Discrete and continuous mechanical systems (3)

single particle dynamics → can be inferred from Lagrange's equation of motion

q_i = generalized coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \qquad 1)$$

which is derived from Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0 \qquad 2)$$

L is the Lagrangian (assumed here not to depend explicitly on time)

$$L = T - V \qquad 3)$$

T = kinetic energy

V = potential energy

The variation from t_1 to t_2 in Eq 2) is to be taken over an arbitrary path $q_i(t)$ such that

$$\delta q_i(t_1) = \delta q_i(t_2) = 0$$

The Hamiltonian of the system can be obtained from the Lagrangian by performing a Legendre transformation

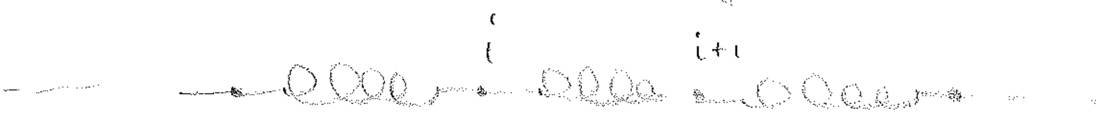
$$H = \sum_i p_i \dot{q}_i - L$$

where p_i is the "canonical conjugate" of q_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

All of these considerations can be generalized to a system with many particles.

Let's consider N particles connected with identical springs with constant k, aligned in 1-dim.



$$L = \frac{1}{2} \sum_{i=1}^{N} \left[m \dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2 \right] =$$

η_i = displacement of the ith particle from its equilibrium position.

$$= \sum_{i=1}^{N} a \, \frac{1}{2} \left[\frac{m}{a} \dot{\eta}_i^2 - ka \left(\frac{\eta_{i+1} - \eta_i}{a} \right)^2 \right] =$$

$$= \sum_i^N a \, \mathcal{L}_i$$

a = separation distance between the equilibrium positions of two neighboring particles

$\mathcal{L}_i$ = linear Lagrangian density, that is Lagrangian density per unit length.

From the discrete to the continuous:

n° of degrees of freedom → +∞ ⇒ separation → infinitesimal

$a \to dx$; $\frac{m}{a} \to \mu$ = linear mass density

$\frac{\eta_{i+1} - \eta_i}{a} \to \frac{\partial \eta}{\partial x}$; $ka \Rightarrow Y$ Young's modulus

Now we have that

$$L = \int \mathcal{L}\, dx$$

where $\mathcal{L} = \frac{1}{2}\left[\mu \dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]$

$\eta = \eta(x,t)$ function of continuous parameters x, t

η = generalized coordinate ↔ q_i in L

• Variational principle in the continuous case

$$\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} dt \int dx\, \mathcal{L}\left(\eta, \dot{\eta}, \frac{\partial \eta}{\partial x}\right)$$