Homework 8

Problem 10.5

- we'll have:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -a \\ Ce^{lx} + De^{-lx}, & -a \leq x \leq 0 \\ 0, & x > 0 \end{cases}$$

- can rewrite due to linearity of SE:

$$Ce^{lx} + De^{-lx} \rightarrow C\sinh(lx) + D\cosh(lx),$$

w/ $k = \sqrt{2mE}/\hbar$, $l = \sqrt{2m(E+V_0)}/\hbar$

- continuity at $x=0$: $D=0$
- continuity at $x=-a$ for ψ & ψ':

$$Ae^{-ika} + Be^{ika} = -C\sinh(la)$$

$$ikAe^{-ika} - ikBe^{ika} = Cl\cosh(la)$$

$$\rightarrow \frac{ik\left[Ae^{-ika} - Be^{ika}\right]}{Ae^{-ika} + Be^{ika}} = -l\coth(la)$$

$$ik\left[Ae^{-ika} - Be^{ika}\right] = -lAe^{-ika}\coth(la) - lBe^{ika}\coth(la)$$

$$lBe^{ika}\coth(la) - ikBe^{ika} = -lAe^{-ika}\coth(la) - ikAe^{-ika}$$

$$B\left[-ik + l\coth(la)\right]e^{ika} = A\left[-ik - l\coth(la)\right]e^{-ika}$$

$$B = A\left[\frac{ik + l\coth(la)}{ik - l\coth(la)}\right]e^{-2ika}$$

$$\Rightarrow \boxed{\psi_r(x) = A\left[\frac{ik + l\coth(la)}{ik - l\coth(la)}\right]e^{i(-2ka-kx)}}$$

b) $$|B|^2 = |A|^2\left(\frac{\alpha + i\beta}{-\alpha + i\beta}\right)\left(\frac{\alpha - i\beta}{-\alpha - i\beta}\right) = |A|^2 \checkmark$$

c) Eq. 10.40: $$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right)$$

- obviously:

$$2\delta = -2ka \rightarrow \boxed{\delta = -ka}$$

Problem 10.6

→ we know $a_\ell = i \dfrac{j_\ell(ka)}{k h_\ell^{(1)}(ka)}$

and $a_\ell = \dfrac{1}{k} e^{i\delta_\ell} \sin(\delta_\ell)$

$$\rightarrow \frac{i}{k} \frac{j_\ell(ka)}{j_\ell(ka) + i n_\ell(ka)} = \frac{1}{k} e^{i\delta_\ell} \sin(\delta_\ell)$$

$$i \frac{j_\ell(ka)}{j_\ell(ka) + i n_\ell(ka)} = \cos(\delta_\ell)\sin(\delta_\ell) + i \sin^2(\delta_\ell)$$

$$i \frac{j_\ell(ka)\left[j_\ell(ka) - i n_\ell(ka)\right]}{[j_\ell(ka)]^2 - [n_\ell(ka)]^2} =$$

$$\frac{i[j_\ell(ka)]^2 + j_\ell(ka) n_\ell(ka)}{[j_\ell(ka)]^2 - [n_\ell(ka)]^2} = \sim$$

$$\frac{i + \left[\dfrac{n_\ell(ka)}{j_\ell(ka)}\right]}{1 - \left[\dfrac{n_\ell(ka)}{j_\ell(ka)}\right]} \equiv \frac{i + \alpha}{1 - \alpha^2} = \cos(\delta_\ell)\sin(\delta_\ell) + i \sin^2(\delta_\ell)$$

- equate real/imaginary components

$$\rightarrow \begin{cases} \dfrac{\alpha}{1-\alpha^2} = \cos(\delta_\ell)\sin(\delta_\ell) \\ \dfrac{1}{1-\alpha^2} = \sin^2(\delta_\ell) \end{cases}$$

$$\rightarrow \frac{1}{\alpha} = \tan(\delta_\ell)$$

$$\Rightarrow \boxed{\delta_\ell = \tan^{-1}\left[\frac{n_\ell(ka)}{j_\ell(ka)}\right]}$$

Problem 16.12

$$-\int_0^\infty e^{-\mu r}\sin(kr)\,dr \qquad u=\sin(kr)\quad dv=e^{-\mu r}dr \qquad du=k\cos(kr)dr\quad v=-\frac{1}{\mu}e^{-\mu r}$$

$$I=-\frac{1}{\mu}\Big[e^{-\mu r}\cancel{\sin(kr)}\Big]_0^\infty{}^{\,0}+\frac{k}{\mu}\int_0^\infty e^{-\mu r}\cos(kr)\,dr$$

$$u=\cos(kr)\quad dv=e^{-\mu r}dr$$
$$du=-k\sin(kr)dr\quad v=-\frac{1}{\mu}e^{-\mu r}$$

$$I=\frac{k}{\mu}\left\{-\frac{1}{\mu}\Big[e^{-\mu r}\cos(kr)\Big]_0^\infty-\frac{k}{\mu}\int_0^\infty e^{-\mu r}\sin(kr)\,dr\right\}$$

$$I=\frac{k}{\mu^2}-\frac{k^2}{\mu^2}I$$

$$\left(1+\frac{k^2}{\mu^2}\right)I=\frac{-k}{\mu^2}$$

$$I=\frac{-k}{\mu^2}\cdot\frac{\mu^2}{k^2+\mu^2}=\frac{-k}{k^2+\mu^2}$$

$$\Rightarrow\ f(\theta)=\frac{-2m\beta}{\hbar^2k}\cdot\frac{-k}{k^2+\mu^2}=\frac{-2m\beta}{\hbar^2(k^2+\mu^2)}$$

$$\frac{d\sigma}{d\Omega}=\left[\frac{2m\beta}{\hbar^2(k^2+\mu^2)}\right]^2$$

$$\sigma=2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\int_0^\pi\frac{\sin\theta\,d\theta}{[4k^2\sin^2\theta/2+\mu^2]^2}$$

$$\sin^2\frac{\theta}{2}=\frac{1}{2}(1-\cos\theta)$$

$$\sigma=2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\int_0^\pi\frac{\sin\theta\,d\theta}{\{2k^2(1-\cos\theta)+\mu^2\}^2}$$

$$u=1-\cos\theta\qquad du=\sin\theta\,d\theta$$

$$\pi\to1-\cos(\pi)=2$$
$$0\to1-\cos(0)=0$$

$$\sigma=2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\int_0^2\frac{du}{[2k^2u+\mu^2]^2}$$

$$=2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{2k^2}\left[\frac{1}{2k^2u+\mu^2}\right]_0^2$$

$$=\sim\left[\frac{1}{4k^2+\mu^2}-\frac{1}{\mu^2}\right]$$

$$=\left(\frac{2m\beta^2}{\hbar^2}\right)\frac{\pi}{k^2}\left[\frac{\mu^2-4k^2-\mu^2}{\mu^2(4k^2+\mu^2)}\right)$$

$$=\left(\frac{2m\beta}{\mu\hbar^2}\right)^2\cdot\frac{4\pi}{\mu^2+4k^2}$$

$$=\pi\left(\frac{4m\beta}{\mu\hbar^2}\right)^2\frac{1}{\mu^2+4\cdot\frac{2mE}{\hbar^2}}$$

$$\boxed{\sigma=\pi\left(\frac{4m\beta}{\mu\hbar}\right)^2\frac{1}{(\mu\hbar)^2+8mE}}$$

Problem 10.20

$$f(\theta) \approx \frac{-2m}{\hbar^2 \kappa} \int_0^\infty r V(r) \sin(\kappa r)\, dr \,,$$

$$\kappa = 2K \sin(\theta/2)$$

$$\rightarrow \quad f(\theta) = \frac{-2mV_0}{\hbar^2 \kappa} \int_0^\infty \underbrace{r e^{-\mu r^2/a^2}} \sin(\kappa r)\, dr$$

- Can try to reduce to get rid of the r:

$$\frac{d}{dr}\left(e^{-\mu r^2/a^2} \right) = -\frac{2\mu r}{a} e^{-\mu r^2/a^2}$$

$$\rightarrow \quad f(\theta) = \frac{2mV_0}{\hbar^2 \kappa} \cdot \frac{a}{2\mu} \int_0^\infty \frac{d}{dr}\left(e^{-\mu r^2/a^2} \right) \sin(\kappa r)\, dr$$

$u = \sin(\kappa r)$ $dv = dr$

$du = \kappa \cos(\kappa r)$ $v = e^{-\mu r^2/a^2}$

$$f(\theta) = \frac{maV_0}{\hbar^2 \kappa \mu} \left\{ \left[\cancel{\sin(\kappa r) e^{-\mu r^2/a^2}} \right]_0^\infty - \kappa \int_0^\infty e^{-\mu r^2/a^2} \cos(\kappa r)\, dr \right\}$$

$$= \frac{maV_0}{\hbar^2 \mu} \int_0^\infty e^{-\mu r^2/a^2} \cos(\kappa r)\, d$$

$$\cos(\kappa r) = \mathrm{Re}\left[e^{i\kappa r} \right]$$

$$\rightarrow \quad = \frac{maV_0}{\hbar^2 \mu} \mathrm{Re} \int_0^\infty e^{-\mu r^2/a^2 + i\kappa r}\, dr$$

$$-\frac{\mu}{a^2} r^2 + i\kappa r \,, \quad \alpha^2 = \frac{a^2}{\mu}$$

$$\rightarrow -\frac{1}{\alpha^2}\left[r^2 - i\kappa\alpha^2 r - \frac{(\kappa\alpha^2)^2}{4} + \frac{(\kappa\alpha^2)^2}{4} \right]$$

$$-\frac{1}{\alpha^2}\left(r - \frac{i\kappa\alpha^2}{2} \right)^2 - \frac{(\kappa\alpha)^2}{4} \qquad \frac{(\kappa a)^2}{4\mu}$$

$$= \frac{maV_0}{\hbar^2 \mu} e^{-(\kappa a)^2/4\mu} \mathrm{Re} \int_0^\infty e^{-u^2/\alpha^2}\, du$$

$$= \frac{maV_0}{2\hbar^2 \mu} e^{-(\kappa a)^2/4\mu} \sqrt{\pi} \left(\frac{\alpha}{2} \right)$$

$$f(\theta) = \frac{ma^2 V_0 \sqrt{\pi}}{4\hbar^2 \mu^{3/2}} e^{-(\kappa a)^2/4\mu}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \rightarrow \sigma = 2\pi \frac{m^2 a^4 V_0^2 \pi}{16\hbar^4 \mu^3} \int_0^\pi e^{-(\kappa a)^2/2\mu} \cdot \sin\theta\, d\theta$$

$$= \frac{1}{\mu}\left(\frac{\pi m a^2 V_0}{4\hbar^2 \mu} \right)^2 \int_0^\pi e^{2(\kappa a)^2 \sin^2(\theta/2)/\mu} \sin\theta\, d\theta$$

$$= \sim \int_0^\pi e^{(\kappa a)^2(1-\cos\theta)/\mu} \sin\theta\, d\theta$$

$u = 1 - \cos\theta$

$du = \sin\theta\, d\theta$

$$= \sim \int_0^2 e^{(\kappa a)^2 u/\mu}\, du$$

$$= \sim \frac{\mu}{(\kappa a)^2} \left[e^{(\kappa a)^2 u/\mu} \right]_0^2$$

$$\boxed{\sigma = \left(\frac{\pi m a V_0}{4\kappa \hbar^2 \mu} \right)^2 \left(e^{2(\kappa a)^2/\mu} - 1 \right)}$$