Balls, Urns, and the Supreme Court

- Supreme Court case: Berghuis v. Smith

If a group is underrepresented in a jury pool, how do you tell?

  - Article by Erin Miller – Friday, January 22, 2010
  - Thanks to (former CS109er) Josh Falk for this article

Justice Breyer [Stanford Alum] opened the questioning by invoking the binomial theorem. He hypothesized a scenario involving **"an urn with a thousand balls, and sixty are red, and nine hundred forty are black, and then you select them at random… twelve at a time."** According to Justice Breyer and the binomial theorem, if the red balls were black jurors then **"you would expect… something like <u>a third to a half</u> of juries would have at least one black person"** on them.

- Justice Scalia's rejoinder: "We don't have any urns here."

Justice Breyer Meets CS109

- Should model this combinatorially (X ~ HypGeo)
  - Ball draws not independent trials (balls not replaced)
- Exact solution:

$$P(\text{draw 12 black balls}) = \binom{940}{12} \Big/ \binom{1000}{12} \approx 0.4739$$

$$P(\text{draw} \geq 1 \text{ red ball}) = 1 - P(\text{draw 12 black balls}) \approx 0.5261$$

- Approximation using Binomial distribution
  - Assume P(red ball) constant for every draw = 60/1000
  - X = # red balls drawn. X ~ Bin(12, 60/1000 = 0.06)
  - $P(X \geq 1) = 1 - P(X = 0) \approx 1 - 0.4759 = 0.5240$

In Breyer's description, should actually expect just <u>over half</u> of juries to have at least one black person on them

Demo

From Discrete to Continuous

- So far, all random variables we saw were *discrete*
  - Have finite or countably infinite values (e.g., integers)
  - Usually, values are binary or represent a count
- Now it's time for *continuous* random variables
  - Have (uncountably) infinite values (e.g., real numbers)
  - Usually represent measurements (arbitrary precision)
    - Height (centimeters), Weight (lbs.), Time (seconds), etc.
- Difference between how *many* and how *much*
- Generally, it means replace $\sum_{x=a}^{b} f(x)$ with $\int_{a}^{b} f(x)dx$

Continuous Random Variables

- *X* is a **Continuous Random Variable** if there is function $f(x) \geq 0$ for $-\infty \leq x \leq \infty$, such that:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

- *f* is a Probability Density Function (PDF) if:

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

Probability Density Functions

- Say f is a **<u>Probability Density Function</u>** (PDF)

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

  - $f(x)$ is **<u>not</u>** a probability, it is probability/units of X
  - Not meaningful without some subinterval over X

$$P(X = a) = \int_{a}^{a} f(x)dx = 0$$

  - Contrast with Probability Mass Function (PMF) in discrete case: $p(a) = P(X = a)$

    where $\sum_{i=1}^{\infty} p(x_i) = 1$ for X taking on values x_1, x_2, x_3, ...

Cumulative Distribution Functions

- For a continuous random variable X, the <u>**Cumulative Distribution Function**</u> (CDF) is:

$$F(a) = P(X < a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$

- Density f is derivative of CDF F: $f(a) = \frac{d}{da} F(a)$
- For continuous f and small ε:

$$P(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}) = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)dx \approx \varepsilon f(a)$$

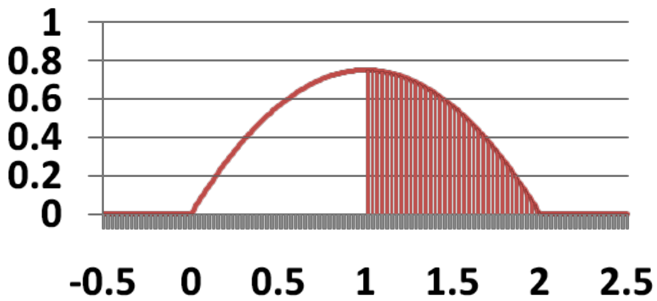
  - So, <u>ratio</u> of probabilities can still be meaningful:
    - P(X = 1)/P(X = 2) ≈ (ε f(1))/(ε f(2)) = f(1)/f(2)

Simple Example

“Support” of PDF

- X is continuous random variable (CRV) with PDF:

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{when } \boxed{0 < x < 2} \\ 0 & \text{otherwise} \end{cases}$$



  - What is C?

$$\int_0^2 C(4x - 2x^2)dx = 1 \quad \Rightarrow \quad C\left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_0^2 = 1$$

$$C\left(\left(8 - \frac{16}{3}\right) - 0\right) = 1 \quad \Rightarrow \quad C\frac{8}{3} = 1 \quad \Rightarrow \quad C = \frac{3}{8}$$

  - What is P(X > 1)?

$$\int_1^\infty f(x)dx = \int_1^2 \frac{3}{8}(4x - 2x^2)dx = \frac{3}{8}\left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_1^2 = \frac{3}{8}\left[\left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)\right] = \frac{1}{2}$$

Disk Crashes

- X = days of use before your disk crashes

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

  - First, determine λ to have actual PDF
    - Good integral to know: $\int e^u du = e^u$

$$1 = \int \lambda e^{-x/100} dx = -100\lambda \int \frac{-1}{100} e^{-x/100} dx = -100\lambda e^{-x/100} \Big|_0^{\infty} = 100\lambda \Rightarrow \lambda = \frac{1}{100}$$

  - What is P(50 < X < 150)?

$$F(150) - F(50) = \int_{50}^{150} \frac{1}{100} e^{-x/100} dx = -e^{-x/100} \Big|_{50}^{150} = -e^{-3/2} + e^{-1/2} \approx 0.383$$

  - What is P(X < 10)?

$$F(10) = \int_{0}^{10} \frac{1}{100} e^{-x/100} dx = -e^{-x/100} \Big|_{0}^{10} = -e^{-1/10} + 1 \approx 0.095$$

Expectation and Variance

For <u>discrete</u> RV X:

$$E[X] = \sum_x x\, p(x)$$

$$E[g(X)] = \sum_x g(x)\, p(x)$$

$$E[X^n] = \sum_x x^n\, p(x)$$

For <u>continuous</u> RV X:

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

$$E[X^n] = \int_{-\infty}^{\infty} x^n\, f(x)\, dx$$

For <u>both</u> discrete and continuous RVs:

$$E[aX + b] = aE[X] + b$$

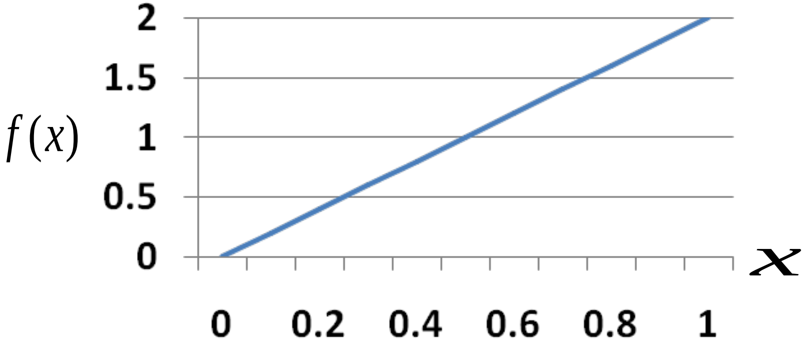
$$\mathrm{Var}(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

$$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$$

Linearly Increasing Density

- X is a continuous random variable with PDF:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



  - What is E[X]?

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^1 2x^2\, dx = \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

  - What is Var(X)?

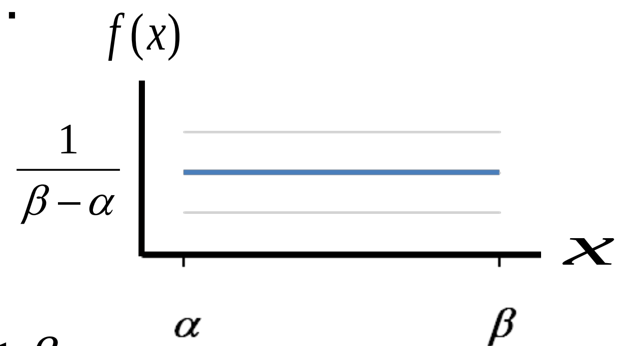
$$E[X^2] = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_0^1 2x^3\, dx = \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Uniform Random Variable

- X is a **Uniform Random Variable**: X ~ Uni(α, β)
  - Probability Density Function (PDF):

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$



    - Sometimes defined over range $\alpha < x < \beta$

  - $$P(a \le x \le b) = \int_a^b f(x)dx = \frac{b-a}{\beta-\alpha}$$ (for $\alpha \le a \le b \le \beta$)

  - $$E[X] = \int_{-\infty}^{\infty} x\,f(x)dx = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}dx = \left.\frac{x^2}{2(\beta-\alpha)}\right|_{\alpha}^{\beta} = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{\alpha+\beta}{2}$$

  - $$Var(X) = \frac{(\beta-\alpha)^2}{12}$$

Fun with the Uniform Distribution

- X ~ Uni(0, 20)

$$f(x) = \begin{cases} \frac{1}{20} & 0 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$$

  - P(X < 6)?

$$P(x < 6) = \int_0^6 \frac{1}{20}\,dx = \frac{6}{20}$$

  - P(4 < X < 17)?

$$P(4 < x < 17) = \int_4^{17} \frac{1}{20}\,dx = \frac{17}{20} - \frac{4}{20} = \frac{13}{20}$$

Riding the Marguerite Bus

- Say the Marguerite bus stops at the Gates bldg. at 15 minute intervals (2:00, 2:15, 2:30, etc.)
  - Passenger arrives at stop uniformly between 2-2:30pm
  - X ~ Uni(0, 30)
- P(Passenger waits < 5 minutes for bus)?
  - Must arrive between 2:10-2:15pm or 2:25-2:30pm

$$P(10 < X < 15) + P(25 < x < 30) = \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

- P(Passenger waits > 14 minutes for bus)?
  - Must arrive between 2:00-2:01pm or 2:15-2:16pm

$$P(0 < X < 1) + P(15 < x < 16) = \int_{0}^{1} \frac{1}{30}\,dx + \int_{15}^{16} \frac{1}{30}\,dx = \frac{1}{30} + \frac{1}{30} = \frac{1}{15}$$