(25.11.13) Computational Physics B

class 1

- reconstruction method
    - the core idea is to use concrete values to reconstruct the slopes and avoid unphysical oscillations.
    - piecewise constant
        - $\left(\frac{\partial a}{\partial x}\right)_i = 0$
        - this is 1st order accurate
    - piecewise linear
        - $\left(\frac{\partial a}{\partial x}\right)_i = \frac{a_{i+1}-a_i}{\Delta x_i}$
        - $a_{i+1/2,l} = a_i + \frac{1}{2}\Delta a_i$
        - this method is 2nd order accurate
    - piecewise parabolic
        - $\left(\frac{\partial a}{\partial x}\right)_i = \frac{3}{2}\frac{a_{i+1}-a_{i-1}}{2\Delta x_i} - \frac{1}{4}\left(\frac{\Delta a_{i+1}}{\Delta x_{i+1}} + \frac{\Delta a_{i-1}}{\Delta x_{i-1}}\right)$
        - this method is 3rd order accurate
    - after the interpolation we get a left and right value at each cell interface.
        - to decide which value to use, we need a Riemann solver.
            - $R(a_l, a_r) = \begin{cases} F(a_l) & \text{if } u > 0 \\ F(a_r) & \text{if } u < 0 \end{cases}$
            - this means that the information flows from left to right if $u > 0$, so we use the left state to calculate the flux, and vice versa.
    - slope limiters
        - to avoid unphysical oscillations near discontinuities, we can use slope limiters to limit the slope.
        - minmod limiter
            - $\left(\frac{\partial a}{\partial x}\right)_i = \text{minmod}\left(\frac{a_i-a_{i-1}}{\Delta x}, \frac{a_{i+1}-a_i}{\Delta x}\right)$
            - where

$$\text{minmod}(a,b) = \begin{cases} \text{sgn}(a)\min(|a|,|b|) & \text{if } \text{sgn}(a) = \text{sgn}(b) \\ 0 & \text{otherwise} \end{cases}$$

            - this limiter helps to maintain stability and accuracy in the presence of discontinuities and can get rid of extremia of slopes.

- reference book
  - "Riemann Solvers and Numerical Methods for Fluid Dynamics" , Springer

class 2

- Burgers' equation
  - $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$
  - this is a non-linear advection equation, and the wave speed depends on the amplitude.
    - when u is a sinusoidal function, the characteristics can cross each other, leading to shock formation.
    - when u is a step function, the characteristics diverge, leading to rarefaction waves.
  - rewrite the equation in conservative form:
    - $\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0$
    - then we can use finite-volume method to solve it.
    - $u^{n+1}_{i+1/2,l}=u^n_i+\frac{\Delta x}{2}(1-\frac{\Delta t}{\Delta x}u_i)\left(\frac{\partial u}{\partial x}\right)^n_i$, and the flux is $f(u)=\frac{u^2}{2}$
    - u_i varied from zone to zone, causing great complexity.
  - the speed of shock
    - Rankine-Hugoniot jump condition
      - $S=\frac{f(u_r)-f(u_l)}{u_r-u_l}=\frac{u_r+u_l}{2}$
      - where S is the shock speed, and u_r and u_l are the right and left states of the shock.
    - the Riemann problem solution for Burgers' equation
      - if $u_l>u_r$, then a shock forms, and the solution is:
        - $$u(x,t)=\begin{cases}u_l & \text{if } x<St\\ u_r & \text{if } x>St\end{cases}$$
      - if $u_l\leq u_r$, then a rarefaction wave forms, and the solution is:
        - $$u(x,t)=\begin{cases}u_l & \text{if } x<u_lt\\ \frac{x}{t} & \text{if } u_lt\leq x\leq u_rt\\ u_r & \text{if } x>u_rt\end{cases}$$
      - then we can compute the flux $F^{n+1/2}_{i+1/2}=\frac{1}{2}\left(u^{n+1/2}_{i+1/2}\right)^2$ based on the above solution.
        - conservative update: $u^{n+1}_i=u^n_i-\frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{i+1/2}-F^{n+1/2}_{i-1/2}\right)$
- Back to Euler equation
  - conservative form

- $\vec{u}_t + [\vec{F}(\vec{u})]_x = 0$
- where $\vec{u} = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}$, $\vec{F}(\vec{u}) = \begin{pmatrix} \rho v \\ \rho v^2 + P \\ v(E+P) \end{pmatrix}$
- $E = e + \frac{1}{2}\rho v^2$, total energy density
- closure relation: $P = (\gamma - 1)\rho e$
- Jacobian $\vec{A} = \frac{\partial \vec{F}}{\partial \vec{u}}$
  - then the equation becomes $\vec{u_t} + \vec{A}\vec{u_x} = 0$ -- eigenvalue problem
  - solve for eigenvalue $|\vec{A} - \lambda \vec{I}| = 0$, we get three eigenvalues:
    - $\lambda_1 = v - c_s$
    - $\lambda_2 = v$
    - $\lambda_3 = v + c_s$
    - where $c_s = \sqrt{\frac{\gamma P}{\rho}}$ is the sound speed.
  - these eigenvalues represent the speeds of the three characteristic waves: left-moving sound wave, contact discontinuity, and right-moving sound wave.
  - the three eigenvalues divide the x-t plane into four regions, and in each region the solution is constant along the characteristics.