Some notes on the fibrations

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Some notes on the conventions used for fibration data within our code.

Contents

1 $SU(2)$ $N_f = 1$

From section 2 of 9706145 (Bilal-Ferrari)

$$y^2 = x^2(x-u) + \frac{m\Lambda^3}{4}x - \frac{\Lambda^6}{64} \tag{1}$$

shifting $z = x + u/3$ brings this into Weierstrass normal form

$$\begin{aligned} y^2 &= z^3 + f(u)z + g(u) \\ f(u) &= \frac{\Lambda^3 m}{4} - \frac{u^2}{3} \\ g(u) &= -\frac{\Lambda^6}{64} + \frac{\Lambda^3 m u}{12} - \frac{2u^3}{27} \end{aligned} \tag{2}$$

which is the form appearing in the code.

2 $SU(2)$ $N_f = 2$

From section 2 of 9706145 (Bilal-Ferrari)

$$y^2 = x^2(x-u) - \frac{\Lambda^4}{64}(x-u) + \frac{\Lambda^2}{4}m_1 m_2 x - \frac{\Lambda^4}{64}(m_1^2 + m_2^2) \tag{3}$$

shifting $z = x + u/3$ brings this into Weierstrass normal form

$$\begin{aligned}
y^2 &= z^3 + f(u)z + g(u) \\
f(u) &= -\frac{\Lambda^4}{64} + \frac{1}{4}\Lambda^2 m_1 m_2 - \frac{u^2}{3} \\
g(u) &= \frac{1}{12}\Lambda^2 m_1 m_2 u - \frac{2u^3}{27} + \frac{\Lambda^4 u}{96} - \frac{1}{64}\Lambda^4\left(m_1^2 + m_2^2\right)
\end{aligned} \tag{4}$$

3 $SU(2)$ $N_f = 3$

From section 2 of 9706145 (Bilal-Ferrari)

$$\begin{aligned}
y^2 = x^2(x-u) - \frac{\Lambda^2}{64}(x-u)^2 - \frac{\Lambda^2}{64}(x-u)(m_1^2 + m_2^2 + m_3^2) \\
+ \frac{\Lambda}{4} m_1 m_2 m_3 x - \frac{\Lambda^2}{64}(m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2)
\end{aligned} \tag{5}$$

shifting $z = x + u/3$ brings this into Weierstrass normal form

$$\begin{aligned}
y^2 &= z^3 + f(u)z + g(u) \\
f(u) &= -\frac{\Lambda^4}{12288} - \frac{1}{64}\Lambda^2\left(m_1^2 + m_2^2 + m_3^2\right) + \frac{1}{4}\Lambda m_1 m_2 m_3 - \frac{u^2}{3} + \frac{\Lambda^2 u}{48} \\
g(u) &= -\frac{1}{27}\left(2u^3\right) - \frac{5\Lambda^2 u^2}{576} + \frac{\Lambda^4 u}{9216} - \frac{\Lambda^6}{3538944} + \frac{1}{12}\Lambda m_1 m_2 m_3 u + \frac{1}{768}\Lambda^3 m_1 m_2 m_3 \\
&\quad + \frac{1}{96}\Lambda^2\left(m_1^2 + m_2^2 + m_3^2\right)u - \frac{\Lambda^4\left(m_1^2 + m_2^2 + m_3^2\right)}{12288} - \frac{1}{64}\Lambda^2\left(m_2^2 m_1^2 + m_3^2 m_1^2 + m_2^2 m_3^2\right)
\end{aligned} \tag{6}$$

4 $SU(2)$ $\mathcal{N} = 2^*$

The theory has two UV parameters: τ, m. The elliptic curve is given in equation (16.24) of SW-2

$$y^2 = \prod_{i=1}^{3}\left(x - e_i\widetilde{u} - \frac{1}{4}e_i^2 m^2\right) \tag{7}$$

where a m, τ-dependent translation of the u-plane has been employed

$$\widetilde{u} = u - \frac{1}{8}e_1 m^2\,. \tag{8}$$

To write this in Weierstrass form, we make a translation on the x-plane by taking

$$x \quad \to \quad x + \frac{1}{12}\left(m^2\left(e_1^2 + e_2^2 + e_3^2\right) + 4\widetilde{u}(e_1 + e_2 + e_3)\right) \equiv x + \frac{1}{12}m^2\left(e_1^2 + e_2^2 + e_3^2\right) \tag{9}$$

and we employ the Weierstrass invariants

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4} \qquad e_1 e_2 e_3 = \frac{g_3}{4} \tag{10}$$

overall we get the simple formula

$$\begin{aligned} y^2 = x^3 &+ \frac{1}{1728}\big(- \frac{9}{4} g_2^2 m^4 - 324 g_3 m^2 \widetilde{u} - 432 g_2 \widetilde{u}^2 \big) x \\ &+ \frac{1}{1728} \left(\frac{1}{32} g_2^3 m^6 - \frac{27}{16} g_3^2 m^6 - \frac{27}{4} g_2 g_3 m^4 \widetilde{u} - 18 g_2^2 m^2 \widetilde{u}^2 - 432 g_3 \widetilde{u}^3 \right) \end{aligned} \tag{11}$$

It would be nice to go ahead and express this in terms of τ directly, using the expressions of g_2, g_3 in terms of Eisenstein series, but on the other hand, any choice of g_2, g_3 does correspond to a τ, so we will always be in the allowed parameter space, although we need to do some work to obtain the value of τ from the invariants.