QM II : Homework 1

Problem 6.5:

- EQ 6.11 is:

$$\psi(x-a) = e^{-iqa}\psi(x),$$

which is the consequence of a system have translational symmetry. Equivalently, we can write

$$\psi(x) = e^{iqx}u(x)$$

for <u>some</u> $u(x)$. Now, via EQ. 6.11:

$$\text{LHS}: \quad \psi(x-a) = e^{iq(x-a)}u(x-a)$$

$$\text{RHS}: \quad e^{-iqa}\psi(x) = e^{iqx}e^{-iqa}u(x)$$

- Equating the two:

$$e^{iq(x-a)}u(x-a) = e^{iq(x-a)}u(x)$$

$$u(x-a) = u(x)$$

- So, by writing our w.f. like this, we find that our generic $u(x)$ must be periodic in the same way the potential is.

Problem 6.6A

- we have $\psi_{nq} = e^{-iqx} u_{nq}(x)$

Plugging into SE:

-

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{nq}}{dx^2} + V(x)\psi_{nq}(x) = E_{nq}\psi_{nq}(x)$$

- Looking at 1^{st} term:

$$\frac{d^2\psi_{nq}}{dx^2} = \frac{d}{dx}\left[-iqe^{-iqx}u_{nq}(x) + e^{-iqx}\frac{du_{nq}}{dx}\right]$$

$$= -q^2e^{-iqx} - iqe^{-iqx}\frac{du_{nq}}{dx} - iqe^{-iqx}\frac{du_{nq}}{dx} + e^{-iqx}\frac{d^2u_{nq}}{dx^2}$$

$\rightarrow$

$$= e^{-iqx}\left[-q^2 - 2iq\frac{du_{nq}}{dx} + \frac{d^2u_{nq}}{dx^2}\right]$$

$$\Rightarrow -\frac{\hbar^2}{2m}e^{-iqx}\left[-q^2 - 2iq\frac{du_{nq}}{dx} + \frac{d^2u_{nq}}{dx^2}\right]$$

$$+ V(x)e^{-iqx}u_{nq}(x) = E_{nq}e^{-iqx}u_{nq}(x)$$

$$= -\frac{\hbar^2}{2m}\frac{d^2u_{nq}}{dx^2} + \frac{i\hbar^2 q}{m}\frac{du_{nq}}{dx} + \frac{\hbar^2 q^2}{2m} + V(x)u_{nq}(x)$$

$$= E_{nq}u_{nq}(x)$$

$$= -\frac{\hbar^2}{2m}\frac{d^2u_{nq}}{dx^2} + \frac{i\hbar^2 q}{m}\frac{du_{nq}}{dx} + V(x)u_{nq} = \left(E_{nq} - \frac{\hbar^2 q^2}{2m}\right)u_{nq}$$

- The only way this is negative is if the exponential is positive, but it isn't, so I really don't know why I'm not getting exactly what's in the book...

Problem 6.7

- Double Taylor expansion, following the same steps as before:

$$\psi(x_1-a, x_2-a) = \left[\sum_n \frac{1}{n!}(-a)^n \frac{\partial^n}{\partial x_1^n}\right]\left[\sum_m \frac{1}{m!}(-a)\frac{\partial^m}{\partial x_2^m}\right]\psi(x_1,x_2)$$

- Since $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_i} = \frac{i}{\hbar}\hat{p}_i$, so

$$= \left[\sum_n \frac{1}{n!}\left(\frac{-ia}{\hbar}\hat{p}_1\right)^n\right]\left[\sum_m \frac{1}{m!}\left(\frac{ia}{\hbar}\hat{p}_2\right)^m\right]\psi(x_1,x_2)$$

- This is just the exponential, so

$$= e^{-\frac{ia}{\hbar}\hat{p}_1} e^{-\frac{ia}{\hbar}\hat{p}_2}\psi(x_1,x_2)$$

- Since $\hat{p}_1$ and $\hat{p}_2$ obviously commute, we can combine the exponentials:

$$= e^{-\frac{ia}{\hbar}(\hat{p}_1+\hat{p}_2)}\psi(x_1,x_2)$$

- or, w/ $\hat{P} \equiv \hat{p}_1 + \hat{p}_2$,

$$\boxed{\hat{T}(a)\psi(x_1,x_2) = e^{-\frac{ia}{\hbar}\hat{P}}\psi(x_1,x_2)}$$

Problem 6.9

a) - True scalar operator has:

$$\hat{\Pi}^{\dagger} \hat{f} \hat{\Pi} = \hat{f},$$

- since Π is Hermitian and unitary,

$$\hat{\Pi}^{\dagger} = \hat{\Pi} = \hat{\Pi}^{-1}$$

- Thus:

$$\underbrace{\hat{\Pi} \hat{\Pi}^{\dagger}}_{1} \hat{f} \hat{\Pi} = \hat{\Pi} \hat{f}$$

$$\rightarrow \quad \hat{f} \hat{\Pi} = \hat{\Pi} \hat{f}$$

$$\rightarrow \quad \underline{[\hat{\Pi}, \hat{f}] = 0}$$

- for a pseudoscalar,

$$\hat{\Pi}^{\dagger} \hat{f} \hat{\Pi} = -\hat{f}$$

$$f \hat{\Pi} = -\hat{\Pi} \hat{f}$$

$$\rightarrow \{\hat{\Pi}, \hat{f}\} = 0$$

- The same follows for vector operators:

True vector: $\hat{\Pi}^{\dagger} \hat{\vec{V}} \hat{\Pi} = -\hat{\vec{V}}$

$$\hat{\vec{V}} \hat{\Pi} = -\hat{\Pi} \hat{\vec{V}}$$

$$\rightarrow \{\hat{\Pi}, \hat{\vec{V}}\} = 0$$

Pseudovector: $\hat{\Pi}^{\dagger} \hat{\vec{V}} \hat{\Pi} = \hat{\vec{V}}$

$$\hat{\vec{V}} \hat{\Pi} = \hat{\Pi} \hat{\vec{V}}$$

$$\rightarrow [\hat{\Pi}, \hat{\vec{V}}] = 0$$

Problem 6.16

- Act on some test function $f(x)$:

$$\hat{x}' f(x) = \hat{\Pi}^{\dagger} \hat{x} \hat{\Pi} f(x)$$

$$= \hat{\Pi}^{\dagger} \hat{x} f(-x) \to \hat{\Pi}^{\dagger} [x f(-x)]$$

$$= -x f(x) \to -\hat{x} f(x)$$

- dropping the test function:

$$\underline{\hat{x}' = \hat{\Pi}^{\dagger} \hat{x} \hat{\Pi} = -x}$$

- this works for all 3 components of position vector:

$$\underline{\hat{\vec{r}}' = \hat{\Pi}^{\dagger} \hat{\vec{r}} \hat{\Pi} = -\hat{\vec{r}}}$$

- for momentum:

$$\hat{p}' f(x) = \hat{\Pi}^{\dagger} \hat{p} \hat{\Pi} f(x)$$

$$= \hat{\Pi}^{\dagger} \hat{p} f(-x)$$

$$\to \hat{\Pi}^{\dagger} [(-i\hbar \partial_x) f(-x)]$$

$$= -(-i\hbar \partial_x) f(x)$$

$$\to -\hat{p} f(x)$$

- so, $$\underline{\hat{p}' = \hat{\Pi}^{\dagger} \hat{p} \hat{\Pi} = -\hat{p}}$$

- similarly,

$$\underline{\hat{\vec{p}}' = \hat{\Pi}^{\dagger} \hat{\vec{p}} \hat{\Pi} = -\hat{\vec{p}}}$$