Higher-index ramification

In coordinates where ram. pt. is @ $z=0$, $x=0$

$$F(x,z) \sim a z + b x^k$$

and k is the number of sheets colliding.

There, $\lambda = x\, dz \cong \left(-\frac{a}{b} z\right)^{1/k} e^{\frac{2\pi i}{k} m}\, dz \qquad m=0,\cdots,k-1$

are the k sheets colliding. The permutation is

$\mathbb{Z}_k$ - cyclic.

<u>S-walls</u>: for each pair of sheets (λ_i, λ_j)

<u>if</u> $\lambda_j - \lambda_i = \langle v_j - v_i, \varphi \rangle = \langle \alpha, \varphi \rangle$ ie if

$v_j - v_i \in \Phi$, there should be a root-type S-wall

Let n be the # of roots (it may not be

maximal, this depends on whether $k \leq \dim \rho$)

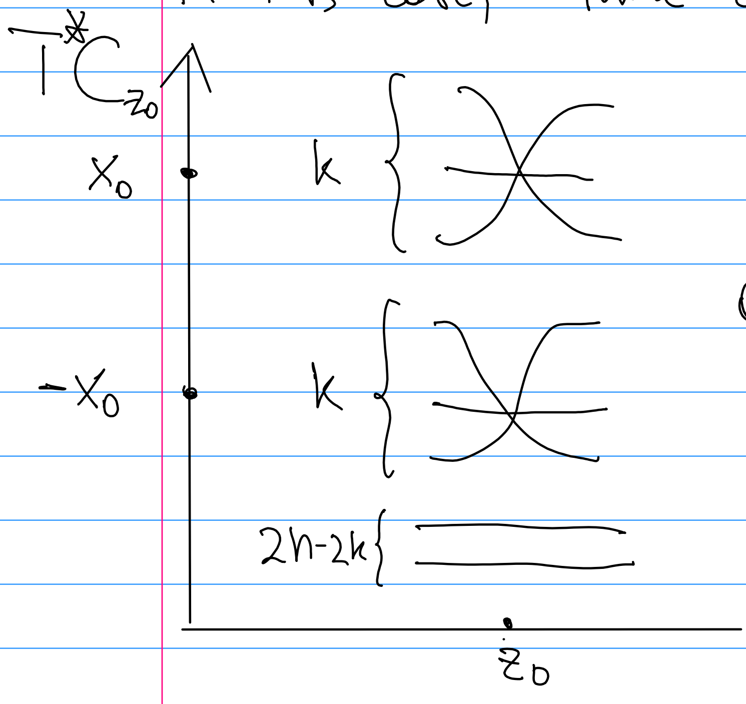
that "vanish" there.

If $\mathfrak{g} = A_r$, and we work in the ρ_1 rep, then $n = \binom{k}{2} \cdot 2 = k(k-1)$, because every pair of weights is separated by a root.

If $\mathfrak{g} = D_r$ instead, all roots are captured by differences/sums of pairs of weights

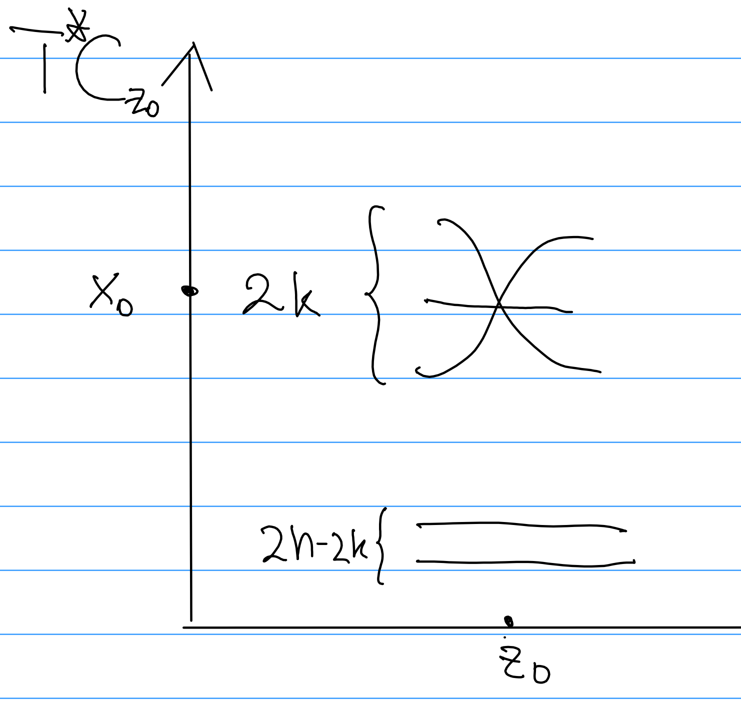
$$\Lambda_\rho = \{\pm e_i\}_{i=1}^{2r} \qquad \Phi = \{\pm(e_i \pm e_j)\}_{i \neq j}$$

in this case, there can be several situations:



$\boxed{x_0 \neq 0}$

OR



$\boxed{x_0 = 0}$

In the 1st case, at x_0 we have sheets

$v_{i_1} \cdots v_{i_k}$ colliding @ x_0

$-v_{i_1} \cdots -v_{i_k}$ colliding @ $-x_0$

In the 2nd case we have

$\pm v_{i_1} \cdots \pm v_{i_k}$ colliding @ x_0.

In the 1st case, each pair of colliding sheets will give rise to roots, by taking

$\pm(v_i - v_j) \in \Phi$ always

and we only need to consider 1/2 of the sheets.

In the 2nd case, we need to pick 1/2 of the sheets, and associate S-walls to sums/differences of them: $\pm(v_i \pm v_j)$ (because now also $x_i + x_j = 0$)

In this case $F(x,z) = az + bx^{2k} = 0$

and to get the right sheets we may

study $y = x^2$ and solve $az + by^k = 0$

and then pick $x_{i_1} \dots x_{i_k} = \sqrt{y_{i_1}} \dots \sqrt{y_{i_k}}$.

NOTE: this also solves the issue with those configurations where $x^2 \left(az + b x^{2k-2} \right) = 0$

where there are 2 sheets which always collide.

Seeds for S walls:

Above we specified which sheets we must use to build S-walls.

Non-degenerate case : $az + bx^k = 0$ (A-type simp.ty)

Type I

Define ϕ_{ij}, ψ_{ij} as follows:

$$x_j - x_i = \left(-\frac{a}{b} z \right)^{1/k} \underbrace{\left(e^{\frac{2\pi i}{k} j} - e^{\frac{2\pi i}{k} i} \right)}_{\phi_{ij}}$$

$$x_j + x_i = \left(-\frac{a}{b} z \right)^{1/k} \underbrace{\left(e^{\frac{2\pi i}{k} j} + e^{\frac{2\pi i}{k} i} \right) \cdot \operatorname{sgn}(j-i)}_{\psi_{ij}}$$

Then, from the S-wall equation

$$e^{i\vartheta}\,\mathbb{R}^{-} \ni \langle \partial_{t}, \lambda_j - \lambda_i \rangle = \frac{dZ}{dt}\,(x_j - x_i)$$

We have:

$\underline{\mathfrak{g} = A_r}$

$$\delta Z = e^{i\vartheta \frac{k}{1+k}} \cdot \left(-\frac{a}{b}\right)^{-\frac{1}{1+k}} \left(-\phi_{ij}\right)^{-\frac{k}{1+k}} e^{2\pi i \frac{k}{1+k}\cdot s}$$

$$s = 0, \dots, k$$

ie. $k+1$ walls for each root-type.

For example, if $k=2$, there is $k(k-1)$ 2 root types $\pm\alpha$, and $k+1 = 3$ S-walls, as usual.

If $k=3$, there are $k(k-1) = 6$ root types $\pm\alpha, \pm\beta, \pm(\alpha+\beta)$ and $6 \times (3+1) = 24$ S-walls total, but some of them will have the same phase.

Type I $\underline{\mathfrak{g} = D_r,\ x_0 \neq 0}$

Again we have

$$\delta Z = e^{i\vartheta \frac{k}{1+k}} \cdot \left(-\frac{a}{b}\right)^{-\frac{1}{1+k}} \left(-\phi_{ij}\right)^{-\frac{k}{1+k}} e^{2\pi i \frac{k}{1+k}\cdot s}$$

$$s = 0, \dots, k$$

Type II $\underline{\mathcal{Y} = D_r,\ X_0 = 0}$ $\qquad y = x^2, \qquad a z + b y^k = 0$

$$Y_s = \left(-\frac{a}{b} z\right)^{\frac{1}{k}} e^{\frac{2\pi i}{k} s} \qquad s = 0, \cdots, k-1 .$$

$$X_s = \left(-\frac{a}{b} z\right)^{1/2k} e^{\frac{2\pi i}{2k} s} \qquad s = 0, \cdots, k-1 .$$

$$\phi_{ij} = \left(e^{\frac{2\pi i}{2k} j} - e^{\frac{2\pi i}{2k} i}\right), \qquad \psi_{ij} = \left(e^{\frac{2\pi i}{2k} j} + e^{\frac{2\pi i}{2k} i}\right) \cdot \mathrm{sgn}(j-i)$$

the seeds will now be at

$$\delta z = e^{i\vartheta \frac{2k}{2k+1}} \left(-\frac{a}{b}\right)^{-\frac{1}{2k+1}} \cdot \left\{ \begin{array}{l} \left(-\phi_{ij}\right)^{-\frac{2k}{2k+1}} \\ \left(-\psi_{ij}\right)^{-\frac{2k}{2k+1}} \end{array} \right\} e^{2\pi i \frac{2k}{2k+1} s}$$

$$s = 0, \cdots, 2k$$

But note: if $X_0 = 0$, then we have sheets $\pm v_i$ colliding. This is non-generic, and should only happen when $\langle \alpha_i, \varphi \rangle = 0$ for all simple roots, ie when $\varphi = \vec{0} \in \mathfrak{t}$. Therefore, when $X_0 = 0$ we expect that $2k = 2r$.

Type III

$\mathfrak{g} = D_r$, degenerate case : $F \sim x^2\left(az + b\, x^{2k-2}\right)$

This may not be the most general, but it appears in $SO(2r)$ SYM @ $u_i = 0$ $\forall i$.

We focus on that specific case, so $X_0 = 0$ here, and $2k = 2r$.

Passing to $y = x^2$, $y\left(az + b\, y^{r-1}\right) = 0$

so $y = 0$ or $y = \left(-\frac{a}{b} z\right)^{\frac{1}{r-1}} e^{\frac{2\pi i}{r-1} s}$

$s = 0, \cdots, r-2$

Therefore we take

$$X_j = \left\{\left(-\frac{a}{b} z\right)^{\frac{1}{2r-2}}, \cdots, \left(-\frac{a}{b} z\right)^{\frac{1}{2r-2}} e^{\frac{2\pi i}{2r-2} j}, \cdots ; 0\right\}$$

$j = 0, \cdots, r-2$

and get 1/2 of the sheets, r of them.

Labeling sheets as above (hence $X_{r-1} = 0$) we have

$$X_j - X_i = \left(-\frac{a}{b} z\right)^{\frac{1}{2r-2}} \phi_{ij}$$

$$X_j + X_i = \quad " \quad \psi_{ij}$$

where $\phi_{j,r-1} = -e^{\frac{2\pi i}{2r-2} j} = -\phi_{r-1,j}$

and similarly for ψ_{ij}.

Therefore, seeds will now be at:

$$\delta z = e^{i\vartheta \frac{2r-2}{2r-1}} \left(-\frac{a}{b}\right)^{-\frac{1}{2r-1}} \cdot \begin{Bmatrix} (-\phi_{ij})^{-\frac{2r-2}{2r-1}} \\ (-\psi_{ij})^{-\frac{2r-2}{2r-1}} \end{Bmatrix} e^{2\pi i \frac{2r-2}{2r-1} \cdot s}$$

$$s = 0, \ldots, 2r-2$$

For example, from the attached Mathematica file, we see that D_5 has n S-walls:

	Non-degenerate $x_0 \neq 0$	Non-degenerate $x_0 = 0$	Degenerate $x_0 = 0$
$n =$	12	22	18