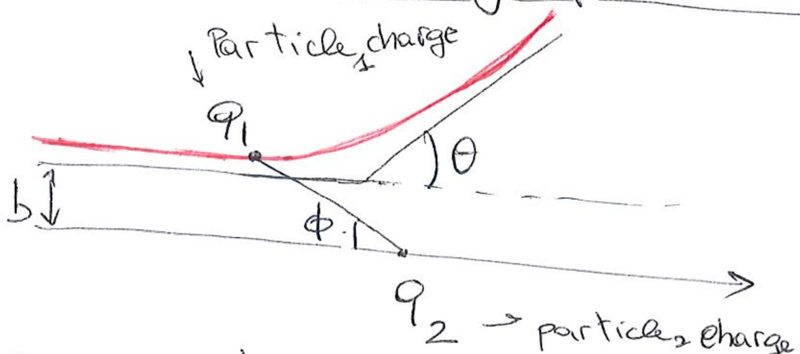
Notes on Scattering theory Phys 4260

Classical theory of scattering: Rutherford formula



$$|\vec{v}| \equiv v = \dot{r} = \frac{dr}{dt}$$
$$|\vec{\omega}| \equiv \omega = \dot{\phi} = \frac{d\phi}{dt}$$

- Conservation of energy $\Rightarrow E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + V(r) = \text{const}$
- Conservation of angular momentum $\Rightarrow J = mr^2\dot{\phi} = \text{const}$

$\Rightarrow \dot{\phi} = \dfrac{J}{mr^2} \Rightarrow$ into the energy equation

$$\dot{r}^2 + \frac{J^2}{m^2r^2} = \frac{2}{m}(E - V)$$

let's try to obtain $r(\phi)$ instead of having $r(t)$

$U \equiv 1/r \qquad \dot{r} = \dfrac{dr}{dt} = \dfrac{d}{dt}r(U(\phi(t))) = \dfrac{dr}{dU}\dfrac{dU}{d\phi}\dfrac{d\phi}{dt}$

$dU = -\dfrac{1}{r^2}dr = -U^2 dr \Rightarrow \dfrac{dr}{dU} = -\dfrac{1}{U^2}$. Therefore,

$$\dot{r} = -\frac{1}{\cancel{U^2}}\frac{dU}{d\phi}\frac{J\cancel{U^2}}{m} = -\frac{J}{m}\frac{dU}{d\phi}$$

From $\dot{r}^2 + \dfrac{J^2}{m^2r^2} = \dfrac{2}{m}(E-V)$

$$\left(-\frac{J}{m}\frac{dU}{d\phi}\right)^2 + \frac{J^2}{m^2}U^2 = \frac{2}{m}(E-V) \quad \text{or}$$

$$\left(\frac{dU}{d\phi}\right)^2 = \frac{2m}{J^2}(E-V) - U^2$$

That is $\dfrac{du}{d\phi} = \sqrt{\dfrac{2m}{J^2}(E-V)-u^2} \Rightarrow$

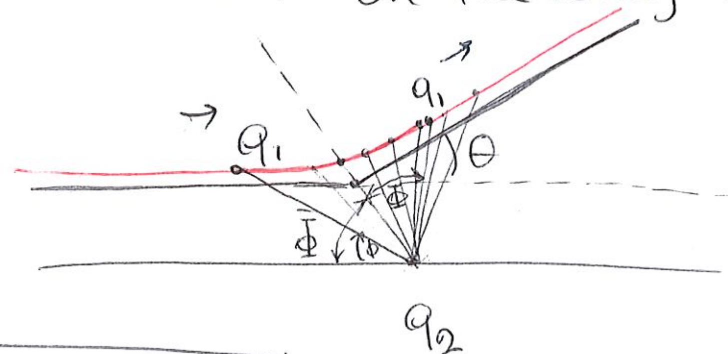
$$d\phi = \frac{du}{\sqrt{\frac{2m}{J^2}(E-V)-u^2}} = \frac{du}{\sqrt{I(u)}}$$

$$I(u) = \frac{2m}{J^2}(E-V)-u^2$$

Particle q_1 starts (unperturbed) at $r = -\infty$ $(u=0)$ and $\phi=0$, and the point of closest approach is $r_{min} \Rightarrow u_{max}$. We can write our integral as

$$\int_0^{\Phi} d\phi = \int_0^{u_{max}} \frac{du}{\sqrt{I(u)}} \Rightarrow \Phi = \int_0^{u_{max}} \frac{du}{\sqrt{I(u)}}$$

Let's observe that on the way out, q_1 swings on a equal angle Φ



$$\boxed{\Phi + \Phi + \Theta = \pi} \Rightarrow \pi - 2\Phi = \Theta \Rightarrow$$

$$\Theta = \pi - 2\int_0^{u_{max}} \frac{du}{\sqrt{I(u)}}$$

Inserting the specific potential $V(r)$

$$I(u) = \frac{2mE}{J^2} - \frac{2m q_1 q_2}{J^2 4\pi\varepsilon_0} u - u^2 = (u_2 - u)(u - u_1)$$

where u_1 and u_2 are the two roots of the quadratic equation defined by $I(u) = 0$.
It's just an equivalent and more convenient way of writing $I(u)$.
As we observed that $\frac{du}{d\phi} = \sqrt{I(u)}$

To find the maximum of $u(\phi)$ we impose

$\frac{du}{d\phi} = 0 \Rightarrow I(u) = 0$ and u_{max} is one of the two roots.

We can set $u_2 > u_1$ and $u_{max} = u_2$

At this point we can write

$$\Theta = \pi - 2\int_0^{u_2} \frac{du}{\sqrt{(u_2 - u)(u - u_1)}} =$$ to solve this we

go back to $I(u) = C - 2uB - u^2 \Rightarrow \boxed{-u_1 u_2 + u(u_2 + u_1) - u^2}$

$C = \frac{2mE}{J^2}$; $B = \frac{m q_1 q_2}{J^2 4\pi\varepsilon_0}$; $\Leftrightarrow \begin{cases} C = -u_1 u_2 \\ 2B = u_2 + u_1 \end{cases}$

and complete the square

$$C + B^2 - B^2 - 2uB - u^2 = C + B^2 - (B + u)^2$$

$$\theta = \pi - 2\int_0^{u_2} \frac{du}{\sqrt{C+B^2-(B+u)^2}}$$

$$t = B+u \Rightarrow dt = du \Rightarrow \qquad t = \begin{cases} B & \text{if } u=0 \\ B+u_2 & \text{if } u=u_2 \end{cases}$$

$$\int_0^{u_2} \frac{du}{\sqrt{C+B^2-(B+u)^2}} = \int_B^{B+u_2} \frac{dt}{\sqrt{\xi^2 - t^2}} \qquad \xi^2 = C+B^2$$

Now this is a known integral

$$t = \xi \sin\beta \quad \Rightarrow \quad dt = \xi \cos\beta \, d\beta$$

$$\int \frac{dt}{\sqrt{\xi^2-t^2}} = \int \frac{\xi \cos\beta \, d\beta}{\sqrt{\xi^2 - \xi^2 \sin^2\beta}} = \int \frac{\xi \cos\beta \, d\beta}{\sqrt{\xi^2(1-\sin^2\beta)}} =$$

$$= \int \frac{\xi \cos\beta \, d\beta}{\sqrt{\xi^2 \cos^2\beta}} \overset{\substack{\xi>0 \\ \cos\beta>0}}{=} \int d\beta = \beta + \text{const} \Rightarrow$$

$$= \arcsin\frac{t}{\xi} + \text{const}$$

where of course we used the fact that $\xi > 0$ and $\cos\beta > 0$. Finally, we can write

$$\theta = \left[\pi + 2\arcsin\left(\frac{-2u+u_1+u_2}{u_2-u_1}\right)\right]_0^{u_2} =$$

$$= \pi + 2\arcsin(-1) - 2\arcsin\left(\frac{u_1+u_2}{u_2-u_1}\right) =$$

$$= \pi - 2\frac{\pi}{2} - 2\arcsin\left(\frac{u_1+u_2}{u_2-u_1}\right) = -2\arcsin\left(\frac{u_1+u_2}{u_2-u_1}\right)$$

$J = mvb \qquad E = \frac{1}{2}mv^2 \qquad v =$ incoming speed

$$J^2 = m^2 b^2 \left(\frac{2E}{m}\right) = 2mb^2E \Rightarrow \quad \frac{2m}{J^2} = \frac{1}{b^2E} \Rightarrow$$

The integrand $I(u)$ can be written as

$$I(u) = \frac{1}{b^2} - \frac{1}{b^2}\left(\frac{1}{E}\frac{q_1q_2}{4\pi\varepsilon_0}\right)u - u^2 .$$ Now we need u_1 ; u_2

if we set $A = \dfrac{q_1q_2}{4\pi\varepsilon_0 E} \Rightarrow I(u) = -\left[u^2 + \dfrac{A}{b^2}u - \dfrac{1}{b^2}\right]$

To get the roots (we need u_1 and u_2) we impose

$$I(u) = 0 \Rightarrow u^2 + \frac{A}{b^2}u - \frac{1}{b^2} = 0 \Rightarrow$$

$$u = \frac{1}{2}\left[-\frac{A}{b^2} \pm \sqrt{\frac{A^2}{b^4} + \frac{A}{b^2}}\right] = \frac{A}{2b^2}\left[-1 \pm \sqrt{1 + \left(\frac{2b}{A}\right)^2}\right]$$

Therefore

$$u_2 = \frac{A}{2b^2}\left[-1 + \sqrt{1 + \left(\frac{2b}{A}\right)^2}\right]; \quad u_1 = \frac{A}{2b^2}\left[-1 - \sqrt{1 + \left(\frac{2b}{A}\right)^2}\right]$$

$$\Rightarrow \left(\frac{u_1 + u_2}{u_2 - u_1}\right) = -\frac{1}{\sqrt{1 + (2b/A)^2}} \Rightarrow \theta = 2\arcsin\left(\frac{1}{\sqrt{1 + (2b/A)^2}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{1 + (2b/A)^2}} = \sin\theta/2 \Rightarrow 1 + \left(\frac{2b}{A}\right)^2 = \frac{1}{\sin^2(\theta/2)}$$

$$\frac{(2b)^2}{A^2} = \frac{1 - \sin^2(\theta/2)}{\sin^2(\theta/2)} = \frac{\cos^2(\theta/2)}{\sin^2(\theta/2)} \Rightarrow \frac{2b}{A} = \cot(\theta/2)$$

$$b = \frac{q_1q_2}{8\pi\varepsilon_0 E}\cot(\theta/2)$$

Now we can calculate $D(\Theta)$. From the notes

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \qquad \frac{db}{d\theta} = \frac{q_1 q_2}{8\pi\varepsilon_0 E}\left(-\frac{1}{2\sin^2(\theta/2)}\right)$$

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

$$D(\theta) = \frac{1}{2\sin(\theta/2)\cos(\theta/2)}\,\frac{q_1 q_2}{8\pi\varepsilon_0 E}\,\frac{\cos(\theta/2)}{\sin(\theta/2)}\left[\frac{q_1 q_2}{8\pi\varepsilon_0 E}\,\frac{1}{2\sin^2(\theta/2)}\right] =$$

$$= \left[\frac{q_1 q_2}{16\pi\varepsilon_0 E\sin^2(\theta/2)}\right]^2$$

Finally, the total cross section is given by

$$\sigma = \int D(\theta)\sin\theta\, d\theta\, d\phi = 2\pi\left(\frac{q_1 q_2}{16\pi\varepsilon_0 E}\right)^2 \int_0^\pi \frac{\sin\theta}{\sin^4(\theta/2)}\, d\theta$$

It is important to note that this integral does not converge when $\theta \to 0$ and $\theta \to \pi$.

In fact $\theta \to 0$; $\theta \to \pi \Rightarrow \sin\theta \approx \theta$

and $\sin(\theta/2) \approx \theta/2 \rightarrow 16\int_0^\varepsilon \theta^{-3} d\theta = \left[\frac{-8}{\theta^2}\right]_0^\varepsilon \rightarrow \infty$