N [30]. When we have a branch point of ramification index N at $t = 0$, the corresponding curve is $t = x^N$, and the differential equation that governs the behavior of each $\mathcal{S}_{ij}$ on the t-plane is

$$\omega_{ij} t^{1/N} \frac{\partial t}{\partial \tau} = \exp(i\theta), \tag{5.6}$$

where $\omega_{ij} = \omega_i - \omega_j$ and

$$\omega_k = \exp\left(\frac{2\pi i}{N} k\right), \; k = 0, 1, \; \ldots, \; N-1. \tag{5.7}$$

Then the solution for an $\mathcal{S}_{ij}$ is

$$t_{ij}(\tau) = \left(\frac{\tau}{\omega_{ij}}\right)^{N/N+1} \exp\left(\frac{N}{N+1} i\theta\right) \tag{5.8}$$

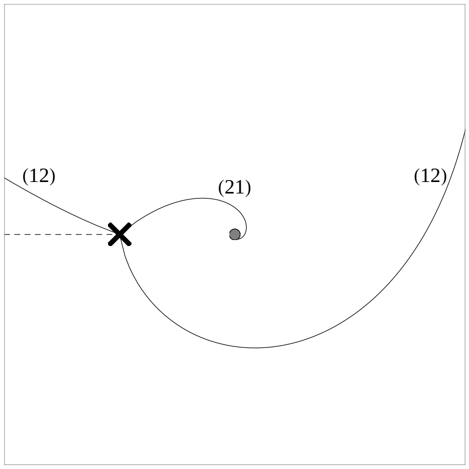
after rescaling τ to absorb a real numerical coefficient. From the factor $1/\omega_{ij}$ we find $N(N-1)$ walls, and the exponent $\frac{N}{N+1}$ makes the angles between the walls to be multiplied by the factor $\frac{N}{N+1}$ from the differences of $\arg(1/\omega_{ij})$'s. As in the $N = 2$ case, the whole spectral network rotates by $\frac{2Nk\pi}{N+1}$ when we change θ from 0 to $2k\pi$. Consistency of a spectral network under this rotation requires $N - 1$ additional walls and we have $N^2 - 1$ $\mathcal{S}$-walls around the branch point. The indices of $\mathcal{S}$-walls are determined by choosing the branch cut. Figures 5.1b, 5.1c shows spectral networks around a branch point of index 3 and 4, respectively.

5.1.3 Around a regular puncture of ramification index N

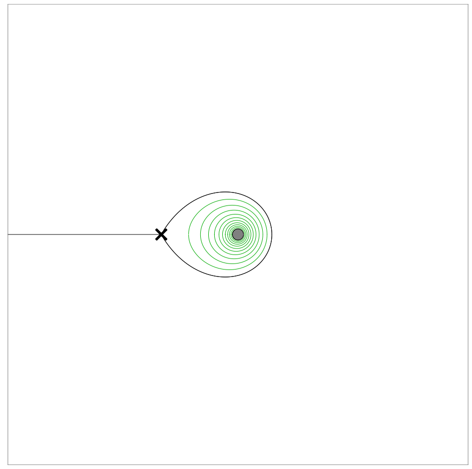
Let us first consider a regular puncture that carries an SU(2) flavor symmetry in the A_1 theory. The residue of the Seiberg-Witten differential at the puncture is the Cartan of the flavor symmetry, in this case a mass parameter m. Consider such a regular puncture at $t = 0$, having $m \neq 0$. Then the corresponding (local) Seiberg-Witten curve is

$$t = (v - m)(v + m) = v^2 - m^2 \tag{5.9}$$

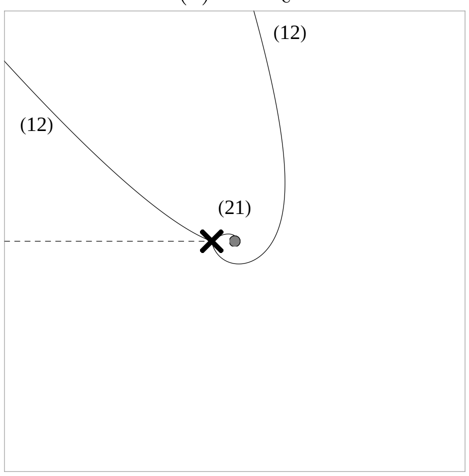
and the Seiberg-Witten differential is $\lambda = \frac{v}{t} dt$. When we project the curve on the t-plane, we have one branch point of index 2 at $t = -m^2$ and one puncture at $t = 0$.



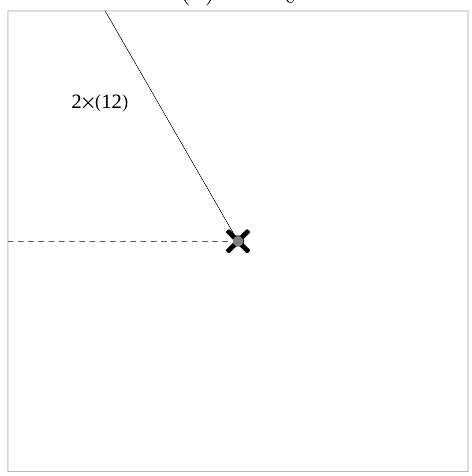
(a) $\theta < \theta_{\mathrm{c}}$



(b) $\theta = \theta_{\mathrm{c}}$



(c) $m \to 0$

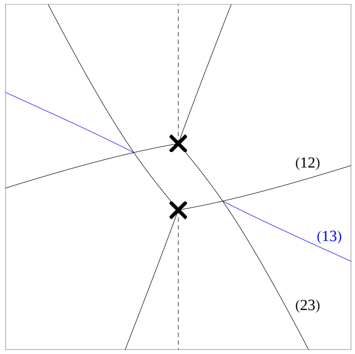


(d) $m = 0$

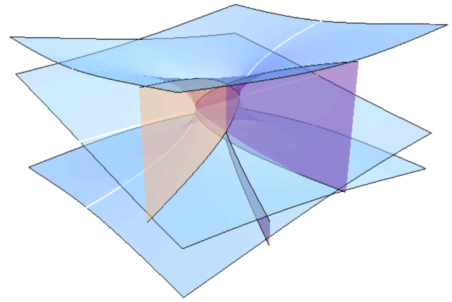
Figure 5.3: $\mathcal{S}$-walls around an SU(2) puncture.

When $m \neq 0$, we can start with a spectral network from a branch point of index 2, as shown in Figure 5.3a. Note that one $\mathcal{S}$-wall flows into the puncture, while the other two escape to infinity [19]. When $\theta = \theta_{\mathrm{c}}$, where $\theta_{\mathrm{c}} = \arg(m_1 - m_2) + \pi/2 = \arg(2m) + \pi/2$, closed $\mathcal{S}$-walls can form around the puncture. This $\mathcal{S}$-wall has a topology of a cylinder, with its boundaries lying along the $\mathcal{S}$-walls on the two sheets. Therefore it corresponds to a BPS state carrying an SU(2) flavor charge. This is consistent with the fact that an $\mathcal{N} = 2$ vector multiplet corresponds to an M2-brane with a topology of a cylinder, and when we gauge the flavor symmetry the $\mathcal{S}$-wall corresponds to a vector multiplet. Now consider the limit of $m \to 0$. Then the branch point moves toward the puncture as shown in Figure 5.3c, and when the two collide, we have a doublet of $\mathcal{S}$-walls emanating from the puncture.

Let us then consider the puncture with an SU(N) flavor symmetry in the A_{N-1} theory.



(a) spectral network



(b) Seiberg-Witten curve and $\mathcal{S}$-walls

Figure 5.4: $\mathcal{S}$-walls forming a joint.

The curve around the puncture is described by

$$t = \prod_{i=1}^{N} (v - m_i), \tag{5.10}$$

where $\sum_i m_i = 0$ and the Seiberg-Witten differential is $\lambda = \frac{v}{t} dt$. Let us focus on the massless limit where $t = 0$ becomes the branch point of index N, in addition to being the puncture. The asymptotic behavior of the $\mathcal{S}$-walls is obtained by solving

$$\int_0^t \omega_{ij} \frac{t'^{1/N}}{t'} dt' = e^{i\theta} \tau, \tag{5.11}$$

where we get $t(\tau) = \left(e^{iN\theta} / \omega_{ij}^N \right) \tau$ after rescaling real parameter τ. There are $N-1$ sets of asymptotic directions for a value of θ due to the factor $1/\omega_{ij}^N$, and along each direction N $\mathcal{S}$-walls of same indices flow from the puncture. In total there are $N(N-1)$ $\mathcal{S}$-walls from the massless puncture.

5.1.4 BPS Joint of $\mathcal{S}$-Walls

When we consider the spectral networks in (the compactification of) the A_{N-1} theory, $N > 2$, then there are more than two types of $\mathcal{S}$-walls. When there is a set of n $\mathcal{S}$-walls $\mathcal{S}_{i_1 i_2}, \mathcal{S}_{i_2 i_3}, \ldots, \mathcal{S}_{i_n i_1}$, there can be a joint of the $\mathcal{S}$-walls. This is because $\lambda_{i_1 i_2} + \lambda_{i_2 i_3} + \cdots + \lambda_{i_n i_1} = 0$ is satisfied at the joint such that it preserves supersymmetry.

Figure 5.4a shows the spectral network of the A_2 theory with two branch points of index 2, where we have $\mathcal{S}_{13}$ coming from the joint of $\mathcal{S}_{12}$ and $\mathcal{S}_{23}$. Figure 5.4b illustrates