Homework 7

Problem 1:

- Not sure what I have to show here... Legendre polynomials are orthogonal, so the coefficients are independently zero:

$$i^l(2l+1)\left[j_l(ka) + ik\, a_l(ka)\, h_l^{(1)}(ka)\right] = 0$$

$$j_l(ka) = -ik\, a_l(ka)\, h_l^{(1)}(ka) = 0$$

$$\rightarrow \boxed{a_l = i \frac{j_l(ka)}{k h_l^{(1)}(ka)} = 0}$$

Problem 10.4

$$V(r) = \alpha\delta(r-a)$$

- We know:

$$\psi(r,\theta) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr) + ika_l h_l^{(1)}(kr)\right]P_l(\cos\theta)$$

for the exterior region.

- Taking $l=0$ only:

$$\psi_{out}(r) = A\left[j_0(kr) + ika_0 h_0^{(1)}(kr)\right]P_0(\cos\theta).$$

$$j_0(kr) = \frac{\sin(kr)}{kr}$$

$$h_0^{(1)}(kr) = -i\frac{e^{ikr}}{kr}$$

$$P_0(\cos\theta) = 1$$

$$\rightarrow \psi_{out}(r) = A\left(\frac{\sin(kr)}{kr} + a_0\frac{e^{ikr}}{r}\right)$$

- The inside w.f. is non-zero this time, but since exponential term in ψ_{out} explodes for $r \to 0$, then:

$$\psi_{in}(r) = B\frac{\sin(kr)}{kr}$$

- of course, $\psi(r)$ must be continuous:

$$\psi_{out}(a) = \psi_{in}(a)$$

$$A\left(\frac{\sin(ka)}{ka} + a_0\frac{e^{ika}}{a}\right) = B\frac{\sin(ka)}{ka}$$

- We also know, from Ch. 2:

$$\Delta\left(\frac{d\psi}{dr}\right) = \frac{2m\alpha}{\hbar^2}\psi(a) = \frac{\beta}{a}\psi_{in}(a)$$

$$\rightarrow \frac{d\psi_{out}}{dr}\bigg|_{r=a} = A\left[-\frac{\sin(kr)}{kr^2} + \frac{\cos(kr)}{r} - a_0\frac{e^{ikr}}{r^2} + ika_0\frac{e^{ikr}}{r}\right]_{r=a}$$

$$= A\left[\frac{\cos(ka)}{a} + ika_0\frac{e^{ika}}{a} - \frac{1}{a}\left(\frac{\sin(ka)}{ka} + a_0\frac{e^{ika}}{a}\right)\right]$$

$$= A\left(\frac{\cos(ka)}{a} + ika_0\frac{e^{ika}}{a}\right) - \frac{1}{a}\psi_{out}(a)$$

$$\rightarrow \frac{d\psi_{in}}{dr}\bigg|_{r=a} = B\left(-\frac{\sin(kr)}{kr^2} + \frac{\cos(kr)}{r}\right)_{r=a}$$

$$= B\left(\frac{\cos(ka)}{a} - \frac{1}{a}\frac{\sin(ka)}{ka}\right)$$

$$= B\frac{\cos(ka)}{a} - \frac{1}{a}\psi_{in}(a)$$

since $\psi_{out}(r) = \psi_{in}(r)$

$$\Rightarrow A\left(\frac{\cos(ka)}{a} + ika_0\frac{e^{ika}}{a}\right) - \frac{1}{a}\cancel{\psi_{out}}(a) - B\frac{\cos(ka)}{a} + \frac{1}{a}\cancel{\psi_{in}}(a) = \frac{\beta}{a}B\frac{\sin(ka)}{ka}$$

$$\Rightarrow \begin{cases} A\left(\cos(ka) + ika_0 e^{ika}\right) = B\left(\beta\frac{\sin(ka)}{ka} + \cos(ka)\right) \\ A\left(\frac{\sin(ka)}{ka} + a_0\frac{e^{ika}}{a}\right) = B\frac{\sin(ka)}{ka} \end{cases}$$

w/ $ka \ll 1$, $\cos(ka) \approx 1$, $\sin(ka) \approx ka$

$$\rightarrow \begin{cases} A(1 + ika_0) = B(\beta + 1) \\ A\left(1 + \frac{a_0}{a}\right) = B \end{cases}$$

$$\rightarrow \frac{1 + ika_0}{1 + \frac{a_0}{a}} = \beta + 1$$

$$1 + ika_0 = (\beta+1)\left(1 + \frac{a_0}{a}\right)$$

$$= \beta + \beta\frac{a_0}{a} + 1 + \frac{a_0}{a}$$

$$ika_0 - \beta\frac{a_0}{a} - \frac{a_0}{a} = \beta$$

$$a_0\left[ik - (\beta+1)\right] = \beta$$

$$a_0 = \frac{\beta}{ik - (\beta+1)}$$

$$= \frac{\beta a}{ika - (\beta+1)}$$

$$\underline{a_0 \approx \frac{-a\beta}{\beta+1}}$$

- Now, $f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l P_l(\cos\theta)$

$$\rightarrow a_0,$$

so $$\underline{f(\theta) = \frac{-a\beta}{\beta+1}}$$

$$\underline{D(\theta) = |f(\theta)|^2 = \left(\frac{a\beta}{\beta+1}\right)^2}$$

$$\rightarrow \sigma = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = 4\pi a_0^2$$

$$\rightarrow \boxed{\sigma = 4\pi\left(\frac{a\beta}{\beta+1}\right)^2}$$