Homework 10

Problem 11.17

$$\left.\begin{aligned}\psi_0(r) &= \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \\ \psi_{f,\vec{k}}(\vec{r}) &= \frac{1}{\sqrt{l^3}} e^{i\vec{k}\cdot\vec{r}}\end{aligned}\right\}$$

- Fermi's Golden Rule:

$$R = \frac{2\pi}{\hbar}\left|\frac{V_{if}}{2}\right|^2 \rho(E_f)$$

- Our potential is an electric field; simply:

$$V(z) = eE_0 z$$

- From the preceding example:

$$\rho(E_f) = \left(\frac{l}{2\pi}\right)^3 \frac{\sqrt{2m^3E_f}}{\hbar^3}\, d\Omega$$

$$\rightarrow \quad V_{if} = \langle \psi_0 | eE_0 z | \psi_f \rangle$$

$$= \frac{eE_0}{\sqrt{\pi a^3 l^3}} \int d^3r\, z e^{-r/a} e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{-ieE_0}{\sqrt{\pi a^3 l^3}} \frac{d}{dk_z} \int d^3r\, e^{-r/a} e^{i\vec{k}\cdot\vec{r}}$$

- which follows from assuming $k = k\hat{z}$

$$\rightarrow \int d^3r\, e^{-r/a} e^{ikr\cos\theta} = \int r^2 e^{-r/a} e^{ikr\cos\theta} \sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\pi \int r^2 e^{-r/a} e^{ikr\cos\theta} \sin\theta\, dr\, d\theta$$

$$u = ikr\cos\theta \qquad du = -ikr\sin\theta\, d\theta$$

$$\rightarrow \quad -\frac{2\pi}{ik} \int r e^{-r/a} \left[\int_{ikr}^{-ikr} e^u\, du\right] dr$$

$$\frac{2\pi i}{k} \int r e^{-r/a} \left(e^{-ikr} - e^{ikr}\right) dr$$

$$\rightarrow \int r e^{-r\left(\frac{1}{a} \pm ik\right)}\, dr = \left(\frac{1}{a} \pm ik\right)^{-2}$$

$$\rightarrow \quad \frac{2\pi i}{k}\left[\frac{1}{\left(\frac{1}{a}+ik\right)^2} - \frac{1}{\left(\frac{1}{a}-ik\right)^2}\right]$$

$$\frac{2\pi a^2 i}{k}\left[\frac{1}{(1+ika)^2} - \frac{1}{(1-ika)^2}\right]$$

$$\frac{1}{[1-(ka)^2] + 2ika} - \frac{1}{[1-(ka)^2] - 2ika}$$

$$\frac{[1-(ka)^2] - 2ika - [1-(ka)^2] - 2ika}{[1-(ka)^2]^2 + 4(ka)^2}$$

$$= \frac{-4ika}{[1+(ka)^2]^2}$$

$$\rightarrow \quad \frac{2\pi a^2 i}{k} \cdot \frac{-4ika}{[1+(ka)^2]^2} = \frac{8\pi a^3}{[1+(ka)^2]^2}$$

$$\rightarrow \quad V_{if} = \frac{-ieE_0}{\sqrt{\pi a^3 l^3}} \frac{d}{dk_z}\left[\ \swarrow\ \right]$$

$$= -ieE_0 \sqrt{\frac{a^3}{\pi l^3}}\, 8\pi \frac{d}{dk}\left[1+(ka)^2\right]^{-2}$$

$$= 16ieE_0\pi \sqrt{\frac{a^3}{\pi l^3}} \left[1+(ka)^2\right]^{-3} \cdot 2a^2k_z$$

$$= i\sqrt{\frac{\pi a^3}{l^3}} \frac{32a^2eE_0k\cos\theta}{[1+(ka)^2]^3}$$

$$\rightarrow R_{i\to d\Omega} = \frac{2\pi}{4\hbar}\left(\ \swarrow\ \right)^2 \left(\frac{l}{2\pi}\right)^3 \frac{\sqrt{2m^3E_f}}{\hbar^3}\, d\Omega$$

$$\frac{l^3\sqrt{2m^3E_f}}{16\pi^2\hbar^4} \cdot \frac{\pi a^3}{l^3} \cdot \frac{32^2 a^4 (eE_0k)^2 \cos^2\theta}{[1+(ka)^2]^6}\, d\Omega$$

$$\frac{64a^7\sqrt{2m^3E_f}}{\pi\hbar^4} \cdot \frac{e^2E_0^2k^2}{[1+(ka)^2]^6} \cos^2\theta\, d\Omega$$

- In the final state we have a photon, so

$$E_f = \frac{\hbar^2k^2}{2m}$$

$$\rightarrow \quad \frac{\sqrt{2m^3E_f}}{\hbar^4} = \frac{m\hbar k}{\hbar^4} = \frac{mk}{\hbar^3}$$

$$\rightarrow \quad \frac{64a^7m}{\pi\hbar^3} \frac{e^2E_0^2k^3}{[1+(ka)^2]^6} \cos^2\theta$$

- we need an ε^0 and to match powers from answer:

$$a = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$$

$$\rightarrow \quad \frac{64a^6m}{\pi\hbar^3} \cdot \frac{4\pi\varepsilon_0\hbar^2}{me^2} \cdot \frac{e^2E_0^2k^3}{[1+(ka)^2]^6} \cos^2\theta\, d\Omega$$

$$\Rightarrow \boxed{R_{i\to d\Omega} = 256 \frac{\varepsilon_0E_0^2a^3}{\hbar} \frac{(ka)^3}{[1+(ka)^2]^6} \cos^2\theta\, d\Omega}$$

b) $\int \cos^2\theta \sin\theta\, d\theta\, d\phi \qquad u = \cos\theta \quad du = -\sin\theta\, d\theta$

$$-2\pi\int_1^{-1} u^2\, du = \frac{4\pi}{3}$$

$$\hbar\omega = \frac{\hbar^2k^2}{2m} - E_1 = \frac{\hbar^2k^2}{2m} + \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$$

$$= \text{〃} + \frac{\hbar^2}{2ma^2}$$

$$= \frac{\hbar^2}{2m}\left(k^2 + \frac{1}{a^2}\right)$$

$$= \frac{\hbar^2}{2ma^2}\left[1+(ka)^2\right]$$

$$\rightarrow \quad \frac{256\varepsilon_0E_0^2a^3}{1/2\,\varepsilon_0E_0^2c\hbar} \frac{(ka)^3}{[1+(ka)^2]^6} \left(\frac{4\pi}{3}\right) \cdot \frac{\hbar^2}{2ma^2}\left[1+(ka)^2\right]$$

$$\sigma(k) = \frac{1024\pi\hbar}{3mc} \frac{k^3a^4}{[1+(ka)^2]^5}$$

Problem 2

- I presume we are to show that w/

$$A^\mu = (\phi, \vec{A}).$$

and the EM equations:

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\phi,$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

- From $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

considering $F^{0i} = \partial^0 A^i - \partial^i A^0$

$$= \left(\frac{\partial \vec{A}}{\partial t}\right)_i + \partial_i A^0$$

$$= \left(\frac{\partial \vec{A}}{\partial t}\right)_i + (\vec{\nabla}\phi)_i = -E_i$$

- Now,

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial A_x}$$

$$F^{13} = \partial^1 A^3 - \partial^3 A^1 = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$F^{23} = \partial^2 A^3 - \partial^3 A^2 = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}$$

- These are cross product components:

$$F^{12} = -(\vec{\nabla} \times \vec{A})_z = -B_z$$

$$F^{13} = (\vec{\nabla} \times \vec{A})_y = B_y$$

$$F^{23} = -(\vec{\nabla} \times \vec{A})_x = -B_x$$

$F^{\mu\mu} = 0$; $F^{\mu\nu} = -F^{\mu\nu}$, so we have:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_y & 0 \end{pmatrix}$$

Problem 3

- We can write $\vec{k}^2 + m^2 \equiv \omega_{\vec{k}}$, so, using Delta fn. props.

$$\delta(k_0^2 - \omega_{\vec{k}}^2) = \frac{\delta(k_0 - \omega_{\vec{k}})}{2k_0} + \frac{\delta(k_0 + \omega_{\vec{k}})}{2k_0}$$

- The $\theta(k_0)$ kills 2nd term.

$$\int \frac{d^4k}{(2\pi)^4} (2\pi) \delta(k_0^2 - \omega_{\vec{k}}^2) \theta(k_0)$$

$$= \int \frac{dk^0 \, d^3k}{(2\pi)^3} \cdot \frac{1}{2k_0} \delta(k_0 - \omega_{\vec{k}})$$

$$= \frac{d^3k}{(2\pi)^3 \, 2\sqrt{\vec{k}^2 + m^2}}$$

- Thus, the two integrals are equivalent.

Problem 4

$$\int d^3r \; f^*_{\vec{k}}(\vec{r}) \, i\overleftrightarrow{\partial_0} \, f_{\vec{k}'}(\vec{r})$$

$$= \frac{i}{(2\pi)^3 2\sqrt{k_0 k_0'}} \int e^{ikr} \overleftrightarrow{\partial_0} e^{-ik'r} \, d^3r$$

$$= \sim \int e^{ik_0 r^0} \overleftrightarrow{\partial_0} e^{-ik_0' r^0} \, e^{-i(k-k')\cdot r} \, d^3r$$

$$= e^{ik_0 r^0}\left(\partial_0 e^{-ik_0' r^0}\right) - \left(\partial_0 e^{ik_0 r^0}\right) e^{-ik_0' r^0}$$

$$-ik_0' e^{-i(k_0'-k_0)r^0} - ik_0 e^{-i(k_0'-k_0)r_0}$$

$$-i(k_0'+k_0) e^{-i(k_0'-k_0)r^0}$$

$$= \frac{(k_0'+k_0) e^{-i(k_0'-k_0)r^0}}{2\sqrt{k_0 k_0'}} \, \delta^{(3)}(\vec{k}-\vec{k}')$$

- This is, inside an integral, effectively a delta fn, leading to

$$\frac{2k_0}{2k_0}\delta^{(3)}(\vec{k}-\vec{k}') = \underline{\delta^{(3)}(\vec{k}-\vec{k}')}$$

Problem 5

$$\vec{A}(\vec{r}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}} \sum_\lambda \vec{\varepsilon}^{(\lambda)}_{\vec{k}}$$

$$\times \left[f_{\vec{k}}(r) a^{(\lambda)}_{\vec{k}} + f^*_{\vec{k}}(r) a^{(\lambda)\dagger}_{\vec{k}} \right]$$

$$\vec{\varepsilon}^{(\lambda)}_{\vec{k}} \vec{A}(\vec{r}) = \frac{1}{\sqrt{(2\pi)^3 2k_0}} \left[f_{\vec{k}}(r) a^{(\lambda)}_{\vec{k}} + f^*_{\vec{k}}(r) a^{(\lambda)\dagger}_{\vec{k}} \right]$$

to isolate one polarization. Then, multiplying by $f^*_{\vec{k}}(r) i \overleftrightarrow{\partial_0}$ and integrating:

$$\sqrt{(2\pi)^3 2k_0} \int d^3r \; f^*_{\vec{k}'} i \overleftrightarrow{\partial_0} \vec{\varepsilon}^{(\lambda)}_{\vec{k}} \cdot \vec{A}(r)$$

$$= \int d^3r \; f^*_{k} i \overleftrightarrow{\partial_0} f_{\vec{k}'} a^{(\lambda)}_{k'}$$

-The other term vanished by orthogonality:

$$\underline{a^{(\lambda)}_{\vec{k}} = \sqrt{(2\pi)^3 2k_0} \int d^3r \; f^*_{\vec{k}}(r) i \overleftrightarrow{\partial_0} \vec{\varepsilon}^{(\lambda)}_{\vec{k}} \cdot \vec{A}(r)}$$