# Example 3: adding $j_1 = 1$ and $j_2 = 1$

In the $\{j_1, j_2, m_1, m_2\}$ basis we have $3 \times 3 = 9$ states.

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

$\Rightarrow j \in \{0, 1, 2\}$

$j = 2 \Rightarrow \quad 2j+1 = 5$ states $\quad m \in \{-2, -1, 0, 1, 2\}$

$j = 1 \Rightarrow \quad 3$ states

$j = 0 \Rightarrow \quad 1$ state

We start with the highest $|1,1;2,2\rangle$ and observe that this is obtained as

$$\underset{j_1\; j_2\; j\; m}{|1,1;2,2\rangle} = |j_1, j_2; m_1, m_2\rangle = |1,1;1,1\rangle$$

We apply $\hat{J}_-$ and use $\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$ with the corresponding eigenvalues

$$\sqrt{(2+2)(2-2+1)}\, |\overset{j_1}{1},\overset{j_2}{1};\overset{j}{2}\; \overset{m-1}{1}\rangle = \sqrt{(1+1)(1-1+1)}\, |1,1;0,1\rangle + \sqrt{(1+1)(1-1+1)}\, |1,1;1,0\rangle$$

$$|1,1;2,1\rangle = \frac{1}{\sqrt{2}} |1,1;0,1\rangle + \frac{1}{\sqrt{2}} |1,1;1,0\rangle$$

We apply $\hat{J}_-$ again on $|1,1;2,1\rangle \Rightarrow$

$$\sqrt{6}\, |1,1;2,0\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}\,|1,1;1,-1\rangle + \sqrt{2}\,|1,1;0,0\rangle\right) + \frac{1}{\sqrt{2}}\left(\sqrt{2}\,|1,1;0,0\rangle + \sqrt{2}\,|1,1;1,-1\rangle\right)$$

Let's obtain $m=-2,-1$. We further apply $\hat{J}_-$ to $|11;20\rangle$. To get $\boxed{m=-1}$ we do:

$$\hat{J}_-|11;20\rangle = (\hat{J}_{1-}+\hat{J}_{2-})\frac{1}{\sqrt{6}}[|1,1;1,-1\rangle+2|1,1;0,0\rangle+|11;-11\rangle] =$$

$$\sqrt{6}|11;2,-1\rangle = [\sqrt{2}|1,1;0,-1\rangle+2\sqrt{2}|1,1;-1,0\rangle]\frac{1}{\sqrt{6}} +$$

$$\frac{1}{\sqrt{6}}[2\sqrt{2}|1,1;0,-1\rangle+\sqrt{2}|1,1;-1,0\rangle] =$$

$$|1,1;2,-1\rangle = \frac{\sqrt{2}}{6}[3|11;0-1\rangle+3|11;-1,0\rangle] =$$

$$\boxed{|1,1;2,-1\rangle = \frac{1}{\sqrt{2}}[|11;0-1\rangle+|1,1;-1,0\rangle]}$$

To obtain $\boxed{m=-2}$ we further apply $\hat{J}_-$

$$\hat{J}_-|1,1;2,-1\rangle = (\hat{J}_{1-}+\hat{J}_{2-})\frac{1}{\sqrt{2}}[|11,0,-1\rangle+|11;-1,0\rangle]$$

$$2|1,1,2,-2\rangle = \frac{1}{\sqrt{2}}[\sqrt{2}|1,1;-1,-1\rangle+\sqrt{2}|11;-1,-1\rangle]$$

$$\Rightarrow \boxed{|1,1,2,-2\rangle = |1,1;-1,-1\rangle \text{ uniquely}}$$

Therefore we have that:

$$|1,1;2,0\rangle = \frac{1}{\sqrt{6}}\left[|1,1;1,-1\rangle + 2|1,1,0,0\rangle + |1,1;-1,1\rangle\right] \quad (*)$$

Let's now consider the state with $j=1$ and $m=1$:

Using orthonormality and the Condon-Shortley convention

$$|1,1;\overset{s}{1},\overset{m}{1}\rangle = \frac{1}{\sqrt{2}}|1,1;1,0\rangle - \frac{1}{\sqrt{2}}|1,1;0,1\rangle$$

which is obtain by observing that this state is orthogonal to $|1,1;2,1\rangle$

We apply $\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$ on $|1,1;1,1\rangle$ and get

$$|1,1,1,0\rangle = \frac{1}{\sqrt{2}}|1,1;1,-1\rangle - \frac{1}{\sqrt{2}}|1,1;-1,1\rangle \quad (**)$$

We now observe that the state $j=0$, $m=0$, that is $|1,1;0,0\rangle$ is orthogonal to both $|1,1;2,0\rangle$ & $|1,1;1,0\rangle$:

$$|1,1;0,0\rangle \perp |1,1;2,0\rangle = (*)$$

$$|1,1;0,0\rangle \perp |1,1;1,0\rangle = (**)$$

We parametrize $|1,1;0,0\rangle$ as follows

$$|1,1;0,0\rangle = a|1,1;1,-1\rangle + b|1,1;0,0\rangle + c|1,1;-1,1\rangle$$

We must determine $a, b, c$ using orthonormality.

$$|1,1;1,-1\rangle = \vec{v}_1 \qquad |1,1;\overset{j}{0},\overset{m}{0}\rangle = \vec{\omega}_1$$

$$|1,1;\overset{m_1}{0},\overset{m_2}{0}\rangle = \vec{v}_2 \qquad |1,1;2,0\rangle = \vec{w}_2$$

$$|1,1;-1,1\rangle = \vec{v}_3 \qquad |1,1;1,0\rangle = \vec{w}_3$$

$$\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$$

$$\vec{\omega}_1 \perp \vec{\omega}_2 \ \wedge \ \vec{\omega}_1 \perp \vec{\omega}_3 \Rightarrow$$

$$\vec{\omega}_1 \cdot \vec{\omega}_2 = 0 \ \wedge \ \vec{\omega}_1 \cdot \vec{\omega}_3 = 0$$

$$\vec{\omega}_1 = a\,\vec{v}_1 + b\,\vec{v}_2 + c\,\vec{v}_3$$

$$\vec{\omega}_2 = \frac{1}{\sqrt{6}}\vec{v}_1 + \frac{2}{\sqrt{6}}\vec{v}_2 + \frac{1}{\sqrt{6}}\vec{v}_3$$

$$\vec{\omega}_3 = \frac{1}{\sqrt{2}}\vec{v}_1 - \frac{1}{\sqrt{2}}\vec{v}_3$$

$$\vec{\omega}_1 \cdot \vec{\omega}_3 = 0 \Rightarrow \frac{a}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0 \Leftrightarrow a = c$$

$$\vec{\omega}_1 \cdot \vec{\omega}_2 = 0 \Rightarrow \frac{a}{\sqrt{6}} + \frac{2b}{\sqrt{6}} + \frac{c}{\sqrt{6}} = 0 \Rightarrow b = -c$$

$$\Rightarrow |1,1;0,0\rangle = a\left[\,|1,1;1,-1\rangle - |1,1;0,0\rangle + |1,1;-1,1\rangle\right.$$

$$|\langle 1,1;0,0|1,1;0,0\rangle|^2 = 1 \Rightarrow a = 1/\sqrt{3}$$

Collecting all the results we obtain that the Clebsch-Gordan coefficients for

$\boxed{m = 2,\ 1}$

$\langle 1,1;2,2 | 1,1;1,1 \rangle = 1$

$\langle 1,1;2,1 | 1,1;1,0 \rangle = 1/\sqrt{2}$

$\langle 1,1;2,1 | 1,1;0,1 \rangle = 1/\sqrt{2}$

$\langle 1,1;1,1 | 1,1;1,0 \rangle = 1/\sqrt{2}$

$\langle 1,1;1,1 | 1,1;0,1 \rangle = -1/\sqrt{2}$

$\boxed{m = 0}$

$\langle 1\,1\,2\,0 | 1\,1\,1\,{-1} \rangle = 1/\sqrt{6}$

$\langle 1\,1\,2\,0 | 1\,1\,1\,0 \rangle = \sqrt{2}/\sqrt{3}$

$\langle 1\,1\,2\,0 | 1\,1\,{-1}\,1 \rangle = 1/\sqrt{6}$

$\langle 1\,1\,1\,0 | 1\,1\,1\,{-1} \rangle = 1/\sqrt{2}$

$\langle 1\,1\,0\,0 | 1\,1\,1\,{-1} \rangle = -1/\sqrt{2}$

$\langle 1\,1\,0\,0 | 1\,1\,1\,{-1} \rangle = 1/\sqrt{3}$

$\langle 1\,1\,0\,0 | 1\,1\,0\,0 \rangle = -1/\sqrt{3}$

$\langle 1\,1\,0\,0 | 1\,1; {-1}\,1 \rangle = 1/\sqrt{3}$

The negative values of $m$ can be obtained by symmetry or using the same machinery (e.g. appeying $\hat{J}_-$ several times) like in 21/a