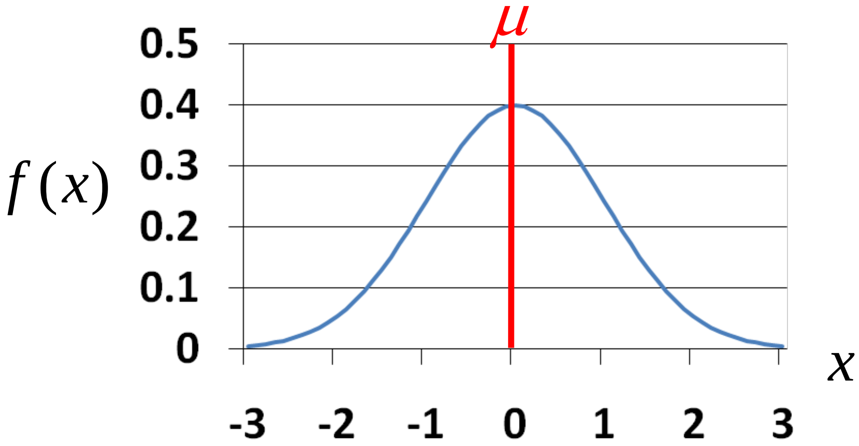
Normal Random Variable

- X is a **Normal Random Variable**: $X \sim \mathrm{N}(\mu, \sigma^2)$
  - Probability Density Function (PDF):

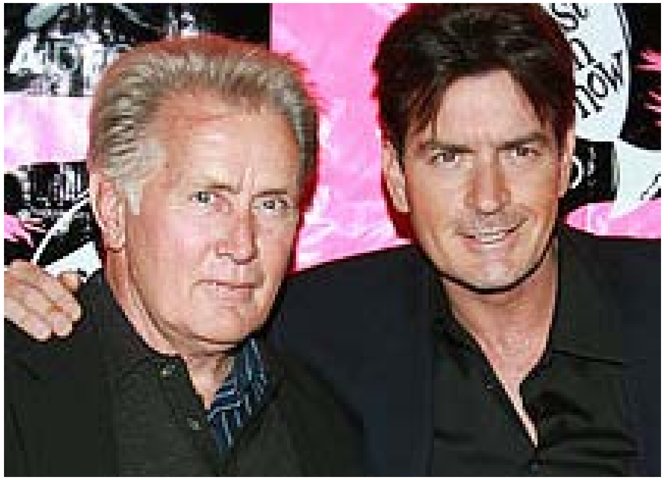
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where} \quad -\infty < x < \infty$$



  - $E[X] = \mu$
  - $Var(X) = \sigma^2$
  - Also called "Gaussian"
  - Note: $f(x)$ is symmetric about μ
  - Common for natural phenomena: heights, weights, etc.
  - Often results from the sum of multiple variables

Carl Friedrich Gauss

- Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician

- Started doing groundbreaking math as teenager
  - Did not invent Normal distribution, but popularized it
- He looked more like Martin Sheen
  - Who is, of course, Charlie Sheen’s father

Properties of Normal Random Variable

- Let X ~ N(μ, σ^2)
- Let Y = aX + b
  - Y ~ N($a\mu$ + b, $a^2\sigma^2$)
  - E[Y] = E[aX + b] = aE[X] + b = $a\mu$ + b
  - Var(Y) = Var(aX + b) = a^2Var(X) = $a^2\sigma^2$

$$F_Y(x) = P(Y \le x) = P(aX + b \le x) = P(X \le \frac{x-b}{a}) = F_X(\frac{x-b}{a})$$

Differentiating $F_Y(x)$ w.r.t. x , yields $f_Y(x)$, the PDF for y:

$$f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{d}{dx} F_X(\frac{x-b}{a}) = \frac{1}{a} f_X(\frac{x-b}{a})$$

- Special case: Z = (X – μ)/σ (a = 1/σ , b = –μ/σ)
  - Z ~ N($a\mu$ + b, $a^2\sigma^2$) = N($\mu/\sigma - \mu/\sigma$, $(1/\sigma)^2\sigma^2$) = N(0, 1)

Standard (Unit) Normal Random Variable

- Z is a **<u>Standard (or Unit) Normal RV</u>**: $Z \sim N(0, 1)$
  - $E[Z] = \mu = 0$ Var(Z) $= \sigma^2 = 1$ SD(Z) $= \sigma = 1$
  - CDF of Z, $F_Z(z)$ does not have closed form
  - We denote $F_Z(z)$ as $\Phi(z)$: "phi of z"

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

  - By symmetry: $\Phi(-z) = P(Z \le -z) = P(Z \ge z) = 1 - \Phi(z)$
- Use Z to compute $X \sim N(\mu, \sigma^2)$, where $\sigma > 0$

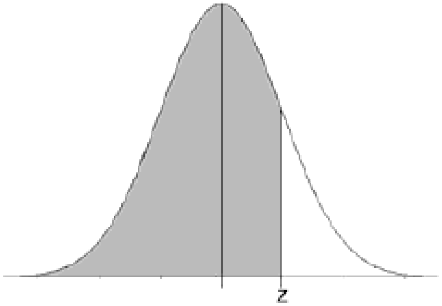
$$F_X(x) = P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

  - Table of $\Phi(z)$ values in textbook, p. 201 and handout

Using Table of Φ(z) Values

Standard Normal Cumulative Probability Table

$\Phi(0.54) = 0.7054$



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Get Your Gaussian On

- X ~ N(3, 16) $\mu = 3$ $\sigma^2 = 16$ $\sigma = 4$
  - What is P(X > 0)?

$$P(X > 0) = P\left(\frac{X-3}{4} > \frac{0-3}{4}\right) = P\left(Z > -\frac{3}{4}\right) = 1 - P\left(Z \leq -\frac{3}{4}\right)$$

$$1 - \Phi\left(-\tfrac{3}{4}\right) = 1 - \left(1 - \Phi\left(\tfrac{3}{4}\right)\right) = \Phi\left(\tfrac{3}{4}\right) = 0.7734$$

  - What is P(2 < X < 5)?

$$P(2 < X < 5) = P\left(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}\right) = P\left(-\frac{1}{4} < Z < \frac{2}{4}\right)$$

$$\Phi\left(\tfrac{2}{4}\right) - \Phi\left(-\tfrac{1}{4}\right) = \Phi\left(\tfrac{1}{2}\right) - \left(1 - \Phi\left(\tfrac{1}{4}\right)\right) = 0.6915 - (1 - 0.5987) = 0.2902$$

  - What is P(|X – 3| > 6)?

$$P(X < -3) + P(X > 9) = P\left(Z < \frac{-3-3}{4}\right) + P\left(Z > \frac{9-3}{4}\right)$$

$$\Phi\left(-\tfrac{3}{2}\right) + \left(1 - \Phi\left(\tfrac{3}{2}\right)\right) = 2\left(1 - \Phi\left(\tfrac{3}{2}\right)\right) = 2(1 - 0.9332) = 0.1336$$

Noisy Wires

- Send voltage of 2 or -2 on wire (to denote 1 or 0)
  - X = voltage sent
  - R = voltage received = X + Y, where noise Y ~ N(0, 1)
  - Decode R: if (R ≥ 0.5) then 1, else 0
  - What is P(error after decoding | original bit = 1)?

$$P(2+Y<0.5)=P(Y<-1.5)=\Phi(-1.5)=1-\Phi(1.5)\approx 0.0668$$

  - What is P(error after decoding | original bit = 0)?

$$P(-2+Y\geq 0.5)=P(Y\geq 2.5)=1-\Phi(2.5)\approx 0.0062$$

Normal Approximation to Binomial

- X ~ Bin(n, p)
  - E[X] = np Var(X) = np(1 – p)
  - Poisson approx. good: n large (> 20), p small (< 0.05)
  - For large n: X ≈ Y ~ N(E[X], Var(X)) = N(np, np(1 – p))
  - Normal approx. good : Var(X) = np(1 – p) ≥ 10

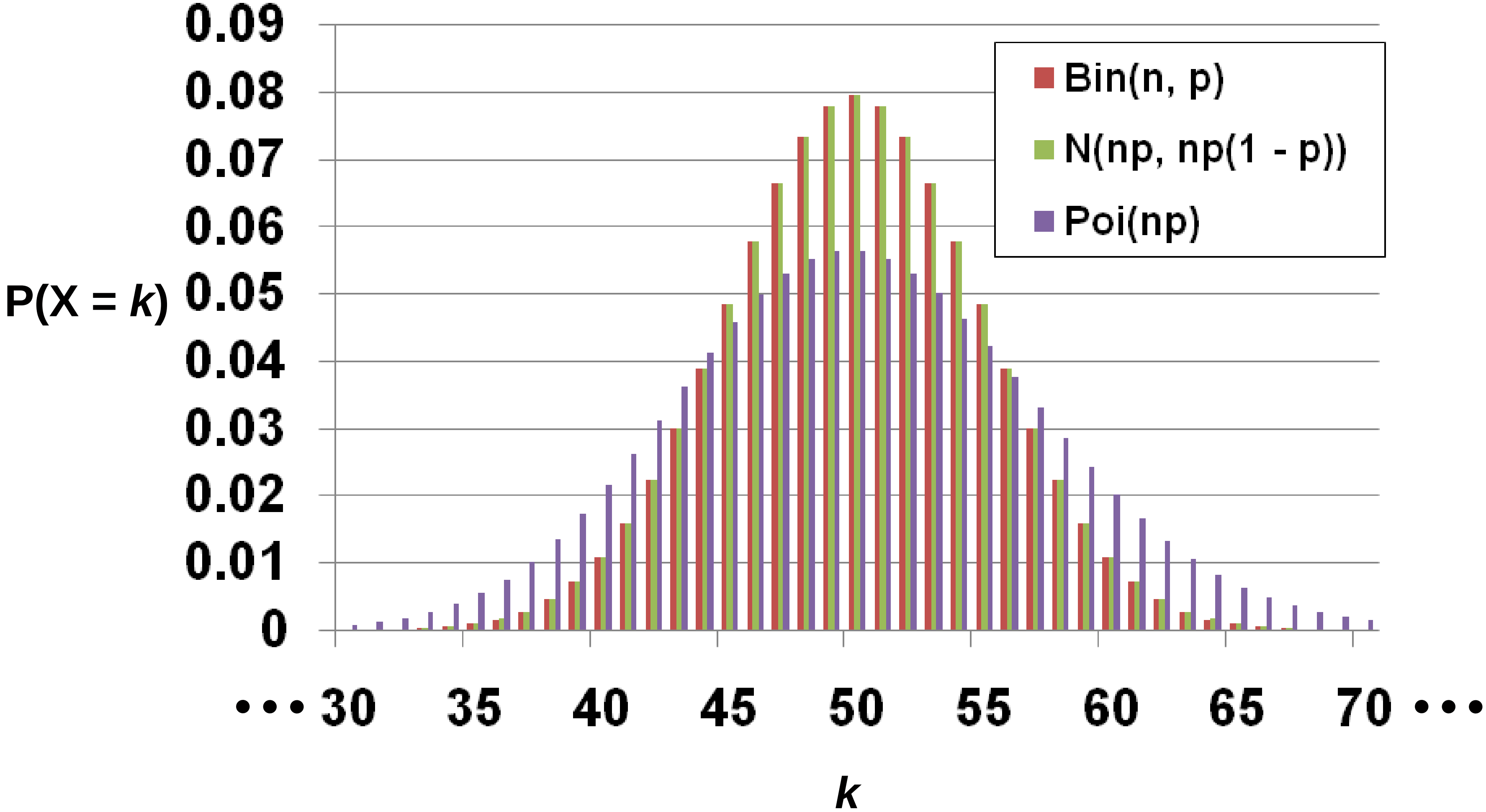
$$P(X = k) \approx P\left(k - \tfrac{1}{2} < Y < k + \tfrac{1}{2}\right) = \Phi\left(\frac{k - np + 0.5}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k - np - 0.5}{\sqrt{np(1-p)}}\right)$$

"Continuity correction"

  - DeMoivre-Laplace Limit Theorem:
    - S_n: number of successes (with prob. p) in n independent trials

$$P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \xrightarrow{n \to \infty} \Phi(b) - \Phi(a)$$

Comparison when $n = 100$, $p = 0.5$



Faulty Endorsements

- 100 people placed on special diet
  - X = # people on diet whose cholesterol decreases
  - Doctor will endorse diet if X ≥ 65
  - What is P(doctor endorses diet | diet has no effect)?
  - X ~ Bin(100, 0.5)

$$np = 50 \quad np(1-p) = 25 \quad \sqrt{np(1-p)} = 5$$

  - Use Normal approximation: Y ~ N(50, 25)

$$P(X \geq 65) \approx P(Y > 64.5)$$

$$P(Y > 64.5) = P\left(\frac{Y-50}{5} > \frac{64.5-50}{5}\right) = P(Z > 2.9) = 1 - \Phi(2.9) \approx 0.0019$$

  - Using Binomial:

$$P(X \geq 65) \approx 0.0018$$

Stanford Admissions (a few years ago)

- Say Stanford accepts 2480 students
  - Each accepted student has 68% chance of attending
  - X = # students who will attend.   X ~ Bin(2480, 0.68)
  - What is P(X > 1745)?

$$np \; = \; 1686.4 \qquad np(1-p) \; \approx \; 539.65 \qquad \sqrt{np(1-p)} \; \approx \; 23.23$$

  - Use Normal approximation: Y ~ N(1686.4, 539.65)

$$P(X > 1745) \approx P(Y > 1745.5)$$

$$P(Y > 1745.5) = P\left(\frac{Y - 1686.4}{23.23} > \frac{1745.5 - 1686.4}{23.23}\right) = 1 - \Phi(2.54) \approx 0.0055$$

  - Using Binomial:

$$P(X > 1745) \approx 0.0053$$

Changes in Stanford Admissions

- Stanford Daily, March 28, 2014

  "Class of 2018 Admit Rates Lowest in University History" by Alex Zivkovic

  "Fewer students were admitted to the Class of 2018 than the Class of 2017, due to the increase in Stanford's yield rate which has increased over 5 percent in the past four years, according to Colleen Lim M.A. '80, Director of Undergraduate Admission."

- Stanford Daily, June 9, 2015

  "Record 81.1 percent yield rate reported for Class of 2019" by Victor Xu

  "The Office of Undergraduate Admission reports that 81.1 percent of admitted undergraduates enrolled as students in the Class of 2019, up from 78.2 percent last year."

  "1,737 students confirmed enrollment out of 2,142 admitted students."

And The Numbers This Year Are…

- Stanford News, March 30, 2018

  “Stanford offers admission to 2,040 students”

  “The Office of Undergraduate Admission announced that 2,040 high school students from across the country and around the world have been admitted to the Class of 2022.”

- Compare that to…

  Stanford News Service, News Release, June 3, 1996:

  “2,608 students offered admission to the Class of 2000 had accepted, a ‘yield rate’ of 61.4 percent, up from 55.1 percent last year.”