Prepared for submission to JHEP

K-walls and K3

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Abstract: September 2, 2014

Contents

1 Wild wall crossing and convergence of tree sums

In [8], on page 4 some assumptions are formulated for establishing a convergence criterion of the sum over trees involved in solving the TBA equations.

Assumption 3: There is a set $\mathcal{S}$ so that if $\Omega(\gamma) \neq 0$ then $\gamma \in \mathcal{S}$, and then $\Omega(d\gamma) = 0$ for all integers d with $|d| > 1$.
This assumption is violated by WWC, for example by the sequence of charges of slope 1, but this is not a crucial assumption.

Assumption 4: We assume that there exist positive constants κ_0, R_0 so that, for $R \geq R_0$ and for all $k = 1, 2, 3..$

$$\sum_{\gamma \in \mathcal{S}} ||\gamma||^{k+1} \Omega(\gamma) e^{-2\pi R|Z_\gamma|} \leq e^{-\kappa_0 R} \tag{1.1}$$

Let us study this condition, under the light of WWC. Consider a sequence $\gamma_n = n\gamma_p$ with γ_p primitive, such that $\gamma_n \in \mathcal{S}$, $\forall n$ and in particular $\Omega(\gamma_n) \sim e^{\beta_{\gamma_p} n}$ as $n \to \infty$. Then, we consider the terms in (1.1) coming associated with this sequence

$$\sum_{\gamma \in \mathcal{S}} ||\gamma||^{k+1} \Omega(\gamma) e^{-2\pi R|Z_\gamma|} \sim ||\gamma_p||^{k+1} \sum_n n^{k+1} e^{n(\beta_{\gamma_p} - 2\pi R|Z_{\gamma_p}|)} \tag{1.2}$$

convergence of this series (assuming $R > 0$ of course) requires

$$\beta_{\gamma_p} - 2\pi R |Z_{\gamma_p}| \geq 0 \tag{1.3}$$

now: β_{γ_p} is a constant (like Weist's coefficient, for charges "of slope 1"), but $Z_{\gamma_p}(u)$ will vary on $\mathcal{B}$. There are two pressing questions about these parameters

1. How large can β_{γ_p} be?
2. How small can $|Z_{\gamma_p}(u)|$ be?

The answer to 1 comes from [7], where it is shown that β_p is unbounded, for example one can choose to consider sequences of slope a/b, for various m-cohorts, then there will be such cohorts for arbitrarily high m, and the coefficient is given by Weist's formula, growing as $\sqrt{ab}\log m^2$. Anyways, we can think of m as a piecewise constant function of u, and then ask whether Z_γ grows faster, but that would require Z_γ to have a pole somewhere.

The answer to 2 is not obvious, on general grounds. However one can ask whether there exists a $u \in \mathcal{B}$ such that $Z_{\gamma_p}(u) = 0$ *and* $\Omega(n\gamma_p, u) \neq 0$. We believe not (and maybe this is proved by someone), simply because this would entail the presence of extra singularities on the Coulomb branch. A simple example of how such a situation is avoided comes from pure $SU(2)$: there the W-boson could become massless on the MS wall of the monopole $(1,0)$ and the dyon $(-1,2)$, but this wall is split into two parts. The W-boson is created on the upper part of the wall, and there $Z_{(1,0)} = -Z_{(-1,2)}$ never happens, on the other hand this could happen on the lower part, however once we parallel transport the $(0,2)$ section of γ there, we have monodromy, hence changing the central charge of the W-boson.

1.1 Is WWC relevant to K3?

Actually it seems that it shouldn't be. The $\mathcal{B}$ we consider is 1-complex-dimensional, resembling the Coulomb branch of an A_1 theory, and in such theories there is no WWC [proved by who?]. I'm not sure if thinking in terms of field theory here is deceiving.

2 Elliptic fibrations of K3

2.1 The Fermat quartic

I'm going to slightly elaborate on the example proposed here. Consider the Fermat quartic (after a couple of "Wick rotations" in y and z)

$$S: \quad x^4 - y^4 - z^4 + t^4 = 0 \quad \subset \mathbb{P}^3 \tag{2.1}$$

we can produce elliptic fibrations over $\mathbb{P}^1$, by considering S to be the intersection of the following 3-folds inside $\mathbb{P}^3 \times \mathbb{P}^1$ (with $[\lambda : \mu] \sim [u : 1]$ parametrizing $\mathbb{P}^1$)

$$\begin{aligned} \lambda(x^2 - y^2) &= \mu(z^2 + t^2) \\ \mu(x^2 + y^2) &= \lambda(z^2 - t^2) \end{aligned} \tag{2.2}$$

eliminating λ and μ returns (2.1) as a simple check.
To exhibit the singularities, notice that when $u = \lambda/\mu = 0, \infty, \pm 1, \pm i$ the equations (2.2) describe four complex lines in $\mathbb{P}^3$ hence we do get 24 singular loci.
Are they A_1 singularities? For example, at $u = \infty$ we have the four lines

$$x = \sigma y \qquad z = \sigma' t \qquad \sigma, \sigma' = \pm \tag{2.3}$$

there is an obvious $\mathbb{Z}_2$ symmetry that leaves each of these lines invariant, however I'm not sure if this example is suitable because these loci can intersect. For example at

$$x = y = 0, \qquad z = t \tag{2.4}$$

we have an intersection of two loci at a point, and the other two similarly intersect at $x = y = 0,\ z = -t$.

Now, to exhibit the elliptic fibers (apparently, there are several possible choices?) we eliminate z from (2.2) and obtain

$$y^2 = \frac{x^2(u^2-1) - 2ut^2}{1+u^2} \tag{2.5}$$

for every fixed $u \in \mathbb{P}^1$ this equation represents a complex-dimension 1 variety inside $\mathbb{P}^2$. If we switch to non-homogeneous coordinates $[x : y : t] \sim [\tilde{x} : \tilde{y} : 1]$ and drop the tildas, we get simply

$$y^2 = \frac{x^2(u^2-1) - 2u^2}{1+u^2} \tag{2.6}$$

but this is a degenerate elliptic curve. Its discriminant is proportional to

$$\frac{8u^2}{-1+u^2} \tag{2.7}$$

therefore, if we look at this fibration on the x plane, we don't exhibit the full set of singularities. A change of coordinates

$$x \to \frac{ax+b}{cx+d} \qquad y \to \frac{ey}{(cx+d)^2} \tag{2.8}$$

would yield the curve

$$y^2 = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \tag{2.9}$$

where a_i are functions of u and $a, \ldots, e$. The discriminant is identically zero (as it should).

[Is there a good parametrization for this fibration, that is nonsingular?? Greg proposed a deformation of the quartic]

2.2 The pedestrian approach

[INCOMPLETE SECTION] As suggested in [8], we want to write down a Weierstrass model for an elliptic fibration over $\mathbb{C}$ such that it exhibits 24 A_1 singularities.

We consider the family of curves

$$\begin{aligned} y^2 &= (x-e_1)(x-e_2)(x-e_3) \\ &= x^3 + g_2 x + g_3 \end{aligned} \tag{2.10}$$

with discriminant

$$\Delta = -(4g_2^3 + 27g_3^2). \tag{2.11}$$

Here

$$g_2 = -(e_1e_2 + e_2e_3 + e_3e_1) \qquad g_3 = e_1e_2e_3 \tag{2.12}$$

and we want them to be of respective degrees 8 and 12 in u. Recall the constraint $e_1+e_2+e_3 = 0$, then let us choose

$$\begin{aligned} e_1 &= \prod_{i=1}^{4}(u-a_i) \\ e_2 &= \prod_{i=1}^{4}(u-b_i) \end{aligned} \tag{2.13}$$

we would now like to study which values of the parameters a_i, b_i produce a discriminant with the correct singularity structure. In particular we want 24 A_1 singularities, there can be characterized as points on $\mathbb{C}$ where $g_2, g_3 \neq 0$ and Δ has a first order zero.

2.3 Explicit example

Consider the Weierstrass parametrization of a family of elliptic curves, with

$$g_2(u) = \left(u^2-1\right)\left(u^2-2\right)\left(u^2+3\right)\left(u^2+4\right) \tag{2.14}$$

$$g_3(u) = \left(u^4+1\right)\left(u^4+2\right)\left(u^4+3\right) \tag{2.15}$$

the discriminant $\Delta = {g_2}^3 - 27{g_3}^2$ has 24 zeros at which $g_{2,3} \neq 0$ and $d\Delta/du \neq 0$.

3 Practical matters about K-walls

A K-wall is the locus

$$\mathcal{KW}(\gamma) = \{(u,\zeta) \in \widetilde{\mathcal{B} \times \mathbb{C}^\times} | Z_\gamma(u)/\zeta < 0 \ \&\ \Omega(\gamma, u) \neq 0\} \tag{3.1}$$

where $\mathcal{B}$ is the "Coulomb branch" (more generally a special kähler manifold[1]), $\widetilde{\ }$ denotes the universal covering, γ is a section of Γ the charge lattice fibration with fiber $H_1(E_u)$ over $u \in \mathcal{B}$, and $Z_\gamma(u)$ is the period of a meromorphic differential $\lambda_{SW} \in \Omega^1(E_u)$.

[1] rigid or local?

3.1 Period matrix of an elliptic curve

We will use two methods for constructing K-walls, the reasons are explained along the way.

For a family of elliptic curves E_u described by a Weierstrass model

$$y^2 = 4x^3 - g_2(u)x - g_3(u) \tag{3.2}$$

$H^1(E_u, \mathbb{C})$ is two dimensional and a basis is provided by

$$\lambda_1 = \frac{dx}{y} \qquad \lambda_2 = \frac{x\,dx}{y} \tag{3.3}$$

the Seiberg-Witten differential is then a linear combination

$$\lambda_{SW} = a_1(u)\lambda_1 + a_2(u)\lambda_2. \tag{3.4}$$

It is generally easy to find such linear decomposition, therefore it will be convenient to compute the periods of $\lambda_{1,2}$, since these will easily entail the perdiods of λ_{SW}. We do this in appendix A.

The SW differential is related to the holomorphic differential λ_1 by

$$\eta_1 = \oint_\alpha \frac{dx}{y} = \frac{\partial a}{\partial u} \qquad \eta_{1,D} = \oint_\beta \frac{dx}{y} = \frac{\partial a_D}{\partial u} \tag{3.5}$$

where α, β form a Darboux basis, and

$$a = \oint_\alpha \lambda_{SW} \qquad a_D = \oint_\beta \lambda_{SW} \qquad \frac{dx}{y} \doteq \frac{d}{du}\lambda_{SW} \tag{3.6}$$

the last equality holding up to exact terms.

Having such explicit expressions for the periods at hand, we can use them to construct K-walls. Fix a $\zeta \in \mathbb{C}^\times$ with phase ϑ, then we denote the projection of a K-wall $\mathcal{KW}(\gamma)$ by $\mathcal{KW}(\gamma, \vartheta)$. We will informally call the latter also a K-wall. This is the locus on $\widetilde{\mathcal{B}}$ where $\arg Z_\gamma(u) = \vartheta + \pi$, and is a real-dimension-one submanifold, let t be an affine coordinate on $\mathcal{KW}(\gamma, \vartheta)$ such that

$$e^{i(\vartheta+\pi)} = \frac{d}{dt} Z(\gamma, u) = \dot{u} \oint_\gamma \lambda_1 = \dot{u}\, \eta_{1,\gamma} \tag{3.7}$$

providing a differential equation for constructing K-walls, this can be obviously iterated, after a suitable rescaling $\hat{u} = u|\eta_1(u)|$ and given some initial conditions

$$\hat{u}(t) = \hat{u}(t_0) + \int_{t_0}^{t} dt' \, \exp i(\vartheta + \pi - \arg \eta_{1,\gamma}(u(t'))) \tag{3.8}$$

The disadvantage of this approach is that, in practice, numerically it is quite difficult to keep track of branch cuts for the meromorphic functions $\eta_1(u)$. We will use the Picard-Fuchs equations to overcome this difficulty, nevertheless this method of constructing K-walls will be useful in a small neighborhood of a singularity, where the PF equations become singular. Notice that near a singular point where some cycle pinches, the holomorphic differential $dx/y \sim dx/x$ ($x = 0$ at the singularity) hence the period integral doesn't vanish, picking up a residue, therefore $\arg \eta_1(u_{sing})$ is actually well defined.

We now come to the second method for constructing K-walls, suggested in [8], using Doran's theorem for an elliptic curve in Weierstrass parametrization (3.2) choose a section $\gamma \in \Gamma$ and consider the periods

$$\mu_i(\gamma, u) = \oint_\gamma \lambda_i(u) \qquad\qquad i = 1, 2 \tag{3.9}$$

they satisfy the PF equations

$$\frac{d}{du}\mu_i = M_{ij}\mu_j \qquad M_{ij} = \begin{pmatrix} -\frac{\Delta'}{12\Delta} & \frac{3\delta}{2\Delta} \\ -\frac{g_2\delta}{8\Delta} & \frac{\Delta'}{12\Delta} \end{pmatrix} \tag{3.10}$$

where

$$\Delta = g_2^3 - 27g_3^2 \qquad \delta = 3g_3g_2' - 2g_2g_3'. \tag{3.11}$$

Again, if we consider the K-wall $\mathcal{K}(\gamma, \vartheta)$ with a suitable affine parameter t, we can consider the following system of equations for explicitly constructing the K-wall

$$\begin{cases} \frac{d}{dt}\mu_i(u(t)) = \dot{u}\, M_{ij}(u(t))\mu_j(u(t)) \\ \dot{u} = \exp i(\vartheta + \pi - \arg\mu_1(u(t))) \end{cases} \tag{3.12}$$

Or equivalently, we can use just one differential (see Appendix B).

Remark Note that from Picard-Fuchs we can obtain a second-order differential equation for η_1

$$\eta_1'' - \left(\frac{\delta'}{\delta} - \frac{\Delta'}{\Delta}\right)\eta_1' - \left[-\frac{3g_2\delta^2}{16\Delta^2} + \frac{\delta'\Delta'}{12\delta\Delta} - \frac{\Delta''}{12\Delta} + \frac{\Delta'^2}{144\Delta^2}\right]\eta_1 = 0 \tag{3.13}$$

There is also a 2nd order PF equation that can be written directly for a, a_D, it can be derived by studying the commutator of the differential operator in (3.13) with $\partial/\partial u$. In some simple cases these 2nd order differential problems can be matched with Riemann's differential equation, thus providing explicit closed form expressions for the periods. We won't need this machinery for elliptic fibrations, but it would probably be useful with more elaborate fibers.

3.2 Intersections of K-walls

Each K-wall is associated to a section of Γ and "carries" the corresponding BPS index. Consider two sections γ_1, γ_2 and suppose that $\mathcal{KW}(\gamma_1) \cap \mathcal{KW}(\gamma_2) \neq \emptyset$. The projection of this locus down on $\mathcal{B}$ is $MS(\gamma_1, \gamma_2$. For convenience, let us work at fixed ϑ. The intersection is then a discrete set of points, for each of these points we need to compute the pairing $\langle \gamma_1, \gamma_2 \rangle$ in order to apply the KSWCF and determine which new K-walls are born.

Ideally, one could just parallel transport sections to some common point $u_0 \in \mathcal{B}$, and compute the pairing there, as the monodromy does not affect the DSZ pairing. Numerically it is instead hard to compute the parallel transport. So what we do instead is the following. Fix a point u_0 near the intersection of the two K-walls (recall we are working at fixed phase here), there choose a Darboux basis α, β for the elliptic curve and compute the periods η_1, $\eta_{1,D}$ as defined in the appendix. Then use the PF equations to parallel transport γ_1, γ_2 to u_0 and compute the periods $\mu_1(\gamma_j)$, $j = 1, 2$ these admit an integer expansion

$$\mu_1(\gamma_j, u_0) = p_j \eta_1(u_0) + q_j \eta_{1,D}(u_0) \qquad p_j, q_j \in \mathbb{Z} \tag{3.14}$$

then

$$\langle \gamma_1, \gamma_2 \rangle = q_1 p_2 - q_2 p_1 \tag{3.15}$$

3.3 Example

We discuss an application of the techniques discussed above. This is not relevant to the code. Let us compute the SW periods for the example proposed in section 8 of [8]

$$\begin{aligned} w^2 &= x^3 - (3+u)x + 2(1-u) \\ y &= 2w \\ y^2 &= 4x^3 - (12+4u)x + 8(1-u) \\ e_1 &= -2, \quad e_2 = 1 - \sqrt{u}, \quad e_3 = 1 + \sqrt{u} \\ \lambda_{SW} &= -\frac{\sqrt{x^2 - 2x + 1 - u}}{\sqrt{x+2}} \end{aligned} \tag{3.16}$$

now, notice that

$$\left[(6x^2 - 6 + 2u) \frac{dx}{y} \right] = [dy(x)] = 0 \tag{3.17}$$

is an exact 1-form. Therefore, rewriting

$$\lambda_{SW} = -2(x^2 - 2x + (1-u)) \tfrac{dx}{y} \tag{3.18}$$

we can substitute $6x^2 - 6 + 2u \simeq 0$, which gives

$$\lambda_{SW} = 4 \left(\frac{u}{3} - 1 \right) \lambda_1 + 4\lambda_2 \tag{3.19}$$

Hence we have the following periods

$$a = \oint_\alpha \lambda_{SW} = 4 \left(\frac{u}{3} - 1 \right) \eta_1 + 4\eta_2 \tag{3.20}$$

$$a_D = \oint_\beta \lambda_{SW} = 4 \left(\frac{u}{3} - 1 \right) \eta_{1,D} + 4\eta_{2,D} \tag{3.21}$$

where the η_i, $\eta_{i,D}$, which are functions of the $e_i(u)$ specified above in (3.16), are given explicitly in appendix A.

- This can be made more general for elliptic curves, of course.

A Some useful formulae

Define an elliptic integrals of first and second kind:

$$\frac{1}{2}\int_0^x \frac{dz}{\sqrt{z(z-x)(z-1)}} = K(x) \tag{A.1}$$

$$\frac{1}{2}\int_0^x dz\sqrt{\frac{z-1}{z(z-x)}} = E(x) \tag{A.2}$$

The relation between the "cross-ratio" x of zeroes of the polynomial and the modular parameter τ of the torus can be summarized in the following formulas [2]

$$\tau = i\frac{K(1-x)}{K(x)}, \quad x = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}, \quad K(x) = \frac{\pi}{2}\theta_3^2(\tau) \tag{A.5}$$

Derivatives of the elliptic integrals can be reexpressed through themselves

$$K'(x) = \frac{1}{2}\left(\frac{E(x)}{x(1-x)} - \frac{K(x)}{x}\right), \quad K'(x) = \frac{E(x)-K(x)}{2x} \tag{A.6}$$

Defined this way the elliptic K-function has a cut from 1 to ∞ and logarithmically diverges near 1

$$K(x) = -\frac{1}{2}\log(1-x) + \dots \tag{A.7}$$

The elliptic curve can be rewritten as

$$y^2 = 4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3) \tag{A.8}$$

$$e_1 = -\frac{\pi^2}{3}(1-2x)\theta_3^4(\tau), \quad e_2 = -\frac{\pi^2}{3}(1+x)\theta_3^4(\tau), \quad e_3 = \frac{\pi^2}{3}(2-x)\theta_3^4(\tau) \tag{A.9}$$

[2] Where we used the Jacobi theta constants:

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}, \quad \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \theta_2(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \quad q = e^{\pi i\tau} \tag{A.3}$$

They satisfy the relation

$$\theta_2^4(\tau) + \theta_4^4(\tau) = \theta_3^4(\tau) \tag{A.4}$$

Suppose near a singularity e_1 and e_2 are close and merge in the singularity at $x = 0$, also suppose the A-cycle is pinched in the singularity, thus

$$\eta_1 = \oint_\alpha \frac{dx}{y} = \frac{2}{(e_2 - e_1)^{\frac{1}{2}}} K\left(\frac{e_2 - e_1}{e_3 - e_1}\right) \tag{A.10}$$

$$\eta_{1,D} = \oint_\beta \frac{dx}{y} = \frac{2i}{(e_2 - e_1)^{\frac{1}{2}}} K\left(\frac{e_3 - e_2}{e_3 - e_1}\right) \tag{A.11}$$

$$\eta_2 = \oint_\alpha \frac{xdx}{y} = \frac{2e_3}{(e_2 - e_1)^{\frac{1}{2}}} K\left(\frac{e_2 - e_1}{e_3 - e_1}\right) - 2(e_2 - e_1)^{\frac{1}{2}} E\left(\frac{e_2 - e_1}{e_3 - e_1}\right) \tag{A.12}$$

$$\eta_{2,D} = \oint_\beta \frac{xdx}{y} = \frac{2ie_3}{(e_2 - e_1)^{\frac{1}{2}}} K\left(\frac{e_3 - e_2}{e_3 - e_1}\right) - 2i(e_2 - e_1)^{\frac{1}{2}} E\left(\frac{e_3 - e_2}{e_3 - e_1}\right) \tag{A.13}$$

Similarly we can derive the periods of a holomorphic differential for a hyperelliptic curve of genus 1

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)(x - e_4) \tag{A.14}$$

$$\eta_1 = \oint_\alpha \frac{dx}{y} = \frac{2}{(-e_{13}e_{24})^{\frac{1}{2}}} K\left(\frac{e_{12}e_{34}}{e_{13}e_{24}}\right) \tag{A.15}$$

$$\eta_{1,D} = \oint_\beta \frac{dx}{y} = \frac{2i}{(-e_{13}e_{24})^{\frac{1}{2}}} K\left(\frac{e_{23}e_{14}}{e_{13}e_{24}}\right) \tag{A.16}$$

B On the Picard-Fuchs equations derivation

Picard-Fuchs equations can be simply derived in two steps. Consider some fibration of hyperelliptic curves

$$y^2 = P_u(x) \tag{B.1}$$

where $P_u(x)$ is a polynomial of degree $2g + 1$ or $2g + 2$, where g is a genus of the curve, and define two kinds of integrals over some fixed closed contour γ:

$$\eta_n = \oint_\gamma \frac{x^{n-1}dx}{y}, \quad \chi_n = \oint_\gamma \frac{x^{n-1}dx}{y^3} \tag{B.2}$$

Now we make two observations. First, the derivative

$$\partial_u \eta_n = -\frac{1}{2} \oint_\gamma \frac{x^{n-1}\partial_u P_u(x)dx}{y^3} \tag{B.3}$$

can be expressed in terms of χ-integrals. Second, we can write a set of identities

$$0 = \oint_\gamma d\frac{x^{n-1}}{y} = (n-1)\eta_{n-1} - \frac{1}{2} \oint_\gamma \frac{x^{n-1}\partial_x P_u(x)dx}{y^3} \tag{B.4}$$

to reexpress χ-integrals back in terms of η-integrals. Consider an example of an elliptic curve in the Weierstrass form

$$y^2 = 4x^3 - g_2(u)x - g_3(u) \tag{B.5}$$

Define two auxiliary functions:

$$\Delta = g_2^3 - 27g_3^2, \quad \delta = 3g_3g_2' - 2g_2g_3' \tag{B.6}$$

The first step is obvious:

$$\partial_u\eta_1 = \frac{g_2'}{2}\chi_2 + \frac{g_3'}{2}\chi_1, \quad \partial_u\eta_2 = \frac{g_2'}{2}\chi_3 + \frac{g_3'}{2}\chi_2 \tag{B.7}$$

The second step gives a bunch of linear equations that can be easily solved:

$$\begin{cases} 12\chi_3 - g_2\chi_1 = 0 \\ 2g_2\chi_2 + 3g_3\chi_1 + \eta_1 = 0 \\ 2g_2\chi_3 + 3g_3\chi_2 - \eta_2 = 0 \end{cases} \Rightarrow \begin{array}{l} \chi_1 = \frac{3(3g_3\eta_1+2g_2\eta_2)}{\Delta} \\ \chi_2 = -\frac{\eta_1 g_2^2+18g_3\eta_2}{2\Delta} \\ \chi_3 = \frac{g_2(3g_3\eta_1+2g_2\eta_2)}{4\Delta} \end{array} \tag{B.8}$$

Substituting these equations we derive

$$\partial_u \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{\Delta'}{12\Delta} & -\frac{3\delta}{2\Delta} \\ \frac{g_2\delta}{8\Delta} & \frac{\Delta'}{12\Delta} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{B.9}$$

Not all the differentials are holomorphic. For a curve of genus g there are g independent holomorphic differentials. In other words for our Weierstrass curve only η_1 is uniquely defined by the homology class of the contour γ. We could want to derive an equation only for a holomorphic differential. To do this we express the integral η_2 from the first equation and substitute it in the second one, the result reads:

$$\eta_1'' - \left(\frac{\delta'}{\delta} - \frac{\Delta'}{\Delta}\right)\eta_1' - \left[-\frac{3g_2\delta^2}{16\Delta^2} + \frac{\delta'\Delta'}{12\delta\Delta} - \frac{\Delta''}{12\Delta} + \frac{\Delta'^2}{144\Delta^2}\right]\eta_1 = 0 \tag{B.10}$$

We can reformulate these equations in a matrix form we used in the numerical evaluation

$$\partial_u \begin{pmatrix} \eta_1 \\ \eta_1' \end{pmatrix} = \hat{F} \begin{pmatrix} \eta_1 \\ \eta_1' \end{pmatrix}, \quad \hat{F} = \begin{pmatrix} 0 & 1 \\ -\frac{3g_2\delta^2}{16\Delta^2} + \frac{\delta'\Delta'}{12\delta\Delta} - \frac{\Delta''}{12\Delta} + \frac{\Delta'^2}{144\Delta^2} & \frac{\delta'}{\delta} - \frac{\Delta'}{\Delta} \end{pmatrix} \tag{B.11}$$

References

[1] GMN1

[2] GMN2

[3] GMN3

[4] GMN4

[5] GMN5

[6] GMN6

[7] GLMMN - Wild Wall Crossing and BPS Giants

[8] GMN - Explicit Metrics on Some Elliptically Fibered Surfaces - August 15, 2011 - private communication

[9] GMN - Comments on a Potential Construction of a Hyperkähler Metric on Elliptically fibered surfaces - private communication