Homework 2

Due Date: Feb 4, 2025

2.1 δ-function Potential.

(a) Show that the reflection coefficient (R) and the transmission coefficient (T) of a quantum particle of mass m and energy E scattering from a potential $V(x) = \lambda\delta(x)$, with $\lambda > 0$, in one dimension are given by

$$R = \frac{1}{1 + (k\alpha)^2} \quad \text{and} \quad T = \frac{(k\alpha)^2}{1 + (k\alpha)^2}\ , \tag{1}$$

where the wavevector $k = \sqrt{2mE}/\hbar$ and $\alpha = \hbar^2/(m\lambda)$.

(b) What are the units of λ, k, and α?

2.2 Position and Momentum Basis.

(a) Starting from $\langle \boldsymbol{x}|\hat{\boldsymbol{p}}|\boldsymbol{p}\rangle$ obtain the differential equation

$$\vec{\nabla}\langle \boldsymbol{x}|\boldsymbol{p}\rangle = \frac{i\boldsymbol{p}}{\hbar}\langle \boldsymbol{x}|\boldsymbol{p}\rangle. \tag{2}$$

(b) Show that

$$\langle \boldsymbol{x}|\boldsymbol{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}. \tag{3}$$

(c) Show that

$$\langle \boldsymbol{k}|\boldsymbol{x}\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{4}$$

2.3 The Harmonic Oscillator.

(a) Consider the one dimensional harmonic oscillator Hamiltonian discussed in class

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\ . \tag{5}$$

Show that this Hamiltonian can be written as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\ , \tag{6}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators.

(b) Show that the eigenkets of the number operator are also eigenkets $\hat{H}$.

(c) Show that $\hat{H}|n\rangle = E_n|n\rangle$ where $|n\rangle$ are the eigenkets of $\hat{N}$ and $E_n = \hbar\omega(n + 1/2)$ with $n = 0, 1, 2, \ldots$

(d) Show that

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{and} \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle. \tag{7}$$

(e) Show that

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle\ . \tag{8}$$

2.4 The Translation Operator. The translation operator for a finite (spatial) displacement a in the x-direction is given by

$$\hat{T}(a) = \exp\left(\frac{-i\hat{p}_x a}{\hbar}\right) . \tag{9}$$

(a) Evaluate the commutator $[\hat{x}, T(a)]$.

(b) Using the result in (a) demonstrate how the expectation value $\langle \hat{x} \rangle$ changes under translation.

2.5 The Born-Oppenheimer Approximation. Consider a system of two particles of masses M (heavy) and m (light), interacting through a potential:

$$V(x_1, x_2) = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 + \frac{1}{2}k_{12}(x_1 - x_2)^2,$$

where x_1 and x_2 are the displacements of the heavy and light particles, respectively. The coupling constant k_{12} links the two oscillators.

(a) Write the Hamiltonian of the system in a real space.

(b) Apply the Born-Oppenheimer approximation by treating x_1 as slow: solve for the light particle, then the heavy particle using the effective potential to find approximate eigenvalues.

(c) Solve the Schrödinger equation to find the exact energy eigenvalues.

(d) Compare the energy levels obtained from the Born-Oppenheimer approximation to the exact energy levels for:

   (1) The weak coupling $k_{12} \ll k_1, k_2$.

   (2) The strong coupling $k_{12} \sim k_1, k_2$.