The quantum pigeonhole paradox

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Abstract

The purpose of this report is to illustrate the quantum pigeonhole principle proposed by Y.Aharonov et.al [1] and a possible experimental implementation. The weak values for the quantum paradox are computed and a short case study based on a trapped ion system is presented.

1 Introduction

In this work we will focus on the computation of the weak values for the quantum pigeonhole principle in [1] and present an alternative experiment which can test the predictions. The report is organized as follows. In Section 2 we review the pigeonhole principle, explain how the weak measurement works and we introduce some basic notions about trapped ion systems needed to perform the experiment. Next, in Section 3 we compute the weak values for the paradox and illustrate how a simulation of the experiment can be done. We conclude the report with Section 4 in which we tackle some experimental issues related to the measurement procedure in trapped ions.

2 Theory

This section provides a theoretical overview of the quantum pigeonhole principle. We also review concepts like weak values explaining the experimental protocol and the conditions under which the weak values can be recovered. Further, we propose an experiment in which we use trapped ions instead of electrons in an interferometer, the two implementations however are perfectly equivalent and they lead to the same answers.

2.1 Quantum pigeonhole principle

The classical pigeonhole principle is one of the fundamental principles in counting theory. It says that if we have more pigeons than holes, and we try to distribute all the pigeons in the available holes, then at least one hole will contain two pigeons. The Fig.1 illustrates this simple classical principle with three balls (pigeons) and two boxes (pigeonholes):

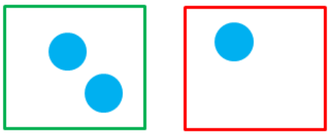

Figure 1: The blue balls represent the pigeon and the green and red boxes are the pigeonholes.

The extension of this principle to quantum mechanics was made in [1] where instead of pigeons and pigeonholes, particles with their associated properties (such as spin, position, momentum etc.) were considered. In the quantum case violations of the classical principle can occur. Let us explain better the last statement with an example.

Example : Consider the case of three particles, each with the Hilbert space spanned by $\{|L\rangle, |R\rangle\}$ (two boxes). Following the two state vector formalism [13], we prepare independently the each particle in the state $|+\rangle = \frac{|L\rangle+|R\rangle}{\sqrt{2}}$ such that at an initial time t_i the total state of the system is a product state:

$$|\psi\rangle_i = |+\rangle_1 |+\rangle_2 |+\rangle_3 \tag{1}$$

According to the eq.(1) we have prepared each particle in the superposition of the two boxes. However, there exist instances of time in which this is not true. In order to find such istances, we perform a second measurement at a later time t_f and post select only the outcomes corresponding to:

$$|\phi\rangle_f = |+i\rangle_1 |+i\rangle_2 |+i\rangle_3 \tag{2}$$

with $|+i\rangle = \frac{|L\rangle+i|R\rangle}{\sqrt{2}}$. Now, we want to know whether two particles are in the same box. Since both the initial and final states are symmetric under permutation, we can query any pair of particles without loss of generality. Thus, at an intermediate time $\tilde{t}$, we perform a joint measurement on the first two particles asking whether the pair of particles was in the same box. Supposing that at such an instance of time the two particles were in the same box, we find that the state of the system (up to a normalization factor) collapsed to:

$$|\psi'\rangle_w = \hat{\Pi}^{same}_{(1,2)} |\psi\rangle_i = \frac{1}{2}(|L,L\rangle_{(1,2)} + |R,R\rangle_{(1,2)}) |+\rangle_3 \tag{3}$$

with $\hat{\Pi}^{same}_{(1,2)} = |L,L\rangle \langle L,L|_{(1,2)} + |R,R\rangle \langle R,R|_{(1,2)}$. The state in the eq.(3) is orthogonal to the post-selected state in the eq.(2):

$$\langle\phi|\psi'\rangle = \langle\phi| \hat{\Pi}^{same}_{(1,2)} |\psi\rangle = 0 \tag{4}$$

Hence, the only case in which the particles can be found in $|\phi\rangle_f$ at the final time t_f, corresponds to the situation in which the two particles are in different boxes at an intermediate time $\tilde{t}$. This example shows how quantum mechanics can violate the classical principle in which we could always find at least one hole with two pigeons.

2.2 Weak measurements

One of the pillars of the standard interpretation of quantum mechanics is the projective measurement postulate, which states that the wavefunction of a state collapses upon measurement of an observable of the theory. A paradigm shift in the understanding of the measurement problem was made by the introduction of the weak measurement and the corresponding weak values [3]. The latter represent effective values of an observable of the system upon asymptotically weak perturbation.

Before going into detailed calculation of weak values for the pigeonhole principle, let us briefly describe the weak measurement protocol:

1. At an initial time t_i, the system that we want to measure is prepared in the state $|\psi_i\rangle_s$. At the same time we prepare independently another system, the pointer, in the state $|\phi_i\rangle_p$. The space representation of the pointer is usually chosen to be a gaussian:

$$\langle x|\phi_i\rangle_p = \frac{1}{(2\pi\Delta^2)^{\frac{1}{4}}} \exp\left\{-\frac{x^2}{4\Delta^2}\right\} \tag{5}$$

   The total initial state is a product state of the pointer and the internal system to be measured:

$$|\Psi_i\rangle_{sp} = |\psi_i\rangle_s \otimes |\phi_i\rangle_p \tag{6}$$

2. Let us assume for simplicity that the free part of the total Hamiltonian of the system and pointer degrees of freedom is zero i.e. $\hat{H}_s = \hat{H}_p = 0$. At an intermediate time t_w, the system weakly interacts with the pointer according to the Von-Neumann's quantum pointer model [4]:

$$\hat{H}_{int}(t) = g(t-t_w)\hat{A}_s \otimes \hat{P}_p \tag{7}$$

   $g(t-t_w)$ is a smooth function with a compact support $|\tau|$ and it is such that $G = \int_{t_w-\frac{\tau}{2}}^{t_w+\frac{\tau}{2}} g(t)dt$ is an effective coupling constant. In eq.(7) $\hat{A}_s$ is an observable acting on the internal system and $\hat{P}_p$ is the canonical momentum acting on the pointer. The weak interaction is such that we can expand the unitary time propagator governing the interaction:

$$\hat{U}_{int} = \exp\left\{-\frac{i}{\hbar} G\hat{A}_s \otimes \hat{P}_p\right\} \tag{8}$$

   up to the 1^{st} order.

3. Eventually, at time t_f a projective measurement is performed on the system. In particular, we measure an observable $\hat{B}_s = \sum_k \lambda_k |b_k\rangle\langle b_k|_s$ and we keep only the outcomes λ_{k_0} corresponding to the eigenstate $|b_{k_0}\rangle_s \equiv |\chi_f\rangle_s$. This step is called post-selection. At this point the pointer state is just:

$$\begin{aligned} |\phi_f\rangle_p &= \langle\chi_f|_s |\Psi_f\rangle_{sp} \\ &= \langle\chi_w| \exp\left\{-\frac{i}{\hbar} G\hat{A}_s \otimes \hat{P}_p\right\} |\psi_w\rangle_s |\phi_i\rangle_p \\ &\sim \langle\chi_w|\psi_w\rangle_s (\mathbb{1} - \frac{i}{\hbar} G \frac{\langle\chi_w| \hat{A}_s |\psi_w\rangle_s}{\langle\chi_w|\psi_w\rangle_s} \hat{P}_p) |\phi_i\rangle_p \end{aligned} \tag{9}$$

We denote the weak value of an observable $\hat{A}_s$ as:

$$A_s^w = \frac{\langle\chi_w|\, \hat{A}_s\, |\psi_w\rangle_s}{\langle\chi_w|\psi_w\rangle_s} \tag{10}$$

Here $|\psi_w\rangle_s$ and $|\chi_w\rangle_s$ stand for the pre- and post- selected states, unitarily propagated under the free Hamiltonians to the time t_w at which the weak interaction occurs. Most importantly, the value of A_s^w in eq.(10) depends on how $|\psi_w\rangle_s$ and $|\chi_w\rangle_s$ are chosen. The weak values A_s^w in general can be complex, negative when the observable is positive definite or can lay outside the eigenspectrum of $\hat{A}_s$ [3]. In order to get these non trivial values one has to choose the pre- and post-selected states such that they are not an eigenstate of the observable $\hat{A}_s$ and make sure that the strongly measured observable $\hat{B}_s$ does not commute with $\hat{A}_s$ [5].

4. The last step of the protocol is the extraction of the weak value. To understand how this can be done, let us consider the space representation of the pointer state at t_f:

$$\begin{aligned} \phi(X, t_f) &\propto \langle x|\exp\left\{-\frac{i}{\hbar} G A_s^w \hat{P}_p\right\}|\phi_i\rangle \\ &\propto \phi(X_i + G A_s^w, t_i) \end{aligned} \tag{11}$$

We see that the wavefunction of the pointer at t_f is only shifted in space by the amount GA_s^w when $\hat{P}_p$ and $\hat{X}_p$ are canonically conjugated variables. However one has to be careful, since A_s^w is in general a complex quantity. We can distinguish two cases when the wavefunction of the pointer has the form indicated in eq.(5):
a)$A_s^w \in \mathbb{R} \Longrightarrow \langle X\rangle_f = \langle X\rangle_i + GA_s^w$
b)$A_s^w \in \mathbb{C} \Longrightarrow \langle X\rangle_f = \langle X\rangle_i + G\,\mathrm{Re}\{A_s^w\}$, $\langle P\rangle_f = \langle P\rangle_i + 2G\sigma^2(P_i)\,\mathrm{Im}\{A_s^w\}$
where $\sigma^2(P_i)$ is the variance of the pointer's momentum at t_i [2]. In the case b), the spatial shift is governed by the $\mathrm{Re}\{A_s^w\}$ while the backaction on the momentum of the pointer is proportional to $\mathrm{Im}\{A_s^w\}$. The mean change in the momentum of the pointer is usually really small since $\sigma^2(P_i) \propto \frac{1}{\Delta^2}$ and $\frac{1}{\Delta^2} \to 0$ in order to fulfill the condition of the expansion in eq.(9) (here Δ^2 is the spread of the gaussian in X variable). More precisely, such expansion holds when the term corresponding to the 2^{nd} order in the expansion of the matrix exponential in eq.(9) is negligibly small compared to the 1^{st} order term. The last statement translates mathematically into a condition for the width of the gaussian wavefunction to be:

$$\Delta \gg G\left|\frac{\langle\chi_w|\, \hat{A}^2\, |\psi_w\rangle}{\langle\chi_w|\, \hat{A}\, |\psi_w\rangle}\right| \tag{12}$$

The Fig.2 summarizes the steps outlined above.

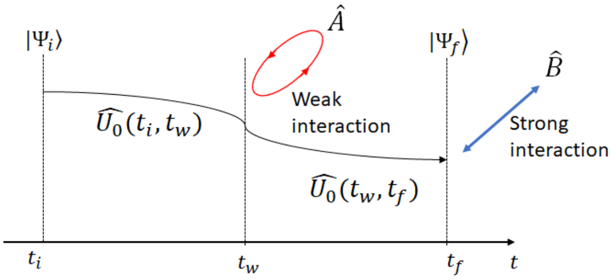

Figure 2: Illustration of how the state of the system evolves in time according to the weak measurement protocol.

2.3 Trapped ion implementation

To begin the discussion, let us consider a system depicted in Fig.3 consisting of one ion with two internal degrees of freedom $|1\rangle$ and $|0\rangle$ coupled to the axial mode of a quantized harmonic motion. Here the two boxes, or arms of the interferometer, are represented by the two properties of the ion $|1\rangle$ and $|0\rangle$, whereas the pointer is physically the state of the harmonic motion the ion is coupled to. Thus, the implementation presented in this work is equivalent to the one proposed in [1].

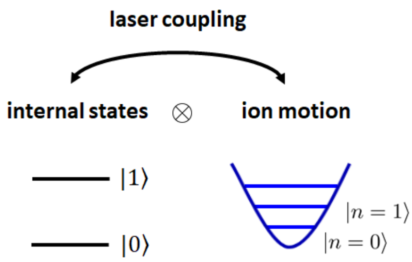

Figure 3: Pictorial representation of the total system. The internal degrees of freedom are coupled to the harmonic motion of the trap via laser field.

Such a system can be cooled down to the ground state of the harmonic motion by means of standard laser cooling techniques such as doppler cooling and sideband cooling (see [7][12]). This allows us to prepare the system in a state:

$$|\Psi_i\rangle = |\psi\rangle_i \otimes |0\rangle_m \tag{13}$$

where $|\psi\rangle_i$ refers to an arbitrary superposition of the internal degrees of freedom and $|0\rangle_m$ stands for the eigenstate of the destruction operator $\hat{a}$ with eigenvalue zero i.e. a coherent state $|\alpha = 0\rangle$. In general we could cool down our system into a different basis e.g squeezed basis [6], but for the sake of the argument let us stick to coherent states. Once we are in the motional ground state, we can use laser pulses to drive transitions which will couple the eigenstates of the free

Hamiltonian of the system. In particular, if we drive the blue sideband and the red sideband with a bi-chromatic laser we get an interaction Hamiltonian which has the same structure as the one in the eq.(7) [7]:

$$\begin{aligned}\hat{H}_{I,SDF} &= \hat{H}_{I,RSB} + \hat{H}_{I,BSB} \\ &\approx \frac{\hbar\Omega_B\eta}{2}(\hat{\sigma}_+ \otimes \hat{a}^\dagger e^{-i(\delta t+\phi_B)} + \hat{\sigma}_- \otimes \hat{a} e^{i(\delta t+\phi_B)}) \\ &+ \frac{\hbar\Omega_R\eta}{2}(\hat{\sigma}_+ \otimes \hat{a} e^{i(\delta t-\phi_R)} + \hat{\sigma}_- \otimes \hat{a}^\dagger e^{-i(\delta t-\phi_R)}) \\ &= \frac{\hbar\Omega\eta}{2} \underbrace{\hat{\sigma}_{\bar{\phi}}}_{system} \otimes \underbrace{(\hat{a}e^{i\Delta\phi} + \hat{a}^\dagger e^{-i\Delta\phi})}_{pointer}\end{aligned} \tag{14}$$

where $\hat{\sigma}_{\bar{\phi}} = \hat{\sigma}_x \cos\bar{\phi} + \hat{\sigma}_y \sin\bar{\phi}$. In the last step we assumed a perfect resonance condition $\delta = 0$, the same strength for the sidebands $\Omega_B = \Omega_R = \Omega$, and we defined $\Delta\phi = \frac{\phi_R - \phi_B}{2}$ and $\bar{\phi} = \frac{\phi_R+\phi_B}{2}$ with $\phi_{R/B}$ being the phase between the laser pulse and the atomic polarization. The Hamiltonian in eq.(14) is called state dependent force (SDF) since it causes displacements of the pointer degrees of freedom depending on the state of the internal degrees of freedom (it was originally proposed by Mølmer and Sørensen in [12] as an entangling gate). Now, we want to choose $\Delta\phi = \frac{\pi}{2}$ such that the pointer operator becomes:

$$\hat{P} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \tag{15}$$

For the spin d.o.f we choose $\bar{\phi} = 0$ and we apply a $\frac{\pi}{2}$ rotation about the y axis such that $\hat{\sigma}_x \to \hat{\sigma}_z$ and we end up with the total interaction Hamiltonian:

$$\hat{H}_{I,SDF} = \frac{\hbar\Omega\eta}{\sqrt{2}}\hat{\sigma}_z \otimes \hat{P} \tag{16}$$

Lastly, we want to consider three ions sharing the same motional state. We cool down the system to the ground state of the center of mass mode (common mode) of the three ion chain. Further, we manipulate with the laser field only a pair of ions which evolve under the interaction Hamiltonian and the third ion does not evolve in the interaction picture. This could be achieved with a sufficiently focused laser beam, such that only two ions at a time are manipulated and the third one experiences only the free evolution. In this case we have to substitute $\hat{\sigma}_z$ with $(\hat{\sigma}_z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\sigma}_z)_{(i,j)}$, this leaves us with the Hamiltonian:

$$\begin{aligned}\hat{H}_{I,SDF(i,j)} &= \sqrt{2}\hbar\Omega\eta(|0,0\rangle\langle 0,0|_{(i,j)} - |1,1\rangle\langle 1,1|_{(i,j)}) \otimes \hat{P} \\ &= g(\hat{\Pi}^{ours}_{(i,j)} \otimes \hat{P})\end{aligned} \tag{17}$$

The Hamiltonian in eq.(17) is not exactly what we were looking for, it differs from the actual observable that we want to weakly measure by a minus sign in front of the projector $|1,1\rangle\langle 1,1|_{(i,j)}$. Since the operators $\hat{\Pi}^{same}_{(i,j)} = (|0,0\rangle\langle 0,0|_{(i,j)} + |1,1\rangle\langle 1,1|_{(i,j)})$ and $\hat{\Pi}^{ours}_{(i,j)} = (|0,0\rangle\langle 0,0|_{(i,j)} - |1,1\rangle\langle 1,1|_{(i,j)})$ have different eignevalues, there exist no unitary operation that can bring one to the other. In order to simulate the paradox in [1], we have to understand the conditions under which the pigeonhole effect appears and relate the weak values of the observables $\hat{\Pi}^{same}_{(i,j)}$ and $\hat{\Pi}^{ours}_{(i,j)}$. This will be the subject of the next section.

3 Simulation of the paradox

In this section we want to enter into a detailed calculation of the weak values of the quantum pigeonhole paradox and explain how to make sense of the analytical results in our experiment.

3.1 Calculation of the weak values

Let us calculate the weak values for the paradox in the case of three qubits (i.e. three particles and two boxes). In general, the two level system up to a global phase can be parametrized as:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \exp\{i\varphi\}\sin\frac{\theta}{2}|1\rangle \tag{18}$$

such that the normalization conditions are automatically satisfied. This allows for the representation of the qubit on the Bloch sphere shown in the figure below:

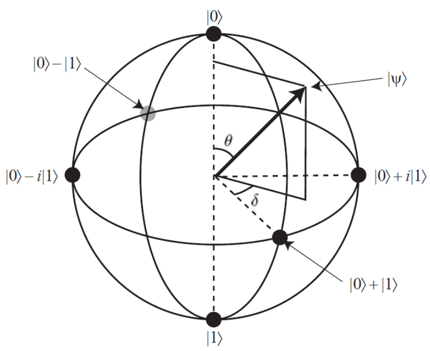

Figure 4: Bloch sphere, the black points denote the bases diagonalizing the Pauli matrices. Image taken from [9].

We will denote the pre-selected state of one qubit as:

$$|I\rangle = \cos\frac{\gamma}{2}|0\rangle + \exp\{i\varphi\}\sin\frac{\gamma}{2}|1\rangle \tag{19}$$

and the post-selected state of the qubit as:

$$|F\rangle = \cos\frac{\theta}{2}|0\rangle + \exp\{i\phi\}\sin\frac{\theta}{2}|1\rangle \tag{20}$$

The goal is to calculate the weak value of the observable $\hat{\Pi}^{same}_{(i,j)}$ acting on a pair of qubits in the case of symmetric pre- and post-selected states. In such a case, when we deal with three qubits we can calculate the weak value for the 1^{st} two qubits and the result will hold for any other pair. By definition we have:

$$\begin{aligned}(\hat{\Pi}^{same}_{(1,2)})^w &= \frac{\langle F,F,F|\,\hat{\Pi}^{same}_{(1,2)}\,|I,I,I\rangle}{\langle F,F,F|I,I,I\rangle} \\ &= \frac{\langle F,F,F|\,(\hat{\Pi}^{00}_{(1,2)} + \hat{\Pi}^{11}_{(1,2)})\,|I,I,I\rangle}{\langle F,F,F|I,I,I\rangle} \\ &= (\hat{\Pi}^{00}_{(1,2)})^w + (\hat{\Pi}^{11}_{(1,2)})^w\end{aligned} \tag{21}$$

where $\hat{\Pi}^{00}_{(1,2)} = |0,0\rangle\langle 0,0|$ and $\hat{\Pi}^{11}_{(1,2)} = |1,1\rangle\langle 1,1|$. From eq.(21) it is clear that the weak value of $\hat{\Pi}^{same}_{(1,2)}$ is the sum of the weak values of the projectors $\hat{\Pi}^{00}_{(1,2)}$ and $\hat{\Pi}^{11}_{(1,2)}$. The weak values of the latter observables have a simple analytical form in the symmetric case:

$$(\hat{\Pi}^{00}_{(1,2)})^w = \frac{\cos^2\frac{\theta}{2}\cos^2\frac{\gamma}{2}}{(\cos\frac{\theta}{2}\cos\frac{\gamma}{2} + e^{i(\varphi-\phi)}\sin\frac{\theta}{2}\sin\frac{\gamma}{2})^2} \tag{22}$$

and:

$$(\hat{\Pi}^{11}_{(1,2)})^w = \frac{e^{i2(\varphi-\phi)}\sin^2\frac{\theta}{2}\sin^2\frac{\gamma}{2}}{(\cos\frac{\theta}{2}\cos\frac{\gamma}{2} + e^{i(\varphi-\phi)}\sin\frac{\theta}{2}\sin\frac{\gamma}{2})^2} \tag{23}$$

We note that with our gauge choice (i.e. putting the phase factor in front of $|1\rangle$ in eq.(19) and eq.(20) instead of putting it in front of the $|0\rangle$) the numerator of $(\hat{\Pi}^{00}_{(1,2)})^w$ is always ≥ 0 while the numerator of $(\hat{\Pi}^{11}_{(1,2)})^w$ can be negative or complex. It is also easy to convince ourselves that these two quantities cannot both yield zero individually, since fixing one of the angles θ or γ fixes automatically the weak values. For example, setting θ or γ to π yields always $(\hat{\Pi}^{11}_{(1,2)})^w = 1$ and $(\hat{\Pi}^{00}_{(1,2)})^w = 0$ regardless of how we choose φ and ϕ. Conversely, if we choose θ or γ to be zero, we will end up in the opposite situation $(\hat{\Pi}^{11}_{(1,2)})^w = 0$ and $(\hat{\Pi}^{00}_{(1,2)})^w = 1$. This is not surprising, choosing such angles leads to a situation in which the pre- and post-selected states are the eigenstates of the projectors $\hat{\Pi}^{11}_{(1,2)}$ and $\hat{\Pi}^{00}_{(1,2)}$ with eigenvalues 0 and 1, this in turn implies that the weak values of such projectors cannot lay outside their eigenspectrum, as mentioned in the point 3) of the protocol in the Section 2. Thus, we conclude by saying that the only case in which the weak value of $\hat{\Pi}^{same}_{(1,2)}$ yields zero, which is the pigeonhole effect, is when the weak values $(\hat{\Pi}^{11}_{(1,2)})^w$ and $(\hat{\Pi}^{00}_{(1,2)})^w$ are equal in magnitude but have opposite signs. This observation allows us to find all the initial and final angles on the Bloch sphere for the qubits. The condition on the opposite signs for the weak values implies:

$$(\varphi - \phi) = (2n+1)\frac{\pi}{2} \tag{24}$$

for $n \in \mathbb{N}$. To obtain the angles θ and γ we have to solve the equation $(\hat{\Pi}^{00}_{(1,2)})^w + (\hat{\Pi}^{11}_{(1,2)})^w = 0$ assuming the eq.(24) holds:

$$\begin{aligned}(\hat{\Pi}^{same}_{(1,2)})^w &= (\hat{\Pi}^{00}_{(1,2)})^w + (\hat{\Pi}^{11}_{(1,2)})^w \\ &= \frac{\cos^2\frac{\theta}{2}\cos^2\frac{\gamma}{2} - \sin^2\frac{\theta}{2}\sin^2\frac{\gamma}{2}}{(\cos\frac{\theta}{2}\cos\frac{\gamma}{2} - i\sin\frac{\theta}{2}\sin\frac{\gamma}{2})^2} \\ &= 0\end{aligned} \tag{25}$$

solving eq.(25) for θ and γ we obtain two solutions:

$$(\theta \pm \gamma) = (2n+1)\pi \tag{26}$$

for $n \in \mathbb{N}$. The solution with the minus sign leads to the situation $\frac{0}{0}$ in the eq.(25), since this is not well defined we want to choose θ and γ such that we avoid it. We therefore impose the following conditions:

$$
\begin{aligned}
\theta - \gamma &= m\pi \\
\theta + \gamma &= (2n+1)\pi
\end{aligned} \tag{27}
$$

for $m \in \mathbb{R}$ and $m \neq 2n+1, \forall n \in \mathbb{N}$. Solving the equations in eq.(27) yields:

$$
\begin{aligned}
\theta &= \frac{\pi}{2}(2n+m+1) \\
\gamma &= \frac{\pi}{2}(2n-m+1)
\end{aligned} \tag{28}
$$

The eq.(28) together with eq.(24) tell us how to pre- and post-select states in order to observe the pigeonhole effect, in the case where both states are symmetric.

3.2 Simulation of the experiment

We want to discuss how to simulate the quantum pigeonhole effect using the Hamiltonian in eq.(17). In order to do this, let us point out a few facts emerging from our analysis in the previous subsection:

1. The weak value of $(\hat{\Pi}^{same}_{(i,j)})^w$ is given by the sum of the weak values of $(\hat{\Pi}^{00}_{(i,j)})^w$ and $(\hat{\Pi}^{11}_{(i,j)})^w$.
2. The numerator of $(\hat{\Pi}^{00}_{(i,j)})^w$ is always ≥ 0 while the numerator of $(\hat{\Pi}^{11}_{(i,j)})^w$ can be negative or complex. Since we make sure that in the eq.(25) the denominator never goes to zero when the numerator does, the only thing we should pay attention to is the numerator.
3. The weak values $(\hat{\Pi}^{00}_{(i,j)})^w$ and $(\hat{\Pi}^{11}_{(i,j)})^w$ cannot yield zero at the same time individually.
4. The quantum pigeonhole effect occurs only when $(\hat{\Pi}^{00}_{(i,j)})^w$ and $(\hat{\Pi}^{11}_{(i,j)})^w$ have the same magnitude but opposite signs. It means that $(\hat{\Pi}^{same}_{(i,j)})^w = 0$ only in such case.

These four facts allow us to map the results of the weak measurement of $\hat{\Pi}^{same}_{(i,j)}$ to the weak measurement of $\hat{\Pi}^{ours}_{(i,j)}$ in an unambiguous way:

1. The weak value of $(\hat{\Pi}^{ours}_{(i,j)})^w$ is given by the difference of the weak values of $(\hat{\Pi}^{00}_{(i,j)})^w$ and $(\hat{\Pi}^{11}_{(i,j)})^w$.
2. The quantum pigeonhole effect for $(\hat{\Pi}^{ours}_{(i,j)})^w$ occurs whenever:

$$
(\hat{\Pi}^{ours}_{(i,j)})^w = (\hat{\Pi}^{00}_{(i,j)})^w - (\hat{\Pi}^{11}_{(i,j)})^w = 2(\hat{\Pi}^{00}_{(i,j)})^w \tag{29}
$$

What we have to do at this point is to look at the pointer's state after the post-selection, and see whether it has been displaced by an amount proportional to $2(\hat{\Pi}^{00}_{(i,j)})^w$. The latter can be analytically calculated using the eq.(22). The displacement of the pointer's wavefunction can occur either in position space if $2(\hat{\Pi}^{00}_{(i,j)})^w \in \mathbb{R}$ or in both position and momentum space if $2(\hat{\Pi}^{00}_{(i,j)})^w \in \mathbb{C}$. For this reason we use the Wigner function (for details see [11]) to represent the state of the pointer after the post-selection. This allows us to observe the displacements of the ponter's state in phase space.

Case study : Consider the simple case in which $\theta = \gamma = \frac{\pi}{2}$, it means that the pre- and post-selected states of individual qubits lay both on the equatorial plane of the Bloch sphere. Let us define $\alpha = \varphi - \phi$ such that the weak values of interest take the form:

$$(\hat{\Pi}^{00}_{(1,2)})^w = \frac{1}{(1+e^{i\alpha})^2} \tag{30}$$

$$(\hat{\Pi}^{11}_{(1,2)})^w = \frac{e^{i2\alpha}}{(1+e^{i\alpha})^2} \tag{31}$$

for $\alpha = \frac{\pi}{2}$ we find that the sum of the weak values in eq.(30) and eq.(31) vanishes $(\hat{\Pi}^{same}_{(i,j)})^w = (\hat{\Pi}^{00}_{(i,j)})^w + (\hat{\Pi}^{11}_{(i,j)})^w = 0$, and we recover the usual quantum pigeonhole effect as expected from the eq.(24). In the simulation of our experiment we will obtain the difference, which for this particular value of α yields $(\hat{\Pi}^{ours}_{(i,j)})^w = (\hat{\Pi}^{00}_{(i,j)})^w - (\hat{\Pi}^{11}_{(i,j)})^w = -i$. If we look at the individual contributions we will find that the real part is zero:

$$\mathrm{Re}\bigg\{\frac{1}{(1+e^{i\frac{\pi}{2}})^2}\bigg\} = \mathrm{Re}\bigg\{\frac{e^{i2\frac{\pi}{2}}}{(1+e^{i\frac{\pi}{2}})^2}\bigg\} = 0 \tag{32}$$

whereas for the imaginary part we have:

$$\mathrm{Im}\bigg\{\frac{1}{(1+e^{i\frac{\pi}{2}})^2}\bigg\} = -\,\mathrm{Im}\bigg\{\frac{e^{i2\frac{\pi}{2}}}{(1+e^{i\frac{\pi}{2}})^2}\bigg\} = -\frac{1}{2} \tag{33}$$

such that everything sums up to $-i$. Once we calculated these quantities we know what to expect in the simulation. The Re part of both $(\hat{\Pi}^{00}_{(i,j)})$ and $(\hat{\Pi}^{11}_{(i,j)})$ is zero, therefore we do not expect to see any shift in the position variable of the pointer state. We prepare the pointer in the coherent vacuum state and the only effect that we will observe is a shift in the momentum variable proportional to the Im part of the weak value $(\hat{\Pi}^{ours}_{(i,j)})$ according to:

$$\langle P\rangle_f = \langle P\rangle_i + 2G\sigma^2(P_i)\,\mathrm{Im}\Big\{(\hat{\Pi}^{ours}_{(i,j)})^w\Big\} \tag{34}$$

The effective coupling strength is given by $G = \sqrt{2}\eta\int_{t_w-\frac{T}{2}}^{t_w+\frac{T}{2}}\Omega(t-t_w) \approx \sqrt{2}\eta\Omega T = gT$, where we taken into account the fact that in general Ω has a time dependent profile and we assumed the latter to have compact support T in time. The variance of the momentum at the initial time, $\sigma^2(P_i)$ is calculated as:

$$\sigma^2(P_i) = \langle\phi|\,\hat{P}^2\,|\phi\rangle - \langle\phi|\,\hat{P}\,|\phi\rangle^2 \tag{35}$$

here $|\phi\rangle$ stands for the pointer state at the initial time and $\hat{P}$ is the canonical momentum operator acting on the pointer state as written in eq.(15). In this work the simulation of the physical system was implemented in Python with QuTiP [12], a quantum toolbox used for simulating the dynamics of open quantum systems. In order to faithfully simulate the paradox, we exactly follow the steps of the weak measurement protocol outlined in the Section 2. In the simulation we have to be careful about several aspects, one of them is the dimension of the Hilbert space. The pointer state in our system, is the coherent vacuum state which in its standard definition [10], lives in an infinite dimensional Hilbert space. In a simulation one always deals with finite dimensions, thus we have to truncate the Hilbert space of the pointer and make sure that it is big enough to not alter the outcomes of the measurement. Another important aspect that we have to take care of is the validity of the expansion of the unitary time propagator up to the first order. To estimate how close we are to the first order expansion we calculate the Hilbert-Schmidt inner product between $\hat{U}(t)_{exact} = e^{-ig\hat{H}_{int}T}$ and $\hat{U}(t)_{approx} \approx (\mathbb{1} - igT\hat{H}_{int})$ which reads:

$$d = \mathrm{Tr}\Big\{\hat{U}^{\dagger}(t)_{approx}\hat{U}(t)_{exact}\Big\} \tag{36}$$

when d is asymptotically close to 1 we assume that the conditions to perform the weak measurement have been fulfilled and we expect to obtain the weak value at the end of the experimental protocol. We set the Plank's constant to be $\hbar = 1$, choose the effective coupling constant $G = gT = 0.2$ and the dimension of the Hilbert space $N = 150$. We find from the eq.(36) that $d \approx 0.99987$. We calculate the expectation value of the position and of the momentum at the post-selection and we find:

$$\langle P \rangle_f \approx -0.1942 \tag{37}$$

$$\langle X \rangle_f = 0 \tag{38}$$

From these equations we extract the Re and Im parts of the weak value of $(\hat{\Pi}^{ours}_{(i,j)})$.

$$\mathrm{Im}\Big\{(\hat{\Pi}^{ours}_{(i,j)})^{w}\Big\} = \frac{\langle P \rangle_f}{2G\sigma^2(P_i)} \approx -0.97 \tag{39}$$

The Re part is obviously zero as one can see from eq.(38) and the Im part is close to -1 which is the value we expected to see for the quantum pigeongole effect.In the simulation $(\hat{\Pi}^{ours}_{(i,j)})$ acts on the last two qubits but since the initial and the final states are symmetric the result holds for all the other pairs. The Fig.(5) shows the Wigner function (for details see for example [11]) of the pointer's stste at the final time:

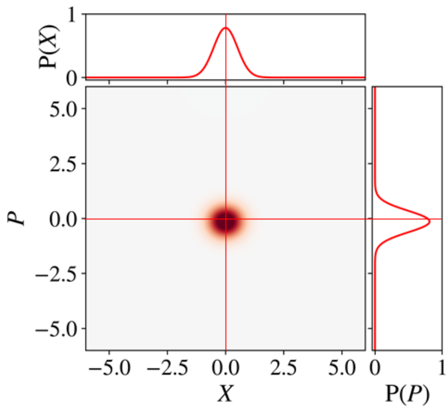

Figure 5: Wigner function of the pointer's state after the weak measurement protocol. The insets show the probability distributions of the X and P variables. The red straight lines have been drawn by hand to emphasize the fact that the peak of the probability distribution in P variable has experienced a shift with respect to the origin by the amount indicated in eq.(37) while the probability distribution in X variable did not experience any shift.

A variant of the experiment could be the following: instead of cooling down the system to the coherent basis, we could cool down the system to squeezed basis as we already mentioned in Section 2. The squeezing should be done with the squeezing parameter $r < 0$ in our case, since we only experience shifts in P variable of the pointer for $\theta = \gamma = \frac{\pi}{2}$ and any angle α (in this case the real part of $(\hat{\Pi}^{ours}_{(i,j)})^w$ is vanishing). In such situation we get a gaussian with smaller variance in P variable while the gaussian in X variable will be very broad (for more details see [6],[7],[11]). This enables us to work with higher coupling constants and therefore it should be more affordable experimentally. We simulate the experiment with squeezed pointer by setting the effective coupling constant $G = gT = 1$ and the dimension of the Hilbert space $N = 270$ (the dimension of the Hilbert space is higher since the squeezed states is broader than the coherent state). The squeezing parameter is set to $r = -1.65dB$. With such parameters we find $d \approx 0.997$ according to eq.(36) and the expectation values for the momentum and the position to be:

$$\langle P \rangle_f \approx -0.0358 \tag{40}$$

$$\langle X \rangle_f \approx 0 \tag{41}$$

As in the case of coherent states we can find the weak value of $(\hat{\Pi}^{ours}_{(i,j)})$:

$$\text{Im}\Big\{(\hat{\Pi}^{ours}_{(i,j)})^w\Big\} = \frac{\langle P \rangle_f}{2G\sigma^2(P_i)} \approx -0.97 \tag{42}$$

$$\text{Re}\Big\{(\hat{\Pi}^{ours}_{(i,j)})^w\Big\} = 0 \tag{43}$$

Thus we recover the pigeonhole effect and obtain $(\hat{\Pi}^{ours}_{(i,j)})^w \approx -i0.97$. The Fig.(6) shows the Wigner function of the pointer at the final time:

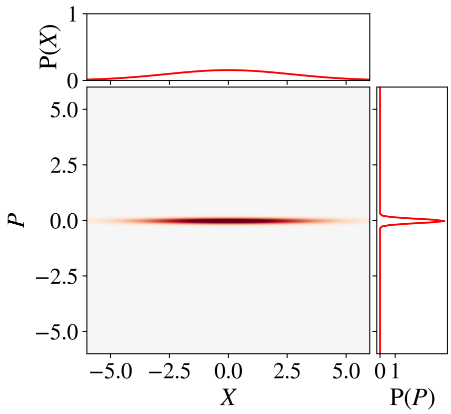

Figure 6: Wigner function of the pointer's state after the weak measurement protocol for the squeezed pointer. The insets show the probability distributions of the X and P variables.

The drawback of such implementation is that in a real experiment we would need a very high resolution in P variable in order to being able to detect displacements as small as the one in eq.(40). On the other hand, using the squeezed states instead of coherent states gives us more resilience to fluctuations in the effective coupling constant.

4 Experimental remarks

We want to spend a few words to point out possible experimental issues with our implementation and propose how to go about them. One of the most important issue with trapped ions, to our knowledge, is the problem which arises when we try to perform a projective measurement on the system. Consider an ion trapped in a harmonic trap whose internal structure is depicted in Fig.(7):

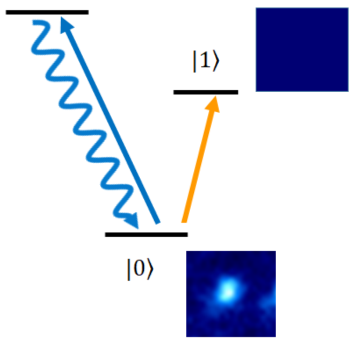

Figure 7: In this schematic representation the yellow arrow indicates the quadrupole allowed transition that we drive to manipulate the two level system optically. The blue lines indicate the dipole allowed transition used for the readout procedure.

The bright and the dark spots in the figure above, represent the outcomes of the fluorescence detection i.e. the projective measurement. If the system was in the ground state we would see a bright spot. Conversely, if the ion was in the excited state we would observe a dark spot. Since the whole system is coupled to the harmonic trap what actually happens when we perform the fluorescence detection goes as follows:

- If the system is in the ground state we scatter many fluorescent photons whose momentum randomizes the pointer's wavefunction and we are not able to see the effect of the weak measurement.
- If the ion is in the excited state we do not scatter fluorescent photons and the state of the motional degrees of freedom is not altered.

The previous points put a very important experimental constraint: we cannot just post-select any state if we want to access the information about the weak measurement "stored" in the motional part of the total state. In an experiment, we always want to post-select the dark counts i.e. we always want to end up with the ions being in the excited state. According to the analytic calculation, post-selecting the state $|F\rangle = |1\rangle$ requires us to choose $\theta = \pi$ (see eq.(20)). Such value leads to find $(\hat{\Pi}^{same}_{(i,j)})^w = 1$ regardless of all the other angles. This in turn leads to the conclusion that we will never be able to observe the pigeonhole effect in $\{|0\rangle, |1\rangle\}$ basis.

We propose a possible solution to this problem. Instead of measuring the weak value $(\hat{\Pi}^{same}_{(i,j)})^w$ in the computational basis, we could measure it in the $\{|+\rangle, |-\rangle\}$. In such case the projector we want to weakly measure would become $\hat{\Pi}^{same}_{(i,j)} = |+,+\rangle\langle+,+|_{(i,j)} + |-,-\rangle\langle-,-|_{(i,j)}$. For example, by pre-selecting $|I,I,I\rangle = |+i,+i,+i\rangle$ (where $|+i\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}}$) and post-selecting the dark counts i.e. $|F,F,F\rangle = |1,1,1\rangle$ we get:

$$\begin{aligned} (\hat{\Pi}^{same}_{(1,2)})^w &= (|+,+\rangle\langle+,+|_{(1,2)} + |-,-\rangle\langle-,-|_{(1,2)})^w \\ &= 2(1+i)^2 + 2(1-i)^2 = 0 \end{aligned} \tag{44}$$

Thus, we can see that we still get the pigeonhole effect and everything we said about the mapping of the $(\hat{\Pi}^{same}_{(i,j)})^w$ to $(\hat{\Pi}^{ours}_{(i,j)})^w$ still holds. Except the fact that now the interaction Hamiltonian will be:

$$\begin{aligned} \hat{H}_{I,SDF(i,j)} &= \sqrt{2}\hbar\Omega\eta(|+,+\rangle\langle+,+|_{(i,j)} - |-,-\rangle\langle-,-|_{(i,j)}) \otimes \hat{P} \\ &= g(\hat{\Pi}^{ours}_{(i,j)} \otimes \hat{P}) \end{aligned} \tag{45}$$

With this being clarified we hope that such experiment could be realized and test the predictions in [1].

5 Conclusion and Outlook

What presented in this work opens the possibility to the experimental verification of the quantum pigeonhole effect in a trapped ion system when the initial and final states are symmetric under permutations. In the latter case, the weak values were computed analytically for three qubits. The simulation using three trapped ions sharing the same motional state was performed and the weak values obtained from the simulation agreed with the analytical calculation. Moreover, we pointed out that performing the experiment in a rotated basis $\{|+\rangle, |-\rangle\}$, allows one to avoid issues related to the projective measurement in trapped ions. A continuation of such project might consist of checking whether the quantum pigeonhole effect still exists in case of pre- and post-selected state with no permutation symmetry. In the case of three qubits, one should measure the operator $\hat{\Pi}^{same}_{(i,j)}$ for every pair (i,j) of qubits in order to know whether the quantum violation of the pigeonhole principle occurred or not.

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A Code Documentation

In the following we explain the main features of our code, and define the most relevant implemented functions. They follow the same structure as the submitted *Jupyter* notebook.

A.1 Plot of the Wigner function

The following function plots the Wigner distribution of a given state

`plot_wig(rho, fig, xvec=xvec):`

INPUT

- `rho`: density matrix of the state (Qutip Qobj)
- `fig`: label for the output figure
- `xvec`: mesh of the plot given by a numpy array (set by default from [-7, 7])

OUTPUT

- Plot of the Wigner distribution with insets representing the marginal proability distributions of the X and P variables of the pointer state

A.2 Pre- and post-selected states parametrized on the Bloch sphere

The definition of the pre- or post-selected state in the case of three qubits (three particles and two boxes)

`state(t,p,l,r):`

INPUT

- `t`: polar angle on the Bloch sphere
- `p`: azimuthal angle on the Bloch sphere
- `l`: $|0\rangle$ in computational basis (Qutip Qobj)
- `r`: $|1\rangle$ in computational basis (Qutip Qobj)

OUTPUT

- tensor product of three qubits in the symmetric case (Qutip Qobj)

A.3 Main function for the pigeonhole simulation

`Pigeon(N,ti,tf,pin,pif,sqzparam,pointer,coupl,sdf_t1)`: Function plotting the Wigner distribution given the pre- and post-selected states

INPUT

- `N`: Dimension of the Hilbert space
- `ti`: polar angle of the pre-selected state
- `tf`: polar angle of the post-selected state
- `pin`: azimuthal angle of the pre-selected state
- `pif`: azimuthal angle of the post-selected state
- `sqzparam`: squeezing parameter expressed in dB
- `pointer`: choice between coherent "c" and squeezed "s" states
- `coupl`: coupling constant of the Hamiltonian
- `sdf_t1`: gate time

OUTPUT

- Prints on the screen:
  - Hilbert-Schmidt inner product according to eq. (36)
  - the expectation value of the position and momentum operator after the post-selection
  - weak value of the observable we are measuring
- Plots the Wigner distribution of the pointer after the post-selection

A.4 Input the simulation parameters and run

Here one has to choose the parameters for the simulation, in the notebook a set of default parameters are already set for reference.