Maxwell Equations in covariant form

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 & a) \\ \vec{\nabla}\cdot\vec{E} = \rho & b) \end{cases} \qquad \begin{cases} \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 & c) \\ \vec{\nabla}\times\vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{j} & d) \end{cases}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}} \leftarrow \text{always} \rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

In c)

$$\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0 \Leftrightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi \Rightarrow \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}}$$ Redefinition of $\vec{E}$

$$\Rightarrow \vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(\underbrace{-\vec{\nabla}\phi}_{0} - \frac{\partial \vec{A}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial \vec{B}}{\partial t}$$

Equations a) and c) are automaticalley satisfied with the redefinitions of $\vec{E}$ & $\vec{B}$.

4-Vector potential & Fied strength tensor

$$A^\mu = (\phi, \vec{A})$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\underbrace{\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi}$$

components of a 4-dim curl

Field strength

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

summarizes a) & c)

Gauge Transformations

$$\vec{A} \to \vec{A} - \vec{\nabla}\chi \qquad \phi \to \phi + \frac{\partial \chi}{\partial t}$$

In covariant form

$$A^\mu \to A^\mu + \partial^\mu \chi$$

χ = arbitrary scalar function.

$\vec{E}$ and $\vec{B}$ unchanged $\Rightarrow F^{\mu\nu}$ unchanged

$$F^{\mu\nu} \to F^{\mu\nu} + \left(\partial^\mu \partial^\nu - \partial^\nu \partial^\mu\right)\chi = F^{\mu\nu}$$

Now

$$\partial_\mu F^{\mu\nu} = j^\nu \Leftrightarrow \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) =$$

$$= \Box A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu$$

$$= \Box A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right) = j^\nu$$

We choose a particular χ so that A^μ satisfies

$$\boxed{\partial_\mu A^\mu = \partial_t \phi + \vec{\nabla} \cdot \vec{A} = 0}$$

This gauge choice is the Lorentz gauge

$$\Box A^\mu = j^\mu$$

$\Box A^\mu = J^\mu \Leftrightarrow \partial_t^2 \phi - \nabla^2 \phi = \rho\,;\ \partial_t^2 \vec{A} - \vec{\nabla}^2 \vec{A} = \vec{J}$

In the vacuum the Maxwell's equations in covariant form are:

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \& \quad \partial_\mu F^{\mu\nu} = 0 \quad \text{or} \quad \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0}$$

These follow from a variational principle with the lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Gauge transf.: $A_\mu \to A'_\mu + \partial_\mu \Lambda(x)$

$$\Box \Lambda(x) = -\partial_\mu A^\mu \Rightarrow \partial_\mu A'_\mu = 0$$

$$\Box \Lambda(x) = 0 \rightarrow \text{Lorentz gauge}$$

Lorentz gauge $\rightarrow A_\mu$ is not unique.

We further impose Λ to satisfy

$$\frac{\partial \Lambda}{\partial t} = -\phi$$

$$\partial_\mu A^\mu = \underbrace{\partial_0 A^0} - \partial_i A^i + \underbrace{\partial_0 \partial^0 \Lambda} - \nabla^2 \Lambda$$

$$\frac{\partial \Lambda}{\partial t} = -\phi \Rightarrow \partial_0 A^0 = -\partial_0 \partial^0 \Lambda \Rightarrow$$

$$\phi' = 0 \quad \& \quad \vec{\nabla} \cdot \vec{A}' = 0$$

Coulomb gauge $\Rightarrow$ $\boxed{\phi = 0 \quad \vec{\nabla} \cdot \vec{A} = 0}$

Coulomb gauge $\Rightarrow$ only 2 independent components of A^μ $\rightarrow$ physical nature of the electromagnetic field evident (two polarization states)

Quantization of the EM field (Canonical)

Conjugate momentum fields

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -\dot{A}^i + \partial^i A^0 = E^i$$

Remember that $A_i = -A^i$

In QM we had

$$[x_i, p_j] = i\delta_{ij} \qquad (\hbar = c = 1)$$

$i, j = 1, 2, 3$

$$[x_i, x_j] = [p_i, p_j] = 0$$

Here $A_i(\vec{r}, t)$ plays the role of x_i

π_i plays the role of p_i

We need the commutation relations for A_i and π_i

$$[A_i(\vec{r},t), \pi^j(\vec{r}',t)] = -[A^i(\vec{r},t), E^j(\vec{r}',t)] = i\delta_{ij}\,\delta^3(\vec{r}-\vec{r}')$$

This is not consistent with $\vec{\nabla}\cdot\vec{A}=0$

Taking $\vec{\nabla}\cdot$ on both sides

$$[\vec{\nabla}\cdot\vec{A}(\vec{r},t), E^i(\vec{r}',t)] = i\partial^i\delta^3(\vec{r}-\vec{r}') \neq 0$$

Note that $\vec{\nabla}=(\partial_x,\partial_y,\partial_z)$ only acts on $\vec{r}=(x,y,z)$ and not on $\vec{r}'=(x',y',z')$.

Need to modify the commutation relation:

1. $\delta_{ij} \to \Delta_{ij}$ rank 2 tensor, $i\,j$ symmetric

2. $\delta^3(\vec{r}-\vec{r}') = \frac{1}{(2\pi)^3}\int d^3\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$ integral form

Then we obtain

$$[A^i(\vec{r},t), E^j(\vec{r}',t)] = -i\Delta^{ij}\frac{1}{(2\pi)^3}\int d^3\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Taking the divergence

$$[\vec{\nabla}\cdot\vec{A}, E^j(\vec{r}',t)] = \frac{1}{(2\pi)^3}\int d^3\vec{k}\left(\sum_i k_i\Delta^{ij}\right)e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

The condition for the commutator to vanish is

$$\Delta_{ij} = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}$$

Therefore, the correct commutator is

$$[A^i(\vec{r},t), E^j(\vec{r}',t)] = i \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2} \right) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} =$$

Fourier transform of $\Rightarrow i \left(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2} \right) \delta^3(\vec{r}-\vec{r}')$

Also, we have that (at equal times)

$$[A^i(\vec{r},t), A^j(\vec{r}',t)] = [E^i(\vec{r},t), E^j(\vec{r}',t)] = 0$$

These commutators together with the fields describe a system with an infinite number of degrees of freedom because at each time t the fields have an independent value at each point in space.

Particle interpretation

$\partial_\mu A^\mu = 0$ is now $\Box A^\mu = 0$ (for each component)

but with the choice $\phi = 0 \Rightarrow \Box \vec{A} = 0$

that is $\Box A^i = 0$ $i = 1, 2, 3$

The solution of this is in terms of e^{ikr}; e^{-ikr} and the coefficients in the linear combination are called polarization vectors $\varepsilon^{(\lambda)}(k)$

$$\vec{A}(r) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{(\lambda)}(k) \left[a^{(\lambda)}(k) e^{-ikr} + a^{(\lambda)\dagger}_{(k)} e^{ikr} \right]$$

r = 4-vector

with $k^2 = 0 \Rightarrow k_0 = |\vec{k}|$ $\qquad kr = k_0 r^0 - \vec{k} \cdot \vec{r}$

$\lambda \rightarrow$ takes only two values because

$$\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \vec{k} \cdot \vec{\varepsilon}^{(\lambda)} = 0$$

$\vec{\varepsilon}^{(\lambda)}$ are chosen to be orthonormal

$$\vec{\varepsilon}^{(\lambda)}_{(k)} \cdot \vec{\varepsilon}^{(\lambda')}_{(k)} = \delta_{\lambda\lambda'}$$