05 Number Theory and Cryptography

CS201 Discrete Mathematics

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Number Theory

- Number theory is a branch of mathematics that explores integers and their properties.
  - It is the basis of many areas, e.g., cryptography, coding theory, computer security, e-commerce, etc.
- At one point, the largest employer of mathematicians in the United States, and probably the world, was the National Security Agency (NSA) (or *No Such Agency*?).
  - NSA is the largest spy agency in the US (larger than CIA, Central Intelligence Agency); it is responsible for code design and breaking.

Fun Story

- G. H. Hardy (1877-1947), UK mathematician
  - In his 1940 autobiography *A Mathematician's Apology*, Hardy wrote "The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, almost as 'useless' as the theory of numbers."
  - If he could see the world now, Hardy would be spinning in his grave :)

Divisibility and Modular Arithmetic

Divisibility

- For integers a, b with $a \neq 0$, we say that a divides b if there is an integer c such that $b = ac$, or equivalently b/a is an integer. In this case, we say a is a factor or divisor of b, and b is a multiple of a.

- Notations: let $a \mid b$ denote a divides b (or b is divisible by a) and let $a \nmid b$ denote a does not divide b (or b is not divisible by a)
  - E.g., we have $4 \mid 24$ and $3 \nmid 7$.

- All integers divisible by $d > 0$ can be enumerated as:

  $\ldots, -kd, \ldots, -2d, -d, 0, d, 2d, \ldots, kd, \ldots$

- How many positive integers $\leq n$ are divisible by $d > 0$?
  - Count the number of integers written as kd such that $0 < kd \leq n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.

Divisibility Properties

- **Theorem:** Let *a, b, c* be integers ($a \neq 0$). Then

  (i) if $a|b$ and $a|c$, then $a|(b + c)$

  (ii) if $a|b$ then $a|bc$ for all integers c

  (iii) if $a|b$ and $b|c$, then $a|c$

- **Corollary:** If *a, b, c* are integers ($a \neq 0$) and $a|b$, $a|c$ hold, then we have $a|(mb + nc)$ for any integers m and n.
  - *proved by applying (i) and (ii)*

The Division Algorithm

- For any integers *a, d* with $d > 0$, there exist unique integers *q, r*, with $0 \le r < d$, such that $a = dq + r$. In this case, *d* is called the divisor, *a* is called the dividend, *q* is called the quotient, and *r* is called the remainder.
  - *proved in later sections*
- Notations: $q = a \ div \ d$ and $r = a \ mod \ d$. * *mod* is short for modulo
- Order of precedence for *mod*: same as multiplication and division
- Example: $17 = 3 \times 5 + 2$
  - $17 \ div \ 3 = ?$

    5
  - $17 \ mod \ 3 = ?$

    2

Computing the *mod* Function

- **Theorem:** For integers *a, b, m* with *m* > *0*, we have:
  - *(a + b) mod m = ((a mod m) + (b mod m)) mod m*
  - *ab mod m = (a mod m)(b mod m) mod m*
- Key observation:
  - *a = m (a div m) + (a mod m) = mq + (a mod m)*
- Example:
  - *78 + 99 mod 5 = ?*

    ((78 mod 5) + (99 mod 5)) mod 5 = (3 + 4) mod 5 = 2
  - *32 x 758 mod 5 = ?*

    (32 mod 5)(758 mod 5) mod 5 = 2 x 3 mod 5 = 1

Arithmetic Modulo *m*

- Let $\boldsymbol{Z}_m = \{0, 1, \ldots, m - 1\}$ be the set of nonnegative integers < m. For $a, b \in \boldsymbol{Z}_m$, addition and multiplication are defined as follows:
  - $+_m : a +_m b = (a + b) \bmod m$
  - $\cdot_m : a \cdot_m b = ab \bmod m$
- Examples:
  - $7 +_{11} 9 = ?$

    $(7 + 9) \bmod 11 = 5$
  - $7 \cdot_{11} 9 = ?$

    $7 \cdot 9 \bmod 11 = 8$

Modular Arithmetic Properties

- **Closure:** if $a, b \in \boldsymbol{Z}_m$, then $a +_m b, a \cdot_m b \in \boldsymbol{Z}_m$
- **Associativity:** if $a, b, c \in \boldsymbol{Z}_m$, then

  $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$
- **Identity elements:** if $a \in \boldsymbol{Z}_m$, then $a +_m 0 = a$ and $a \cdot_m 1 = a$
- **Additive inverses:** if $a \neq 0$ and $a \in \boldsymbol{Z}_m$, then $m - a$ is an additive inverse of a modulo m, i.e., $m - a \in \boldsymbol{Z}_m$ and $a +_m (m - a) = 0$
- **Commutativity:** if $a, b \in \boldsymbol{Z}_m$, then $a +_m b = b +_m a$
- **Distributivity:** if $a, b, c \in \boldsymbol{Z}_m$, then

  $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$

  $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$

Integer Representations

Integer Representations

- There are many ways to represent integers: decimal (base *10*) or binary (base *2*) or octal (base *8*) or hexadecimal (base *16*) or other notations.

- Let $b > 1$ be an integer. Any positive integer n can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0,$$

  where k, a_i are nonnegative integers and $0 \leq a_i < b$.

- This representation of n is called the base-b expansion of n, denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.

Base-*b* Expansions

- Recall that the base-*b* expansion of $n = (a_k a_{k-1} \ldots a_1 a_0)_b$ means:

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

- Getting the decimal expansion is easy.
  - Examples:

$$(101011111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$$

$$(7016)_8 = 7 \cdot 8^3 + 1 \cdot 8 + 6 = 3598$$

- Conversions between binary, octal, hexadecimal expansions are also easy.
  - Examples:

$$(101011111)_2 = (101011111)_2 = (537)_8$$

$$(7016)_8 = (111000001110)_2 = (111000001110)_2 = (E0E)_{16}$$

Constructing Base-*b* Expansions

- Base-*b* expansion can be derived from *mod b* operations:

$$\begin{aligned} n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\ &= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + a_0 \\ &= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + a_1) + a_0 \\ &= \cdots \end{aligned}$$

- Algorithm: constructing the base-*b* expansion of an integer *n*

1. Divide *n* by *b* to get $a_0 = n \bmod b$, with $n = bq_0 + a_0$, $0 \le a_0 < b$, then set a_0 as the rightmost digit in the base-*b* expansion of *n*.
2. Divide q_0 by *b* to get $a_1 = q_0 \bmod b$, with $q_0 = bq_1 + a_1$, $0 \le a_1 < b$, then set a_1 as the second digit from the right.
3. Continue this process by successively mod the quotients by *b* until the quotient q_i is 0.

Constructing Base-*b* Expansions

ALGORITHM 1 Constructing Base *b* Expansions.

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while $q \neq 0$
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return $(a_{k-1}, \ldots, a_1, a_0)$ {$(a_{k-1} \ldots a_1 a_0)_b$ is the base b expansion of n}

- Example: $(12345)_{10} = (30071)_8$

  $12345 = 8 \cdot 1543 + 1$

  $1543 = 8 \cdot 192 + 7$

  $192 = 8 \cdot 24 + 0$

  $24 = 8 \cdot 3 + 0$

  $3 = 8 \cdot 0 + 3$

Binary Addition of Integers

- **Add** $a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2$ and $b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2$
  - start from right to left and maintain a carry bit c
  - $O(n) = O(\log a) = O(\log b)$ bit additions/subtractions

ALGORITHM 2 Addition of Integers.

procedure *add*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$
and $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$, respectively}
$c := 0$
for $j := 0$ **to** $n - 1$
$d := \lfloor (a_j + b_j + c)/2 \rfloor$
$s_j := a_j + b_j + c - 2d$
$c := d$
$s_n := c$
return $(s_0, s_1, \ldots, s_n)$ {the binary expansion of the sum is $(s_ns_{n-1}\ldots s_0)_2$}

Binary Multiplication of Integers

- **Multiply** $a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2$ by $b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2$
  - $ab = a(b_0 2^0 + \cdots + b_{n-1}2^{n-1}) = a(b_0 2^0) + \cdots + a(b_{n-1}2^{n-1})$
  - $O(n^2) = O(\log a \log b)$ bit operations: n-round binary shifts/adds

ALGORITHM 3 Multiplication of Integers.

procedure *multiply*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$
and $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$, respectively}
for $j := 0$ **to** $n - 1$
    if $b_j = 1$ **then** $c_j := a$ shifted j places
    else $c_j := 0$
{$c_0, c_1, \ldots, c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ **to** $n - 1$
    $p := add(p, c_j)$
return p {p is the value of ab}

Binary Division of Integers

- **Divide** a by $d > 0$: compute $q = a \text{ div } d$ and $r = a \bmod d$
  - $O(q \log a)$ bit operations: q iterations of binary adds/subs on q, r
  - $O(q \log a) = O(2^{\log_2 q} \log a)$ is exponential in the input size $\Theta(\log ad)$! (Note that $\log_2 q = \Theta(\log ad)$ for small d.) There are efficient division algorithms that run in $O(\log a \log d)$, or $O(n^2)$ ($n = max(\log a, \log d)$)

ALGORITHM 4 Computing div and mod.

procedure *division algorithm*(a: integer, d: positive integer)
$q := 0$
$r := |a|$
while $r \geq d$
    $r := r - d$
    $q := q + 1$
if $a < 0$ and $r > 0$ **then**
    $r := d - r$
    $q := -(q + 1)$
return (q, r) {$q = a$ **div** d is the quotient, $r = a$ **mod** d is the remainder}

Binary Division of Integers (fast)

- **Divide** a by $d > 0$: compute $q = a \text{ div } d$ and $r = a \bmod d$
  - Key observation: $a = 2\lfloor a/2 \rfloor + (a \bmod 2)$ for any $a \geq 0$
  - $O((\log a) \max(\log q, \log d))$ bit operations: $\log_2 a$ iterations of binary shifts/adds/subs on q, r (with sizes $O(\log q)$ and $O(\log d)$ resp.)

procedure *division2* $(a, d \in \mathbb{N},\ d \geq 1)$
if $a < d$
  return $(q, r) = (0, a)$
$(q, r) = division2(\lfloor a/2 \rfloor, d)$
$q = 2q,\ r = 2r$
if a is odd
  $r = r + 1$
if $r \geq d$
  $r = r - d$
  $q = q + 1$
return (q, r)

Fast Modular Exponentiation

- Compute $b^n \bmod m$ (where $n = (a_{k-1}\ldots a_1 a_0)_2$)
  - $b^n = b^{a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0} = b^{a_{k-1} \cdot 2^{k-1}} \cdots b^{a_1 \cdot 2} \cdot b^{a_0}$
  - Compute successively (squared each time): $b \bmod m$, $b^2 \bmod m$, $b^{2^2} \bmod m$, …, $b^{2^{k-1}} \bmod m$, and multiply together b^{2^j} where $a_j = 1$.
  - $O((\log m)^2 \log n)$ bit operations: $\log n$ iterations of $\bmod m$ mults

ALGORITHM 5 Fast Modular Exponentiation.

procedure *modular exponentiation*(b: integer, $n = (a_{k-1}a_{k-2} \ldots a_1 a_0)_2$,
m: positive integers)
$x := 1$
$power := b$ **mod** m
for $i := 0$ **to** $k - 1$
  if $a_i = 1$ **then** $x := (x \cdot power)$ **mod** m
  $power := (power \cdot power)$ **mod** m
return $x\{x$ equals b^n **mod** $m\}$

Primes and Greatest Common Divisors

Primes and Prime Factorization

- **Prime:** a positive integer p that is greater than or equal to 2 and has only two positive factors 1 and p
  - E.g., 13 is a prime, with only two positive factors 1 and 13.
  - We already proved before that there are infinite many primes.
- **Composite:** a positive integer ≥ 2 that is not a prime
  - E.g., 14 is a composite, divisible by 2 and 7.
- **Fundamental Theorem of Arithmetic:** Every integer ≥ 2 can be written uniquely as a prime or as the product of multiple primes, where the prime factors are written in nondecreasing order.
  - E.g., $12 = 2 \cdot 2 \cdot 3$
  - The above is also known as the prime factorization theorem.
  - It is not hard to see the existence of a prime factorization, but its uniqueness is not easy to prove.

Uniqueness of Prime Factorization

- **Lemma:** If p is prime and $p|a_1a_2 \cdots a_n$, then $p|a_i$ for some i.
  - proved in later sections by Bézout's theorem and induction
- **Theorem:** a prime factorization of a positive integer where the primes are in nondecreasing order is unique.
- Proof (by contradiction):
  - Suppose that the positive integer n can be written as a product of primes in two distinct ways:

    $n = p_1p_2 \cdots p_s$ and $n = q_1q_2 \cdots q_t$
  - Remove all common primes from the factorizations to get

    $p_{i_1}p_{i_2} \cdots p_{i_u} = q_{j_1}q_{j_2} \cdots q_{j_v}$
  - Since p_{i_1} divides the left side, it must also divides the right side. Then, by **Lemma**, we have p_{i_1} divides q_{j_k} for some k, which contradicts the assumption that p_{i_1} and q_{j_k} are distinct primes.

Primality Tests

- A primality test is an algorithm for determining whether a number is prime or composite.
  - Approach 1: test if each integer $2 \le x < n$ divides n.
  - Approach 2: test if each prime number $x < n$ divides n.
  - Trivial Division: test if each prime number $x \le \sqrt{n}$ divides n. *Why?*

- **Theorem:** If n is composite, then n has a prime divisor $\le \sqrt{n}$

- Proof: If n is composite, then by definition it has a positive integer factor a such that $2 \le a < n$. This means that $n = ab$, where b is an integer such that $2 \le b < n$. If $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab > n$, which contradicts $n = ab$. Therefore, we have $a \le \sqrt{n}$ or $b \le \sqrt{n}$ and hence n has a divisor $\le \sqrt{n}$. By the Fundamental Theorem of Arithmetic, this divisor is either prime or is a product of multiple prime factors. In either case, n has a prime divisor $\le \sqrt{n}$.

The Sieve of Eratosthenes

- Find all primes ≤ 100:
  - delete integers divisible by 2
  - delete integers divisible by 3
  - delete integers divisible by 5
  - delete integers divisible by 7

- Why?
  - 7 is the largest prime $\leq \sqrt{100}$

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Mersenne Primes

- **Mersenne Prime:** a prime of the form $2^p - 1$, where p is a prime
- Examples:
  - $2^2 - 1 = 3$
  - $2^3 - 1 = 7$
  - $2^5 - 1 = 37$
  - $2^{11} - 1 = 2047 = 23 \cdot 89$ is not a Mersenne prime

Marin Mersenne

- The largest known prime numbers are Mersenne primes.
  - https://www.mersenne.org/

51st Known Mersenne Prime Found!

December 21, 2018 — The Great Internet Mersenne Prime Search (GIMPS) has discovered the largest known prime number, $2^{82,589,933}-1$, having 24,862,048 digits. A computer volunteered by Patrick Laroche from Ocala, Florida made the find on December 7, 2018. The new prime number, also known as M82589933, is calculated by multiplying together 82,589,933 twos and then subtracting one. It is more than one and a half million digits larger than the previous record prime number.

Conjectures about Primes

- **Goldbach's Conjecture (1 + 1):** Every even integer that is greater than *2* is the sum of two primes.
  - "3 + 4", "3 + 3", "2 + 3" – Y. Wang, 1956
  - "1 + 5" – C. Pan, 1962
  - "1 + 4" – Y. Wang, 1962
  - "1 + 2" – J. Chen, 1973

    Every sufficiently large even number can be written as the sum of a prime and a semiprime (the product of two primes)

- **Twin-prime Conjecture:** There are infinitely many twin primes.
  - A twin prime is a prime number that is either *2* less or *2* more than another prime number, e.g., either of the twin prime pair *(41, 43).*

Greatest Common Divisor (GCD)

- Let *a* and *b* be integers, not both *0*. The largest integer *d* such that *d*|*a* and *d*|*b* is called the greatest common divisor of *a* and *b*, denoted by *gcd(a, b)*.
  - E.g., *gcd(12, -21) = ?*
- A systematic way to find the *gcd* is factorization.
  - Let $|a| = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $|b| = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Then,

    $gcd(a, b) = p_1^{min(a_1, b_1)} p_2^{min(a_2, b_2)} \cdots p_n^{min(a_n, b_n)}$
  - E.g., $12 = 2^2 \times 3^1$, $|-21| = 3^1 \times 7^1$, $gcd(12, -21) = 2^0 \times 3^1 \times 7^0 = 3$
- Two integers *a* and *b* are relatively prime (or coprime) if their greatest common divisor *gcd(a, b) = 1*.

Least Common Multiple (LCM)

- Let *a* and *b* be non-zero integers. The smallest positive integer that is divisible by both *a* and *b* is called the least common multiple of *a* and *b*, denoted by *lcm(a, b)*.
  - E.g., *lcm(12, -21) = ?*
- We can also use factorization to find the *lcm* systematically.
  - Let $|a| = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $|b| = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$. Then,

    $lcm(a, b) = p_1^{max(a_1, b_1)} p_2^{max(a_2, b_2)} \cdots p_n^{max(a_n, b_n)}$
  - E.g., $12 = 2^2 \times 3^1$, $|-21| = 3^1 \times 7^1$, $lcm(12, -21) = 2^2 \times 3^1 \times 7^1 = 84$

The Euclidean Algorithm

- Computing *gcd* using factorization can be cumbersome and time consuming since we need to find all factors of the two integers.

- Fortunately, we have an efficient algorithm, called the Euclidean algorithm. This algorithm has been known since ancient times and is named after the ancient Greek mathematician Euclid.

- Example: finding *gcd(287, 91)*

  Step 1: $287 = 91 \cdot 3 + 14$

  Step 2: $91 = 14 \cdot 6 + 7$

  Step 3: $14 = 7 \cdot 2 + 0$

  $gcd(287, 91) = gcd(91, 14) = gcd(14, 7) = 7$

SUSTech

The Euclidean Algorithm

ALGORITHM 1 The Euclidean Algorithm.

procedure $gcd(a, b$: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x$ **mod** y
    $x := y$
    $y := r$
return $x\{\gcd(a, b)$ **is** $x\}$

Works like magic :)

number of divisions: O(log min(a, b))

** will be proved later*

- Example: finding *gcd(287, 91)*

  Step 1: *287* = *91* · 3 + *14*

  Step 2: *91* = *14* · 6 + *7*

  Step 3: *14* = *7* · 2 + *0*

  gcd(287, 91) = gcd(91, 14) = gcd(14, 7) = 7

Correctness of the Euclidean Algorithm

- **Lemma:** Let $a = bq + r$, where a, b, q and r are integers, then $gcd(a, b) = gcd(b, r)$.

- Proof:

  - For any d such that $d|a$ and $d|b$, we have d also divides $a - bq = r$. Hence, any common divisor of a and b must also be a common divisor of b and r.

  - For any d such that $d|b$ and $d|r$, we have d also divides $bq + r = a$. Hence, any common divisor of b and r must also be a common divisor of a and b.

  - Therefore, $gcd(a, b) = gcd(b, r)$.

Correctness of the Euclidean Algorithm

- Proof (correctness of Euclidean algorithm):
  - Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$. We obtain the following divisions from the algorithm:

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \leq r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 && 0 \leq r_3 < r_2, \\
&\vdots \\
r_{n-2} &= r_{n-1} q_{n-1} + r_n && 0 \leq r_n < r_{n-1}, \\
r_{n-1} &= r_n q_n.
\end{aligned}
$$

- The number of divisions is finite because $r_0 > r_1 > \cdots > r_{n-1} > r_n \geq 0$. Recall **Lemma**: "Let $a = bq + r$, where a, b, q and r are integers, then $gcd(a, b) = gcd(b, r)$". We have:

  $gcd(a, b) = gcd(r_0, r_1) = \cdots = gcd(r_{n-1}, r_n) = gcd(r_n, 0) = r_n$

Bézout's Theorem

- **Bézout's Theorem:** If *a* and *b* are positive integers, then there exist integers *s* and *t* such that *gcd(a, b) = sa + tb*.
  - *s, t* are called Bézout coefficients of *a* and *b*
  - *gcd(a, b) = sa + tb* is called Bézout's identity
- Solution: (two-pass) Euclidean algorithm (e.g., *gcd(503, 286) = 1*)

$$
\begin{aligned}
503 &= 1 \cdot 286 + 217 & 1 &= 10 - 1 \cdot 9 \\
286 &= 1 \cdot 217 + 69 & &= 7 \cdot 10 - 1 \cdot 69 \\
217 &= 3 \cdot 69 + 10 & &= 7 \cdot 217 - 22 \cdot 69 \\
69 &= 6 \cdot 10 + 9 & &= 29 \cdot 217 - 22 \cdot 286 \\
10 &= 1 \cdot 9 + 1 & &= 29 \cdot 503 - 51 \cdot 286
\end{aligned}
$$

substitute for the smaller number at each step

- Better solution: (one-pass) extended Euclidean algorithm
  - *see the textbook for details and will be proved in later sections*

Corollaries of Bézout's Theorem

- **Corollary 1:** If *a, b, c* are positive integers such that $gcd(a, b) = 1$ and $a|bc$, then $a|c$.
  - Proof: Since $gcd(a, b) = 1$, by Bézout's theorem, there exist *s* and *t* such that $1 = sa + tb$. This yields $c = sac + tbc$. Since $a|bc$, we have $a|tbc$. It is also clear that $a|sac$. Therefore, we have $a|(sac + tbc) = c$.

- **Corollary 2:** If *p* is prime and $p|a_1a_2 \cdots a_n$, then $p|a_i$ for some *i*.
  - can be used to prove the uniqueness of prime factorization
  - *proved by induction in later sections*

Congruences

Congruences

- If *a* and *b* are integers and *m* is a positive integer, then *a* is congruent to *b* modulo *m* if *m* divides *a* – *b*, denoted by $a \equiv b \pmod{m}$. (In Chinese, this is called “同余”.)
  - $a \equiv b \pmod{m}$ is called a congruence and *m* is its modulus.
- Examples:
  - $15 \equiv 3 \pmod{6}$
  - $-1 \equiv 11 \pmod{6}$
- Let *m* be a positive integer. The integers *a* and *b* are congruent modulo *m* if and only if there is an integer *k* such that $a = b + km$.
  - prove the “if” part and “only if” part

Notations: *mod m* vs ≡ *(mod m)*

- The notations $a = b \bmod m$ and $a \equiv b \pmod m$ are different:
  - *mod m* denotes a function "*mod m*": $\boldsymbol{Z} \to \boldsymbol{Z}_m = \{0, 1, \ldots, m - 1\}$
  - ≡ *(mod m)* is a relation between two integers

- **Theorem:** Let *a* and *b* be integers, and *m* be a positive integer. Then, $a \equiv b \pmod m$ if and only if $a \bmod m = b \bmod m$.
  - prove the "if" part and "only if" part

Congruences of Sums and Products

- **Theorem:** Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$
  - proof by definition
- If $a \equiv b \bmod m$ and $c > 0$, then:
  - $ca \equiv cb \pmod{m}$?

    Yes
  - $c + a \equiv c + b \pmod{m}$?

    Yes
  - $a/c \equiv b/c \pmod{m}$?

    No, *e.g.,* $14 \equiv 8 \pmod{6}$ *but* $7 \not\equiv 4 \pmod{6}$

Dividing Congruences by an Integer

- **Theorem:** Let *m* be a positive integer and let *a, b, c* be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

- Proof: By the definition of $ac \equiv bc \pmod{m}$, we have $m \mid (ac - bc)$, i.e., $m \mid c(a - b)$. Since $\gcd(c, m) = 1$, it follows that $m \mid (a - b)$.

- Example:
  - $20 \equiv 56 \pmod{9}$ and $\gcd(4, 9) = 1$, then $20/4 \equiv 56/4 \pmod{9}$.

Linear Congruences

- A congruence of the form $ax \equiv b \pmod{m}$ is called a linear congruence, where m is a positive integer, a and b are integers, and x is a variable.

- The solutions to a linear congruence $ax \equiv b \pmod{m}$ are all integers x that satisfy the congruence.

- Linear congruences have been studied since ancient times.
  - About 1500 years ago, the Chinese mathematician Sun-Tsu asked: "There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?"

    有物不知其数，三三数之剩二，五五数之剩三，七七数之剩二。问物几何？
  - Translation: $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$, $x \equiv 2 \pmod{7}$, $x = ?$

Modular Multiplicative Inverse

- An integer $\bar{a}$ such that $\bar{a}a \equiv 1 \pmod{m}$ is said to be a modular multiplicative inverse (or inverse for simplicity) of a modulo m.

- If an inverse of a *modulo* m exists, say $\bar{a}$, then one can solve the linear congruence $ax \equiv b \pmod{m}$ for x by multiplying $\bar{a}$ on both sides, i.e., $x \equiv \bar{a}ax \equiv \bar{a}b \pmod{m}$.
  - Note that $x \equiv \bar{a}ax \pmod{m}$ follows from $1 \equiv \bar{a}a \pmod{m}$.
  - When does an inverse of a *mod* m exist?

- **Theorem:** If $\gcd(a, m) = 1$ and $m > 1$, then there exists an inverse of a modulo m. Furthermore, the inverse is unique modulo m.
- Proof: Since $\gcd(a, m) = 1$, from Bézout's theorem there are integers s and t such that $sa + tm = 1$, i.e., $sa + tm \equiv 1 \pmod{m}$. Since $tm \equiv 0 \pmod{m}$, it follows that $sa \equiv 1 \pmod{m}$. This means that s is an inverse of a modulo m.
  - *the proof of uniqueness is left as an exercise*

How to Find Inverses?

- Use the (extended) Euclidean algorithm
- Example: Find an inverse of *101* modulo *4620*.

$$4620 = 45 \cdot 101 + 75$$
$$101 = 1 \cdot 75 + 26$$
$$75 = 2 \cdot 26 + 23$$
$$26 = 1 \cdot 23 + 3$$
$$23 = 7 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$
$$2 = 2 \cdot 1.$$

$$\begin{aligned}
1 &= 3 - 1 \cdot 2 \\
&= 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3 \\
&= -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23 \\
&= 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = -9 \cdot 75 + 26 \cdot 26 \\
&= -9 \cdot 75 + 26 \cdot (101 - 1 \cdot 75) = 26 \cdot 101 - 35 \cdot 75 \\
&= 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101) = -35 \cdot 4620 + \boxed{1601} \cdot 101.
\end{aligned}$$

** see the textbook for the use of the extended Euclidean algorithm*

Solving Linear Congruences

- Solve $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.
- Example:
  - What are the solutions of the congruence $3x \equiv 4 \pmod{7}$?
  - Solution: First, find an inverse of *3* modulo *7*, e.g., *-2* is an inverse because $3 \cdot -2 = -6 \equiv 1 \pmod{7}$. Then, multiply both sides of the congruence by -2, we have $x \equiv -6x \equiv -8 \equiv 6 \pmod{7}$.

Number of Congruence Solutions

- **Theorem:** Let $d = gcd(a, m)$. The linear congruence $ax \equiv b \pmod m$ has solutions if and only if $d|b$. If $d|b$, then there exist exactly d solutions in $\boldsymbol{Z}_m$. Let $m' = m/d$. If x_0 is a solution, then the other $d - 1$ solutions are given by $x_0 +_m m', x_0 +_m 2m', \ldots, x_0 +_m (d - 1)m'$.

- Proof: ("only if" + "if" + "exactly d solutions")
  - "only if": If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.
  - "if": Suppose that $d|b$. Let $b = kd$. There exist integers s, t such that $d = as + mt$. Multiplying both sides by k, we have $b = ask + mtk$. Let $x_0 = sk$ and we have $ax_0 \equiv b \pmod m$.
  - "# = d": $ax_0 \equiv b \pmod m$ and $ax_1 \equiv b \pmod m$ imply that $m|a(x_1 - x_0)$. Dividing both sides by d results in $m'|a'(x_1 - x_0)$, where $a' = a/d$. Since d is the GCD of a and m, we have $gcd(m', a') = 1$ and hence $m'|(x_1 - x_0)$, i.e., $x_1 = x_0 + km'$. This relation implies that there are at most $d = m/m'$ distinct solutions in $\boldsymbol{Z}_m$. Then, it is not hard to see that if x_0 is a solution $x_0 +_m km'$ is also a solution.

The Chinese Remainder Theorem

- **The Chinese Remainder Theorem:** Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers ≥ 2 and let $a_1, a_2, \ldots, a_n$ be arbitrary integers. Then, the system of congruences

$$x \equiv a_1 \pmod{m_1}$$
$$x \equiv a_2 \pmod{m_2}$$
$$\ldots$$
$$x \equiv a_n \pmod{m_n}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

- Proof: Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$. Since $\gcd(m_k, M_k) = 1$, there exists an integer y_k, an inverse of M_k modulo m_k, such that $M_k y_k \equiv 1 \pmod{m_k}$. Let $x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n$. It is not hard to check that x is a solution to the n congruences.
  - *the proof of uniqueness is left as an exercise*

The Chinese Remainder Theorem

- Example:

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

三人同行七十稀，五树梅花廿一枝，
七子团圆正月半，除百零五便得知。
-- 程大位 《算法统要》 (1593年)

- Solution (using the Chinese remainder theorem):
  - Let $m = 3 \cdot 5 \cdot 7 = 105$
  - $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$

$35 \cdot 2 \equiv 1 \pmod 3$ $\quad y_1 = 2$

$21 \equiv 1 \pmod 5$ $\quad y_2 = 1$

$15 \equiv 1 \pmod 7$ $\quad y_3 = 1$

  - $x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \pmod{105}$

** What if the moduli are not pairwise coprime? (left as an exercise)*

Back Substitution

- Example:

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

- Solution (using back substitution):
  - there exists an integer t such that $x = 3t + 2$
  - substitute this into the 2nd congruence: $3t + 2 \equiv 3 \pmod 5$ and solve it as $t \equiv 2 \pmod 5$, *i.e.*, $t = 5u + 2$ for some integer u
  - substitute this back into $x = 3t + 2$ shows $x = 15u + 8$
  - substitute this into the 3rd congruence: $15u + 8 \equiv 2 \pmod 7$ and we can solve it as $u \equiv 1 \pmod 7$, *i.e.*, $u = 7v + 1$ for some integer v
  - substitute this back into $x = 15u + 8$ tells us $x = 105v + 23$
  - therefore, $x \equiv 23 \pmod{105}$

Fermat's Little Theorem

- **Fermat's Little Theorem:** If p is prime and $a \not\equiv 0 \pmod{p}$, then

  $$a^{p-1} \equiv 1 \pmod{p}$$

  Furthermore, for every integer a we have

  $$a^{p} \equiv a \pmod{p}$$

  - *the proof is left as an exercise*

- Example: $7^{222} \equiv ? \pmod{11}$
  - $7^{222} = 7^{22 \cdot 10+2} = (7^{10})^{22} 7^{2} \equiv 1^{22} \cdot 49 \equiv 5 \pmod{11}$

- *What is **Fermat's Last Theorem**?*
  - *The equation $a^n + b^n = c^n$, where the integer $n > 2$, has no integer solutions a, b, c such that $abc \neq 0$. * the first proof found in 1990s*

Euler's Theorem

- Euler's totient function $\phi(n)$ maps a positive integer n to the number of positive integers coprime to n in $\boldsymbol{Z}_n$.

- Examples: (p, q are prime)
  - $\phi(p) = p - 1$
  - $\phi(pq) = (p - 1)(q - 1)$
  - $\phi(p^i) = p^i - p^{i-1}$

  ** count $\phi(n)$ by excluding the integers divisible by n's prime factors*

- **Euler's Theorem:** Let a, n be positive coprime integers. Then

  $$a^{\phi(n)} \equiv 1 \pmod{n}$$

  - when n is prime, this becomes Fermat's Little Theorem for $a > 0$
  - *the proof is very similar to that of Fermat's Little Theorem*

Primitive Roots

- A primitive root modulo a prime p is an integer $r \in \boldsymbol{Z}_p$ such that every nonzero element of $\boldsymbol{Z}_p$ is a power of $r \bmod p$.

- Examples:
  - Is *3* is a primitive root modulo *5*?

    Yes. $3 \equiv 3^1 \pmod 5$, $4 \equiv 3^2 \pmod 5$, $2 \equiv 3^3 \pmod 5$, $1 \equiv 3^4 \pmod 5$

  - Is *2* a primitive root modulo *7*?

    No. $2 \equiv 2^1 \pmod 7$, $4 \equiv 2^2 \pmod 7$, $1 \equiv 2^3 \pmod 7$, $2 \equiv 2^4 \pmod 7$…

    ** already cycles so will never reach 3, 5, 6*

Primitive Roots

- A primitive root modulo a prime p is an integer $r \in \boldsymbol{Z}_p$ such that every nonzero element of $\boldsymbol{Z}_p$ is a power of $r \bmod p$.

- Let $\boldsymbol{Z}_n^* = \{k \in \boldsymbol{Z}_n \mid gcd(k, n) = 1\}$ and consider an arbitrary integer $n \geq 2$. A primitive root modulo n is an integer $r \in \boldsymbol{Z}_n^*$ such that every element of $\boldsymbol{Z}_n^*$ is a power of $r \bmod n$.
  - When n is prime, this definition becomes the above one.

- **Theorem:** There exists a primitive root modulo n $(n \geq 2)$ if and only if $n = 2, 4, p^k$ or $2p^k$, where p is an odd prime and $k \in \boldsymbol{Z}^+$.
  - *the proof is advanced and beyond the scope of this course*

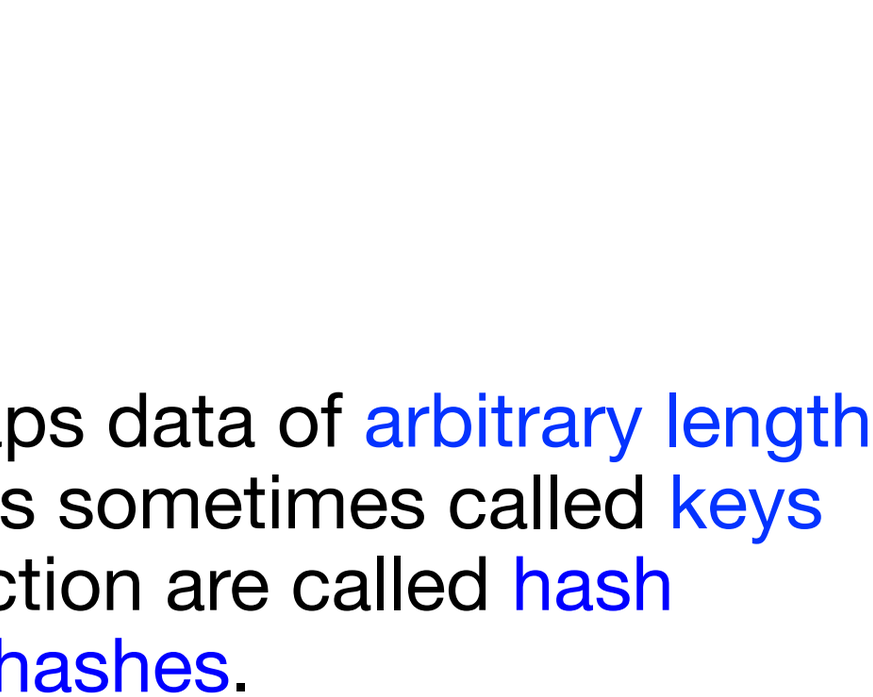
- Primitive roots are very useful in cryptography (as shown later).

Applications of Modular Arithmetic

Hash Functions

- A hash function *h* is a function that maps data of arbitrary length to fixed-length values. The input data is sometimes called keys and the values returned by a hash function are called hash values, hash codes, digests or simply hashes.
  - example use case: hashing the identifiers (keys) of data records to their memory locations (hash values)
  - example function: $h(k) = k \bmod m$
- How to handle hash collisions? (two keys hashed to the same value)
  - move to the next available hash value
  - add a secondary structure

    e.g., hash pointing to a linked list



SUSTech

Pseudorandom Number Generators

- Pseudorandom numbers are generated by systematic methods to approximate truly random numbers. A pseudorandom number generator (PRNG) is an algorithm for generating them.
  - PRNGs are widely used in simulation and cryptography.

- The most commonly used procedure for generating pseudorandom numbers is the linear congruential method:
  - choose *4* numbers: modulus m, multiplier a, increment c, seed x_0
  - generate a sequence of pseudorandom numbers $\{x_n\}$ in $\boldsymbol{Z}_m$:

    $x_{n+1} = (ax_n + c) \bmod m$

Check Digits

- Congruences are used to check for errors in digit strings.

- A parity check bit is an extra bit appended to each data block that is stored or transmitted. The parity check bit x_{n+1} for the bit string $x_1x_2\cdots x_n$ is defined as:

$$x_{n+1} = (x_1 + x_2 + \cdots + x_n) \bmod 2$$

- All books are identified by an International Standard Book Number (ISBN-10), a 10-digit code $x_1x_2\cdots x_{10}$, assigned by the publisher. An ISBN-10 is valid if and only if the check digit x_{10} is computed as:

$$x_{10} = (x_1 + 2x_2 + \cdots + 9x_9) \bmod 11$$

  - What is the check digit for the ISBN-10 starting with *007288008*?

    2

  - Is *084930149X* a valid ISBN-10 (where $X = 10$)?

    No

Introduction to Cryptography

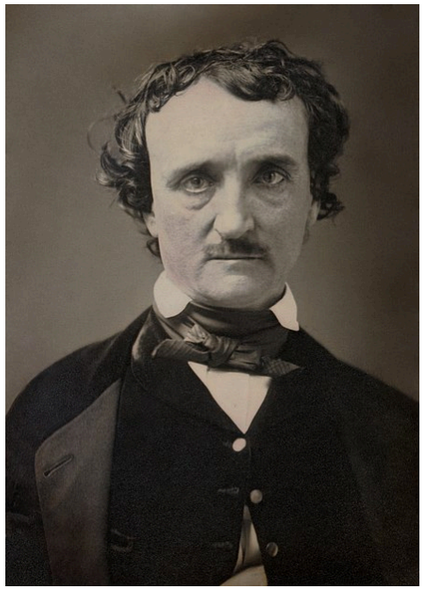
Cryptography and Number Theory

- Roughly, cryptography is the subject of transforming information so that it cannot be easily recovered without special knowledge.
  - "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called adversaries." – *Ronald L. Rivest* (Turing Award winner)

- Number theory plays an important role in cryptography:
  - classical ciphers (before modern cryptography)
  - public-key cryptosystems

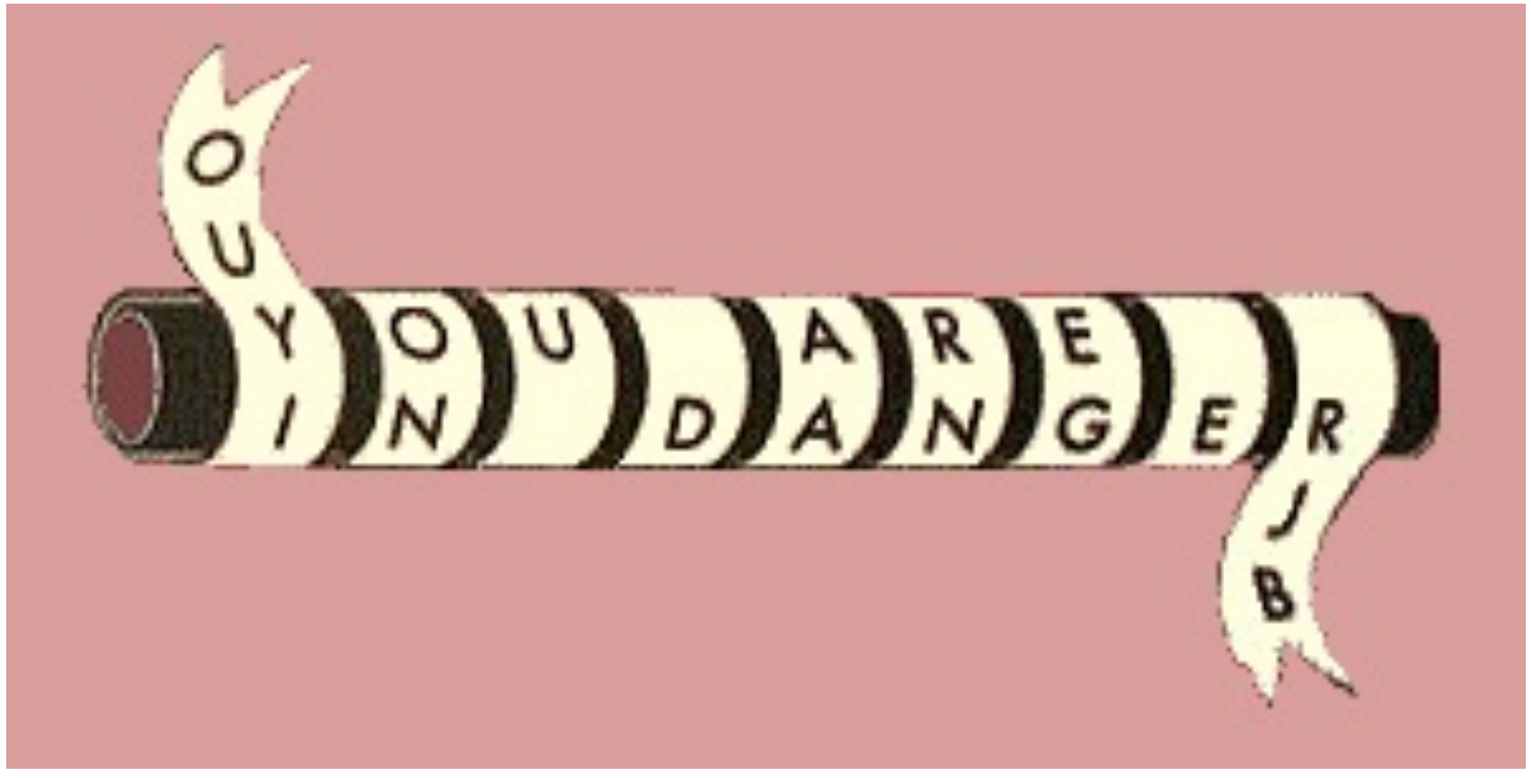
Origin of Cryptography

- History of almost 4000 years (from 1900 B.C.)
- Cryptography = kryptos (secret) + graphos (writing)
- This term was first used in the short story *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).
- “Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve.” – 1841



Classic Cryptography

- In 405 BC, the Spartan general Lysander was sent a coded message written on the inside of a servant's belt.



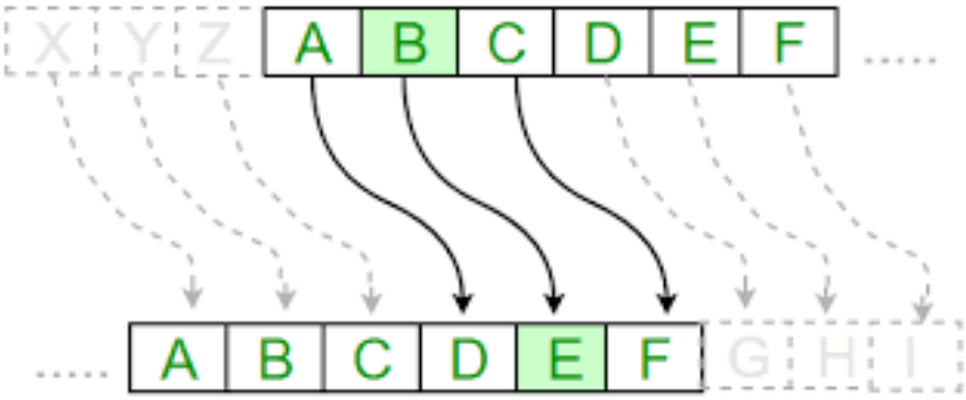
Classic Cryptography

- The Greeks invented a cipher which changed letters to numbers. A form of this code was still being used during World War I.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Classic Cryptography

- Caesar Cipher (named after Roman general Julius Caesar)

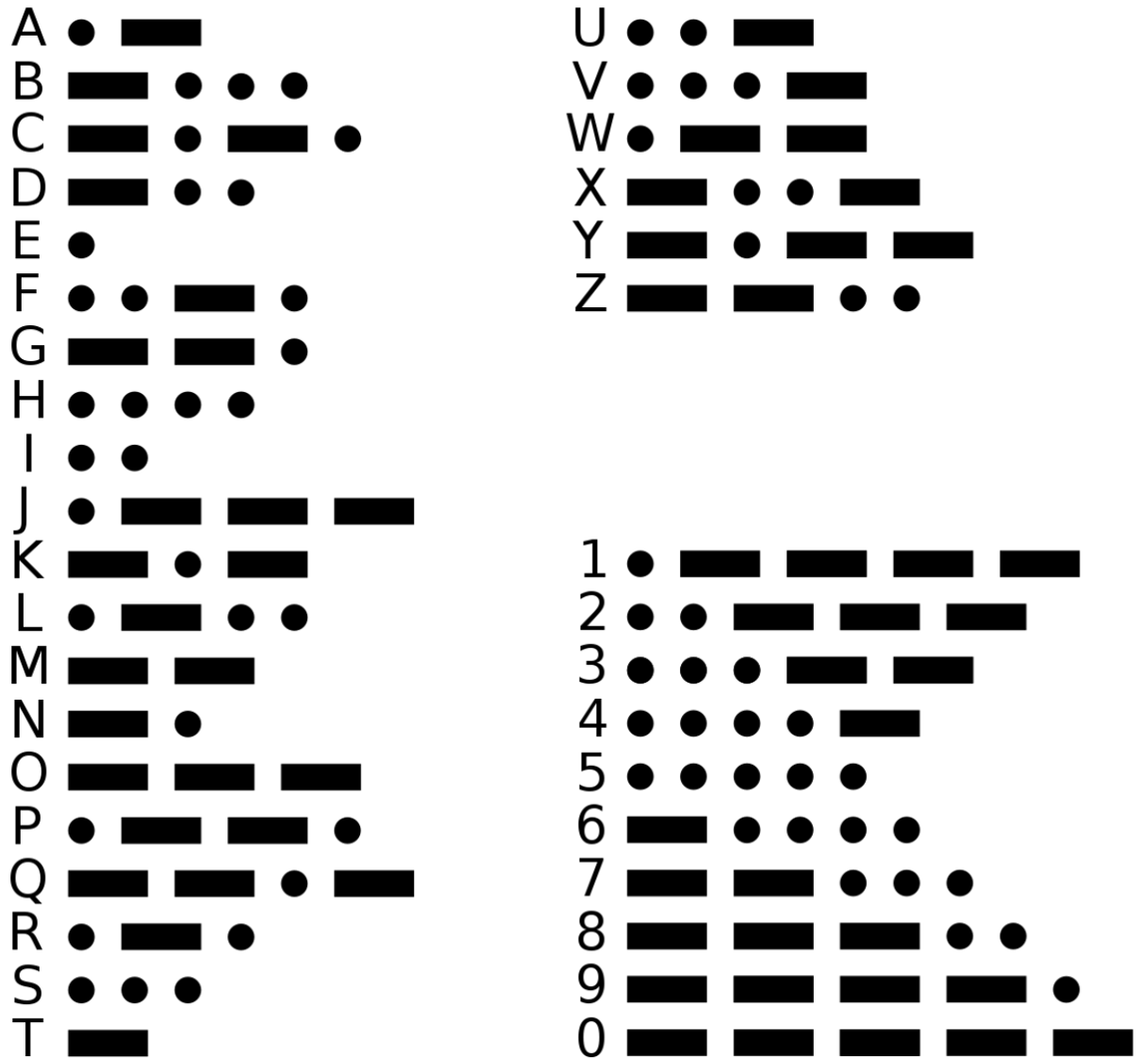


VENI, VIDI, VICI

YHQL YLGL YLFL

Classic Cryptography

- Morse Code: created by Samuel Morse in 1838



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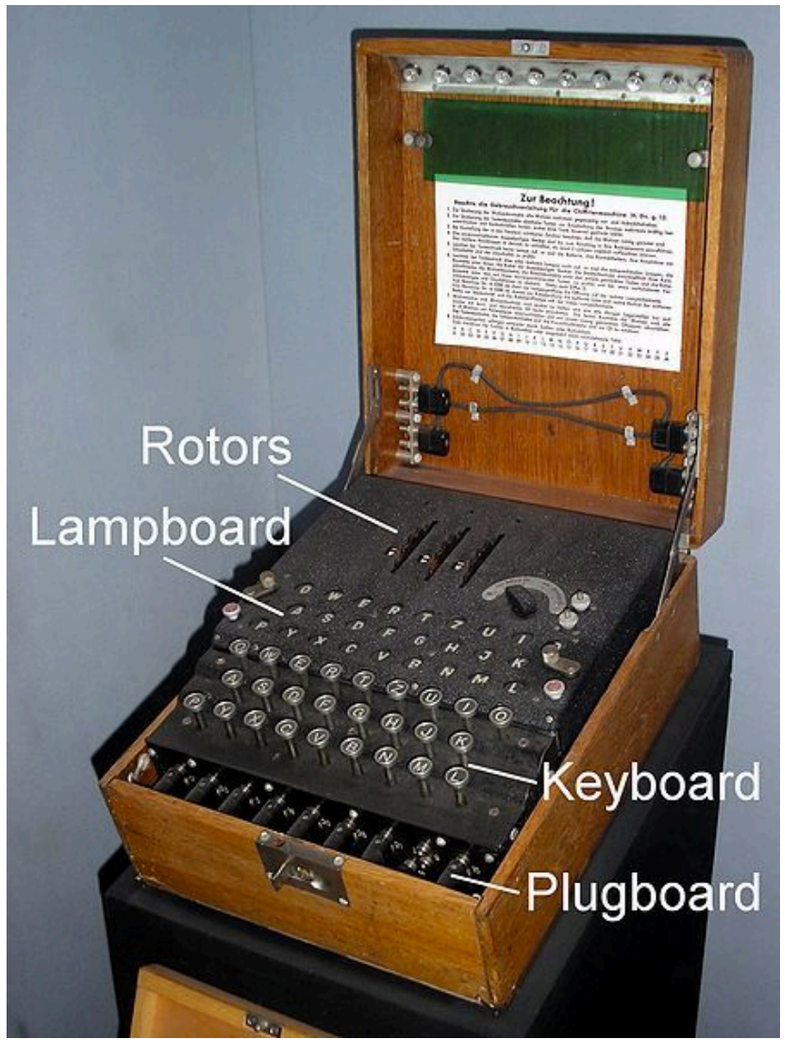
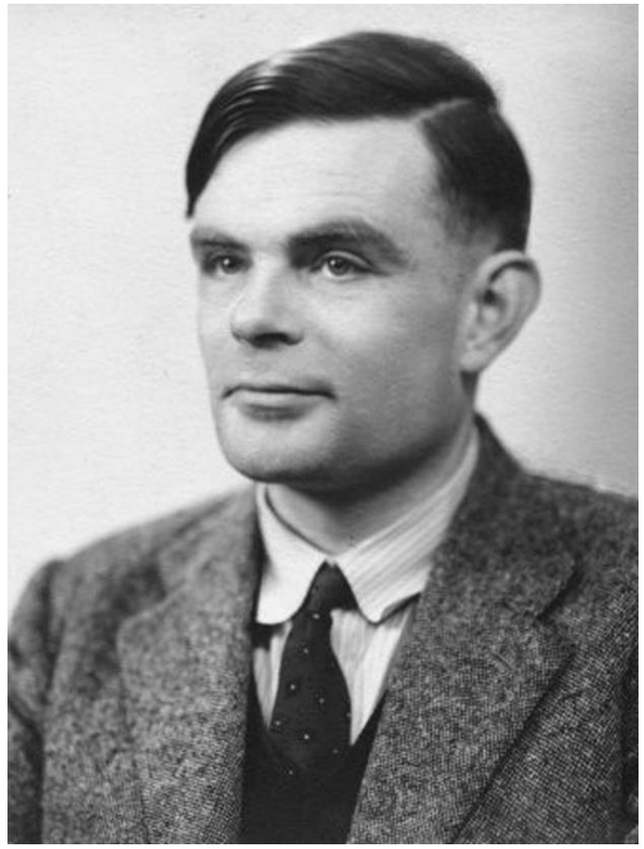
Classic Cryptography

- Cryptograms from the Chinese gold bars
  - http://www.iacr.org/misc/china/china.html

Classic Cryptography

- Enigma: German coding machine in World War II.





Alan Turing
(1912-1954)

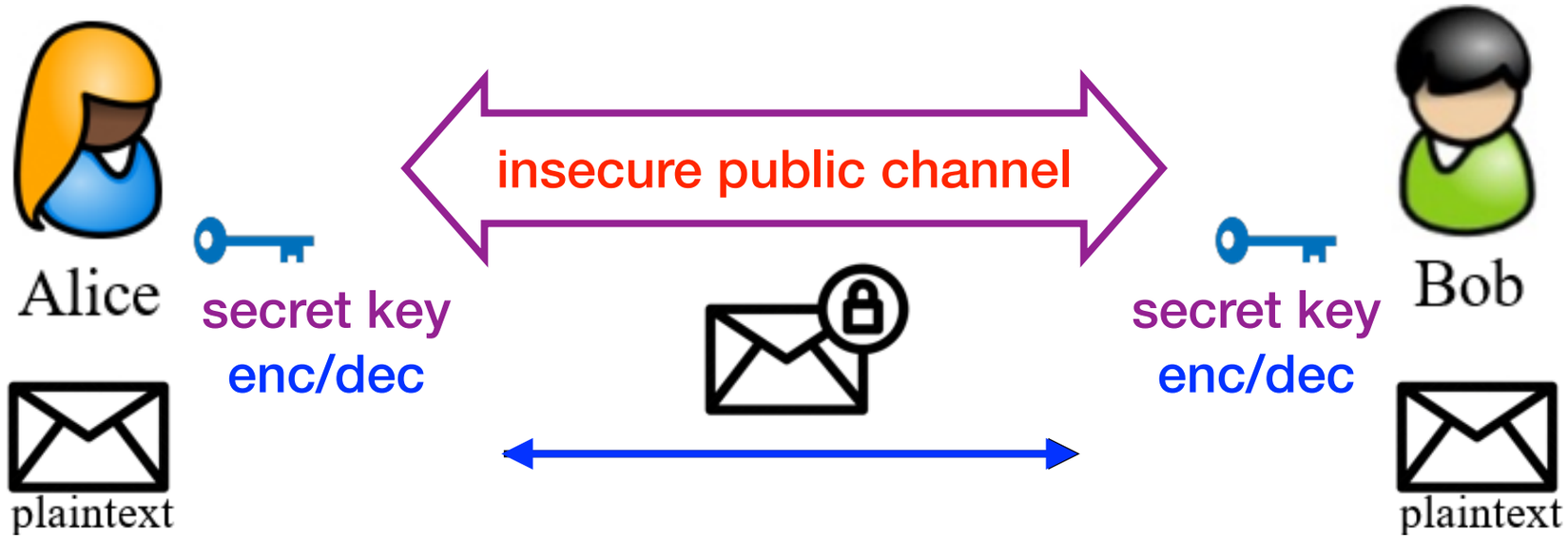
Modern Cryptography

- Modern cryptography (since the early 20th century) makes extensive use of math and mainly consists of two parts:
  - symmetric cryptography (also called secret-key cryptography)
  - asymmetric cryptography (also called public-key cryptography)
- Kerckhoffs's principal (1883): a cryptosystem should be secure, even if everything about the system, except the key, is public knowledge.
  - security through obscurity does not work

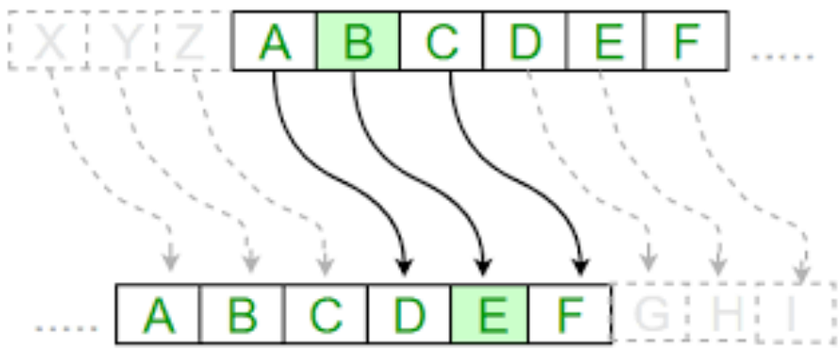
Symmetric Cryptography

Symmetric Cryptography

- Symmetric cryptography focuses on cryptosystems where the message sender and receiver share the same secret key for both encryption and decryption.



Caesar Cipher



- Key: *k = 0,1, …, 25*
- Encryption: encrypt *m* as *(m + k) mod 26*
- Decryption: decrypt *c* as *(c – k) mod 26*
- Example: *k = 2*
  - plaintext: SEND REINFORCEMENT
  - ciphertext: UGPF TGKPHQTEGOGPV
- Problem: only *26* possible keys!

Substitution Cipher

- Key: table mapping each letter to another letter

A	B	C																					Z
V	R	E																					D

- Encryption & Decryption: letter by letter according to table

- Number of possible keys: $26! \approx 4 \times 10^{26}$

- However, substitution cipher is still insecure!

- Key observation: one can recover the plaintext by analyzing the letter frequencies

Substitution Cipher

Table 1: Relative frequencies of the letters of the English language

Letter	Relative Frequency (%)	Letter	Relative Frequency (%)
a	8.167	n	6.749
b	1.492	o	7.507
c	2.782	p	1.929
d	4.253	q	0.095
e	12.702	r	5.987
f	2.228	s	6.327
g	2.015	t	9.056
h	6.094	u	2.758
i	6.966	v	0.978
j	0.153	w	2.360
k	0.772	x	0.150
l	4.025	y	1.974
m	2.406	z	0.074

Substitution Cipher

Table 2: Number of Diagraphs Expected in 2,000 Letters of English Text

th	-	50	at	-	25	st	-	20
er	-	40	en	-	25	io	-	18
on	-	39	es	-	25	le	-	18
an	-	38	of	-	25	is	-	17
re	-	36	or	-	25	ou	-	17
he	-	33	nt	-	24	ar	-	16
in	-	31	ea	-	22	as	-	16
ed	-	30	ti	-	22	de	-	16
ne	-	30	to	-	22	rt	-	16
ha	-	26	it	-	20	ve	-	16

Table 3: The 15 Most Common Trigraphs in the English Language

1	-	the	6	-	tio	11	-	edt
2	-	and	7	-	for	12	-	tis
3	-	tha	8	-	nde	13	-	oft
4	-	ent	9	-	has	14	-	sth
5	-	ion	10	-	nce	15	-	men

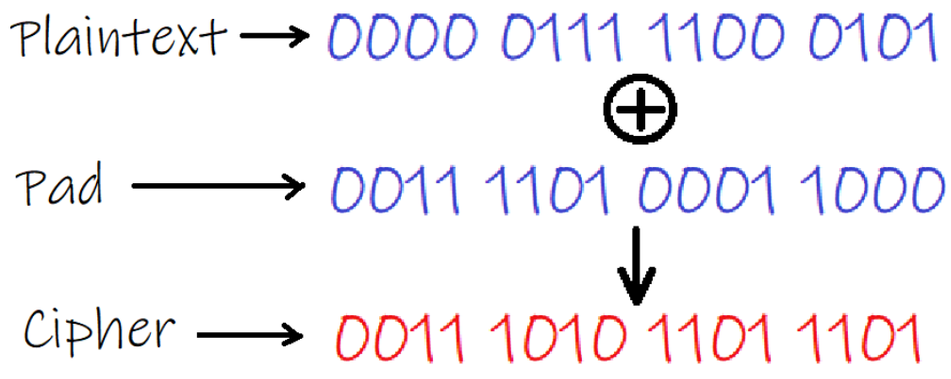
Substitution Cipher

- Ciphertext: LIVITCSWPIYVEWHEVSRIQMXLEYVEOIEWHRXEXIP FEMVEWHKVSTYLXZIXLIKIIXPIJVSZEYPERRGERIMWQLMGLM XQERIWGPSRIHMXQEREKI

- Frequency analysis:
  - I – most common letter I = e
  - LI – most common pair L = h
  - XLI – most common triple X = t
  - LIVI = he?e V = r
  - …

- Plaintext: HereUpOnLeGrandAroseWithAGraveAndStatelyAirAnd BroughtMeTheBeetleFromAGlassCaseInWhichItWasEnclosedItW asABe

One-Time Pad (OTP)



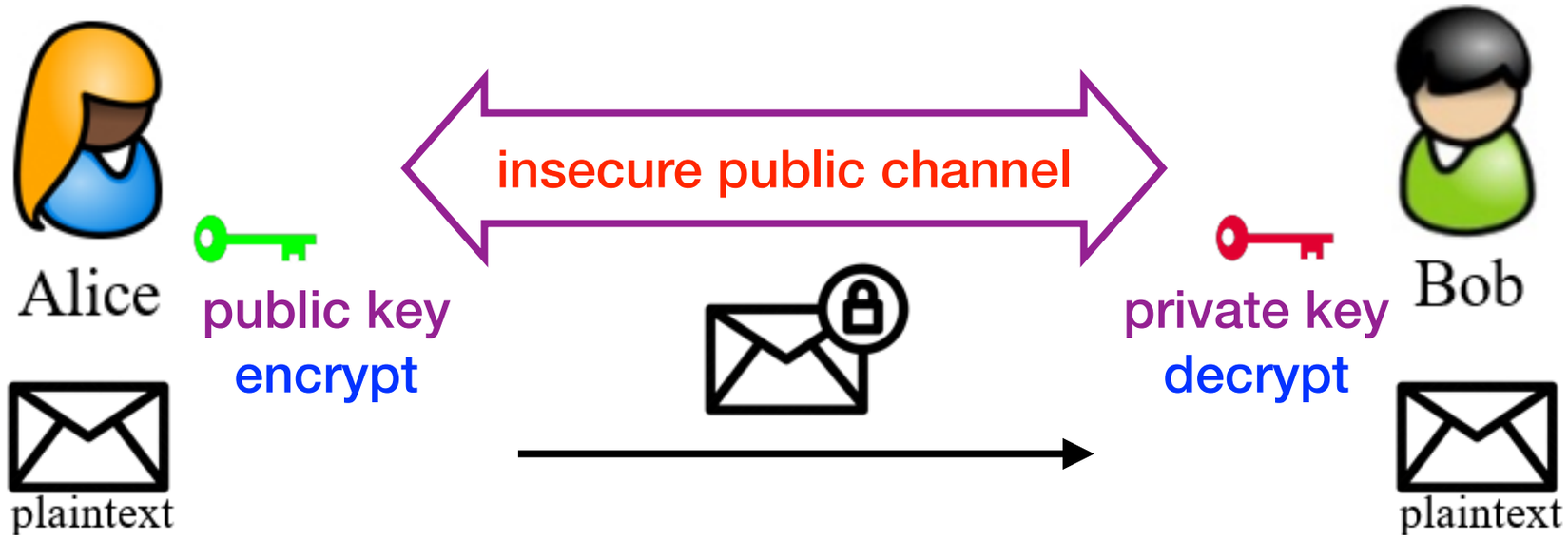
- Key (or pad): k = random binary string as long as the plaintext
- Encryption & Decryption: xor with the one-time pad (key) k
- Perfect secrecy: secure against even unlimited computing power
- End of story for symmetric cryptography?
  - ***No!*** E.g., no integrity, very long random and one-time keys, etc.

** How to solve these issues? Take a cryptography course :)*

Asymmetric Cryptography

Asymmetric Cryptography

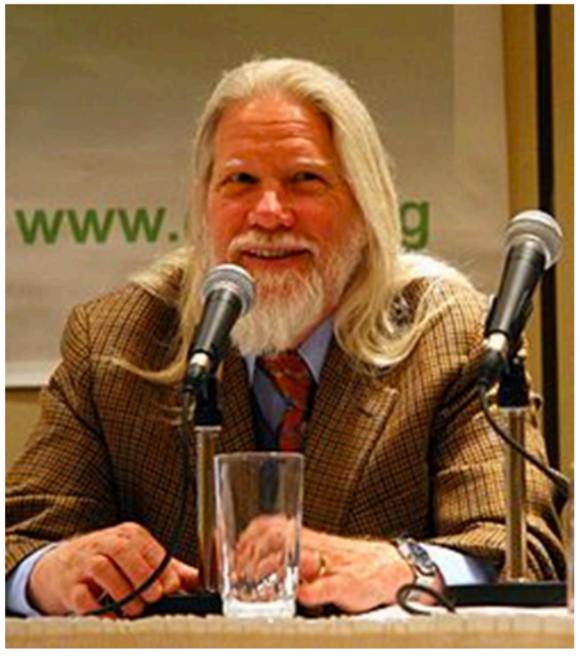
- Asymmetric cryptography focuses on cryptosystems where the message sender holds a pubic encryption key and the message receiver holds a private decryption key.



Sounds magic, right?

Public-Key Cryptography

- Public-key cryptography becomes real since the important breakthrough by Diffie and Hellman in 1976.
  - W. Diffie, M. Hellman, "*New directions in cryptography*", IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.
  - "We stand today on the brink of a revolution in cryptography."



Bailey W. Diffie

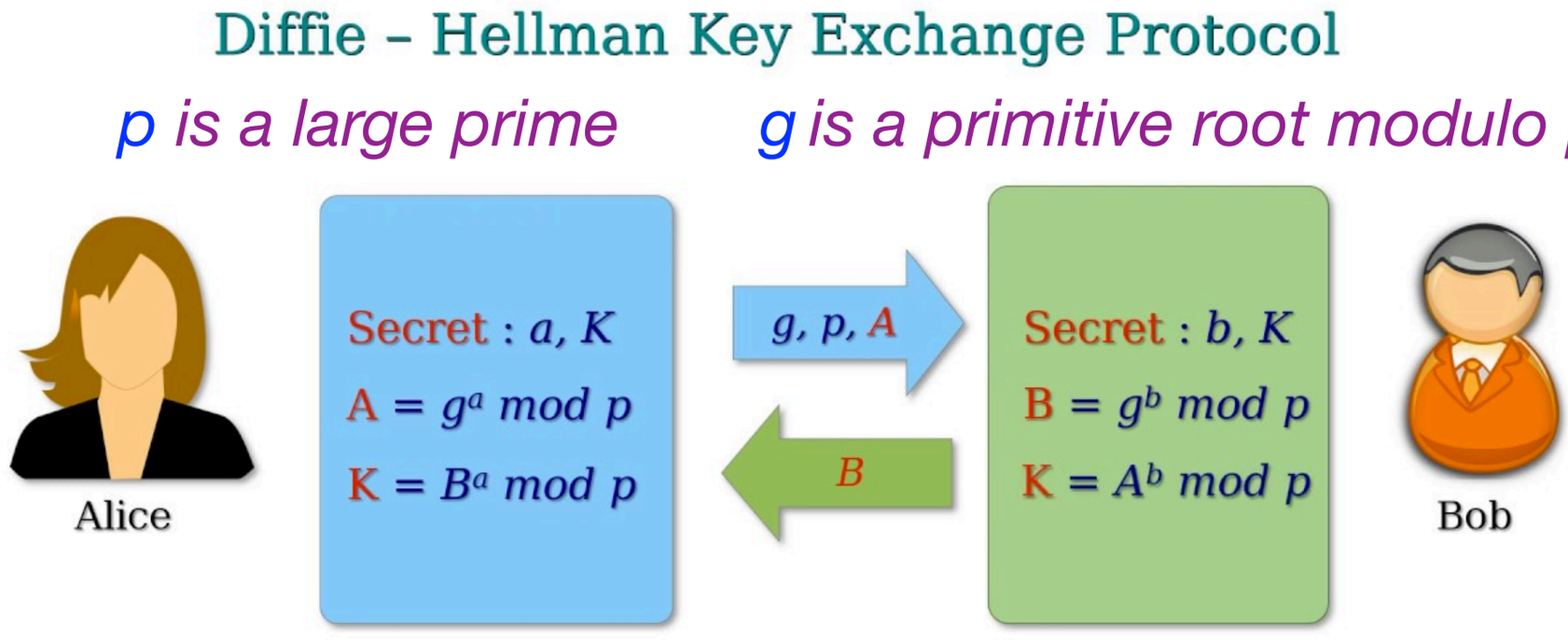
Martin E. Hellman

2015 Turing Award

for fundamental contributions to modern cryptography

Diffie-Hellman (DH) Key Exchange

- How to securely exchange keys (i.e., establish a shared secret) between two users over an insecure public channel?



Why is this secure?

under modular arithmetic:

knowing $A = g^a$ *and* $B = g^b$ *does not help derive* $K = g^{ab}$

The Discrete Logarithm Problem

- The discrete logarithm of an integer y to the base b modulo m is an integer x such that $b^x \equiv y \pmod{m}$.

- The discrete logarithm problem (DLP) is defined as "given m, b, y, find the discrete logarithm x".
  - In cryptography, usually m is a prime p and b is a primitive root g.
  - It is believed that DLP is very hard and cannot be solved by any polynomial-time algorithms, i.e., DLP is not in class ***P***.

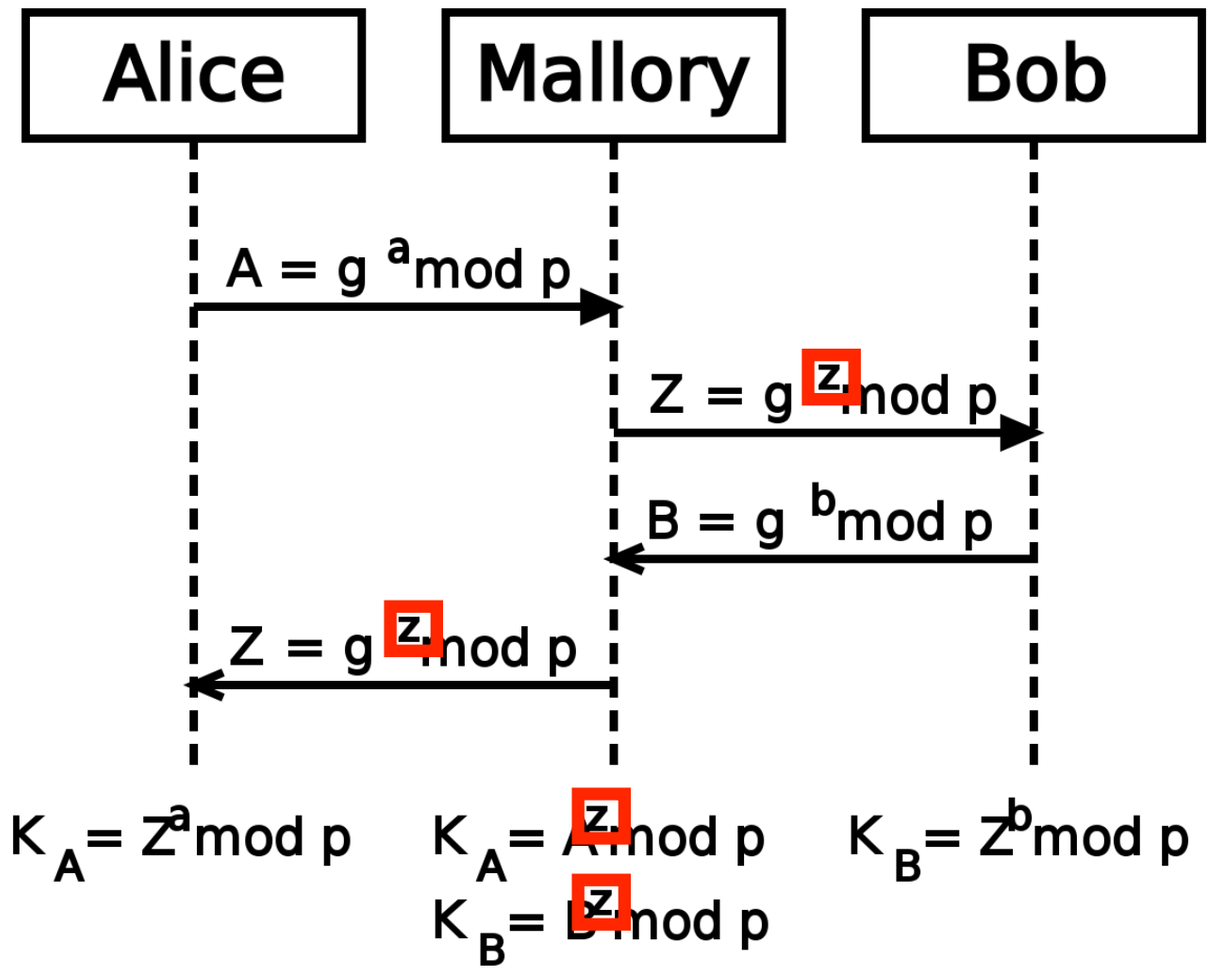
- Security of DH key exchange is based on the hardness of DLP.
  - actually based on stronger hardness assumptions (omitted here)
  - actually only secure against passive attackers is guaranteed:

    knowing $A = g^a$ and $B = g^b$ does not help derive $K = g^{ab}$

** What if the attacker is active (also called man-in-the-middle attacks)?*

Man-In-The-Middle Attacks

- Diffie-Hellman key exchange is insecure against active man-in-the-middle (MITM) attackers (e.g., see Mallory below).



Wonder how this can be prevented? Take a cryptography course :)

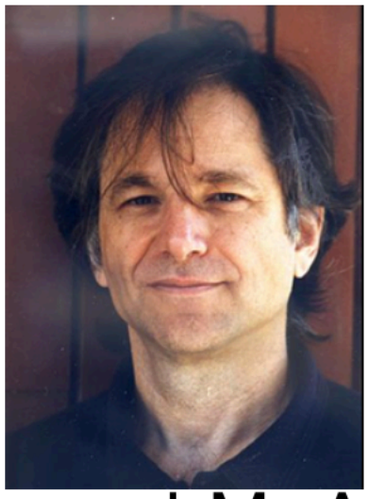
SUSTech

The RSA Cryptosystem

- The widely used public-key cryptosystem RSA was invented by Rivest, Shamir and Adleman in 1977.
  - R. Rivest, A. Shamir, L. Adleman, “A method for obtaining digital signatures and public-key cryptosystems”, Communications of the ACM, vol. 21-2, pages 120-126, 1978.

Ronald L. Rivest

Adi Shamir



Leonard M. Adleman

2002 Turing Award

for their ingenious contribution for making public-key cryptography useful in practice

RSA Encryption and Decryption

- Pick two large primes, p and q. Let $n = pq$, so $\phi(n) = (p - 1)(q - 1)$. The public encryption key e and private decryption key d are selected such that $gcd(e, \phi(n)) = 1$ and $ed \equiv 1 \pmod{\phi(n)}$.

- Encryption: encrypt m as $c = m^e \bmod n$ * *(n, e) is the public key*

- Decryption: decrypt c as $m = c^d \bmod n$ * *d is the private key*

- Correctness: For each integer $m \in \boldsymbol{Z}_n$ we have $m^{ed} \equiv m \pmod{n}$.
  - the proof is quite straightforward by applying Euler's theorem

- Security of RSA is based on the hardness of factoring big integers.
  - actually based on stronger hardness assumption (omitted here)

- Note that, besides d, the values p, q, $\phi(n)$ must be kept secret!
  - e.g., finding $\phi(n)$ is equivalent to factoring $n = pq$

RSA Digital Signature

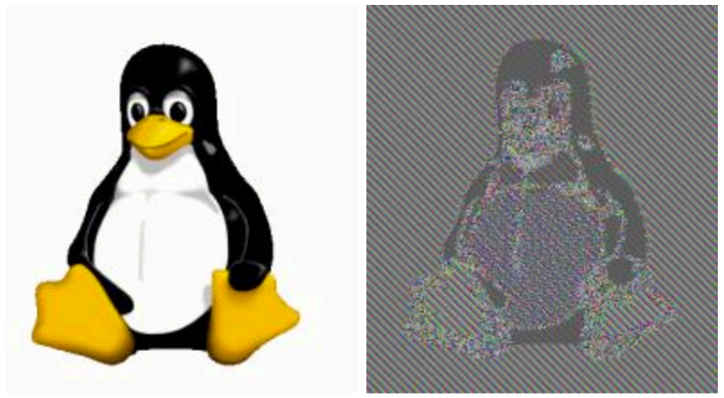
- Pick two large primes, p and q. Let $n = pq$, so $\phi(n) = (p - 1)(q - 1)$. The private signing key d and public verification key e are selected such that $gcd(e, \phi(n)) = 1$ and $ed \equiv 1 \pmod{\phi(n)}$.
- Signing: sign m as $s = m^d \bmod n$ * *d is the private key*
- Verification: verify (m, s) as $m = s^e \bmod n$ * *(n, e) is the public key*
- Correctness: For each integer $m \in \boldsymbol{Z}_n$ we have $m \equiv m^{de} \pmod{n}$.
  - the proof is quite straightforward by applying Euler's Theorem

Basically sign with decryption and verify with encryption.

- Security of RSA is based on the hardness of factoring big integers.
  - actually based on stronger hardness assumption (omitted here)
- Note that, besides d, the values p, q, $\phi(n)$ must be kept secret!
  - e.g., finding $\phi(n)$ is equivalent to factoring $n = pq$

Security of RSA

- In practice, RSA keys are typically 1024 to 2048 bits long and the random large primes p and q are sampled in a good way.
- The textbook/plain RSA (that we described) is not secure!
  - deterministic encryption: same ciphertext for same message



  - forgeable signature: new signatures derived from existing ones
  - …

Again, please take a cryptography course to know more :)

Cryptography Topics and Applications

Symmetric Cryptography

- Encryption
- Stream ciphers
- Block ciphers
- Chosen plaintext attacks
- Message integrity
- Message integrity from universal hashing
- Message integrity from collision resistant hashing
- Authenticated encryption
- …

Asymmetric Cryptography

- Public key tools
- Public key encryption
- Chosen ciphertext secure public-key encryption
- Digital signatures
- Fast signatures from one-way functions
- Elliptic curve cryptography and pairings
- Post-quantum cryptography: lattices and isogenies
- Analysis of number theoretic assumptions
- …

Cryptographic Protocols

- Protocols for identification and login
- Identification and signatures from sigma protocols
- Proving properties in zero-knowledge
- Modern proof systems
- Authenticated key exchange
- Two-party and multi-party secure computation
- …

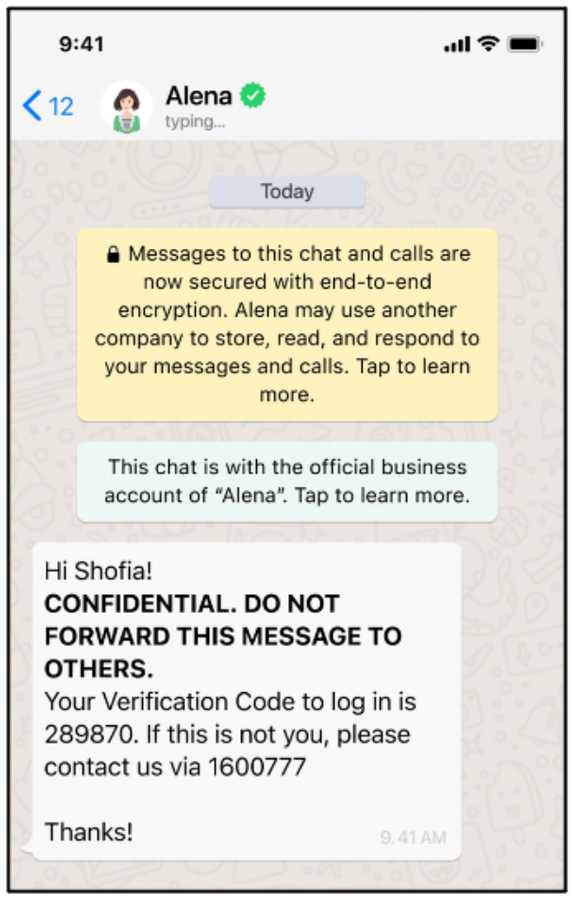
Real-World Cryptography

- Cryptography is widely used in the real world! ** my research area*

Secure Connections

Secure Authentication

Secure Messaging



Searchable Encryption

Blockchain Technology

Cryptography is a very cool and young area with lots of fascinating topics. If you like mathematics and algorithms, cryptography is a great choice!

SUSTech

06 Induction and Recursion

To be continued…