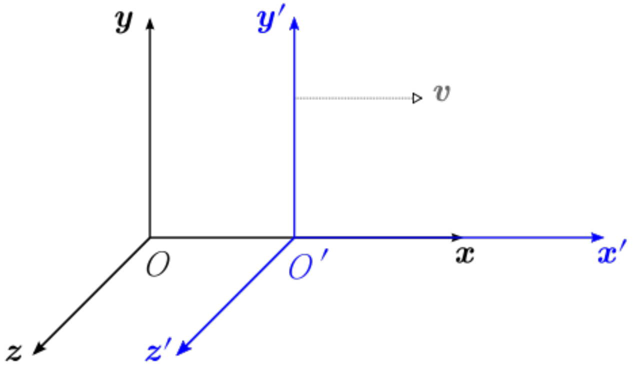
Special Relativity Part II: 4-vectors

Einstein's postulates of special relativity:

- The laws of Physics (*i.e.*, Mechanics & Electromagnetism) are the same in all inertial ref. frames
- The speed of light is the same in all inertial ref. frame

Lorentz transformations are the correct transformations to be used to relate physical quantities as measured in inertial ref. frames

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - \frac{v}{c^2}x) \end{cases} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$



$$\begin{cases} x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \\ ct' = \gamma(ct - \beta x) \end{cases}$$

it's a distance: [meter/second * second]

Relativistic mechanics

$\vec{p} = m\gamma\vec{u}$ Relativistic definition of linear momentum

$E = m\gamma c^2$ Relativistic definition of energy

$$\vec{p}\cdot\vec{p} \quad and \quad E^2 \Rightarrow E^2 = p^2c^2 + m^2c^4$$

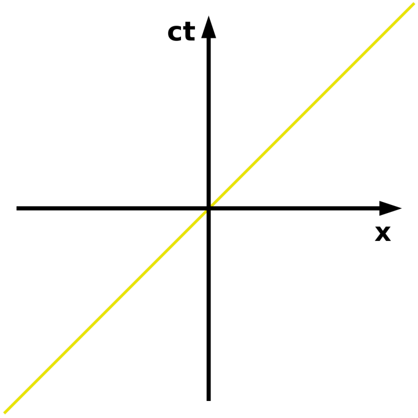
$\frac{\vec{p}}{E} = \frac{\vec{u}}{c^2}$; $\vec{\beta} \equiv \frac{\vec{u}}{c} = \frac{\vec{p}c}{E}$ where u is the velocity of the relativistic particle.

$\vec{F} = \frac{d\vec{p}}{dt}$ 2nd Newton's law

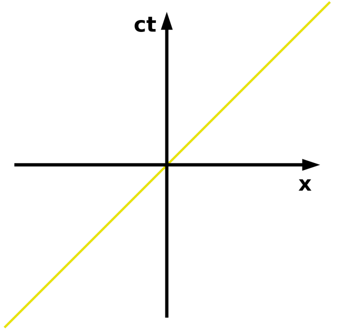
Space-time

- A formulation of Physics which incorporates explicitly the two relativity postulates is called ``**covariant**''.
- According to Lorentz transf., space and time form a single entity called space time: **3-dim Euclidean space + time.**
- This is a 4-dim manifold in which the distance between two points is generalized to the interval between two ``events''.
- The interval between 2 events is invariant (or independent) w.r.t. the inertial reference frame.
- This 4dim space is known as Minkowski space-time

Minkowski spacetime
in 1 spatial and 1 time dim



Space-time



$$(\Delta s)^2 = c^2\,(time\ interval)^2 - (space\ interval)^2$$

Δs = distance between 2 events in the Minkowski spacetime, which is left invariant by Lorentz transformations

Let's consider 2 points (aka 2 events) in the Minkowski spacetime :

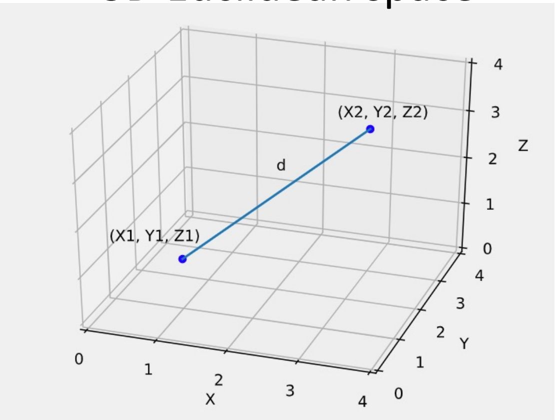
$$\mathcal{A} = (ct_1, x_1, y_1, z_1)\,; \quad \mathcal{B} = (ct_2, x_2, y_2, z_2)$$

The distance (invariant quantity under Lorentz transf.) is given by

$$||\mathcal{AB}||^2 = \pm\left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right]$$

What are the changes that we see here w.r.t. the Euclidean case? Let's have a look:

3D Euclidean space



$|\vec{r}_2 - \vec{r}_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ = distance between 2 points

$\vec{r} = (x, y, z)$ = vector position: position of point P = (x,y,z) w.r.t. the origin

$|\vec{r}|^2 = \vec{r}\cdot\vec{r} = x^2 + y^2 + z^2 = \sum_{i,j=1}^{3} \delta_{ij} x_i x_j = x^T x$ distance of point P from the origin

$x_1 = x; \quad x_2 = y; \quad x_3 = z$

Space-time

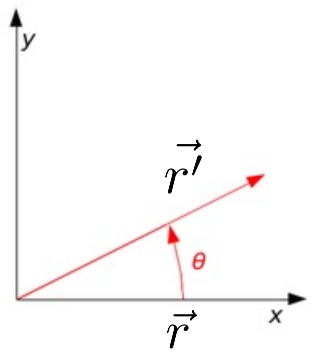
Kronecker delta symbol $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$

$$|\vec{r}|^2 = \vec{r}\cdot\vec{r} = x^2 + y^2 + z^2 = \sum_{i,j=1}^{3} \delta_{ij} x_i x_j = x^T x$$

$$\textit{Identity matrix } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|\vec{r}|^2$ is nothing but the length of a vector. This quantity is left invariant by rotations.

δ_{ij} is a tensor (for now, think about it as a n x n identity matrix) which allows us to define the concept of distance (aka ``metric'') between two points in a Euclidean space. It is called ``metric tensor''.



In 2dim, when we rotate a vector $\vec{r}$ from the x-axis to $\vec{r}$' at angle θ, we have $\vec{r'} = (x', y')$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|\vec{r}|^2 = |\vec{r'}|^2 = x'^2 + y'^2 = x^2 + y^2$$ Invariant under rotations!

That is, the length of the vector is left unchanged in a rotation.

An infinitesimal line element $(ds)^2 = dx^2 + dy^2 + dz^2$ (back in 3D now) is left invariant by rotations. The metric is (+++).

What happens in the Minkowski spacetime? The question we might want to ask is:
What is the infinitesimal line element that is left invariant by ``rotations'' (Lorentz transformations) in the Minkowski spacetime ?

$$(ds)^2 = (cdt)^2 - dx^2 - dy^2 - dz^2$$

If we pass from a ref. frame S to a ref. frame S' in motion with constant velocity, then the ``length'' does not change $(ds)^2 = (ds')^2$

Space-time: 4-vectors

Let's analyze the concept of invariance at a deeper level:

A 4-vector, is an element of the Minkowski spacetime which transforms according to Lorentz transformations when we go from one inertial ref. frame to another.

$$\mathcal{A}^{\mu} = (ct, x, y, z) = (ct, \vec{r}) = (\mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3)$$ Spacetime 4-vector

Time-like component — Space-like component

$\mu = 0, 1, 2, 3$ A Greek index is used to identify the components of a 4 vector

$i = 1, 2, 3$ A Latin index is normally used to identify only the spatial components

4-vectors can be written in various equivalent notations. **Contravariant representation: upper indices** A^{μ}

$$\begin{aligned}\mathbf{A} &= (A^0, A^1, A^2, A^3)\\ &= A^0\mathbf{E}_0 + A^1\mathbf{E}_1 + A^2\mathbf{E}_2 + A^3\mathbf{E}_3\\ &= A^0\mathbf{E}_0 + A^i\mathbf{E}_i\\ &= A^{\alpha}\mathbf{E}_{\alpha}\\ &= A^{\mu}\end{aligned}$$

$\mathbf{E}_i$ = basis vector

$$\mathbf{E}_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \mathbf{E}_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Einstein's convention on the repeated indices:

Contravariant and Covariant 4-vectors

$$\mathbf{E}_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \mathbf{E}_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix}A^0\\A^1\\A^2\\A^3\end{pmatrix} = A^\mu$$

Contravariant

Covariant representation: lower indices A_μ

$$\begin{aligned}\mathbf{A} &= (A_0,\, A_1,\, A_2,\, A_3)\\ &= A_0\mathbf{E}^0 + A_1\mathbf{E}^1 + A_2\mathbf{E}^2 + A_3\mathbf{E}^3\\ &= A_0\mathbf{E}^0 + A_i\mathbf{E}^i\\ &= A_\alpha\mathbf{E}^\alpha\end{aligned}$$

$$\mathbf{E}^0 = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}, \quad \mathbf{E}^1 = \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}, \quad \mathbf{E}^2 = \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}, \quad \mathbf{E}^3 = \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix}A_0 & A_1 & A_2 & A_3\end{pmatrix} = A_\mu$$

Relation between Contravariant and covariant representation of a 4-vector: **Metric tensor in the Minkowski spacetime**

$$A_\mu = \eta_{\mu\nu}A^\nu \qquad |A|^2 = A_\mu A^\mu = A_0A^0 - A_1A^1 - A_2A^2 - A_3A^3$$

The metric is (+---).

Generalization of the concept of length in the Minkowski spacetime

Metric tensor in the Minkowski spacetime

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathbf{E}^{\alpha} \cdot \mathbf{E}^{\beta} = \mathbf{E}_{\alpha} \cdot \mathbf{E}_{\beta}$$ This realizes the metric in the Minkowski spacetime

$$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta$$

Metric = (+---)

Scalar product of two 4-vecs A and B

$$\mathbf{A} \cdot \mathbf{B} = A^{\mu}\eta_{\mu\nu}B^{\nu} = A_{\nu}B^{\nu} = A^{\mu}B_{\mu}$$

$$\mathbf{A} \cdot \mathbf{B} = A_{\mu}\eta^{\mu\nu}B_{\nu}$$

$$\mathbf{A} \cdot \mathbf{B} = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3$$

$$\mathbf{A} \cdot \mathbf{A} = A_{\mu}A^{\mu} = A_0A^0 - A_1A^1 - A_2A^2 - A_3A^3$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} A^0 & A^1 & A^2 & A^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$\eta^{\mu\rho}\eta_{\rho\nu} = \delta^{\mu}{}_{\nu}$$ Generalization of the Kronecker Symbol

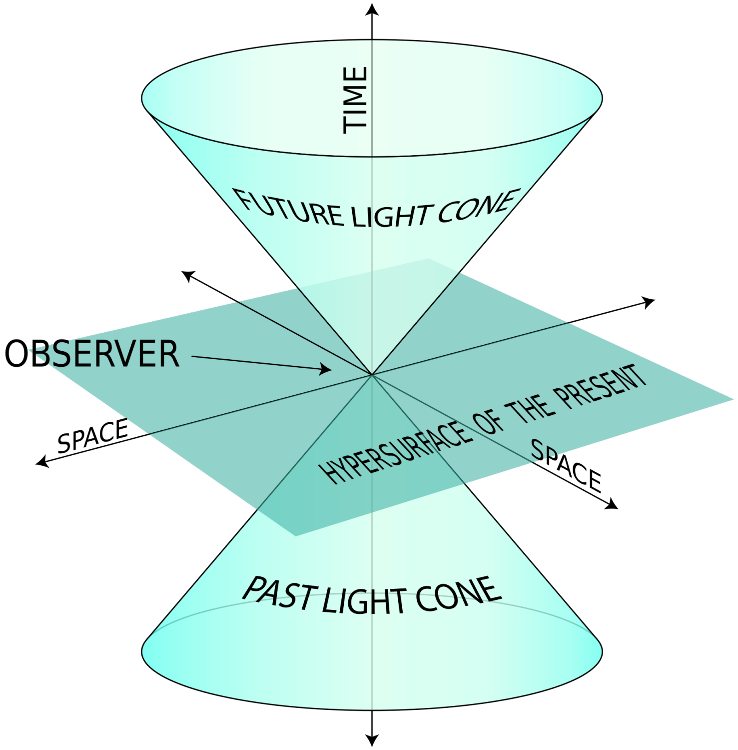
Causal structure

$$\mathbf{A}\cdot\mathbf{A} = A_\mu A^\mu = A_0A^0 - A_1A^1 - A_2A^2 - A_3A^3$$

$\mathbf{A}\cdot\mathbf{A} < \mathbf{0}$ Space-like 4-vectors: they live outside the cone

$\mathbf{A}\cdot\mathbf{A} > \mathbf{0}$ Time-like 4-vectors: they live inside the cone

$\mathbf{A}\cdot\mathbf{A} = \mathbf{0}$ Null 4-vectors: they live on the cone surface



Null vectors fall into three classes:

1. the zero vector, whose components in any basis are $(0, 0, 0, 0)$ (origin),
2. future-directed null vectors whose first component is positive (upper light cone), and
3. past-directed null vectors whose first component is negative (lower light cone).

Let's consider two events X and Y in the Minkowski spacetime:

1. X **chronologically** precedes Y if $\mathrm{Y}-\mathrm{X}$ is future directed time-like: $\|\mathrm{X}\| < \|\mathrm{Y}\|$
2. X **causally** precedes Y if $\mathrm{Y}-\mathrm{X}$ is future directed null or future directed time-like: $\|\mathrm{X}\| \leq \|\mathrm{Y}\|$
3. Reversed Tringle inequality holds $\|\mathrm{X}+\mathrm{Y}\| \geq \|\mathrm{X}\| + \|\mathrm{Y}\|$

Lorentz Transformation in terms of 4 vectors

Given two inertial or rotated frames of reference, a four-vector is defined as a quantity which transforms according to the Lorentz transformation matrix $\boldsymbol{\Lambda}$:

$$\mathbf{A}' = \boldsymbol{\Lambda}\mathbf{A}$$

In index notation, the contravariant and covariant components transform according to, respectively:

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu}\,, \quad A'_{\mu} = \Lambda_{\mu}{}^{\nu} A_{\nu}$$

in which the matrix $\boldsymbol{\Lambda}$ has components $\Lambda^{\mu}{}_{\nu}$ in row μ and column ν, and the inverse matrix $\boldsymbol{\Lambda}^{-1}$ has components $\Lambda_{\mu}{}^{\nu}$ in row μ and column ν.

$$\begin{cases} x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \\ ct' = \gamma(ct - \beta x) \end{cases} \Rightarrow \underbrace{\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}}_{A'^{\mu}} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^{\mu}{}_{\nu}} \underbrace{\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}}_{A^{\nu}}$$

This is the case of boost transformations which we have extensively studied

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$$ Inverse of the Lorentz transformation matrix

Hyperbolic rotations in the x-ct plane

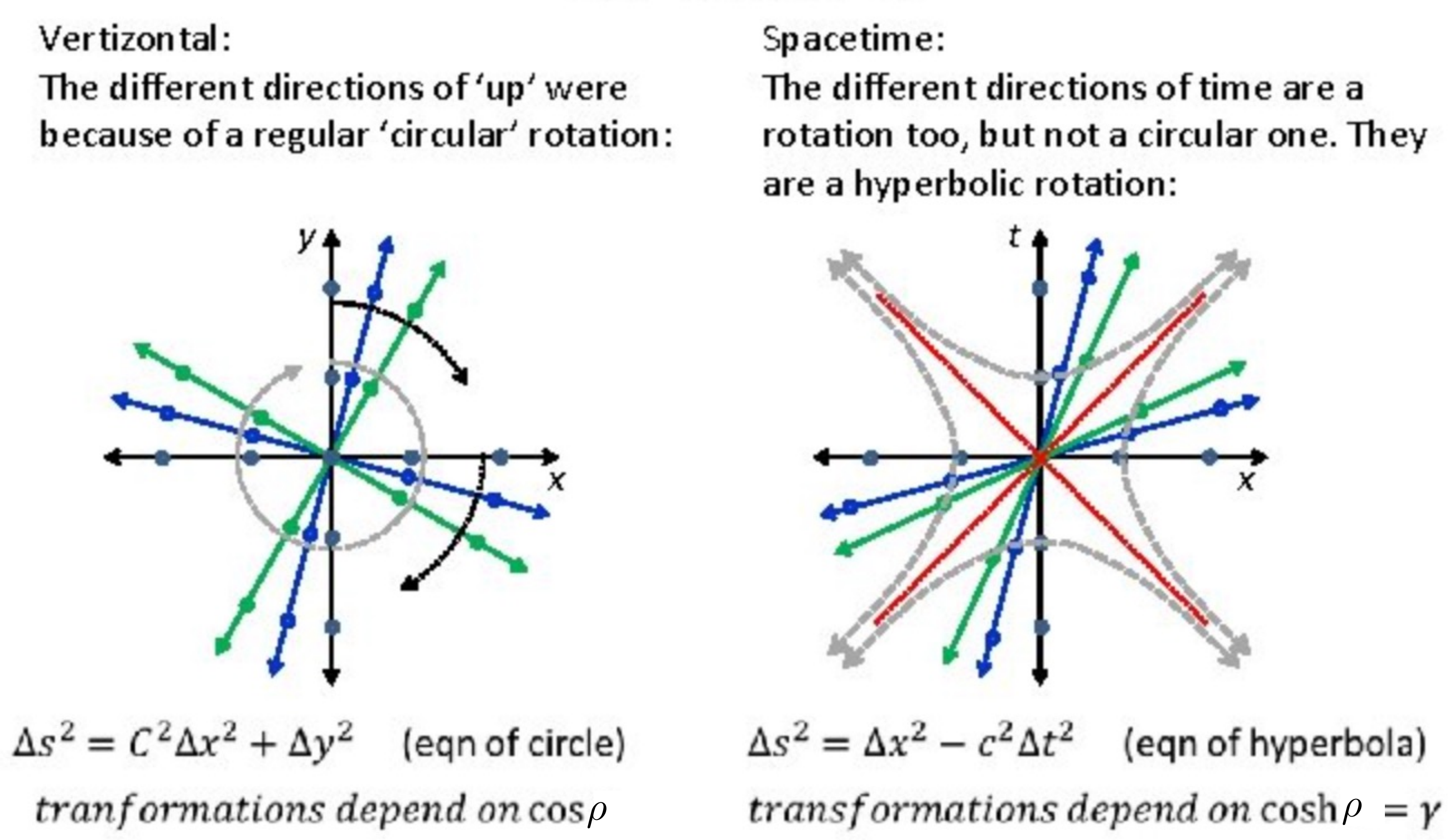
$$\cosh\rho = \frac{e^{\rho} + e^{-\rho}}{2} \qquad \sinh\rho = \frac{e^{\rho} - e^{-\rho}}{2} \qquad \cosh^2\rho - \sinh^2\rho = 1$$

$$ct' = ct\cosh\rho - x\sinh\rho$$
$$x' = x\cosh\rho - ct\sinh\rho$$
$$y' = y$$
$$z' = z$$

$$\tanh\rho = \beta \Rightarrow 1 - \beta^2 = \frac{1}{\cosh^2\rho} \Rightarrow \begin{aligned} \gamma &= \cosh\rho \\ \beta\gamma &= \sinh\rho \end{aligned}$$

ρ = Rapidity. Rapidity is commonly used as a measure for relativistic velocity. Mathematically, rapidity can be defined as the hyperbolic angle that differentiates two frames of reference in relative motion

Lorentz transformation could be seen as simply a hyperbolic rotation of the spacetime coordinates



Special Relativity is the discovery that the geometry of space is hyperbolic. All the results of special relativity can be derived and calculated using hyperbolic geometry / spacetime diagrams.

Tensors

An n^{th}-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space.
However, the dimension of the space is largely irrelevant in most tensor equations (with the notable exception of the contracted Kronecker delta). Tensors are generalizations of scalars (that have no indices),
vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices.

For example, properties that require one direction (first rank) can be fully described by
a 3×1 column vector, and properties that require two directions (second rank tensors), can be described by 9 numbers, as a 3×3 matrix. As such, in general an n^{th} rank tensor can be described by 3^n coefficients.
The need for second rank tensors comes when we need to consider more than one direction
to describe one of these physical properties.

$A = scalar$ Transformation law: scalars are invariant under any Lorentz transformation

$A^\mu = vector\ aka\ rank\ 1\ tensor$ Transformation law $A'^\mu = \Lambda^\mu{}_\rho A^\rho$

$A^{\mu\nu} = rank\ 2\ tensor$ Transformation law $A'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\delta A^{\rho\delta}$

…

4-momentum: linear momentum and energy

$$p_\mu = m\frac{dr_\mu}{d\tau}$$ $$dt = \gamma d\tau$$

dt is the time elapsed in the LAB frame
$d\tau$ is the time elapsed in the particle frame

$$p_\mu = m\frac{dr^\mu}{d\tau} = m\frac{dt}{d\tau}\frac{dr_\mu}{dt} = m\gamma\frac{d}{dt}(ct, x, y, z)$$

$$p_\mu = m\gamma(c, u_x, u_y, u_z) = m\gamma(c, \vec{u}) = (m\gamma c, \vec{p})$$

$E = m\gamma c^2$ Relativistic energy definition ⇨ $p_\mu = \left(\frac{E}{c}, \vec{p}\right)$ ⇨ 4-vector. It transforms as $p'_\mu = \Lambda_\mu{}^\nu p_\nu$

$$\begin{cases} \frac{E'}{c} = \gamma(\frac{E}{c} - \beta p_x) \\ p_{x'} = \gamma(-\beta\frac{E}{c} + p_x) \\ p_{y'} = p_y \\ p_{z'} = p_z \end{cases}$$

$$p^2 = p^\mu p_\mu = p^\mu \eta_{\mu\nu} p^\nu = \left(\frac{E}{c}\right)^2 - (p_x^2 + p_y^2 + p_z^2) = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2$$

p^2 is invariant: it's the same in all inertial ref. frames. This means that if we determine p^2 in the ref. of the particle where $\vec{p} = 0$ and $E = mc^2$ then we have:

$$p^2 = p^\mu p_\mu = m^2c^2$$

$p^2 = 0$ For photons!!!

4-Force definition

$$F_\mu = \frac{dp_\mu}{d\tau} \qquad p_\mu = \left(\frac{E}{c}, \vec{p}\right) \qquad E = \sqrt{p^2c^2 + m^2c^4} \qquad \vec{F} = \frac{d\vec{p}}{dt}$$

$$F_\mu = \frac{dp_\mu}{d\tau} = \gamma \frac{dp_\mu}{dt} = \left(F_0, \vec{F}\right)$$

According to our previous results:

$$\frac{dE}{dt} = \sum_{i=1}^{3} \frac{\partial E}{\partial p_i}\frac{\partial p_i}{\partial t} = \sum_{i=1}^{3} \frac{p_i c^2}{E}\frac{dp_i}{dt} = \frac{\vec{p}c^2}{E} \cdot \frac{d\vec{p}}{dt} = \frac{\vec{u}c^2}{c^2} \cdot \frac{d\vec{p}}{dt} = \vec{u} \cdot \frac{d\vec{p}}{dt} = \vec{u} \cdot \vec{F}$$

$$\frac{dE}{dt} = \frac{dm\gamma c^2}{dt} = \vec{u} \cdot \vec{F}$$

The definition of the 4 force in special relativity

Finally, we obtain

$$F_\mu = \gamma \left(\frac{\vec{u} \cdot \vec{F}}{c}, \vec{F}\right)$$

where $\vec{F} = (F_1, F_2, F_3)$

4-operators definition

The 4-gradient covariant components compactly written in four-vector and Ricci calculus notation are:

$$\frac{\partial}{\partial X^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_0, \partial_i) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) = \left(\frac{\partial_t}{c}, \vec{\nabla}\right) = \left(\frac{\partial_t}{c}, \partial_x, \partial_y, \partial_z\right) = \partial_\mu = {}_{,\mu}$$

The *comma* in the last part above ${}_{,\mu}$ implies the *partial differentiation* with respect to 4-position X^μ.

The contravariant components are:

$$\partial = \partial^\alpha = \eta^{\alpha\beta}\partial_\beta = (\partial^0, \partial^1, \partial^2, \partial^3) = (\partial^0, \partial^i) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) = \left(\frac{\partial_t}{c}, -\vec{\nabla}\right) = \left(\frac{\partial_t}{c}, -\partial_x, -\partial_y, -\partial_z\right)$$

Example

The 4-divergence of the 4-position $X^\mu = (ct, \vec{\mathbf{x}})$ gives the dimension of spacetime:

$$\partial \cdot \mathbf{X} = \partial^\mu \eta_{\mu\nu} X^\nu = \partial_\nu X^\nu = \left(\frac{\partial_t}{c}, -\vec{\nabla}\right) \cdot (ct, \vec{x}) = \frac{\partial_t}{c}(ct) + \vec{\nabla} \cdot \vec{x} = (\partial_t t) + (\partial_x x + \partial_y y + \partial_z z) = (1) + (3) = 4$$

D'Alambertian operator: wave equation compact

$$\Box = \partial \cdot \partial = \partial^\mu \partial_\mu = \partial^\mu \eta_{\mu\nu} \partial^\nu = \partial_\nu \partial^\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \left(\frac{\partial_t}{c}\right)^2 - \nabla^2$$

$$\Box \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0$$ Wave equation

- **Invariant**: means it's the same in all inertial ref. frames
- **Covariant**: (applied to 4-vec quantities) means that it's the mathematical structure of an equation that is invariant
- **Conserved**: means ``it does not change with time'' or also, ``the same before and after''

-Rest mass: is Lorentz invariant, but it is not conserved

-Energy: is conserved, but it is not Lorentz invariant

Maxwell Equations

The previous relativistic transformations suggest the electric and magnetic fields are coupled together, in a mathematical object with 6 components: an antisymmetric rank-2 tensor. This is called the electromagnetic field tensor, usually written as $F^{\mu\nu}$. In matrix form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Maxwell Equations
from classical electromagnetism

Differential equations
$\nabla \cdot \mathbf{E} = 4\pi\rho$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)$

Maxwell Equations in covariant form

The four-current is the contravariant four-vector which combines electric charge density ρ and electric current density **j**:

$$J^\alpha = (c\rho, \mathbf{j})\,.$$

The two inhomogeneous Maxwell's equations, Gauss's Law and Ampère's law (with Maxwell's correction) combine into (with (+ − − −) metric):[3]

Gauss–Ampère law

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

while the homogeneous equations – Faraday's law of induction and Gauss's law for magnetism combine to form $\partial_\sigma F^{\mu\nu} + \partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} = 0$, which may written using Levi-Civita duality as:

Gauss–Faraday law

$$\partial_\alpha\left(\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) = 0$$

where $F^{\alpha\beta}$ is the electromagnetic tensor, J^α is the four-current, $\varepsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, and the indices behave according to the Einstein summation convention.