Lecture Notes 1: Matrix Algebra
Part A: Vectors and Matrices

Peter J. Hammond
My email is p.j.hammond@warwick.ac.uk
or hammond@stanford.edu
A link to these lecture slides can be found at
https://web.stanford.edu/~hammond/pjhLects.html

revised 2020 September 14th

Outline

Example of Two Equations in Two Unknowns

It is easy to check that

$$\left.\begin{aligned} x + y &= 10 \\ x - y &= 6 \end{aligned}\right\} \Longrightarrow x = 8,\; y = 2$$

More generally, one can:

1. add the two equations, to eliminate y;
2. subtract the second equation from the first, to eliminate x.

This leads to the following transformation

$$\left.\begin{aligned} x + y &= b_1 \\ x - y &= b_2 \end{aligned}\right\} \Longrightarrow \begin{cases} 2x = b_1 + b_2 \\ 2y = b_1 - b_2 \end{cases}$$

of the two equation system with general right-hand sides.

Obviously the solution is

$$x = \tfrac{1}{2}(b_1 + b_2),\; y = \tfrac{1}{2}(b_1 - b_2)$$

Using Matrix Notation, I

Matrix notation allows the two equations

$$1x + 1y = b_1$$
$$1x - 1y = b_2$$

to be expressed as

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or as $\mathbf{Az} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Here $\mathbf{A}, \mathbf{z}, \mathbf{b}$ are respectively: (i) the coefficient matrix; (ii) the vector of unknowns; (iii) the vector of right-hand sides.

Using Matrix Notation, II

Also, the solution $x = \frac{1}{2}(b_1 + b_2),\ y = \frac{1}{2}(b_1 - b_2)$ can be expressed as

$$\begin{aligned} x &= \tfrac{1}{2} b_1 + \tfrac{1}{2} b_2 \\ y &= \tfrac{1}{2} b_1 - \tfrac{1}{2} b_2 \end{aligned}$$

or as

$$\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{Cb}, \quad \text{where} \quad \mathbf{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Two General Equations

Consider the general system of two equations

$$\begin{aligned} ax + by &= u = 1u + 0v \\ cx + dy &= v = 0u + 1v \end{aligned}$$

in two unknowns x and y, filled in with some extra 1s and 0s.

In matrix form, these equations can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

For simplicity, we assume throughout the subsequent analysis that the coefficients a, b, c, d on the left-hand sides are all $\neq 0$.

Three Different Cases

We are considering the two equations

$$ax + by = u \quad \text{and} \quad cx + dy = v$$

They correspond to the two straight lines

$$y = (u - ax)/b \quad \text{and} \quad y = (v - cx)/d$$

The two lines have respective slopes $-a/b$ and $-c/d$.

These two slopes are equal iff $a/b = c/d$, or iff $ad = bc$,
or iff $D := ad - bc = 0$.

We will distinguish three cases:

(A) If $D \neq 0$, the two lines have different slopes,
so their intersection consists of a single point.

(B) If $D = 0$ and $u/b \neq v/d$, then the two lines
are parallel but distinct, so their intersection is empty.

(C) If $D = 0$ and $u/b = v/d$, then the two lines are identical,
so their intersection consists of all the points on either line.

First Steps

We assume that $a \neq 0$ in the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We can eliminate x from the second equation by adding $-c/a$ times the first row to the second.

Given $D = ad - bc$, we obtain the new equality

$$\begin{pmatrix} a & b \\ 0 & D/a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Now multiply the second row by a to obtain

$$\begin{pmatrix} a & b \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Two General Equations: Case A

The equations are

$$\begin{pmatrix} a & b \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

In **Case A** when $D := ad - bc \neq 0$,
we can add $-b/D$ times the second row to the first, which yields

$$\begin{pmatrix} a & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + (bc/D) & -ab/D \\ -c & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Recognizing that $1 + (bc/D) = (D + bc)/D = ad/D$,
then dividing the two rows/equations by a and D respectively,
we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

This implies the unique solution

$$x = (1/D)(du - bv) \quad \text{and} \quad y = (1/D)(av - cu)$$

Two General Equations: Cases B and C

Beyond Case A, when $D := ad - bc = 0$,
the multiplier $-ab/D$ is undefined and the system

$$\begin{pmatrix} a & b \\ 0 & D/a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

collapses to $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v - cu/a \end{pmatrix}$.

This leaves us with two cases:

Case B) If $cu \neq av$, there is no solution.

We are trying to find the intersection of two distinct parallel lines.

Case C) If $cu = av$, then the second equation reduces to $0 = 0$.

There is a continuum of solutions
satisfying the one remaining equation $ax + by = u$,
or $x = (u - by)/a$ where y is any real number.

Outline

Vectors and Inner Products

Let $\mathbf{x} = (x_i)_{i=1}^m \in \mathbb{R}^m$ denote a column m-vector of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

Its transpose is the row m-vector

$$\mathbf{x}^\top = (x_1, x_2, \ldots, x_m).$$

Given a column m-vector $\mathbf{x}$ and row n-vector $\mathbf{y}^\top = (y_j)_{j=1}^n \in \mathbb{R}^n$ where $m = n$, the dot or scalar or inner product is defined as

$$\mathbf{y}^\top \mathbf{x} := \mathbf{y} \cdot \mathbf{x} := \sum_{i=1}^n y_i x_i.$$

But when $m \neq n$, the scalar product is not defined.

Exercise on Quadratic Forms

Exercise

Consider the **quadratic form** $f(\mathbf{w}) := \mathbf{w}^\top \mathbf{w}$
as a function $f : \mathbb{R}^n \to \mathbb{R}$ *of the column n-vector* $\mathbf{w}$.

Explain why $f(\mathbf{w}) \geq 0$ *for all* $\mathbf{w} \in \mathbb{R}^n$,
with equality if and only if $\mathbf{w} = \mathbf{0}$,
where $\mathbf{0}$ *denotes the zero vector of* $\mathbb{R}^n$.

Net Quantity Vectors

Suppose there are n commodities numbered from $i = 1$ to n.

Each component q_i of the net quantity vector $\mathbf{q} = (q_i)_{i=1}^n \in \mathbb{R}^n$ represents the quantity of the ith commodity.

Often each such quantity is non-negative.

But general equilibrium theory, following Debreu's *Theory of Value*, often uses only the sign of q_i to distinguish between

- a consumer's demands and supplies of the ith commodity;
- or a producer's outputs and inputs of the ith commodity.

This sign is taken to be

positive for demands or outputs;

negative for supplies or inputs.

In fact, q_i is taken to be

- the consumer's net demand for the ith commodity;
- the producer's net supply or net output of the ith commodity.

Then $\mathbf{q}$ is the net quantity vector.

Price Vectors

Each component p_i of the (row) price vector $\mathbf{p}^\top \in \mathbb{R}^n$ indicates the price per unit of commodity i.

Then the scalar product

$$\mathbf{p}^\top \mathbf{q} = \mathbf{p} \cdot \mathbf{q} = \sum_{i=1}^{n} p_i q_i$$

is the total value of the net quantity vector $\mathbf{q}$ evaluated at the price vector $\mathbf{p}$.

In particular, $\mathbf{p}^\top \mathbf{q}$ indicates

- the net profit (or minus the net loss) for a producer;
- the net dissaving for a consumer.

Outline

Definitions

Consider any two n-vectors $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$ in $\mathbb{R}^n$.

Their sum $\mathbf{s} := \mathbf{x} + \mathbf{y}$ and difference $\mathbf{d} := \mathbf{x} - \mathbf{y}$ are constructed by adding or subtracting the vectors component by component — i.e., $\mathbf{s} = (s_i)_{i=1}^n$ and $\mathbf{d} = (d_i)_{i=1}^n$ where

$$s_i = x_i + y_i \quad \text{and} \quad d_i = x_i - y_i$$

for $i = 1, 2, \ldots, n$.

The scalar product $\lambda \mathbf{x}$ of any scalar $\lambda \in \mathbb{R}$ and vector $\mathbf{x} = (x_i)_{i=1}^n \in \mathbb{R}^n$ is constructed by multiplying each component of the vector $\mathbf{x}$ by the scalar λ — i.e.,

$$\lambda \mathbf{x} = (\lambda x_i)_{i=1}^n$$

Algebraic Fields

Definition

An algebraic field $(\mathbb{F}, +, \cdot)$ of scalars is a set $\mathbb{F}$ that, together with the two binary operations $+$ of addition and $\cdot$ of multiplication, satisfies the following axioms for all $a, b, c \in \mathbb{F}$:

1. $\mathbb{F}$ is closed under $+$ and $\cdot$:
   — i.e., both $a + b$ and $a \cdot b$ are in $\mathbb{F}$.
2. $+$ and $\cdot$ are associative:
   both $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. $+$ and $\cdot$ both commute:
   both $a + b = b + a$ and $a \cdot b = b \cdot a$.
4. There are identity elements $0, 1 \in \mathbb{F}$ for $+$ and $\cdot$ respectively, with $0 \neq 1$, such that: (i) $a + 0 = a$; (ii) $1 \cdot a = a$.
5. There are inverse operations $-$ for $+$ and ${}^{-1}$ for $\cdot$ such that: (i) $a + (-a) = 0$; (ii) provided $a \neq 0$, also $a \cdot a^{-1} = 1$.
6. The distributive law: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Three Examples of Real Algebraic Fields

Exercise

Verify that the following well known sets are algebraic fields:

- *the set* $\mathbb{R}$ *of all real numbers,*
  with the usual operations of addition and multiplication;
- *the set* $\mathbb{Q}$ *of all rational numbers*
  — i.e., those that can be expressed as the ratio $r = p/q$
  of integers $p, q \in \mathbb{Z}$ *with* $q \neq 0$.
  (Check that $\mathbb{Q}$ *is closed*
  under the usual operations of addition and multiplication,
  and that each non-zero rational
  has a rational multiplicative inverse.)
- *the set* $\mathbb{Q} + \sqrt{2}\mathbb{Q} := \{r_1 + \sqrt{2} r_2 \mid r_1, r_2 \in \mathbb{Q}\} \subset \mathbb{R}$
  of all real numbers that can be expressed as the sum of:
  (i) a rational number;
  (ii) a rational multiple of the irrational number $\sqrt{2}$.

Two Examples of Complex Algebraic Fields

Exercise

Verify that the following well known sets are algebraic fields:

- $\mathbb{C}$, *the set of all complex numbers*
  — *i.e., those that can be expressed as* $c = a + ib$,
  where $a, b \in \mathbb{R}$ *and* i *is defined to satisfy* $i^2 = -1$.

  Note that $\mathbb{C}$ *is effectively the set* $\mathbb{R} \times \mathbb{R}$ *of ordered pairs* (a, b)
  satisfying $a, b \in \mathbb{R}$, *together with the operations of:*

  (i) addition defined by $(a, b) + (c, d) = (a + c, b + d)$
  because $(a + bi) + (c + di) = (a + c) + (b + d)i$;

  (ii) multiplication defined by
  $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
  because $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- *the set of all rational complex numbers*
  — *i.e., those that can be expressed as* $c = a + ib$,
  where $a, b \in \mathbb{Q}$ *and* i *is defined to satisfy* $i^2 = -1$.

General Vector Spaces

Definition

A vector (or linear) space V over an algebraic field $\mathbb{F}$ is a combination $\langle V, \mathbb{F}, +, \cdot \rangle$ of:

- a set V of vectors;
- the field $\mathbb{F}$ of scalars;
- the binary operation $V \times V \ni (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v} \in V$ of vector addition
- the binary operation $\mathbb{F} \times V \ni (\alpha, \mathbf{u}) \mapsto \alpha \mathbf{u} \in V$ of multiplication by a scalar.

These are required to satisfy all of the following eight vector space axioms.

Eight Vector Space Axioms

1. Addition is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
2. Addition is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. Additive identity: There exists a zero vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
4. Additive inverse: For every $\mathbf{v} \in V$, there exists an additive inverse $-\mathbf{v} \in V$ of $\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. Multiplication by a scalar is distributive w.r.t. vector addition: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
6. Multiplication by a scalar is distributive w.r.t. field addition: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$
7. Multiplication by a scalar and field multiplication are compatible: $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
8. The unit element $1 \in \mathbb{F}$ is an identity element for scalar multiplication: $1\mathbf{v} = \mathbf{v}$

Multiplication by the Zero Scalar

Exercise

Prove that $0\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{v} \in V$.

Hint: Which three axioms justify the following chain of equalities

$$0\mathbf{v} = [1 + (-1)]\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = \mathbf{v} - \mathbf{v} = \mathbf{0} \text{ ?}$$

A General Class of Finite Dimensional Vector Spaces

Exercise

Given an arbitrary algebraic field $\mathbb{F}$, let $\mathbb{F}^n$ denote the space of all lists $\langle a_i \rangle_{i=1}^n$ of n elements $a_i \in \mathbb{F}$ — i.e., the n-fold Cartesian product of $\mathbb{F}$ with itself.

1. *Show how to construct the respective binary operations*

$$\mathbb{F}^n \times \mathbb{F}^n \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in \mathbb{F}^n$$
$$\mathbb{F} \times \mathbb{F}^n \ni (\lambda, \mathbf{x}) \mapsto \lambda \mathbf{x} \in \mathbb{F}^n$$

   of addition and scalar multiplication so that $(\mathbb{F}^n, \mathbb{F}, +, \times)$ is a vector space.

2. *Show too that subtraction and division by a (non-zero) scalar can be defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-1)\mathbf{w}$ and $\mathbf{v}/\alpha = (1/\alpha)\mathbf{v}$.*

Two Particular Finite Dimensional Vector Spaces

From now on we mostly consider real vector spaces
over the real field $\mathbb{R}$,
and especially the space $(\mathbb{R}^n, \mathbb{R}, +, \times)$ of n-vectors over $\mathbb{R}$.

We will consider, however, the space $(\mathbb{C}^n, \mathbb{C}, +, \times)$
of n-vectors over $\mathbb{C}$ — the complex plane — when considering:

- eigenvalues and diagonalization of square matrices;
- systems of linear difference and differential equations;
- the characteristic function of a random variable.

Outline

Linear Combinations

Definition

A linear combination of vectors is the weighted sum $\sum_{h=1}^{k} \lambda_h \mathbf{x}^h$, where $\mathbf{x}^h \in V$ and $\lambda_h \in \mathbb{F}$ for $h = 1, 2, \ldots, k$.

Exercise

By induction on k, show that the vector space axioms imply that any linear combination of vectors in V must also belong to V.

Linear Functions

Definition

A function $V \ni \mathbf{u} \mapsto f(\mathbf{u}) \in \mathbb{F}$ is linear provided that

$$f(\lambda \mathbf{u} + \mu \mathbf{v}) = \lambda f(\mathbf{u}) + \mu f(\mathbf{v})$$

for every linear combination $\lambda \mathbf{u} + \mu \mathbf{v}$ of two vectors $\mathbf{u}, \mathbf{v} \in V$, with $\lambda, \mu \in \mathbb{F}$.

Exercise

Prove that the function $V \ni \mathbf{u} \mapsto f(\mathbf{u}) \in \mathbb{F}$ is linear if and only if both:

1. *for every vector $\mathbf{v} \in V$ and scalar $\lambda \in \mathbb{F}$ one has $f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$;*
2. *for every pair of vectors $\mathbf{u}, \mathbf{v} \in V$ one has $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$.*

Key Properties of Linear Functions

Exercise

Use induction on k to show that if the function $f : V \to \mathbb{F}$ is linear, then

$$f\left(\sum_{h=1}^{k} \lambda_h \mathbf{x}^h\right) = \sum_{h=1}^{k} \lambda_h f(\mathbf{x}^h)$$

for all linear combinations $\sum_{h=1}^{k} \lambda_h \mathbf{x}^h$ in V — i.e., f preserves linear combinations.

Exercise

In case $V = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$, show that any linear function is homogeneous of degree 1, meaning that $f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$ for all $\lambda \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^n$.

In particular, putting $\lambda = 0$ gives $f(\mathbf{0}) = 0$.

What is the corresponding property in case $V = \mathbb{Q}^n$ and $\mathbb{F} = \mathbb{Q}$?

Affine Functions

Definition

A function $g : V \to \mathbb{F}$ is said to be affine
if there is a scalar additive constant $\alpha \in \mathbb{F}$
and a linear function $f : V \to \mathbb{F}$ such that $g(\mathbf{v}) \equiv \alpha + f(\mathbf{v})$.

Exercise

Under what conditions is an affine function $g : \mathbb{R} \to \mathbb{R}$ linear when its domain $\mathbb{R}$ is regarded as a vector space?

An Economic Aggregation Theorem

Suppose that a finite population of households $h \in H$
with respective non-negative incomes $y_h \in \mathbb{Q}_+$ ($h \in H$)
have non-negative demands $x_h \in \mathbb{R}$ ($h \in H$)
which depend on household income via a function $y_h \mapsto f_h(y_h)$.

Given total income $Y := \sum_h y_h$, under what conditions
can their total demand $X := \sum_h x_h = \sum_h f_h(y_h)$
be expressed as a function $X = F(Y)$ of Y alone?

The answer is an implication of Cauchy's functional equation.

In this context the theorem asserts that this aggregation condition
implies that the functions f_h ($h \in H$) and F must be co-affine.

This means there exists a common multiplicative constant $\rho \in \mathbb{R}$,
along with additive constants α_h ($h \in H$) and A, such that

$$f_h(y_h) \equiv \alpha_h + \rho y_h \ (h \in H) \text{ and } F(Y) \equiv A + \rho Y$$

Cauchy's Functional Equation: Proof of Sufficiency

Theorem

Except in the trivial case when H has only one member, Cauchy's functional equation $F(\sum_{h \in H} y_h) \equiv \sum_{h \in H} f_h(y_h)$ is satisfied for functions $F, f_h : \mathbb{Q} \to \mathbb{R}$ if and only if:

1. *there exists a single function $\phi : \mathbb{Q} \to \mathbb{R}$ such that*

   $$F(q) = F(0) + \phi(q) \text{ and } f_h(q) = f_h(0) + \phi(q) \text{ for all } h \in H$$

2. *the function $\phi : \mathbb{Q} \to \mathbb{R}$ is linear, implying that the functions F and f_h are co-affine.*

Proof.

Suppose $f_h(y_h) \equiv \alpha_h + \rho y_h$ for all $h \in H$, and $F(Y) \equiv A + \rho Y$. Then Cauchy's functional equation $F(\sum_{h \in H} y_h) \equiv \sum_{h \in H} f_h(y_h)$ is obviously satisfied provided that $A = \sum_{h \in H} \alpha_h$. □

Cauchy's Equation: Necessity in the Differentiable Case

Suppose that $\# H \geq 2$ and that $F(\sum_{h \in H} y_h) \equiv \sum_{h \in H} f_h(y_h)$ where each $\mathbb{R} \ni y_h \mapsto f_h(y_h) \in \mathbb{R}$ is differentiable.

For any pair $j, k \in H$, consider the effect of transferring a small amount η from k to j, with y_h fixed for all $h \in H \setminus \{j, k\}$.

Because both $\sum_{h \in H} y_h$ and $\sum_{h \in H \setminus \{j,k\}} f_h(y_h)$ are unchanged, equating the changes to the two sides of the Cauchy equation gives $0 = f_j(y_j + \eta) + f_k(y_k - \eta) - f_j(y_j) - f_k(y_k)$.

Because we assume that $f_j'(y_j)$ and $f_k'(y_k)$ both exist, we can differentiate the last equation to get $0 = f_j'(y_j) - f_k'(y_k)$.

It follows that there exists a constant c such that $f_h'(y) = c$ for all $h \in H$ and all real y, so $f_h(y) = \alpha_h + cy$.

But then one has

$$F\left(\sum_{h \in H} y_h + \eta\right) - F\left(\sum_{h \in H} y_h\right) = f_j(y_j + \eta) - f_j(y_j) = c\eta$$

It follows that $F(Y)$ is the affine function $A + cY$, for some real A.

Cauchy's Equation: Beginning the Proof of Necessity

Lemma

The mapping $\mathbb{Q} \ni q \mapsto \phi(q) := F(q) - F(0) \in \mathbb{R}$ *must satisfy;*

1. $\phi(q) \equiv f_i(q) - f_i(0)$ *for all* $i \in H$ *and* $q \in \mathbb{Q}$*;*
2. $\phi(q + q') \equiv \phi(q) + \phi(q')$ *for all* $q, q' \in \mathbb{Q}$*.*

Proof.

To prove part 1, consider any $i \in H$ and all $q \in \mathbb{Q}$.

Note that Cauchy's equation $F(\sum_h y_h) \equiv \sum_h f_h(y_h)$
implies that $F(q) = f_i(q) + \sum_{h \neq i} f_h(0)$
and also $F(0) = f_i(0) + \sum_{h \neq i} f_h(0)$.

Now define the function $\phi(q) := F(q) - F(0)$ on the domain $\mathbb{Q}$.

Then subtract the second equation from the first to obtain

$$\phi(q) = F(q) - F(0) = f_i(q) - f_i(0)$$

$\square$

Cauchy's Equation: Continuing the Proof of Necessity

Proof.

To prove part 2, consider any $i, j \in H$ with $i \neq j$, and any $q, q' \in \mathbb{Q}$.

Note that Cauchy's equation $F(\sum_h y_h) \equiv \sum_h f_h(y_h)$ implies that

$$\begin{aligned} F(q + q') &= f_i(q) + f_j(q') + \textstyle\sum_{h \in H \setminus \{i,j\}} f_h(0) \\ F(0) &= f_i(0) + f_j(0) + \textstyle\sum_{h \in H \setminus \{i,j\}} f_h(0) \end{aligned}$$

Now subtract the second equation from the first, then use the equation $\phi(q) = F(q) - F(0) = f_i(q) - f_i(0)$ derived in the previous slide, to obtain successively

$$\begin{aligned} \phi(q + q') &= F(q + q') - F(0) \\ &= f_i(q) - f_i(0) + f_j(q') - f_j(0) \\ &= \phi(q) + \phi(q') \end{aligned}$$

$\square$

Cauchy's Equation: Resuming the Proof of Necessity

Because $\phi(q + q') \equiv \phi(q) + \phi(q')$,
for any $k \in \mathbb{N}$ one has $\phi(kq) = \phi((k-1)q) + \phi(q)$.

As an induction hypothesis, which is trivially true for $k = 2$, suppose that $\phi((k-1)q) = (k-1)\phi(q)$.

Confirming the induction step, the hypothesis implies that

$$\phi(kq) = \phi((k-1)q) + \phi(q) = (k-1)\phi(q) + \phi(q) = k\phi(q)$$

So $\phi(kq) = k\phi(q)$ for every $k \in \mathbb{N}$ and every $q \in \mathbb{Q}$.

Putting $q' = kq$ implies that $\phi(q') = k\phi(q'/k)$.

Interchanging q and q', it follows that $\phi(q/k) = (1/k)\phi(q)$.

Cauchy's Equation: Completing the Proof of Necessity

So far we have proved that, for every $k \in \mathbb{N}$ and every $q \in \mathbb{Q}$, one has both $\phi(kq) = k\phi(q)$ and $\phi(q/k) = (1/k)\phi(q)$.

Hence, for every rational $r = m/n \in \mathbb{Q}$ one has $\phi(mq/n) = m\phi(q/n) = (m/n)\phi(q)$ and so $\phi(rq) = r\phi(q)$.

In particular, $\phi(r) = r\phi(1)$, so ϕ is linear on its domain $\mathbb{Q}$ (though not on the whole of $\mathbb{R}$ without additional assumptions such as continuity or monotonicity).

The rest of the proof is routine checking of definitions.

Outline

Euclidean Norm as Length

Pythagoras's theorem implies that the length of the typical vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ is $\sqrt{x_1^2 + x_2^2}$ or, perhaps less clumsily, $(x_1^2 + x_2^2)^{1/2}$.

In $\mathbb{R}^3$, the same result implies that the length of the typical vector $\mathbf{x} = (x_1, x_2, x_3)$ is

$$\left[\left((x_1^2 + x_2^2)^{1/2}\right)^2 + x_3^2\right]^{1/2} = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

An obvious extension to $\mathbb{R}^n$ is the following:

Definition

The length of the typical n-vector $\mathbf{x} = (x_i)_{i=1}^n \in \mathbb{R}^n$ is its (Euclidean) norm

$$\|\mathbf{x}\| := \left(\sum_{i=1}^n x_i^2\right)^{1/2} = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

Unit n-Vectors, the Unit Sphere, and Unit Ball

Definition

A unit vector $\mathbf{u} \in \mathbb{R}^n$ is a vector with unit norm
— i.e., its components satisfy $\sum_{i=1}^n u_i^2 = \|\mathbf{u}\|^2 = 1$.

The set of all such unit vectors forms a surface called the unit sphere of dimension $n - 1$ (one less than n because of the defining equation).

It is defined as the hollow set (like a football or tennis ball)

$$S^{n-1} := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum\nolimits_{i=1}^n x_i^2 = 1 \right\}$$

The unit ball $B \subset \mathbb{R}^n$ is the solid set (like a cricket ball or golf ball)

$$B := \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum\nolimits_{i=1}^n x_i^2 \leq 1 \right\}$$

of all points bounded by the surface of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Cauchy–Schwartz Inequality

Theorem

For all pairs $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, *one has* $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$.

Proof.

Define the function $\mathbb{R} \ni \xi \mapsto f(\xi) := \sum_{i=1}^n (a_i \xi + b_i)^2 \in \mathbb{R}$.

Clearly f is the quadratic function $f(\xi) \equiv A\xi^2 + B\xi + C$ where $A := \sum_{i=1}^n a_i^2 = \|\mathbf{a}\|^2$, $B := 2 \sum_{i=1}^n a_i b_i = 2\mathbf{a} \cdot \mathbf{b}$, and $C := \sum_{i=1}^n b_i^2 = \|\mathbf{b}\|^2$.

There is a trivial case when $A = 0$ because $\mathbf{a} = \mathbf{0}$.

Otherwise $A > 0$, so we can complete the square to get

$$f(\xi) \equiv A\xi^2 + B\xi + C = A[\xi + (B/2A)]^2 + C - B^2/4A$$

But the definition of f implies that $f(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, including $\xi = -B/2A$, so $0 \leq f(-B/2A) = C - B^2/4A$.
Hence $\frac{1}{4}B^2 \leq AC$,
implying that $|\mathbf{a} \cdot \mathbf{b}| = \left|\frac{1}{2}B\right| \leq \sqrt{AC} = \|\mathbf{a}\| \|\mathbf{b}\|$. $\square$

Outline

The Angle Between Two Vectors

Consider the triangle in $\mathbb{R}^n$ whose vertices are the three disjoint vectors $\mathbf{x}, \mathbf{y}, \mathbf{0}$.

Its three sides or edges have respective lengths $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, $\|\mathbf{x} - \mathbf{y}\|$, where the last follows from the parallelogram law.

Note that $\|\mathbf{x} - \mathbf{y}\|^2 \lesseqqgtr \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ according as the angle at $\mathbf{0}$ is: (i) acute; (ii) a right angle; (iii) obtuse. But

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 &= \sum_{i=1}^n (x_i - y_i)^2 - \sum_{i=1}^n (x_i^2 + y_i^2) \\ &= \sum_{i=1}^n (-2x_i y_i) = -2\mathbf{x} \cdot \mathbf{y} \end{aligned}$$

So the three cases (i)–(iii) occur according as $\mathbf{x} \cdot \mathbf{y} \gtreqqless 0$.

Using the Cauchy–Schwartz inequality, one can define the angle between $\mathbf{x}$ and $\mathbf{y}$ as the unique solution $\theta = \arccos\left(\mathbf{x} \cdot \mathbf{y} / \|\mathbf{x}\| \|\mathbf{y}\|\right)$ in the interval $[0, \pi)$ of the equation $\cos\theta = \mathbf{x} \cdot \mathbf{y} / \|\mathbf{x}\| \|\mathbf{y}\| \in [-1, 1]$.

Orthogonal and Orthonormal Sets of Vectors

Case (ii) suggests defining two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
as orthogonal just in case $\mathbf{x} \cdot \mathbf{y} = 0$,
which is true if and only if $\theta = \arccos 0 = \frac{1}{2}\pi = 90^\circ$.

A set of k vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is said to be:

- pairwise orthogonal just in case $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ whenever $j \neq i$;
- orthonormal just in case, in addition, each $\|\mathbf{x}_i\| = 1$
  — i.e., all k elements of the set are vectors of unit length.

Define the Kronecker delta function

$$\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \ni (i, j) \mapsto \delta_{ij} \in \{0, 1\}$$

on the set of pairs $i, j \in \{1, 2, \ldots, n\}$ by

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Then the set of k vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is orthonormal
if and only if $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$ for all pairs $i, j \in \{1, 2, \ldots, k\}$.

Outline

The Canonical Basis of $\mathbb{R}^n$

Example

A prominent orthonormal set is the canonical basis of $\mathbb{R}^n$, defined as the set of n different n-vectors $\mathbf{e}^i$ $(i = 1, 2, \ldots, n)$ whose respective components $(e^i_j)^n_{j=1}$ satisfy $e^i_j = \delta_{ij}$ for all $j \in \{1, 2, \ldots, n\}$.

Exercise

Show that each n-vector $\mathbf{x} = (x_i)^n_{i=1}$ *is a linear combination*

$$\mathbf{x} = (x_i)^n_{i=1} = \sum\nolimits_{i=1}^{n} x_i \mathbf{e}^i$$

of the canonical basis vectors $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n\}$, *with the multiplier attached to each basis vector* $\mathbf{e}^i$ *equal to the respective component* x_i $(i = 1, 2, \ldots, n)$.

The Canonical Basis in Commodity Space

Example

Consider the case when each vector $\mathbf{x} \in \mathbb{R}^n$ is a quantity vector, whose components are $(x_i)_{i=1}^n$, where x_i indicates the net quantity of commodity i.

Then the ith unit vector $\mathbf{e}^i$ of the canonical basis of $\mathbb{R}^n$ represents a commodity bundle that consists of one unit of commodity i, but nothing of every other commodity.

In case the row vector $\mathbf{p}^\top \in \mathbb{R}^n$ is a price vector for the same list of n commodities, the value $\mathbf{p}^\top \mathbf{e}^i$ of the ith unit vector $\mathbf{e}^i$ must equal p_i, the price (of one unit) of the ith commodity.

Linear Functions

Theorem

The function $\mathbb{R}^n \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$ *is linear*
if and only if there exists $\mathbf{y} \in \mathbb{R}^n$ *such that* $f(\mathbf{x}) = \mathbf{y}^\top \mathbf{x}$.

Proof.

Sufficiency is easy to check.

Conversely, note that $\mathbf{x}$ equals the linear combination $\sum_{i=1}^n x_i \mathbf{e}^i$ of the n canonical basis vectors.

Hence, linearity of f implies that

$$f(\mathbf{x}) = f\left(\sum_{i=1}^n x_i \mathbf{e}^i\right) = \sum_{i=1}^n x_i f(\mathbf{e}^i) = \sum_{i=1}^n f(\mathbf{e}^i)\, x_i = \mathbf{y}^\top \mathbf{x}$$

where $\mathbf{y}$ is the column vector whose components are $y_i = f(\mathbf{e}^i)$ for $i = 1, 2, \ldots, n$. $\square$

Linear Transformations: Definition

Definition

The vector-valued function

$$\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = (F_i(\mathbf{x}))_{i=1}^m \in \mathbb{R}^m$$

is a linear transformation just in case
each component function $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{F}_i(\mathbf{x}) \in \mathbb{R}$ is linear
— or equivalently, iff $\mathbf{F}(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\mathbf{F}(\mathbf{x}) + \mu\mathbf{F}(\mathbf{y})$
for every linear combination $\lambda\mathbf{x} + \mu\mathbf{y}$ of every pair $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Characterizing Linear Transformations

Theorem

The mapping $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) \in \mathbb{R}^m$ *is a linear transformation if and only if there exist vectors* $\mathbf{y}_i \in \mathbb{R}^m$ *for* $i = 1, 2, \ldots, n$ *such that each component function satisfies* $F_i(\mathbf{x}) = \mathbf{y}_i^\top \mathbf{x}$*.*

Proof.

Sufficiency is obvious.

Conversely, because $\mathbf{x}$ equals the linear combination $\sum_{i=1}^n x_i \mathbf{e}_i$ of the n canonical basis vectors $\{\mathbf{e}_i\}_{i=1}^n$ and because each component function $\mathbb{R}^n \ni \mathbf{x} \mapsto F_i(\mathbf{x})$ is linear, one has

$$F_i(\mathbf{x}) = F_i\left(\sum_{j=1}^n x_j \mathbf{e}_j\right) = \sum_{j=1}^n x_j F_i(\mathbf{e}_j) = \mathbf{y}_i^\top \mathbf{x}$$

where $\mathbf{y}_i^\top$ is the row vector whose components are $(\mathbf{y}_i)_j = F_i(\mathbf{e}_j)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. $\square$

Representing a Linear Transformation

Definition

A matrix representation
of the linear transformation $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) \in \mathbb{R}^m$
relative to the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$
is an $m \times n$ array whose n columns
are the m-vector images $\mathbf{F}(\mathbf{e}_j) = (F_i(\mathbf{e}_j))_{i=1}^m \in \mathbb{R}^m$
of the n canonical basis vectors $\{\mathbf{e}_j\}_{j=1}^n$ of $\mathbb{R}^n$.

Outline

Linear Combinations and Dependence: Definitions

Definition

A linear combination of the finite set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ of vectors is the scalar weighted sum $\sum_{h=1}^k \lambda_h \mathbf{x}^h$, where $\lambda_h \in \mathbb{R}$ for $h = 1, 2, \ldots, k$.

Definition

The finite set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ of vectors is linearly independent just in case the only solution of the equation $\sum_{h=1}^k \lambda_h \mathbf{x}^\mathbf{h} = \mathbf{0}$ is the trivial solution $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Alternatively, if the equation has a non-trivial solution, then the set of vectors is linearly dependent.

Characterizing Linear Dependence

Theorem

The finite set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ of k n-vectors is linearly dependent if and only if at least one of the vectors, say $\mathbf{x}^1$ after reordering, can be expressed as a linear combination of the others
— i.e., there exist scalars α^h $(h = 2, 3, \ldots, k)$ such that $\mathbf{x}^1 = \sum_{h=2}^{k} \alpha_h \mathbf{x}^h$.

Proof.

If $\mathbf{x}^1 = \sum_{h=2}^{k} \alpha_h \mathbf{x}^h$, then $(-1)\mathbf{x}^1 + \sum_{h=2}^{k} \alpha_h \mathbf{x}^h = \mathbf{0}$,
so $\sum_{h=1}^{k} \lambda_h \mathbf{x}^h = \mathbf{0}$ has a non-trivial solution.

Conversely, suppose $\sum_{h=1}^{k} \lambda_h \mathbf{x}^h = \mathbf{0}$ has a non-trivial solution.

After reordering, we can suppose that $\lambda_1 \neq 0$.

Then $\mathbf{x}^1 = \sum_{h=2}^{k} \alpha_h \mathbf{x}^h$,
where $\alpha_h = -\lambda_h / \lambda_1$ for $h = 2, 3, \ldots, k$. $\square$

Orthogonality Implies Linear Independence

Theorem

If the finite set $S = \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ of k non-zero n-vectors is pairwise orthogonal, then it is linearly independent.

Proof.

Let $\mathbf{s}$ denote the linear combination $\sum_{h=1}^{k} \alpha_h \mathbf{x}^h$.

Then for each $j = 1, \ldots, k$ one has $\mathbf{s} \cdot \mathbf{x}^j = \sum_{h=1}^{k} \alpha_h \mathbf{x}^h \cdot \mathbf{x}^j$.

In case S is pairwise orthogonal, one has $\mathbf{x}^h \cdot \mathbf{x}^j = 0$ for all $h \neq j$, and so $\mathbf{s} \cdot \mathbf{x}^j = \alpha_j \mathbf{x}^j \cdot \mathbf{x}^j$.

So when $\mathbf{s} = \mathbf{0}$, it follows that $\alpha_j \mathbf{x}^j \cdot \mathbf{x}^j = 0$ for all $j = 1, \ldots, k$.

Then, because we assumed that $\mathbf{x}^j \neq \mathbf{0}$,
we have $\alpha_j = 0$ for all $j = 1, \ldots, k$.

This proves that S is linearly independent. $\square$

Dimension

Definition

The dimension of a vector space V is the size of any maximal set of linearly independent vectors, if this number is finite.

Otherwise, if there is an infinite set of linearly independent vectors, the dimension is infinite.

Exercise

Show that the canonical basis of $\mathbb{R}^n$ *is linearly independent.*

Example

The previous exercise shows that the dimension of $\mathbb{R}^n$ is at least n.

Later results will imply that any set of $k > n$ vectors in $\mathbb{R}^n$ is linearly dependent.

This implies that the dimension of $\mathbb{R}^n$ is exactly n.

Outline

Matrices as Rectangular Arrays

An $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$ is a (rectangular) array

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = ((a_{ij})_{i=1}^m)_{j=1}^n = ((a_{ij})_{j=1}^n)_{i=1}^m$$

Note that in a_{ij}, we write the row number i before the column number j.

An $m \times 1$ matrix is a column vector with m rows and 1 column.

A $1 \times n$ matrix is a row vector with 1 row and n columns.

The $m \times n$ matrix $\mathbf{A}$ consists of:

n columns in the form of m-vectors
$\mathbf{a}_j = (a_{ij})_{i=1}^m \in \mathbb{R}^m$ for $j = 1, 2, \dots, n$;

m rows in the form of n-vectors
$\mathbf{a}_i^\top = (a_{ij})_{j=1}^n \in \mathbb{R}^n$ for $i = 1, 2, \dots, m$.

The Transpose of a Matrix

The transpose of the $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$
is defined as the $n \times m$ matrix

$$\mathbf{A}^\top = (a_{ij}^\top)_{n \times m} = (a_{ji})_{n \times m} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}$$

Thus the transposed matrix $\mathbf{A}^\top$ results from transforming each column m-vector $\mathbf{a}_j = (a_{ij})_{i=1}^m$ ($j = 1, 2, \dots, n$) of $\mathbf{A}$ into the corresponding row m-vector $\mathbf{a}_j^\top = (a_{ji}^\top)_{i=1}^m$ of $\mathbf{A}^\top$.

Equivalently, for each $i = 1, 2, \dots, m$,
the ith row n-vector $\mathbf{a}_i^\top = (a_{ij})_{j=1}^n$ of $\mathbf{A}$
is transformed into the ith column n-vector $\mathbf{a}_i = (a_{ji})_{j=1}^n$ of $\mathbf{A}^\top$.

Either way, one has $a_{ij}^\top = a_{ji}$ for all relevant pairs i, j.

Rows Before Columns

VERY Important Rule: Rows before columns!

This order really matters.

Reversing it gives a transposed matrix.

Exercise

Verify that the double transpose of any $m \times n$ *matrix* $\mathbf{A}$ *satisfies* $(\mathbf{A}^\top)^\top = \mathbf{A}$
— i.e., transposing a matrix twice recovers the original matrix.

Outline

Multiplying a Matrix by a Scalar

A scalar, usually denoted by a Greek letter,
is simply a member $\alpha \in \mathbb{F}$ of the algebraic field $\mathbb{F}$
over which the vector space is defined.

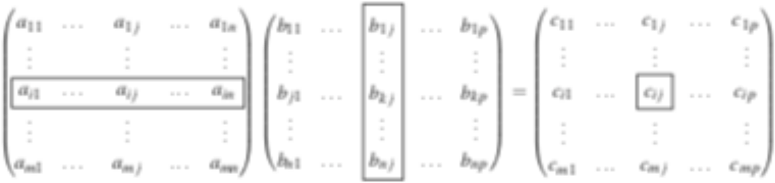
So when $\mathbb{F} = \mathbb{R}$, a scalar is a real number $\alpha \in \mathbb{R}$.

The product of any $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$
and any scalar $\alpha \in \mathbb{R}$
is the new $m \times n$ matrix denoted by $\alpha \mathbf{A} = (\alpha a_{ij})_{m \times n}$,
each of whose elements αa_{ij} results
from multiplying the corresponding element a_{ij} of $\mathbf{A}$ by α.

Matrix Multiplication

The matrix product of two matrices $\mathbf{A}$ and $\mathbf{B}$
is defined (whenever possible) as the matrix $\mathbf{C} = \mathbf{AB} = (c_{ij})_{m \times n}$
whose element c_{ij} in row i and column j
is the inner product $c_{ij} = \mathbf{a}_i^\top \mathbf{b}_j$ of:

- the ith row vector $\mathbf{a}_i^\top$ of the first matrix $\mathbf{A}$;
- the jth column vector $\mathbf{b}_j$ of the second matrix $\mathbf{B}$.



$\mathbf{a}_i^\top \quad \cdot \quad \mathbf{b}_j \quad = \quad c_{ij}$

Compatibility for Matrix Multiplication, I

Again: rows before columns!

Note that the resulting matrix product $\mathbf{C}$ must have:

- as many rows as the first matrix $\mathbf{A}$;
- as many columns as the second matrix $\mathbf{B}$.

Yet again: rows before columns!

Compatibility for Matrix Multiplication, II

Question: when is this definition of the matrix product $\mathbf{C} = \mathbf{AB}$ possible?

Answer: if and only if $\mathbf{A}$ has as many columns as $\mathbf{B}$ has rows.

This condition ensures that every inner product $\mathbf{a}_i^\top \mathbf{b}_j$ is defined, which is true iff (if and only if) every row of $\mathbf{A}$ has exactly the same number of elements as every column of $\mathbf{B}$.

In this case, the two matrices $\mathbf{A}$ and $\mathbf{B}$ are compatible for multiplication.

Specifically, if $\mathbf{A}$ is $m \times \ell$ for some m, then $\mathbf{B}$ must be $\ell \times n$ for some n.

Then the product $\mathbf{C} = \mathbf{AB}$ is $m \times n$, with elements $c_{ij} = \mathbf{a}_i^\top \mathbf{b}_j = \sum_{k=1}^{\ell} a_{ik} b_{kj}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Laws of Matrix Multiplication

Exercise

Verify that the following laws of matrix multiplication hold whenever the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are compatible for multiplication.*

associative law for matrices: $\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$;

distributive: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}$ *and* $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C}$;

transpose: $(\mathbf{A}\mathbf{B})^\top = \mathbf{B}^\top \mathbf{A}^\top$.

associative law for scalars: $\alpha(\mathbf{A}\mathbf{B}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$ *(all* $\alpha \in \mathbb{R}$*).*

Exercise

Let $\mathbf{X}$ *be any* $m \times n$ *matrix, and* $\mathbf{z}$ *any column n-vector.*

1. *Show that the matrix product* $\mathbf{z}^\top \mathbf{X}^\top \mathbf{X}\mathbf{z}$ *is well-defined, and that its value is a scalar.*
2. *By putting* $\mathbf{w} = \mathbf{X}\mathbf{z}$ *in the previous exercise regarding the sign of the quadratic form* $\mathbf{w}^\top \mathbf{w}$*, what can you conclude about the value of the scalar* $\mathbf{z}^\top \mathbf{X}^\top \mathbf{X}\mathbf{z}$*?*

Exercise for Econometricians I

Exercise

An econometrician has access to data series (such as time series) involving the real values

- y_t $(t = 1, 2, \ldots, T)$ *of one endogenous variable;*
- x_{ti} $(t = 1, 2, \ldots, T$ *and* $i = 1, 2, \ldots, k)$
  of k *different exogenous variables*
  — sometimes called explanatory variables or regressors.

The data is to be fitted into the linear regression model

$$y_t = \sum_{i=1}^{k} b_i x_{ti} + e_t$$

whose scalar constants b_i $(i = 1, 2, \ldots, k)$
are unknown regression coefficients,
and each scalar e_t *is the error term or residual.*

Exercise for Econometricians II

1. Discuss how the regression model can be written in the form $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ for suitable column vectors $\mathbf{y}, \mathbf{b}, \mathbf{e}$.
2. What are the dimensions of these vectors, and of the exogenous data matrix $\mathbf{X}$?
3. Why do you think econometricians use this matrix equation, rather than the alternative $\mathbf{y} = \mathbf{b}\mathbf{X} + \mathbf{e}$?
4. How can the equation $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$ accommodate the constant term α in the alternative equation $y_t = \alpha + \sum_{i=1}^{k} b_i x_{ti} + e_t$?

Matrix Multiplication Does Not Commute I

The two matrices $\mathbf{A}$ and $\mathbf{B}$ commute just in case $\mathbf{AB} = \mathbf{BA}$.

Note that typical pairs of matrices DO NOT commute, meaning that $\mathbf{AB} \neq \mathbf{BA}$ — i.e., the order of multiplication matters.

Indeed, suppose that $\mathbf{A}$ is $\ell \times m$ and $\mathbf{B}$ is $m \times n$, as is needed for $\mathbf{AB}$ to be defined.

Then the reverse product $\mathbf{BA}$ is undefined except in the special case when $n = \ell$.

Hence, for both $\mathbf{AB}$ and $\mathbf{BA}$ to be defined, where $\mathbf{B}$ is $m \times n$, the matrix $\mathbf{A}$ must be $n \times m$.

But then $\mathbf{AB}$ is $n \times n$, whereas $\mathbf{BA}$ is $m \times m$.

Evidently $\mathbf{AB} \neq \mathbf{BA}$ unless $m = n$.

Thus all four matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{AB}$ and $\mathbf{BA}$ are $m \times m = n \times n$.

We must be in the special case when all four are square matrices of the same dimension.

Matrix Multiplication Does Not Commute II

Even if both $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices,
implying that both $\mathbf{AB}$ and $\mathbf{BA}$ are also $n \times n$,
one can still have $\mathbf{AB} \neq \mathbf{BA}$.

Example

Here is a 2×2 example:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Exercise

For matrix multiplication, explain why there are two different versions of the distributive law — namely

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \textit{ and } (\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

More Warnings Regarding Matrix Multiplication

Exercise

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *denote three general matrices.*

Give examples showing that:

1. *The matrix* $\mathbf{AB}$ *might be defined, even if* $\mathbf{BA}$ *is not.*
2. *One can have* $\mathbf{AB} = \mathbf{0}$ *even though* $\mathbf{A} \neq \mathbf{0}$ *and* $\mathbf{B} \neq \mathbf{0}$*.*
3. *If* $\mathbf{AB} = \mathbf{AC}$ *and* $\mathbf{A} \neq \mathbf{0}$*, it does not follow that* $\mathbf{B} = \mathbf{C}$*.*