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M2 Course GTA
«Game Theory and Applications»

Lecture slides and exercises

Academic year 2023-2024

Contents

Course organization

Jour	Heure	Cours
11/09/2023	9h45-11h15	Course introduction, and some background mathematics Bruno Tuffin **I54 anglais**
14/09/2023	16h45-18h15	Game Theory basics (strategy, equilibrium) Bruno Tuffin **I53**
18/09/2023	9h45-11h15	Exercise session Bruno Tuffin **I56**
18/09/2023	16h45-18h15	Exercise session Bruno Tuffin **I53**
25/09/2023	9h45-11h15	Repeated games, Bayesian games, value of information Bruno Tuffin **I53**
28/09/2023	16h45-18h15	Exercises on repeated games and Bayesian games Bruno Tuffin **I53**
2/10/2023	9h45-11h15	Mechanism design and auctions Patrick Maillé **I55 anglais**
2/10/2023	16h45-18h15	Exercise session on mechanism design Patrick Maillé **I53**
5/10/2023	16h45-18h15	Adword auctions, GSP/VCG mechanisms Bruno Tuffin **I53**
9/10/2023	9h45-11h15	Exercise session on GSP/VCG mechanisms Patrick Maillé **I55 anglais**
12/10/2023	16h45-18h15	Routing games (1/2) Patrick Maillé **I57**
23/10/2023	9h45-11h15	Routing games (2/2) Patrick Maillé **I58**
26/10/2023	16h45-18h15	Practical session on routing games Patrick Maillé **I57**
6/11/2023	16h45-18h15	Course Exam Patrick Maillé **I53**
9/11/2023	16h45-18h15	Oral presentation Patrick Maillé and Bruno Tuffin **I59**

Game theory and Optimization for Economics: Modern Information and Telecommunication Networks General introduction

Patrick Maillé & Bruno Tuffin

IMT & Inria

GTA, Univ. Rennes 1

Outline

1. Course overview/organization
2. About us
3. Evolution of telecommunications and associated economic models
4. Need for modeling and analysis/Examples of application

Overview of the course

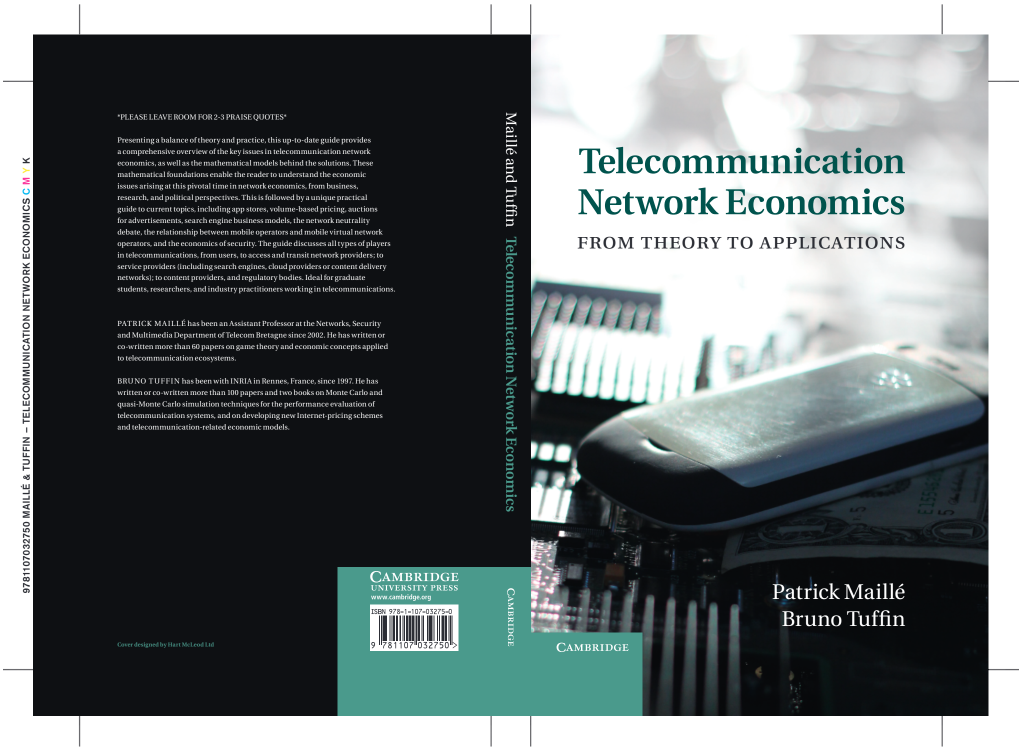
1. 22.5h overall
   - 13 times 1.5 hours
   - Mixing lectures and exercises
   - Exam: 1.5 hours on desk (exercises) + presentation of an article.
2. Introduction to the basics on game theory (and applications to networking)
3. Goals:
   - Model actors' preferences with a *utility function*
   - Represent the interactions with a matrix for a finite game
   - Determine the output of an interaction via the concept of *Nash equilibrium*
   - Analyze the impact of a decision on this output and optimize this decision

Content (subject to adaptation!)

1. Definition of a game from telecommunication problems (jamming game, social networks, collusion between operators)
   - Notion of player, strategy, utility
   - Simple games: prisoner dilemma, battle of sexes, avoidance game
   - Examples in many domains
   - Notion of (pure and mixed) Nash equilibrium
   - Notion of best response
   - Existence results
2. Routing games
   - Notion of Wardrop equilibrium
   - Notion of price of anarchy: what is lost due to selfishness?
3. "Google" auctions (how is Google making money?)
   - Different types of auctions: first price, second price
   - Properties of incentive compatibility, efficiency, Vickrey-Clarke-Groves mechanism
   - Notion of mechanism design
   - Currently in use: Generalized Second Price

Course based on the book

P. Maillé and B. Tuffin. **Telecommunication Network Economics: From Theory to Applications**. Cambridge University Press, 2014.



Outline

1. Course overview/organization
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4. Need for modeling and analysis/Examples of application

About us

- Patrick Maillé
  - Professor, IMT.
  - PhD in Computer Science in 2005.
  - Performance and reliability analysis of communication networks.
  - Application of Game theory.
  - Web page: `http://perso.telecom-bretagne.eu/patrickmaille/`
- Bruno Tuffin
  - Research Director at Inria.
  - PhD in Probability and statistics in 1997.
  - Performance and reliability analysis of communication networks.
  - Monte Carlo and quasi-Monte Carlo simulation methods, with applications.
  - Telecommunication Network economics and Game theory.
  - Web page: `https://people.rennes.inria.fr/Bruno.Tuffin/`

Outline

From centralization to decentralization

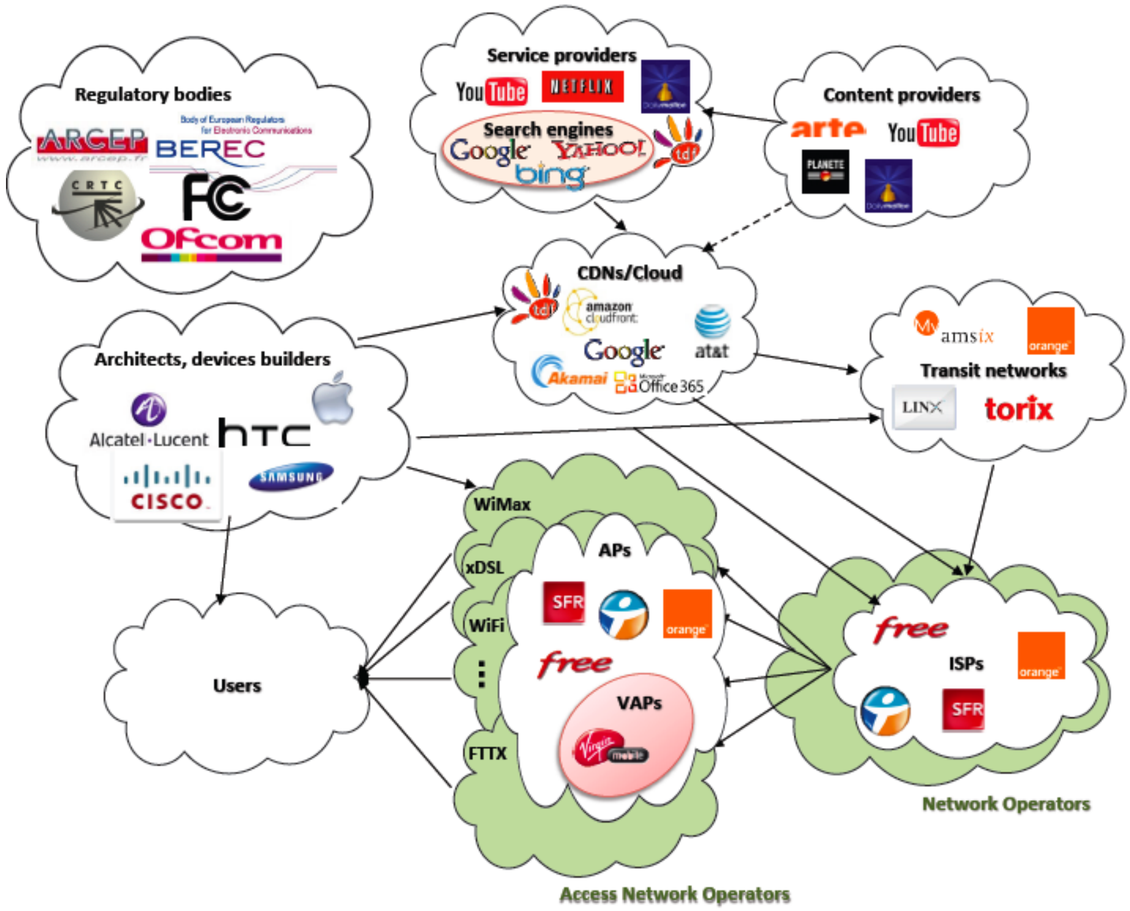
- Networking has switched from the centralized telephone network to the decentralized Internet (scalability reason).
- Decentralization (or deregulation) is a key factor.
- Illustration: "failure" of ATM networks.
- In such a situation:
  - From the decentralization, there is a general envisaged/advised behavior
  - But each *selfish* user can try to modify his behavior at his benefits and at the expense of the network performance.
  - How to analyze this, and how to control and prevent such a thing?
- It is the purpose of non-cooperative game theory.

Competitive actors: not only users

- The Internet has also evolved from an academic to a commercial network with providers in competition for customers and services.
- As a consequence, users are not the only *competitive* actors, but also
  - network providers: several providers propose the same type of network access
  - applications/services providers: the same type of application can be proposed by several entities (ex: search engines...)
  - platforms/technologies: you may access the Internet from ADSL, WiFi, 3G, WiMAX...

All those interacting actors have to be considered.

The actors of the Internet



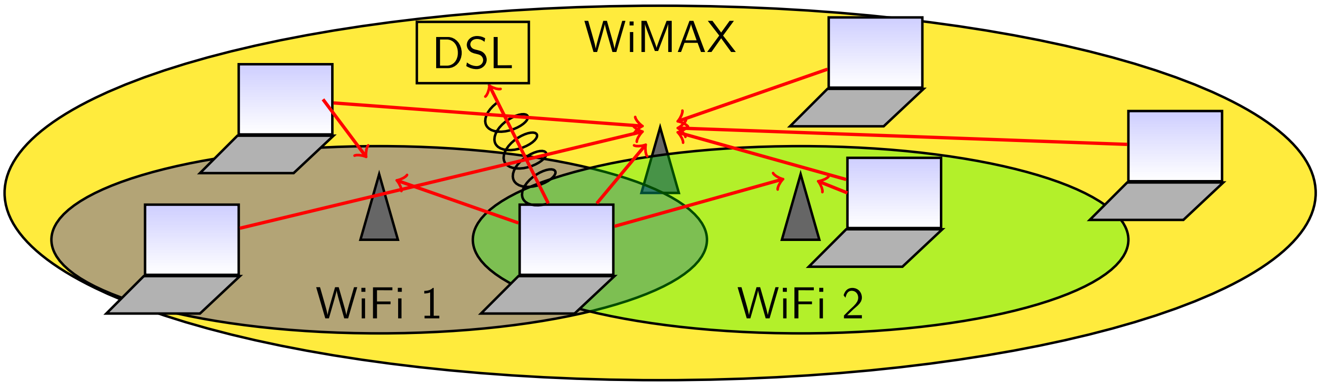
Short description of the set of actors

- End-users
- Access network providers
- Transit providers
- Content providers
- Service providers
- Content delivery networks
- Network architects and device constructors
- Regulators

Some goals in network economics

- Determining the most relevant and profitable access network pricing scheme for end users
- Determining the best investments for network providers
- Defining the economic relations between network operators
- Understanding the relations between content providers and ISPs.
- Defining the economic model of content and (application) service providers

Illustration of an intricate competition model



Outline

The tragedy of the commons

Definition

A *common* is a resource owned by no one, but to which all have access, i.e., somehow a public good.

Principle: Several individuals acting independently, rationally and selfishly can actually deplete a shared limited resource.

- Surprising at first sight since not in the interest of individuals
- The result of non-cooperative game theory.
- Observation in practice: company managing coaches
  - all coaches assigned to a specific driver are in very good shape...
  - ... while those in a shared pool are dirty and deteriorated
- Most standard examples: fishery, global warming, etc.
- or the Internet, wireless spectrum.

The tragedy of the commons: videos

- https://www.youtube.com/watch?v=bs2P0wRod8U
- https://www.khanacademy.org/economics-finance-domain/ap-microeconomics/ap-consumer-producer-surplus/ap-externalities-topic/v/tragedy-of-the-commons
- https://www.youtube.com/watch?v=6wNj9hOIt9g

First very simple model (for the tragedy of the commons)

A pool of N participants share a resource

- Each participant i $(1 \leq i \leq N)$ makes a contribution $a_i \in [0,1]$ to the resource
- Demand from customers is a function of the average effort $D(\sum_i a_i/N)$, assumed to be $\sum_i a_i/N$.
- Gains are shared among participants: $R_i = \frac{1}{N} D(\sum_i a_i/N) = \frac{\sum_i a_i}{N^2}$.

- What is the optimal contribution from participant i?
  His/her *utility* is

$$U_i = R_i - \alpha a_i = K + a_i \left(\frac{1}{N^2} - \alpha \right).$$

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$$U_i = R_i - \alpha a_i = K + a_i \left(\frac{1}{N^2} - \alpha \right).$$

- Optimal not to contribute $(a_i^* = 0)$ if $\frac{1}{N^2} - \alpha < 0$, i.e., if there are enough participants.
- $a_i^* = 1$ if "very few" participants.

Question then: what can be done ? Introduce rules.

The Braess paradox

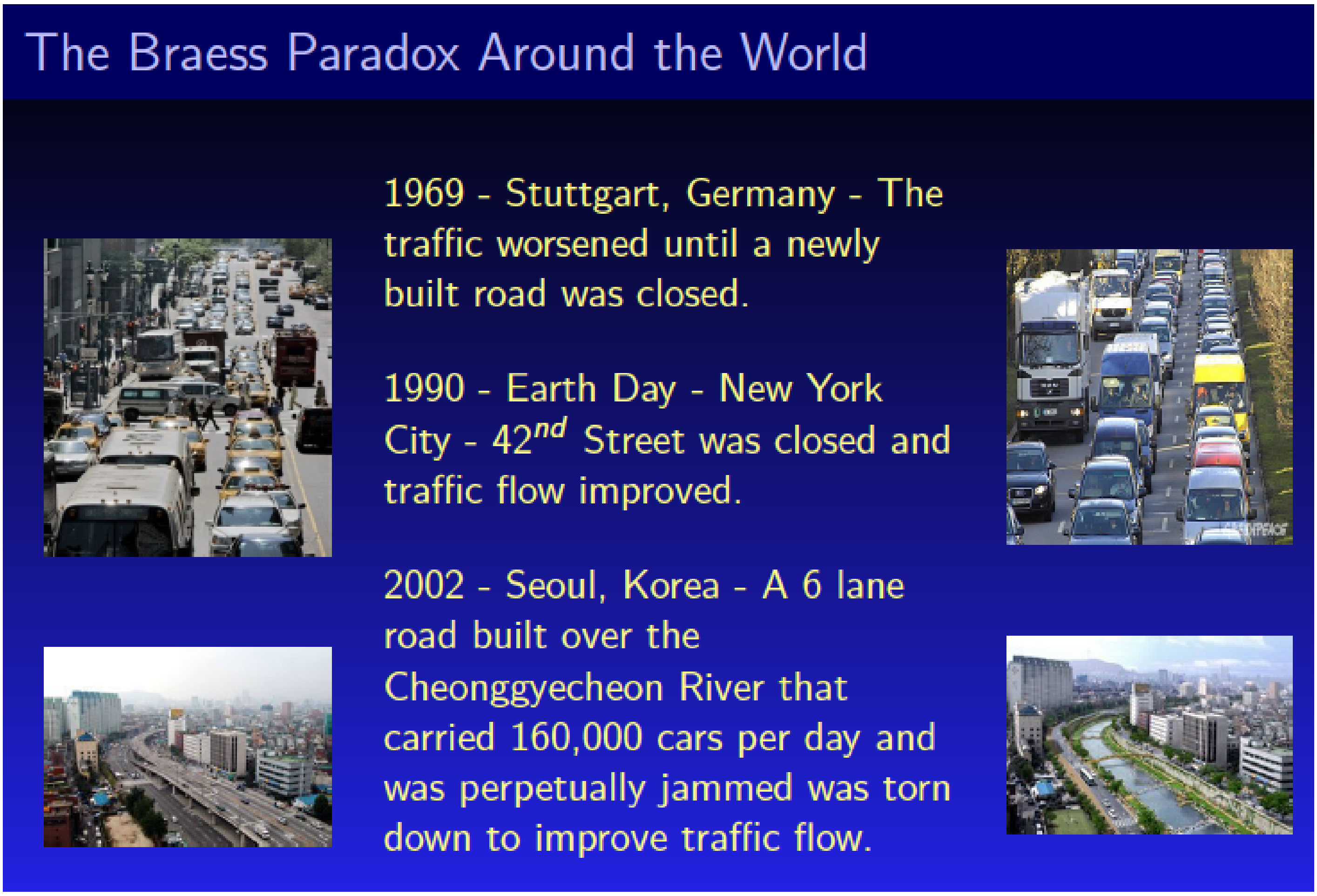
Comes from road traffic

Definition

Adding extra capacity to a network when the entities selfishly choose their route, can reduce the overall performance of the network.

- Highly counter-intuitive
- Explained by non-cooperative game theory
- Similarly, closing roads can sometimes increase efficiency and allow drivers to travel faster
- Application in telecommunications: each mobile may be willing to transmit information to more than one base station at a time thanks to multi-homing, at the expense of others' quality.

Slide from Ana Nagurney



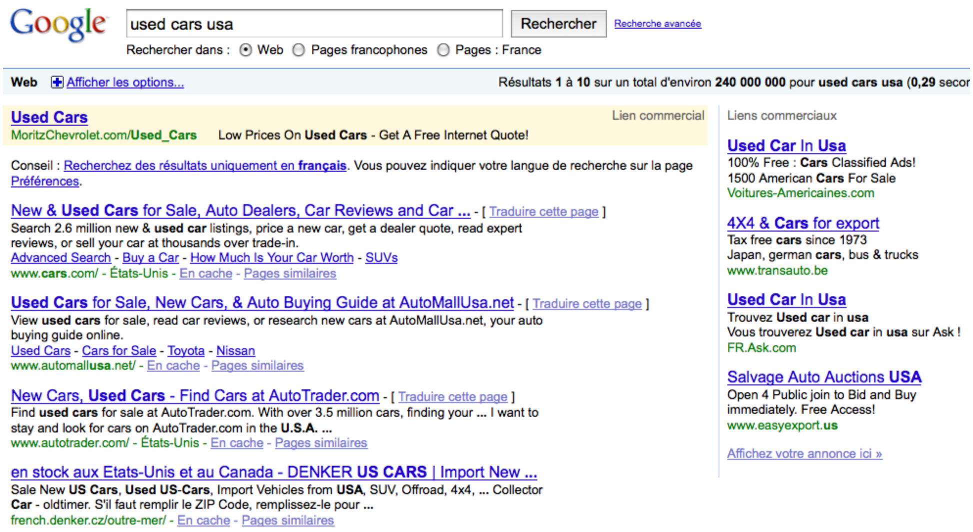
Spectrum auctions

- Radio spectrum: scarce resource
- Licences: avoids the tragedy of the commons
- How to earn money from spectrum allocation?
  - "Beauty success": takes time and lacks transparence
  - Lotteries (random choice of winners): inefficient
  - Auctions!
- 93 spectrum auctions in the USA between 1994 and 2002 ($80 billion in revenues)
- Have to be properly designed to avoid collusion, or when the winners can eventually not bear the costs of the bids.

"Google" auctions

Services on the Internet free of charge. Money made thanks to advertisements

- Ex: web pages
- Ex: search engines



How does it work?

Network neutrality debate

- Dispute between Content Providers (CP) and Internet Service Providers (ISP).
  - ISPs want distant CPs to pay fees for the use of their network
  - Threat of blocking or slowing down traffic: those CPs need to pay for the network upgrade (while getting an increasing proportion of Internet revenue).
- Vivid debate worldwide; critic because the outcome will fashion the future of the Internet.
- Numerous public consultations in 2009, 2010, and 2016.
- How does game theory help?
  - represents actors, their strategies and the impact of the outcome of the game
  - Determines if a (and what) regulation procedure needs to be introduced.

Mathematical foundations and basic games

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Outline

1 Basic economic theory

2 Optimization, fixed point theory

3 Game theory

General modeling tools

- In telecommunications: stakeholders with various goals
  - users mainly focus on the Quality of Service and price
  - service providers are interested in the revenue they get and infrastructures costs, etc.
- Interactions of stakeholders through network performance
  - users impacted by other users traffic and providers capacities
  - service providers impacted by competitors market share and own users traffic.
- The action of a player has an impact on the output of other players, and therefore on their own strategies.
- They all have to play strategically.
- Tools for mathematical modeling:
  - queueing analysis or
  - signal processing.

Representing actor preferences

- Denote a_i the action of player/stakeholder i.
- The actions profile is $\mathbf{a} = (a_i)_i$
- Each player i (usually) represented by its utility function $u_i(\mathbf{a})$ representing quantitatively its level of satisfaction
  - It ranks the outcomes: for $\mathbf{a}$ and $\mathbf{a}'$, $u_i(\mathbf{a}) > u_i(\mathbf{a}')$ means a preference for $\mathbf{a}$.
  - $u_i(\mathbf{a}) = u_i(\mathbf{a}')$ means it is indifferent between the two outcomes
  - Remark: if f strictly increasing function $f \circ u_i$ expresses the same preferences.
  - It can be expressed in monetary units for instance

Valuation function

Definition

Valuation function $V_i(q)$: Maximum price he would be willing to pay to prefer the (non-monetary) outcome q

The (quasi-linear) utility is then the difference between valuation and price charged c.

$$U_i(q) = V_i(q) - c.$$

Next slide: the opposite view: given a utility function, define the valuation function.

Valuation function: definition *from* the utility function

- If money exchanges are involved, separate the other aspects q of the outcome (resource allocation, quality of service, ...)
- The prices paid are represented in a vector $\mathbf{p} = (p_i)_i$
- Outcome decomposed into $o = (q, \mathbf{p})$
- Assumptions:
  - impact of price on utility is independent of non-monetary components: if $o = (q, \mathbf{p})$ and $\tilde{o} = (\tilde{q}, \tilde{\mathbf{p}})$,
  $$\forall x \in \mathbb{R}, \quad U_i(q, p_i + x) > U_i(q, p_i) \quad \text{iff } U_i(\tilde{q}, \tilde{p}_i + x) > U_i(\tilde{q}, \tilde{p}_i).$$
  - Assume a reference outcome without monetary transfer q_0 and the price effect can always exceed the non-monetary part of an outcome: $\lim_{p_i \to -\infty} U_i(q, p_i) > U_i(q_0, 0)$, and $\lim_{p_i \to +\infty} U_i(q, p_i) < U_i(q_0, 0)$.

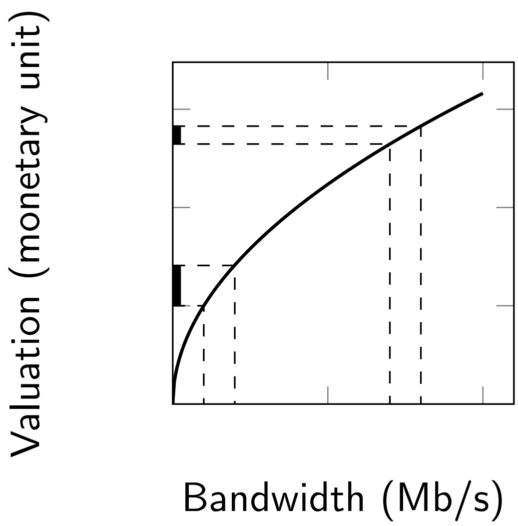
Definition (Valuation function (or willingness to pay))

Maximum price he would be willing to pay to prefer the (non-monetary) outcome q over the reference situation $(q_0, 0)$

$$V_i(q) := \sup\{p_i : U_i(q, p_i) - U_i(q_0, 0) \geq 0\}.$$

Relation between valuation and demand

- Let user i with *willingness to pay* (that is, max) $V_i(q_i)$ for quantity q_i (*here, uni dimensional*).
- Marginal valuation (increase in valuation per extra unit of good) $p_i = V_i'(q_i)$ for q_i.
- V_i reasonably non-decreasing and concave



- If p actual unit price (total price paid $c = pq_i$), User i chooses to maximize its utility $u_i(q_i) = V_i(q_i) - q_i p$.
- From a simple derivation (exercise):

$$\begin{cases} q_i^* = 0 & \text{if } V_i'(0) < p \\ q_i^* = +\infty & \text{if } V_i'(\infty) > p \\ q_i^* \in (V_i')^{-1}(p) & \text{otherwise,} \end{cases}$$

It is the demand function *for user i*.

Demand function (for a population) and example

- Consider a very large population of users, modeled as a continuum of total mass normalized to 1
- They decide whether to subscribe or not to a service (say, the ADSL service) charged at a flat fee p.
- Let $F(\cdot)$ be the cdf of the willingness-to-pay among the population: a proportion $F(p)$ of the population is willing to pay p or less for the service.
- Aggregated demand function then:

$$D(p) = 1 - F(p).$$

- Remark: with a total mass of users is m instead of 1, $D(p) = m(1 - F(p))$.
- Numerical illustration: if the willingness-to-pay is uniformly distributed over an interval $[\theta_{\min}, \theta_{\max}]$

$$D(p) = 1 - F(p) = 1 - \frac{[\min(p, \theta_{\max}) - \theta_{\min}]^+}{\theta_{\max} - \theta_{\min}} = \min\left(1 \quad , \quad \frac{[\theta_{\max} - p]^+}{\theta_{\max} - \theta_{\min}}\right).$$

Pareto optimality

Consider I stakeholders in interaction, with actions $(a_i)_{1\leq i\leq I}$.

Definition: Pareto optimum

An outcome with stakeholders utilities $(u_i(a_1^*,\ldots,a_I^*))$ is Pareto-optimal if and only if for any action profile $(a_i)\in\prod A_i$,

$$\exists i : u_i(a_1,\ldots,a_I) > u_i(a_1^*,\ldots,a_I^*) \Rightarrow \exists j : u_j(a_1,\ldots,a_I) < u_j(a_1^*,\ldots,a_I^*)$$

At a Pareto optimum, there is no way of improving the utility of any player without deteriorating the utility of another one.
Said otherwise, there is no way to make all actors happier.

- Ex: a given amount Q of resource has to be shared among several users. The Pareto optimal outcomes are simply all outcomes such that all the capacity Q is allocated

There can be a lot of Pareto-optimal situations.

The social Optimum

Definition: social optimum

An outcome is a social optimum if the average utility (i.e., the sum of utilities) for all players is optimal. (It is particular Pareto optimal.)

Remark: We will see that it is not necessarily a Nash equilibrium (ex: prisoner's dilemma)
But we can change the rules to reach this type of situation. This is the goal of mechanism design.

Outline

Brief reminder on continuous optimization

Trying to maximize the objective function $f : \mathbb{R}^d \to \mathbb{R}$. We look at *local* optima.

- For any $\mathbf{x} = (x_1, ..., x_d)$ the *gradient* of f at $\mathbf{x}$ is the vector

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_d}(\mathbf{x}) \end{bmatrix}.$$

- From Taylor expansion

$$f(\mathbf{x} + \delta) = f(\mathbf{x}) + \delta \cdot \nabla f(\mathbf{x}) + o(||\delta||),$$

where $\cdot$ stands for the inner product in $\mathbb{R}^d$, and $||\ ||$ is any norm in $\mathbb{R}^d$.

Theorem

A necessary condition for getting an extremum is $\nabla f(\mathbf{x}) = \mathbf{0}$.

Sufficient condition

Let the *Hessian matrix* of f at $\mathbf{x}$ be, with $\mathbf{v}'$ the transpose vector of $\mathbf{v}$),

$$\mathbf{H}_f(\mathbf{x}) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_d}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_d \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_d \partial x_d}(\mathbf{x}) \end{bmatrix}.$$

From Taylor expansion

$$f(\mathbf{x}+\delta) = f(\mathbf{x}) + \delta \cdot \nabla f(\mathbf{x}) + \frac{1}{2}\delta \cdot \mathbf{H}_f(\mathbf{x}) \cdot \delta' + o(||\delta||^2),$$

Theorem (At $\mathbf{x}^*$ such that $\nabla f(\mathbf{x}^*) = \mathbf{0}$)

- *if $\mathbf{H}_f(\mathbf{x}^*)$ is positive definite (i.e., $\forall \mathbf{v} \in \mathbb{R}^d \setminus 0$, $\mathbf{v}'\mathbf{H}_f(\mathbf{x}^*)\mathbf{v} > 0$) , then $\mathbf{x}^*$ is a local minimum;*
- *if $\mathbf{H}_f(\mathbf{x}^*)$ is negative definite (i.e., $-\mathbf{H}_f(\mathbf{x}^*)$ is positive definite) then $\mathbf{x}^*$ is a local maximum;*
- *otherwise $\mathbf{x}^*$ may not be a local extremum.*

Optimization with equality constraints: Lagrange multipliers

We want to solve the problem

$$\begin{array}{lll} \text{minimize} & f(\mathbf{x}) & \\ \text{subject to} & g_j(\mathbf{x}) = 0, & 1 \le j \le k. \end{array}$$

- Define the *Lagrangian* function L

$$\begin{array}{rcl} L : \mathbb{R}^{d+k} & \mapsto & \mathbb{R} \\ (x_1, ..., x_d, \lambda_1, ..., \lambda_k) & \mapsto & f(x_1, ..., x_d) - \sum_{j=1}^{k} \lambda_j g_j(x_1, ..., x_d), \end{array}$$

  where the scalar λ_j is called the *Lagrange multiplier* associated to const. j.
- A solution $\mathbf{x}^*$ is necessary such that there exists $\lambda^* = (\lambda_1^*, \ldots, \lambda_d^*)$ with

$$\begin{array}{rcll} \frac{\partial L}{\partial x_i}(\mathbf{x}^*, \lambda^*) & = & 0 & i = 1, ..., d \\ \frac{\partial L}{\partial \lambda_j}(\mathbf{x}^*, \lambda^*) & = & 0 & j = 1, ..., k. \end{array}$$

  Last line: ensures that constraints are satisfied.

Algorithm

- **Input:** A differentiable function $f : \mathbb{R}^D \mapsto \mathbb{R}$, some differentiable constraint functions $g_j : \mathbb{R}^D \mapsto \mathbb{R}$ for $1 \leq j \leq k$.
- **Output:** a set of candidate solutions $\mathbf{x}^*$ of the optimization problem

$$\begin{array}{lll} \text{minimize} & f(\mathbf{x}) & \\ \text{subject to} & g_j(\mathbf{x}) = 0, & 1 \leq j \leq k. \end{array}$$

1. **Define the Lagrangian** over $\mathbb{R}^{d+k}$ as

$$L(x_1, ..., x_d, \lambda_1, ..., \lambda_k) = f(x_1, ..., x_d) - \sum_{j=1}^{k} \lambda_j g_j(x_1, ..., x_d),$$

2. **Find the set $\tilde{S}$ (if any) of solutions $(\tilde{\mathbf{x}}, \tilde{\lambda})$ of** $\nabla L(\mathbf{x}, \lambda) = 0$
3. **Return** $\{\tilde{\mathbf{x}} : (\tilde{\mathbf{x}}, \tilde{\lambda}) \in \tilde{S}\}$.

Sufficient conditions: look at the second order with Hessian $\mathbf{H} := \mathbf{H}_f(\mathbf{x}^*) + \sum_{j=1}^{k} \lambda_j \mathbf{H}_{g_i}(\mathbf{x}^*)$.

Optimization with inequality constraints: the Karush-Kuhn-Tucker conditions

$$\begin{array}{lll} \text{minimize} & f(\mathbf{x}) & \\ \text{subject to} & g_j(\mathbf{x}) \leq 0, & 1 \leq j \leq k. \end{array}$$

Theorem

If $\mathbf{x}^*$ is a local minimum such that $\{\nabla g_j(\mathbf{x}^*) : 1 \leq j \leq k, g_j(\mathbf{x}^*) = 0\}$ is a set of linearly indep. vectors, then there exist nonnegative constants $(\mu_j)_{1 \leq j \leq k}$, called Karush-Kuhn-Tucker multipliers, such that

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{j=1}^{k} \mu_j \nabla g_j(\mathbf{x}^*) &= 0, & \\ g_j(\mathbf{x}^*) &\leq 0, & \forall j = 1, ..., k \\ \mu_j g_j(\mathbf{x}^*) &= 0, & \forall j = 1, ..., k \end{aligned}$$

Sufficient for a global minimum if f and g_j are continuously differentiable and vex.

Fixed point theory

For a function f, a fixed point is an element x_0 such that $x_0 = f(x_0)$.

- Existence result:

Theorem (Brouwer fixed-point theorem)

If f is a continuous function from a convex compact subset K of an Euclidean space (e.g., $\mathbb{R}^d$ with $d \in \mathbb{N}$) onto K, then f has a fixed point.

- Uniqueness result:

Theorem (Banach fixed-point theorem)

Let $\mathcal{S}$ complete metric space, with distance denoted by dist(x, y). Let $f : \mathcal{S} \mapsto \mathcal{S}$ a contracting function, i.e. there exists $\rho < 1$ such that

$$\forall x, y \in \mathcal{S}, \quad \mathrm{dist}(f(x), f(y)) \leq \rho \mathrm{dist}(x, y)$$

Then f has one and only one fixed point x_0.
Moreover, that fixed point can be found as the limit of $(y_n)_{n\in\mathbb{N}}$ defined by y_0 taken arbitrarily in $\mathcal{S}$ and $y_{n+1} = f(y_n)$.

Outline

Introduction to game theory in telecommunications

- While before optimization was the tool for routing, QoS provisioning, interactions between players has to be taken into account.
- Game theory: distributed optimization: individual users make their own decisions. "Easier" than to solve NP-hard problems (approximation).
- We need to look at a stable point (*Nash equilibrium*) for interactions.
- Tools used before in Economics, Transportation...
- and has recently appeared in telecommunications.
- We may have paradoxes (Braess paradox) that can be studied that way.
- A way to control things: to introduce pricing incentives/discouragements (TBC).

Examples

In many situations, the satisfaction level of agents/individuals depends not only on their own actions, but also on the actions of the others. Frequently, their objectives diverge...

- Transportation: which route to select?
- Exam:
  - on the student side: which part(s) to study?
  - on the teacher side: which part(s) to include in the exam?
- Operators: on which technologies to invest? What charging price for the service?

Is there an equilibrium situation? in the sense that everybody is "happy" since no one can improve his/her own situation if the others do not move.

Basic definitions

- Game theory: set of tools to understand the behavior of interacting *decision makers* or players.
- Classical assumption: players are rational: they have well-defined objectives, and they take into account the behavior of others.
- In most of this course: strategic or normal games, players play (simultaneously) once and for all.
- There are also branches called
  - extensive games, for which players play sequentially;
  - repeated games for which they can change their choices over time;
  - Bayesian games, evolutionary games...

Rationality

Players/agents **rational** more exactly means

$$\begin{cases} \text{Each agent has a utility function ranking preferences over the game} \\ \text{ outputs (each can rank all issues by order of preference)} \\ \text{Each agent optimizes its utility function} \end{cases}$$

Strategy

Definition : Strategy

Each agent has a set of strategies describing the set of actions he can play with given the available information (on other agents choice, on the game rules, etc.)

Questions

- How to define the equilibrium situation?
- Does an equilibrium exist?
- What are the decisions of each agent at an equilibrium?
- To what extent do the game rules affect the equilibrium outcome?
- Is it possible to improve the payoff of some (all) agents by changing the rules of the game?

Typical networking applications

- P2P networks: a node tries to benefit from others, but limits its available resource (free riding)
- Grid computing: same issue, try to benefit from others' computing power, while limiting its own contribution.
- Routing games: each sending node tries to find the route minimizing dealy, but intermediate links shared with other flows (interactions).
- Ad hoc networks: what is the incentive of nodes to forward traffic of neighbors? If no one does, no traffic is successfully sent.
- Congestion control game (TCP...): why reducing your sending rate when congestion is detected?
- Power control in wireless networks: maximizing your power will induce a better QoS, but at the expense of others' interferences.
- Transmission games (WiFi...): if collision, when to resubmit packets?

Strategic (or normal form) Games

- A strategic game Γ consists of:
  - A finite set of players, N.
  - A set A_i of actions available to each player $i \in N$. and $A = \prod_{i \in N} A_i$.
  - For each player a utility function, (payoffs) $u_i : A \to \mathbb{R}$, characterizing the gain/utility from a state of the game.
- Players make decisions independently, without information about the choice of other players.
- We note $\Gamma = \{N, A_i, u_i\}$.
- For two players: description via a table, with payoffs corresponding to the strategic choices of users:

	C_1	C_2
F_1	$b_{11}\ c_{11}$	$b_{12}\ c_{12}$
F_2	$b_{21}\ c_{21}$	$b_{22}\ c_{22}$

$N = \{1, 2\},\ A_1 = \{F_1, F_2\},\ A_2 = \{C_1, C_2\},\ u_1(F_j, C_k) = b_{jk},\ u_2(F_j, C_k) = c_{jk}$.

Example: association game

- Two users have the choice to connect to the Internet through WiFi and 3G
- If they both select the same technology, there will be interferences.
- They may get different throughput due to heterogeneous terminals and/or radio conditions
- Table of payoffs (obtained throughputs):

	3G	WiFi
3G	3; 3	6; 4
WiFi	5; 6	1; 1

- What is the best strategy for both players? Is there an "equilibrium" choice?

Nash equilibrium

- Most important equilibrium concept in game theory.
- Let $a \in A$ strategy profile, $a_i \in A_i$ player i's action, and a_{-i} denote the actions of the other players.
- Each player makes his own maximization.
- A Nash equilibrium is an action profile at which no user may gain by unilaterally deviating.

Definition

A N.E of a strategic game Γ is a profile $a^* \in A$ such that for every player $i \in N$:

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \;\; \forall a_i \in A_i$$

How to look for a Nash equilibrium?

- For each player i, look for the *best response* a_i in terms of a_{-i}.
- A Nash equilibrium: a point such that no one can deviate (i.e. improve his utility), that is, a strategy profile such that each player's action is a best response
- In a table with two players (can be generalized):
  1. Write in bold the best response of a player for each choice of the opponent;
  2. A Nash equilibrium is a profile where both actions are in bold.
  3. Example (blue is also used here):

	C_1	C_2
F_1	b_{11} $\mathbf{c_{11}}$	$\mathbf{b_{12}}$ c_{12}
F_2	$\mathbf{b_{21}}$ $\mathbf{c_{21}}$	b_{22} c_{22}

  4. Remark: on this example, *dominant strategies* so that the table can be simplified.

Classical illustration: The Battle of the Sexes

- *Bach or Stravinsky ?* Married people want to go together to a concert of Bach or Stravinsky. Their main concern is to go together, but one person prefers Stravinsky and the other Bach.

	B	S
B	2; 1	0; 0
S	0; 0	1; 2

Classical illustration: The Battle of the Sexes

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	B	S
B	2; 1	0; 0
S	0; 0	1; 2

$\Rightarrow$

	B	S
B	**2; 1**	0; 0
S	0; 0	**1; 2**

- The game has two N.E.: (B, B) *and* (S, S).

Nash equilibrium in our association game

- Two users have the choice to connect to the Internet through WiFi and 3G
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$\Rightarrow$

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WiFi	**5; 6**	1; 1

- Nash equilibria: (5; 6) and (6; 4).

Prisonner's Dilemma

- Suspects in a crime are in separate cells.
- If they both confess (betray their partner), each will be sentenced to a three years of prison.
- If only one betrays, he will be free and the other will be sentenced four years.
- If they both stay silent the sentence will be a year in prison for each one.
- Goal here: to minimize years in prison.
- Utility $u_i = 4-$number of years in jail.

	stays silent	*betrays*
stays silent	3; 3	0; **4**
betrays	**4**; 0	**1; 1**

- Best outcome: stay silent, but this requires cooperation.
- But, (betrays, betrays) is the unique N.E.
- Not optimal!

Prisonner's Dilemma in wireless networks

Gaoning He PhD thesis, Eurecom, 2010

- Two players sending information at a base station.
- Two power levels: High or Normal.
- Payoff table:

	Normal	High
Normal	Win; Win	Lose much; Win much
High	Win much; Lose much	Lose; Lose

- Best outcome: Normal, but this requires cooperation.
- But, (High, High) is the unique N.E.
- Not socially optimal here too, and not Pareto optimal either.

A Nash equilibrium does not always exist

- Game where 2 players play odd and even:

	Odd	*Even*
Odd	$\mathbf{1}; -1$	$-1; \mathbf{1}$
Even	$-1; \mathbf{1}$	$\mathbf{1}; -1$

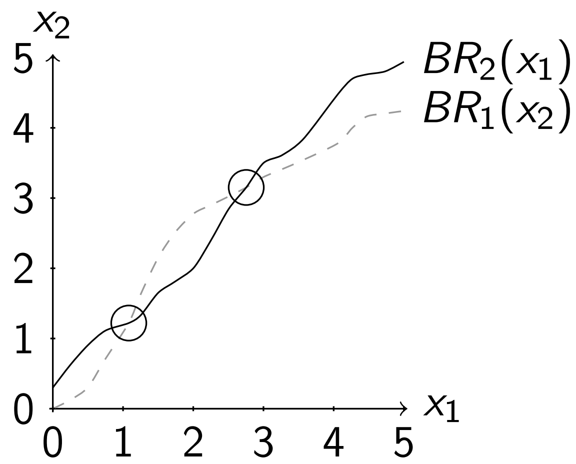
- This game does not have a N.E.
- So in general, games may have no, one, or several Nash equilibria...

Case of continuous set of actions

- In the case of a continuous set of strategies, simple derivation can be used to determine the Nash equilibrium (always simpler!).
- For two players 1 and 2: draw the best-response in terms

$$BR_1(x_2) = \text{argmax}_{x_1} u_1(x_1, x_2) \text{ and } BR_2(x_1) = \text{argmax}_{x_2} u_2(x_1, x_2).$$

  A Nash equilibrium is an intersection point of the best-response curves:



Application to power control in 3G networks Mandayam & Goodman

- In CDMA-based networks each user can play on transmission power.
- QoS based on the signal-to-interference-and-noise ratio (SINR):

$$SINR_i = \gamma_i = \frac{W}{R} \frac{h_i p_i}{\sum_{j \neq i} h_j p_j + \sigma^2}$$

  with W spread-spectrum bandwdith, R rate of transmission, p_i power transmission, h_i path gain, σ^2 background noise.
- Different utility functions found in the litterature. Ex: the number of bits transmitted per Joule

$$u_j(p_i, \gamma_i) = \frac{R}{p_i}(1 - 2BER(\gamma_i))^L = \frac{R}{p_i}(1 - e^{-\gamma_i/2})^L$$

  where $BER(\gamma_i)$ bit error rate and L length of symbols (packets).
- Increasing *alone* your own power increases your QoS, but decreases the others'.
  $\Rightarrow$ Game theory.
- A Nash eq. exists, but its efficiency can be improved through pricing.

Mixed strategies

- Previous Nash equilibrium also called *pure Nash equilibrium*.
- A mixed strategy is a probability distribution over pure strategies: $\pi_i(a_i)\ \forall a_i \in A_i$.
- Player i utility function is the expected value over distributions

$$\mathbb{E}_\pi[u_i] = \sum_{a\in A} u_i(a) \left(\prod_i \pi_i(a_i)\right).$$

- A Nash equilibrium is a set of distribution functions $\pi^* = (\pi_i^*)_i$ such that no user i can unilaterally improve his expected utility by changing alone his distribution π_i.
  Formally,

$$\forall i, \forall \pi_i, \quad \mathbb{E}_{\pi^*}[u_i] \geq \mathbb{E}_{(\pi_i,\pi^*_{-i})}[u_i].$$

Theorem

Advantage (proved by John Nash): for every finite game, there always exist a (Nash) equilibrium in mixed strategies.

Interpretation of mixed strategies

- Concept of mixed strategies known as "intuitively problematic".
- Simplest and most direct view: randomization, from a 'lottery".
- Other interpretation: case of a large population of agents, where each of the agent chooses a pure strategy, and the payoff depends on the fraction of agents choosing each strategy. This represents the distribution of pure strategies (does not fit the case of individual agents).
- Or comes from the game being played several times *independently*.
- Other interpretation: purification. Randomization comes from the lack of knowledge of the agent's information.

A Real-World Example

- **Penalty kicks in soccer**
  - A kicker and a goalie in a penalty kick
  - Kicker can kick left or right
  - Goalie can jump to left or right
  - Kicker scores iff he/she kicks to one side and goalie jumps to the other
  - Analogy to Matching Pennies
    - If you use a pure strategy and the other agent uses his/her best response, the other agent will win
    - If you kick or jump in either direction with equal probability, the opponent can't exploit your strategy

Illustration of mixed strategies: jamming game

- Consider two mobiles wishing to transmit at a base station: a regular transmitter (1) and a jammer (2)
- Two channels, c_1 and c_2 for transmission, collision if they transmit on the same channel, success otherwise
- For the regular transmitter: reward for success 1, -1 if collision
- For the jammer: reward 1 if collision, -1 if missed jamming.
- payoff table

	c_1	c_2
c_1	$-1; 1$	$1; -1$
c_2	$1; -1$	$-1; 1$

- No pure Nash equilibrium.

Mixed strategy equilibrium for the jamming game

- The transmitter (resp. jammer) chooses a probability p_t (resp. p_j) to transmit on channel c_1.
- Utilities (average payoff values):

$$\begin{aligned}
u_t(p_t, p_j) &= -1(p_t p_j + (1-p_t)(1-p_j)) + 1(p_t(1-p_j) + (1-p_t)p_j) \\
&= -1 + 2p_t + 2p_j - 4p_t p_j \\
u_j(p_t, p_j) &= 1(p_t p_j + (1-p_t)(1-p_j)) + -1(p_t(1-p_j) + (1-p_t)p_j) \\
&= 1 - 2p_t - 2p_j + 4p_t p_j
\end{aligned}$$

- For finding the Nash equilibrium:

$$\begin{aligned}
\frac{\partial u_t(p_t, p_j)}{\partial p_t} &= 2 - 4p_j = 0 \\
\frac{\partial u_j(p_t, p_j)}{\partial p_j} &= 2 - 4p_t = 0.
\end{aligned}$$

- $(p_t = 1/2, p_j = 1/2)$ mixed Nash equilibrium (sufficient conditions verified too).

Other interpretation

- The transmitter will choose both options iff they are indifferent for him in terms of utility (otherwise, there is a pure strategy)
  - Expected utility if choosing c_1: $-1p_j + 1(1-p_j) = 1 - 2p_j$
  - Expected utility if choosing c_2: $1p_j - 1(1-p_j) = 2p_j - 1$
  - Indifferent means equality of the two, and $p_j = 1/2$.
- Same thing for the jammer, leading to $p_t = 1/2$.

Price of Anarchy (PoA)

- The optimal social utility function happens when we have a single authority who dictates every agent what to do.
- When agents choose their own action, we should study their behavior and compare the obtained social utility with the optimal one.

Definition (Price of Anarchy)

It is the ratio of optimal social utility divided by the worst social utility at a Nash equilibrium.

- A price of Anarchy of 1 corresponds to the optimal case where decentralization does not bring any loss of efficiency (that may happen).
- Research activity for computing bounds for the price of Anarchy in specific games. We will detail the case of nonatomic routing games.

Ex: PoA in our association game

- If they both select the same technology, there will be interferences.
- They may get different throughput due to heterogeneous terminals and/or radio conditions
- Recall the table of payoffs on the choice to connect to the Internet through WiFi and 3G:

	3G	WiFi
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- PoA: $11/10$.

Other notion: Stackelberg game

- Decision maker (network administrator, designer, service provider...) wants to optimize a utility function.
- His utility depends on the reaction of users (who want to maximize their own utility, minimize their delay...)
- Hierarchical relationship: *leader-follower problem* called *Stackelberg game*.
  - For a set of parameters provided by the leader, followers (users) respond by seeking a new algorithm between them.
  - The leader has to find out the parameters that lead to the equilibrium yielding the best outcome for him.
- Typical application: the provider plays on prices, capacities, users react on traffic rates...

Stackelberg game: formal problem

- Say that there are N users
- Let $u(x) = (u_1(x), \dots, u_N(x))$ the utility function vector for users for the set of parameters x set by the leader.
- Denote by $R(u(x), x)$ the utility of the leader.
- Define $u^*(x)$ as the (Nash) equilibrium (if any) corresponding to x.
- Goal: find x^* such that

$$R(u(x^*), x^*) = \max_x R(u(x), x).$$

- Works fine if $u^*(x)$ is unique
- If not, and if $U^*(x)$ is the set of equilibria, we may want to maximize the *worst case*: find x^* such that

$$R(u(x^*), x^*) = \max_x \min_{u^*(x) \in U^*(x)} R(u^*(x), x).$$

Simple illustration of Stackelberg game

- leader: service provider fixing its price p
- followers: users, modeled by a demand function $D(p)$ representing the *equilibrium* population accepting the service for a given price.
- Equilibrium among users therefore already included in the model.
- The provider chooses the price p to maximize its revenue

$$R(p) = pD(p).$$

- Obtained by computing the derivative of $R(p)$.

Exercise 1. Quizz on game theory basics

A few questions for you to check that you have understood the notions of the course.

The questions proposed here are taken from the MOOC "Game Theory" on Coursera, created by Matthew Jackson, Kevin Leyton-Brown and Yoav Shoham. Many other interesting resources are available in that MOOC, including full lecture videos

Question 1 Consider the following normal-form game:

Player 1 \ Player 2	Movie	Theater
Movie	a, b	$0, 0$
Theater	$0, 0$	c, d

- N=1, 2
- A_i={Movie, Theater}. Each player chooses an action of either going to a movie or going to the theater.
- Player 1 prefers to see a movie with Player 2 over going to the theater with Player 2.
- Player 2 prefers to go to the theater with Player 1 over seeing a movie with Player 1.
- Players get a payoff of 0 if they end up at a different place than the other player.

Which restrictions should a, b, c and d satisfy?

a). $a > c, b > d$

b). $a > d, b < c$

c). $a > c, b < d$

d). $a < c, b < d$

Question 2 n people guess an integer between 1 and 100, and the winner is the player whose guess is closest to the mean of the guesses + 1 (ties broken randomly). Which of the following is an equilibrium?

a). All announce 1.

b). All announce 50.

c). All announce 75.

d). All announce 100.

Question 3 Consider the collective-action game:

Player 1 \ Player 2	Revolt	Not
Revolt	2,2	-1,1
Not	1,-1	0,0

When Player 1 plays "Not", for Player 2

a). "Revolt" is a best response.

b). "Not" is a best response.

c). "Revolt" and "Not" are both best responses.

d). There is no best response.

Question 4 Consider the following game in which two firms must decide whether to open a new plant or not:

Firm 1 \ Firm 2	Build	Not
Build	1,1	3,0
Not	0,3	2,2

Find all pure strategy Nash equilibria:

a). (Build, Not)

b). (Not, Not)

c). (Build, Build)

d). (Not, Build)

Question 5 Consider the game:

Player 1 Player 2	Left	Right
Up	2,1	1,1
Down	0,1	0,2

Which of the players has a strictly dominant strategy?

a). Player 1

b). Player 2

c). Both players

d). Neither player

Question 6 Consider the game:

Player 1 Player 2	Left	Right
Left	3,3	1,1
Right	1,4	1,1

Which of the following outcomes is Pareto-optimal? (There might be more than one, or none.)

a). (3,3)

b). (1,1)

c). (1,4)

Exercises

Patrick Maillé and Bruno Tuffin

1 Game Theory

See also

http://mibe.unipv.it/attach/AdvMicroSolutions.pdf

http://www.matthew-hoelle.com/1/75/resources/document_691_1.pdf

http://www.maa.org/sites/default/files/pdf/ebooks/GTE_sample.pdf

http://press.princeton.edu/chapters/sm10001.pdf

Exercise 1 (Tragedy of the commons, by Dylan Selterman) *https://www.washingtonpost.com/posteverything/wp/2015/07/20/why-i-give-my-students-a-tragedy-of-the-commons-extra-credit-challenge/*

Imagine you are a student and your teacher poses this challenge to the entire class:

> *You can each earn some extra credit on your term paper. You get to choose whether you want 2 points added to your grade, or 6 points. But there's a catch: if more than 10% of the class selects 6 points, then no one gets any points. All selections are anonymous, and the course grades are not curved.*

Exercise 2 *The game of Three Fingers is played as followed: Alice and Bob simultaneously hold up one, two, or three fingers. Alice wins in case of a match (both players show the same number of fingers), and Bob wins if there is a nonmatch. If Alice wins, Bob must give her an amount equal to the total number of fingers held up. Otherwise, Alice must give Bob an amount equal to the number of fingers that he held up.*

1. *Write this game in normal form.*
2. *Does this game have a Nash equilibrium (of pure strategies) ?*

Exercise 3 *We consider a modified version of the* game of nim*. There are three rows of matches on the table in front of the two players, containing respectively 1, 2,and 3 matches. In turn, the players take any (positive) number of matches from one row. The player taking the last match loses.*

1. *Sketch a game tree (you must not expand every branch of the tree because several subtrees are similar).*
2. *Show that the second player has a sure win.*

3. *How would you advise the first player to play if the amount paid by the loser to the winner is equal to the total number of turns that have been played until the last match is taken?*

Exercise 4 (Coordination game) *Two operators can choose to invest in two technologies. Operator 1 already runs WiMAX, Operator 2 already runs LTE. But if different technologies are chosen, the gain of operators is null (customers refuse to choose a new technology if no norm has emerged). Operators have to coordinate on the same technology given that no one manages the same. We assume the following matrix of gains::*

		Operator II	
		WiMAX	*LTE*
Operator	*WiMAX*	$(3,1)$	$(0,0)$
I	*LTE*	$(0,0)$	$(1,4)$

1. *Compute the Nash equilibrium (equilibria) if any*
2. *What will be the outcome?*
3. *Is there an equilibrium in mixed strategy where Operator 1 invests x on WiMAX and $(1-x)$ on LTE, and Operator II invests y on WiMAX and $(1-y)$ on LTE?*
4. *What are the corresponding expected gains?*

Exercise 5 (Bertrand duopoly) *Two operators are in competition and play with their service price. Each operator experiences high fixed cost C and a zero variable cost. Market demand is*

$$D(\bar{p}) = D_0 - \beta\frac{\bar{p}}{2}.$$

All demand goes with the operator proposing the smallest price. If prices are the same demand is distributed equally between operators. Each operator chooses a prices p_i maximizing its net benefit.

1. *Compute the Nash equilibrium.*
2. *What is the price in case of collusion (i.e., maximizing the total net benefit)?*
3. *How do operators share the surplus?*
4. *Why are oligopolies stable?*

Exercise 6 () **This exercise has initially been proposed by François Marini and Françoise Forges (université Paris-Dauphine).**

We consider the following game under strategic form:

		Player 2	
		L	*R*
Player 1	*T*	*0,2*	*3,0*
	B	*2,1*	*1,3*

1. *Express the best-reply correspondences of each player.*

2. *Show that there exists a unique Nash equilibrium in mixed strategies. Compute the expected payment of each player at that equilibrium*

Exercise 7 () **This exercise has initially been proposed by François Marini and Françoise Forges (université Paris-Dauphine).**

The game is played between Andre and Betsy, with the exterior help of a third person who tosses a coin. The coin is biased, it has a probability 0.8 to give "head" (that bias is known by both players).

The third person is isolated to toss the coin, and only informs Andre of the result. Then Andre announces "head" or "tail" to Betsy (he may lie). Betsy then has to guess and announce the real result of the toss.

1. *Describe the pure strategies of each player.*

   The player payoffs are defined as follows:

   (a) *Betsy gets 10 if she guesses correctly, and 0 otherwise.*

   (b) *Andre's payoff is the sum of two amounts:*

   - *he gets 20 if Betsy announced "head" and 0 otherwise,*
   - *additionally he gets 10 if he tells the truth and 0 otherwise.*

2. *Represent the game under strategic form, where Betsy's utility is the expected value with regard to the coin toss.*

3. *Check whether the game has pure strategy Nash equilibria, and if so, determine them.*

Exercise 8 () **This exercise has initially been proposed by François Marini and Françoise Forges (université Paris-Dauphine).**

We consider N farmers who can (each) produce as much wheat as they want, at zero cost. If the k^{th} farmer produces a quantity q_k, the total quantity produced is $Q = q_1 + q_2 + \cdots + q_N$. The price of the wheat is given by $p = e^{-Q}$.

1. *Study the variations of the function $f(x) = xe^{-x}$ for $x \geq 0$.*

2. *Using the previous results, show that the strategy consisting in producing a unit of wheat is a dominant strategy for each farmer. Deduce from that that the benefit of a farmer is e^{-N}.*

3. *Assume that the farmers agree that each one produces $1/N$ unit of wheat. Still using the results of the first question, show that the total profit is then maximized. Check that the benefit of each farmer is $\frac{1}{eN}$. Can such an agreement be respected without an explicit contract?*

4. *Why is that N-player game a generalization of the prisoner's dilemma?*

Exercise 9 (A war of attrition) *Two players are involved in a dispute over an object. The value of the object to player i is $v_i > 0$ with $v_1 > v_2$. Time is modeled as a continuous variable that starts at 0 and runs indefinitely. Each player chooses when to concede the object to the other player; if the first player to concede does so at time t, the other player obtains the object at that time. If both players concede simultaneously, the object is split equally between them, player i receiving a payoff of $v_i/2$. Time is valuable: until the first concession each player loses one unit of payoff per unit of time.*

Formulate this situation as a strategic game and show that in all Nash equilibria one of the players concedes immediately.

Exercise 10 (A location game) *Each of n people chooses whether or not to become a political candidate, and if so which position to take. There is a continuum of citizens, each of whom has a favorite position; the distribution of favorite positions is given by a density function f on $[0, 1]$ with $f(x) > 0$ for all $x \in [0, 1]$. A candidate attracts the votes of those citizens whose favorite positions are closer to his position than to the position of any other candidate; if k candidates choose the same position then each receives the fraction $1/k$ of the votes that the position attracts. The winner of the competition is the candidate who receives the most votes. Each person prefers to be the unique winning candidate than to tie for first place, prefers to tie for first place than to stay out of the competition, and prefers to stay out of the competition than to enter and lose.*

1. *Formulate this situation as a strategic game, find the set of Nash equilibria when $n = 2$, and show that there is no Nash equilibrium when $n = 3$.*

2. *Consider a variant of that game, in which there are two players, the distribution of the citizens' favorite positions is uniform, and each player is restricted to choose a position of the form ℓ/m for some $\ell \in \{0, ..., m\}$, where m is even. Show that the only outcome that survives iterated elimination of weakly dominated actions is that in which both players choose the position 1/2.*

Exercise 11 *From "Wireless Network Pricing" by Jianwei Huang and Lin Gao Consider the following communication congestion game.Two mobile users transmit on the same channel, each deciding whether or not to transmit its data at a particular time slot. Each user will incur a cost c from each transmission, which mainly includes the power cost, channel access fee, etc. Each user can achieve a revenue $R > c$ from each successful transmission, and a zero revenue if a collision occurs (i.e., if both users transmit at the same time). The above congestion game can be represented by the following payoff matrix, where rows denote the actions of user 1, and columns denote the actions of user 2.*

	Transmit	*Not transmit*
Transmit	$-c; -c$	$R - c; 0$
Not transmit	$0; R - c$	$0; 0$

1. *Determine whether the game has pure strategy equilibria or not. If it does, find the pure strategy Nash equilibria.*

2. *Find the set of all mixed-strategy Nash equilibria.*

Exercise 12 (An exchange game) *Each of two players receives a ticket on which there is a number in some finite subset S of the interval [0, 1]. The number on a player's ticket is the size of a prize that he may receive. The two prizes are identically and independently distributed, with distribution function F. Each player is asked independently and simultaneously whether he wants to exchange his prize for the other player's prize. If both players agree then the prizes are exchanged; otherwise each player receives his own prize. Each player's objective is to maximize his expected payoff.*

Model this situation as a Bayesian game (i.e., taking into account the expected utilities) and show that in any Nash equilibrium the highest prize that either player is willing to exchange is the smallest possible prize.

Exercise 13 (Guessing right) *Players 1 and 2 each choose a member of the set $\{1, ..., K\}$. If the players choose the same number then player 2 pays*

\$1 to player 1; otherwise no payment is made. Each player maximizes his expected monetary payoff.

Find the mixed strategy Nash equilibria of this game.

Exercise 14 (Air strike) *Army A has a single plane with which it can strike one of three possible targets. Army B has one anti-aircraft gun that can be assigned to one of the targets. The value of target k is v_k, with $v_1 > v_2 > v_3 > 0$. Army A can destroy a target only if the target is undefended and A attacks it. Army A wishes to maximize the expected value of the damage and army B wishes to minimize it.*

Formulate the situation as a strategic game and find its mixed strategy Nash equilibria.

Exercise 15 (Guess the average) *Each of n people announces a number in the set $\{1, ..., K\}$. A prize of \$1 is split equally between all the people whose number is closest to 2/3 of the average number.*

Show that the game has a unique mixed strategy Nash equilibrium, in which each player's strategy is pure.

Repeated games, Bayesian games, value of information

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Repeated games

Game repeated n times:
We repeat the game n times. At each date $t = 1 \dots n$, Player i plays a_i^t. The final utility is the average of the utilities at each step (other choices possible):

$$u_i^n((a_i^t)_{t=0 \dots n-1}) = \frac{1}{n} \sum_{t=0}^{n-1} u_i((a_i^t)_i)$$

Infinitely repeated game:
Idem, but the final utility i the limit (if it exists):

$$u_i^\infty((a_i^t)_{t=0 \dots}) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} u_i((a_i^t)_i)$$

Infinitely (discounted) repeated game:(Two possible interpretations)
with a discount factor $0 < \delta < 1$: normalized expected utility

$$u_i^\infty((a_i^t)_{t=0 \dots}) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i((a_i^t)_i)$$

Strategies

- Player strategy: a function that at each t associates an action depending on the history played so far
  - $h_t = ((a_i^0)_i, \ldots, (a_i^{t-1})_i)$ history
  - $H_t = \{h_t\}$ set of possible histories
- At each t: each player can play a correlated strategy, depending on some public signal
- Ex: in the transmission power game

	Normal	High
Normal	Win; Win	Lose much; Win much
High	Win much; Lose much	Lose; Lose

Player 2 can play the same power than Player 1 in previous period to incentivize Player 1 to keep his power level to *Normal*. Strategy *tit-for-tat*.

Minimax utility

- In the *one-shot* game with I players
- Recall the notations a_i action of Player i and A_i set of actions.

Definition

$$\underline{u}_i = \min_{a_{-i}\in A_{-i}} \max_{a_i\in A_i} u_i(a_i, a_{-i}).$$

It is the minimal utility that can be ensure if others try to minimize his/her own utility.

Basic form of the Folk Theorem (discounted case)

(a similar version exists for the average case)
Notations:

- Let $Conv(X)$, the convex hull of X, be the smallest convex set that contains X

Folk Theorem / Characterizing equilibria (discounted case)

Any point

$$\begin{cases} (u_i^*) \in Conv\{(u_i(a_1 \ldots, a_I)),\ a_k \in A_k\} \\ \forall i,\ u_i^* \geq \underline{u}_i \end{cases}$$

is the output of a Nash equilibrium of the infinitely repeated discounted game if δ is sufficiently close to 1.

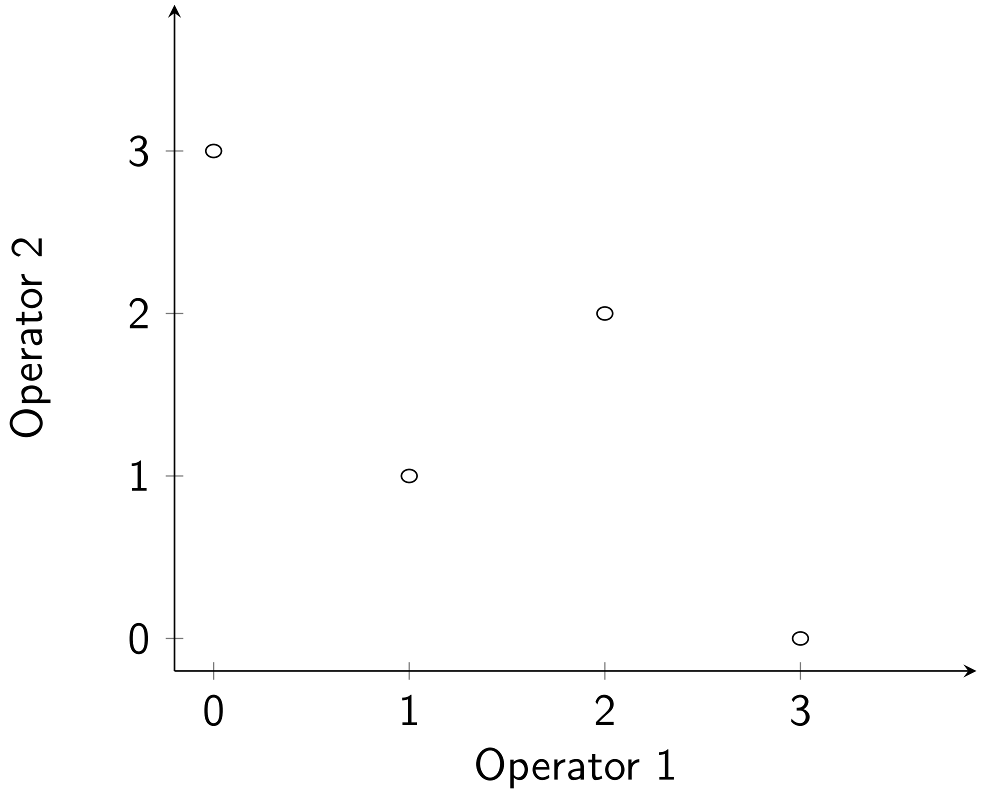
An infinity of such equilibria: those points can be reached under the threat of punishment from other players.

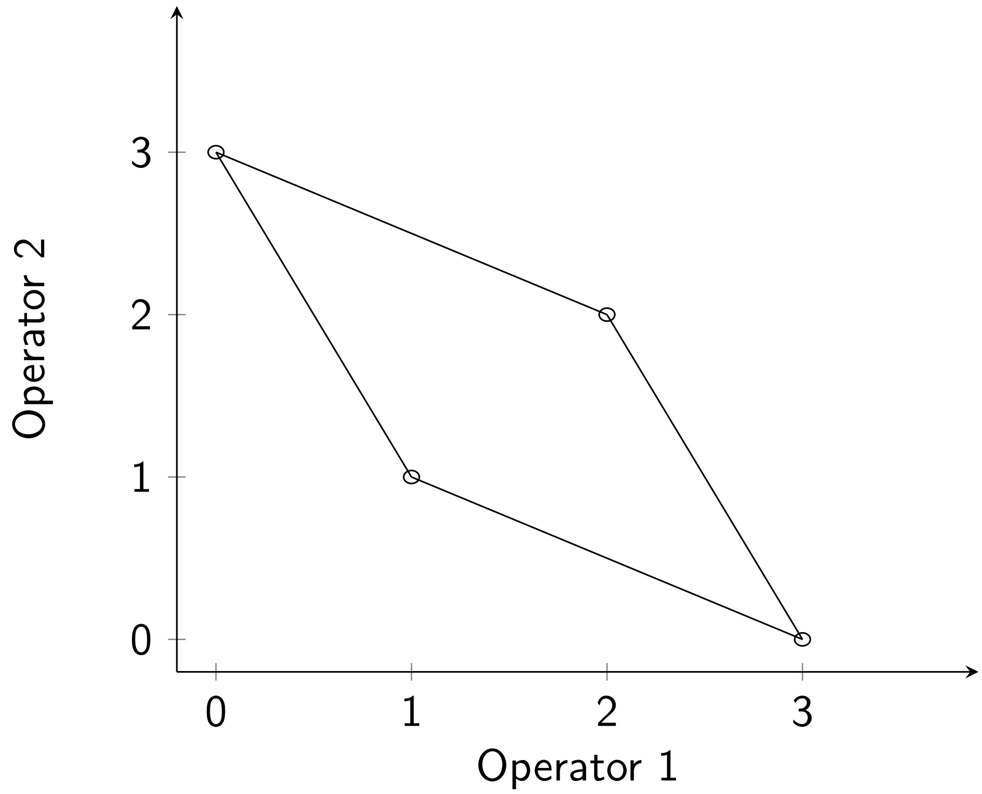
In particular: any point in the convex hull with payoffs above a Nash equilibrium can be reached.

Application: prisoner's repeated dilemma

		Operator II: Low price	Operator II: High price
Operator I	Low price	$(1,1)$	$(3,0)$
	High price	$(0,3)$	$(2,2)$

Exercise: draw the convex hull and the minimax value to get the set of possible equilibria.





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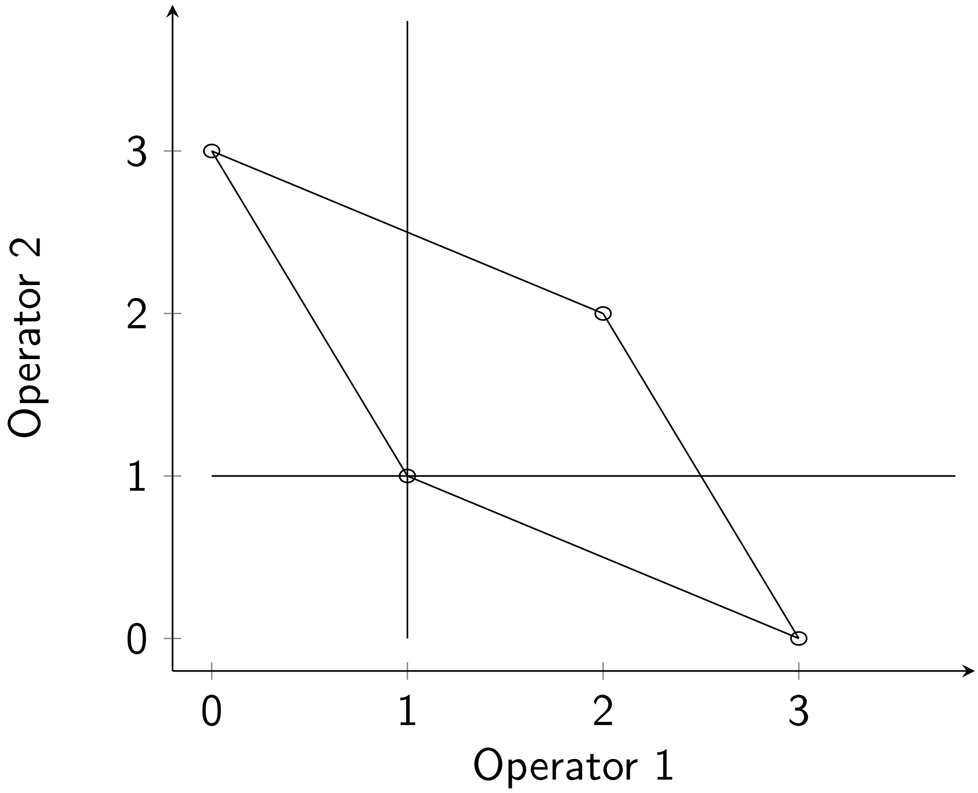
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2
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Operator 2
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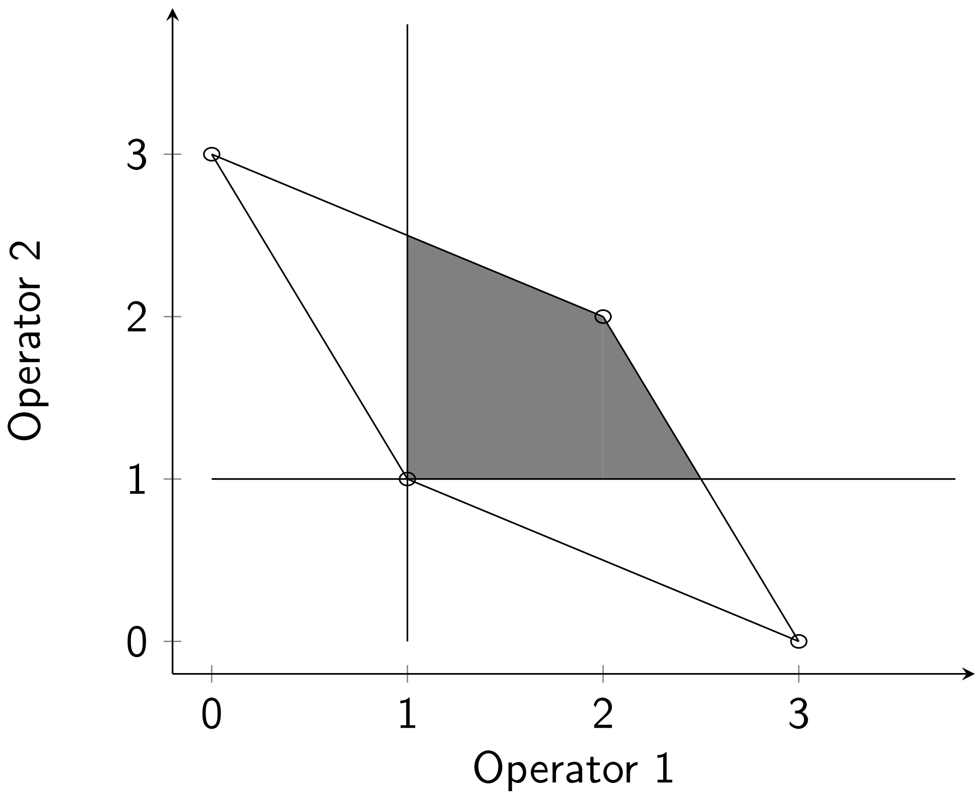
3
Operator 1


3
2
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Operator 2
0
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2
3
Operator 1

Application : prisoner's repeated dilemma

In particular, the social optimum $(2, 2)$ becomes a Nash equilibrium
The equilibria are stable because of retaliation threat: if one player deviates, the other can punish him by deviating too during the next steps. Unilateral deviation is therefore not profitable. (We'll see an exercises on this, with threshold on δ.)

Be cautious! Some threats are not credible when the retaliation is more costly than just accepting the unilateral deviation f. (ex: Bertrand duopoly)

A "good" strategy in prisoner's repeated dilemma is strategy "Tit for tat" which balances cooperation and retaliation.

Bayesian games

Game where the information of at least one player is incomplete, but each player has some prior probability distribution he believes the lacking information to follow.
The information in question concerns in general:

- the player(s)' payoffs in the different possible outcomes (possibly including one's own payoff),
- the beliefs of the other players regarding those payoffs,
- and/or the available strategies of the other players.

Can be formalized.

Next slides: an illustration.

The value of information

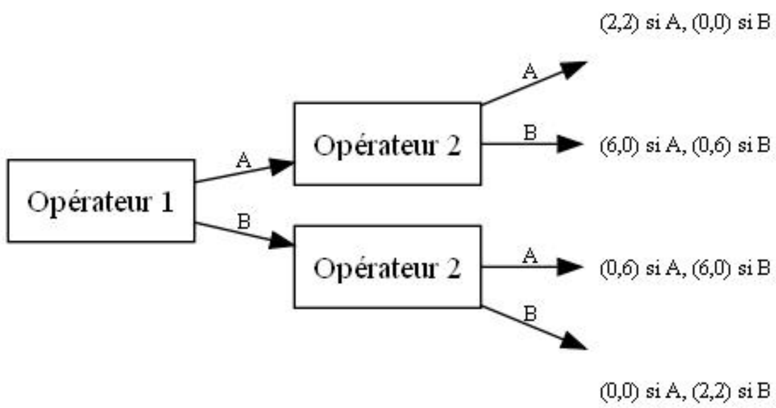
We consider a coordination game on an innovating service.
We assume that Operator 1 invests first in a service (A or B), then Operator 2 invests in A or B knowing the choice of Operator 1 (thus Operator 1 is a leader).

We assume that users will finally adopt only one service, each one with probability 1/2.

The payoff of each operator depends on the final choice of users, that is only known in probability.

The value of information

Given the time difference between both player choices, the game can be represented as a tree:



We consider that each operator maximizes his expected payoff.
What is the equilibrium?

The value of information

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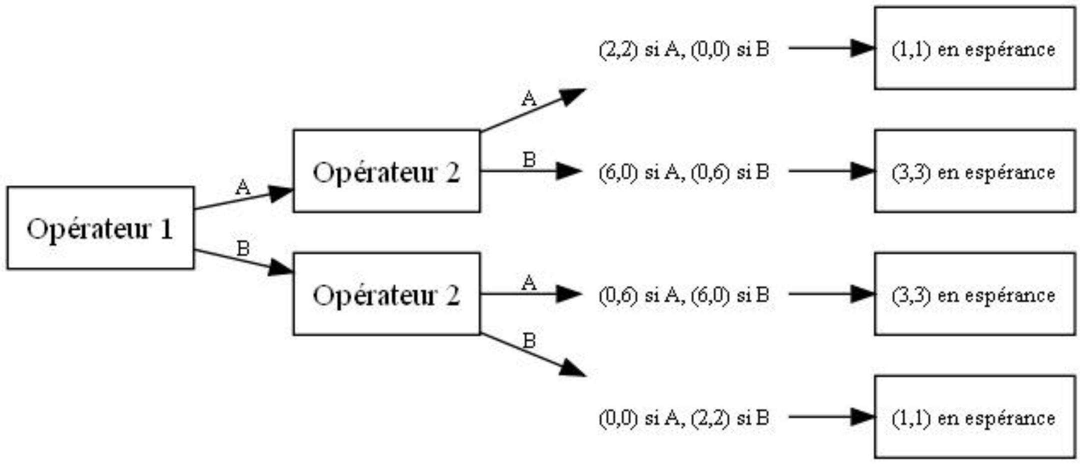
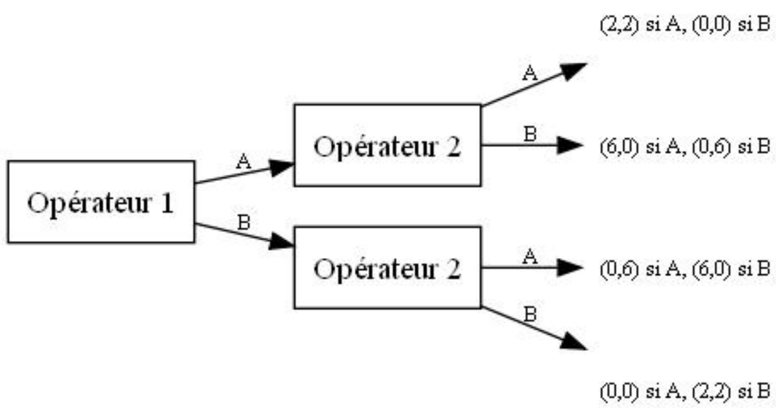


We consider that each operator maximizes his expected payoff.
What is the equilibrium? $\Rightarrow$Operator 2 maximizes his payoff by making a choice different from Operator 1.
Expected payoffs at equilibrium: $(3, 3)$

Incomplete information equilibrium

The same game is played, but Operator 1 knows the service that will be adopted, and Operator 2 knows that Operator 1 has that information:



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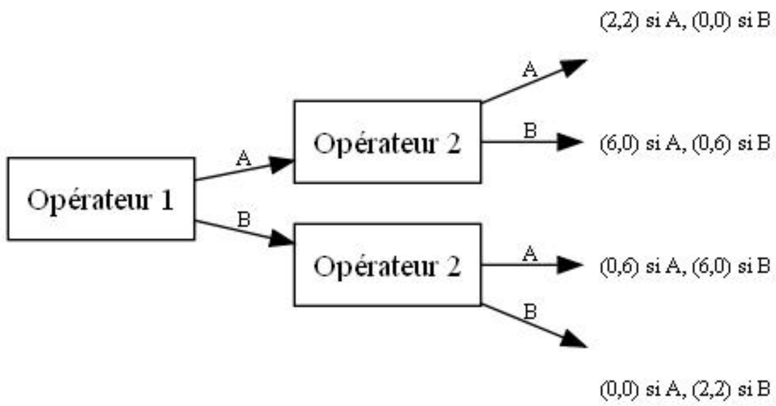
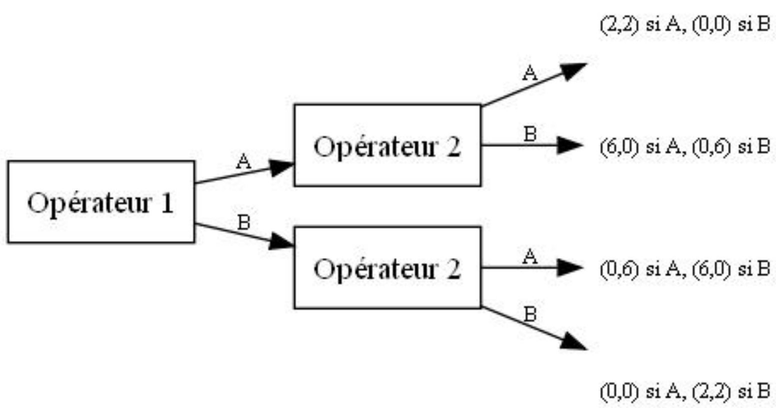


Expected payoffs at equilibrium: $(2,2) < (3,3)$

Incomplete information equilibrium

The same game is played, but Operator 1 knows the service that will be adopted, and Operator 2 knows that Operator 1 has that information:



Expected payoffs at equilibrium: $(2,2) < (3,3)$
Particular case when information has a negative value!

Other types of game (not studied here)

- Potential games: utility function have a specific structure, derived from a potential function
- Coalitional games: players decide if they should create a coalition and with who to cooperate
  - avoids the effects of competition
  - Question of how to share the "revenue" to be decided then.
- Evolutionary games:
  - comes from biology
  - studies the evolution of some characteristics within an ecosystem
  - notion of "evolutionary stable strategy", stable when there is some small deviation/mutation.

Things to remember

1. Selfishness **does** need to be taken into account in telecommunication networks.
2. The Nash equilibrium is a notion that helps predict the possible rational outcomes of a game.
3. It is often not Pareto-optimal, and different from the social optimum.
4. The outcome of the game strongly depends on the information that each player has.
5. The social optimum may be reached by changing the rules of the game (e.g., via additional payments)
6. However, designing a mechanism with a given set of desirable properties is not always doable (as will be seen later).

Exercise 1. Quizz on repeated games

A few questions for you to check that you have understood the notions of the course.

The questions proposed here are taken from the MOOC "Game Theory" on Coursera, created by Matthew Jackson, Kevin Leyton-Brown and Yoav Shoham. Many other interesting resources are available in that MOOC, including full lecture videos

Question 1 Consider a repeated game such that with probability p the game continues to the next period and with probability $(1-p)$ it ends. The game starts in period 1 and in odd periods both players play L and in even periods both players play R. The stage game payoffs are listed below

1 2	L	R
L	3,3	-1,4
R	4,-1	1,1

What is the expected total future payoff (starting at the beginning of the game) for each player, when the game is forecast to be played as described as above:

a). $3 + 3p + 3p^2 + 3p^3 + ...$

b). $4 - 1p + 4p^2 - 1p^3 + ...$

c). $3 + 1p + 3p^2 + 1p^3 + ...$

d). $4 + 3p + 4p^2 + 3p^3 + ...$

Question 2 Consider the rock-paper-scissors game:

1 2	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

How many elements are there in H^2 (the set of histories of two plays of the game):

a). 2^3

b). 9^2

c). 3^2

d). 3^3

Question 3 Consider the following two-player game:

Player 1 Player 2	Movie	Home
Movie	3,0	1,2
Home	2,1	0,3

Which per-period payoff is not enforceable:

a). (0,3)

b). (3,0)

c). (2,1)

d). All of above.

Exercises on repeated games

Patrick Maillé and Bruno Tuffin

Exercise 1 (Taken from David K. Levine) *Consider the following 2-player game with output matrix:*

	a	b	c	d
A	$3,4$	$3,2$	$-1,5$	$0,1$
B	$1,3$	$2,4$	$0,3$	$0,1$
C	$4,0$	$3,1$	$1,2$	$0,1$
D	$0,1$	$0,1$	$0,1$	$-1,0$

1. *For the normal form of the game (one-shot), compute the best responses and Nash equilibria, if any.*
2. *If Player 1 plays first, what is the equilibrium strategy (Stackelberg game)?*
3. *Idem of Player 2 is the leader. Note: if there is a tie, we assume the Stackelberg follower plays the strategy most favorable to the leader.*
4. *Compute the Minmax payoffs for both players.*
5. *Does profile (A,a) strictly Pareto dominates the Nash eq.?*
6. *For a repeated game: Given a strategy profile that strictly Pareto dominates the static Nash equilibrium, you are expected to be able to construct a grim-strategy equilibrium that sustains that outcome on the equilibrium path. Simple: play first "how expected" and continue like this as long as nobody deviates. If someone deviates, play as in the Nash eq. then.*
   - (a) *What are the average payoffs of players always play (A,a)?*
   - (b) *What is the best response of each player given that the other is playing A or a?*
   - (c) *Compute the payoff of players if they deviate at the first step and the other applies the grim strategy*

(d) *For which value of δ does the grim strategy lead to a Nash equilibrium of the repeated game?*

Exercise 2 *We come back to the Prisonner's dilemma problem. Apply also the grim (retaliation) strategy to determine the value δ^* such that for $\delta > \delta^*$ the two suspects will collaborate. Interpret it.*

Exercise 3 *Let there be 2 depositors each with a deposit of $D > 0$ in a bank. The bank has invested the deposit of both depositors in a project which can be liquidated in the short- or long-run. If the project is liquidated in the short run the project is worth $2r$, while a liquidation in the long run yields a dividend of $2R$. We assume that $r < D < R$. The game proceed as follows:*

Stage 1:

- *if both withdraw their deposit, the bank must liquidate and both gets a payoff of r, and the game ends*
- *if one withdraw her deposit, but the second keeps the deposit, the bank must liquidate and the withdrawer gets D, and the other gets $2r - D$, and the game ends*
- *if neither withdraws, the game proceed to the next stage*

Stage 2

- *if both withdraw their deposit, the bank must liquidate and both gets a payoff of R, and the game ends*
- *if one withdraw her deposit, but the second keeps the deposit, the bank must liquidate and the withdrawer gets D, and the other gets $2R - D$, and the game ends*
- *if neither withdraws, their deposit is now R each*

Determine the Nash equilibria of the game.

Mechanism design and auctions

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Outline

Definition of mechanism design

A decision maker (e.g., a network owner or a regulator) wants to optimize some global objective.

A mechanism is a set of rules chosen by the decision maker to that end:

- a set of available strategies for each agent,
- an outcome rule, that maps the strategy profiles of agents to an outcome (ex: allocation of resource).

The rules have to be designed so that a game played by selfish agents "naturally" reaches the expected outcome.

- We are in a Stackelberg situation, where the leader is the *designer* of the game played by the followers.
- We are also in the framework of Bayesian games: all agents reason on a common prior distribution of the *types* of the others.

Direct and indirect mechanisms

The difficulty in mechanism design:

Direct and indirect mechanisms

The difficulty in mechanism design:

- Designer ignores the preferences of agents (followers)

  agent i's preferences are characterized by its *type* $\theta_i \in \Theta_i$

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- Agents act to maximize their own utility

  agent i's strategy depends on its *type*: denote it by $s_i(\theta_i)$

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Direct mechanism: agents are asked to reveal their type
(one-shot simultaneous declaration, strategy space for i is Θ_i)

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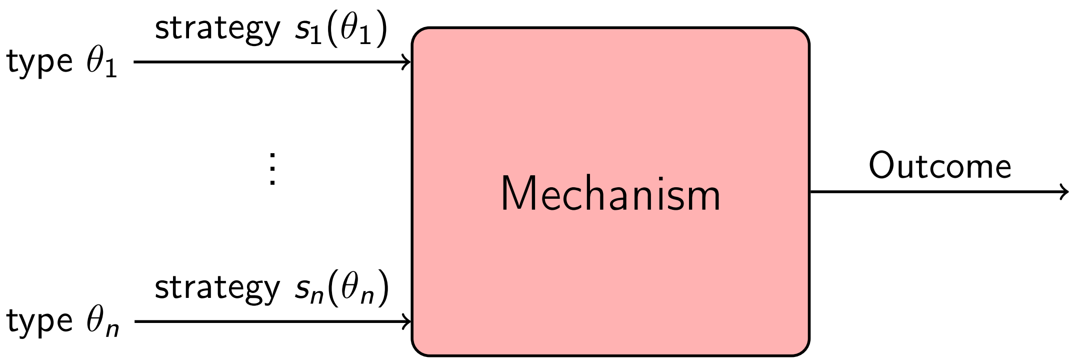
Direct mechanism: agents are asked to reveal their type
(one-shot simultaneous declaration, strategy space for i is Θ_i)

Indirect mechanism: any other mechanism
(can involve several steps, any strategy space)

High-level view of a mechanism



Imposing “rules" helps to reach the social optimum. Ex: pricing

Consider the following game:
N states negotiate their pollution levels $s_i \geq 0$.
State i utility is assumed to be $u_i(s_1, \ldots s_N) = \ln(s_i) - \sum_{j=1}^{N} s_j$.

Nash equilibrium:

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Nash equilibrium:

$$\frac{1}{s_i} - 1 = 0$$

$$\text{so that } (s^*) = \vec{1}$$

Social optimum for this example

We maximize $\sum_{i=1}^{N} u_i(s) = \sum_{j=1}^{N} \ln(s_j) - N \sum_{j=1}^{N} s_j$
So that the social optimum is

$$\frac{1}{\bar{s}_i} - N = 0$$

$$(\bar{s}) = \frac{\vec{1}}{N}$$

Meaning that the Nash equilibrium is inefficient with respect to the social optimum.

Pricing/taxing to reach the optimum

We impose a tax of rate t on the level of pollution. Benefits from the tax is uniformly distributed over all the states. Now each state maximizes:

$$u_i(s_1, \dots s_N) = \ln(s_i) - ts_i - \sum_{j=1}^{N} s_j + \frac{1}{N}\sum_{j=1}^{N} ts_j$$
$$= \ln(s_i) - ts_i - (1 - \frac{t}{N})\sum_{j=1}^{N} s_j$$

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The optimal strategy becomes: $\frac{1}{s_i} - t - (1 - \frac{t}{N}) = 0$

$$\text{so that } (s_i^*)(t) = \frac{\vec{1}}{1+\frac{N-1}{N}t}$$

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$$\text{so that } (s_i^*)(t) = \frac{\vec{1}}{1+\frac{N-1}{N}t}$$

meaning that we reach the social optimum if we choose the tax level to be $t^* = N$

Illustration of pricing interest

Courcoubetis & Weber, 2003

- User i buying a service quantity x_i at unit price p.
- $u_i(x_i, y)$ utility for using quantity x_i, where $y = \sum_j x_j/k$ with k resource capacity.
- u_i assumed decreasing in y: negative externality because of congestion.
- Net benefit of user i:
$$u_i(x_i, y) - px_i$$
- Benefit of provider: $p\sum_i x_i - c(k)$.
- Social welfare: sum of benefits of all actors in the game (provider + users):
$$SW = \sum_i u_i(x_i, y) - c(k).$$
- Optimal SW determined by maximizing over $x_1, \ldots; x_n$. Leads to (by differentiating over each x_i)
$$\frac{\partial u_i(x_i^*, y^*)}{\partial x_i} + \frac{1}{k}\sum_j \frac{\partial u_j(x_j^*, y^*)}{\partial y} = 0 \quad \forall i.$$

Illustration of pricing interest (2)

Courcoubetis & Weber, 2003

- Define the price as the marginal decrease in SW due to a marginal increase in congestion, at the SW optimum,
$$p_E = -\frac{1}{k} \sum_j \frac{\partial u_j(x_j^*, y^*)}{\partial y}$$
(positive thanks to the decreasingness of u_i in y)
- With this price, a user acting selfishly tries to optimize his net benefit
$$\max_{x_i} u_i(x_i, y) - p_E x_i.$$
- Differentiating with respect to x_i, this gives
$$\frac{\partial u_i}{\partial x_i} + \frac{1}{k}\frac{\partial u_i}{\partial y} - p_E = 0$$
- For a large n, assuming $\left|\frac{\partial u_i}{\partial y}\right| << \left|\sum_j \frac{\partial u_j}{\partial y}\right|$, we get approximately the same system of equations than when optimizing SW.
- Pricing can therefore help to drive to an optimal situation.

Relevant (desirable) properties when designing a mechanism

- Individual rationality: ensures that participating to the game will give non-negative utility
- Incentive compatibility or truthfulness: players' best interest is to declare their real preferences
(Possibilities for "best": dominant strategy, or just equilibrium)
- Efficiency: mechanism results in a maximized sum of utilities
- Budget Balance: sum of money exchanged is null
- Decentralized: decentralized implementation of the mechanism
- Collusion robustness: no incentive to collusion among players

A question we will ask:
Is there a mechanism verifying the whole set or a given set of properties?

The Revelation Principle

Theorem

Any outcome that can be attained (as an equilibrium) by any mechanism can also be attained by a **direct**, **truthful** *mechanism*

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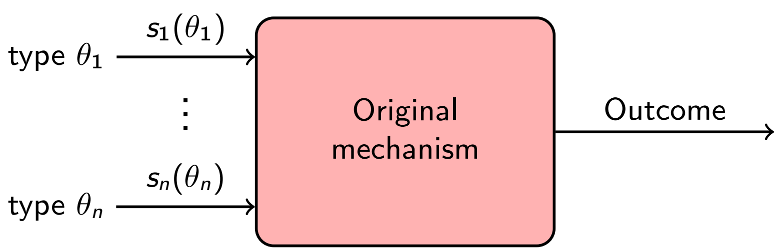

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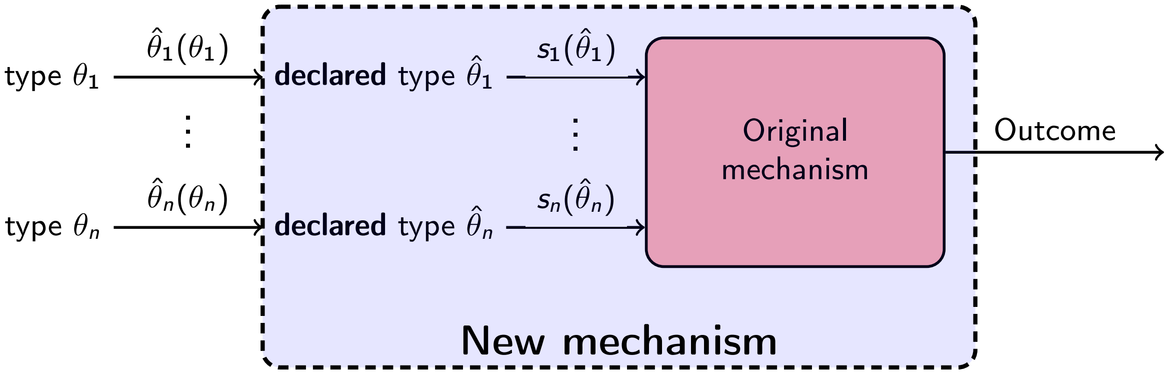

Outline

1 Mechanism design

2 Auctions

Auctions: definition

Definition (McAfee and McMillan)

An auction is a market institution with an explicit set of rules determining resource allocation and prices on the basis of bids from the market participants.

Three ingredients:

- under what form the participants will be able to submit **bids** (in most cases the amount of money that the participant is willing to pay for some good);
- what **allocation rule** to apply;
- and finally what **pricing rule** to implement.

A lot of applications in telecommunications.

- Spectrum auctions at the operator level
- advertisements (web pages, search engines)
- Ebay, etc.

First-price auctions

Consider a single *indivisible* item (ex: a license to transmit on a specific radio frequency band).

In a *first-price auction*

- Each buyer i is asked to submit a (sealed) bid with a price b_i
- The item is awarded to the highest bidder
- He pays his bid, while losing bidders do not pay anything.

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Under this scheme no buyer has an interest in bidding above his valuation, but given his knowledge of other buyers, there might be an incentive to *under-bid*.

Iterative open auctions

Auctions with *unsealed* (open) bids, that each bidder can update over time.

- Ascending auctions (also called English auctions): buyers iteratively increase their bid until only one interested buyer remains, the item being allocated to that bidder at his final bid price.
- Descending auctions (also called Dutch auctions): a large starting price is advertised, and is progressively decreased until one buyer claims to be interested, and gets the item at the currently advertised price.

- In ascending auctions, the highest-valuation bidder avoids revealing his true valuation (bidding just above the second bid);
- In descending auctions, like in first-price auctions, some knowledge of others' valuations can be used.

Second-price (or Vickrey) auctions

- Each buyer i is asked to submit a (sealed) bid with a price b_i
- The item is awarded to the highest bidder
- but the price charged equals the *second-highest* bid value.

Theorem

Bidding one's real valuation is a dominant strategy.

- over-bidding creates the risk of paying more than one's valuation
- under-bidding leads to the risk of losing the auction and getting utility 0 in some cases when bidding truthfully would have led to a strictly positive utility.

Theorem (Revenue-equivalence theorem)

Consider two auction mechanisms such that

- *bidders are risk-neutral;*
- *bidder valuations are independently distributed over a given interval, with a finite and strictly positive density;*
- *the bidder with the lowest possible valuation expects a null utility;*
- *the bidder with the highest valuation always wins the item.*

Then both schemes yield the same expected revenue to the seller **at equilibrium**, *and each bidder gets the same utility.*

With the definition

Definition (Risk neutrality)

Given the choice between a guaranteed return r and a gamble with expected return also equal to r, the bidder is completely indifferent. If preferring s the guaranteed return is said to be "risk-averse".

Vickrey-Clarke-Groves (VCG) auctions

1. each player i declares his whole valuation function V_i (define $\hat{V}_i$ the declared valuation);
2. the allocation $\mathbf{a}^{\mathrm{VCG}}$ is chosen among those that maximize *social welfare* with declared valuations:
$$\mathbf{a}^{\mathrm{VCG}} \in \arg\max_{\mathbf{a}} \sum_{i\in\mathcal{N}} \hat{V}_i(\mathbf{a});$$
3. Price p_i^{VCG} charged equals to i: *social opportunity cost*, i.e. the total loss of (declared) value his presence imposes on the others:
$$p_i^{\mathrm{VCG}} = \underbrace{\max_{\mathbf{a}_{-i}} \sum_{j\in\mathcal{N}\setminus\{i\}} \hat{V}_j(\mathbf{a}_{-i})}_{:=h_i(\hat{\mathbf{V}}_{-i})} - \sum_{j\in\mathcal{N}\setminus\{i\}} \hat{V}_j(\mathbf{a}^{\mathrm{VCG}}).$$

Theorem

Playing truthfully is a dominant strategy. Only mechanism to be jointly incentive compatible, individually rational and efficient.

VCG and Second-price auctions

For a single item VCG and second-price auctions are exactly the same.

VCG and Second-price auctions

For a single item VCG and second-price auctions are exactly the same.
If i is the winner

1. For $j \in \mathcal{N} \setminus \{i\}$, $V_j(\mathbf{a}^{\mathrm{VCG}}) = 0$ because getting nothing, so that $\sum_{j\in\mathcal{N}\setminus\{i\}} \hat{V}_j(\mathbf{a}^{\mathrm{VCG}}) = 0$.
2. If i (the winner) not here, the item is allocated to the second highest bidder, and the other get nothing, so that $\max_{\mathbf{a}_{-i}} \sum_{j\in\mathcal{N}} \hat{V}_j(\mathbf{a}_{-i})$ is the valuation of the second highest bidder
3. We thus end up with the valuation of the second highest bidder.

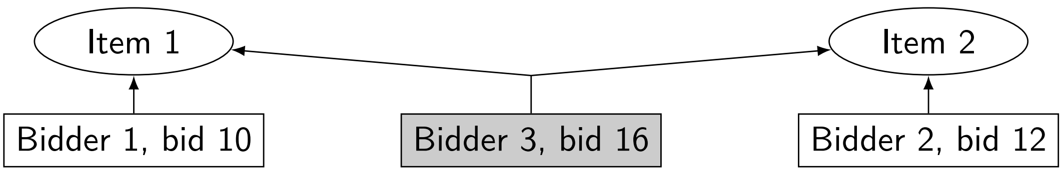
We will see how it behaves for more than one item.

Combinatorial auctions

Bid for bundles of items.

- Computationally challenging: the number of combinations grows exponentially with the number of items.

Example: Bids and corresponding VCG allocations and prices (exercise).



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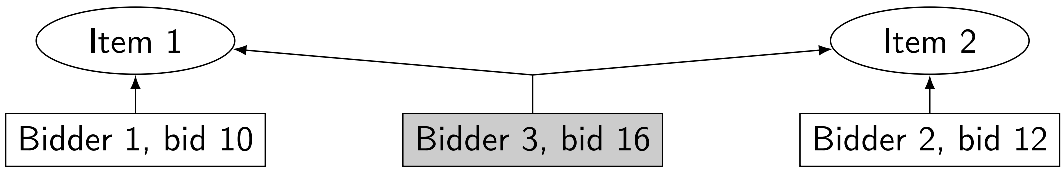

gets Item 1, pays $p_1^{VCG} = 4$ *gets no item* *gets Item 2, pays* $p_2^{VCG} = 6$

Bidder 3 gets nothing while the highest bidder.

Double-sided auctions

Auctions where both sellers and buyers are participants.

- sellers declare the minimum price they are willing to be paid to provide (possibly different quantities of) resource
- buyers reveal the maximum price they are willing to pay (possibly for different quantities).

A matching scheme has to be defined.

Double-sided VCG auction for a single good

- Single indivisible good produced by a seller (Player 0) for a cost $C > 0$ (or equivalently, a valuation $V_0 = -C$);
- n buyers. Buyer i valuation V_i, and assume $V_1 > V_2 > ... > V_n$.

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- *Seller revenue*:

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- *Seller revenue*: Only Buyer 1 affected by the absence of the seller and would lose V_1 . Therefore the seller should receive a payment of $-p_0^{\text{VCG}} = V_1$.
- *Buyer* 1 *payment*:

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- *Buyer* 1 *payment*: In the absence of Buyer 1, good allocated to Buyer 2 if $V_2 \geq C$, and not produced if $C > V_2$ (the seller then saving C, no buyer being affected). As a result, Bidder 1 should pay $p_1^{\text{VCG}} = \max(V_2, C)$.

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But not budget-balanced: $\sum_{i=0}^{n} p_i = p_0 + p_1 = \max(V_2, C) - V_1 < 0$.

Exercises on auctions

Patrick Maillé and Bruno Tuffin

Exercise 1 *We consider a given set* $\{1, ..., n\}$ *of bidders interested by an indivisible good to be auctioned. The value that the good has to bidder* i *(his willingness-to-pay) is* v_i*. We assume that* $v_1 > v_2 > ... > v_n$*. Each player* i *is asked to submit a bid* b_i*; the highest bidder obtains the good and is charged depending on the pricing rule selected.*

Formulate a first price auction as a strategic game and analyze its Nash equilibria. In particular, show that in all equilibria the player with the highest valuation obtains the object.

Assumption: in the event of a tie for the highest bid the winner is the player with the lowest index.

Exercise 2 (Second-price auction) *In a second price auction the payment that the winner makes is the highest bid among those submitted by the players who do not win (so that if only one player submits the highest bid then the price paid is the second highest bid).*

Show that in a second price auction the bid v_i *of any player* i *is a weakly dominant action: player* i*'s payoff when he bids* v_i *is at least as high as his payoff when he submits any other bid, regardless of the actions of the other players.*

Exercise 3 (From Felix Munoz-Garcia, Washington State University) *Consider an auction with five participants, each of them with the following (privately observed) valuation of the object for sale: Person A ($10), Person B ($6), Person C ($45), Person D ($81), and Person E ($62).*

1. *If the seller organizes a second-price auction, who will be the winner? What will be his winning bid? What price he will pay for the object?*

2. *Suppose now that bidders can observe each other's valuations, but the seller cannot. The seller, however, only knows that bidder's valuations are in the range* $\{0, 1, \ldots, \$90\}$*. If players participate in a first-price auction, who will be the winner? What is his winning bid?*

Exercise 4 (From Felix Munoz-Garcia, Washington State University) *Consider a third-price auction, where the winner is the bidder who submits the highest bid, but he/she only pays the third highest bid. Assume that you compete against two other bidders, whose valuations you are unable to observe, and that your valuation for the object is $10. Show that bidding above your valuation (with a bid of, for instance, $15) can be a best response to the other bidder's bid, while submitting a bid that coincides with your valuation ($10) might not be a best response to your opponent's bids.*

Exercise 5 *Suppose you are one of two bidders in a first price sealed bid auction for Super Bowl tickets. Each bidder (including you) knows its own value for the Super Bowl tickets, but not the values of the other bidders. All a bidder knows about the values of the other bidder is that it is uniformly distributed between $200 and $300. Your value for the tickets is $280. What is the optimal bid that you should place (in a risk-neutral case)?*

Next exercises on auctions are from the book Networks, Crowds, and Markets: Reasoning about a Highly Connected World. By David Easley and Jon Kleinberg. Cambridge University Press, 2010.

Exercise 6 *In this question we will consider an auction in which there is one seller who wants to sell one unit of a good and a group of bidders who are each interested in purchasing the good. The seller will run a sealed-bid, second-price auction. Your firm will bid in the auction, but it does not know for sure how many other bidders will participate in the auction. There will be either two or three other bidders in addition to your firm. All bidders have independent, private values for the good. Your firm's value for the good is c. What bid should your firm submit, and how does it depend on the number of other bidders who show up? Give a brief (1-3 sentence) explanation for your answer*

Exercise 7 *In this problem we will ask how the number of bidders in a second-price, sealed-bid auction affects how much the seller can expect to receive for his object. Assume that there are two bidders who have independent, private values v_i which are either 1 or 3.*

For each bidder, the probabilities of 1 and 3 are both $1/2$. *(If there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.)*

1. *Show that the seller's expected revenue is* $3/2$.

2. *Now let's suppose that there are three bidders who have independent, private values v_i which are either 1 or 3. For each bidder, the probabilities of 1 and 3 are both* $1/2$. *What is the seller's expected revenue in this case?*

3. *Briefly explain why changing the number of bidders affects the seller's expected revenue.*

Exercise 8 *In this problem we will ask how much a seller can expect to receive for his object in a second-price, sealed-bid auction. Assume that all bidders have independent, private values v_i which are either 0 or 1. The probability of 0 and 1 are both* $1/2$.

1. *Suppose there are two bidders. Then there are four possible pairs of their values (v_1, v_2):* $(0,0); (0,1), (1,0), (1,1)$. *Each pair of values has probability* $1/4$. *Show that the seller's expected revenue is* $1/4$. *(Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.)*

2. *What is the seller's expected revenue if there are three bidders?*

3. *This suggests a conjecture that as the number of bidders increases the seller's expected revenue also increases. In the example we are considering the seller's expected revenue actually converges to 1 as the number of bidders grows. Explain why this should occur. You do not need to write a proof; an intuitive explanation is fine.*

Exercise 9 *A seller will run a second-price, sealed-bid auction for an object. There are two bidders, a and b, who have independent, private values v_i which are either 0 or 1. For both bidders the probabilities of both values are* $1/2$. *Both bidders understand the auction, but bidder b sometimes makes a mistake about his value for the object. Half of the time his value is 1 and he is aware that it is 1; the other half of the time his value is 0 but occasionally he mistakenly believes that his value is 1. Let's suppose that when b's value is 0 he acts as if it is 1 with probability* $1/2$ *and as if it is 0 with probability* $1/2$. *So in effect bidder b sees value 0 with probability* $1/4$ *and value 1 with probability* $3/4$. *Bidder a never makes mistakes about his value for the object, but he is aware of the mistakes that bidder b makes. Both bidders bid optimally given their perceptions of the value of the object. Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.*

1. *Is bidding his true value still a dominant strategy for bidder a? Explain briefly*

2. *What is the seller's expected revenue? Explain briefly*

Exercise 10 *In this question we will consider the effect of collusion between bidders in a second-price, sealed-bid auction. There is one seller who will sell one object using a second-price sealed-bid auction. The bidders have independent, private values drawn from a distribution on* $[0, 1]$. *If a bidder with value* v *gets the object at price* p, *his payoff is* $v - p$*; if a bidder does not get the object his payoff is 0. We will consider the possibility of collusion between two bidders who know each others' value for the object. Suppose that the objective of these two colluding bidders is to choose their two bids as to maximize the sum of their payoffs. The bidders can submit any bids they like as long as the bids are in* $[0, 1]$.

1. *Let's first consider the case in which there are only two bidders. What two bids should they submit? Explain.*

2. *Now suppose that there is a third bidder who is not part of the collusion. Does the existence of this bidder change the optimal bids for the two bidders who are colluding? Explain.*

Exercise 11 *In this problem we will ask how irrational behavior on the part of one bidder affects optimal behavior for the other bidders in an auction. In this auction the seller has one unit of the good which will be sold using a second-price, sealed-bid auction. Assume that there are three bidders who have independent, private values for the good,* v_1, v_2, v_3, *which are uniformly distributed on the interval* $[0, 1]$.

1. *Suppose first that all bidders behave rationally; that is they submit optimal bids. Which bidder (in terms of values) wins the auction and how much does this bidder pay (again in terms of the bidder's values)?*

2. *Suppose now that bidder 3 irrationally bids more than his true value for the object; in particular, bidder 3's bid is* $(v_3 + 1)/2$. *All other bidders know that bidder 3 is irrational in this way, although they do not know bidder 3's actual value for the object. How does this affect the behavior of the other bidders?*

3. *What effect does bidder 3's irrational behavior have on the expected payoffs of bidder 1? Here the expectation is over the values of* v_2 *and*

v_3 which bidder 1 does not know. You do not need to provide an explicit solution or write a proof for your answer; an intuitive explanation of the effect is fine. [Remember a bidder's payoff is the bidder's value for the object minus the price, if the bidder wins the auction; or 0, if the bidder does not win the auction.]

Exercise 12 *In this problem we will ask how much a seller can expect to receive for his object in a second-price, sealed-bid auction. Assume that there are two bidders who have independent, private values v_i which are either 1 or 2. For each bidder, the probabilities of each value is $1/2$. Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x. We also assume that the value of the object to the seller is 0.*

1. *Show that the seller's expected revenue is $5/4$.*

2. *Now let's suppose that the seller sets a reserve price of R with $1 < R < 2$: that is, the object is sold to the highest bidder if her bid is at least R, and the price this bidder pays is the maximum of the second highest bid and R. If no bid is at least R, then the object is not sold, and the seller receives 0 revenue. Suppose that all bidders know R. What is the seller's expected revenue as a function of R?*

3. *Using the previous part, show that a seller who wants to maximize expected revenue would never set a reserve price, R, that is more than 1 and less than 1.5.*

Exercise 13 *Consider the following example of the VCG mechanism. The government is proposing a new policy with three possible alternatives, X, Y and Z. There are three lobbying groups which have submitted bids on the three alternatives. If we run VCG on these bids, what alternative is chosen and what prices are charged?*

	X	Y	Z	
Lobby 1	12	9	2	paie 3
Lobby 2	5	8	10	paie 2
Lobby 3	4	5	6	paie 0

Exercise 14 *Consider a combinatorial auction with two items, A and B, and three bidders. The first bidder has valuation 1 for receiving both items ($v_1(AB) = 1$) and 0 otherwise. The 2nd bidder has valuation 1 for item A*

($v_2(AB) = 1$ and $v_2(A) = 1$) and 0 otherwise. The 3rd bidder has valuation 1 for B, $v \geq 1$ for AB and 0 otherwise.

Determine the allocation and prices in terms of v when a VCG mechanism is applied.

Adword auctions, GSP/VCG mechanisms

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Outline

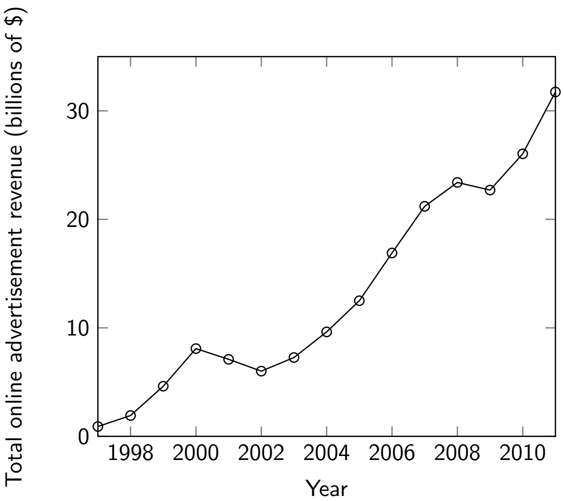
Illustrating the intertwining of economic roles: Google

- Created in 1998
- Search engine: software on a web page giving as output a list of documents corresponding to keywords
- Based on a powerful algorithm called PageRank
  - assigns weights to documents to represent their importance
  - quick and efficient: quasi-monopoly in some countries and 92% of searches worldwide in June 2021.
- How do they make money? Sponsored links: selling slots *declared* as ads
- Business extended:
  - Google AdSense
  - Gmail, Google Calendar, Google Maps, YouTube Google News, Chrome, Android...
  - Most are free, but ads and/or search tools can sometimes be placed
- Other reason? Ex. Android:
  - Operating system industry more competitive, hence more demand and more searches/ads.
  - Better control of the whole supply chain, Google more likely to be used
  - Indirect: gain from Google Play (Google application store for Android).

Outline

On the Internet, most content is free

- Advertisement a way to yield a return on investment for content producers.
  - incorporated banner or video
  - pop-up: new page open in front of the current one
  - Interstitial ads: full page appearing before the expected content for some time.
- Becoming targeted using users' preferences (cookies: files stored recording the history of your browsing)
- Evolution of advertisement revenue in the USA:



Advertisement where?

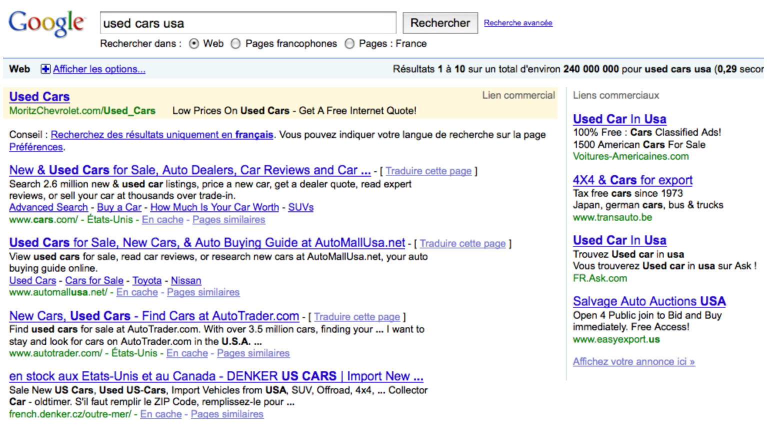
- Banners on web pages

- (Online) games

- Videos placed before or within content

Search engines: Introduction to adword auctions

- Search engines play a crucial role in the Internet.
- Revenue through advertising slots, usually displayed at the top or right of the search page.
- Advertisers submit bids for relevant keywords only.



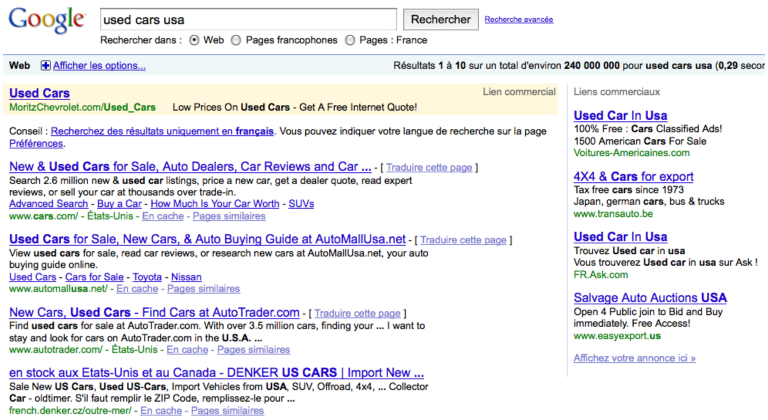
- Allocation of slots thanks to adword auctions.
  - "search and other" revenue of Google in 2020: $104 billion
  - 71% of Google advertising revenue.

Auction principle (single keyword, K slots)

- Advertisers submit bids for specific keywords.
- Each time there is a search on that keyword:
  - advertisers are ranked and allocated slots according to a prespecified criterion:
    - bid value (initially for Yahoo!)
    - the revenue they will generate (more or less Google), taking into account the (learned) click-through rate (CTR).
  - Possible payment rules:
    - *Pay-Per-Impression* (PPI): advertisers charged every time their ad is displayed
    - *Pay-Per-Click* (PPC): advertisers is charged only when the ad is clicked
    - *Pay-Per-Transaction* (PPT): advertisers charged when the click results in a real sell.
  - Amount to be paid each time?
    - First Price: advertisers pay their bid
    - Generalized Second Price (GSP): they pay the bid/revenue of advertiser below them in the ranking
    - Vickrey-Clarke-Groves (VCG) auctions: you pay the opportunity cost that your presence introduce to all other advertisers.
- In use: PPC and GSP.

More details on auctions for ads

- n advertisers, $k(< n)$ advertisement slots
- v_i: valuation of Advertiser i for the "considered action" (impression, click, sale): maximum price i is willing to pay
- b_i: bid submitted by i (not necessarily equal to v_i)
- $\mathbf{b} = (b_1, \ldots, b_n)$ bid profile
- But the k slots do not have the same probability to be looked at



  Ads at the top more seen than those at the bottom.
- Different interest for n advertisers depending on the search too.

Click-Through-Rate (CTR)

Definition (Click-Through-Rate)

Probability that a given ad will be clicked when displayed.

CTR $w_{i,s}$ for advertiser i at slot s.

CTR often assumed separable:

$$w_{i,s} = q_i \theta_s$$

- q_i: attractiveness of Advertiser i
- θ_s: probability that a user considers the ad on slot s. (Slots ordered $\theta_1 \geq \cdots \geq \theta_n$.)

How to rank for slot allocations?

- At the beginning (Yahoo!), ranked according to bids
- More efficient (Google): rank according to $w_{i,s}b_i$. More exactly
  1. Ranked first: the advertiser maximizing $w_{i,1}b_i$
  2. Ranked second: advertiser maximizing $w_{i,2}b_i$ (excluding the first)
  3. ...
- Generalizes ranking per bid: just consider $w_{i,s} = 1\ \forall i, s$
- If the charge is p_s at the s-th slot,revenue generated with a pay-per-click scheme: $\sum_{s=1}^{k} w_{(s)s}p_s$,
  with (s) the s-th ranked advertiser.

Charging rule: GSP

- First price auction could be considered
- ... or VCG, but
- In practice GSP: Generalized Second-Price

Definition (Generalized Second-Price)

you pay not what you have declared, but a price equivalent to the minimum bid to maintain your position in the ranking

Explicitly: :

- if we rank by bid, the winner of slot $s \leq k$ is charged $b_{(s+1)}$, because bidding less would mean losing the s-th slot.
- If we rank by revenue,
  - revenue associated to slot s: $w_{(s)s}p_s$
  - under the separability assumption, price p_s charged such that bidding less than p_s would make you lose the slot : $q_{(s)}\theta_s p_s \geq q_{(s+1)}\theta_s b_{(s+1)}$. This gives
  $$p_s = b_{(s+1)}\frac{q_{(s+1)}}{q_{(s)}}.$$
  - Intuition: some advertisers with low CTR q would generate a low revenue even if their bids are high.

Example: $k = 3$ slots, $n = 5$ advertisers with $\theta_1 = 1/2$, $\theta_2 = 1/4$ and $\theta_3 = 1/5$

Advertiser i	Bid b_i	CTR q_i	
1	10	0.05	
2	9	0.1	
3	6	0.12	
4	5	0.15	
5	4	0.2	

- Ranking per bid:
- Ranking per revenue:

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- Ranking per bid:
  - The three slots are allocated to first three advertisers
  - $p_1 = b_2 = 9, \; p_2 = b_3 = 6, \; p_3 = b_4 = 5$
  - Expected revenue $\sum_{s=1}^{3} \theta_s q_{(s)} p_s = \sum_{s=1}^{3} \theta_s q_{(s)} b_{(s+1)} = \frac{1}{2}0.45 + \frac{1}{4}0.6 + \frac{1}{5}0.6 = 0.52.$
- Ranking per revenue:

Example: $k = 3$ slots, $n = 5$ advertisers with $\theta_1 = 1/2$, $\theta_2 = 1/4$ and $\theta_3 = 1/5$

Advertiser i	Bid b_i	CTR q_i	Product $b_i q_i$
1	10	0.05	0.5
2	9	0.1	0.9
3	6	0.12	0.72
4	5	0.15	0.75
5	4	0.2	0.8

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  - The three slots are allocated to first three advertisers
  - $p_1 = b_2 = 9$, $p_2 = b_3 = 6$, $p_3 = b_4 = 5$
  - Expected revenue $\sum_{s=1}^{3} \theta_s q_{(s)} p_s = \sum_{s=1}^{3} \theta_s q_{(s)} b_{(s+1)} = \frac{1}{2}0.45 + \frac{1}{4}0.6 + \frac{1}{5}0.6 = 0.52.$
- Ranking per revenue:
  - Advertiser 2 is allocated the first slot, Advertiser 5 is allocated the second, and Advertiser 4 the third
  - $p_1 = 8$, $p_2 = 3.75$ and $p_3 = 4.8$ (with $p_s = b_{(s+1)} \frac{q_{(s+1)}}{q_{(s)}}$)
  - Revenue $\sum_{s=1}^{3} \theta_s q_{(s)} p_s = \sum_{s=1}^{3} \theta_s q_{(s+1)} b_{(s+1)} = \frac{1}{2}0.8 + \frac{1}{4}0.75 + \frac{1}{5}0.72 = 0.7315.$

GSP and VCG

Proposition

In the case of a single slot, VCG and GSP are equivalent.

VCG procedure:

- Maximizes the the declared valuation (bid) of the winner for the bid-based ranking
  - hence selects the largest bidder like GSP.
  - Price paid: loss of declared value, second highest bid;
- Maximizes the (declared) generated revenue for the revenue-based ranking
  - hence advertiser maximizing $q_i b_i$ like GSP.
  - *Total charge*: loss of declared revenue of other players, value $\theta_1 q_{(2)} b_{(2)}$. Idem.

But not true for more than one slot

Back to our previous example, focusing on revenue-based ranking:

- Allocations for VCG are the same
- Payments:

But not true for more than one slot

Back to our previous example, focusing on revenue-based ranking:

- Allocations for VCG are the same
- Payments:
  - For Advertiser 2, winner of the first slot: loss of (declared) revenue due to his presence:

$$b_3q_3(\theta_3 - 0) + b_4q_4(\theta_2 - \theta_3) + b_5q_5(\theta_1 - \theta_2)$$
$$= \frac{0.72}{5} + 0.75\left(\frac{1}{4} - \frac{1}{5}\right) + 0.8\left(\frac{1}{2} - \frac{1}{4}\right) = 0.3815,$$

But not true for more than one slot

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  - For Advertiser 5,winner of the second slot:

$$b_3q_3(\theta_3 - 0) + b_4q_4(\theta_2 - \theta_3) = 0.72\frac{1}{5} + 0.75\left(\frac{1}{4} - \frac{1}{5}\right) = 0.1815;$$

But not true for more than one slot

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- Payments:
  - For Advertiser 2, winner of the first slot: loss of (declared) revenue due to his presence:

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$$b_3q_3(\theta_3 - 0) + b_4q_4(\theta_2 - \theta_3) = 0.72\frac{1}{5} + 0.75\left(\frac{1}{4} - \frac{1}{5}\right) = 0.1815;$$

  - For Advertiser 4,winner of the third slot:

$$b_3q_3(\theta_3 - 0) = 0.72\frac{1}{5} = 0.144.$$

But not true for more than one slot

Back to our previous example, focusing on revenue-based ranking:

- Allocations for VCG are the same
- Payments:
  - For Advertiser 2, winner of the first slot: loss of (declared) revenue due to his presence:

$$b_3q_3(\theta_3 - 0) + b_4q_4(\theta_2 - \theta_3) + b_5q_5(\theta_1 - \theta_2)$$
$$= \frac{0.72}{5} + 0.75\left(\frac{1}{4} - \frac{1}{5}\right) + 0.8\left(\frac{1}{2} - \frac{1}{4}\right) = 0.3815,$$

  - For Advertiser 5,winner of the second slot:

$$b_3q_3(\theta_3 - 0) + b_4q_4(\theta_2 - \theta_3) = 0.72\frac{1}{5} + 0.75\left(\frac{1}{4} - \frac{1}{5}\right) = 0.1815;$$

  - For Advertiser 4,winner of the third slot:

$$b_3q_3(\theta_3 - 0) = 0.72\frac{1}{5} = 0.144.$$

- Expected revenue 0.707, less than the 0.7315 when using GSP.

Why GSP instead of VCG?

- GSP does not satisfy the incentive compatibility property in general (exercise)
  - VCG prices unique truthful prices
  - But at least verifies properties such as "every bidder allocated position s has no incentive to switch to positions $s-1$ or $s+1$ through a bid change";
- GSP more "complicated" in terms of strategy and resulting equilibrium
- But yields a larger expected revenue than VCG .

  Let $p_j^{(\mathrm{GSP})}$ and $p_j^{(\mathrm{VCG})}$ charges per click of GSP and VCG for slot j.

  Our induction assumption is $p_{s+1}^{(\mathrm{GSP})} \geq p_{s+1}^{(\mathrm{VCG})}$ (equal for the last slot).
  Then diff in charging between slots s and $s+1$ with VCG is the difference of opportunity costs between s and $s+1$:

$$\begin{aligned}
\theta_s q_{(s)} p_s^{(\mathrm{VCG})} - \theta_{s+1} q_{(s+1)} p_{s+1}^{(\mathrm{VCG})} &= b_{(s+1)} q_{(s+1)} (\theta_s - \theta_{s+1}) \\
&\leq b_{(s+1)} q_{(s+1)} \theta_s - b_{(s+2)} q_{(s+2)} \theta_{s+1} \\
&= \theta_s q_{(s)} p_s^{(\mathrm{GSP})} - \theta_{s+1} q_{(s+1)} p_{s+1}^{(\mathrm{GSP})}.
\end{aligned}$$

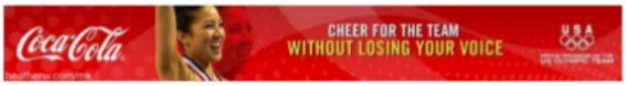
  Therefore $\theta_s q_{(s)} p_s^{(\mathrm{GSP})} \geq \theta_s q_{(s)} p_s^{(\mathrm{VCG})} + \theta_{s+1} q_{(s+1)} (p_{s+1}^{(\mathrm{GSP})} - p_{s+1}^{(\mathrm{VCG})}) \geq \theta_s q_{(s)} p_s^{(\mathrm{VCG})}$
- And payment rule simpler to understand.

Ads on web pages: per-click or pay-per-view?

- The previous analysis focusing on pay-per-click
- Pay-per-view: given cost per mile impressions
- In 2008, 39% of the ads were priced by pay-per-view, 57% by pay-per-click, and the rest by a mixture
- Google AdSense and Facebook propose advertisers to choose
- Pay-per-click often seen as better by advertisers since clicks might measure relevance

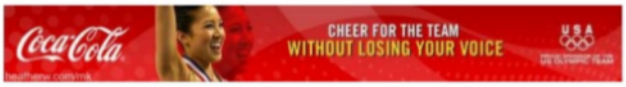
Choice depending on the goal(?)

- Some advertisers are more interested in *brand awareness* –not related to clicks–. Ex: Coca-Cola; no direct sale from clicks
  - pay-per-impression, appropriate for “brand awareness” (ex: Coca-Cola)

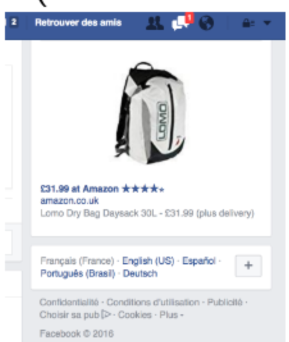


Choice depending on the goal(?)

- Some advertisers are more interested in *brand awareness* –not related to clicks–. Ex: Coca-Cola; no direct sale from clicks
  - pay-per-impression, appropriate for “brand awareness” (ex: Coca-Cola)



  - pay-per-click, appropriate for direct sales (ex: online shops)



- Therefore depends on the advertiser’s goal.
- But also a question from the web publisher point of view (models can be built).

Learning

- CTR have to be learned
- The advertiser has to trust the publisher: some advertisers filed lawsuits claiming to be victims of overcharging by lying (increasing) the real CTR. Ex in 2006:

Postscript 1: Google sent this statement:

> *We are proposing a settlement with the plaintiffs in this case. The proposal would allow advertisers to apply for credits for clicks they believe were not valid. Specific details of the settlement will remain confidential until it is presented to the judge. We do not know how many advertisers will apply and receive credits, but the total amount, including the legal fees determined by the judge, will not exceed $90 million.*

- Statistical tools to estimate the CTR.

Existing tools/companies

Tools helping advertisers to place their ads on search results or on banners on web pages

- Google AdWords (and Google AdSense to place ads on other web sites than Google ones), Yahoo! Search Marketing and Microsoft adCenter
- Advantage of such big companies: big market (opportunities)
- They "do the work" for advertisers
- Registration fee ($5) and reserve price per click (minimum amount)
- Tools (ex: Google analytics) to help to determine the best-performing keywords for your ads
- Possibility to get reports on ads
- Companies differ in the way they assist advertisers, their payment models which can consist in charges after an amount of clicks (Google) or a beforehand deposit that will be then used (Yahoo!).

Some links

- Google AdSense
  https://www.google.fr/adsense/start/
- Facebook ads
  https://www.facebook.com/business/products/ads
- Google Analytics
  https://www.google.fr/intl/fr/analytics/
- FaceBook Analytics
  https://analytics.facebook.com/

Outline

1. Illustrative example: Google
2. Advertising
3. Paid applications versus free applications with advertisement

Paid or free with ads?

- Internet users have historically been reluctant to pay for content or applications
- But it changes with tablets
- Problem for application/web content owners: should we develop paid access or free with advertisements?
  - Typical case of news web sites: minimal information, and subscription required for more content
  - Issue particularly true for applications on smartphones where the two possibilities are existing
    - 89% of applications free in 2012
    - but most paid applications cost less than $3.
    - 30% of revenues taken by the application stores (from ads or payments).
- Notion of *freemium* (mixture of free and paid):
  - Free basic service to incentivize users to pay for additional features
  - Ex: Skype, Candy Crush Saga, Spotify, web sites of newspapers.

Simple model, example

- Paid application with unit price p
  - α proportion of revenue to the store
  - c_{ap} application developing cost
  - n the number of applications sold

$$
\begin{aligned}
R_{\text{st}}(p) &= \alpha n p \\
R_{\text{ap}}(p) &= (1-\alpha) n p - c_{\text{ap}}.
\end{aligned}
$$

  - $F(p)$ expected proportion (or complementary distribution) of users downloading the application at a price p
  - Λ the total number, or mass, of potentially interested users, so that $n = \Lambda F(p)$
  - Exercise: Optimize $F(p)p$, provided that the solution p^* verifies $R_{\text{ap}}(p^*) \geq 0$, i.e., $p^* F(p^*) \geq c_{\text{ap}}/(\Lambda(1-\alpha))$.

- Free applications with ads:
  - α' proportion of revenue to the store
  - p' advertisement revenue from the application over its whole life
  - $n = \Lambda F(0)\mu = \Lambda\mu$ with $\mu \in [0,1]$ proportion of users accepting to see ads

$$\begin{aligned} R'_{\mathrm{st}} &= \alpha'\mu\Lambda p' \\ R'_{\mathrm{ap}} &= (1-\alpha')\mu\Lambda p' - c_{\mathrm{ap}}. \end{aligned}$$

- Which option to prefer?
  - From the store viewpoint advertisement is preferred if $R'_{\mathrm{st}} \geq R_{\mathrm{st}}(p^*)$, i.e., if $\alpha'\mu/\alpha \geq F(p^*)p^*/p'$.
  - From the application designer view point, advertisement is preferred if $R'_{\mathrm{ap}} \geq R_{\mathrm{ap}}(p^*)$, i.e., if $(1-\alpha')\mu/(1-\alpha) \geq F(p^*)p^*/p'$.
- Assume $\alpha = 0.3$, $\mu = 0.8$ and $\alpha' = 0.4$, and that $F(p^*)p^* = 1$ (wlog; unit redefining factor).
  - App designer will prefer advertisement if $p' \geq 1.45833$
  - Store will prefer it if and only if $p' \geq 0.9375$.

Exercises on VCG/GSP auctions

Patrick Maillé and Bruno Tuffin

Exercise 1 Ranking per revenue does not always produce the largest revenue!!

Consider the example considered in the class slides with a small variation. Here, $k = 3$ slots, where $\theta_1 = 1/2$, $\theta_2 = 1/4$ and $\theta_3 = 1/5$, and $n = 5$ advertisers with bids and CTRs given in Table 1. The modification is: q_2 is increased to 1.

Advertiser i	Bid b_i	CTR q_i
1	10	0.05
2	9	1
3	6	0.12
4	5	0.15
5	4	0.2

Table 1: Example with 5 bidders.

1. *Determine the winners of the slots for the ranking based on bids. Compute the paid prices and expected revenue.*

2. *Determine the winners of the slots for the ranking based on revenue. Compute the paid prices and expected revenue.*

3. *Compare them and conclude.*

Exercise 2 (GSP not incentive compatible) *Consider three advertisers competing for two slots, such that with truthful bids Advertiser 1 is allocated the first slot and Advertiser 2 the second slot. Thus $q_1 v_1 \geq q_2 v_2 \geq q_3 v_3$. Check whether Advertiser 1 has an interest in changing his bid to a value $b_1 < v_1$.*

Nonatomic network routing games
An application of Game Theory

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Context (1/2)

Game theory is used to model interactions between agents with different objectives.
Interactions: what I get depends not only on my actions but also on the others' actions.

Examples:

- (transportation) my expected commute time depends on my route choice but also on the choices of the other people,
- (telephony) my call blocking probability depends on my provider choice and on the other users' choices,
- (Internet) my QoS depends -among others- on the number of persons connected to the same service or access network as me,

Context (2/2)

We focus here on some particular games where

- user strategies can be represented as a route choice on a graph,
- the number of players is very large,
- only massive decisions affect the payoffs: each individual player has a negligible impact⇒nonatomic game.

Benefits of that approach

- valid in several domains
- simplifies the study of Nash equilibria

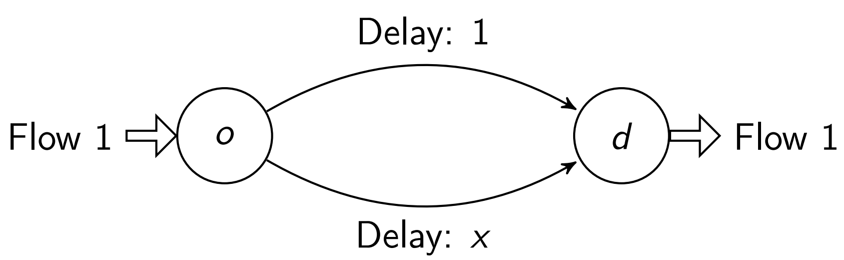
Applications: network dimensioning, pricing, demand prediction

An example: Pigou's instance (1/2)

Interpretation: imagine one unit (million, thousand) of commuters willing to go from the suburbs to the city center to work.

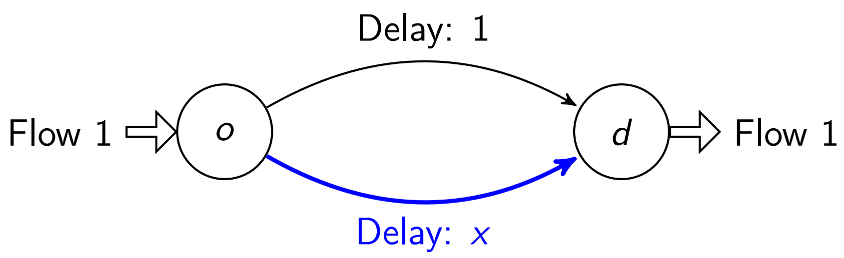
Two choices:

- take public transport⇒fixed commute time (1h)
- take one's car⇒commute time depends on the number x of people taking their car (congestion dependence), assume commute time is x.



An example: Pigou's instance (2/2)

Only one equilibrium: everybody takes his car and experiences a commute time of 1h!



That outcome is **not Pareto-efficient**: we could strictly decrease the commute time of some users without increasing that of the others by making some users switch to public transport.

- Enforce people to take public transport⇒badly perceived
- Give *incentives* to take public transport instead of one's car: taxes on roads, subsidies on public transport.

- Recall that we are using mathematical *models*, which are in general simplifications of reality
- For example, considering the cost of public transport to be independent of the load is not always perfectly realistic...

video: public transport in Japan

Outline

1. Formal presentation of routing games
   - Traffic routing problem
   - Feasible flow, arc flow
2. Equilibria of a routing game
   - Wardrop's principle and Nash equilibria
   - Wardrop equilibrium: existence and uniqueness
   - Variational inequality
3. Efficiency considerations
   - Total cost and social optimum
   - The Price of Anarchy
   - How to enforce coordination among users?
4. Conclusion: application examples
   - Partially optimal routing
   - Wireless networks

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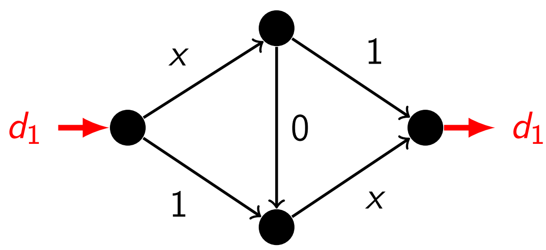
Traffic routing problem

We consider

- a network, represented by a **directed graph** $G = (N, \mathcal{A})$, with N the set of nodes and $\mathcal{A}$ the set of arcs;
- a set $K \subset N \times N$ of **origin-destination pairs**: for all $k = (s_k, t_k) \in K$, a flow of rate d_k must be routed from s_k to t_k,
- for each arc $a \in \mathcal{A}$, a nonnegative and nondecreasing **latency function** ℓ_a mapping the flow on a to the time to traverse a.

The triplet $(G, (\ell_a)_{a\in\mathcal{A}}, (d_k)_{k\in K})$ is an *instance* of the traffic routing problem.
Example: Braess' Instance

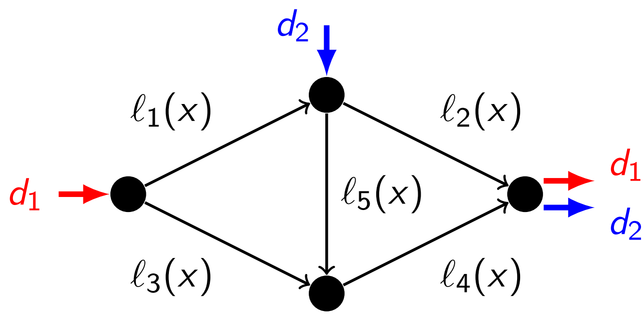


Paths

We denote by

- $P \subset \mathcal{A}$ a **path** (set of arcs);
- $\mathcal{P}_k$ the set of paths from s_k to t_k for $k = (s_k, t_k) \in K$;
- $\mathcal{P} = \bigcup_{k\in K} \mathcal{P}_k$ the set of paths susceptible of being used.

Example:



$$\mathcal{P} = \big\{\{1,2\};\{3,4\};\{1,5,4\};\{2\};\{5,4\}\big\}$$

Feasible flow, arc flow (1/2)

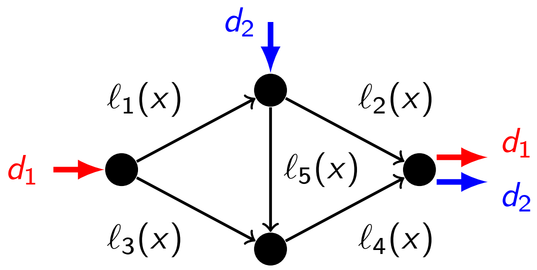
Definition

A *feasible flow* is a vector $(x_P)_{P\in\mathcal{P}}$ that meets the demand, i.e.,

$$\begin{cases} \sum_{P\in\mathcal{P}_k} x_P &= d_k & \forall k\in K \\ x_P &\geq 0 & \forall P\in\mathcal{P}. \end{cases}$$

Given a path flow, the corresponding *arc flow* can be found easily:

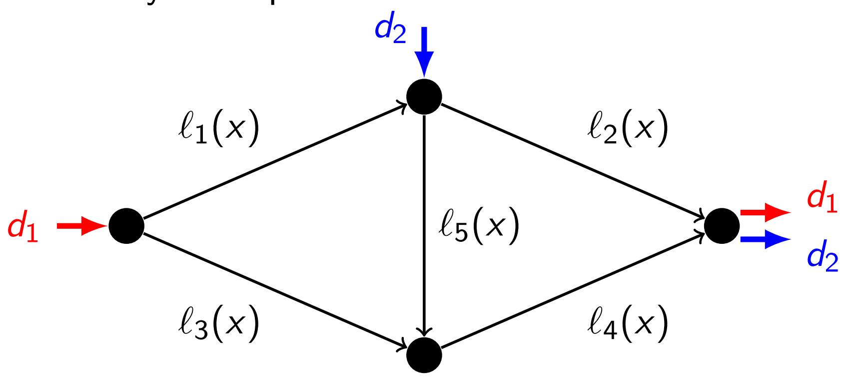
$$x_a = \sum_{P:a\in P} x_P \quad \forall a\in\mathcal{A}.$$



Exercise: check that $\begin{cases} x_{\{1,2\}} = 0.5; x_{\{3,4\}} = 0.2; x_{\{1,5,4\}} = 0.3; \\ x_{\{2\}} = 1.5; x_{\{5,4\}} = 0.5 \end{cases}$ is feasible for $d_1 = 1$, $d_2 = 2$, and compute the load on each link $a\in\mathcal{A}$.

Feasible flow, arc flow (2/2)

Careful: several path flows may correspond to the same arc flow



Example: compute the arc flows for the following path flows

$$(a)\quad \begin{cases} x_{\{1,2\}} = 0.5; x_{\{3,4\}} = 0.2; x_{\{1,5,4\}} = 0.3; \\ x_{\{2\}} = 1.5; x_{\{5,4\}} = 0.5 \end{cases}$$

$$\text{and}\quad (b)\quad \begin{cases} x_{\{1,2\}} = 0; x_{\{3,4\}} = 0.2; x_{\{1,5,4\}} = 0.8; \\ x_{\{2\}} = 2; x_{\{5,4\}} = 0 \end{cases}$$

Outline

Wardrop's principle and Nash equilibria

Definition

The cost of a path $P \in \mathcal{P}$ is the sum of the costs on the path links:

$$\ell_P(x) := \sum_{a \in P} \ell_a(x_a).$$

General principle: at a stable situation (equilibrium), for each origin-destination pair only cheapest paths are chosen.

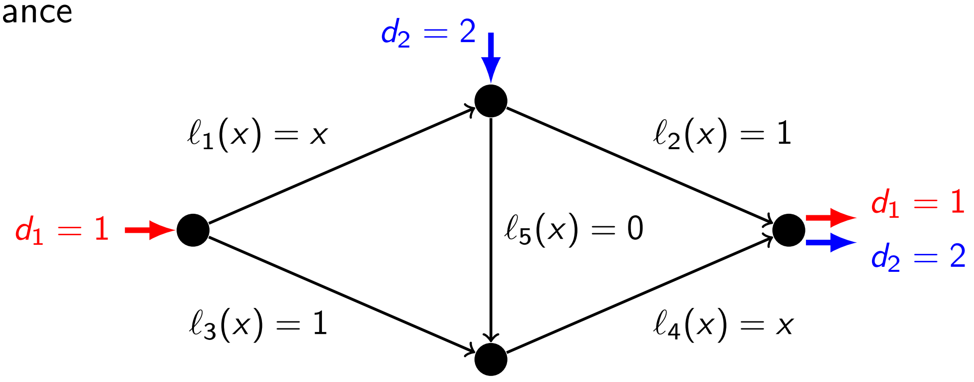
Definition (Wardrop, 1952 [11])

A feasible flow x is a *Wardrop equilibrium* if

$$\ell_P(x) \leq \ell_Q(x) \quad \text{for all } k \in K \text{ and } P, Q \in \mathcal{P}_k \text{ such that } x_P > 0.$$

Wardrop equilibrium: illustration

Consider the instance



Exercise: check that $\begin{cases} x_{\{1,2\}} = 0.5; x_{\{3,4\}} = 0.2; x_{\{1,5,4\}} = 0.3; \\ x_{\{2\}} = 1.5; x_{\{5,4\}} = 0.5 \end{cases}$ is not a Wardrop equilibrium,

but that $\begin{cases} x_{\{1,2\}} = 0.6; x_{\{3,4\}} = 0; x_{\{1,5,4\}} = 0.4; \\ x_{\{2\}} = 1.4; x_{\{5,4\}} = 0.6 \end{cases}$ is one.

Equilibria as a solution of an optimization problem

Consider nondecreasing derivable convex functions $(f_a)_{a\in\mathcal{A}}$, and a solution x^* of the convex optimization problem

$$\min_x \quad \sum_{a\in\mathcal{A}} f_a(x_a) \tag{1}$$

$$\text{such that} \quad \begin{cases} \sum_{P\in\mathcal{P}_k} x_P = d_k & \forall k \in K \\ x_P \geq 0 & \forall P \in \mathcal{P}. \end{cases}$$

x^* is feasible and verifies

$$\sum_{a\in P} f_a'(x_a^*) \leq \sum_{a\in Q} f_a'(x_a^*) \quad \text{for all } k \in K \text{ and } P, Q \in \mathcal{P}_k \text{ such that } x_P^* > 0.$$

$\Rightarrow$ *x^* is a Wardrop equilibrium for an instance where $\ell_a(x_a^*) = f_a'(x_a^*), \forall a \in \mathcal{A}$*

It can be proved that the reverse is also true, because of the convexity of the objective function (admitted here, see [1]): a Wardrop equilibrium x^* is a solution of (1) when $f'_a(x^*_a) = \ell_a(x^*_a), \forall a \in \mathcal{A}$.

The following result is then proved.

Theorem

The set of Wardrop equilibria of a routing game instance with nondecreasing latency functions is the set of solutions of the problem

$$\begin{array}{ll} \min\limits_{x} & \sum\limits_{a\in\mathcal{A}} \int_0^{x_a} \ell_a(y)dy \\ \text{such that} & \left\{ \begin{array}{rcll} \sum_{P\in\mathcal{P}_k} x_P & = & d_k & \forall k \in K \\ x_P & \geq & 0 & \forall P \in \mathcal{P}. \end{array} \right. \end{array} \tag{2}$$

Wardrop equilibrium: existence and uniqueness

Proposition (classical optimization result)

The problem of minimizing a convex function over a compact convex set always has a solution, that is unique if the objective function is strictly convex.

Theorem (Beckmann, McGuire, and Winsten 1956 [1])

Consider an instance with continuous and nondecreasing latency functions. A user equilibrium always exists and is essentially unique.
If in addition latency functions are strictly increasing, then arc flows are unique.

Essential uniqueness: under different equilibria (if any), each user experiences the same travel time.

Example: for the instance of slide 15, $\left\{ \begin{array}{l} x_{\{1,2\}} = 0; x_{\{3,4\}} = 0; x_{\{1,5,4\}} = 1; \\ x_{\{2\}} = 2; x_{\{5,4\}} = 0 \end{array} \right.$ is also a Wardrop equilibrium.

Variational inequality

The reasoning of slide 16 can also be applied to the functions

$$f_a(x_a) = x_a \ell_a(x_a^{\text{WE}}) \quad \forall a \in \mathcal{A},$$

where x^{WE} is a Wardrop equilibrium with latencies $\ell_a, a \in \mathcal{A}$.

Those functions are linear and therefore convex, thus we have

Theorem (Smith 1979 [8])

A feasible flow $\hat{x}$ for an instance with continuous and nondecreasing latency functions is a Wardrop equilibrium if and only if

$$\sum_{a \in \mathcal{A}} (x_a - \hat{x}_a)\ell_a(\hat{x}_a) \geq 0 \quad \text{for all feasible flow } x.$$

This relation is called *variational inequality*.

Outline

Total cost and social optimum

Definition

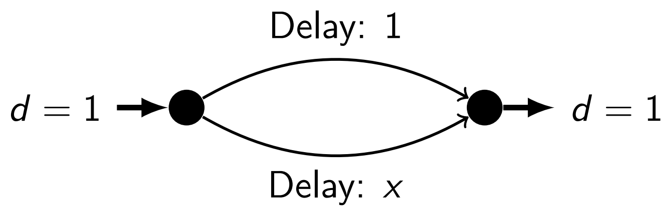
For a feasible flow x, the total cost (latency) experienced by the whole population is

$$C(x) := \sum_{P \in \mathcal{P}} x_P \ell_P(x) = \sum_{a \in \mathcal{A}} x_a \ell_a(x_a).$$

(For transportation networks, it is the total time spent on roads for the population.)

Exercise: for Pigou's instance,

1. compute the Wardrop equilibrium and the associated total cost.
2. What is the minimal cost among feasible flows?



Definition

A feasible flow that minimizes the total cost is called a *social optimum*.

$$x^{\mathrm{SO}} \in \arg\min_x \sum_{a \in \mathcal{A}} x_a \ell_a(x_a) \quad (3)$$

$$\text{such that} \quad \begin{cases} \sum_{P \in \mathcal{P}_k} x_P = d_k & \forall k \in K \\ x_P \geq 0 & \forall P \in \mathcal{P}. \end{cases}$$

It is naturally interesting to determine the optimal performance of the system, and thus to compute a social optimum.

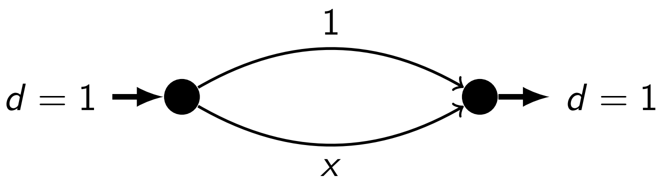
Social optimum as an equilibrium

Apply again the reasoning of slide 16 with the functions $f_a(x) = x\ell_a(x), a \in \mathcal{A}$:

Theorem

If for all $a \in \mathcal{A}$, the function $x \mapsto x\ell_a(x)$ is convex, then the set of socially optimal routings is the set of Wardrop equilibria for an instance with arc cost functions $\bar{\ell}_a(x) := \ell_a(x) + x\ell'_a(x)$ instead of ℓ_a.

Exercise: apply this method to find the social optimum for Pigou's instance.



The Braess paradox

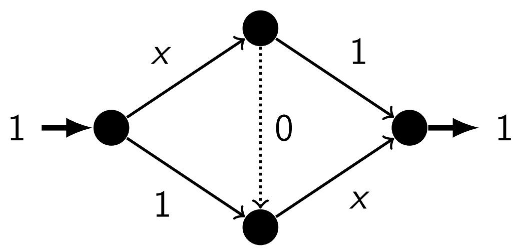
We could reasonably expect that adding a resource (or reducing the cost of existing resources) always improves the total cost.

$\Rightarrow$It is false, due to participants' selfishness!

Definition

A situation where adding a resource increases total cost is called a *Braess paradox*.

Example (Braess, 1969 [2, 3]):



Without link between north and south nodes:

$$C^{\text{WE}} = 3/2.$$

With a zero-cost link between north and south nodes:

$$C^{\text{WE}} = 2.$$

$\Rightarrow$**Adding the link has worsened the cost for all users.**

Braess paradox in “real life”

In the New York Times, Dec. 25, 1990, p38, *What if They Closed 42d Street and Nobody Noticed?*, by Gina Kolata :

On Earth Day this year, New York City’s Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn’t need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem." But to everyone’s surprise, Earth Day generated no historic traffic jam. *Traffic flow actually improved when 42d Street was closed.*

The Price of Anarchy

We wish to answer the question: **How bad is selfish routing?**

Definition (Koutsoupias & Papadimitriou [5])

We call *Price of Anarchy* (**POA**) the maximum ratio of the total cost at equilibrium to the optimal total cost.

$$\textbf{POA} := \max_{\text{instances}} \frac{C^{\text{WE}}}{C^{\text{SO}}}$$

POA measures the "price" of not having central coordination in system:

- If **POA** small, owner may want to let participants choose: not much to gain from dictating what people should do!
- If **POA** large, owner may want to re-design system or to give incentives to achieve more efficient results.

Price of Anarchy – Affine Costs

- For unrestricted cost functions, **POA** is unbounded.
- We will assume a fixed set of cost functions $\mathcal{L}$, e.g., affine.

Theorem (Roughgarden & Tardos, 2002 [7])

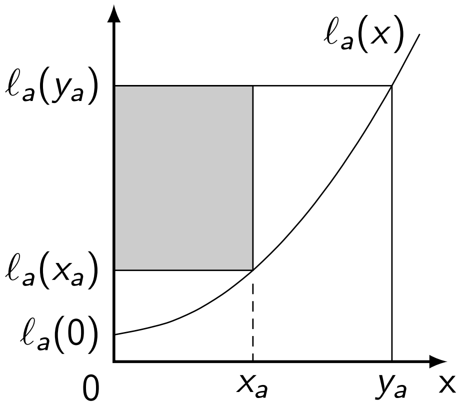
In networks with affine costs,

$$C^{WE} \leq \frac{4}{3} C^{SO}$$

Selfishness drives the system close to optimality

Corollary. Braess' and Pigou's instances are worst possible.

Price of Anarchy – General Costs



$$\text{Let } \beta(\mathcal{L}) := \sup_{\ell\in\mathcal{L}} \left\{ \frac{\text{shaded area}}{\text{big rectangle}} \right\}$$

$$\text{i.e., } \beta(\mathcal{L}) = \sup_{\ell\in\mathcal{L}} \sup_{x,y\geq 0} \frac{x(\ell(y)-\ell(x))}{y\ell(y)}.$$

We then have for all $x, y \geq 0$,

$$x(\ell(y)-\ell(x)) \leq \beta(\mathcal{L})y\ell(y).$$

Theorem (2003,2004 [6, 4])

If costs are drawn from a family of continuous *costs* $\mathcal{L}$,

$$C^{WE} \leq (1-\beta(\mathcal{L}))^{-1} \;\; C^{SO}$$

Exercise: prove that theorem

Use the variational inequality at equilibrium (slide 19).

Bounds on the Price of Anarchy

Theorem (2002, 2003, 2004)

For polynomials of maximum degree p, **POA** $= C^{WE}/C^{SO}$ *is bounded by*

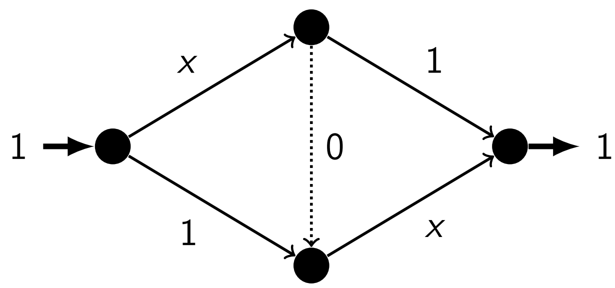
degree	*1*	*2*	*3*	*4*	...	*p*
POA	4/3	1.626	1.896	2.151	...	$\Omega(p/\ln p)$

Game and system objectives are partially 'aligned'.

How to enforce coordination among users?

Stackelberg games: the *leader* (network manager) determines his actions, predicting that users *(followers)* will react selfishly

- Removing some edges: **network design**



$\Rightarrow$ finding the best set of edges to remove is NP-hard...

- Controlling a given **proportion of the demand**

Coordinating through Network Pricing

Most well-known policy: add on each arc $a \in \mathcal{A}$ *Pigovian tax*

$$t_a = x_a^{SO} \ell_a'(x_a^{SO}), \tag{4}$$

where ℓ_a' is the derivative of ℓ_a, and x^{SO} is a socially optimal flow.

Users now perceive $\ell_a(x_a) + t_a$ on each arc a.

Exercise: prove the following result

Theorem

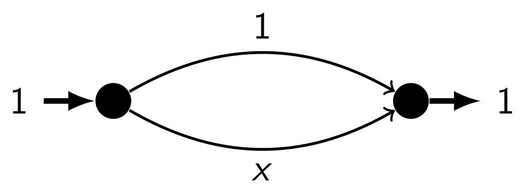
The optimal flow x^{SO} used in (4) is a Wardrop equilibrium for the game with prices, and the only one if cost functions $(\ell_a)_{a \in \mathcal{A}}$ are strictly increasing.

$\Rightarrow$ pigovian taxes enforce an optimal behavior!

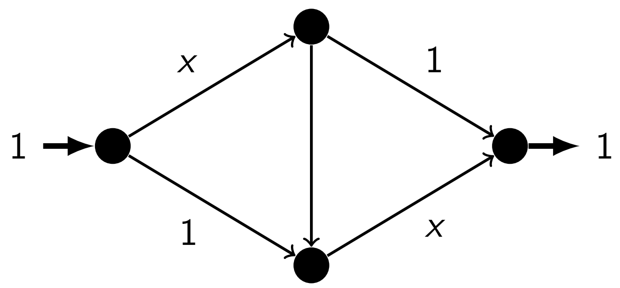
Application: exercise

Compute some optimal taxes for

- Pigou's instance



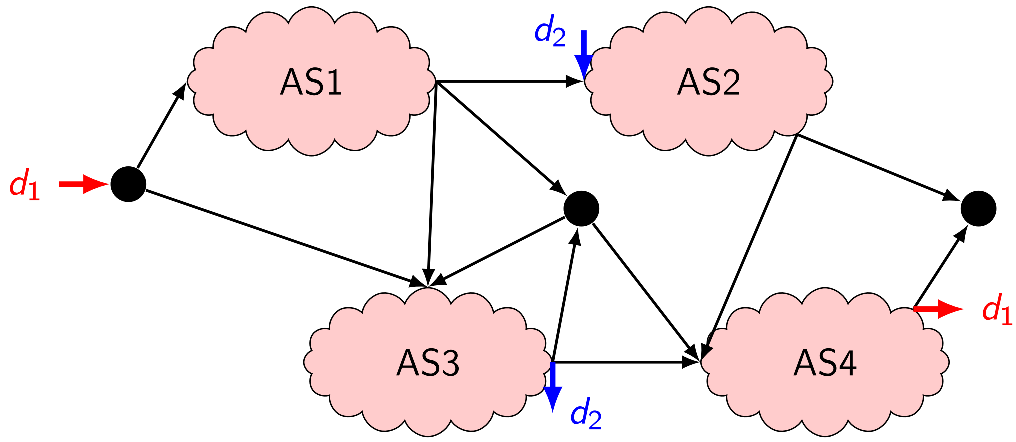
- Braess' instance



Outline

Partially optimal routing (1/2)

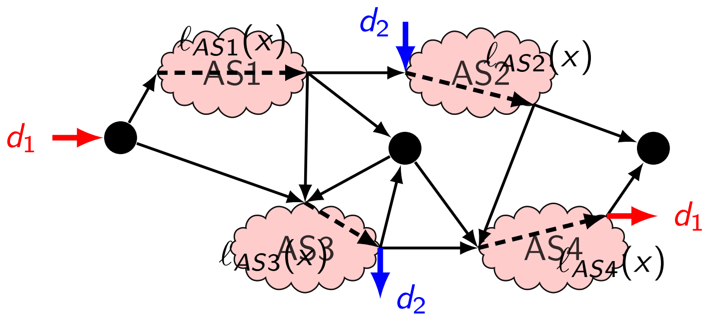
Some Autonomous Systems (ASs) may control the routing inside their domain.



What happens if ASs optimize their internal routing?

Partially optimal routing (2/2)

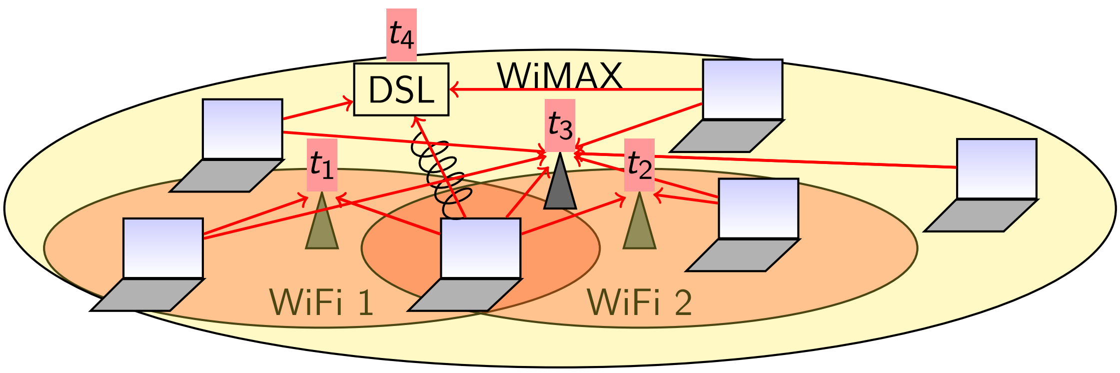
If ASs optimize their internal routing, but flows choose the overall routing (Acemoglu, Johari & Ozdaglar 2007),



1. If each AS has a single entry and exit point:
   - the outcome may be worse than when flows choose the whole routing
   - for polynomial cost functions, the **POA** is unchanged
2. If ASs can have several entry and exit points:
   - even with affine cost functions, the outcome may be infinitely inefficient (**POA** $= +\infty$).

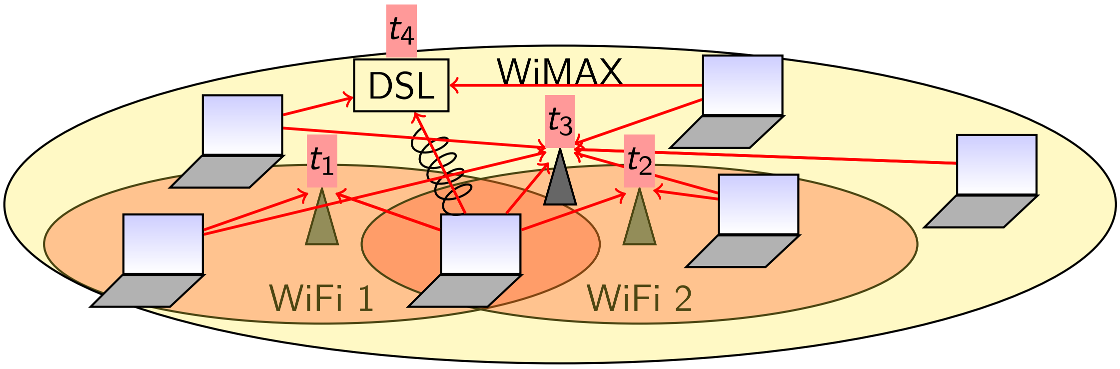
In any case, an AS is not even sure to experience less total cost, since the optimization may make it more attractive...

Wireless networks



- Interactions among non-cooperative consumers: **game**
- Congested networks provide poorer quality (delay, packet losses)

Wireless networks



Situation: a two-level noncooperative game.

1. *Higher level*: **networks** set prices (taxes) to minimize total cost or maximize revenue,
2. *Lower level*: **consumers** choose their provider

We considered a simplified model!

There exist recent work aiming to relax some assumptions.
We assumed that users are

- **Nonatomic** (i.e., infinitely small): what if some users have nonnegligible demand?
- **Homogeneous**: what if user sensitivity to prices differ among users?
- **Perfectly rational**: what if users make (limited) errors in determining their optimal strategy?
- **With perfect information**: what if users do not know at all times the congestion level of all links?

Moreover we focused here on **efficiency** issues, ignoring **fairness** (do similar users experience similar costs?).

Conclusions

- Networks involve several kinds of interacting agents
- The strategic choice of agents can affect the other ones
- Most agents are selfish

⇒ Game Theory seems to be an appropriate framework

Routing games (and the notion of Wardrop equilibrium) have recently made a breakthough, and apply in numerous domains:

- transportation networks
- demand modelling in economy
- telecommunication networks

For a better written version...

Most of the concepts and results presented in this document are included in Chapter 2 (pp 33-49) of Nicolàs E. Stier-Moses' PhD dissertation [9], available here: `https://dspace.mit.edu/bitstream/handle/1721.1/16650/56430658-MIT.pdf?sequence=2`

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Practical work: non-atomic routing games

Patrick Maillé
IMT Atlantique

The objective of this practical session is to define all parameters that characterize a non-atomic routing game instance, to compute numerically the equilibrium of that game, and to compare the performance of that equilibrium to the optimal allocation, in order to illustrate the differences between the two outcomes, and the concept of Price of Anarchy. The results will also be used to calculate the link taxes that would induce an optimal behavior from selfish users.

To that extent, we will use a numerical computing tool like Python to implement our algorithms.

Getting started

Create a folder for that practical session. Download there the file that we will use as the skeleton for our code, available here:
`https://partage.imt.fr/index.php/s/rmX2bLYgtipEP5C`

Stuck at some place?

A completed `.py` file, that can be used as a correction for the different questions, is available here:
`https://partage.imt.fr/index.php/s/E4MB9wk4jm7jxiC`

Try to limit its use!

1 Parameters of the routing game

We first recall all the elements that constitute a non-atomic routing game, and we will look for a way to store them in our programming language. Those elements are:

- the network topology, i.e., the set of links (arcs) and how they are connected;
- the congestion function of each arc;
- the origin-destination pairs and the corresponding demand (flow) levels.

1.1 The network topology

We avoid indexing the vertices of the network, and only focus on links, that we index from `0`, as suggested in Figure 1. Therefore we define the network topology by indicating the *connections* between links: we suggest to store that information in a list of lists called `Next`, where `Next[i]` is a list, that contains `j` if and only if link `j` can "follow" link `i`, i.e., the ending point of link `i` coincides with the starting point of link `j`.

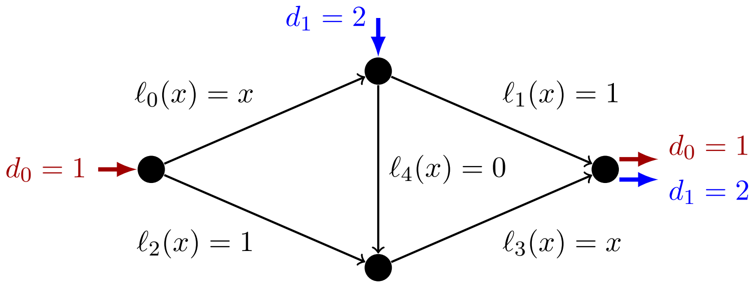

Figure 1: Our first game instance

Question 1 Complement the file `routing.py` so that `Next` corresponds to the instance of Figure 1.

1.2 Link congestion functions

Link congestion functions can be defined by a single function, that we can call `latency(a,x)`, that takes as arguments the index `a` of the considered link and the amount `x` of flow using that link.

Question 2 Edit the file `routing.py` by complementing the function `latency` so that it returns the latency values for the five links corresponding to the instance represented in Figure 1.
Then you can execute your script using the command

```
python3 -i routing.py
```

and check for example what `latency(3,5)` gives.

1.3 Origin-destination pairs and corresponding demands

Again, we use only the links to represent the OD pairs. We represent each OD pair `k` by three parameters, namely:

- the demand level
- the set of links that start from the origin point of `k`
- the set of links that terminate at the destination point of `k`

We propose to access those data, using a **list of dictionaries** named `OD`, where for each origin-destination pair index `k`, the dictionary `OD[k]` contains the values for the fields

- `"dem"`: the demand level
- `"start"`: a list containing the indices of links that start from the origin of the demand
- `"term"`: a list containing the indices of links that terminate at the destination of the demand

Question 3 Complement the file to define that object `OD`, still for the instance of Figure 1.

2 Computation of a Wardrop equilibrium

We now intend to use the instance data we have defined in order to compute some equilibria of the game. Those same data will also be used to find an optimal allocation of flows, and to estimate the effect of prices.

A first step consists in finding the feasible paths for each origin-destination pair. Those paths will constitute the choices for the population of that OD pair, we will store them in a list once for the whole analysis.

2.1 Finding the feasible paths for each origin-destination pair

The objective here is to find all possible ways for an OD pair `k` to route its traffic. Formulating that in terms of links, we are looking for sequences of link indices $(a_1, a_2, ..., a_n)$ such that

- a_1 starts from the origin of the OD pair `k`, i.e., it is in the list `"start"`,
- a_n ends at the destination of the OD pair `k`, i.e., it is in the list `"term"`,
- the successive links are connected to form a path, i.e., forall $i = 1, ..., n-1$ we have a_{i+1} `in Next[`a_i`]`.

To that extent, we propose to implement a simple algorithm, that explores each possible path starting with a link in `"start"`, avoiding to use the same link twice, until a link in `"term"` is found (the path is then valid). To be sure we do not miss any path, we treat simultaneously all the candidate paths of the same length, starting with length 1 (the links in `"start"`). A path is represented as a list containing the indices of the links in it, in the order we should take those links.

We intend to store all feasible paths for an OD pair in a list (hence, a **list of lists** since each element is a path).

We want to be generic when defining that algorithm, so as to be able to use it for any instance described by the function `latency`, and the objects `Next` and `OD`.

Question 4 Continue editing the file `routing.py` by complementing the function `compute_paths`, so that `compute_paths(k)` returns a list of all feasible paths for origin-destination pair `k`,

without any doublon.
Then load again your new functions using

```
python3 -i routing.py
```

and check that the algorithm gives the right paths, i.e., that

- `compute_paths(0)` gives `[[0,1], [2,3], [0,4,3]]`
- `compute_paths(1)` gives `[[1], [4,3]]`

Question 5 Finally, build the list `Paths` containing all paths for each OD pair, i.e., so that `Paths[k]` gives the list of paths for OD pair `k`.

2.2 Algorithm to approximate the Wardrop equilibrium

We now use our previous results to approach numerically the Wardrop equilibrium.

Question 6 We will need to compute the load on each link, given the loads on paths. Complement the function `compute_linkloads(pathloads)` where `pathloads` is a list of lists: the element `j` in `pathloads[k]` being the load on path `j` of OD pair `k`, following the order in `Paths`. The function should return a list, whose value at position `a` is the load of link `a`.

Question 7 We will also need to be able to compute the total cost of a path, depending on the load on all links and possibly on taxes that may be added to each link. To get that value, write down a function `pathcost(path,linkloads,linktaxes)` that takes as arguments a path `path` (i.e., a list with the link indices of the path), `linkloads` (a list containing the load on each link), and `linktaxes` (a list containing the tax value of each link).

Question 8 Look at the code of the function `Wardrop`, and describe in a few sentences what the algorithm does.
Why do we take a `min` when computing `switch_amount`?

Question 9 Run the function `Wardrop` with the command `Wardrop([0,0,0,0,0])` (that is, without taxes), and check that the outcome is an equilibrium.

3 Optimal flow repartition and Price of Anarchy

3.1 Computing an optimal flow repartition

Question 10 From the theoretical results on optimal flows, would it be possible to use the algorithms defined so far in order to compute an optimal (i.e., cost-minimizing) flow repartition instead of an equilibrium?

Question 11 Add a new boolean variable `changed_latencies`, that will be a global variable, and modify the function `pathcost` so that it returns

- if `changed_latencies` is true: the cost that the path would have if the latency of each link a were $\bar{\ell}_a(x) = \ell_a(x) + x\ell'_a x)$ instead of $\ell_a(x)$. (You can approximate the differential by taking finite differences with a small `eps`.)
- Else, the real cost of the path, as the function was returning before.

Question 12 Then define a function `optimal()`, that returns an optimal allocation by temporarily changing `changed_latencies` and calling the `Wardrop` function. Do not forget to declare `changed_latencies` as a global variable in your function, and to set it back to `False` before leaving the function.

Question 13 Compute an optimal allocation for the instance of Figure 1, by calling the function `optimal()` (do not forget to first run `exec(open("routing.py").read())` to execute your new code).

3.2 Computation of the Price of Anarchy

Question 14 Write down a function `TotalCost(pathloads)` that returns the total cost for the repartition `pathloads`. You may need to call the function `compute_linkloads` for that.

Question 15 Write down a function `calcPoA()` that returns in a tuple the total cost at equilibrium, the optimal total cost, and the Price of Anarchy for the instance characterized by `Next`, `OD`, and the `latency` function.

Question 16 Compute the equilibrium cost, the minimum (optimal) cost and the Price of Anarchy for the instance of Figure 1.

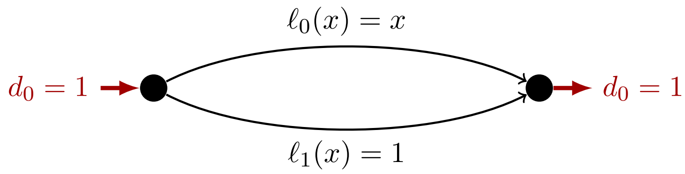

Figure 2: The Pigou instance

Question 17 Test your functions on the Pigou instance (the very first example seen in class, depicted in Figure 2), by just re-defining `Next` and `OD` (do not delete the values for Figure 1, we will use them again).
What is the Price of Anarchy for that instance?

4 Effect of link taxes, optimal taxes

4.1 Influence of taxes on the equilibrium

Question 18 Go back to the initial example of Figure 1. Based on the difference between the equilibrium and the optimal flow allocations obtained for that instance, select a link where adding a tax could improve the situation.

Question 19 Test that program by setting a high tax value on the link you have chosen, and by checking that the equilibrium is the one you expected.

Question 20 Complement the function `vary_tax()`, that varies continuously the tax on that link from `0` to `1`, and plots the corresponding Price of Anarchy (we recall that taxes are not taken into account when computing the total cost). Test that function by calling it after executing your new code.

4.2 Computing optimal taxes

Question 21 Write down a function `opttaxes()` that returns a list whose `i`th term is an optimal tax on link `i` (for example, you can compute the Pigovian taxes). As before, you may use finite differences with a small `eps` to approximate the derivatives.

Question 22 To check that those taxes are indeed optimal, compute the equilibrium cost when they are applied.

Presenting and explaining a scientific paper

M2 SIF, GTA course, 2023-2024

Objectives

The objective of this work is to be able to:

- understand a scientific paper published in an international conference or journal,
- explain it to the teachers and your fellow students,
- highlight strengths and weaknesses of the model and results.

List of papers

For the 2023-2024 year, we will focus on the works of Nicolas Stier (formerly with Columbia University then Torcuato Di Tella University in Buenos Aires, now with Facebook Core Data Science). The list of his publications is available here:

`https://sites.google.com/site/nicostier/home/publications?authuser=0`

Please select a paper related to the topics addressed in the course.

Organization

- Before Oct 15: indicate your group and your chosen paper by e-mail to bruno.tuffin@inria.fr and patrick.maille@imt.fr
  A different paper for each group (first come, first served: if your chosen paper is already taken, we'll ask you to pick another one)
- On Nov 11 (16:45-18:15): presentation + questions