Particle interpretation

$\partial_\mu A^\mu = 0$ is now $\Box A^\mu = 0$ (for each component)

but with the choice $\phi = 0 \Rightarrow \Box \vec{A} = 0$

that is $\Box A^i = 0 \quad i = 1, 2, 3$

The solution of this is in terms of e^{ikx}; e^{-ikx} and the coefficients in the linear combination are called polarization vectors $\varepsilon^{(\lambda)}(k)$

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{(\lambda)}(k) \left[a^{(\lambda)}(k) e^{-ikx} + a^{(\lambda)\dagger}(k) e^{ikx} \right]$$

$\boxed{x = \text{4-vector}}$

with $k^2 = 0 \Rightarrow k_0 = |\vec{k}|$ $\qquad kx = k_0 x^0 - \vec{k} \cdot \vec{x}$

$\lambda \to$ takes only two values because

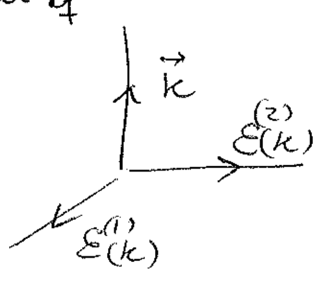
$$\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \vec{k} \cdot \vec{\varepsilon}^{(\lambda)} = 0$$

$\vec{\varepsilon}^{(\lambda)}$ are chosen to be orthonormal

$$\vec{\varepsilon}^{(\lambda)}(k) \cdot \vec{\varepsilon}^{(\lambda')}(k) = \delta_{\lambda\lambda'}$$

$\vec{\varepsilon}^{(\lambda)}$ are real unit vectors whose direction depends on $\vec{k}$ (propagation direction)

$\left(\vec{\varepsilon}^{(1)}, \vec{\varepsilon}^{(2)}, \frac{\vec{k}}{|\vec{k}|} \right) \Rightarrow$ right-handed set of mutually orthogonal unit vectors



Now we can calculate the <u>commutation relations</u> of the operators $a^{(\lambda)}_{(k)}$ and $a^{(\lambda)+}_{(k)}$

We define

$$f_k(r) = \frac{1}{\left[(2\pi)^3\, 2k_0\right]^{1/2}}\, e^{-ikr} \qquad kr = k_0 r^0 - \vec{k}\cdot\vec{r}$$

$$\downarrow$$

t when $c=\hbar=1$

and we have

$$\vec{A}(r) = \int \frac{d^3k}{\left[(2\pi)^3\, 2k_0\right]^{1/2}} \sum_\lambda \vec{\mathcal{E}}^{(\lambda)}_{(k)} \left[f_k(r)\, a^{(\lambda)}_{(k)} + f^*_k(r)\, a^{(\lambda)+}_{(k)} \right]$$

$f_k(r)$ and $f^*_k(r)$ are orthonormal $\Rightarrow$

$$a^{(\lambda)}(k) = \int d^3\vec{r}\, \left[(2\pi)^3\, 2k_0\right]^{1/2} f^*_k(r)\, i\, \overleftrightarrow{\partial_0}\, \vec{\mathcal{E}}^{(\lambda)}_{(k)} \cdot \vec{A}(r) \quad \underline{\text{annihilation}}$$

$$a^{(\lambda)+}(k) = -\int d^3\vec{r}\, \left[(2\pi)^3\, 2k_0\right]^{1/2} f_k(r)\, i\, \overleftrightarrow{\partial_0}\, \vec{\mathcal{E}}^{(\lambda)}_{(k)} \cdot \vec{A}(r) \quad \underline{\text{creation}}$$

where the orthonormality condition is

$$\int f^*_k(r)\, i\, \overleftrightarrow{\partial_0}\, f_{k'}(r)\, d^3\vec{r} = \delta^3(\vec{k}-\vec{k}')$$

and

$$A(t)\, \overleftrightarrow{\partial_0}\, B(t) = A(t)\frac{\partial B(t)}{\partial t} - \frac{\partial A(t)}{\partial t} B(t)$$

With these definitions we can calculate the equal time commutators:

$$[a^{(\lambda)}(k),\, a^{(\lambda')\dagger}(k')] = 2k_0 (2\pi)^3 \delta_{\lambda\lambda'} \delta^3(\vec{k}-\vec{k}')$$

$$[a^{(\lambda)}(k),\, a^{(\lambda')}(k')] = [a^{(\lambda)\dagger}(k),\, a^{(\lambda')\dagger}(k')] = 0$$

Annihilation and creation operators for photons.

We construct the operator $\boxed{N^{(\lambda)}(k) = A\, a^{(\lambda)\dagger}(k)\, a^{(\lambda)}(k)}$

$N^{(\lambda)}(k)$ = particle number operator.

A is a normalization constant $= [(2\pi)^3 2k_0]^{-1}$

The field energy is given by the Hamiltonian

$$H = \frac{1}{2}\int d^3\vec{r}\left(|\vec{E}|^2 + |\vec{B}|^2\right) = \frac{1}{2}\int d^3\vec{r}\left(|\dot{\vec{A}}|^2 + |\vec{\nabla}\times\vec{A}|^2\right)$$

remember that in the radiation (or Coulomb) gauge $\vec{\nabla}\cdot\vec{A} = 0$ $\phi = 0$ $\Rightarrow$

$-\dot{A}^i = E^i$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

We observe that

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, \hat{x}_i\, \partial_j\, A_k \equiv \varepsilon_{ijk}\, \hat{x}_i\, \partial_j A_k =$$

$$= \hat{x}_1(\partial_2 A_3 - \partial_3 A_2) + \hat{x}_2(\partial_3 A_1 - \partial_1 A_3) + \hat{x}_3(\partial_1 A_2 - \partial_2 A_1)$$

$x_i = 1, 2, 3 \qquad x_1 = x \quad x_2 = y \; - - -$

$$(\vec{\nabla} \times \vec{A})^2 = (\vec{\nabla} \times \vec{A})_i\, (\vec{\nabla} \times \vec{A})_i = \varepsilon_{ijk}\, \varepsilon_{imn}\, \partial_j A_k\, \partial_m A_n$$

We use the identity

$$\varepsilon_{ijk}\, \varepsilon_{imn} = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})$$

which gives

$$(\vec{\nabla} \times \vec{A})^2 = (\partial_j A_k)(\partial_j A_k) - (\partial_j A_k)(\partial_k A_j)$$

$(\partial_j A_k)(\partial_k A_j) \Rightarrow$ can be written as a total divergence

$$\partial_j (A_k \partial_k A_j) = (\partial_j A_k)(\partial_k A_j) + A_k(\partial_j \partial_k A_j) =$$

$$= (\partial_j A_k)(\partial_k A_j) + A_k \partial_k \overbrace{(\vec{\nabla} \cdot \vec{A})}^{\partial_j A_j}$$

$\vec{\nabla} \cdot \vec{A} = 0$ in the radiation gauge $\Rightarrow$

$$(\partial_j A_k)(\partial_k A_j) = \partial_j (A_k \partial_k A_j) \rightsquigarrow \text{total div}$$

But this would lead to the integral of a total divergence which vanishes $\Rightarrow$

$$(\vec{\nabla}\times\vec{A})^2 = (\partial_j A_k)(\partial_j A_k)$$

Also, $\quad \partial_j (A_k \partial_j A_k) = (\partial_j A_k)(\partial_j A_k) + A_k \nabla^2 A_k$

$\hookrightarrow 0$ under integration

And

$$\int (\partial_j A_k)(\partial_j A_k) d^3\vec{r} = -\int \vec{A}\cdot\nabla^2\vec{A}\, d^3\vec{r}$$

$$H = \frac{1}{2}\int\left(\dot{\vec{A}}^2 - \vec{A}\cdot\nabla^2\vec{A}\right) d^3\vec{r}$$

Substituting the expansion of $\vec{A}(r)$ in this integral we obtain (after some algebra)

$$H = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 2k_0}\frac{k_0}{2}\left[a^{(\lambda)+}(k)a^{(\lambda)}(k) + a^{(\lambda)}(k)a^{(\lambda)+}(k)\right]$$

$$a^{(\lambda)}(k)a^{(\lambda)+}(k) = a^{(\lambda)+}(k)a^{(\lambda)}(k) + 2k_0(2\pi)^3\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')$$

from the commutation relation.

$$N^{(\lambda)}(k) = \left((2\pi)^3 k_0\right)^{-1} a^{(\lambda)+}(k)\, a^{(\lambda)}(k) \quad\Rightarrow$$

$$H = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} k_0 \left[N^{(\lambda)}(k) + \frac{1}{2}\right]$$

$H \to$ total energy of a collection of photons with transverse polarization.

$H \to$ positive definite

only the transverse degrees of freedom are quantized. (sacrificed Lorentz invariance)

$$H = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} k_0 \left[N^{(\lambda)}(k) + \frac{1}{2} \right] \qquad k_0 = \omega_k$$

zero point energy

$$a^{(\lambda)}(k)\,|0\rangle = 0 \qquad [N^{(\lambda)}(k), a^{+(\lambda')}(k')] = \delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}')\,a^{+(\lambda)}(k)$$

$$N^{(\lambda)}(k)\,|0\rangle = a^{(\lambda)+}(k)\,a^{(\lambda)}(k)\,|0\rangle = 0 \qquad [N^{(\lambda)}(k), a^{(\lambda')}(k')] = -\delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}')\,a^{(\lambda)}(k)$$

$$N^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle = n^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle$$

$$N^{(\lambda)}(k)\,a^{(\lambda)+}(k)\,|n^{(\lambda)}(k)\rangle = a^{(\lambda)+}(k)\,N^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle + a^{(\lambda)+}(k)\,|n^{(\lambda)}(k)\rangle =$$

$$= \left(n^{(\lambda)}(k) + 1\right) a^{(\lambda)+}(k)\,|n^{(\lambda)}(k)\rangle$$

$$N^{(\lambda)}(k)\,a^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle = a^{(\lambda)}(k)\,N^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle - a^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle$$

$$= \left(n^{(\lambda)}(k) - 1\right) a^{(\lambda)}(k)\,|n^{(\lambda)}(k)\rangle$$

The zero point energy is arbitrary and can be subtracted with no physical consequences.
We may use the energy scale in which the energy of the vacuum state is zero

$$\langle 0 | H | 0 \rangle = \int d^3k \, k_0 \, \langle 0 | a^{(\lambda)+}_{(k)} a^{(\lambda)}_{(k)} | 0 \rangle = 0$$

This is equivalent to writing all annihilation operators on the right of the creation operators.

$\Rightarrow$ normal ordering $\Rightarrow$ $: \quad :$

$$H = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3 k_0} \frac{k_0}{2} \left[a^{(\lambda)+}_{(k)} a^{(\lambda)}_{(k)} \right]$$

The role of $a^{(\lambda)+}_{(k)}$ and $a^{(\lambda)}_{(k)}$ is clear:

$$a^{(\lambda)+} | n^{(\lambda)} \rangle = C_+ | n^{(\lambda)} + 1 \rangle$$

$$a^{(\lambda)} | n^{(\lambda)} \rangle = C_- | n^{(\lambda)} - 1 \rangle$$

Or more precisely

$$a^{(\lambda)+}(k_i) | n^{(\lambda)}(k_1), \ldots n^{(\lambda)}(k_i), \ldots \rangle = C_+(n^{(\lambda)}(k_i)) | n^{(\lambda)}(k_1), \ldots n^{(\lambda)}(k_i)+1, \ldots \rangle$$

etc..

To determine $C_\pm$ we require that all states are normalized:

$$|C_+(n^{(\lambda)}(k))|^2 \langle n^{(\lambda)}(k)+1 | n^{(\lambda)}(k)+1\rangle =$$

$$\langle n^{(\lambda)}(k) | a^{(\lambda)}(k)\, a^{(\lambda)\dagger}(k) | n^{(\lambda)}(k)\rangle =$$

$$\langle n^{(\lambda)}(k) | n^{(\lambda)}\rangle [n^{(\lambda)}(k)+1](2\pi)^3 2k_0$$

for C_- we can proceed analogously.

$$C_+(n^{(\lambda)}(k)) = \{[n^{(\lambda)}(k)+1](2\pi)^3 2k_0\}^{1/2}$$

$$C_-(n^{(\lambda)}(k)) = \{n^{(\lambda)}(k)(2\pi)^3 2k_0\}^{1/2} \Rightarrow$$

$$a^{(\lambda)}(k_i)\,|n^{(\lambda)}(k_1)\ldots n^{(\lambda)}(k_i)\ldots\rangle = \left[(2\pi)^3 k_{0,i}\, n^{(\lambda)}(k_i)\right]^{1/2} |n^{(\lambda)}(k_1)\ldots n^{(\lambda)}(k_i)-1,\ldots\rangle$$

etc..

$N^{(\lambda)}(k)$ → state vector for a state with a definite number of photons in state $(\vec{k}, \lambda)$

Situation in which there are many types of photons

$$|n^{(\lambda_1)}(k_1), n^{(\lambda_2)}(k_2), n^{(\lambda_3)}(k_3)\ldots\rangle = |n^{(\lambda_1)}(k_1)\rangle\, |n^{(\lambda_2)}(k_2)\rangle \ldots$$

Interpreted as a direct product of states.