PHYS4260 SPRING 2025. HW 11
DUE DATE: MONDAY APRIL 28 2025 BY 12PM.

The structure of the Hamiltonian reported below

$$H = \frac{p^2}{2m} + V(q) \tag{1}$$

allows us to obtain a more compact form of the path-integral for the transition amplitude $\langle q_f t_f | q_i t_i \rangle$ (see April 22 lecture notes) by simply integrating out the p variables.

Prove the following equation

$$\langle q_f t_f | q_i t_i \rangle = \lim_{n\to\infty} \int \prod_{j=1}^{n} dq_j \prod_{j=1}^{n} \frac{dp_j}{h} \exp\left\{ \frac{i}{\hbar} \sum_{j=0}^{n} \left[p_j (q_{j+1} - q_j) - \tau H(p_j, \bar{q}_j) \right] \right\} = N \int \mathcal{D}q \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} L(q, \dot{q}) dt \right] \tag{2}$$

where τ is the step used to divide the $(t_f - t_i)$ interval such that $\tau = (t_f - t_i)/(n+1)$, and identify the coefficient N. Use the result for the gaussian integral given below

$$\int_{-\infty}^{+\infty} e^{-ax^2 + bx + c}\, dx = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{4a} + c} \tag{3}$$

Problem 2 The transition amplitude (propagator) for a non-relativistic quantum mechanical system represented by a particle of mass m evolving from initial time t_i to final time t_f in the path integral formalism can be written as follows:

$$\langle q_f t_f | q_i t_i \rangle = N \int \mathcal{D}q \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} L(q, \dot{q})\, dt\right]$$

where $L = T - V$. Let us assume a 1-dim system with coordinate $q = x$, and that the potential $V(x)$ is small, i.e., the time integral of $V(x, t)$ is small compared with $\hbar$. $V(x)$ can then be expanded in series:

$$\exp\left[\frac{-i}{\hbar}\int_{t_i}^{t_f} V(x, t)\, dt\right] = 1 - \frac{i}{\hbar}\int_{t_i}^{t_f} V(x, t)\, dt - \frac{1}{2!\hbar^2}\left[\int_{t_i}^{t_f} V(x, t)\, dt\right]^2 + \cdots$$

and substituting this in the propagator above, we obtain the following series expansion:

$$K = K_0 + K_1 + K_2 + \ldots$$

where K_0 represent the free propagator and is given as

$$K_0 = N\int\left[\exp\left(\frac{i}{\hbar}S\right)\right]\mathcal{D}x = N\int\left[\exp\left(\frac{i}{\hbar}\int \tfrac{1}{2}m\dot{x}^2\, dt\right)\right]\mathcal{D}x$$

Using the discretized form of K_0 and the integral identity given in the formula sheet, prove that:

$$K_0(x_f t_f; x_i t_i) = \theta(t_f - t_i)\left(\frac{m}{i\hbar(t_f - t_i)}\right)^{1/2} \exp\left[\frac{im(x_f - x_i)^2}{2\hbar(t_f - t_i)}\right]$$