Phys 4260 Spring 2025

Symmetry $\Leftrightarrow$ Conservation laws

↳ conserved quantity

Time Translations $\longrightarrow$ Energy

Spatial Translations $\longrightarrow$ Momentum

Rotation invariance $\longrightarrow$ Angular momentum

$\Downarrow$

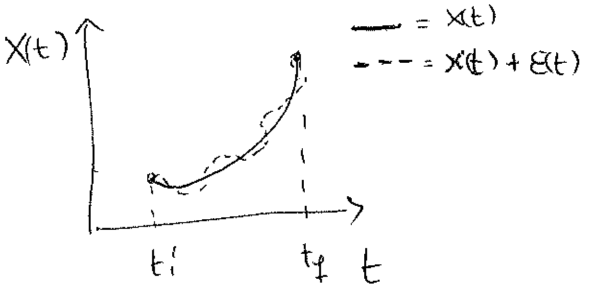
$\boxed{\text{Noether's theorem}}$

We will start looking at examples in 1-dim and 2-dim to develop our intuition.
Then, we will concentrate on the more formal and general aspects.

Single particle moving in 1-dim

$m \longrightarrow x \qquad L = T - V = \frac{1}{2} m \dot{x}^2 - U(x)$

$S = \int L \, dt$ classically defined Action



Principle of minimal action:
the particle is going to choose the path for which S is minimized

$x(t) \rightarrow x(t) + \varepsilon(t)$ little deformation caused by the introduction of an infinitesimal function ε of the time t.

$\varepsilon(t_i) = \varepsilon(t_f) = 0$ ⇒ we do not want to change the endpoints of the path.

* S → stationary along the physical path ⇒ $\Delta S = 0$

At the minimum of S, it should not change at all under this deformation at leading order in ε (that's the definition of extremal point for a function)

$dL = 2 \frac{1}{2} m \dot{x} \dot{\varepsilon} - U'(x) \varepsilon$

think about ε like $\delta x(t) \rightarrow df = f' \delta x$

$$m\dot{x}\dot{\varepsilon} = m\dot{x}\frac{d\varepsilon}{dt} = \frac{d}{dt}(m\dot{x}\varepsilon) - m\ddot{x}\varepsilon$$ (integration by parts)

$$dL = -\overbrace{(m\ddot{x} + U'(x))}^{0}\varepsilon + \frac{d}{dt}(m\dot{x}\varepsilon)$$

When we integrate this to obtain the change in the action, the 2nd term contributes to zero because ε vanishes at t_i and t_f. The first term has to be zero for all $\varepsilon(t)$ values $\Rightarrow$ gives us the equations of motion

$$m\ddot{x} = \underbrace{-U'(x)}_{F(x)} \rightarrow \text{EOM}$$

$F(x) \rightarrow$ force acting on m

$F(x) = -\frac{dU}{dx}$ in 1-dim

ε = completely general, no assumption except for it has to be infinitesimal and $\varepsilon(t_i) = \varepsilon(t_f) = 0$

for a physical trajectory

$$X \rightarrow X + \eta(t,x) \Rightarrow \underset{\overset{?}{0} \,\rightarrow\, \text{symmetry}}{dL} = -(\text{EOM})\eta + \underbrace{\frac{d}{dt}(m\dot{x}\eta)}_{\text{conserved } Q}$$

(*) A symmetry is an infinitesimal transformation of the coordinates tha leaves the lagrangian L invariant.

*) For every infinitesimal symmetry of the lagrangian we obtain a conserved quantity Q. $\frac{dQ}{dt} = 0$

More formally, in 1-dim the Euler Lagrange eqns (2a)

$$J[f] = \int_a^b L(x, f(x), f'(x))\,dx \qquad \text{functional}$$

If f extremizes J, perturbations of f preserving boundary conds. must either increase or decrease J.

$f + \varepsilon\eta$ = result of $\varepsilon\eta$ perturbation. ε is small & $\eta(a) = \eta(b) = 0$

$$\Phi(\varepsilon) = J[f + \varepsilon\eta] = \int_a^b L(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x))\,dx$$

The total derivative of $\Phi(\varepsilon)$ wrt ε is

$$\frac{d\Phi}{d\varepsilon} = \frac{d}{d\varepsilon}\int_a^b L(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x))\,dx = \int_a^b \frac{d}{d\varepsilon} L(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x))\,dx$$

$$= \int_a^b \left[\frac{\partial L}{\partial f}\eta + \frac{\partial L}{\partial f'}\eta'\right] dx \qquad \text{here we have that } \frac{dx}{d\varepsilon} = 0$$

$$\left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left[\eta(x)\frac{\partial L}{\partial f}(x, f(x), f'(x)) + \eta'(x)\frac{\partial L}{\partial f'}(x, f(x), f'(x))\right] dx = 0$$

Using IBP $\frac{d}{dx}\left[\eta\frac{\partial L}{\partial f'}\right] = \frac{d}{dx}\frac{\partial L}{\partial f'}\eta + \eta'\frac{\partial L}{\partial f'} \Rightarrow$

$$\left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0} = \int_a^b \left[\eta(x)\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'}\eta(x)\right] dx + \left[\eta(x)\frac{\partial L}{\partial f'}\right]_a^b = 0$$

$$\Rightarrow \boxed{\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0}$$

Free particle

$$L = \frac{1}{2} m \dot{x}^2$$

A simpe immediate symmetry is the spatial translation

$$x \to x + \eta_0 \Rightarrow L \to L$$

$\hookrightarrow$ constant

spatial translation invarrance $\Rightarrow$ $m \dot{x} \eta_0 = \text{constant}$

$\Rightarrow$ $m \dot{x} = \text{constant}$ $\Rightarrow$ $p = \text{constant}$ $\quad \frac{dp}{dt} = 0$

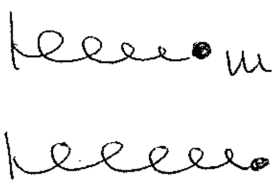
$\Rightarrow$ momentum conservation!

If we now turn on a potential $U(x)$, L will in general no longer be invariant for $x \to x + \eta_0$

$dL = - U'(x) \eta_0$ $\Rightarrow$ Translational invariance is broken

$$F(x) = -\frac{dU}{dx} = \frac{dp}{dt} \neq 0$$

Simple harmonic motion



$$U(x) = \frac{1}{2} k (x-\ell)^2$$

no translational symmetry
$\Rightarrow$ force on our particle.

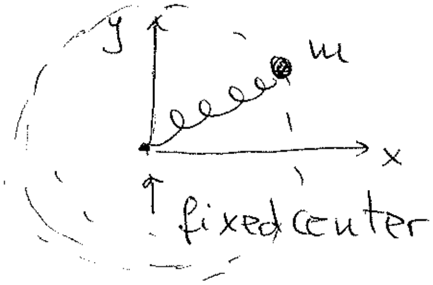
Two masses and a spring system

The whole system can translate!
Translation invariant system $\Rightarrow$ total momentum is conserved.

$$\sum_i F_{ext} = \frac{dP_{tot}}{dt} \rightsquigarrow 0$$

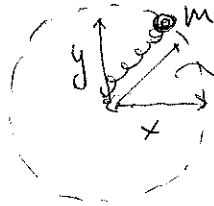
$\Downarrow$
0 isolated

Oscillator in 2-dim



the momentum is not going to be conserved because the system has no translation invariance!

We have rotational symmetry instead!



$\rightsquigarrow x^2+y^2 \Rightarrow U(x,y) = \frac{1}{2} k \left(\sqrt{x^2+y^2} - \ell\right)^2$

Using polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$U = \frac{1}{2} k (r-\ell)^2$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \overbrace{\frac{1}{2} k (r-\ell)^2}^{U(r)}$$

<u>Observation</u>: no thetas everywhere! $\Rightarrow \theta \to \theta + \eta_0 \Rightarrow$ rotation symmetry

From the Euler-Lagrange equations

$$\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}}_{mr^2\dot{\theta}} = \frac{\partial L}{\partial \theta} \rightsquigarrow 0$$

$mr^2\dot{\theta} \Rightarrow$ Angular momentum L is conserved

In a similar fashion, one can show that time translation invariance leads to Energy conservation

All these relations are very general!