4-operators definition

The 4-gradient covariant components compactly written in four-vector and Ricci calculus notation are:

$$\frac{\partial}{\partial X^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_0, \partial_i) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) = \left(\frac{\partial_t}{c}, \vec{\nabla}\right) = \left(\frac{\partial_t}{c}, \partial_x, \partial_y, \partial_z\right) = \partial_\mu = {}_{,\mu}$$

The *comma* in the last part above ${}_{,\mu}$ implies the *partial differentiation* with respect to 4-position X^μ.

The contravariant components are:

$$\partial = \partial^\alpha = \eta^{\alpha\beta}\partial_\beta = (\partial^0, \partial^1, \partial^2, \partial^3) = (\partial^0, \partial^i) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right) = \left(\frac{\partial_t}{c}, -\vec{\nabla}\right) = \left(\frac{\partial_t}{c}, -\partial_x, -\partial_y, -\partial_z\right)$$

Example

The 4-divergence of the 4-position $X^\mu = (ct, \vec{\mathbf{x}})$ gives the dimension of spacetime:

$$\partial \cdot \mathbf{X} = \partial^\mu \eta_{\mu\nu} X^\nu = \partial_\nu X^\nu = \left(\frac{\partial_t}{c}, -\vec{\nabla}\right) \cdot (ct, \vec{x}) = \frac{\partial_t}{c}(ct) + \vec{\nabla} \cdot \vec{x} = (\partial_t t) + (\partial_x x + \partial_y y + \partial_z z) = (1) + (3) = 4$$

D'Alambertian operator: wave equation compact

$$\Box = \partial \cdot \partial = \partial^\mu \partial_\mu = \partial^\mu \eta_{\mu\nu} \partial^\nu = \partial_\nu \partial^\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \left(\frac{\partial_t}{c}\right)^2 - \nabla^2$$

$$\Box\phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi = 0$$ Wave equation

- **Invariant**: means it's the same in all inertial ref. frames
- **Covariant**: (applied to 4-vec quantities) means that it's the mathematical structure of an equation that is invariant
- **Conserved**: means ``it does not change with time'' or also, ``the same before and after''

-Rest mass: is Lorentz invariant, but it is not conserved

-Energy: is conserved, but it is not Lorentz invariant

Maxwell Equations

The previous relativistic transformations suggest the electric and magnetic fields are coupled together, in a mathematical object with 6 components: an antisymmetric rank-2 tensor. This is called the electromagnetic field tensor, usually written as $F^{\mu\nu}$. In matrix form

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Maxwell Equations
from classical electromagnetism

Differential equations
$\nabla \cdot \mathbf{E} = 4\pi\rho$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)$

Maxwell Equations in covariant form

The four-current is the contravariant four-vector which combines electric charge density ρ and electric current density $\mathbf{j}$:

$$J^\alpha = (c\rho, \mathbf{j})\,.$$

The two inhomogeneous Maxwell's equations, Gauss's Law and Ampère's law (with Maxwell's correction) combine into (with (+ − − −) metric):[3]

Gauss–Ampère law

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

while the homogeneous equations – Faraday's law of induction and Gauss's law for magnetism combine to form $\partial_\sigma F^{\mu\nu} + \partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} = 0$, which may written using Levi-Civita duality as:

Gauss–Faraday law

$$\partial_\alpha\left(\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) = 0$$

where $F^{\alpha\beta}$ is the electromagnetic tensor, J^α is the four-current, $\varepsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, and the indices behave according to the Einstein summation convention.