Homework 3

Problem 6.18

a) $\hat{\Pi}\hat{T}(a) f(x) = \hat{\Pi} f(x-a) = f(-x+a)$

$\hat{T}(a)\Pi f(x) = \hat{T}(a) f(-x) = f(-x-a)$

- in general, $f(-x-a) \neq f(-x+a)$, so

$$\underline{[\hat{\Pi}, \hat{T}(a)] = 0}$$

b) $f_{\vec{p}}(\vec{x})$ has $\hat{H} f_{\vec{p}}(\vec{x}) = E f_{\vec{p}}(\vec{x})$ and

$\hat{\vec{p}} f_{\vec{p}}(x) = \vec{p} f_{\vec{p}}(\vec{x})$.

we know what this function is: an exponential

$$f_{\vec{p}}(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left[\frac{-i\vec{p}\cdot\vec{x}}{\hbar}\right]$$

so, $\hat{\pi} f_{\vec{p}}(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left[\frac{i\vec{p}\cdot\vec{x}}{\hbar}\right] = \underline{f_{-\vec{p}}(\vec{x})}$

The Hamiltonian is a fn only of $\hat{\vec{p}}^2$, so this state has the same energy, as expected of a system w/ inversion symmetry.

c) Can also choose $f(x)$ (1-dim) s.t. $\hat{H} f(x) = E f(x)$, &

$\hat{\Pi} f(x) = \pm f(x)$

- such fns are sin / cos, as the book states:

$$f^{(1)}(x) = \frac{1}{\sqrt{\pi\hbar}} \cos\left(\frac{px}{\hbar}\right) \quad f_p^{(2)}(x) = \frac{1}{\sqrt{\pi\hbar}} \sin\left(\frac{px}{\hbar}\right)$$

$$\to \hat{T}(a) f^{(1)}(x) = \frac{1}{\sqrt{\pi\hbar}} \cos\left(\frac{p(x-a)}{\hbar}\right)$$

$$= \frac{1}{\sqrt{\pi\hbar}} \cos\left(\frac{px}{\hbar} - \frac{pa}{\hbar}\right)$$

$$= \frac{1}{\sqrt{\pi\hbar}} \left[\cos\frac{px}{\hbar}\cos\frac{pa}{\hbar} + \sin\frac{px}{\hbar}\sin\frac{pa}{\hbar}\right]$$

$$\to \hat{T}(a) f^{(2)}(x) = \frac{1}{\sqrt{\pi\hbar}} \sin\left(\frac{p(x-a)}{\hbar}\right)$$

$$= \frac{1}{\sqrt{\pi\hbar}} \sin\left(\frac{px}{\hbar} - \frac{pa}{\hbar}\right)$$

$$= \frac{1}{\sqrt{\pi\hbar}} \left[\sin\frac{px}{\hbar}\cos\frac{pa}{\hbar} - \cos\frac{px}{\hbar}\sin\frac{pa}{\hbar}\right]$$

- both now contain terms proportional to both $f^{(1)}$ and $f^{(2)}$ $\to$ $\hat{T}$ mixes the two states together.

Problem 6.20

- we start at Eq. (6.45) but w/ $[\hat{L}_-, \hat{f}]$ instead of $\hat{L}_+$:

$$\langle n', l', m' | [\hat{L}_-, \hat{f}] | n, l, m \rangle = 0$$

$$\langle n', l', m' | \hat{L}_- \hat{f} | n, l, m \rangle - \langle n', l', m' | \hat{f} \hat{L}_- | n, l, m \rangle = 0$$

$$A_{l'}^{m'} \langle n', l', (m'+1) | \hat{f} | n, l, m \rangle - B_l^m \langle n', l', m' | \hat{f} | n, l, (m-1) \rangle = 0,$$

where $A_{l'}^{m'} = \hbar \sqrt{l'(l'+1) - m'(m'+1)}$, and

$$B_l^m = \hbar \sqrt{l(l+1) - m(m-1)}.$$

- we know from examining $\hat{L}_z$ and $\hat{L}^2$ that we must have $\Delta m = 0$ and $l = l'$, so $m' = m-1$.
- w/ this:

$$A_l^{m-1} \langle n', l, m | \hat{f} | n, l, m \rangle - B_l^m \langle n', l, (m-1) | \hat{f} | n, l, (m-1) \rangle = 0$$

now, $A_l^{m-1} = \hbar \sqrt{l(l+1) - (m-1)(m-1+1)}$

$$= \hbar \sqrt{l(l+1) - m(m-1)} = B_l^m,$$

so:

$$\langle n', l, m | \hat{f} | n, l, m \rangle = \langle n', l, (m-1) | \hat{f} | n, l, (m-1) \rangle,$$

which is exactly the same as the $\hat{L}_+$ result.

Problem 6.21

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_{211} + \psi_{21-1}\right)$$

- $\vec{r}$ is a scalar operator, so we can use our selection rules:

$$\rightarrow \langle r \rangle = \frac{1}{2}\Big[\langle 211|r|211\rangle + \langle 210|r|210\rangle +$$
$$\langle 211|r|210\rangle + \langle 210|r|211\rangle\Big]$$

- 2^{nd} two matrix elements have $\Delta m \neq 0$, so they vanish, and 1^{st} two are identical:

$$\langle r \rangle = \langle 21\|r\|21\rangle$$

- Choose easiest to compute: $n=2,\ l=1,\ m=0$:

$$\langle 210|r|210\rangle = \int d^3r\ \psi_{210}\, r\, \psi_{210}$$

$$\psi_{210} = R_{21} Y_1^0 = \frac{1}{2\sqrt{6}} a^{-3/2}\left(\frac{r}{a}\right)\exp(-r/2a)$$
$$\times \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$= \frac{a^{-5/2}}{2}\sqrt{\frac{1}{8\pi}}\ r\cos\theta\ \exp(-r/2a)$$

$$\rightarrow \langle r \rangle = \frac{a^{-5}}{32\pi}\int d^3r\ r^3\cos^2\theta\ \exp(-r/a)$$

$$= \frac{a^{-5}}{32\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\ \sin\theta\cos^2\theta \int_0^{\infty} r^5 \exp(-r/a)$$

$$= \frac{a^{-5}}{16}\int_0^{\pi} d\theta\ \sin\theta\cos^2\theta \int_0^{\infty} r^5 \exp(-r/a)$$

- use formula from book:

$$\int_0^{\infty} r^n e^{-r/a}\, dr = n!\, a^{n+1} \rightarrow 5!\, a^6$$

$$\langle r \rangle = 5a \cdot \frac{3}{2}\int_0^{\pi} d\theta\ \sin\theta\cos^2\theta$$

$$u = \cos\theta\ ,\ du = -\sin\theta\, d\theta\ ,\quad \begin{matrix}\pi \rightarrow \cos\pi = -1 \\ 0 \rightarrow \cos 0 = 1\end{matrix}$$

$$= 5a\,\frac{3}{2}\int_{-1}^{1} du\ u^2 = 5a\,\frac{3}{2}\cdot\left[\frac{u^3}{3}\right]_{-1}^{1}$$

$$\langle r \rangle = 5a$$

Problem 6.26

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$\rightarrow \hat{P}_H(t)\psi_n(x) = \hat{U}^\dagger(t)\,\hat{p}\,U(t)\,\psi_n(x)$

- we just follow the same steps as in the example, using

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_+ - \hat{a}_-\right)$$

$$\rightarrow \quad \hat{P}_H(t)\psi_n(x) = \left(i\sqrt{\frac{\hbar m\omega}{2}}\right)e^{i\hat{H}t/\hbar}\left(\hat{a}_+ - \hat{a}_-\right)e^{-iE_nt/\hbar}\psi_n(x)$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-iE_nt/\hbar}\cdot e^{i\hat{H}t/\hbar}\left(\hat{a}_+\psi_n(x) - \hat{a}_-\psi_n(x)\right)$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-iE_nt/\hbar}\cdot e^{i\hat{H}t/\hbar}\left[\sqrt{n+1}\,\psi_{n+1}(x) - \sqrt{n}\,\psi_{n-1}(x)\right]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}e^{-iE_nt/\hbar}\left[\sqrt{n+1}\,e^{iE_{n+1}t/\hbar}\psi_{n+1}(x) - \sqrt{n}\,e^{iE_{n-1}t/\hbar}\psi_{n-1}(x)\right]$$

- we know:

$$E_{n+1} - E_n = \left(n+1+\frac{1}{2}\right)\hbar\omega - \left(n+\frac{1}{2}\right)\hbar\omega$$

$$= \hbar\omega$$

$$\rightarrow \quad \omega = (E_{n+1} - E_n)/\hbar = -(E_{n-1} - E_n)/\hbar$$

- we can thus write exponentials as:

$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\,e^{i\omega t}\psi_{n+1}(x) - \sqrt{n}\,e^{-i\omega t}\psi_{n-1}(x)\right]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[e^{i\omega t}\hat{a}_+ - e^{-i\omega t}\hat{a}_-\right]$$

- using definition of ladder op:

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega x\right)$$

$$\rightarrow = \frac{i}{2}\Big[(\cos\omega t + i\sin\omega t)(-i\hat{p} + m\omega\hat{x})$$

$$- (\cos\omega t - i\sin\omega t)(i\hat{p} + m\omega\hat{x})\Big]$$

$$= \frac{i}{2}\left[-2i\hat{p}\cos(\omega t) + 2im\omega\hat{x}\sin(\omega t)\right]$$

$$= \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)$$

$$\Rightarrow \quad \boxed{\hat{P}_H(t) = \hat{P}_H(0)\cos(\omega t) - m\omega\hat{x}_H(0)\sin(\omega t)}$$

Problem 7.1

$$\hat{H}' = \alpha\delta(x-a/2)$$

- we know $E_n' = \langle \psi_n^0 | \hat{H}' | \psi_n^0 \rangle$

$$= \alpha |\psi_n^0(x=\tfrac{a}{2})|^2$$

- Solutions to infinite square well are

$$\psi_n^0(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

if $x=\frac{a}{2}$, we have a $\sin\left(\frac{n\pi}{2}\right)$, which is of course zero for odd n.

b) $$\psi_1' = \sum_{m\neq 1} \frac{\langle \psi_m^0 | \hat{H}' | \psi_1^0 \rangle}{(E_1^0 - E_m^0)} \psi_m^0$$

- first, $E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$, so, switching to n as the loop variable to not collide w/ mass:

$$\psi_1' = \sum_{\substack{n\neq 1,\\ \text{odd}}} \frac{\frac{2\alpha}{a}\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{\pi^2\hbar^2}{2ma^2} - E_n^0\right)} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$m=3 \rightarrow \frac{-\frac{2\alpha}{a}}{\frac{\pi^2\hbar^2}{2ma^2}(1-9)} \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)$$

$$m=5 \rightarrow \frac{\frac{2\alpha}{a}}{\frac{\pi^2\hbar^2}{2ma^2}(1-25)} \sqrt{\frac{2}{a}} \sin\left(\frac{5\pi x}{a}\right)$$

$$m=7 \rightarrow \frac{-\frac{2\alpha}{a}}{\frac{\pi^2\hbar^2}{2ma^2}(1-49)} \sqrt{\frac{2}{a}} \sin\left(\frac{5\pi x}{a}\right)$$

$$\rightarrow \psi_1' \approx \frac{4\alpha m}{\pi^2\hbar^2}\sqrt{\frac{2}{a}}\left[\frac{1}{8}\sin\left(\frac{3\pi x}{a}\right) - \frac{1}{24}\sin\left(\frac{3\pi x}{a}\right) + \frac{1}{48}\sin\left(\frac{5\pi x}{a}\right)\right]$$

$$\psi_1' \approx \frac{\alpha m}{\pi^2\hbar^2}\sqrt{\frac{2}{a}}\left[\frac{1}{2}\sin\left(\frac{3\pi x}{a}\right) - \frac{1}{6}\sin\left(\frac{3\pi x}{a}\right) + \frac{1}{12}\sin\left(\frac{5\pi x}{a}\right)\right]$$

Problem 7.2

$$V(x) = \frac{1}{2}kx^2$$

$$E_n = (n+\tfrac{1}{2})\hbar\omega \ , \quad \omega = \sqrt{\frac{k}{m}}$$

a) This involves just changing $k \to (1+\epsilon)k$:

$$E_n = (n+\tfrac{1}{2})\hbar\omega, \quad \omega = \sqrt{\frac{(1+\epsilon)k}{m}}$$

$$= \hbar\sqrt{\frac{(1+\epsilon)k}{m}}\,(n+\tfrac{1}{2})$$

$$= \sqrt{(1+\epsilon)} \cdot \hbar\sqrt{\frac{k}{m}}\,(n+\tfrac{1}{2})$$

$$= \sqrt{(1+\epsilon)} \cdot \hbar\omega(n+\tfrac{1}{2})$$

- expansion of $\sqrt{1+\epsilon} = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + O(\epsilon^3)$:

$$\underline{E_n = \hbar\omega(n+\tfrac{1}{2})\left(1+\frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \dots\right)}$$

b)

$$E_n' = \langle \psi_n^0 | \hat{H}' | \psi_n^0 \rangle$$

$$\rightarrow \hat{H}' = \hat{H} - \hat{H}^0 = \frac{1}{2}(1+\epsilon)kx^2 - \frac{1}{2}kx^2 = \frac{\epsilon}{2}kx^2$$

$$E_n' = \epsilon\langle \psi_n^0 | \tfrac{1}{2}kx^2 | \psi_n^0 \rangle = \epsilon\langle V^0 \rangle_{\psi_n^0}$$

- This is the expectation value of the potential for the unperturbed states of the SHO; from Example 2.5,

$$\langle V^0 \rangle_{\psi_n^0} = \frac{1}{2}E_n^0 = \frac{1}{2}\hbar\omega(n+\tfrac{1}{2}),$$

so

$$E_n' = \frac{\epsilon}{2}\hbar\omega(n+1)$$

- this is exactly the ϵ^1 contribution from (a)!