Homework 6

Problem 9.1

$$\phi(x) = \frac{i}{\hbar}\int_0^a p(x)\,dx, \quad \text{with}$$

$$p(x) = \sqrt{2m(E - V(x))}$$

- Based on $V(x)$:

$$\phi(x) = \frac{i}{\hbar}\sqrt{2m}\left[\int_0^{a/2}\sqrt{E-V_0}\,dx + \int_{a/2}^{a}\sqrt{E}\,dx\right]$$

$$= \frac{i}{\hbar}\sqrt{2m}\,\frac{a}{2}\left(\sqrt{E-V_0} + \sqrt{E}\right)$$

- Now,

$$\psi(x) = \frac{1}{\sqrt{p(x)}}e^{i\phi(x)}$$

$$= \frac{1}{\sqrt{p(x)}}\left[C_1\cos\phi + C_2\sin\phi\right]$$

- For our case, boundary conditions:

$$\underset{①}{\psi(0)} = \underset{②}{\psi(a)} = 0$$

- ① implies $C_1 = 0$; ② implies:

$$\phi(x) = \frac{a\sqrt{2m}}{2\hbar}\left(\sqrt{E-V_0} + \sqrt{E}\right) = n\pi$$

$$= \frac{2ma^2}{4\hbar^2}\left(E - V_0 + E + 2\sqrt{E(E-V_0)}\right) = n^2\pi^2$$

$$= \frac{1}{4}\left(2E - V_0 + 2\sqrt{E(E-V_0)}\right) = E_n^0$$

$$\rightarrow \quad 2\sqrt{E(E-V_0)} = 4E_n^0 + V_0 - 2E$$

$$4(E^2 - EV_0) = 16(E_n^0)^2 + V_0^2 + 4E^2 + 8E_n^0V_0 - 16E_n^0E - 4V_0E$$

$$0 = 16(E_n^0)^2 + V_0^2 + 8E_n^0V_0 - 16E_n^0E$$

$$16E_n^0E = 16(E_n^0)^2 + V_0^2 + 8E_n^0V_0$$

$$\rightarrow \boxed{E = E_n^0 + \frac{V_0}{2} + \frac{V_0^2}{16E_n^0}}$$

- If n is large or V_0 is small, V_0^2 term vanishes, exactly matching our result from perturbation theory.

Problem 9.2

- Given $\psi(x) = e^{if(x)/\hbar}$,

we use $\psi''(x) = -\frac{p^2}{\hbar^2}\psi$

- First: $\psi'(x) = \frac{if'(x)}{\hbar} e^{if(x)/\hbar}$

$$\to \psi''(x) = \frac{i}{\hbar} f''(x)\, e^{if(x)/\hbar} + \left[\frac{i}{\hbar} f'(x)\right]^2 e^{if(x)/\hbar}$$

$$\psi''(x) = \frac{i}{\hbar} f''(x) - \frac{1}{\hbar^2}[f'(x)]^2 \psi = -\frac{p^2}{\hbar^2}\psi$$

$$\to \underline{i\hbar f''(x) - [f'(x)]^2 + p^2 = 0}$$

If we expand f in a power series:

$$\begin{cases} f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) \\ f'(x) = f_0'(x) + \hbar f_1'(x) + \hbar^2 f_2'(x) \\ \ldots = \ldots \end{cases}$$

- Plugging on:

$$i\hbar\left[f_0''(x) + \hbar f_1''(x)\right] - \left[f_0'(x) + \hbar f_1'(x) + \hbar^2 f_2'(x)\right]^2 + p^2 = 0$$

$$i\hbar f_0''(x) + \hbar^2 f_1''(x) - [f_0'(x)]^2 - \hbar^2[f_1'(x)]^2$$
$$- 2\hbar f_0'(x) f_1'(x) - 2\hbar^2 f_0'(x) f_2'(x) + p^2 = 0$$

$$\hbar^0\left[p^2 - (f_0')^2\right] + \hbar\left[if_0'' - 2f_0'f_1'\right] + \hbar^2\left[f_1'' - (f_1')^2 - 2f_0'f_2'\right] = 0$$

$$\Rightarrow \begin{cases} p^2 - (f_0')^2 = 0 \\ if_0'' - 2f_0'f_1' = 0 \\ f_1'' - (f_1')^2 - 2f_0'f_2' = 0 \end{cases}$$

- obviously, $f_0' = p$, since p can run negative.

so, $f_0 = \int p\,dx$ and $f_0'' = p'$

- from 2nd EQ:

$$ip' = 2p f_1' \to f_1' = \frac{i}{2}\frac{p'}{p}$$

$$\Rightarrow f_1 = \frac{i}{2}\int \frac{p'}{p}\,dx$$
$$= \frac{i}{2}\int \frac{d}{dx}[\ln p]\,dx$$
$$= \frac{i}{2}[p(x) + c]$$

- w/ this, to order $\hbar$:

$$f(x) = \int p\,dx + \frac{i\hbar}{2}p(x) + c$$

- thus:

$$\psi(x) = \exp\left\{\frac{i}{\hbar}\left[\int p\,dx + \frac{i\hbar}{2}p(x) + c\right]\right\}$$
$$= C\exp\left[\frac{i}{\hbar}\int p\,dx - \frac{1}{2}p(x)\right]$$

$$\underline{\psi(x) = \frac{C}{\sqrt{p(x)}}\exp\left[\frac{i}{\hbar}\int p\,dx\right]},$$

Problem 9.3

- we know

$$T = e^{-2\gamma}, \quad \gamma = \frac{1}{\hbar}\int_0^{2a} |p(x)|\, dx$$

- since our barrier $= V_0$ for $[0, 2a]$

$$\Rightarrow \gamma = \frac{1}{\hbar}\int_0^{2a} \sqrt{2m(V_0 - E)}\, dx$$

$$= \frac{2a}{\hbar}\sqrt{2m(V_0 - E)}$$

$$\Rightarrow \underline{T = \exp\left[\frac{-4a}{\hbar}\sqrt{2m(V_0 - E)}\right]}$$

- The answer to 2.33 is:

$$T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)$$

- If $T \ll 1$, T^{-1} is very large; the $+1$ can be ignored

- Also: $\sinh x = \frac{1}{2}(e^x - e^{-x})$ - for us, $x > 0$, so the e^{-x} term would be small, meaning

$$T^{-1} = \frac{V_0^2}{16E(V_0 - E)} \exp\left[\frac{4a}{\hbar}\sqrt{2m(V_0 - E)}\right]$$

$$\rightarrow T = \frac{16E(V_0 - E)}{V_0^2} \exp\left[\frac{-4a}{\hbar}\sqrt{2m(V_0 - E)}\right]$$

- The coefficient is ~ 1, so the behavior is dominated by the exponential; thus

$$T \approx \exp\left[\frac{-4a}{\hbar}\sqrt{2m(V_0 - E)}\right]$$

which matches the WKB Approx. result.

Problem 9.4

U^{238} has $Z = 92$ and $A = 238$. First:

$m_p = 238.05078826\ u$,

$m_d = m_{Th^{234}} = 234.0436\ u$,

$m_\alpha = 4.001506\ u$, with

$1\ u = 931.4941\ MeV/c^2$

$\rightarrow E = m_p c^2 - m_\alpha c^2 - m_x c^2 = 5.2930$

$\rightarrow r_1 = 1.07 \sqrt[3]{238}\ fm = 6.63\ fm$

$\rightarrow \gamma = 1.980 \cdot \frac{92}{\sqrt{5.2930}} - 1.485 \sqrt{(92)(6.63)}$

$= 42.502$

$\rightarrow V = \sqrt{\frac{2E}{m_\alpha}} = \sqrt{\frac{(2)(5.2930)}{(4.001506)(931.4941)}} = 0.05329\ c$

$= 1.599 \times 10^7\ m/s = 1.599 \times 10^{22}\ fm/s$

$\Rightarrow \tau = \frac{2 \cdot 6.63}{1.599 \times 10^{22}} \exp[85.004]\ s$

$\approx 6.846 \times 10^{15}\ s \cdot \frac{1\ yr}{3.154 \times 10^7\ s}$

$\boxed{\tau \approx 2.171 \times 10^8\ yr}$

- For Po^{212}, $Z = 84$, $A = 212$, $m = 211.9889\ u$
- daughter is Pb^{208} $m = 207.9767$

$\rightarrow E = 211.9889 - 207.9767 - 4.001506\ u$

$\approx 9.96\ MeV$

$r_1 = 1.07 \sqrt[3]{212} \approx 6.38\ fm$

$\gamma = 1.98 \frac{84}{\sqrt{9.96}} - 1.485 \sqrt{(84)(6.38)}$

≈ 18.323

$V = \sqrt{\frac{2E}{m_\alpha}} = \sqrt{\frac{(2)(9.96)}{(4.001506)(931.4941)}}$

$\approx 0.0731\ c = 2.193 \times 10^7\ m/s$

$= 2.193 \times 10^{22}\ fm/s$

$\tau = \frac{2 \cdot 6.38}{2.193 \times 10^{22}} \exp[36.646]$

$\approx 4.786 \times 10^{-6}\ s$

$\boxed{\tau \approx 4.786\ \mu s}$

Problem 9.5

- need functional form of $V(x)$
- at first, it's just a wall of $V(x) = E_g$, but with the decreasing factor from electric field:
- if we also consider the electron w/ energy E, then the potential looks like:

$$V(x) = E + E_g - eE_0x$$

$$\rightarrow \gamma = \frac{1}{\hbar}\int_0^a \left|\sqrt{2m[E-(E+E_g-eE_0x)]}\right| dx$$

$$= \frac{1}{\hbar}\int_0^a \sqrt{2m(E_g - eE_0x)}\, dx \qquad u = E_g - eE_0x$$

$$du = -eE_0\, dx$$

$$= \frac{\sqrt{2m}}{\hbar}\int \frac{1}{-eE_0} u^{1/2}\, du$$

$$= -\frac{\sqrt{2m}}{eE_0} \cdot \frac{2}{3}\left[(E_g - eE_0x)^{3/2}\right]_0^a$$

$$= \frac{-2\sqrt{2m}}{3eE_0}\left[(E_g - eE_0a)^{3/2} - E_g^{3/2}\right]$$

- But, as this is the classical region, we know

$$V(a) = E$$

$$\rightarrow E + E_g - eE_0a = E \Rightarrow E_g = eE_0a,$$

so

$$\gamma = \frac{2\sqrt{2m}}{3eE_0}E_g^{3/2},$$

meaning:

$$T \approx \exp\left[\frac{-4\sqrt{2m}}{3eE_0}E_g^{3/2}\right]$$