Homework 5

Problem 7.17

- we have that

$$E_r^1 = -\frac{1}{2mc^2}\left[E^2 - 2E\langle V\rangle + \langle V^2\rangle\right]$$

- for SHO $\rightarrow$ $E = (n+\frac{1}{2})\hbar\omega$, $V = \frac{1}{2}m\omega^2 x^2$, so

$$E_r^1 = -\frac{1}{2mc^2}\left[(n+\tfrac{1}{2})^2(\hbar\omega)^2 - (n+\tfrac{1}{2})\hbar m\omega^3\langle x^2\rangle + \left(\frac{m\omega^2}{2}\right)^2\langle x^4\rangle\right]$$

- $\langle x^2\rangle = \int \psi_n^* \hat{x}^2 \psi_n \, dx$

- using the fact that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-),$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int \psi_n^*(\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2)\psi_n \, dx$$

$$= \frac{\hbar}{2m\omega}\int \psi_n^*(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)\psi_n$$

$$= \frac{\hbar}{2m\omega}\int\left[n\,\psi_{n-1}^*\psi_{n-1} + (n+1)\psi_{n+1}^*\psi_{n+1}\right]dx$$

$$= \frac{\hbar}{2m\omega}(2n+1) = (n+\tfrac{1}{2})\frac{\hbar}{m\omega}$$

$$\langle x^4\rangle = \left(\frac{\hbar}{2m\omega}\right)^2\int dx\, \psi_n^*(\hat{a}_+ + \hat{a}_-)^4\psi_n$$

- all terms w/ non-equal #'s of $\hat{a}_+$'s & $\hat{a}_-$'s vanish by orthogonality
- left w/ essentially all 4-permutations (there are 6)

$$\rightarrow (\hat{a}_+ + \hat{a}_-)^4 = \overset{①}{a_+a_+a_-a_-} + \overset{②}{a_-a_-a_+a_+} + \overset{③}{a_+a_-a_-a_+} + \overset{④}{a_-a_+a_+a_-} + \overset{⑤}{a_+a_-a_+a_-} + \overset{⑥}{a_-a_+a_-a_+}$$

$$\hat{a}_+\hat{a}_+\psi_n = \sqrt{n+1}\,\hat{a}_+\psi_{n+1} \rightarrow \sqrt{n+1}\sqrt{n+2}$$

$$\hat{a}_-\hat{a}_-\psi_n = \sqrt{n}\,\hat{a}_-\psi_{n-1} \rightarrow \sqrt{n}\sqrt{n-1}$$

$$a_+a_-\psi_n = \sqrt{n}\,a_+\psi_{n-1} = \sqrt{n}\sqrt{n-1+1} = n$$

$$a_-a_+\psi_n = \sqrt{n+1}\,a_-\psi_{n+1} = \sqrt{n+1}\sqrt{n+1} = n+1$$

① $\rightarrow n(n-1)$ ② $\rightarrow (n+1)(n+2)$

③ $\rightarrow n(n+1)$ ④ $\rightarrow n(n+1)$

⑤ n^2 ⑥ $\rightarrow (n+1)^2$

$$= n^2 - n + n^2 + 3n + 2 + n^2 + n + n^2 + n + n^2 + n^2 + 2n + 1$$

$$= 6n^2 + 6n + 3$$

$$\rightarrow E_r^1 = -\frac{1}{2mc^2}\left[(n+\tfrac{1}{2})^2(\hbar\omega)^2 - (n+\tfrac{1}{2})\hbar m\omega^3(n+\tfrac{1}{2})\frac{\hbar}{m\omega} + \left(\frac{m\omega^2}{2}\right)^2\left(\frac{\hbar}{2m\omega}\right)^2(6n^2+6n+3)\right]$$

$$= \frac{-1}{2mc^2}\cdot\frac{m^2\omega^4}{4}\cdot\frac{\hbar^2}{4m^2\omega^2}(6n^2+6n+3)$$

$$= \frac{-\hbar^2\omega^2}{32mc^2}(6n^2+6n+3)$$

$$\Rightarrow \boxed{-\frac{3\hbar^2\omega^2}{32mc^2}(2n^2+2n+1)}$$

Problem 7.19

$[J_i, J_j] = i\hbar J_k \varepsilon_{ijk}$

a) $[\vec{L}\cdot\vec{S}, \vec{L}]$

$\rightarrow [L_x S_x + L_y S_y + L_z S_z, \vec{L}]$

x: $[L_y S_y, L_x] + [L_z S_z, L_x]$

$= S_y [L_y, L_x] + S_z [L_z, L_x]$

$= i\hbar [-S_y L_z + S_z L_y]$

(y, z follow similarly)

y: $[L_x S_x, L_y] + [L_z S_z, L_y]$

$= i\hbar [S_x L_z - S_z L_x]$

z: $[L_x S_x, L_z] + [L_y S_y, L_z]$

$= i\hbar [-S_x L_y + S_y L_x]$

- Final result is a vector; looks like cross prod:

$$[\vec{L}\cdot\vec{S}, \vec{L}] = i\hbar (\vec{S} \times \vec{L})$$

b) Since $[\vec{L}, \vec{S}]$, this is the same but swapping $\vec{L}, \vec{S}$

$$[\vec{L}\cdot\vec{S}, \vec{S}] = i\hbar (\vec{L} \times \vec{S})$$

c) $[\vec{L}\cdot\vec{S}, \vec{J}] = [\vec{L}\cdot\vec{S}, \vec{L}+\vec{S}]$

$= [\vec{L}\cdot\vec{S}, \vec{L}] + [\vec{L}\cdot\vec{S}, \vec{S}]$

$= i\hbar [(\vec{S}\times\vec{L}) + (\vec{L}\times\vec{S})]$

- but, for any $\vec{A}, \vec{B}$, $(\vec{A}\times\vec{B}) = -(\vec{B}\times\vec{A})$, so

$$[\vec{L}\cdot\vec{S}, \vec{J}] = 0$$

d) $\vec{L}^2$ commuts w/ $\vec{L}$ and obviously $\vec{S}$, so

$$[\vec{L}\cdot\vec{S}, L^2] = 0$$

e) similar to d):

$$[\vec{L}\cdot\vec{S}, \vec{S}^2] = 0$$

f) $\vec{J}^2 = L^2 + 2\vec{L}\cdot\vec{S} + S^2$. L^2 & S^2 commute, $\vec{L}\cdot\vec{S}$ commutes w/ itself, so:

$$[\vec{L}\cdot\vec{S}, \vec{J}^2] = 0$$

Problem 7.22

- we need to do a ton of power series expansions to make this manageable...

$$\left(\frac{\alpha}{n-(j+1/2)+\underbrace{\sqrt{(j+1/2)^2-\alpha^2}}}\right)^2$$

- handle this first

$$(j+1/2)\sqrt{1-\alpha^2/(j+1/2)^2} \approx (j+1/2)\left[1+\frac{\alpha^2}{2(j+1/2)^2}+\cdots\right]$$

$$= (j+1/2)+\frac{\alpha^2}{2(j+1/2)}$$

$$\rightarrow \left(\frac{\alpha}{n-\frac{\alpha^2}{2(j+1/2)}}\right)^2 = \left[\frac{\alpha}{n\left(1-\frac{\alpha^2}{2n(j+1/2)}\right)}\right]^2$$

$$= \left[\frac{\alpha}{n}\left(1-\frac{\alpha^2}{2n(j+1/2)}\right)^{-1}\right]^2 \approx \left[\frac{\alpha}{n}\left(1+\frac{\alpha^2}{2n(j+1/2)}\right)\right]^2$$

$$= \left(\frac{\alpha}{n}\right)^2\left(1+\frac{\alpha^2}{2n(j+1/2)}\right)^2 \approx \left(\frac{\alpha}{n}\right)^2\left(1+\frac{\alpha^2}{n(j+1/2)}\right)$$

$$\Rightarrow E_{nj} = mc^2\left\{\underbrace{\left[1+\left(\frac{\alpha}{n}\right)^2\left(1+\frac{\alpha^2}{n(j+1/2)}\right)\right]}^{-1/2}-1\right\}$$

- we can now expand to order α^4:

$$= mc^2\left\{1-\frac{1}{2}\left(\frac{\alpha}{n}\right)^2\left(1+\frac{\alpha^2}{n(j+1/2)}\right)+\frac{3}{8}\left(\frac{\alpha}{n}\right)^4\left(1+\frac{\alpha^2}{n(j+1/2)}\right)^2-1\right\}$$

- only to order α^4, can neglect most of term

$$= mc^2\left\{-\frac{1}{2}\left(\frac{\alpha}{n}\right)^2\left(1+\frac{\alpha^2}{n(j+1/2)}\right)+\frac{3}{8}\left(\frac{\alpha}{n}\right)^4\right.$$

$$= -\frac{mc^2}{2}\left\{\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^3(j+1/2)}-\frac{3}{4}\frac{\alpha^4}{n^4}\right.$$

$$= \frac{-mc^2\alpha^2}{2n^2}\left\{1+\frac{\alpha^2}{n(j+1/2)}-\frac{3}{4}\frac{\alpha^2}{n^2}\right\}$$

$$= \frac{-mc^2\alpha^2}{2n^2}\left\{1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right\}$$

- the quantity $\frac{1}{2}mc^2\alpha^2$ is the Rydberg energy

$$R_E = \frac{1}{2}mc^2\alpha^2 = 13.6\text{ eV},$$

so

$$E_{nj} \approx \frac{-13.6\text{ eV}}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right],$$

which is exactly EQ. (7.69).

Problem 7.24

- total energy for weak-field Zeeman effect is sum of $H_{Bohr} + H'_{fs}$ as the unperturbed w/

$$E_{nj} = \frac{-13.6\,eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]$$

- the $B_{ext} \ll B_{int}$ gives energy corrections:

$$E'_Z = \mu_B g_J B_{ext} m_j$$

$$\Rightarrow \quad E_{tot} = \frac{-13.6\,eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right] + \mu_B g_J B_{ext} m_j$$

- for $n=2$, $l=0$, $j=1/2$, $m=\pm 1/2$, we have

→ unperturbed energy:

$$E_{nj} = -\frac{13.6}{4}\left[1 + \frac{\alpha^2}{4}\left(2 - \frac{3}{4}\right)\right]$$

$$= \frac{-13.6}{4}\left[1 + \frac{5\alpha^2}{16}\right]$$

$$\Rightarrow \quad g_J = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]$$

$$= \left[1 + \frac{2 \cdot 1/2(1/2+1)}{2 \cdot 1/2(1/2+1)}\right] = 2$$

- total energy (2 states):

$$E_{tot} = \frac{-13.6}{4}\left[1 + \frac{5\alpha^2}{16}\right] \pm \mu_B B_{ext}$$

- for $l=1$, $j=\frac{3}{2}$ or $\frac{1}{2}$; considering $\frac{1}{2}$ first, can see unperturbed energy stays the same, but

$$g_J = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]$$

$$= \left[1 + \frac{3/4 - 2 + 3/4}{3/2}\right]$$

$$= \left[1 + \frac{-1/2}{3/2}\right] = \frac{2}{3}$$

- total therefore is:

$$\frac{-13.6}{4}\left[1 + \frac{5\alpha^2}{16}\right] \pm \frac{1}{3}\mu_B B_{ext}$$

- for $n=2$, $l=1$, $j=\frac{3}{2}$, we have four states w/ $m_j \in \left\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right\}$

- first, unperturbed energy:

$$E_{nj} = \frac{-13.6}{4}\left[1 + \frac{\alpha^2}{4}\left(\frac{2}{2} - \frac{3}{4}\right)\right]$$

$$= \frac{-13.6}{4}\left[1 + \frac{\alpha^2}{16}\right]$$

$$\rightarrow \quad g_J = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]$$

$$= \left[1 + \frac{15/4 - 2 + 3/4}{15/2}\right]$$

$$= \left[1 + \frac{9/2 - 2}{15/2}\right] = \left[1 + \frac{1}{3}\right] = \frac{4}{3}$$

for $m = \pm\frac{1}{2}$:

$$E_{tot} = \frac{-13.6}{4}\left[1 + \frac{\alpha^2}{16}\right] \pm \frac{2}{3}\mu_B B_{ext}$$

for $m = \pm\frac{3}{2}$:

$$E_{tot} = \frac{-13.6}{4}\left[1 + \frac{\alpha^2}{16}\right] \pm 2\mu_B B_{ext}$$