$\mathcal{L}_i$ = linear Lagrangian density, that is Lagrangian density per unit length.

From the discrete to the continuous:

n° of degrees of freedom $\rightarrow +\infty$ $\Rightarrow$ separation $\rightarrow$ infinitesimal

$a \rightarrow dx$; $\quad \frac{m}{a} \rightarrow \mu$ = linear mass density

$\frac{\eta_{i+1} - \eta_i}{a} \rightarrow \frac{\partial \eta}{\partial x}$; $\quad ka \Rightarrow Y$ Young's modulus

Now we have that

$$L = \int \mathcal{L}\, dx$$

where $\quad \mathcal{L} = \frac{1}{2}\left[\mu \dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]$

$\eta = \eta(x,t)$ function of continuous parameters x, t

η = generalized coordinate $\leftrightarrow q_i$ in L

- Variational principle in the continuous case

$$\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} dt \int dx\, \mathcal{L}\left(\eta, \dot{\eta}, \frac{\partial \eta}{\partial x}\right) = \delta S = 0$$

$$\delta \int L\,dt = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta} \delta \eta + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial x}\right)} \underbrace{\delta\left(\frac{\partial \eta}{\partial x}\right)}_{\frac{\partial \delta \eta}{\partial x}} + \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial t}\right)} \underbrace{\delta\left(\frac{\partial \eta}{\partial t}\right)}_{\frac{\partial \delta \eta}{\partial t}} \right\}$$

Using integration by parts on the last two terms

$$\delta \int L\,dt = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta} \delta \eta - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial x}\right)} \right) \delta \eta - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial t}\right)} \right) \delta \eta \right\}$$

where in the integration by parts we used the fact that $\delta\eta$ vanishes at the end points of the space-time interval.

Therefore we obtain that

$\delta \int L\,dt = 0$ for any arbitrary variation as

Euler-Lagrange equation $\Rightarrow$ $$\boxed{\frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial x}\right)} + \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \eta}{\partial t}\right)} - \frac{\partial \mathcal{L}}{\partial \eta} = 0}$$

For the particular example $\mathcal{L} = \frac{1}{2}\left[\mu \dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right]$

we obtain that the Euler-Lagrange equation is the "equation of motion"

$$Y \frac{\partial^2 \eta}{\partial x^2} - \mu \frac{\partial^2 \eta}{\partial t^2} = 0 .$$

This is the wave equation in 1-dim.

It represents the 1-dim propagation of a disturbance with velocity $\sqrt{Y/\mu}$.
In analogy with

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

one can define the Hamiltonian density $\mathcal{H}$

$$\mathcal{H} = \dot{\eta} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \mathcal{L}$$

which, for the previous example, gives

$$\mathcal{H} = \frac{1}{2} \mu \dot{\eta}^2 + \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x} \right)^2$$

$\frac{\partial \mathcal{L}}{\partial \dot{\eta}}$ = canonical momentum conjugate to η.
it is often denoted by π.