Homework 9

Problem 11.1

a) if k is just a constant, we simply:

$$\int \frac{1}{f} df = k \int dt$$

$$\ln f = kt + c$$

$$\rightarrow f(t) = Ae^{kt}, \quad A \text{ is constant}$$

b) if $k = k(t)$:

$$\int \frac{1}{f} df = \int k dt$$

$$\rightarrow f(t) = C\exp\left(\int k dt\right),$$

which is not so simple; we need the functional form of k.

c) We cannot, in general, go from

$$\Psi(t) = \Psi(0) \exp\left\{-\frac{i}{\hbar}\left[H_1\tau + H_2(t-\tau)\right]\right\}$$

to $$\Psi(t) = \Psi(0) \exp\left(\frac{-i}{\hbar}H_1\tau\right)\exp\left[\frac{-i}{\hbar}H_2(t-\tau)\right]$$

because this would require that H_1 and H_2 commute, something that doesn't have to be true, in general. $k = k(t)$ is annoying!

Problem 11.2

- first, from Laporte's rule,

$$\langle n'l'm' | z | nlm \rangle = 0 \quad \text{if } l' + l = \text{even}$$

Thus, H_{ii} implies $l' = l$, and $l' + l = 2l$ = even, so $H'_{ii} = 0$. So too is $\langle 200 | z | 100 \rangle$, so we only care about:

$$\psi_{100} = 2a^{-3/2} e^{-r/a} \sqrt{\frac{1}{4\pi}}$$

$$\psi_{210} = \underbrace{\frac{1}{2\sqrt{6}} a^{-3/2} \left(\frac{r}{a}\right) e^{-r/2a}}_{} \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$\psi_{21\pm1} = \swarrow \cdot \mp \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

but in $\langle \psi_{21\pm1} | z | \psi_{100} \rangle$, we get

$$\int d\phi \, e^{\pm i\phi}$$, which is zero, so

only integral is

$$\langle 210 | z | 100 \rangle = eE \left(2a^{-3/2} \sqrt{\frac{1}{4\pi}} \right) \left(\frac{1}{2\sqrt{6}} a^{-5/2} \sqrt{\frac{3}{4\pi}} \right)$$

$$\times \int d^3r \; z e^{-r/a} \, r e^{-r/2a} \cos\theta$$

$$= \frac{eE}{4\pi a^4 \sqrt{2}} \int d^3r \; r^2 \cos^2\theta \; e^{-3r/2a}$$

$$= \downarrow \; \underbrace{\left(\int_0^\infty dr \; r^4 e^{-3r/2a} \right)}_{①} \underbrace{\left(\int_0^\pi d\theta \; \cos^2\theta \sin\theta \right)}_{②} \cdot 2\pi$$

① $\alpha \equiv \frac{2a}{3} \rightarrow \int_0^\infty dr \; r^4 e^{-r/\alpha} = 4! \left(\frac{2a}{3} \right)^5$

$$= 24 \left(\frac{2^5}{3^5} \right) a^5$$

② $u \equiv \cos\theta \quad du = -\sin\theta \, d\theta \qquad \pi \rightarrow -1, \; 0 \rightarrow 1$

$$\Rightarrow \int_{-1}^{1} du \; u^2 = \frac{1}{3} \left[u^3 \right]_{-1}^{1} = \frac{2}{3}$$

$$\rightarrow \frac{eE}{4\pi a^4 \sqrt{2}} \cdot 24 \left(\frac{2^5}{3^5} \right) a^5 \cdot \frac{2}{3} \cdot 2\pi$$

$$= \left(\frac{2^5}{3^5} \right) \frac{8eEa}{\sqrt{2}} = \boxed{\left(\frac{2^8}{3^5} \right) \frac{eEa}{\sqrt{2}}}$$

This is the only non-zero matrix element.

Problem 11.13

- From selection rules, if $l = l'$, the matrix element

$$\langle \psi_{100} | \vec{r} | \psi_{200} \rangle$$

vanishes, so $A = 0$ and $\tau = \frac{1}{A} \to$ infinite.

- In the other case,

$$\langle \psi_{100} | \vec{r} | \psi_{210} \rangle$$

has x and y components $= 0$; only z remains: we calculated it already:

$$\langle \psi_{100} | z | \psi_{210} \rangle = \frac{a}{\sqrt{2}} \left(\frac{2^8}{3^5} \right) = 2a \left(\frac{2^7}{3^5} \right)$$

- For $\langle \psi_{100} | \vec{r} | \psi_{21\pm1} \rangle$, the z component is 0, and x, y are related.

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$\psi_{21-1} = \frac{1}{2\sqrt{6a^5}} \sqrt{\frac{3}{8\pi}} \, r e^{-r/2a} \sin\theta \, e^{-i\phi}$$

$$= \frac{1}{8\sqrt{\pi a^5}} r e^{-r/2a} \sin\theta \, e^{-i\phi}$$

$$\to \langle \psi_{100} | x | \psi_{21-1} \rangle = \frac{1}{8\pi a^4} \int d^3r \, r e^{-3r/2a} \sin\theta \, e^{-i\phi} \, x$$

$$= \frac{1}{8\pi a^4} \underbrace{\int_0^\infty dr \, r^4 e^{-3r/2a}}_{①} \underbrace{\int_0^\pi d\theta \, \sin^3\theta}_{②} \underbrace{\int_0^{2\pi} d\phi \, e^{-i\phi} \cos\phi}_{③}$$

$$① = 4! \left(\frac{2a}{3} \right)^5 = 24 a^5 \left(\frac{2^5}{3^5} \right)$$

$$② = \frac{1}{4} \int_0^\pi (3\sin\theta - \sin 3\theta) \, d\theta$$

$$= \frac{1}{4} \left[-3\cos\theta + \frac{1}{3}\cos 3\theta \right]_0^\pi = \frac{1}{2} \left(3 - \frac{1}{3} \right) = \frac{4}{3}$$

$$③ = \frac{1}{2} \int_0^{2\pi} e^{-i\phi} (e^{i\phi} + e^{-i\phi}) \, d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} (1 + \underbrace{e^{-2i\phi}}_{=0}) \, d\phi = \pi$$

$$\to \frac{1}{8\pi a^4} \cdot 24 a^5 \left(\frac{2^5}{3^5} \right) \cdot \frac{4}{3} \cdot \pi = a \left(\frac{2^7}{3^5} \right)$$

- Now, $\langle 100 | x | 21-1 \rangle = -i \langle 100 | y | 21-1 \rangle$, so

$$|\langle 100 | \vec{r} | 21-1 \rangle|^2 = 2a^2 \left(\frac{2^7}{3^5} \right)^2$$

- Since $\langle 100 | \vec{r} | 211 \rangle$ differs only by a minus,

$$|\langle 100 | \vec{r} | 211 \rangle|^2 = 2a^2 \left(\frac{2^7}{3^5} \right)^2$$

- Similarly: $|\langle 100 | \vec{r} | 210 \rangle| = \uparrow$; all equal!

$$\to |\vec{p}|^2 = 2e^2 a^2 \left(\frac{2^7}{3^5} \right)^2 = \frac{(ea)^2 2^{15}}{3^{10}}$$

$$\to A = \frac{\omega_0^3 |\vec{p}|^2}{3\pi\epsilon_0 \hbar c^3}$$

- Here, $\omega_0 = \frac{1}{\hbar}(E_2 - E_1) = -\frac{1}{\hbar}\left(\frac{E_1}{4} - E_1 \right) = \frac{-3E_1}{4\hbar}$

$$\Rightarrow A = -\frac{3^3}{4^3} \frac{E_1^3}{\hbar^3} \frac{e^2 a^2 2^{15}}{3^{10}} \frac{1}{3\pi\epsilon_0 \hbar c^3}$$

$$= -\frac{2^9}{3^8} \frac{E_1^3 e^2 a^2}{\pi\epsilon_0 \hbar^4 c^3}$$

- Plugging everything in, we get a unit of s^{-1}:

$$A \approx 6.244 \times 10^8 \, s^{-1}$$

$$\to \tau = \frac{1}{A} \approx 1.6 \times 10^9 \, s$$

Problem 11.14

$$\langle n'l'm'|[L_z, x]|nlm\rangle = i\hbar\langle n'l'm'|y|nlm\rangle$$

$$\langle n'l'm'|L_z x|nlm\rangle - \langle n'l'm'|x L_z|nlm\rangle = \text{—}$$

- $L_z = L_z^\dagger$ so

$$(m'-m)\langle n'l'm'|x|nlm\rangle = i\langle n'l'm'|y|nlm\rangle \quad ①$$

- Similarly

$$(m'-m)\langle n'l'm'|y|nlm\rangle = -i\langle n'l'm'|x|nlm\rangle \quad ②$$

- Lastly: $(m'-m)\langle n'l'm'|z|nlm\rangle = 0$ ③

- If $m=m'$, from (1) and ② we can obviously tell the first line of 11.76 (but can't say anything about z)
- If $m' = m \pm 1$, from ③ the z-component must be zero, and we can easily derive the 2nd line of 11.76 from (2).
- Lastly, if $\Delta m = \pm k$ w/ $k > 1$, then from ②:

$$\langle n'l'm'|x|nlm\rangle = \pm k i\langle n'l'm'|y|nlm\rangle,$$

but from ①:

$$\langle n'l'm'|x|nlm\rangle = \pm\frac{i}{k}\langle n'l'm'|y|nlm\rangle$$

which is only true if the matrix element vanishes; thus, $\Delta m = 0$ or ± 1 is required.

Problem 11.15

- First: $[L^2, z] = [L_x^2, z] + [L_y^2, z] + \cancel{[L_z^2, z]}^{0}$

$$= \underbrace{L_x[L_x, z] + [L_x, z]L_x} + (x \to y)$$

$$\to L_x(-i\hbar y) + (-i\hbar y)L_x = -i\hbar(L_x y + y L_x)$$

- For $(x \to y)$, add a minus since $[L_y, z] = i\hbar x$:

$$[L^2, z] = i\hbar(L_y x + x L_y - L_x y - y L_x)$$

- Now, $[L_y, x] = L_y x - x L_y$

$$\to L_y x = [L_y, x] + x L_y = -i\hbar z + x L_y$$

$$L_x y = i\hbar z + y L_x$$

$$\Rightarrow [L^2, z] = 2i\hbar(x L_y - y L_x - i\hbar z)$$

- Next,

$$[L^2, [L^2, z]] = 2i\hbar([L^2, x L_y] - [L^2, y L_x] - i\hbar[L^2, z])$$

- Generically:

$$[A, BC] = ABC - BCA$$
$$= ABC - BAC + BAC - BCA$$
$$= [A,B]C + B[A,C]$$

$$\to [L^2, x L_y] = [L^2, x]L_y + x[L^2, L_y]$$
$$= [L^2, x]L_y$$

$$\to 2i\hbar([L^2, x]L_y - [L^2, y]L_x - i\hbar(L^2 z - zL^2))$$

$\Rightarrow$ x, y, z are cyclic; from a)

$$[L^2, x] = 2i\hbar(y L_z - z L_y - i\hbar x)$$
$$[L^2, y] = 2i\hbar(z L_x - x L_z - i\hbar y)$$

$$\to 2i\hbar\Big[2i\hbar(yL_z - zL_y - i\hbar x)L_y - 2i\hbar(zL_x - xL_z - i\hbar y)L_x$$
$$- i\hbar(L^2 z + zL^2) + 2i\hbar z L^2\Big]$$

$$= 2\hbar^2(zL^2 + L^2 z) - 4\hbar^2\big[yL_zL_y - zL_y^2 - i\hbar x L_y$$
$$- zL_x^2 + xL_zL_x + i\hbar y L_x + zL^2$$

$$= \sim [(yL_z - i\hbar x)L_y + (xL_z + i\hbar y)L_x + zL_z^2$$

- Now, $[L_z, y] = L_z y - y L_z = -i\hbar x$

$$\to yL_z - i\hbar x = L_z y \quad \text{and}$$
$$xL_z + i\hbar y = L_z x$$

$$\to \sim [L_z(yL_y + xL_x + zL_z)]$$
$$= L_z \vec{r}\cdot\vec{L} = 0$$

$$\Rightarrow [L^2, [L^2, z]] = 2\hbar^2(zL^2 + L^2 z) \checkmark$$

$\to$ obviously, in 3d we will have

$$= 2\hbar^2(\vec{r}L^2 + L^2\vec{r})$$

- Plugging in:

$$\langle n'l'm'|[L^2[L^2, \vec{r}]]|nlm\rangle =$$
$$= \langle n'l'm'|L^2[L^2, \vec{r}]|nlm\rangle - \langle n'l'm'|[L^2, \vec{r}]L^2|nlm\rangle$$
$$= \hbar^2[l'(l'+1) - l(l+1)]\langle n'l'm'|[L^2, \vec{r}]|nlm\rangle$$
$$= \hbar^4[l'(l'+1) - l(l+1)]^2\langle n'l'm'|\vec{r}|nlm\rangle$$

also $= 2\hbar^2\langle n'l'm'|\vec{r}L^2 + L^2\vec{r}|nlm\rangle$

$$= 2\hbar^4[l'(l'+1) + l(l+1)]\langle n'l'm'|\vec{r}|nlm\rangle$$

- So, we must have

$$2[l'(l'+1) + l(l+1)] = [l'(l'+1) - l(l+1)]^2$$

or else the matrix element is zero.

- Rearranging first term:

$$2l'^2 + 2l' + 2l^2 + 2l$$

$$l'^2 + 2l' + 2l'l + 2l + l^2 + 1 + l'^2 - 2l'l + l^2 - 1$$

$$\to (l' + l + 1)^2 + (l' - l)^2 - 1$$

- 2nd: $l'^2 + l' - l^2 - l$

$$(l'^2 - l^2) + (l' - l)$$
$$(l' - l)(l' + l) + (l' - l)$$
$$(l' - l)(l' + l + 1)$$

- Now, let $a \equiv l' + l + 1$, $b \equiv l' - l$

$$\to a^2 + b^2 - 1 = a^2 b^2$$
$$a^2 - a^2b^2 + b^2 - 1 = 0$$
$$a^2(1 - b^2) - 1(1 - b^2) = 0$$
$$(a^2 - 1)(1 - b^2) = 0$$

- Thus, $(l' - l)^2 = 1$

$$l' - l = \pm 1$$

$$l' = l \pm 1$$