Homework 4

Problem 7.6

$$V = \tfrac{1}{2}m\omega^2x^2 \underbrace{-qEx}$$

a) we calculate E_1:

$$E_1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle$$

$$= -qE \cdot \frac{2}{a} \langle \psi_n^0 | \hat{x} | \psi_n^0 \rangle$$

w/ $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$:

$$= -qE \frac{2}{a}\sqrt{\frac{\hbar}{2m\omega}} \langle \psi_n^0 | (\hat{a}_+ + \hat{a}_-) | \psi_n^0 \rangle$$

both terms will be different n, so $E_1 = 0$.

- for 2^{nd} order case:

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | H' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

in general, $\langle \psi_m^0 | H' | \psi_n^0 \rangle = -qE\sqrt{\frac{\hbar}{2m\omega}} \langle \psi_m^0 | \hat{a}_+ + \hat{a}_- | \psi_n^0 \rangle$

- only non-zero for $m-1 = n$, $m+1 = n$

- denominator is $(n+\frac{1}{2})\hbar\omega - (m+\frac{1}{2})\hbar\omega = (n-m)\hbar\omega$

$$\Rightarrow \frac{q^2E^2\hbar}{2m\omega}\left\{ \frac{[\langle \psi_m^0 | \hat{a}_+ + \hat{a}_- | \psi_n^0 \rangle]^2}{\hbar\omega(n-m)}\delta_{n,m-1} + \frac{[\langle \psi_m^0 | \hat{a}_+ + \hat{a}_- | \psi_n^0 \rangle]^2}{\hbar\omega(n-m)}\delta_{n,m+1} \right\}$$

$$= \frac{q^2E^2}{2m\omega^2}\left\{ \frac{n}{n-(n-1)} + \frac{n+1}{n-(n+1)} \right\}$$

$$= \frac{q^2E^2}{2m\omega^2}\left(n - n - 1\right) = \boxed{\frac{-q^2E^2}{2m\omega^2} = E_n^2}$$

b) $$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2}m\omega^2x^2 - qEx$$

taking $x \to x' \equiv x - \left(\frac{qE}{m\omega^2}\right)$, we find

$$\to \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{dx'}{dx} = \frac{\partial f}{\partial x'},$$

$$\to \tfrac{1}{2}m\omega^2\left[x' + \frac{qE}{m\omega^2}\right]^2 - qE\left(x' + \frac{qE}{m\omega^2}\right)$$

$$\tfrac{1}{2}m\omega^2\left[x'^2 + \frac{2qE}{m\omega^2}x' + \frac{q^2E^2}{m^2\omega^4}\right] - qEx' - \frac{q^2E^2}{m\omega^2}$$

$$= \tfrac{1}{2}m\omega^2x'^2 + qEx' + \frac{1}{2}\frac{q^2E^2}{m\omega^2} - qEx' - \frac{q^2E^2}{m\omega^2}$$

$$= \tfrac{1}{2}m\omega^2x'^2 - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

$$\to \hat{H}' = \frac{\hat{p}'^2}{2m} + \tfrac{1}{2}m\omega^2x'^2 - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

- This is normal harmonic oscillator w/ a constant term, so we have normal sols plus this constant term:

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega - \frac{1}{2}\frac{(qE)^2}{m\omega^2}$$

Problem 7.10

- EQ. (7.34) $\rightarrow$ $E' = \pm \epsilon \frac{\hbar\omega}{2}$

- we are considering exact solution for

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2) + \epsilon m\omega^2 xy$$

which, in rotated/normal coords, have energies:

$$E_{mn} = \left(m+\frac{1}{2}\right)\hbar\omega_+ + \left(n+\frac{1}{2}\right)\hbar\omega_-$$

$\rightarrow$ $\omega_\pm = \sqrt{1\pm\epsilon}\,\omega$

- Plugging in:

$$E_{mn} = \left(m+\frac{1}{2}\right)\hbar\sqrt{1+\epsilon}\,\omega + \left(n+\frac{1}{2}\right)\hbar\sqrt{1-\epsilon}\,\omega$$

$$= E_m\sqrt{1+\epsilon} + E_n\sqrt{1-\epsilon}$$

$\sqrt{1+\epsilon} \rightarrow 1+\frac{\epsilon}{2}+\cdots$, $\sqrt{1-\epsilon} \rightarrow 1-\frac{\epsilon}{2}+\cdots$

$$E_{mn} = E_m\left(1+\frac{\epsilon}{2}\right) + E_n\left(1-\frac{\epsilon}{2}\right)$$

$$= E_m + \frac{\epsilon E_m}{2} + E_n - \frac{\epsilon}{2}E_n$$

$$= E_m + E_n + \frac{\epsilon}{2}(E_m - E_n)$$

- As in Ex 7.2, we consider $m=1, n=0$ (1) and $m=0, n=1$ (2):

(1) $E_1 + E_0 + \frac{\epsilon}{2}(E_1 - E_0)$, $E_1 = \frac{3\hbar\omega}{2}$, $E_0 = \frac{\hbar\omega}{2}$

$2\hbar\omega + \frac{\epsilon}{2}\hbar\omega$

(2) $E_0 + E_1 + \frac{\epsilon}{2}(E_0 - E_1)$

$2\hbar\omega - \frac{\epsilon}{2}\hbar\omega$

- the 1^{st}-order corrections are

$$\pm\epsilon\frac{\hbar\omega}{2},$$

just like the approximation!

Problem 7.11

$$H' = a^3 V_0\, \delta(x-\tfrac{a}{4})\,\delta(y-\tfrac{a}{2})\,\delta(z-\tfrac{3a}{4})$$

$$\to \psi_n^0 = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{a}\right)$$

$$E_n' = \langle \psi_n^0 | H' | \psi_n^0 \rangle$$

- delta fns kill all integrals!

$$= \left(\frac{2}{a}\right)^3 a^3 V_0 \sin^2\left(\frac{\pi}{a}\cdot\frac{a}{4}\right) \sin^2\left(\frac{\pi}{a}\cdot\frac{a}{2}\right) \sin^2\left(\frac{\pi}{a}\cdot\frac{3a}{4}\right)$$

$$= 8V_0 \sin^2\left(\frac{\pi}{4}\right)\sin^2\left(\frac{\pi}{2}\right)\sin^2\left(\frac{3\pi}{4}\right)$$

$$= 8V_0 \cdot \left(\frac{1}{2}\right)\cdot 1 \cdot \left(\frac{1}{2}\right) = \underline{2V_0}$$

- three first excited states: $\underset{\psi_1}{\{2,1,1\}}, \underset{\psi_2}{\{1,2,1\}}, \underset{\psi_3}{\{1,1,2\}}$.

- looking to diagonalize W:

$$W_{aa} = \langle \psi_1^0 | H' | \psi_1^0 \rangle =$$

$$= 8V_0 \sin^2\left(\frac{2\pi}{a}\cdot\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\frac{a}{2}\right)\sin^2\left(\frac{\pi}{a}\cdot\frac{3a}{4}\right)$$

$$= 8V_0 \sin^4\left(\frac{\pi}{2}\right)\sin^2\left(\frac{3\pi}{4}\right) = \frac{8V_0}{2} = \underline{4V_0}$$

$$W_{bb} = \langle \psi_2^0 | H' | \psi_2^0 \rangle =$$

$$= 8V_0 \sin^2\left(\frac{\pi}{a}\cdot\frac{a}{4}\right)\sin^2\left(\frac{2\pi}{a}\cdot\frac{a}{2}\right)\sin^2\left(\frac{\pi}{a}\cdot\frac{3a}{4}\right) = \underline{0}$$

$$W_{cc} = \langle \psi_3^0 | H' | \psi_3^0 \rangle$$

$$= 8V_0 \sin^2\left(\frac{\pi}{a}\cdot\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\cdot\frac{a}{2}\right)\sin^2\left(\frac{2\pi}{a}\cdot\frac{3a}{4}\right)$$

$$= 8V_0 \left(\frac{1}{2}\right)\cdot 1\cdot(-1)^2 = \underline{4V_0}$$

$$W_{ab} = W_{ba} = \langle \psi_1^0 | H' | \psi_2^0 \rangle$$

$$= 8V_0 \sin\left(\frac{2\pi}{a}\cdot\frac{a}{4}\right)\sin\left(\frac{\pi}{a}\cdot\frac{a}{4}\right)\sin\left(\frac{2\pi}{a}\cdot\frac{a}{2}\right)\sin\left(\frac{\pi}{a}\cdot\frac{a}{2}\right)$$
$$\times \sin^2\left(\frac{\pi}{a}\cdot\frac{3\pi}{4}\right) = 0$$

$$W_{ac} = W_{ca} = \langle \psi_1^0 | H' | \psi_3^0 \rangle$$

$$= 8V_0 \sin\left(\frac{2\pi}{a}\cdot\frac{a}{4}\right)\sin\left(\frac{\pi}{a}\frac{a}{4}\right)\sin^2\left(\frac{\pi}{a}\cdot\frac{a}{2}\right)\sin\left(\frac{2\pi}{a}\cdot\frac{3a}{4}\right)\sin\left(\frac{\pi}{a}\cdot\frac{3a}{4}\right)$$

$$= 8V_0 (1)\left(\frac{1}{\sqrt{2}}\right)(1)(-1)\left(\frac{1}{\sqrt{2}}\right) = \underline{-4V_0}$$

$$W_{bc} = W_{cb} = \langle \psi_2^0 | H' | \psi_3^0 \rangle$$

$$= 8V_0 \sin^2\left(\frac{\pi}{a}\cdot\frac{a}{4}\right)\sin\left(\frac{2\pi}{a}\cdot\frac{a}{2}\right) \cancel{\sin} = 0$$

$$\to W = 4V_0 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\to \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & -\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\to (1-\lambda)(-\lambda)(1-\lambda) - 1\left(-1(-\lambda \cdot -1)\right) = 0$$

$$-\lambda(1-\lambda)^2 + \lambda = 0$$

$$\lambda\left(-(1-\lambda)^2 + 1\right) = 0$$

$$\lambda\left(-\lambda^2 + 2\lambda\right) = 0 \to \lambda^2(\lambda - 2) = 0 \to \lambda = 0, 0, 2$$

$$\Rightarrow \underline{E' = 0, 0, 8V_0}$$

Problem 7.13

$$\psi^0 = \sum_{j=1}^{n} \alpha_j \psi_j^0$$

- Following Section 7.2.1, we first have that:

$$\hat{H}\psi_i^0 = E_i^0 \psi_i^0, \quad \langle \psi_i^0 | \psi_j^0 \rangle = \delta_{ij}$$

$$\rightarrow \hat{H}\psi^0 = E^0 \psi^0$$

- Everything is general up until EQ 7.26:

$$H^0 \psi' + H' \psi^0 = E^0 \psi' + E' \psi^0$$

- Inner product w/ $\langle \psi_i^0 |$:

$$\langle \psi_i^0 | H^0 | \psi' \rangle + \langle \psi_i^0 | H' | \psi^0 \rangle = E^0 \langle \psi_i^0 | \psi' \rangle + E' \langle \psi_i^0 | \psi^0 \rangle$$

- Again, $H^0 = (H^0)^\dagger$, so and cancel:

$$\langle \psi_i^0 | H' | \psi^0 \rangle = E' \langle \psi_i^0 | \psi^0 \rangle$$

$$\rightarrow \langle \psi_i^0 | H' \left(\sum_{j=0}^{n} \alpha_j | \psi_j^0 \rangle \right) = E' \alpha_i$$

$$\rightarrow \sum_{j=0}^{n} \alpha_j \langle \psi_i^0 | H' | \psi_j^0 \rangle = E' \alpha_i$$

$$\rightarrow \underline{\sum_{j=0}^{n} \alpha_j W_{ij} = E' \alpha_i}$$

- In the $n=2$ case, we had

$$\begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E' \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

from EQ 7.27 & 7.29, our EQ is the generalization of this, meaning the same conclusion applies, that we have an eigenval equation which yields 1st-order energy corrections.