# Spectral Curves, their trivialization and the role of Weyl groups

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The main purpose of this note is to present an algorithm for identifying the sheets $\{x_i\}_i$ of a spectral curve $\Sigma_\rho$, associated with a representation $\rho$, above a generic point $z$ of the base $C$.

## 1 Preliminaries

This kind of problem arises immediately in concrete approaches to the study of these covers, where one makes a choice of branch cuts, and trivializes the cover. The input data is given simply by a set of complex numbers, marking the positions of the sheets in the fiber $\mathbb{C}$ above $z$, the task is then to identify consistently each of these with a weight. Once this is accomplished for a single point (the basepoint of the trivialization, in particular), the trivialization then assigns a weight to each point of a sheet uniquely.

We will present techniques for accoomplishing this task for all representations of ADE Lie groups. A basic fact which holds in general is that the positions of the sheets are linear functions of the weights.

$$x_i = \langle \mu_i, \varphi(z) \rangle \tag{1}$$

This entails a great simplification: for each Lie algebra, it is sufficient to solve the problem for any faithful representation, the solution will then extend to all other representations by linearity.

A bit more concretely, if we solve the problem of identifying the weights of a certain rep $\rho$ with the sheets of the cover $\Sigma_\rho$, we may then pick a basis for $\mathfrak{t}^*$ among the $\{\mu_i\}_i$, and expand all the weights of any other rep $\rho'$ in that basis

$$\mu'_j = \sum_i c_{ij} \mu_i \tag{2}$$

this data being easily recovered from the knowledge of the weight systems themselves. By linearity, this gives the desired identification of the sheets of $\Sigma'_\rho$ as

$$x'_j = \sum_i c_{ij} x_i \tag{3}$$

where $x_i$ are the sheets of $\Sigma_\rho$.

## 2 $A_n$ systems

We choose the first fundamental representation, and label the weights $\mu_1 \dots \mu_{n+1}$. The residual gauge freedom after diagonalizing $\varphi(z)$ consists in the action of the Weyl group $W = S_{n+1}$, which permutes all sheets $x_i \mapsto x_{\sigma(i)}$. Given this freedom, we can choose to assign any weight to any sheet, as long as no repetitions are made.

## 3 $D_n$ systems

We choose again the first fundamental representation, a.k.a. the vector rep. This consists of $2n$ weights, subject to the relations

$$\mu_i + \mu_{i+n} = 0 \qquad (i \in \mathbb{Z}_{2n})\,. \tag{4}$$

The Weyl group is $H_{n-1} \rtimes S_n$, of order $2^{n-1}n!$. Here $H_{n-1} \subset \mathbb{Z}_2^n$ is the kernel of the product homomorphism $\{\epsilon_1, \dots, \epsilon_n\} \to \epsilon_1 \dots \epsilon_n$ where $\epsilon_i = \pm 1$. In other words, $H_{n-1}$ is the subgroup of elements with an even number of $-1$'s.

The action of $W$ is as follows: it permutes all the pairs

$$\{\mu_i, \mu_{i+n}\} \mapsto \{\mu_{\sigma(i)}, \mu_{\sigma(i)+n}\} \qquad \sigma \in S_n \tag{5}$$

and independently switches an *even* number of signs

$$\mu_i \mapsto -\mu_i = \mu_{i+n}\,. \tag{6}$$

The identification of sheets and weights can be carried out as follows: first identify the $n$ pairs of opposite sheets such that $x + x' = 0$. Then, the permutation symmetry $S_n$ allows us to match any pair of opposite sheets with any pair of opposite weights. The problem thus boils down to identifying consistently a 'positive' and a 'negative' sheet in *each* pair.

The choices are not independent: the $H_{n-1}$ freedom however allows us to choose freely the positive sheet from the first $n-1$ pairs. Only the choice of positive vs negative within the last pair

$$\begin{array}{c} x_n \leftrightarrow \mu_n \\ x_{2n} \leftrightarrow \mu_{2n} \end{array} \qquad \text{vs} \qquad \begin{array}{c} x_n \leftrightarrow \mu_{2n} \\ x_{2n} \leftrightarrow \mu_n \end{array} \tag{7}$$

is constrained. This can be seen as follows: suppose we knew a 'reference' weight-sheet identification that works, then we can compare it to ours by first permuting the sheet pairs suitably, then by 'flipping' the positive/negative role within each sheet pair, for the first $n-1$ pairs. If we had to perform an even number of switches, then we should make the same positive/negative identification for the last pair as in the reference one; if instead we had to perform an odd number of switches, we should invert the last identification with respect to the reference one.

Given however that we don't have a reference identification to compare with, we cannot determine how to make the positive/negative identification in the last pair. All we can do is evaluate the consequence of a wrong choice. As it turns out,the wrong choice corresponds to acting with the outer automorphism which exchanges the spinor weights of the $D_n$ diagram. We can therefore randomly make the last choice, keeping in mind that we cannot tell whether a spinor cover corresponds to one spinor rep, or the other.

# 4 $E_n$ systems

## 4.1 $E_6$

We choose to work with the $\rho_1$ representation, with Dynkin indices $(1,0,0,0,0,0)$, of dimension 27. The Weyl group is of order $51840 \ll 27!$.

The 27 weights of $\rho_1$ can be arranged into *null triples*, obeying

$$\mu_i + \mu_j + \mu_k = 0 \tag{8}$$

There are 45 such triples (up to $S_3^{\times 45}$ permutations within each triple), each weight appears in exactly 5 of them.

Similarly, by linearity, the sheets must also organize into 45 null triples, and each sheet will feature in exactly 5 of them.

Choose a labeling for the sheets $x_0, \ldots, x_{26}$, and make an ansatz by identifying $\mu_0 \leftrightarrow x_0$. We consider the quintet $Q_0$ of null triples to which $\mu_0$ belongs, they are:

| $t_1^{(0)}$ | $t_2^{(0)}$ | $t_3^{(0)}$ | $t_4^{(0)}$ | $t_5^{(0)}$ |
|---|---|---|---|---|
| $(0,12,26)$ | $(0,15,26)$ | $(0,17,24)$ | $(0,19,23)$ | $(0,21,22)$ |

(9)

In notation where $(i,j,k)$ stands for $(\mu_i, \mu_j, \mu_k)$.

Given that we chose an ansatz that fixes $x_0$, we still have the remaining freedom given by the stabilizer subgroup

$$W_0 := \{w \in W \,|\, w(\mu_0) = \mu_0\} \tag{10}$$

The order of $W_0$ is 1920. The action of $W_0$ on the *quintet of ordered triples* $(t_1^{(0)}, \ldots, t_5^{(0)})$ gives 1920 distinct quintets of ordered triples. For clarity, two quintets

$$\begin{gathered}(t_1^{(0)}, t_2^{(0)}, t_3^{(0)}, t_4^{(0)}, t_5^{(0)}) \\ (t_2^{(0)}, t_1^{(0)}, t_3^{(0)}, t_4^{(0)}, t_5^{(0)})\end{gathered} \tag{11}$$

are considered different. Moreover, also two quintets differing by a triplet changing from $(0,12,26)$ to $(0,26,12)$ are considered different in our count. Note that the index 0 is left invariant by $W_0$, by definition.

In fact $1920 = 5! \cdot 2^4$ is the number of such ordered quintets obtained by considering all their permutations by $S_5$, and by flipping an *even* number of pairs within each triple (in other words, $W_0 = H_4 \rtimes S_5$).

This freedom means the following. Consider the null triples of sheets in which $x_0$ features. This singles out 5 pairs of sheets $(x_i, x_j)$ such that $x_i + x_j = -x_0$. We choose to label the sheets of those five triples as follows:

$$\tilde{Q}_0 := \left\{(x_0, x_{12}, x_{26})\ (x_0, x_{15}, x_{26})\ (x_0, x_{17}, x_{24})\ (x_0, x_{19}, x_{23})\ (x_0, x_{21}, x_{22})\right\} \tag{12}$$

where the identification of $x_0$ is fixed by our ansatz, and the $W_0$ gauge freedom accounts for possible permutations among the five pairs of other sheets, as well as for an *even* number of 'flips' of the pairs of sheets $x_i \leftrightarrow x_j$ within each triple $(x_0, x_i, x_j)$. More precisely, we have the freedom to choose who is $x_i$ vs $x_j$ (where we associate $x_{i,j} \to \mu_{i,j}$), in 4 out of 5 pairs, but the last choice is constrained by the first four ones (the reasoning is the same as for $D_n$, and likewise a wrong choice on the last pair likely corresponds to the $\mathbb{Z}_2$ outer automorphism of the Lie algebra).

Having identified the first 11 sheets, the subgroup of $W$ which stabilizes all of our choices is precisely of order 1. Thus, the Weyl freedom has been used exhaustively at this point.

How to identify the remaining 16 sheets? We have identified the 11 sheets

$$x_0, x_{12}, x_{26}, x_{15}, x_{25}, x_{17}, x_{24}, x_{19}, x_{23}, x_{21}, x_{22} \tag{13}$$

with the corresponding weights ($x_i \leftrightarrow \mu_i$). We can therefore construct their quintets:

$$\tilde{Q}_0, \tilde{Q}_{12}, \tilde{Q}_{26}, \tilde{Q}_{15}, \tilde{Q}_{25}, \tilde{Q}_{17}, \tilde{Q}_{24}, \tilde{Q}_{19}, \tilde{Q}_{23}, \tilde{Q}_{21}, \tilde{Q}_{22} \tag{14}$$

Again, each quintet will contain 5 triples of the form $(x_i, x_j, x_k)$ with $x_i$ being one of the sheets we identified, and $x_j, x_k$ being generally sheets that we have yet to identify with a weight.

Now, from the weight data, we know that each one of the missing weights has a unique pattern of whether it belongs or not to each of these quintets. The same must be true for the weights, by linearity [1], and therefore the problem of identifying the remaining weights and sheets is completely solved. We give below the data table: each row tells whether one of the missing weights belongs or not to the quintets which label the columns. A '0' stands for 'not contained', while a '1' stands for 'contained'. The same analysis can be carried out for the remaining sheets, since we know their coordinates, and we know the sheet quintets corresponding to the weight quintets $Q_i \leftrightarrow \tilde{Q}_i$. The requirement that the pattern in table of $(x_i, \tilde{Q}_j)$ matches with the pattern of the table $(\mu_i, Q_j)$ uniquely fixes all the remaining $x_i$'s.

(15)

| | $Q_0$ | $Q_{12}$ | $Q_{26}$ | $Q_{15}$ | $Q_{25}$ | $Q_{17}$ | $Q_{24}$ | $Q_{19}$ | $Q_{23}$ | $Q_{21}$ | $Q_{22}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_1$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| $\mu_2$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| $\mu_3$ | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| $\mu_4$ | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $\mu_5$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| $\mu_6$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\mu_7$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| $\mu_8$ | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| $\mu_9$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| $\mu_{10}$ | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\mu_{11}$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| $\mu_{13}$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\mu_{14}$ | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| $\mu_{16}$ | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| $\mu_{18}$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| $\mu_{20}$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |

[1] Expand a bit here on this assertion...

### 4.2 $E_7$

We choose to work with the $\rho_7$ representation, with Dynkin indices $(0,0,0,0,0,0,1)$, of dimension 56. The Weyl group is of order $2903040 \ll 56!$.

The 56 weights of $\rho_7$ can be arranged into *null quartets*, obeying

$$\mu_i + \mu_j + \mu_k + \mu_l = 0 \tag{16}$$

There are 1008 such quartets (up to $S_4^{\times 1008}$ permutations within each quartet), each weight appears in exactly 72 of them.

Similarly, by linearity, the sheets must also organize into 1008 null quartets, and each sheet will feature in exactly 72 of them.

Choose a labeling for the sheets $x_0, \ldots, x_{55}$, and make an ansatz by identifying $\mu_0 \leftrightarrow x_0$. We consider the 72-plet $Q_0$ of null quartets to which $\mu_0$ belongs, they are:

$$\begin{array}{c|c|c} t_1^{(0)} & \ldots & t_{72}^{(0)} \\ \hline (0,1,34,46) & \ldots & (0,44,46,50) \end{array} \tag{17}$$

In notation where $(i,j,k,l)$ stands for $(\mu_i, \mu_j, \mu_k, \mu_l)$.

The stabilizer of $\mu_0$ is $W_0$, of dimension My laptop can't compute this. But I'd expect it to be $72! \cdot (3!)^{72}$ or $72! \cdot (3!)^{71}$. If that is true, then probably we can proceed as before, using this freedom to identify all sheets with weights appearing in $\tilde{Q}_0, Q_0$, and the rest by studying their patter of which 72-plets they belong to.