

Electron Dynamics in Crystals

Consider H_0 as the perfect crystal Hamiltonian, then the motion of an electron in an additional potential $U(\vec{r})$ within the non-interacting picture is described by

$$H_{\text{eff}} = [H_0 + U(\vec{r})] \Psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t).$$

The static perturbation $U(\vec{r})$ may arise from electric or magnetic fields, or from imperfections in the crystal.

Effective Hamiltonian Treatment

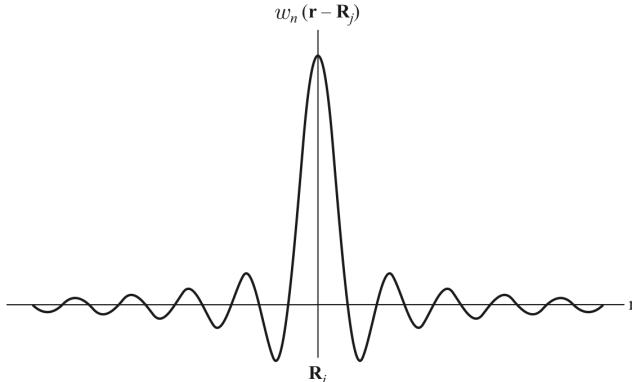
Let us start by recalling $[\vec{r}, \vec{p}] = -i\hbar$, $\vec{p} = -i\hbar \frac{\partial}{\partial \vec{r}}$ and $\vec{r} = i\hbar \frac{\partial}{\partial \vec{p}}$.

The Bloch functions of a wavevector \vec{k} and energy band n can be written as

$$\Phi_{n\vec{k}}(\vec{r}) = \frac{1}{\sqrt{N}} \sum_j e^{i\vec{k} \cdot \vec{R}_j} u_n(\vec{r} - \vec{R}_j), \text{ where}$$

$\langle u_n(\vec{R}_j) | u_n(\vec{R}_j) \rangle = S_{nn} S_{jj}^*$, $u_n(\vec{r} - \vec{R}_j)$ is localized about \vec{R}_j ,

$u_n(\vec{r} - \vec{R}_j) \equiv \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{R}_j} \Phi_{n\vec{k}}(\vec{r})$ are the Wannier functions.



Schematic diagram of a Wannier function localized around the lattice site \vec{R}_j .

Hence $\vec{R} |W_n(\vec{R}_j)\rangle = \vec{R}_j |W_n(\vec{R}_j)\rangle$ and we have

$$[\vec{K}, \vec{R}] = -\varepsilon, \quad \vec{R} = i \frac{\partial}{\partial \vec{K}} \quad \text{and} \quad \vec{K} = -i \frac{\partial}{\partial \vec{R}}.$$

Also, we can write H_{eff} as

$$H_{\text{eff}} = E_n \left(-i \frac{\partial}{\partial \vec{R}} \right) + V(\vec{R}) \quad (\vec{R}\text{-representation})$$

$$H_{\text{eff}} = E_n(\vec{K}) + V\left(i \frac{\partial}{\partial \vec{K}}\right) \quad (\text{Bloch-representation}).$$

Motion in External fields

In quantum mechanics

$$\frac{d\hat{A}}{dt} = \frac{d\hat{A}}{dt} + \frac{[\hat{A}, H]}{\varepsilon \hbar}.$$

Then

$$\begin{aligned} i\hbar \frac{d\vec{R}}{dt} &= [\vec{R}, H_{\text{eff}}] \\ &= \left[\vec{R}, E_n \left(-i \frac{\partial}{\partial \vec{R}} \right) + V(\vec{R}) \right] \\ &= \left[\vec{R}, E_n \left(-i \frac{\partial}{\partial \vec{R}} \right) \right] \\ &= \left[i \frac{\partial}{\partial \vec{K}}, E_n(\vec{K}) \right] \\ &= i \frac{\partial}{\partial \vec{K}} E_n(\vec{K}). \end{aligned}$$

Then

$\vec{V} = \frac{d\vec{R}}{dt} = \frac{1}{\hbar} (\nabla_{\vec{K}} E_n(\vec{K}))$ and \vec{V} does not depend on $U(\vec{R})$. Hence, \vec{V} is a property of the band structure.

Now,

$$\begin{aligned} i\hbar \frac{d\vec{k}}{dt} &= [\vec{k}, H_{\text{eff}}] \\ &= \left[\vec{k}, U\left(\frac{i\hbar}{\partial \vec{k}}\right) \right] \\ &= \left[-\frac{i\hbar}{2\vec{k}}, U(\vec{R}) \right] \\ &= -e (\nabla_{\vec{R}} U(\vec{R})) \end{aligned}$$

$$i\hbar \frac{d\vec{k}}{dt} = -\nabla_{\vec{R}} U(\vec{R})$$

Since $-\nabla_{\vec{R}} U(\vec{R})$ is the force \vec{F} on an electron, we have

$$i\hbar \frac{d\vec{k}}{dt} = \vec{F}$$

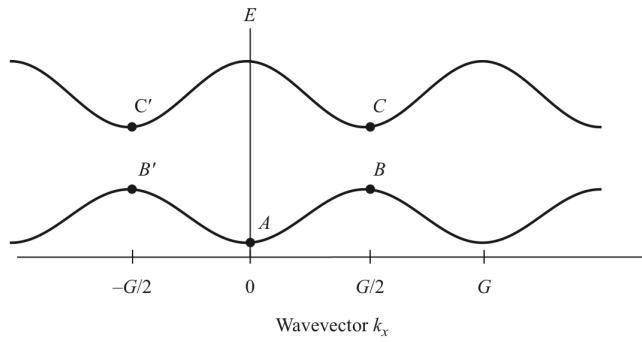
If the force is constant $\vec{k}(t) = \vec{k}(t=0) + \frac{\vec{F}t}{i\hbar}$.

With the equations

$\vec{V} = \frac{d\vec{R}}{dt} = \frac{1}{\hbar} (\nabla_{\vec{K}} E_n(\vec{K}))$ and $i\hbar \frac{d\vec{k}}{dt} = \vec{F}$ we can follow the motion

of an electron starting at time $t=0$ in an individual Bloch state with a wavevector \vec{k}_0 , or of a wavepacket centered about a given \vec{k}_0 .

Example : Consider a crystal with the band structure



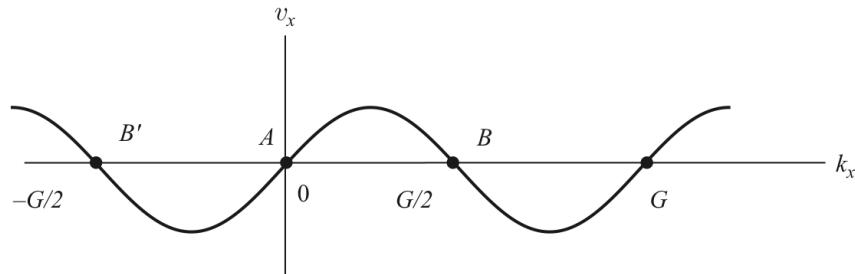
Schematic of a band structure in a repeated zone scheme along the k_x direction, where G is the smallest reciprocal lattice vector along this direction.

subjected to uniform electric field $\vec{E} = E \hat{x}$.

The wavevector of a state with an initial wavevector $k_0 = 0$ at $t=0$ will change according to

$$k(t) = -\frac{|e|Et}{\hbar} \hat{x}.$$

The group velocity is given by $v_g = \frac{\partial E_n}{\partial k_x}$

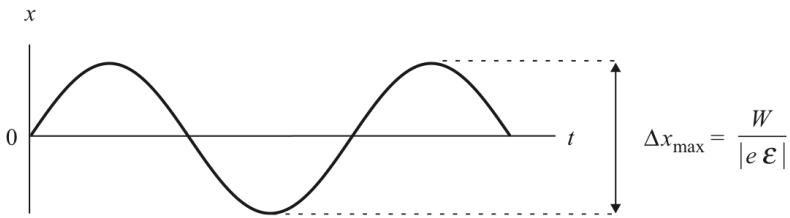


Group velocity v_x as a function of k_x for the lower band

from $v_x(t)$ we can find $x(t)$

$$x(t) = \int_{-G/2}^t v_x(t') dt' = \int_{-G/2}^t \frac{dt}{\hbar} \frac{\partial E_n}{\partial k_x} = \frac{1}{\hbar} \int_{-G/2}^t \frac{dt}{dk_x} \frac{\partial E_n}{\partial k_x} dk_x$$

$$x(t) = [E_n(k_x(t)) - E_n(k_x(t=0))] / (-|e|\varepsilon).$$

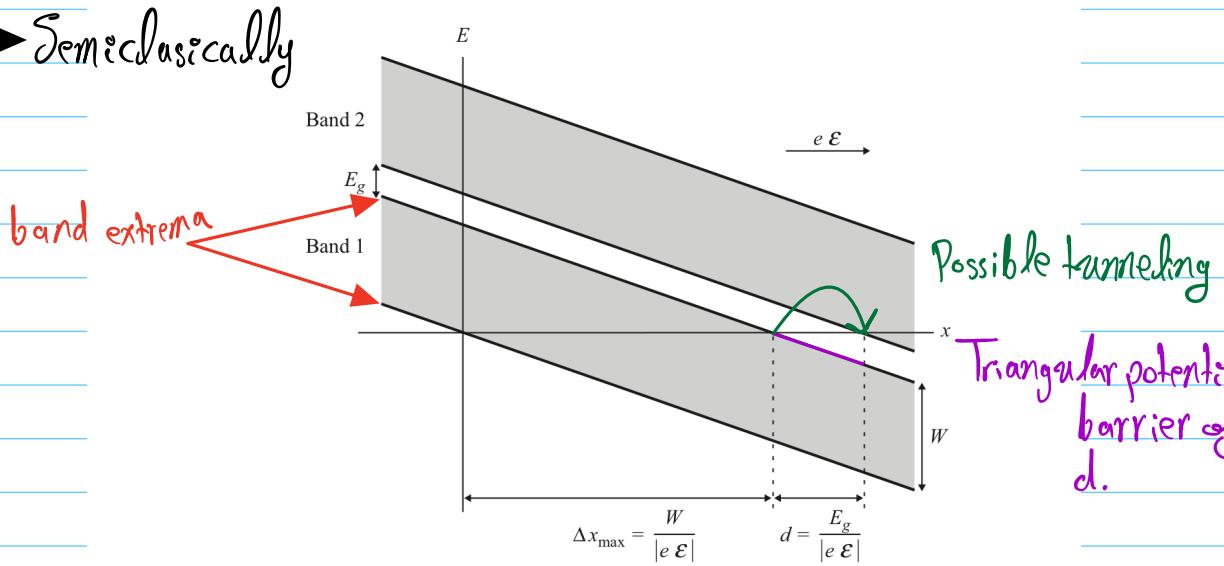


Schematic of $x(t)$ of a crystalline electron in a uniform field which shows Bloch oscillation where W is the band width.

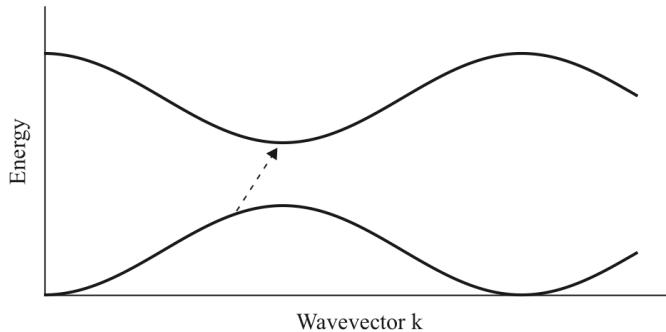
- Physically this means that the wavepacket's center of mass wavevector in \vec{k} -space and its center of mass in real space are changing as a function of time.
- When the center of mass wavevector reaches the B.Z. boundary, the wavepacket is Bragg reflected, leading to an oscillatory behaviour in $\vec{r}(t)$.
- In this idealized situation there is **no net current for a perfect crystal**.
- For real materials, impurities, and zero-point motion of various excitations will introduce scattering that would give rise to a net current even at $T=0$.
- The oscillations in the motion of the electrons are called **Bloch Oscillations**.
- For usual materials Bloch oscillations are very hard to observe since their observation requires the electron traveling many periods. For half a period, the distance traveled is

$$\Delta x_{\max} = \frac{W}{|e\epsilon|}$$
 $W \sim \text{few eV (bandwidth)}, \epsilon \sim \text{V/mm}, \Delta x_{\max} \sim \text{few mm}$. This is much larger than the mean free path of electrons.
- Bloch oscillations have been observed in optical lattices and semiconductor superlattices.

→ Semiclassically



Semiclassical picture of Bloch oscillations with \mathcal{E} an applied uniform electric field and W the bandwidth.



Schematic of Zener tunneling with an electron tunneling from the lower band to the upper band under the influence of a strong external field.

Topology and Berry Phase

Geometry and topology are playing an increasingly important role in modern quantum mechanics.

Euclidean plane geometry can be characterized as the study of invariants under linear scale changes and rotations.

Topology is the study of invariants under arbitrary continuous transformations.

There are fewer invariants under arbitrary distortions than there is under linear

scale changes. Topological invariants are robust and they are special and fundamental.

In recent years we have recognized that certain physical observables are universal and invariant under changes in the Hamiltonian.

This universality is mathematically described by a topological invariant.

Topological protection means that a physical quantity remains invariant even when the Hamiltonian undergoes changes, unless the system goes through a quantum phase transition transition that results in a discontinuous change in the topological invariant.

Since we learn geometry in high-school and topology in undergrad or grad level courses, we will start with the study of the geometry of the Hilbert space and then we move to topological invariants in quantum physics.

To explore the geometry of the Hilbert space we need to move inside the Hilbert space.

In the Hilbert space we have a well-defined notion of length and distance through the inner product between vectors.

However the fact that this is a complex vector space adds a new twist.

Physical states are represented by rays, not vectors in the Hilbert space, and this leads to very interesting and novel properties.

A ray is the equivalence class of vectors all related to a given vector by a multiplication by an overall phase $e^{i\theta}$ for any real θ . More generally, since overall normalization does not matter, a ray is defined as the equivalence class of vectors all related by the multiplication by any complex $\# \lambda$.

Now we are going to adiabatically move around the Hilbert space while

following a particular eigen state of the Hamiltonian.

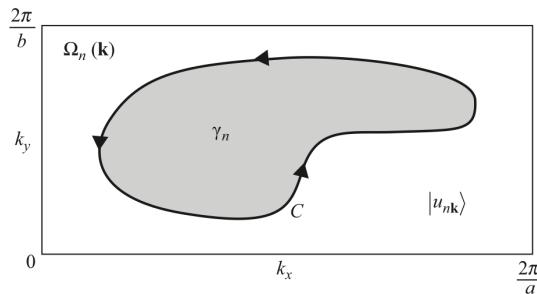
Berry Phase and Berry Curvature

Consider an electron on the Bloch state $|n\vec{k}\rangle$,

$$\langle \vec{r} | n\vec{k} \rangle = \psi_{n\vec{k}}(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} u_{n\vec{k}}(\vec{r}).$$

We also assume that this state is constrained to a single band n .

In the presence of a slowly varying static potential $U(\vec{r})$, the electron will adiabatically drift in \vec{k} -space along a path C .



Berry curvature in \vec{k} -space. Here $|u_{n\vec{k}}\rangle$ is the periodic part of a Bloch state and $\Omega_n(\vec{k})$ is the Berry curvature of band n . C denotes a closed loop over which an electron in the n th band picks up a Berry phase of γ_n .

The electron is at an initial state $|\Psi(t=0)\rangle = |n\vec{k}(0)\rangle$, then at some later time the state will evolve into the state $|\Psi(t)\rangle$, which is $|n\vec{k}(t)\rangle$ apart from a phase factor according to the adiabatic theorem in Q.M. (See the quantum review notes for details).

Our target is to find the phase factor that relates $|\Psi(t)\rangle$ to $|n\vec{k}(t)\rangle$.

The periodic part of the Bloch wavefunction satisfies

$$\left[\frac{(\vec{p} + \hbar\vec{k})^2}{2m} + V(\vec{r}) \right] U_{n\vec{k}}(\vec{r}) = E_n(\vec{k}) U_{n\vec{k}}(\vec{r}) , \text{ or}$$

$$H(\vec{k}) |U_{n\vec{k}}\rangle = E_n(\vec{k}) |U_{n\vec{k}}\rangle.$$

We use $|U_{n\vec{k}}\rangle$ as the instantaneous orthonormal basis to expand $|\Psi(t)\rangle$.

$H(\vec{k})|U_{n\vec{k}}\rangle = E_n(\vec{k})|U_{n\vec{k}}\rangle$ allows for an arbitrary \vec{k} -dependent phase factor. To remove this arbitrariness we make a phase choice (graye) that requires the phase of $|U_{n\vec{k}}\rangle$ to be smooth and single valued along the path C in the parameter space or \vec{k} -space.

Treating $\vec{k}(t)$ as a parameter along C , the periodic part of the state of the electron evolves according to the time dependent Schrödinger eqn.

$$H(\vec{k}(t))|\Phi(t)\rangle = i\hbar \frac{d}{dt}|\Phi(t)\rangle, \text{ where}$$

$$\Phi(\vec{r}, t) = e^{i\vec{k} \cdot \vec{r}} \Psi(\vec{r}, t).$$

Adiabatically a system prepared in $|U_{n\vec{k}(0)}\rangle$ will evolve with $H(\vec{k}(t))$ and be in $|U_{n\vec{k}(t)}\rangle$ at t . Then

$$|\Phi(t)\rangle = \exp \left[-\frac{i}{\hbar} \int_0^t dt' [E_n(\vec{k}(t'))] \right] \underbrace{\exp(i\gamma_n(t))}_{\text{Dynamical Phase}} |U_{n\vec{k}(t)}\rangle.$$

non-integrable phase that cannot be written as a function of \vec{k} .

We determine $\gamma_n(t)$. We require that $|\Phi(t)\rangle$ satisfies the t -dependent S.E, and we get

$$\frac{d}{dt} \gamma_n(t) = i \langle U_{n\vec{k}}(t) | \vec{\nabla}_{\vec{k}} U_{n\vec{k}}(t) \rangle \cdot \frac{d\vec{k}}{dt},$$

where

$$\langle \vec{\nabla}_{\vec{k}} U_{n\vec{k}}(t) | = \oint \left[\frac{(|U_{n(\vec{k}+d\vec{k})}\rangle - |U_{n\vec{k}}\rangle)}{d\vec{k}} \right] \vec{k} \cdot$$

Integrating along the path C from a point \vec{k}_0 with respect to time we get

$$\gamma_n(\vec{k}) = i \int_{\Gamma_0}^{\vec{k}} \langle u_{n\vec{k}_1} | \vec{\nabla}_{\vec{k}} u_{n\vec{k}_1} \rangle \cdot d\vec{k}_1.$$

The normalization of $|u_{n\vec{k}}\rangle$ requires that $\langle u_{n\vec{k}} | \vec{\nabla}_{\vec{k}} u_{n\vec{k}} \rangle = -\langle \vec{\nabla}_{\vec{k}} u_{n\vec{k}} | u_{n\vec{k}} \rangle$, then $\gamma_n(\vec{k})$ is real.

We can define the Berry vector potential $\vec{A}_n(\vec{k})$ [Berry connection] as

$$\vec{A}_n(\vec{k}) = i \langle u_{n\vec{k}} | \vec{\nabla}_{\vec{k}} u_{n\vec{k}} \rangle = -\text{Im} \langle u_{n\vec{k}} | \vec{\nabla}_{\vec{k}} u_{n\vec{k}} \rangle.$$

\vec{A} has dimensions of length and is always real.

Then $|0(t)\rangle$ gains a phase $\gamma_n(\vec{k})$ in addition to the dynamical phase during its adiabatic evolution along C ,

$$\gamma_n(\vec{k}) = \int_C^{\vec{k}} d\vec{k}_1 \cdot \vec{A}_n(\vec{k}_1)$$

This phase may be regarded as an Aharonov-Bohm phase caused by an effective "magnetic field"

$$\vec{J}_n(\vec{k}) = \vec{\nabla}_{\vec{k}} \times \vec{A}_n(\vec{k}),$$

which lives in \vec{k} -space. This is called the Berry curvature.

The Berry vector potential is gauge dependent, i.e.,

$$|u_{n\vec{k}}\rangle = e^{iU(\vec{k})} |u_{n\vec{k}}\rangle,$$

where $U(\vec{k})$ is an arbitrary smooth function \vec{k} ,

$$\vec{A}_n(\vec{k}) \rightarrow \vec{A}_n(\vec{k}) - \vec{\nabla}_{\vec{k}} U(\vec{k}).$$

Then $\gamma_n(\vec{k})$ is gauge dependent and changed by $U(\vec{k}(0)) - U(\vec{k}(t))$.

However, for a closed loop C in the parameter space the extra phase acquires

ered by $|0\rangle$ after it traversed back to its initial position

$$\gamma_n = \oint_C d\vec{k} \cdot \vec{A}_n(\vec{k})$$

is gauge invariant and it is a physical observable

γ_n is the Berry phase or geometric phase. ($\gamma_n(\vec{k})$ sometimes is also called the Berry phase).

The Berry curvature is also a gauge invariant quantity, since

$$\vec{\nabla}_{\vec{k}} \times (\vec{\nabla}_{\vec{k}} M(\vec{k})) = 0$$

for any smooth function $M(\vec{k})$.

Using Stokes theorem we can also write

$$\gamma_n = \oint_C d\vec{k} \cdot \vec{A}_n(\vec{k}) = \int_S d\vec{s} \cdot \vec{J}_n(\vec{k})$$

where S is the surface enclosed by the loop C . The Berry curvature is observable.

The physical origin of Berry curvature are the residual interactions with the states in the bands that are not considered in the effective Hamiltonian.

The Berry curvature $\vec{J}_n(\vec{k})$ is an intrinsic local property in \vec{k} -space of the band structure of a crystal, and modifies the motion of the electron.

The Spin 1/2 Berry Phase

Consider a spin 1/2 system, whose most general Hamiltonian is given by

$$H = \vec{h} \cdot \vec{G} = \sum_{i=1}^3 h_i G_i$$

where G_i are the Pauli matrices ($G_1=x$, $G_2=y$, and $G_3=z$). In this case the

parameter $\vec{h} = (h_1, h_2, h_3)$ is the familiar Euclidean Space (E_3). Also notice that we can write the Hamiltonian of any 2-band system as the Hamiltonian above.

It is convenient to write the position vector in parameter space, \vec{h} , using spherical coordinates, such that

$$\vec{h} = h(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)).$$

The two eigenstates of \vec{h} with energies $\pm h$ are

$$|-\rangle = \begin{pmatrix} \sin(\theta/2) e^{-i\phi} \\ -\cos(\theta/2) \end{pmatrix}, \text{ and } |+\rangle = \begin{pmatrix} \cos(\theta/2) e^{-i\phi} \\ \sin(\theta/2) \end{pmatrix}.$$

It is interesting to notice that the eigenvalues of \vec{h} depend on the magnitude h , and the eigenstates depend on the orientation of the "quantization axis", which depends on two parameters of \vec{h} , these are θ and ϕ and they form the space of S^2 (the surface of unit sphere).

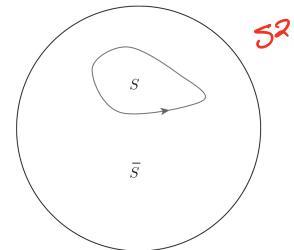
The Berry connection for the ground state $|-\rangle$ is

$$\bar{A}_\theta = \langle - | i \partial_\theta | - \rangle = 0, \quad \bar{A}_\phi = \langle - | i \partial_\phi | - \rangle = \sin^2(\theta/2), \text{ and}$$

$$\bar{A}_{\theta\phi} = \partial_\phi \bar{A}_\theta - \partial_\theta \bar{A}_\phi = \frac{1}{2} \sin(\theta) \quad (1)$$

Then the Berry flux through an element of S^2 is

$$\bar{\mathcal{F}}_{\theta\phi} d\theta d\phi = \frac{1}{2} \sin(\theta) d\theta d\phi = \frac{1}{2} d\Omega$$



where $\sin(\theta) d\theta d\phi$ is the solid angle subtended by the element when viewed from the origin.

This means that "the magnetic field" (Berry Curvature) is constant everywhere on the surface of a sphere in parameter space. Then we have

$$\int \bar{\mathcal{F}}_{\theta\phi} d\theta d\phi = 1/2 \int d\Omega = 2\pi$$

and

$$\tilde{\theta}_S = \frac{1}{2} \int_S d\mathcal{A} = \frac{1}{2} \pi S$$

or $1/2$ the solid angle subtended by the trajectory, as viewed from the center of the sphere.

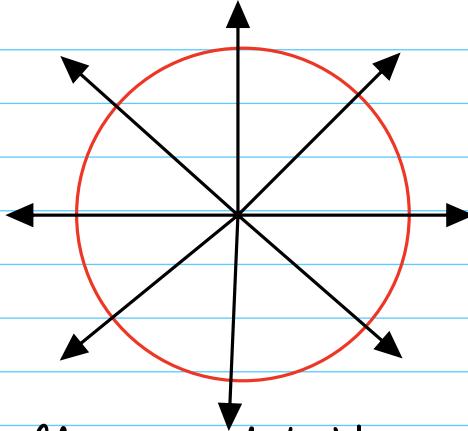
Going back to the cartesian coordinate parameter space the Berry curvature seen by $\vec{l} \rightarrow \vec{e}_S$ is found by

$$\frac{\partial}{\partial h} = \frac{2}{2h} \hat{h} + \frac{1}{h} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{h \sin(\theta)} \frac{\partial}{\partial \phi} \hat{\phi}$$

$$\vec{A_h} = \langle -1 | \vec{d}/d\vec{h} | 1 \rangle = \frac{1}{2h} \frac{(1 - \cos(\theta))}{\sin(\theta)} \hat{\phi}, \text{ and}$$

$$\vec{J_h} = \vec{\nabla}_{\vec{h}} \times \vec{A_h} = \frac{1}{2} \frac{\vec{h}}{h^3} \quad (\text{This could have been obtained from (1)})$$

which has the form of a magnetic monopole located at the origin (with a charge $+1/2$ in this case).



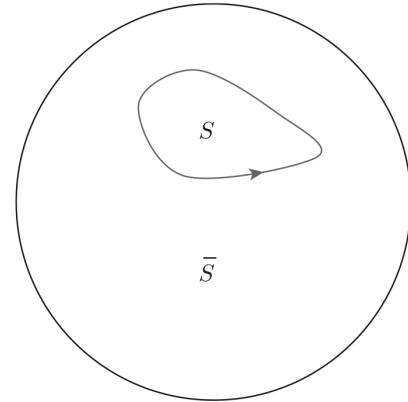
The magnetic field generated by the monopole

is equal to the spin ($1/2$) times the local Gaussian curvature of a sphere with radius h , and its direction is normal to the direction of the surface.

Adiabatic evolution assumes the existence of an excitation gap, which in this model is $E_+ - E_- = 2|\vec{h}|$. We see that the gap vanishes at the origin of the

parameter space and we can think of the degeneracy at that point as radiating Berry curvature \vec{N} just as a magnetic monopole is a point source of magnetic field.

For a closed loop on S^2 there is an ambiguity of which part is the interior and which part is the exterior. A loop with right-hand orientation enclosing S can be viewed as a left-hand oriented loop enclosing the area \bar{S} which is the complementary of S . Then there are 2 possibilities for the Berry phase $\gamma = \eta \oint_S$ and $\bar{\gamma} = -\eta \oint_{\bar{S}}$ (η = arbitrary spin).



To have the Berry phase well defined we require

$$\gamma = \bar{\gamma} \pmod{2\pi}$$

but $\oint_S + \oint_{\bar{S}} = 4\pi$ (Solid angle of a sphere)

$$\text{then } \gamma - \bar{\gamma} = \eta(4\pi) = 0 \pmod{2\pi}$$

if η = integer multiple of $1/2$. (Shows the quantization of spin).

Anomalous Velocity

Go back to the effects of Berry phase in materials. Consider the effective Hamiltonian

$$H = E_n(\vec{R}) + U(\vec{R}),$$

where $\vec{R} = i\vec{r}_K$ is the Wannier coordinate indexing the lattice sites and $E_n(\vec{r})$ is the unperturbed band structure.

We can perform a gauge transformation to remove the effective Berry phase at the expense of adding the Berry vector potential to $i\vec{r}_K$ in the argument

of \vec{U} . Then

$$H^1 = E_n(\vec{k}) + U(c \nabla_{\vec{k}} + \vec{A}_n(\vec{k})),$$

Then the position operator in this gauge is

$$\vec{x} = c \nabla_{\vec{k}} + \vec{A}_n(\vec{k}),$$

and the operator components of \vec{x} do not commute with each other.

The commutation between the components of \vec{x} are

$$[x_i, x_j] = i \epsilon_{ijk} \nabla_k^n(\vec{k})$$

ϵ_{ijk} is the Levi-Civita symbol in 3D.

Now we can derive the equations of motion of H^1

$$\hbar \frac{d\vec{k}}{dt} = -i [\vec{k}, H^1] = -\nabla_{\vec{k}} U(\vec{k})$$

which is unchanged by the geometrical phase. However

$$\hbar \vec{v} = -i [\vec{x}, H^1] = \nabla_{\vec{k}} E_n(\vec{k}) + \underbrace{\nabla_{\vec{x}} U \times \vec{A}_n(\vec{k})}_{\text{extra term arising from the Berry curvature.}}$$

The new term is known as Luttinger anomalous velocity.

for example, a crystal with a static electric field and a weak \vec{B} field we have

$$\hbar \frac{d\vec{k}}{dt} = e \vec{E} + \frac{e}{c} \vec{v} \times \vec{B} \quad \text{and}$$

$$\hbar \vec{v} = \nabla_{\vec{k}} E_n(\vec{k}) - e \vec{E} \times \vec{A}_n(\vec{k}).$$

The contribution of $\sqrt{g(\vec{k})}$ to the dynamics is important in a wide range of materials and systems, such as the quantum Hall effect (QHE), the anomalous QHE, topological insulators, ...

We can neglect the effects of $\sqrt{g_n(\vec{k}^*)}$ if a crystal preserves both, time reversal

$$g_n(-\vec{k}) = -g_n(\vec{k}^*)$$

and inversion symmetry

$$g_n(-\vec{k}) = g_n(\vec{k}).$$

Topological Quantization of the Hall Conductance of Magnetic Bloch Bands.

for a filled magnetic band in 2D, we have Hall current density from electrons in this band

$$\vec{J} = -\frac{e}{A} \sum_{\vec{k} \in B.Z} \vec{v}_a(n, \vec{k}) = \sigma_{xy}^n \hat{z} \times \vec{\mathcal{E}},$$

here A is the area, \vec{v}_a the anomalous velocity and σ_{xy}^n is the x-y-component of the conductivity. Hence

$$-\frac{e^2}{Ah} \sum_{\vec{k} \in B.Z} \vec{\mathcal{E}} \times \vec{J}_n(\vec{k}) = \sigma_{xy}^n \hat{z} \times \vec{\mathcal{E}} \quad (\vec{J}_n(\vec{k}) \text{ points in } z).$$

$$\sigma_{xy}^n = \frac{e^2}{Ah} \sum_{\vec{k} \in B.Z} \vec{J}_n(\vec{k}) = \frac{e^2}{h 2\pi} \int_{B.Z} \vec{J}_n(\vec{k}) dK_x dK_y$$

$$\sigma_{xy}^n = \left(\frac{e^2}{h} \right) \frac{1}{2\pi} \int_{B.Z} \vec{J}_n(\vec{k}) dK_x dK_y$$

Then the Hall conductance of filled bands is proportional to the total Berry curvature of the band.

$$\frac{1}{2\pi} \int_{\vec{k} \in B.Z.} f_n(\vec{k}) dK_x dK_y = C_n$$

where C_n is the (first) Chern Number. This # is an integer and this leads to the quantization of the Hall conductance in units of e^2/h .

A related result in differential geometry is the Gauss-Bonnet theorem, which states that the total curvature of a closed 2D surface is quantized in units of 2π

$$\frac{1}{2\pi} \int_M K dA = 2 - 2g_M,$$

where K is the Gaussian curvature, g_M is the genus number of the surface M .

The Chern Insulator

for a closed surface in \mathbb{R}^3 with $\vec{n}(\vec{r})$ a unique 3-component unit vector pointing outward, normal to the surface at \vec{r} , we expect that the solid angle subtained by a small loop on the surface to be

$$d\mathcal{V} = K(\vec{r}) dA.$$

$K(\vec{r})$ is a measure of how curved is the surface at \vec{r} . Quantitatively it can be verified that

$$K_{ij}(\vec{r}) = \vec{n}(\vec{r}) \cdot [\partial_{x_i} \vec{n}(\vec{r}) \times \partial_{x_j} \vec{n}(\vec{r})]$$

Then for these kind of systems the Chern # (Skyrmion # in real space) is

$$C = \frac{1}{4\pi} \int \vec{n}(\vec{r}) \cdot (\partial_{x_i} \vec{n}(\vec{r}) \times \partial_{x_j} \vec{n}(\vec{r})) d\lambda_i d\lambda_j$$

In momentum space 2-band insulating systems in 2D, described by

$$H(\vec{k}) = \sum_i g_i d_i(\vec{k}) = \vec{e} \cdot \vec{d}(\vec{k})$$

are called Chern insulators. These insulators are topologically classified according to the Chern #

$$C = \frac{1}{4\pi} \int d(\vec{k}) \cdot [d_{kx} d(\vec{k}) \times d_{ky} d(\vec{k})] dk_y dk_x$$

Then their Berry curvature is $\mathcal{A}(\vec{k}) = \frac{1}{2} d \cdot (d_{kx} d \times d_{ky} d)$, and are characterized by

$$g_{xy} = \frac{e^2}{h} C \text{ when } E_F \text{ is in the gap.}$$

C for the Chern insulator takes 2-values

$$C = \begin{cases} 0 & \rightarrow \text{trivial} \\ \pm 1 & \rightarrow \text{topological.} \end{cases}$$