'_2}) \dots \varphi_{N-1}^*(z_{\sigma'_N}).\end{aligned} \quad (169)$$

By using $\int d^2 z \varphi_n^*(z) \varphi_m(z) = \delta_{nm}$, we must have $\sigma'_i = \sigma_i$. Thus the product of the N body wave function is

$$\int d^2 z_1 \dots d^2 z_N |\psi(z_1, \dots, z_N)|^2 = \sum_{\sigma_1 \sigma_2 \dots \sigma_N} (\epsilon_{\sigma_1 \sigma_2 \dots \sigma_N})^2 = N!, \quad (170)$$

since $\epsilon_{\sigma_1 \sigma_2 \dots \sigma_N}$ is either 1 or -1 . Similarly we have

$$\int d^2 z_3 \dots d^2 z_N |\psi(0, z, z_3, \dots, z_N)|^2 = (N-2)! \sum_{i < j} |\varphi_i(0) \varphi_j(z) - \varphi_i(z) \varphi_j(0)|^2. \quad (171)$$

Consequently, the pair distribution function is

$$g(z) = \frac{1}{n^2} \sum_{i < j} |\varphi_i(0)\varphi_j(z) - \varphi_i(z)\varphi_j(0)|^2. \quad (172)$$

Now we apply it to 1D free Fermi gas constrained inside a strip of width L . The normalized wave function is $\psi(x) = e^{ikx}/\sqrt{L}$, where x is the coordinate and k is the wave vector. By employing the periodic boundary condition $\psi(x+L) = \psi(x)$, we have

$$k = \frac{2\pi m}{L}, \quad (173)$$

where m is a positive integer. Note that $dk = \frac{2\pi}{L} dm$. Thus we have the single particle wave function

$$\varphi_m(x) = \frac{1}{\sqrt{L}} e^{i2\pi mx/L}. \quad (174)$$

Substituting this into Eq. (172), we get

$$\begin{aligned} g(x) &= \frac{1}{n^2 L^2} \sum_{m < p} \left(2 - 2 \cos \frac{2\pi(p-m)x}{L} \right) \\ &= \frac{(N-1)N}{n^2 L^2} - \frac{2}{n^2 L^2} \sum_{m < p} \cos \frac{2\pi(p-m)x}{L}. \end{aligned} \quad (175)$$

Now we can replace the summation by an integration in momentum space if we choose the length of the strip to be infinite:

$$\sum_{m < p} \cos \frac{2\pi(p-m)x}{L} = \frac{L^2}{(2\pi)^2} \int_{-k_F}^{k_F} dk_m \int_{k_m}^{k_F} dk_p \cos(k_p - k_m)x, \quad (176)$$

where k_F is the Fermi wave vector. Note electron number density is $n = N/L = 2k_F/(2\pi)$, we finally obtain

$$g(x) = 1 - \frac{2(1 - \cos k_F x)}{k_F^2 x^2}. \quad (177)$$

Check that $g(0) = 0$ and $g(+\infty) = 1$ as expected. Also note its oscillatory nature as shown in Fig. 7.

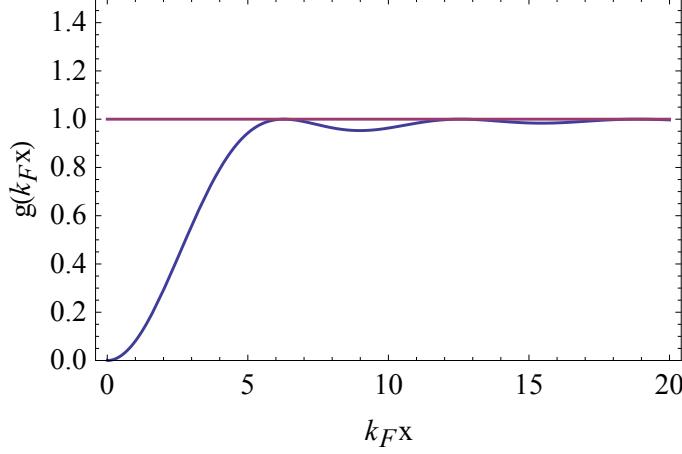


Figure 7: The pair distribution function $g(x)$ for 1D Fermi gas.

Ex. 16.14

Now we apply Eq. 172 to the case of completely filled lowest Landau level. Since $\varphi_i(0) = 0$ if $i \geq 1$, we have

$$\begin{aligned}
 g(z) &= \frac{1}{n^2} \sum_{j=1}^{N-1} |\varphi_0(0)|^2 |\varphi_j(z)|^2 \\
 &= e^{-\frac{|z|^2}{2}} \sum_{j=1}^{N-1} \frac{|z|^{2j}}{2^j j!} \\
 &= e^{-\frac{|z|^2}{2}} \left(\sum_{j=0}^{+\infty} \frac{|z|^{2j}}{2^j j!} - 1 \right) \\
 &= 1 - e^{-\frac{|z|^2}{2}},
 \end{aligned} \tag{178}$$

in which we have taken the thermodynamic limit $N \rightarrow +\infty$.

Note that in contrast to the oscillatory nature of pair distribution function for Fermi gas, here the pair distribution function monotonically increases from 0 to 1 as $|z|$ increases from 0 to $+\infty$.

Ex. 16.15

We have the Schrödinger equation for the eigenstate of Hamiltonian in Eq. (16.93),

$$H'\psi' = E\psi'. \tag{179}$$

Now we make a gauge transformation

$$\psi' = U\psi, \tag{180}$$

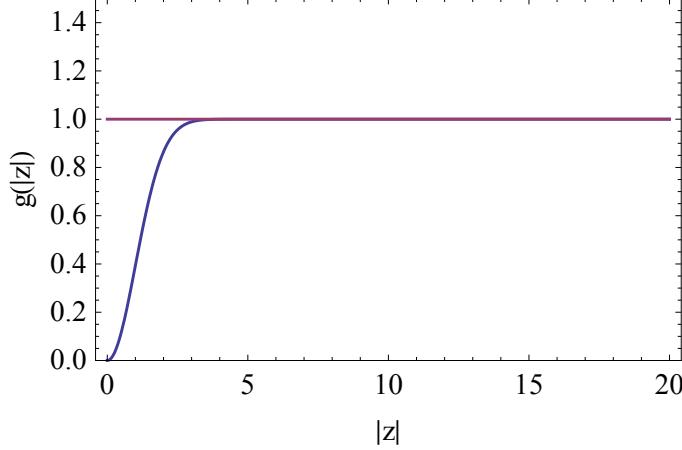


Figure 8: The pair distribution function $g(|z|)$ for filled lowest Landau level.

where $U = e^{-i\frac{\theta\varphi}{\pi}}$. We act U^{-1} on Eq. (179) to get

$$U^{-1}H'UU^{-1}\psi' = EU^{-1}\psi'. \quad (181)$$

By using Baker-Hausdorff formula, we have $U^{-1}H'U = H$, where H is the Hamiltonian in Eq. (16.91). Thus we arrive at the Schrödinger equation for the original Hamiltonian,

$$H\psi = E\psi. \quad (182)$$

\therefore the vector potential term $\frac{\theta}{\pi}$ is eliminated by the gauge transformation.

The boundary condition in Eq. (16.90) is
 $\psi(0) = \psi(\pi)$.

$$\psi'(\phi) = U\psi(\phi)$$

$$\psi'(0) = \psi(0)$$

$$\psi'(\pi) = e^{-i\frac{\theta}{\pi} \times \pi} \psi(\pi) = e^{-i\theta} \psi(0)$$

$$\therefore \psi'(\pi) = e^{-i\theta} \psi'(0)$$

the boundary condition changes and the wave function picks up a phase $-i\theta$ when the two particles exchange. This is the same as the AB phase picked up by the system when the two particles exchanges.

Ex. 16.16

Eq. (16.105):

$$\psi_{Z_1, Z_2}(z) = [(Z_1 - Z_2)^* (Z_1 - Z_2)]^{\frac{1}{2m}} e^{-\frac{(|z_1|^2 + |z_2|^2)}{4m}} \prod_{j=1}^N (z_j - Z_1) (z_j - Z_2) \psi_m(z)$$

To prove the wave function of Eq. (16.105) has a constant norm, we take the advantage of the plasma analogy. Note that $\langle \psi(z) | \psi(z) \rangle$ can be regarded as the partition function of a plasma system. The partition function is equal to $e^{-\beta U(Z_1, Z_2)}$ where $U(Z_1, Z_2)$ is the free energy of the system with two impurities at Z_1 and Z_2 . We choose $\beta = \frac{2}{m}$.

In the plasma analogy, we have $|\psi|^2 = e^{-\beta U'}$ where

$$U' = -m^2 \sum_{i < j=1}^N \ln |z_i - z_j| + \frac{m}{4} \sum_i |z_i|^2 - m \sum_{j=1}^N (\ln |z_j - Z_1| + \ln |z_j - Z_2|) + \frac{1}{4} (|Z_1|^2 + |Z_2|^2) - \ln |Z_1 - Z_2|,$$

The first two terms represent the interaction between mobile particles with charge m and that of each particle with the “jelly-like” uniform background. The third term describes the interactions between particles and the two impurities, each with charge 1. Finally, the last two terms represent the interactions between the impurities and the background, and that between the two impurities.

Complete screening (which is realized as long as the impurities are far away from the edge of the plasma) implies the free energy $U(Z_1, Z_2)$ is independent of the distance between Z_1 and Z_2 beyond the screening length, which is of order magnetic length.

Ex. 16.17

(a)

$$\psi_{Z_1, Z_2}(z) = [(Z_1 - Z_2)^* (Z_1 - Z_2)]^{\frac{1}{2m}} \cdot e^{-\frac{(|Z_1|^2 + |Z_2|^2)}{4m}} \times \prod_{j=1}^N (z_j - Z_1) (z_j - Z_2) \psi_m(z)$$

Set $Z_2 = 0, Z_1 = Z$:

$$\psi_{Z_1, Z_2}(z) = \psi_{Z, 0}(z) = (Z^* Z)^{\frac{1}{2m}} \cdot e^{-\frac{(Z^* Z)}{4m}} \prod_{j=1}^N (z_j - Z) z_j \psi_m(z)$$

$$\bar{A}_{Z, 0} = A_Z + a_Z \quad \text{and} \quad \bar{A}_{Z^*, 0} = A_{Z^*} + a_{Z^*}$$

where A_Z and A_{Z^*} are defined in Eq. (16.99).

$$\begin{aligned}
\bar{A}_{Z^*,0} &= i \langle \psi_{Z,0}(z) | \partial_{Z^*} \psi_{Z,0}(z) \rangle \\
&= i \langle \psi_{Z,0}(z) | \frac{1}{2m} \frac{1}{Z^*} - \frac{Z}{4m} | \psi_{Z,0}(z) \rangle \\
&= \frac{i}{2m} \frac{1}{Z^*} - \frac{iZ}{4m} \\
&= \frac{i}{2m} \frac{Z}{R^2} - \frac{iZ}{4m} \\
\therefore a_{Z^*} &= \frac{iZ}{2mR^2}
\end{aligned}$$

$$\begin{aligned}
\bar{A}_{Z,0} &= i \langle \psi_{Z,0}(z) | \partial_Z \psi_{Z,0}(z) \rangle \\
&= -i (\partial_Z \langle \psi_{Z,0}(z) |) | \psi_{Z,0}(z) \rangle \\
&= \frac{-i}{2m} \frac{1}{Z} + \frac{iZ^*}{4m} \\
\therefore a_Z &= \frac{-iZ^*}{2mR^2}
\end{aligned}$$

$$\begin{aligned}
a_X = a_Z + a_{Z^*} &= \frac{-Y}{mR^2} \\
a_Y = i(a_Z - a_{Z^*}) &= \frac{X}{mR^2} \\
\therefore \vec{a}(\vec{r}) &= \frac{-1}{mR^2} \vec{R} \times \hat{Z} = \frac{1}{mR} \hat{\varphi}.
\end{aligned}$$

(b)

$$\psi_{Z_1, Z_2}(z) = [(Z_1 - Z_2)^* (Z_1 - Z_2)]^{\frac{1}{2m}} e^{-\frac{(|Z_1|^2 + |Z_2|^2)}{4m}} \times \prod_{j=1}^N (z_j - Z_1)(z_j - Z_2) \psi_m(z)$$

let $Z \equiv Z_1 - Z_2$ and $K \equiv Z_1 + Z_2$

$$\psi_{Z_1, Z_2}(z) = \psi_{Z, K}(z) = (Z^* Z)^{\frac{1}{2m}} e^{-\frac{(|Z|^2 + |K|^2)}{8m}} \times \prod_{j=1}^N \left(z_j - \frac{(Z+K)}{2} \right) \left(z_j - \frac{(K-Z)}{2} \right) \psi_m(z)$$

$$\bar{A}_{Z^*} = i \langle \psi_{Z,K}(z) | \partial_{Z^*} \psi_{Z,K}(z) \rangle = \frac{i}{2m} \frac{1}{Z^*} - \frac{iZ}{4m}$$

$$\bar{A}_Z = i \langle \psi_{Z,K}(z) | \partial_Z \psi_{Z,K}(z) \rangle = \frac{-i}{2m} \frac{1}{Z} + \frac{iZ^*}{4m}$$

$$\therefore a_{Z^*} = \frac{i}{2mZ^*} = \frac{i}{2mR} \hat{\varphi}$$

$$a_Z = \frac{-i}{2mZ} = \frac{i}{2mR} \hat{\varphi}$$

$$\theta = \frac{1}{2} \oint (dZ a_Z + dZ^* a_{Z^*}) = \frac{1}{2} \frac{1}{2mR} \cdot R \times 2 \int_0^{2\pi} d\varphi = \frac{\pm\pi}{m}$$

Ex. 16.18 & Ex. 16.19

The vector potential of the flux quantum Φ_0 at Z is

$$\mathbf{a} = \frac{\Phi_0}{2\pi} \nabla \text{Arg}(z - Z). \quad (183)$$

Check: $\int_Z \mathbf{a} \cdot d\mathbf{l} = \Phi_0$. Now we have the Schrödinger equation for the eigenstate $\tilde{\psi}_Z[z]$ of the Hamiltonian $H = \sum_i (\mathbf{P}_i + e\mathbf{A}_i + e\mathbf{a}_i)^2/2m + \sum_{i < j} V(\mathbf{r}_i - \mathbf{r}_j)$:

$$H\tilde{\psi}_Z[z] = E\tilde{\psi}_Z[z]. \quad (184)$$

We know from the textbook of quantum mechanics that

$$U^\dagger (\mathbf{P}_i + e\mathbf{A}_i + e\nabla_i \Lambda) U = \mathbf{P}_i + e\mathbf{A}_i, \quad (185)$$

where $U = e^{-i\frac{e\Lambda}{\hbar}}$. So we let

$$\Lambda = \sum_j \frac{\Phi_0}{2\pi} \text{Arg}(z_j - Z), \quad (186)$$

to get

$$U^\dagger H U = H_0, \quad (187)$$

where $H_0 = \sum_i (\mathbf{P}_i + e\mathbf{A}_i)^2/2m + V(\mathbf{r}_i - \mathbf{r}_j)$ is the Hamiltonian with only the original magnetic field, absent the flux quantum at Z . Now we act U^\dagger on Eq. (184) to obtain

$$H_0 U^\dagger \tilde{\psi}_Z[z] = E U^\dagger \tilde{\psi}_Z[z]. \quad (188)$$

Thus $U^\dagger \tilde{\psi}_Z[z]$ is just the eigenstate of H_0 ,

$$U^\dagger \tilde{\psi}_Z[z] = \psi_Z[z]. \quad (189)$$

Therefore, we get

$$\tilde{\psi}_Z[z] = U \psi_Z[z] = \prod_j \frac{|z_j - Z|}{z_j - Z} \psi_Z[z] = e^{-\frac{|Z|^2}{4m}} \prod_j |z_j - Z| \Psi_m[z]. \quad (190)$$

Ex. 16.20

The Hamiltonian for the fractional quantum Hall system is

$$H = \sum_i \frac{\mathbf{\Pi}_i^2}{2m_e} + \sum_{i < j} V(\mathbf{r}_i - \mathbf{r}_j). \quad (191)$$

Since the mechanical momentum $\mathbf{\Pi}$ commutes with guiding center coordinate \mathbf{R} , we have

$$\left[T_\alpha, \sum_i \frac{\mathbf{\Pi}_i^2}{2m_e} \right] = 0. \quad (192)$$

Now we prove T_α and the interaction potential in the Hamiltonian commute with each other. To do that we start with its first term, $V_{12} = V(\mathbf{r}_1 - \mathbf{r}_2)$. To prove $[T, V_{12}] = 0$ we only need to prove $[V_{12}, e^{i\vec{q}\cdot(\vec{R}_1 + \vec{R}_2)}] = 0$, which follows simply from the fact $[\mathbf{r}_1 - \mathbf{r}_2, \vec{q}\cdot(\vec{R}_1 + \vec{R}_2)] = 0$. We have thus proven Eq. (16.174).

Now we verify Eq. (16.175). We have

$$T_1 T_2 = \prod_{j=1}^N e^{i\mathbf{Q}_1 \cdot \mathbf{R}_j / N_\Phi} e^{i\mathbf{Q}_2 \cdot \mathbf{R}_j / N_\Phi}, \quad (193)$$

since $[\mathbf{R}_i, \mathbf{R}_j] = 0$ for $i \neq j$.

Now we use the fact $e^A e^B = e^B e^A e^{[A, B]}$ when $[A, B]$ commutes with A and B to get

$$e^{i\mathbf{Q}_1 \cdot \mathbf{R}_j / N_\Phi} e^{i\mathbf{Q}_2 \cdot \mathbf{R}_j / N_\Phi} = e^{i\mathbf{Q}_2 \cdot \mathbf{R}_j / N_\Phi} e^{i\mathbf{Q}_1 \cdot \mathbf{R}_j / N_\Phi} e^{\frac{i\pi^2}{N_\Phi^2} \mathbf{Q}_1 \wedge \mathbf{Q}_2}, \quad (194)$$

where the phase

$$e^{\frac{i\pi^2}{N_\Phi^2} \mathbf{Q}_1 \wedge \mathbf{Q}_2} = e^{\frac{i}{l^2 N_\Phi^2} \mathbf{L}_1 \wedge \mathbf{L}_2} = e^{\frac{i}{l^2 N_\Phi^2} \mathbf{L}_1 \wedge \mathbf{L}_2} = e^{i \frac{2\pi}{N_\Phi}}. \quad (195)$$

Thus we have

$$T_1 T_2 = \prod_{j=1}^N e^{i\mathbf{Q}_2 \cdot \mathbf{R}_j / N_\Phi} e^{i\mathbf{Q}_1 \cdot \mathbf{R}_j / N_\Phi} e^{i \frac{2\pi}{N_\Phi}} = T_2 T_1 e^{i \frac{2\pi N}{N_\Phi}} = T_2 T_1 e^{i 2\pi \nu}, \quad (196)$$

where the filling factor $\nu = N/N_\Phi$.

Ex. 16.21

From the last exercise, we find that T_1 and T_2 all commute with H . However, T_1 does not commute with T_2 :

$$T_1 T_2 = T_2 T_1 e^{i 2\pi \frac{q}{p}}, \quad (197)$$

where q/p is the filling factor.

Consider a simultaneous eigenstate of H and T_1 , $|\psi_0\rangle$, whose T_1 eigenvalue is η . Since T_2 commutes with H we know $T_2|\psi_0\rangle$ is also an eigenstate of H with the same energy, but from the equation above we find its T_1 eigenvalue is $e^{i 2\pi \frac{q}{p}} \eta \neq \eta$, thus orthogonal to $|\psi_0\rangle$. It is now clear by applying T_2 repeatedly we can generate a set of p degenerate energy eigenstates from $|\psi_0\rangle$.

If there is another simultaneous eigenstate of H and T_1 , $|\psi'_0\rangle$, that has the same energy but cannot be written as a linear combination of these states, we can project it out of the subspace spanned by these states, and generate another set of p energy eigenstates with the same energy that are orthogonal to each other as well as the previous set of states. As a result all energy eigenstates must have a degeneracy that is an integer multiple of p .

Chapter 17

Ex. 17.1

The free energy is

$$F = -k_B T \ln Z, \quad (198)$$

where Z is the partition function,

$$Z = \sum_{m=-l}^l e^{-\beta \hbar \omega_c m / 2} = \frac{e^{l \beta \hbar \omega_c / 2} (1 - e^{-\beta \hbar \omega_c (2l+1) / 2})}{1 - e^{-\beta \hbar \omega_c / 2}}, \quad (199)$$

where $\beta = 1/(k_B T)$. The magnetic moment is $-\frac{\partial F}{\partial B}$ so the susceptibility is

$$\begin{aligned} \chi &= \frac{\partial}{\partial B} \left(-\frac{\partial F}{\partial B} \right) \\ &= \frac{1}{\beta} \left(\frac{e}{m_e} \right)^2 \frac{\partial}{\partial \omega_c} \left(\frac{1}{Z} \frac{\partial Z}{\partial \omega_c} \right). \end{aligned} \quad (200)$$

In the limit $k_B T \gg \hbar \omega_c$, we can Taylor expand $\frac{\partial}{\partial \omega_c} \left(\frac{1}{Z} \frac{\partial Z}{\partial \omega_c} \right)$ in $\beta \hbar \omega_c$ to get

$$\chi = \left(\frac{e}{m_e} \right)^2 \frac{\hbar^2 l(l+1)}{12} \beta + \beta \mathcal{O}(\beta \hbar \omega_c)^2. \quad (201)$$

Thus the susceptibility at finite temperature is proportional to $\frac{1}{T}$.

Ex. 17.2

See the solution to Ex. 17.10.

$$\begin{aligned} H &= -|J| \sum_{j=1}^3 \vec{S}_j \cdot \vec{S}_{j+1} \\ &= -|J| (\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1) \\ &= -|J| \sum_{j=1}^3 \frac{1}{2} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ + S_j^z S_{j+1}^z) \end{aligned}$$

First eigenket and its corresponding eigenenergy is the following:

$$\begin{aligned} |\psi_1\rangle &= |\uparrow\uparrow\uparrow\rangle \\ H|\psi_1\rangle &= -\frac{3}{4}|J| \end{aligned}$$

To find other eigenkets we use spin wave vector operator:

$$S_q^- = \frac{1}{\sqrt{3}} \sum_{j=1}^3 e^{-iqR_j} S_j^-$$

In the case of periodic boundary condition (PBC) we have $S_{N+1} = S_1$, which yields to $e^{-iqNa} = 1$ (a is lattice constant) and we have the following:

$$\begin{aligned} qNa &= 2\pi N \implies q = \frac{2\pi}{Na}n \quad (n \text{ is integer}) \\ q &= -\frac{2\pi}{3a}, 0, +\frac{2\pi}{3a} \end{aligned}$$

Thus we have the following as the eigenkets of the Hamiltonian:

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{\sqrt{3}} \sum_{j=1}^3 S_j^- |\uparrow\uparrow\uparrow\rangle \\ &= \frac{1}{\sqrt{3}} (|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\ |\psi_3\rangle &= \frac{1}{\sqrt{3}} \sum_{j=1}^3 e^{i(2\pi/3)j} S_j^- |\uparrow\uparrow\uparrow\rangle \\ &= \frac{1}{\sqrt{3}} (e^{i(2\pi/3)} |\downarrow\uparrow\uparrow\rangle + e^{i(4\pi/3)} |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \\ |\psi_4\rangle &= \frac{1}{\sqrt{3}} \sum_{j=1}^3 e^{-i(2\pi/3)j} S_j^- |\uparrow\uparrow\uparrow\rangle \\ &= \frac{1}{\sqrt{3}} (e^{-i(2\pi/3)} |\downarrow\uparrow\uparrow\rangle + e^{-i(4\pi/3)} |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle) \end{aligned}$$

For the other four eigenkets we reverse all spins and complex conjugate the amplitudes as following:

$$\begin{aligned}
|\psi_5\rangle &= |\downarrow\downarrow\downarrow\rangle \\
|\psi_6\rangle &= \frac{1}{\sqrt{3}}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\
|\psi_7\rangle &= \frac{1}{\sqrt{3}}(e^{-i(2\pi/3)}|\uparrow\downarrow\downarrow\rangle + e^{-i(4\pi/3)}|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \\
|\psi_8\rangle &= \frac{1}{\sqrt{3}}(e^{i(2\pi/3)}|\uparrow\downarrow\downarrow\rangle + e^{i(4\pi/3)}|\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle)
\end{aligned}$$

Ex. 17.4

$$M = \frac{N}{2} - \frac{V}{(2\pi)^3} 4\pi \int_0^\infty \frac{q^2}{e^{\beta A q^2} - 1} dq \quad (202)$$

We change the variable in the integrand as:

$$\beta A q^2 = x \implies dq = \frac{dx}{2\sqrt{x}\sqrt{\beta A}}$$

Then we have:

$$M = \frac{N}{2} - \frac{V}{(2\pi)^3} 4\pi \frac{1}{(\beta A)^{3/2}} \frac{1}{2} \overbrace{\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx}^{=\frac{\sqrt{\pi}}{2}\zeta(3/2)}$$

By writing $V = Na^3$ (assuming a simple cubic lattice), we then have:

$$\begin{aligned}
M &= \frac{N}{2} \left[1 - \frac{a^3}{4\pi^{3/2}} \zeta(3/2) \frac{1}{(\beta A)^{3/2}} \right] \\
&= \frac{N}{2} \left[1 - \frac{a^3}{4\pi^{3/2}} \zeta(3/2) \left(\frac{k_B}{A} \right)^{3/2} T^{3/2} \right]
\end{aligned}$$

Thus we can write T_0 as

$$T_0 = \frac{\pi}{a^2} \left(\frac{4}{\zeta(3/2)} \right)^{2/3} \frac{A}{k_B} \quad (203)$$

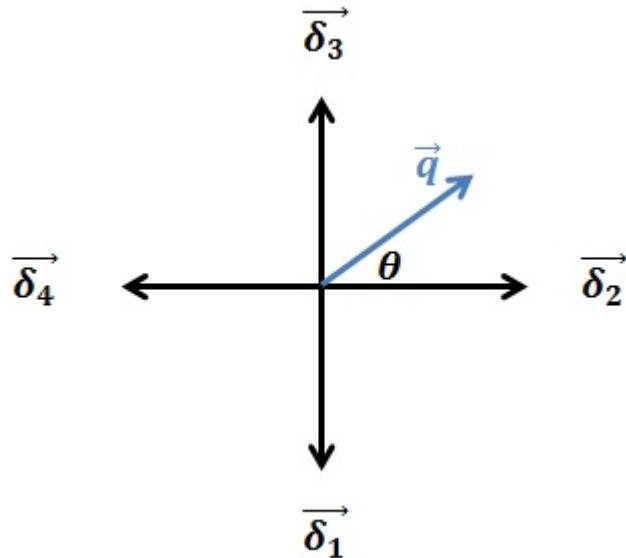
Now we find A in terms of coordination number and J for a simple cubic lattice:

$$\begin{aligned}
\varepsilon_{\vec{q}} &= \frac{z}{2} |J| (1 - \gamma_{\vec{q}}) \\
\gamma_{\vec{q}} &= 1 - \frac{1}{z} q^2 a^2 \\
\varepsilon_{\vec{q}} &= \frac{1}{2} |J| q^2 a^2 = A q^2 \implies A = \frac{1}{2} |J| a^2 \\
\implies T_0 &= \frac{|J| \pi}{2} \left(\frac{4}{\zeta(3/2)} \right)^{2/3} \frac{1}{k_B}
\end{aligned}$$

Ex. 17.5

$$\begin{aligned}
 E_{\vec{q}} &= \frac{\hbar^2 z J}{2} \sqrt{1 - \gamma_{\vec{q}}^2} \\
 \gamma_{\vec{q}} &= \frac{1}{z} \sum_{\vec{\delta}} e^{-i\vec{q}\cdot\vec{\delta}} = \frac{1}{z} \sum_{\vec{\delta}} (1 - i\vec{q}\cdot\vec{\delta} + \frac{1}{2}(-i\vec{q}\cdot\vec{\delta})^2) \\
 &= \frac{1}{z} \left(z - \frac{1}{2} \sum_{\vec{\delta}} (\vec{q}\cdot\vec{\delta})^2 \right)
 \end{aligned}$$

2D square lattice:



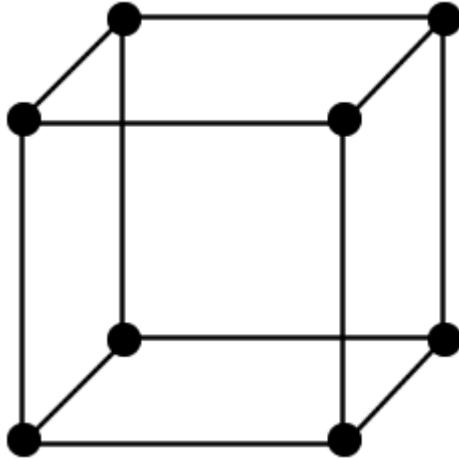
we consider an arbitrary vector \vec{q} in two dimensional space and θ is the angel between \vec{q} and horizontal axis. Also we set the origin to be at a lattice point.

$$\begin{aligned}
 \sum_{\vec{\delta}} (\vec{q}\cdot\vec{\delta})^2 &= (\vec{q}\cdot\vec{\delta}_1)^2 + (\vec{q}\cdot\vec{\delta}_2)^2 + (\vec{q}\cdot\vec{\delta}_3)^2 + (\vec{q}\cdot\vec{\delta}_4)^2 \\
 &= q^2 a^2 (\cos^2(\theta) + \cos^2(\pi/2 - \theta) + \cos^2(\pi - \theta) + \cos^2(\pi/2 + \theta)) \\
 &= 2q^2 a^2
 \end{aligned}$$

$$\gamma_{\vec{q}} = \frac{1}{z} \left(z - \frac{1}{2} 2q^2 a^2 \right) = 1 - \frac{1}{4} q^2 a^2$$

$$\begin{aligned}
E_{\vec{q}} &= \frac{\hbar^2 z J}{2} \sqrt{1 - (1 - \frac{1}{4} q^2 a^2)^2} \\
&\approx \frac{\hbar^2 z J}{2} \sqrt{1 - 1 + \frac{1}{2} q^2 a^2 + \mathcal{O}(q^4)} \\
&\approx \frac{\hbar^2 z J}{2} \frac{qa}{\sqrt{2}} \\
\implies \alpha &= \frac{1}{\sqrt{2}}
\end{aligned}$$

3D simple cubic lattice:



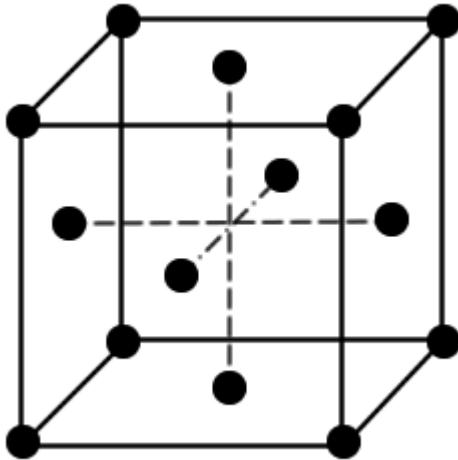
We put the origin at one of the cubic lattice points and each of the angles $\theta_i, i = x, y, z$ are the angle between \vec{q} and x, y, z axis respectively.

$$\begin{aligned}
\sum_{\vec{\delta}} (\vec{q} \cdot \vec{\delta})^2 &= (\vec{q} \cdot \vec{\delta}_1)^2 + (\vec{q} \cdot \vec{\delta}_2)^2 + (\vec{q} \cdot \vec{\delta}_3)^2 + (\vec{q} \cdot \vec{\delta}_4)^2 + (\vec{q} \cdot \vec{\delta}_7)^2 + (\vec{q} \cdot \vec{\delta}_8)^2 \\
&= q^2 a^2 (\cos^2(\theta_1) + \cos^2(\pi - \theta_1) + \cos^2(\theta_2) + \cos^2(\pi - \theta_2) + \cos^2(\theta_3) + \cos^2(\pi - \theta_3)) \\
&= 2q^2 a^2
\end{aligned}$$

$$\gamma_{\vec{q}} = \frac{1}{z} (z - \frac{1}{2} 2q^2 a^2) = 1 - \frac{1}{6} q^2 a^2$$

$$\begin{aligned}
E_{\vec{q}} &= \frac{\hbar^2 z J}{2} \sqrt{1 - (1 - \frac{1}{6} q^2 a^2)^2} \\
&\approx \frac{\hbar^2 z J}{2} \sqrt{1 - 1 + \frac{1}{3} q^2 a^2 + \mathcal{O}(q^4)} \\
&\approx \frac{\hbar^2 z J}{2} \frac{qa}{\sqrt{3}} \\
\implies \alpha &= \frac{1}{\sqrt{3}}
\end{aligned}$$

3D FCC cubic lattice:



We put the origin at one of the FCC lattice points ($z = 12, \vec{q} = (q_x, q_y, q_z)$)

$$\begin{aligned}
\delta_1 &= \frac{a}{2}(1, -1, 0), & \delta_5 &= \frac{a}{2}(0, -1, 1), & \delta_9 &= \frac{a}{2}(0, -1, -1) \\
\delta_2 &= \frac{a}{2}(1, 1, 0), & \delta_6 &= \frac{a}{2}(1, 0, 1), & \delta_{10} &= \frac{a}{2}(1, 0, -1), \\
\delta_3 &= \frac{a}{2}(-1, 1, 0), & \delta_7 &= \frac{a}{2}(0, 1, 1), & \delta_{11} &= \frac{a}{2}(0, 1, -1), \\
\delta_4 &= \frac{a}{2}(-1, -1, 0), & \delta_8 &= \frac{a}{2}(-1, 0, 1), & \delta_{12} &= \frac{a}{2}(-1, 0, -1),
\end{aligned}$$

$$\begin{aligned}
\sum_{\vec{\delta}} (\vec{q} \cdot \vec{\delta})^2 &= \frac{a^2}{4} (8q_x^2 + 8q_y^2 + 8q_z^2) \\
&= \frac{a^2}{4} 8q^2 = 2q^2 a^2
\end{aligned}$$

$$\gamma_{\vec{q}} = \frac{1}{z} (z - \frac{1}{2} 2q^2 a^2) = 1 - \frac{1}{12} q^2 a^2$$

$$\begin{aligned}
E_{\vec{q}} &= \frac{\hbar^2 z J}{2} \sqrt{1 - (1 - \frac{1}{12} q^2 a^2)^2} \\
&\approx \frac{\hbar^2 z J}{2} \sqrt{1 - 1 + \frac{1}{6} q^2 a^2 + \mathcal{O}(q^4)} \\
&\approx \frac{\hbar^2 z J}{2} \frac{qa}{\sqrt{6}} \\
\implies \alpha &= \frac{1}{\sqrt{6}}
\end{aligned}$$

Ex. 17.6

$$\gamma_{\vec{q}} \equiv \frac{1}{z} \sum_{\vec{\delta}} e^{-i\vec{q}\cdot\vec{\delta}} \quad (204)$$

$$\Delta M_s = \frac{L^d}{2} \int_{1BZ} \frac{d^d \vec{q}}{(2\pi)^d} \left(\frac{1}{\sqrt{1 - (\gamma_{\vec{q}})^2}} - 1 \right)$$

For 2D, $z = 4$ and $\vec{\delta} = (a, 0), (0, a), (-a, 0),$ and $(0, -a).$ According to Eq. 204, $\gamma_{\vec{q}} = \frac{1}{2} (\cos q_x a + \cos q_y a).$

Therefore,

$$\begin{aligned}
\Delta M_s &= \frac{L^2}{2(2\pi)^2} \int_{-\pi/a}^{\pi/a} dq_x \int_{-\pi/a}^{\pi/a} dq_y \left(\frac{1}{\sqrt{1 - (\cos q_x a + \cos q_y a)^2 / 4}} - 1 \right) \\
&= \frac{L^2}{8\pi^2} \int_{-\pi/a}^{\pi/a} dq_x \int_{-\pi/a}^{\pi/a} dq_y \frac{1}{\sqrt{1 - (\cos q_x a + \cos q_y a)^2 / 4}} - \frac{L^2}{2a^2}
\end{aligned}$$

(through Mathematica)

$$\begin{aligned}
&= \frac{L^2}{a^2} \left[\frac{55.00157}{8\pi^2} - \frac{1}{2} \right] \\
&= 0.196602 \times \frac{L^2}{a^2}
\end{aligned}$$

For 3D, $z = 6$ and $\vec{\delta} = (a, 0, 0), (0, a, 0), (-a, 0, 0), (0, -a, 0), (0, 0, a)$ and $(0, 0, -a).$ According to Eq. 204, $\gamma_{\vec{q}} = \frac{1}{3} (\cos q_x a + \cos q_y a + \cos q_z a).$

Therefore,

$$\begin{aligned}
\Delta M_s &= \frac{L^3}{2(2\pi)^3} \int_{-\pi/a}^{\pi/a} dq_x \int_{-\pi/a}^{\pi/a} dq_y \int_{-\pi/a}^{\pi/a} dq_z \left(\frac{1}{\sqrt{1 - (\cos q_x a + \cos q_y a + \cos q_z a)^2 / 9}} - 1 \right) \\
&= \frac{L^3}{16\pi^3} \int_{-\pi/a}^{\pi/a} dq_x \int_{-\pi/a}^{\pi/a} dq_y \int_{-\pi/a}^{\pi/a} dq_z \frac{1}{\sqrt{1 - (\cos q_x a + \cos q_y a + \cos q_z a)^2 / 9}} - \frac{L^3}{2a^3} \\
&\quad (\text{through Mathematica }) \\
&= \frac{L^3}{a^3} \left[\frac{286.924}{16\pi^3} - \frac{1}{2} \right] \\
&= 0.0783576 \times \frac{L^3}{a^3}
\end{aligned}$$

Ex. 17.7

$$\begin{aligned}
\Delta M_s &= \sum_{\vec{q}} \left[\cosh^2(\theta) \langle \beta_{\vec{q}}^\dagger \beta_{\vec{q}} \rangle + \sinh^2(\theta) \langle \beta_{\vec{q}} \beta_{\vec{q}}^\dagger \rangle \right] \\
&= \sum_{\vec{q}} \left[(\cosh^2(\theta) + \sinh^2(\theta)) \langle \beta_{\vec{q}}^\dagger \beta_{\vec{q}} \rangle + \sinh^2(\theta) \right] \\
&= \sum_{\vec{q}} \left[\frac{1}{\sqrt{1 - \tanh^2(2\theta)}} \langle \beta_{\vec{q}}^\dagger \beta_{\vec{q}} \rangle + \sinh^2(\theta) \right] \\
&= \sum_{\vec{q}} \left[\frac{1}{\sqrt{1 - \gamma_{\vec{q}}^2}} \langle \beta_{\vec{q}}^\dagger \beta_{\vec{q}} \rangle + \sinh^2(\theta) \right]
\end{aligned}$$

And we also know:

$$\begin{aligned}
\langle \beta_{\vec{q}}^\dagger \beta_{\vec{q}} \rangle &= \frac{1}{e^{\beta E_{\vec{q}}} - 1} \\
E_{\vec{q}} &= \frac{\hbar^2 z J}{2} \sqrt{1 - \gamma_{\vec{q}}^2}
\end{aligned}$$

The contribution from the 1st term in square brackets is

$$\begin{aligned}
\Delta M_{s1} &= \sum_{\vec{q}} \frac{1}{\sqrt{1 - \gamma_{\vec{q}}^2}} \frac{1}{e^{\beta \frac{\hbar^2 z J}{2}} \sqrt{1 - \gamma_{\vec{q}}^2} - 1} \\
&= L^2 \int_{1BZ} \frac{d^2 q}{(2\pi)^2} \frac{1}{\sqrt{1 - \gamma_{\vec{q}}^2}} \frac{1}{e^{\beta \frac{\hbar^2 z J}{2}} \sqrt{1 - \gamma_{\vec{q}}^2} - 1} \\
&= L^2 \int_{1BZ} \frac{dq d\theta}{(2\pi)^2} q \frac{1}{\sqrt{1 - \gamma_{\vec{q}}^2}} \frac{1}{e^{\beta \frac{\hbar^2 z J}{2}} \sqrt{1 - \gamma_{\vec{q}}^2} - 1}
\end{aligned}$$

For small q (near the zone center) $\sqrt{1 - \gamma_q^2} \approx \sqrt{q^2} = q$. Thus for small q , the radial part of the integrand is:

$$\approx q \frac{1}{q} \frac{1}{e^{\beta \frac{\hbar^2 z J}{2} q} - 1} \approx q \frac{1}{q} \frac{1}{\beta \frac{\hbar^2 z J}{2} q} \propto \frac{1}{q}$$

So, the integral diverges logarithmically at the zone center $q = 0$.

Ex. 17.8

First let us calculate $O_1^\dagger \mathbf{S}_p \cdot \mathbf{S}_q O_1$ by using the formula

$$e^A B e^{-A} = B + \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots . \quad (205)$$

We obtain

$$\begin{aligned} O_1^\dagger \mathbf{S}_p \cdot \mathbf{S}_q O_1 &= \mathbf{S}_p \cdot \mathbf{S}_q + (S_p^x S_q^y - S_p^y S_q^x) \sin\left(\frac{2\pi}{N}(q-p)\right) \\ &\quad + (S_p^x S_q^x + S_p^y S_q^y) \left(\cos\left(\frac{2\pi}{N}(q-p)\right) - 1 \right). \end{aligned} \quad (206)$$

Thus the upper bound for the variation gap is

$$\begin{aligned} \langle 0 | \sum_{p,q} J_{pq} \left[(S_p^x S_q^y - S_p^y S_q^x) \sin\left(\frac{2\pi}{N}(q-p)\right) \right. \\ \left. + (S_p^x S_q^x + S_p^y S_q^y) \left(\cos\left(\frac{2\pi}{N}(q-p)\right) - 1 \right) \right] | 0 \rangle. \end{aligned} \quad (207)$$

When we Taylor expand this in the order of $\frac{1}{N}$, the sine term is expanded as $\frac{1}{N}, \frac{1}{N^3}$ and so on while the cosine -1 term is expanded as $\frac{1}{N^2}, \frac{1}{N^4}$ and so on.

The first term in the sine term contributes

$$\begin{aligned} &\frac{2\pi}{N} \langle 0 | \sum_{p,q} J_{pq} (S_p^x S_q^y - S_p^y S_q^x) (q-p) | 0 \rangle \\ &\propto \frac{1}{N} \langle 0 | [H, \sum_{j=1}^N j S_j^z] | 0 \rangle \\ &= 0. \end{aligned} \quad (208)$$

Let us study the first term in cosine -1 term,

$$\begin{aligned} &-\frac{1}{2} \left(\frac{2\pi}{N} \right)^2 \langle 0 | \sum_{p,q} J_{pq} (S_p^x S_q^x + S_p^y S_q^y) (q-p)^2 | 0 \rangle \\ &\propto \frac{1}{N^2} \sum_{p,q} \frac{1}{(q-p)^{\beta-2}} \\ &\sim \frac{1}{N} \sum_{x=1}^N \frac{1}{x^{\beta-2}}. \end{aligned} \quad (209)$$

Thus β must be larger than 3 for the summation $\sum_{x=1}^N \frac{1}{x^{\beta-2}}$ to converge to a finite value.

Other terms in sine and cosine are of higher order in $\frac{1}{N}$ than the above term.

In conclusion, for the Lieb-Schultz-Mattis theorem to remain valid, β must be larger than 3.

Ex. 17.9

The Hamiltonian of J_1 - J_2 model breaks translational symmetry. As a result, the ground state is not an eigenstate of the lattice translation operator T . Since the proof of Lieb-Schultz-Mattis theorem hinges on the fact that the ground state is an eigenstate of lattice translation operator T , this theorem does not apply to the J_1 - J_2 model. More specifically, the variational state generated by the twist operator O_1 is not orthogonal to the original ground state.

Ex. 17.10

For a rotationally invariant system, the coupling Hamiltonian commutes with the total spin $\vec{S}_{total} = \vec{S}_1 + \vec{S}_2$. Thus Hamiltonian eigenstates have of S_{total} as a good quantum number. We also know the scalar product of these two vectors, $(\vec{S}_1 \cdot \vec{S}_2)$, commutes with \vec{S}_{total} . As a result its eigenstates are identical to Hamiltonian eigenstates. Thus the Hamiltonian can be written as a summation of any power of $(\vec{S}_1 \cdot \vec{S}_2)$. On the other hand, total spin of two spin- s , can be $0, 1, \dots, 2s$. Thus to characterize these $2s+1$ energy levels of these two spins, we need $2s$ parameters, up to an overall constant. We can thus choose them to be the coefficients of $(\vec{S}_1 \cdot \vec{S}_2)^k$, with $k = 1, 2, \dots, 2s$. For example, for two spin- $\frac{1}{2}$, total spin can be 0 or 1, we have two energy levels, so we write the coupling Hamiltonian as $J_0 + J_1(\vec{S}_1 \cdot \vec{S}_2)$. For the case of two spin-1, total spin can be 0, 1, 2, so write the coupling Hamiltonian as $J_0 + J_1(\vec{S}_1 \cdot \vec{S}_2) + J_2(\vec{S}_1 \cdot \vec{S}_2)^2$. In the above J_0 is this overall constant.

Ex. 17.11

First, take spin-2 for example. We decompose it into four half spins, two forming singlets with two of the half spins on its left neighbor, and the remaining two forming singlets with two of the half spins on its right neighbor, as illustrated in the figure. Then symmetrize the four half spins on each site to ensure they form net spin two. This results in a spin-2 chain VBS state that is a translationally invariant.

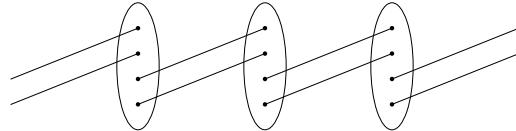


Figure 9: Illustration of a translationally invariant VBS state formed by a spin-2 chain. Dots are virtual spin- $\frac{1}{2}$ particles; a solid line represents a single state. Four virtual particles in an ellipse form a spin-2 particle after symmetrization.

For general integer spin s , decompose it into $2s$ half spins, and follow exactly the same procedures.

Ex. 17.12

From the construction of Ex. 17.11 it is clear the total spin of the two neighboring spins can only take values $S = 0, 1, \dots, s$, while $S > s$ is not possible. As a result [cf. Eq. (17.150) for $s = 1$] the VBS state is the exact zero energy ground state of the following Hamiltonian:

$$H = \sum_j \sum_{l=0}^s [2J_l P_{j,j+1}^{S=l}],$$

as long as $J_l > 0$. Obviously this Hamiltonian is not unique for $s > 1$.

Ex. 17.13

A $+$ (in the S^z basis) in the AKLT state is made of two up one-half spins $\uparrow\uparrow$. The right \uparrow must form a singlet with one \downarrow on its right; thus the spin on the right of $+$ must be 0 or $-$. If it is 0, it must be made of $\downarrow\uparrow$. Again only 0 or $-$ are allowed on the right of this 0 because of singlet constraint. As a result, $+$ cannot follow $+$, even with 0 between them.

Another way to see a configuration like $+0\dots0+$ is impossible is S_{tot}^z of the spins involved is two is this case. On the other hand from the VBS construction it is clear S_{tot}^z for any contiguous segment can only be 0 or ± 1 .

Ex. 17.14

Let us consider two neighboring spins of $z/2$ in the VBS state. Each spin can be considered as composed of z one-half spins. Thus among these $2z$ one-half spins on the neighboring two sites, there is one spin singlet with vanishing total spin. The total spin S_{total} of remaining $2z - 2$ one-half spins is $S_{total} = 0, 1, \dots, z - 1$. On the other hand, $\mathbf{S}_1 \cdot \mathbf{S}_2$ for these two neighboring spins of integer s is $S_{total}(S_{total} + 1)/2 - \frac{z}{2}(\frac{z}{2} + 1)$, thus having a one-to-one correspondence with the total spin S_{total} . The two body part of Hamiltonian $H = \sum_{k=1}^{\tilde{z}} J_k (\mathbf{S}_1 \cdot \mathbf{S}_2)^k$ should have this state as the ground state. Thus we have $H(S_{total} = 0) = H(S_{total} = 1) = \dots = H(S_{total} = z - 1)$. We have z unknown variables in the Hamiltonian with $z - 1$ constraints. The number of independent variables is 1.

- (i) $z = 3$ for 2D honeycomb lattice. $H(S_{total} = 0) = H(S_{total} = 1) = H(S_{total} = 2)$ lead to $J_2 = \frac{116}{243} J_1$ and $J_3 = \frac{16}{243} J_1$.
- (ii) $z = 4$ for 2D square lattice. $H(S_{total} = 0) = H(S_{total} = 1) = H(S_{total} = 2) = H(S_{total} = 3)$ lead to $J_2 = \frac{7}{10} J_1$, $J_3 = \frac{7}{45} J_1$ and $J_4 = \frac{1}{90} J_1$.

Ex. 17.15

(i) We first state formulas relevant to this exercise.

$$\langle A \rangle = \text{Tr}(A\rho_c) \quad (210)$$

with A being a general operator and ρ_c being the density matrix of the system.

$$S^x |\uparrow\rangle = \frac{1}{2} |\downarrow\rangle, S^x |\downarrow\rangle = \frac{1}{2} |\uparrow\rangle, S^y |\uparrow\rangle = \frac{i}{2} |\downarrow\rangle, S^y |\downarrow\rangle = -\frac{i}{2} |\uparrow\rangle, \quad (211)$$

where we have set $\hbar = 1$. Now we use the above formulas to calculate the average of spin component,

$$\begin{aligned} \langle S_1^x \rangle &= \text{Tr}(S_1^x \rho_c) \\ &= \text{Tr}\left(\frac{1}{4} |\downarrow\downarrow\rangle \langle \uparrow\downarrow| + \frac{1}{4} |\uparrow\uparrow\rangle \langle \downarrow\uparrow|\right) \\ &= 0, \\ \langle S_1^y \rangle &= \text{Tr}(S_1^y \rho_c) \\ &= \text{Tr}\left(\frac{i}{4} |\downarrow\downarrow\rangle \langle \uparrow\downarrow| - \frac{i}{4} |\uparrow\uparrow\rangle \langle \downarrow\uparrow|\right) \\ &= 0, \\ \langle S_1^z \rangle &= \text{Tr}(S_1^z \rho_c) \\ &= \text{Tr}\left(\frac{1}{4} |\uparrow\downarrow\rangle \langle \uparrow\downarrow| - \frac{1}{4} |\downarrow\uparrow\rangle \langle \downarrow\uparrow|\right) \\ &= \frac{1}{4} - \frac{1}{4} \\ &= 0. \end{aligned} \quad (212)$$

Similarly, we have

$$\langle S_2^x \rangle = 0, \langle S_2^y \rangle = 0, \langle S_2^z \rangle = 0. \quad (213)$$

(ii) It is easy to see that

$$\begin{aligned}
\langle S_1^z S_2^z \rangle &= \text{Tr}(S_1^z S_2^z \rho_c) \\
&= \text{Tr} \left(-\frac{1}{8} |\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \frac{1}{8} |\downarrow\uparrow\rangle\langle\downarrow\uparrow| \right) \\
&= -\frac{1}{8} - \frac{1}{8} \\
&= -\frac{1}{4}, \\
\langle S_1^x S_2^x \rangle &= \text{Tr}(S_1^x S_2^x \rho_c) \\
&= \text{Tr} \left(\frac{1}{8} |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{8} |\uparrow\downarrow\rangle\langle\downarrow\uparrow| \right) \\
&= 0, \\
\langle S_1^y S_2^y \rangle &= \text{Tr}(S_1^y S_2^y \rho_c) \\
&= \text{Tr} \left(\frac{1}{8} |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{8} |\uparrow\downarrow\rangle\langle\downarrow\uparrow| \right) \\
&= 0.
\end{aligned} \tag{214}$$

Thus x and y components of two spins are completely uncorrelated while the z components of two spins are maximally anti-correlated in this classical system.

(iii) For a system described by the density matrix of singlet state, we have

$$\langle S_1^x \rangle = 0, \langle S_1^y \rangle = 0, \langle S_1^z \rangle = 0, \langle S_2^x \rangle = 0, \langle S_2^y \rangle = 0, \langle S_2^z \rangle = 0. \tag{215}$$

$$\langle S_1^x S_2^x \rangle = \langle S_1^y S_2^y \rangle = \langle S_1^z S_2^z \rangle = -\frac{1}{4}. \tag{216}$$

Therefore x , y and z components of two spins in the singlet state are maximally anti-correlated.

Ex. 17.16

We have

$$\begin{aligned}
|(12), (34)\rangle &= \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle) \frac{1}{\sqrt{2}}(|\uparrow_3\downarrow_4\rangle - |\downarrow_3\uparrow_4\rangle) \\
&= \frac{1}{2}(|\uparrow\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle),
\end{aligned} \tag{217}$$

where in the last line the spins in each term are in the order of 1234. Likewise, we have

$$\begin{aligned}
|(13), (24)\rangle &= \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_3\rangle - |\downarrow_1\uparrow_3\rangle) \frac{1}{\sqrt{2}}(|\uparrow_2\downarrow_4\rangle - |\downarrow_2\uparrow_4\rangle) \\
&= \frac{1}{2}(|\uparrow\uparrow\downarrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle).
\end{aligned} \tag{218}$$

It is easy to see that $|\uparrow\downarrow\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\downarrow\uparrow\rangle$ are unique to $|(12), (34)\rangle$, $|\uparrow\uparrow\downarrow\downarrow\rangle$ and $|\downarrow\downarrow\uparrow\uparrow\rangle$ are unique to $|(13), (24)\rangle$. Thus two states are independent. Their overlap is easily calculated to be $\frac{1}{2}$.

Ex. 17.17

It is easy to see that $[A_s, A_{s'}] = 0$ and $[B_p, B_{p'}] = 0$ since $[\sigma_j^z, \sigma_{j'}^z] = [\sigma_j^x, \sigma_{j'}^x] = 0$. It is also easy to see that $[A_s, B_p] = 0$ if s and p do not share bonds. s and p share bonds only when the common endpoint of s is a vertex of the square p . In this case, they share two bonds. Let us name these two shared bonds as j and k , the other two bonds in s as n and q , and the remaining bonds in p as l and m . Then

$$\begin{aligned} [A_s, B_p] &= [\sigma_j^z \sigma_k^z \sigma_n^z \sigma_q^z, \sigma_j^x \sigma_k^x \sigma_l^x \sigma_m^x] \\ &= \sigma_n^z \sigma_q^z [\sigma_j^z \sigma_k^z, \sigma_j^x \sigma_k^x] \sigma_l^x \sigma_m^x. \end{aligned} \tag{219}$$

It is easy to prove $[\sigma_j^z \sigma_k^z, \sigma_j^x \sigma_k^x] = 0$ by using the identity $\{\sigma_j^z, \sigma_j^x\} = 0$.

By using the fact that every bond is shared by two vertices and two squares, and the identities $(\sigma_j^x)^2 = (\sigma_j^y)^2 = (\sigma_j^z)^2 = 1$, we immediately have

$$\prod_s A_s = \prod_p B_p = 1. \tag{220}$$

Ex. 17.18

The fact B_p has eigenvalues ± 1 implies $B_p^2 = 1$. As a result $B_p \left(\frac{1+B_p}{2} \right) = \frac{1+B_p}{2} = P_p$. Eq. (17.175) follows.

Ex. 17.19

Let us calculate $\prod_{p \in \mathcal{A}} B_p$. It is obvious that those bonds along the loop contribute once to $\prod_{p \in \mathcal{A}} B_p$ and those bonds inside the region enclosed by the loop contribute twice to $\prod_{p \in \mathcal{A}} B_p$, since each bond is shared by two neighboring plaquettes. Now with the identity $(\sigma_j^x)^2 = 1$, we immediately see that only σ_j^x on the bonds along the loop matter. Thus we have

$$\prod_{p \in \mathcal{A}} B_p = \prod_{j \in \mathcal{L}} \sigma_j^x. \tag{221}$$

Ex. 17.20

There are two kinds of excitations for the toric code: charge excitation and flux excitation. The number of charge excitation or flux excitation should be even due to the constraint $\prod_s A_s = \prod_p B_p = 1$. Thus the minimum number of charge excitation or flux excitation is 2. Now consider charge excitations at sites s_1 and s_2 . The gap for these charge excitations is $2J_{s_1} + 2J_{s_2}$. For the flux excitations at plaquettes p_1 and p_2 , the gap is $2K_{p_1} + 2K_{p_2}$. Therefore the low bounds for the gap is the minimum of $2J_{s_1} + 2J_{s_2}$ or minimum of $2K_{p_1} + 2K_{p_2}$, whichever is smaller.

Ex. 17.21

If J_s can take negative values, the ground state must have all $A_s = -1$, thus we have the constraint $\prod_s A_s = (-1)^{N_s}$, where N_s is the number of sites. Similarly, if K_p can take negative values, the ground state must have all $B_p = -1$, thus we have the constraint $\prod_p B_p = (-1)^{N_p}$, where N_p is the number of plaquettes. To make these consistent with the constraint $\prod_s A_s = \prod_p B_p = 1$, the number of sites and plaquettes must be even. Now consider the ground state degeneracy. It is still determined by the parity of non-trivial loops wrapping around x and y directions; thus it is still 4.

Ex. 17.22

Hitting the ground state with σ_k^x flips the spin on the bond k , and creates an exact excited state with two charge excitations on s_1 and s_2 which are the end points of k . So the energy denominator is $J_{s_1} + J_{s_2}$. Hitting this excited state with σ_k^x flips the spin on the bond k back, and the system returns to the original ground state (in the same topological sector). Namely no other excited states are involved. We can thus diagonalize $\lambda\sigma_k^x$ in this 2×2 Hilbert subspace to obtain the exact new ground state and its energy shift:

$$\Delta E = \frac{1}{2} \left[J_{s_1} + J_{s_2} - \sqrt{(J_{s_1} + J_{s_2})^2 + 4\lambda^2} \right],$$

which is the same for all topological sectors. So the topological degeneracy remains exact.

Ex. 17.23

In this case hitting the ground state with V repeatedly will generate more and more excitations with complicated superpositions. To start from one topological sector and reach another, however, requires creating a string of spin flips that wraps around the torus. The order in perturbation theory this happens is $\min(L_x, L_y)$.

Ex. 17.24

The proof mimics the manipulations around Eqs. (17.180-17.183), with σ^x replaced by σ^z that moves the flux around.

Ex. 17.25

When we generalize the toric code to 3D cubic lattice, the form of the Hamiltonian is still Eq. (17.172), except the definition of A_s in (17.173) need to include all 6 bonds that have s as an end point on a cubic lattice, while definition of B_p remains the same with p standing for a square shaped loop in a cubic lattice. It is also easy to prove that all A_s and B_p still commute among themselves. Since A_s and B_p still have ± 1 as eigenvalues, the ground state must have all A_s and B_p equal to 1. Charge excitation corresponds to points with $A_s = -1$, which are identical to the 2D case. Flux

excitations now correspond to *closed loops* made of squares with $B_p = -1$. The way to see why fluxes must form loops is the following. Take the product of all B 's corresponding to the 6 faces of a cube, and it must be 1. This means the net flux entering the cube must be 0 mod 2. The Berry phase picked up by the charge when circulating around a flux loop is -1.

Now consider the ground state degeneracy. If we consider the spin $\sigma_z = 1$ as background, again for the ground state all the bonds on which $\sigma_z = -1$ will form a loop. Any two loops along the same direction can be annihilated by operations of B_p . Consequently the ground state degeneracy is determined by the parity numbers of non-contractible loops along the x, y, z directions. We have the ground state degeneracy $2^3 = 8$.

Ex. 17.26

Electron energy in 2D in the absence of the magnetic field is $\epsilon = \hbar^2 k^2 / (2m)$, so the change in energy $\delta\epsilon = \hbar^2 k \delta k / m$ when momentum k is changed by δk . The number of states changes by $\delta N = \frac{A}{(2\pi)^2} 2\pi k \delta k = \frac{Am\delta\epsilon}{2\pi\hbar^2}$, where A is area of the 2D system. Thus the density of states per spin is $D = \frac{\delta N}{\delta\epsilon} = \frac{Am}{2\pi\hbar^2}$, which is a constant. We have $D\hbar\omega_c = D\hbar \frac{eB}{mc} = \frac{BA}{hc/e} = \frac{\Phi}{\Phi_0} = N_\Phi$. The filling factor is $\nu = \frac{N_e}{N_\Phi} = \frac{D\epsilon_F}{D\hbar\omega_c} = \frac{\epsilon_F}{\hbar\omega_c}$

Chapter 18

Ex. 18.1

$$\begin{aligned}
 b_0|\alpha\rangle &= e^{\frac{-|\alpha|^2}{2}} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} b_0|N\rangle \\
 &= e^{\frac{-|\alpha|^2}{2}} \sum_{N=1}^{\infty} \frac{\alpha^N}{\sqrt{N!}} \sqrt{N}|N-1\rangle \\
 &= \alpha e^{\frac{-|\alpha|^2}{2}} \sum_{N'=0}^{\infty} \frac{\alpha^{N'}}{\sqrt{N'!}} |N'\rangle = \alpha |\alpha\rangle
 \end{aligned}$$

Ex. 18.2

$$\begin{aligned}
 U_\alpha &= e^{\alpha b_0^\dagger - \alpha^* b_0} = e^{\alpha b_0^\dagger} e^{-\alpha^* b_0} e^{\frac{-|\alpha|^2}{2} [b_0, b_0^\dagger]} = e^{\frac{-|\alpha|^2}{2}} e^{\alpha b_0^\dagger} e^{-\alpha^* b_0} \\
 \text{So } U_\alpha |0\rangle &= e^{\frac{-|\alpha|^2}{2}} e^{\alpha b_0^\dagger} e^{-\alpha^* b_0} |0\rangle = e^{\frac{-|\alpha|^2}{2}} e^{\alpha b_0^\dagger} |0\rangle = |\alpha\rangle
 \end{aligned}$$

Ex. 18.3

Choose $\alpha = e^{i\varphi}$. Then,

$$\frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{-iN\varphi} |\alpha\rangle = \frac{1}{\sqrt{N!}} |N\rangle,$$

or

$$|N\rangle = \sqrt{N!} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{-iN\varphi} |\alpha\rangle$$

Since $\{|N\rangle\}$ is a complete set, $\{|\alpha\rangle\}$ must also be complete.

Ex. 18.4

$$\overline{N} = e^{-|\alpha|^2} \sum_{N=0}^{\infty} \frac{|\alpha|^{2N}}{N!} N = e^{-|\alpha|^2} \sum_{N'=0}^{\infty} \frac{|\alpha|^{2(N'+1)}}{N'!} = |\alpha|^2$$

$$\overline{N^2} = e^{-|\alpha|^2} \sum_{N=0}^{\infty} \frac{|\alpha|^{2N}}{N!} N^2 = |\alpha|^4 + |\alpha|^2 = \overline{N}^2 + \overline{N}$$

$$\sqrt{\overline{N^2} - (\overline{N})^2} = \sqrt{\overline{N}} \ll \overline{N} \quad \text{for } \overline{N} \gg 1$$

Ex. 18.5

Choose $\vec{r}' = 0$ without loss of generality.

$$G(r) = \langle\langle \psi^\dagger(\vec{r}) \psi(0) \rangle\rangle = \frac{2\pi}{(2\pi)^3} \int_0^\infty k^2 dk \int_0^\pi \sin\theta d\theta \frac{e^{-ikr \cos\theta}}{e^{\alpha(k^2 + k_0^2)} - 1}$$

where $\frac{\hbar^2 k_0^2}{2m} = \mu$, and $\alpha = \frac{\hbar^2}{2mk_B T}$. So,

$$\begin{aligned} G(r) &= \frac{2}{(2\pi)^2} \int_0^\infty \frac{k^2 \sin kr}{e^{\alpha(k^2 + k_0^2)} - 1} dk \\ &= \frac{1}{(2\pi)^2} \frac{1}{r} \int_{-\infty}^\infty dk \frac{k \sin kr}{e^{\alpha(k^2 + k_0^2)} - 1} \end{aligned}$$

Using residue theorem, we find $\rho(r) \sim e^{-\kappa r}$ at large r , where κ is given by the position of residue: $e^{\alpha(-\kappa^2 + k_0^2)} - 1 = 0$ or $\kappa = k_0 = \left(\frac{2m\mu}{\hbar^2}\right)^{\frac{1}{2}} = \frac{\sqrt{2m\mu}}{\hbar}$. $\frac{1}{\kappa}$ is the decay length.

Ex. 18.6

(i)

$$\begin{aligned} \langle n(\vec{r}) \rangle &= \frac{1}{V} \sum_{\vec{k}, \vec{k}'} \langle b_{\vec{k}}^\dagger b_{\vec{k}'} \rangle e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}} \\ &= \frac{1}{V} \left(\langle b_{\vec{k}}^\dagger b_{\vec{k}} \rangle + \langle b_{\vec{k}'}^\dagger b_{\vec{k}'} \rangle \right) = \frac{2M}{V} \end{aligned}$$

(ii)

$$\begin{aligned}
\langle n(\vec{r}) n(\vec{r}') \rangle &= \frac{1}{V^2} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}'_1, \vec{k}'_2} e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{r}} e^{i(\vec{k}'_2 - \vec{k}'_1) \cdot \vec{r}'} \langle b_{\vec{k}_1}^\dagger b_{\vec{k}_2} b_{\vec{k}'_1}^\dagger b_{\vec{k}'_2} \rangle \\
&= \langle n(\vec{r}) \rangle \langle n(\vec{r}') \rangle + \frac{1}{V^2} \left[\langle b_{\vec{k}_+}^\dagger b_{\vec{k}_+} \rangle \langle b_{\vec{k}_-} b_{\vec{k}_-}^\dagger \rangle \cdot e^{i(\vec{k}_+ - \vec{k}_-) \cdot (\vec{r}' - \vec{r})} + \langle b_{\vec{k}_-}^\dagger b_{\vec{k}_-} \rangle \langle b_{\vec{k}_+}^\dagger b_{\vec{k}_+} \rangle \cdot e^{i(\vec{k}_+ - \vec{k}_-) \cdot (\vec{r} - \vec{r}')} \right] \\
&= \langle n(\vec{r}) \rangle \langle n(\vec{r}') \rangle + \frac{2M(M+1)}{V^2} \cos \left[(\vec{k}_+ - \vec{k}_-) \cdot (\vec{r} - \vec{r}') \right]
\end{aligned}$$

We thus have

$$\langle n(\vec{r}) n(\vec{r}') \rangle - \langle n(\vec{r}) \rangle \langle n(\vec{r}') \rangle \sim \cos[2k_+(x - x')].$$

(iii)

For each shot one should see the same wave vector $\vec{k}_+ - \vec{k}_-$ for the fringe due to the correlation calculated in (ii), but the phase is random in order to have no oscillation in $\langle n(\vec{r}) \rangle$ itself.

Ex. 18.7

$$\begin{aligned}
\langle N | H | N \rangle &= \langle N | \hat{V} | N \rangle \\
&= \frac{V_0}{2V} \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \langle N | b_{\vec{k}_1}^\dagger b_{\vec{k}_2}^\dagger b_{\vec{k}_3} b_{\vec{k}_1 + \vec{k}_2 - \vec{k}_3} | N \rangle \\
&= \frac{V_0}{2V} \langle N | \left(b_0^\dagger \right)^2 b_0^2 | N \rangle \\
&= \frac{V_0}{2V} \langle N | b_0^\dagger \left(b_0 b_0^\dagger - 1 \right) b_0 | N \rangle \\
&= \frac{V_0}{2V} \left(\langle N | \left(b_0^\dagger b_0 \right)^2 | N \rangle - \langle N | b_0^\dagger b_0 | N \rangle \right) \\
&= \frac{V_0}{2V} (N^2 - N) \\
&= \frac{V_0}{2V} N (N - 1)
\end{aligned}$$

Ex. 18.8

We focus on a particular \vec{k} . The terms involved are of the form

$$\begin{aligned}
& A b_{\vec{k}}^\dagger b_{\vec{k}} + B \left(b_{\vec{k}}^\dagger b_{-\vec{k}}^\dagger + b_{\vec{k}} b_{-\vec{k}} \right) \\
&= A \left(\cosh \theta_{\vec{k}} \beta_{\vec{k}}^\dagger + \sinh \theta_{\vec{k}} \beta_{-\vec{k}} \right) \left(\cosh \theta_{\vec{k}} \beta_{\vec{k}} + \sinh \theta_{\vec{k}} \beta_{-\vec{k}}^\dagger \right) \\
&+ B \left[\left(\cosh \theta_{\vec{k}} \beta_{\vec{k}}^\dagger + \sinh \theta_{\vec{k}} \beta_{-\vec{k}} \right) \left(\cosh \theta_{\vec{k}} \beta_{-\vec{k}}^\dagger + \sinh \theta_{\vec{k}} \beta_{\vec{k}} \right) + \left(\cosh \theta_{\vec{k}} \beta_{\vec{k}} + \sinh \theta_{\vec{k}} \beta_{-\vec{k}}^\dagger \right) \left(\cosh \theta_{\vec{k}} \beta_{-\vec{k}} + \sinh \theta_{\vec{k}} \beta_{\vec{k}}^\dagger \right) \right] \\
&= A \left(\cosh^2 \theta_{\vec{k}} \beta_{\vec{k}}^\dagger \beta_{\vec{k}} + \sinh^2 \theta_{\vec{k}} \beta_{-\vec{k}} \beta_{-\vec{k}}^\dagger \right) + B \sinh \theta_{\vec{k}} \cosh \theta_{\vec{k}} \left(\beta_{\vec{k}}^\dagger \beta_{\vec{k}} + \beta_{\vec{k}} \beta_{\vec{k}}^\dagger + \beta_{-\vec{k}}^\dagger \beta_{-\vec{k}} + \beta_{-\vec{k}} \beta_{-\vec{k}}^\dagger \right) \\
&+ [(A \sinh \theta_{\vec{k}} \cosh \theta_{\vec{k}}) + B (\cosh^2 \theta_{\vec{k}} + \sinh^2 \theta_{\vec{k}})] \left(\beta_{\vec{k}}^\dagger \beta_{-\vec{k}}^\dagger + \beta_{\vec{k}} \beta_{-\vec{k}} \right)
\end{aligned}$$

$$\text{Choose } A_{\vec{k}} \sinh \theta_{\vec{k}} \cosh \theta_{\vec{k}} + B (\cosh^2 \theta_{\vec{k}} + \sinh^2 \theta_{\vec{k}}) = 0 \Rightarrow \tanh 2\theta_{\vec{k}} = \frac{-B}{A_{\vec{k}}} = \frac{-nV_0}{(\epsilon_{\vec{k}} + nV_0)}$$

Apart from constants, we have

$$\begin{aligned}
H &= \sum_{\vec{k}} [A_{\vec{k}} (\cosh^2 \theta_{\vec{k}} + \sinh^2 \theta_{\vec{k}}) + 2B \sinh \theta_{\vec{k}} \cosh \theta_{\vec{k}}] \beta_{\vec{k}}^\dagger \beta_{\vec{k}} \\
&= \sum_{\vec{k}} (A_{\vec{k}} \cosh 2\theta_{\vec{k}} + B \sinh 2\theta_{\vec{k}}) \beta_{\vec{k}}^\dagger \beta_{\vec{k}} \\
&= \sum_{\vec{k}} \cosh 2\theta_{\vec{k}} (A_{\vec{k}} + B \tanh 2\theta_{\vec{k}}) \beta_{\vec{k}}^\dagger \beta_{\vec{k}} \\
&= \sum_{\vec{k}} \frac{A_{\vec{k}}}{\sqrt{A_{\vec{k}}^2 - B^2}} \left(A_{\vec{k}} - B \frac{B}{A_{\vec{k}}} \right) \beta_{\vec{k}}^\dagger \beta_{\vec{k}} \\
&= \sum_{\vec{k}} \sqrt{A_{\vec{k}}^2 - B^2} \beta_{\vec{k}}^\dagger \beta_{\vec{k}} \\
&= \sum_{\vec{k}} \sqrt{\epsilon_{\vec{k}}^2 + 2\epsilon_{\vec{k}} n V_0} \beta_{\vec{k}}^\dagger \beta_{\vec{k}}
\end{aligned}$$

Ex. 18.9

The state being considered is

$$|N-1, 1\rangle = \frac{1}{\sqrt{(N-1)!}} b_{\vec{k}}^\dagger b_0^\dagger |0\rangle.$$

Obviously, $\Delta T = \frac{-\hbar^2 k^2}{2m}$.

$$\begin{aligned}\langle N | \hat{V} | N \rangle &= \frac{N(N-1)}{2V} v_0, \\ \langle N-1, 1 | \hat{V} | N-1, 1 \rangle &= \frac{1}{2} \frac{v_0}{V} \langle N-1, 1 | b_{\vec{k}}^\dagger b_{\vec{k}}^\dagger b_{\vec{k}} b_{\vec{k}} + 4 b_{\vec{k}}^\dagger b_{\vec{k}} b_0^\dagger b_0 | N-1, 1 \rangle \\ &= \frac{1}{2} \frac{v_0}{V} [(N-1)(N-2) + 4(N-1)] = \frac{v_0}{2V} (N-1)(N+2).\end{aligned}$$

Thus $\Delta E_1 = \frac{-\hbar^2 k^2}{2m} + \frac{(N-1)v_0}{V} \approx \frac{-\hbar^2 k^2}{2m} + nv_0$.

Now consider the state $|N_1, N_2\rangle = \frac{1}{\sqrt{N_1!N_2!}} \left(b_{\vec{k}}^\dagger\right)^{N_1} (b_0)^{N_2} |0\rangle$ with $N_1 + N_2 = N$. We find

$$\begin{aligned}\langle N_1 N_2 | \hat{V} | N_1 N_2 \rangle &= \frac{1}{2} \frac{v_0}{V} \langle N_1 N_2 | b_{\vec{k}}^\dagger b_{\vec{k}}^\dagger b_{\vec{k}} b_{\vec{k}} + b_0^\dagger b_0^\dagger b_0 b_0 + 4 b_{\vec{k}}^\dagger b_{\vec{k}} b_0^\dagger b_0 | N_1 N_2 \rangle \\ &= \frac{1}{2} \frac{v_0}{V} [N_1(N_1-1) + N_2(N_2-1) + 4N_1N_2]\end{aligned}$$

For $N_2 \ll N_1 \approx N$, we have

$$\langle N_1 N_2 | \hat{V} | N_1 N_2 \rangle \approx \frac{v_0}{2V} N(N-1) + nv_0 \cdot N_2,$$

Thus $\Delta E_1 \approx \Delta E_2 \approx \Delta E_3 \approx \dots$. More generally, $\Delta E(N_2) = -N_2 \frac{\hbar^2 k^2}{2m} + N_2(nv_0 - \frac{\hbar^2 k^2}{2m}) - \frac{v_0 N_2^2}{V}$ whose maximum value is $\frac{V}{4v_0} (nv_0 - \frac{\hbar^2 k^2}{2m})^2 \propto V$

Ex. 18.10

(i) $E_{moving} = E_{lab} + \vec{v} \cdot \vec{P}_{tot} = E_{lab} + \hbar \vec{v} \cdot \vec{k}$

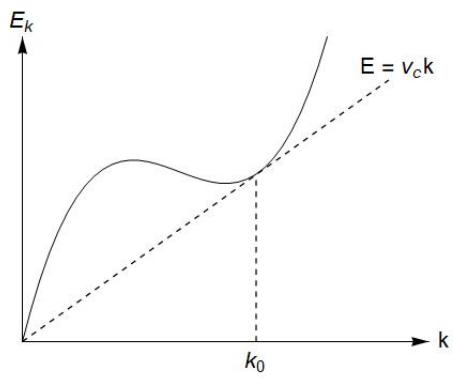
where $\hbar \vec{k}$ is the momentum of the state in the comoving frame. The state itself does not change because it is an eigenstate of \vec{P}_{tot} .

(ii) In the moving frame,

$$E_{\vec{k}} \rightarrow E'_{\vec{k}} = E_{\vec{k}} + \hbar \vec{v} \cdot \vec{k} = E_{\vec{k}} - \hbar v k$$

for $\vec{k} \parallel -\vec{v}$. Thus $E_{\vec{k}} < 0$ for $v > v_c = v_s$, signaling instability of the moving ground state.

(iii) In this case v_c is the slope of the dashed line that is tangential to the E_k curve, as for $v > v_c$, $E'(k_0) = E(k_0) - v k_0 < 0$ signaling instability.



Chapter 19

Ex. 19.1

The change in heat capacity from the normal to the superconducting phase is given by

$$\begin{aligned} C_{\text{super}} - C_{\text{normal}} &= T \left(\frac{\partial}{\partial T} S_{\text{super}} - \frac{\partial}{\partial T} S_{\text{normal}} \right) = T \frac{1}{4\pi} \frac{\partial}{\partial T} \left(H_c(T) \frac{dH_c(T)}{dT} \right) . \\ &= \frac{1}{4\pi} T \left(\frac{dH_c(T)}{dT} \right)^2 + \frac{1}{4\pi} TH_c(T) \frac{d^2H_c(T)}{dT^2} . \end{aligned}$$

The question asks for the “zero field phase transition point”. This means at the point where the phase transition occurs at zero applied field. In this case, $H_c(T) = 0$. So

$$C_{\text{super}} - C_{\text{normal}} = \frac{1}{4\pi} T \left(\frac{dH_c(T)}{dT} \right)^2 \Big|_{T=T_c} = \frac{H_c^2}{\pi T_c} .$$

Ex. 19.2

For an amount of flux $\Phi_0\delta$ going through a quantum rotor:

$$H = -\frac{\hbar^2}{2I} \left(-i \frac{\partial}{\partial \phi} + \frac{eR}{\hbar c} A_\phi \right)^2$$

(this is the “symmetric gauge”). Here $A_\phi = \beta \times \left(\frac{eR}{\hbar c}\right)^{-1}$. We saw in Ex. 10.1 that the eigenfunctions are given by $\psi_n(\phi) = e^{in\phi}$ independent of the flux amount δ , for positive and negative integers n . For small β the ground state has $n = 0$, independent of β . This is the wave function rigidity.

paramagnetic current \hat{J}_p

The paramagnetic current is proportional the derivative of wave function. Since the ground state wave function is a constant for $n = 0$, the paramagnetic current is zero.

diamagnetic current \hat{J}_d

So the entire current equals diamagnetic current. From part (iv) of Ex. 10.1 we find $J \propto \beta \propto A_\phi$. This is the analog of London equation.

Ex. 19.3

$$\begin{aligned}
\frac{\delta F[\vec{A}, \psi]}{\delta \vec{A}(\vec{R})} &= \frac{\delta}{\delta \vec{A}(\vec{R})} \int d\vec{r} f[\vec{A}(\vec{r}), \psi(\vec{r})] \\
&= \frac{\partial f}{\partial \vec{A}(\vec{R})} = \frac{1}{m^*} \frac{e^*}{c} \psi^* \left(\vec{p} + \frac{e^*}{c} \vec{A} \right) \psi + \frac{1}{4\pi} \frac{\partial}{\partial \vec{A}(\vec{R})} (\nabla \times \vec{A})^2 \\
&= \frac{e^*}{c} \psi^* \frac{\left(\vec{p} + \frac{e^*}{c} \vec{A}(\vec{R}) \right)}{m^*} \psi - \frac{1}{4\pi} \nabla \times (\nabla \times \vec{A}) = 0 \\
\Rightarrow \quad \nabla \times \vec{B} &= \nabla \times (\nabla \times \vec{A}) = \frac{4\pi}{c} e^* \psi^* \frac{\left(\vec{p} + \frac{e^*}{c} \vec{A} \right)}{m^*} \psi = \frac{4\pi}{c} \vec{J}
\end{aligned}.$$

In the above \vec{J} should be understood as QM/Thermal ensemble average.

Ex. 19.4

n flux quanta

The wave function is $\psi(x, y, z) = \psi_\infty f(r_\perp) e^{in\phi(x, y)}$. We are seeking to minimize the free energy density f :

$$f(\vec{A}, \psi) = \frac{1}{2m^*} \left| \left(\vec{p} + \frac{e^*}{c} \vec{A} \right) \psi \right|^2 + \dots$$

In this case we have $\vec{p}\psi = -i\hbar\nabla(\psi_\infty f(r_\perp) e^{in\phi(x, y)})$. In a region of large r_\perp , $f \approx 1$, so we get

$$\vec{p}\psi = n\hbar \left(\psi_\infty f(r_\perp) e^{in\phi(x, y)} \right) \nabla\phi(x, y) = n\hbar \frac{\hat{\phi}}{r_\perp} \psi.$$

We find that space integral of this term $\left| \left(\vec{p} \right) \psi \right|^2$ becomes

$$\psi^2 \propto \hbar^2 n^2 L_z \int_0^{L_\perp} dr_\perp 2\pi r_\perp \frac{f^2(r_\perp)}{r_\perp^2}.$$

Again this is logarithmically divergent in L_\perp . Therefore, we expect the \vec{A} term in the free energy to cancel this. For large r_\perp we must have

$$\vec{A} = -n \left(\hbar \frac{c}{e^*} \right) \frac{\hat{\phi}}{r_\perp} = -n\Phi_0 \frac{\hat{\phi}}{r_\perp}.$$

This means that

$$\oint \vec{A} \cdot d\vec{r} = \int d^2 r_\perp \vec{b} \cdot \hat{z} = -n\Phi_S$$

so this vortex holds n flux quanta.

energy of vortex with $n=2$ vs two $n=1$ vortices far apart

The cancellation discussed above is incomplete in the core region of the vortex. For simplicity we neglect A in this region, then the first term in the free energy goes like

$$\psi^2 \propto \hbar^2 n^2 L_z \int_0^{L_\perp} dr_\perp 2\pi r_\perp \frac{f^2(r_\perp)}{r_\perp^2}.$$

The n^2 for $n = 2$ is 4, which is more than $n + n = 2$ for $n = 1$, which would be the energy for two widely separated vortices.

It should be noted that we assumed the same $f(r_\perp)$ for different n in the above. Relaxing this constraint may change the conclusion, which is what happens in a type-I superconductor.

Ex. 19.5

(i)

Order parameter takes the form

$$\psi(x, y, z) = \psi_\infty f(r_\perp) e^{i\phi(x, y)}$$

$$r_\perp \geq \xi \Rightarrow f(r_\perp) \rightarrow 1.$$

$r_\perp < \xi \Rightarrow f(r_\perp)$ decays to zero as shown in Fig.1.

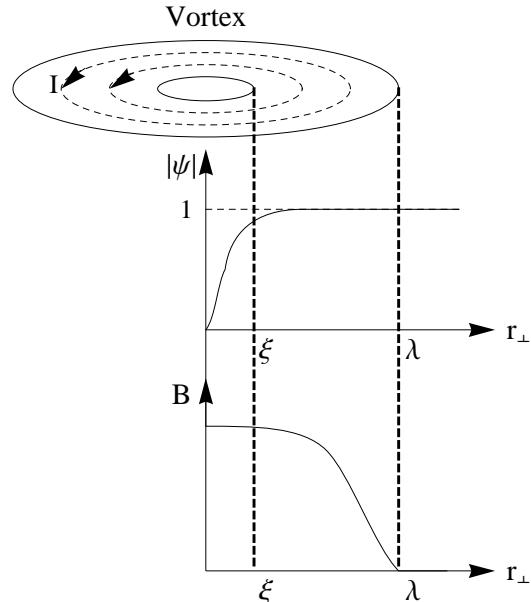


Figure 10

The energy per unit length of a vortex comes from two parts: the kinetic energy of matter corresponding to the energy of the paramagnetic current and the energy of magnetic field.

Now we estimate the energy contribution from the kinetic energy. As shown in Fig.1, the wave function within ξ decays to zero. For $r_\perp > \lambda$, B field vanishes so that there is little screening current. Therefore, the kinetic energy is dominated by the region $\xi \leq r_\perp \leq \lambda$. This region allows for two simplifications in the limit $\kappa \gg 1$. (i) The wave function magnitude can be regarded as being uniform and taking the bulk value $|\psi|$. (ii) We may neglect the magnetic field, and diamagnetic contribution to current and kinetic energy.

$$\therefore T \approx \int d\vec{r}_\perp \frac{|\vec{p}\psi|^2}{2m^*} \approx \frac{|\psi|^2}{2m^*} \int_0^{2\pi} d\phi \int_\xi^\lambda dr_\perp r_\perp \frac{\hbar^2}{r_\perp^2} \left(y \equiv \frac{r_\perp}{\xi} \right) = \frac{\hbar^2 |\psi|^2}{4\pi m^*} \int_1^{\frac{\lambda}{\xi}} dy \frac{1}{y} = \left(\frac{\Phi_S}{4\pi\lambda} \right)^2 \ln \kappa$$

where $\lambda^2 = \frac{m^* c^2}{4\pi e^* |\psi|^2}$ [cf. Eq.(19.70)] and $\Psi_S = \frac{hc}{e^*}$.

(ii) If $\lambda = \infty$, the upper limit of the integral will be system size, resulting in logarithmic divergence in it.

Ex. 19.6

$$\vec{J} = \frac{e^*}{m^*} \Psi^* \left(\vec{p} + \frac{e^*}{c} \vec{A} \right) \Psi = 0$$

$$\vec{A} = \frac{-\Psi^* \vec{p} \Psi}{|\Psi|^2} \frac{c}{e^*} = \frac{\hbar c}{e^*} \nabla \varphi$$

where φ is phase of Ψ . Thus,

$$\oint \nabla \varphi d\vec{r} = 2\pi n = \frac{e^*}{\hbar c} \oint \vec{A} \cdot d\vec{r} = \frac{e^*}{\hbar c} \Phi$$

$$\Rightarrow \Phi = n \frac{hc}{e^*} = n\Phi_s$$

Ex. 19.7

(a) From Eq. (19.128), the ‘magnetic field’ for a particle of ‘charge’ (i.e. mass) m in a frame rotating with angular velocity $\vec{\Omega}$, is $\vec{B} = 2\vec{\Omega}$. Thus for a charge e , the effective magnetic field is $\vec{B} = \frac{m}{e} 2\vec{\Omega}$. For $\Omega = 2\pi \times 100\text{Hz}$, $B = \frac{4\pi m}{e} \times 10^2\text{Hz}$. Recall that in SI units, the Bohr magneton is

$$\frac{\mu_B}{h} = \frac{e}{4\pi m_e} \approx 13.996 \text{ GHz/T.} \quad (222)$$

Thus for a rotation rate of 100 Hz,

$$B = \frac{m}{m_e} \frac{10^2 \text{ Hz}}{13.996 \times 10^9 \text{ Hz}} T. \quad (223)$$

As a specific example, consider ^{87}Rb with atomic mass $m \approx 86.9m_{\text{H}} \approx 1.6 \times 10^5 m_{\text{e}}$ for which

$$\mathcal{B} \approx 1.14 \times 10^{-3} \text{ T}. \quad (224)$$

(b) The rotation rate needed to achieve an effective magnetic field of 10 T is extremely large:

$$f = \frac{10\text{T}}{1.14 \times 10^{-3} \text{ T}} \times 100 \text{ Hz} \approx 8.77 \times 10^5 \text{ Hz}. \quad (225)$$

Ex. 19.8

$$\begin{aligned} \hat{I}|\varphi\rangle &= -i\frac{e}{\hbar}E_J \sum_{m,m'} (|m\rangle\langle m+1| - |m+1\rangle\langle m|) e^{im'\varphi} |m'\rangle \\ &= -i\frac{e}{\hbar}E_J \sum_m \left(|m\rangle e^{i(m+1)\varphi} - |m+1\rangle e^{im\varphi} \right) \\ &= -i\frac{e}{\hbar}E_J \sum_m |m\rangle e^{im\varphi} (e^{i\varphi} - e^{-i\varphi}) \\ &= \frac{2e}{\hbar}E_J \sin\varphi |\varphi\rangle \\ &= I_c \sin\varphi |\varphi\rangle \end{aligned}$$

Ex. 19.9

$$\begin{aligned} i\hbar\partial_t|\Psi(t)\rangle &= (i\hbar\partial_t e^{+\frac{i}{\hbar}E_J \int_0^t d\tau \cos\varphi(\tau)})|\varphi(t)\rangle + e^{+\frac{i}{\hbar}E_J \int_0^t d\tau \cos\varphi(\tau)}i\hbar\partial_t|\varphi(t)\rangle \\ &= -E_J \cos\varphi(t) e^{+\frac{i}{\hbar}E_J \int_0^t d\tau \cos\varphi(\tau)}|\varphi(t)\rangle - e^{+\frac{i}{\hbar}E_J \int_0^t d\tau \cos\varphi(\tau)} \sum_{m=-\infty}^{+\infty} e^{im\varphi(t)} m\hbar\dot{\varphi} |m\rangle \\ &= \cos\varphi(t) e^{+\frac{i}{\hbar}E_J \int_0^t d\tau \cos\varphi(\tau)} H_T |\varphi(t)\rangle + \cos\varphi(t) e^{+\frac{i}{\hbar}E_J \int_0^t d\tau \cos\varphi(\tau)} U |\varphi(t)\rangle \\ &= (H_T + U)|\Psi(t)\rangle. \end{aligned}$$

Ex. 19.10

(i)

The commutation relation (19.156) is similar to the canonical commutation relation $[x, p] = i\hbar$. Thus by analogy we can write

$$\hat{n} = i\frac{\partial}{\partial\varphi}$$

in the φ representation. It should be kept in mind though φ is a periodic variable with periodicity 2π . As a result eigen wave functions of \hat{n} takes the form

$$\psi_m(\varphi) = \sqrt{\frac{1}{2\pi}} e^{-im\varphi},$$

with eigenvalue m being an integer due to the periodicity of φ . Since $|\psi_m(\varphi)|^2 = \frac{1}{2\pi}$ is independent of φ , it has maximum uncertainty.

(ii) The eigenstates of $\hat{\varphi}$ are given in Eq. (19.133). Obviously the probability of finding the state in a given m is independent of m , thus \hat{n} has maximum uncertainty.

Ex. 19.11

Define $\Delta\varphi = \varphi_2 - \varphi_1$. Then

$$\begin{aligned} H &= -E_{J_1} \cos(\Delta\varphi - \theta) - E_{J_2} \cos\Delta\varphi \\ &= -(E_{J_1} \cos\theta + E_{J_2}) \cos\Delta\varphi - E_{J_1} \sin\theta \sin\Delta\varphi \\ &= -[(E_{J_1} \cos\theta + E_{J_2})^2 + E_{J_1}^2 \sin^2\theta]^{\frac{1}{2}} \cos(\Delta\varphi - \theta') \end{aligned},$$

where $\theta' = \tan^{-1} \left(\frac{E_{J_1} \sin\theta}{E_{J_1} \cos\theta + E_{J_2}} \right)$, and $\theta = 2\pi \frac{\Phi}{\Phi_s}$

$$\text{So } I_c = (I_{c_1}^2 + I_{c_2}^2 + 2I_{c_1}I_{c_2} \cos\theta)^{\frac{1}{2}}$$

Ex. 19.12

$$H_0 = +4E_c \hat{n}^2 + \frac{E_J}{2} \hat{\varphi}^2 \quad (226)$$

$$V \approx -E_J \hat{\varphi}^4 \quad (227)$$

$$(228)$$

where $\hat{n} = +i\partial_\varphi$ implies the commutation relation $[\hat{n}, \hat{\varphi}] = +i$. If $[a, a^\dagger] = +1$, then the following representation yields the required commutation relation

$$\hat{\varphi} = \varphi_{\text{ZPF}}(a + a^\dagger) \quad (229)$$

$$\hat{n} = \frac{i}{2\varphi_{\text{ZPF}}}(a - a^\dagger), \quad (230)$$

for any real constant φ_{ZPF} .

H_0 is the Hamiltonian of a harmonic oscillator with ‘mass’ M , ‘spring constant’ k , and frequency Ω_J given by

$$\frac{1}{2M} = \frac{4E_c}{\hbar^2} \quad (231)$$

$$k = E_J \quad (232)$$

$$\Omega_J = \sqrt{\frac{k}{M}} = \sqrt{\frac{8E_J}{E_c}}. \quad (233)$$

The quadratic Hamiltonian can be rewritten

$$H_0 = \frac{1}{2}\hbar\Omega_J \left\{ \sqrt{\frac{8E_c}{E_J}} \hat{n}^2 + \sqrt{\frac{E_J}{8E_c}} \dot{\varphi}^2 \right\} \quad (234)$$

$$= \frac{1}{2}\hbar\Omega_J \left\{ 2\varphi_{ZPF}^2 \hat{n}^2 + \frac{1}{2\varphi_{ZPF}^2} \dot{\varphi}^2 \right\}, \quad (235)$$

where we have defined the quantity

$$\varphi_{ZPF}^2 \equiv \sqrt{\frac{2E_c}{E_J}}. \quad (236)$$

It follows that

$$H_0 = \frac{\hbar\Omega_J}{4} \left\{ -(a - a^\dagger)^2 + (a + a^\dagger)^2 \right\} = \frac{\hbar\Omega_J}{2} (a^\dagger a + \frac{1}{2}). \quad (237)$$

The ground state wave function obeys $a\psi_0(\varphi) = 0$ which implies $\partial_\varphi\psi = -\frac{\varphi}{2\varphi_{ZPF}^2}\psi$. The wave function given in Eq. (19.177) in the textbook solves this equation.

From the above results it follows that the anharmonic term can be written

$$V = -\frac{E_J}{4!} \varphi_{ZPF}^4 (a + a^\dagger)^4 \quad (238)$$

$$= -\frac{E_c}{12} (a + a^\dagger)^4. \quad (239)$$

Expanding the last term in parentheses, keeping only terms with equal numbers of raising and lowering operators (i.e., making the RWA), and then normal ordering using the commutation relation yields

$$V \approx -\frac{E_c}{2} (a^\dagger a^\dagger a a + 2a^\dagger a). \quad (240)$$

The quadratic term renormalizes the harmonic oscillator frequency to $\omega_{01} = \Omega_J - E_c/\hbar$ and the first term gives the anharmonicity $\omega_{12} - \omega_{01} = -E_c$.

Chapter 20

Ex. 20.1

For simplicity we suppress the \vec{k} index.

$$\begin{aligned}
 [S^+, S^-] &= [C_\uparrow^\dagger C_\downarrow^\dagger, C_\downarrow C_\uparrow] \\
 &= C_\uparrow^\dagger C_\downarrow^\dagger C_\downarrow C_\uparrow - C_\downarrow C_\uparrow C_\uparrow^\dagger C_\downarrow^\dagger \\
 &= C_\uparrow^\dagger C_\uparrow C_\downarrow^\dagger C_\downarrow - C_\downarrow C_\downarrow^\dagger C_\uparrow C_\uparrow^\dagger \\
 &= n_\uparrow n_\downarrow - (1 - n_\downarrow) (1 - n_\uparrow) \\
 &= n_\uparrow + n_\downarrow - 1 = 2S^z \\
 [S^z, S^+] &= \frac{1}{2} [n_\uparrow + n_\downarrow, C_\uparrow^\dagger C_\downarrow^\dagger] \\
 &= \frac{1}{2} [n_\uparrow, C_\uparrow^\dagger] C_\downarrow^\dagger + \frac{1}{2} C_\uparrow^\dagger [n_\downarrow, C_\downarrow^\dagger] \\
 &= C_\uparrow^\dagger C_\downarrow^\dagger = S^z
 \end{aligned}$$

Similarly $[S^z, S^-] = -S^-$.

So these are indeed spin (1/2) operators.

Ex. 20.2

Again we suppress \vec{k} index.

$$\{\alpha, \alpha^\dagger\} = |u|^2 \{C_\uparrow, C_\uparrow^\dagger\} + |v|^2 \{C_\downarrow, C_\downarrow^\dagger\} = |u|^2 + |v|^2 = 1.$$

Similarly $\{\beta, \beta^\dagger\} = 1$.

$$\{\alpha, \beta\} = \{u C_\uparrow - v C_\downarrow^\dagger, u C_\downarrow + v C_\uparrow^\dagger\} = u v \{C_\uparrow, C_\uparrow^\dagger\} - u v \{C_\downarrow^\dagger, C_\downarrow\} = u v - u v = 0.$$

$$\{\alpha, \beta^\dagger\} = \{u C_\uparrow - v C_\downarrow^\dagger, u^* C_\downarrow^\dagger + v^* C_\uparrow\} = 0.$$

Obviously $\{\alpha, \alpha\} = \{\beta, \beta\} = 0$.

So these are fermion creation/annihilation operators.

Ex. 20.3

Consider a specific term in (20.39) and suppress \vec{k} index. We have (assuming all u 's and v 's are real)

$$\begin{aligned} H &= \epsilon [(u\alpha^\dagger + v\beta) (u\alpha + v\beta^\dagger) + (u\beta^\dagger - v\alpha) (u\beta - v\alpha^\dagger)] \\ &\quad - \Delta_0 [(u\alpha^\dagger + v\beta) (u\beta^\dagger - v\alpha) + (u\beta - v\alpha^\dagger) (u\alpha + v\beta^\dagger)] \\ &= [\epsilon (u^2 - v^2) + 2uv\Delta_0] (\alpha^\dagger \alpha + \beta^\dagger \beta) + [2uv\epsilon - \Delta_0 (u^2 - v^2)] (\alpha^\dagger \beta^\dagger + \beta \alpha) + \text{const.} \end{aligned}$$

Choose $\frac{uv}{u^2 - v^2} = \frac{\Delta_0}{2\epsilon}$, the $(\alpha^\dagger \beta^\dagger + \beta \alpha)$ term vanishes.

Neglecting const., we find

$$H = \sqrt{\epsilon^2 + \Delta_0^2} (\alpha^\dagger \alpha + \beta^\dagger \beta) = E (\alpha^\dagger \alpha + \beta^\dagger \beta).$$

Eq. (20.42) follows.

Ex. 20.4

$$D_s(E) dE = D(0) d\epsilon$$

$$D_s(E) = \frac{D(0)}{dE/d\epsilon} = \frac{D(0)}{\epsilon/\sqrt{\epsilon^2 + \Delta_0^2}} = D(0) \frac{E}{\sqrt{E^2 - \Delta_0^2}}$$

Ex. 20.5

$$\begin{aligned} \langle \Psi_R | C_{\gamma\uparrow}^\dagger C_{\bar{\gamma}\downarrow}^\dagger | \Psi_R \rangle &= \langle \Psi_R | (u_\gamma \alpha_\gamma^\dagger + v_r^* \beta_{\bar{\gamma}}) (u_\gamma \beta_{\bar{\gamma}}^\dagger - v_\gamma^* \alpha_\gamma) | \Psi_R \rangle \\ &= v_\gamma^* u_\gamma = \frac{\Delta_R^*}{2E_r} \end{aligned}$$

Similarly $\langle \Psi_L | C_{\bar{\rho}\downarrow} C_{\rho\uparrow} | \Psi_L \rangle = u_\rho^* v_\rho = \frac{\Delta_L}{2E_\rho}$.

Thus Eq. (20.56) and (20.57) follow.

Ex. 20.6

Let us first focus on the k integral and call it $I'(\theta) = \int k^2 dk$. Obviously in order for it to have a non-zero range, we need $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. The range is given by

$$|\vec{k} - \frac{\vec{q}}{2}|^2 = k^2 + \frac{q^2}{4} - kq \cos\theta < k_F^2$$

and

$$k^2 + \frac{q^2}{4} + kq \cos\theta > k_F^2$$

To $O(q^2)$, we find the range is

$$k_- = \left(k_F - \frac{q^2}{8k_F} \sin^2\theta \right) - q \cos\theta < k < \left(k_F - \frac{q^2}{8k_F} \sin^2\theta \right) + q \cos\theta = k_+$$

Thus,

$$\begin{aligned} I'(\theta) &= \int_{k_-}^{k_+} k^2 dk = \frac{1}{3} (k_+^3 - k_-^3) \\ &= q \left[k_F^2 \cos\theta - q^2 \left(\frac{\sin^2\theta \cos\theta}{4} - \frac{\cos^3\theta}{12} \right) \right] + O(q^5) \end{aligned}$$

and

$$I = \frac{L^2}{(2\pi)^2} \frac{1}{q} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I'(\theta) d\theta = \frac{k_F^2 L^2}{8\pi} \left[1 - \frac{1}{6} \frac{q^2}{k_F^2} + O(q^4) \right]$$

Ex. 20.7

$$\prod_{\vec{k}} \left(u_{\vec{k}} + e^{i\phi} v_{\vec{k}} C_{\vec{k}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger \right) |0\rangle \propto e^{i\phi \sum_{\vec{k}} g_{\vec{k}} C_{\vec{k}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger} |0\rangle = \sum_{N'=0}^{\infty} e^{iN' \phi} \left(\sum_{\vec{k}} g_{\vec{k}} C_{\vec{k}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger \right)^{N'} |0\rangle$$

Thus,

$$\int_0^{2\pi} d\phi e^{-iN\phi} \prod_{\vec{k}} \left(u_{\vec{k}} + e^{i\phi} v_{\vec{k}} C_{\vec{k}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger \right) |0\rangle \propto \left(\sum_{\vec{k}} g_{\vec{k}} C_{\vec{k}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger \right)^N |0\rangle$$

Ex. 20.8

Since there is no spin, we have to be very careful with the pairing term, as the term $\Delta(k)$ and $\Delta(-k) = -\Delta(k)$ are the same. To avoid double counting, we restrict to modes with $0 \leq k < \pi$:

$$\hat{H}_K = \sum_{0 \leq k < \pi} \left[\epsilon_k \left(C_k^\dagger C_k + C_{-k}^\dagger C_{-k} \right) - 2\Delta(k) C_k^\dagger C_{-k}^\dagger - 2\Delta^*(k) C_{-k} C_k \right]$$

Define $S_k^z = (C_k^\dagger C_k + C_{-k}^\dagger C_{-k} - 1)/2$, $S_k^+ = C_k^\dagger C_{-k}^\dagger$, and $S_k^- = C_{-k} C_k$. We find

$$\begin{aligned}\hat{H}_K &= \sum_{0 \leq k < \pi} [2\epsilon_k S_k^z - 2\Delta(k) S_k^+ - 2\Delta^*(k) S_k^-] + const. \\ &= - \sum_{0 \leq k < \pi} \vec{B}_k \cdot \vec{S}_k,\end{aligned}$$

with

$$\begin{aligned}B_k^z &= -2\epsilon_k = 4t \cos k + 2\mu \\ B_k^x &= 2\Delta(k) + 2\Delta^*(k) = 0 \\ B_k^y &= 2i\Delta(k) - 2i\Delta^*(k) = -4\Delta \sin k\end{aligned}$$

Obviously \vec{B}_k is restricted to the $y-z$ plane. Now restore k to the full 1BZ such that \vec{B}_k is a loop. We find this loop encloses the origin $\vec{B} = 0$ for $|\frac{\mu}{t}| < 2$ (non-trivial) phase, and it does not enclose the origin for $|\frac{\mu}{t}| > 0$ (trivial) phase.

Ex. 20.9

Let us choose \hat{e} to be \hat{x} . Then

$$E(\delta \vec{k}) = \sqrt{v_g^2 \delta k_x^2 + v_F^2 \delta k_y^2} = \sqrt{v_g v_F} \left(\frac{v_g}{v_F} \delta k_x^2 + \frac{v_F}{g} \delta k_y^2 \right)^{\frac{1}{2}}$$

Perform a rescaling: $\delta k'_x = \sqrt{\frac{v_g}{v_F}} \delta k_x$ and $\delta k'_y = \sqrt{\frac{v_F}{v_g}} \delta k_y$. We obtain an isotropic dispersion:

$$E(\delta \vec{k}') = \sqrt{v_g v_F} |\delta \vec{k}'|$$

just like in graphene with $\sqrt{v_g v_F}$ replacing v_F . Note $d\delta k_x d\delta k_y = d\delta k'_x d\delta k'_y$. Using results of Ex.7.31, we find its contribution to C_v is $\frac{9\xi(3)}{\pi} \frac{k_B^3}{\hbar^2 v_F v_g} T^2$.

Ex. 20.10

$$\begin{aligned}\chi_0(\vec{Q}, T) &= 2 \sum_{\vec{k}} \frac{f_{\vec{k}} - f_{\vec{k}+\vec{Q}}}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{Q}}} \\ &= 2 \int_{-\epsilon_m}^{\epsilon_m} D(\epsilon) \frac{f(\epsilon) - f(-\epsilon)}{\epsilon - (-\epsilon)} d\epsilon = 2 \int_0^{\epsilon_m} \frac{f(\epsilon) - f(-\epsilon)}{\epsilon} D(\epsilon) d\epsilon\end{aligned}$$

where ϵ_m is maximum band energy. At low T , the integral is dominated by small ϵ ; it would diverge logarithmically at $T = 0$ as long as $D(\epsilon = 0)$ is finite, which is the case here. At finite (small) T , it is cut off by $k_B T$, resulting in

$$\chi_0(\vec{Q}, T) \simeq -2D(0) \ln \frac{\epsilon_m}{k_B T}$$

Ex. 20.11

1. Let us start with $T = 0$, where the pseudospin mapping is appropriate, and we can study the response for each \vec{k} separately.

$$H_{\vec{k}} = 2\epsilon_{\vec{k}} S_{\vec{k}}^z - 2\Delta S_{\vec{k}}^x + \text{const}$$

$$\langle C_{\vec{k}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger \rangle = \langle S_{\vec{k}}^\dagger \rangle = \langle S_{\vec{k}}^x \rangle = \frac{\Delta}{\sqrt{\epsilon_{\vec{k}}^2 + \Delta^2}} \simeq \frac{\Delta}{|\epsilon_{\vec{k}}|}.$$

Thus, $\sum_{\vec{k}'} \langle C_{\vec{k}'\uparrow}^\dagger C_{-\vec{k}'\downarrow}^\dagger \rangle = \chi \cdot \Delta = \Delta \int_{-\epsilon_m}^{\epsilon_m} \frac{D(\epsilon)}{|\epsilon|} d\epsilon$. Here ϵ_m is maximum energy in the \vec{k} sum, which is divergent logarithmically if $D(\epsilon = 0)$ finite. At finite T , the response is cut off for $|\epsilon| \simeq k_B T$, as a result

$$\chi(T) \simeq 2D(0) \int_{k_B T}^{\epsilon_m} \frac{d\epsilon}{\epsilon} = 2D(0) \ln \frac{\epsilon_m}{k_B T}.$$

2. We again consider each \vec{k} separately.

$$H_{\vec{k}} = \epsilon_{\vec{k}+\vec{q}\uparrow} C_{\vec{k}+\vec{q}\uparrow}^\dagger C_{\vec{k}+\vec{q}\uparrow} + \epsilon_{-\vec{k}\downarrow} C_{-\vec{k}\downarrow}^\dagger C_{-\vec{k}\downarrow} - \Delta_{\vec{q}} \left(C_{\vec{k}+\vec{q}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger + C_{-\vec{k}\downarrow} C_{\vec{k}+\vec{q}\uparrow} \right),$$

$$\epsilon_{\vec{k}\uparrow,\downarrow} = \frac{\hbar^2 k^2}{2m} - \mu_{\uparrow,\downarrow}.$$

For $\epsilon_{\vec{k}+\vec{q}\uparrow} \cdot \epsilon_{-\vec{k}\downarrow} < 0$, $\Delta_{\vec{q}}$ has no effect due to Pauli blocking. We can thus focus on the \vec{k} 's with $\epsilon_{\vec{k}+\vec{q}\uparrow} \cdot \epsilon_{-\vec{k}\downarrow} > 0$, where we can again use the pseudo spin mapping:

$$S_{\vec{k}}^z = \left(C_{\vec{k}+\vec{q}\uparrow}^\dagger C_{\vec{k}+\vec{q}\uparrow} + C_{-\vec{k}\downarrow}^\dagger C_{-\vec{k}\downarrow} \right) / 2$$

$$S_{\vec{k}}^+ = C_{\vec{k}+\vec{q}\uparrow}^\dagger C_{-\vec{k}\downarrow}^\dagger$$

$$S_{\vec{k}}^- = C_{-\vec{k}\downarrow} C_{\vec{k}+\vec{q}\uparrow}$$

As a result,

$$H_{\vec{k}} = \left(\epsilon_{\vec{k}+\vec{q}\uparrow} + \epsilon_{-\vec{k}\downarrow} \right) S_{\vec{k}}^z - 2\Delta_{\vec{q}} S_{\vec{k}}^x + \text{const.}$$

$$+ \left(\epsilon_{\vec{k}+\vec{q}\uparrow} - \epsilon_{-\vec{k}\downarrow} \right) \left(C_{\vec{k}+\vec{q}\uparrow}^\dagger C_{\vec{k}+\vec{q}\downarrow} - C_{-\vec{k}\downarrow}^\dagger C_{-\vec{k}\downarrow} \right)$$

The last term is 0 for $\epsilon_{\vec{k}+\vec{q}\uparrow} \cdot \epsilon_{-\vec{k}\downarrow} > 0$ because we have $C_{\vec{k}+\vec{q}\uparrow}^\dagger C_{\vec{k}+\vec{q}\downarrow} = C_{-\vec{k}\downarrow}^\dagger C_{-\vec{k}\downarrow}$ at $T = 0$. As a result, we find

$$\chi(\vec{q}) = \sum_{\vec{k}}'' \frac{2}{\epsilon_{\vec{k}+\vec{q}\uparrow} + \epsilon_{-\vec{k}\downarrow}}$$

where \sum'' is over \vec{k} 's that have $\epsilon_{\vec{k}+\vec{q}\uparrow} \cdot \epsilon_{-\vec{k}\downarrow} > 0$, and also within the range that $\Delta \neq 0$. The first restriction renders the density of such \vec{k} 's going to zero as the denominator goes to zero, rendering the sum/integral finite. So $\chi(q)$ is finite at $T = 0$. Also this density goes to zero the slowest for $q = k_F\uparrow - k_F\downarrow$, as a result the leading pairing in stability is of the FFLO type.

Ex. C.1

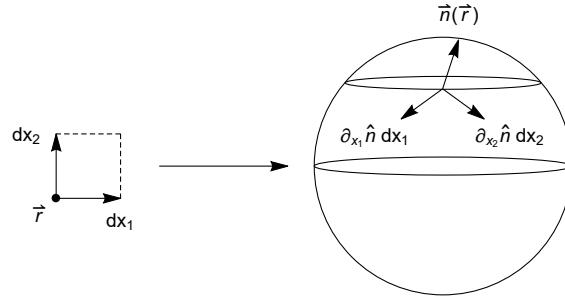
For non-interacting electrons, many-body eigenstates $|i\rangle$ and $|f\rangle$ differ by occupation of a single-particle eigenstate $|\psi_\alpha\rangle$, whose energy is ϵ_α , measured from chemical potential μ . The first and second terms in (C.17) correspond to cases with $|\psi_\alpha\rangle$ being occupied and empty in $|f\rangle$ respectively, and share the same $\delta(\epsilon - \epsilon_\alpha)$ factor; as a result

$$\sum_f \{ \dots \} = \sum_\alpha |\langle l | \psi_\alpha \rangle|^2 \delta(\epsilon - \epsilon_\alpha),$$

and the summation of initial state i simply cancel the normalization factor $1/Z$, resulting in agreement with (11.160).

Ex. D.1

Consider $dA = dx_1 dx_2$ is the area of an infinitesimal rectangle on the surface S . Then under the mapping $\vec{n}(\vec{r})$, it maps onto a parallelogram on unit sphere spanned by $\frac{\partial \hat{n}(\vec{r})}{\partial x_1} dx_1$ and $\frac{\partial \hat{n}(\vec{r})}{\partial x_2} dx_2$, whose area is $\hat{n} \cdot [\partial_{x_1} \hat{n}(\vec{r}) \times \partial_{x_2} \hat{n}(\vec{r})] dx_1 dx_2$. The sign of this area is determined by the orientation of the parallelogram compared with that of the rectangle $dx_1 dx_2$. Thus $d\Omega = \hat{n} \cdot [\partial_{x_1} \hat{n}(\vec{r}) \times \partial_{x_2} \hat{n}(\vec{r})] dx_1 dx_2$ and (D.14) follows from (D.13).



Ex. F.1

$$1. (MM^\dagger)_{ij} = \sum_\rho M_{i\rho} M_{\rho j}^\dagger = \sum_\rho M_{i\rho} M_{j\rho}^* = \sum_\rho M_{j\rho}^* \left(M_{\rho i}^\dagger \right)^* = \left[(MM^\dagger)_{ji} \right]^*$$

2. Assume $|a\rangle$ is a normalized eigenvector of $M^\dagger M$, with eigenvalue α . Then

$$\alpha = \langle a | M^\dagger M | a \rangle = \langle a' | a' \rangle \geq 0$$

where $|a\rangle = M|a\rangle$. Similar for MM^\dagger .

$$\begin{aligned}
3. \langle \Psi | \Psi \rangle &= 1 = \sum_{i,j} |M_{ij}|^2 \\
&= \sum_i \sum_j M_{ij} M_{ij}^* = \sum_i \sum_j M_{ij} (M^\dagger)_{ij} \\
&= \sum_i (MM^\dagger)_{ii} = \text{tr}(MM^\dagger)
\end{aligned}$$

Ex. F.2

Intuitively it is clear the entropy is maximized when $p_\rho = |S_\rho|^2$ is the same for all $\rho = 1, \dots, r$, and from the normalization $\sum_{\rho=1}^r p_\rho = 1 \Rightarrow p_\rho = \frac{1}{r}$, thus $S_E = \sum_{\rho=1}^r \frac{1}{r} \ln r = \ln r$.

More formally we need to maximize

$$S_E = - \sum_{\rho=1}^r p_\rho \ln p_\rho$$

under the constraint $\sum_\rho p_\rho = 1$. Thus

$$\frac{\partial S_E}{\partial p_\rho} = -\ln p_\rho - 1 = \epsilon,$$

where ϵ is a Lagrange multiplier. Thus all p_ρ 's are equal, leading to the same result.

Ex. F.3

Since P_A is positive semi-definite, we can write

$$\rho_A = \sum_\ell \rho_\ell |\ell\rangle\langle\ell| = \sum_\ell e^{-\ln \rho_\ell} |\ell\rangle\langle\ell|$$

where $|\ell\rangle$ is an eigenstate of ρ_A and $\rho_\ell \geq 0$ is its eigenvalue. Thus we can define

$$H_A = \sum_\ell -\ln \rho_\ell |\ell\rangle\langle\ell| = \sum_\ell \epsilon_\ell |\ell\rangle\langle\ell|, \text{ resulting in}$$

$$\rho_A = e^{-H_A}.$$

$\epsilon_\ell = -\ln \rho_\ell$ is known as entanglement energy, and $\{\epsilon_\ell\}$ form the entanglement spectrum. For $\rho_\ell = 0$, we define ϵ_ℓ to be $+\infty$.

Ex. I.1

Up to (I.20) bosons and fermions are the same. Things start to change from (I.21), as two parallel spin bosons can occupy the same site. We find $\tilde{H}|\uparrow\uparrow\rangle = -\frac{4t^2}{U}|\uparrow\uparrow\rangle$, $\tilde{H}|\downarrow\downarrow\rangle = -\frac{4t^2}{U}|\downarrow\downarrow\rangle$, $\langle\uparrow\downarrow|\tilde{H}|\uparrow\downarrow\rangle = \langle\downarrow\uparrow|\tilde{H}|\downarrow\uparrow\rangle = -\frac{2t^2}{U} = \langle\uparrow\downarrow|\tilde{H}|\downarrow\uparrow\rangle = \langle\downarrow\uparrow|\tilde{H}|\uparrow\downarrow\rangle$.

As a result,

$$\begin{aligned}
\tilde{H} &= -\frac{2t^2}{U} \{\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+ + \frac{1}{2} (\sigma_1^z \sigma_2^z + 3)\} = -\frac{t^2}{U} \{\vec{\sigma}_1 \cdot \vec{\sigma}_2 + 3\} = -J \vec{S}_1 \cdot \vec{S}_2 - C \\
\text{with } J &= \frac{4t^2}{\hbar^2 U} \text{ and } C = -\frac{3t^2}{U}.
\end{aligned}$$