Genetic Inheritance (Binomial review)

- Person has 2 genes for trait (eye color)
  - Child receives 1 gene (equally likely) from each parent
  - Child has brown eyes if either (or both) genes brown
  - Child only has blue eyes if both genes blue
  - Brown is "dominant" (d) , Blue is "recessive" (r)
  - Parents each have 1 brown and 1 blue gene
- 4 children, what is P(3 children with brown eyes)?
  - Child has blue eyes: p = (½) (½) = ¼   (2 blue genes)
  - P(child has brown eyes) = 1 – (¼) = 0.75
  - X = # of children with brown eyes. X ~ Bin(4, 0.75)

$$P(X=3)=\binom{4}{3}(0.75)^3(0.25)^1 \approx 0.4219$$

Whither the Binomial…

- Recall example of sending bit string over network
  - $n = 4$ bits sent over network where each bit had independent probability of corruption $p = 0.1$
  - X = number of bits corrupted. $X \sim \text{Bin}(4, 0.1)$
  - In real networks, send large bit strings (length $n \approx 10^4$)
  - Probability of bit corruption is very small $p \approx 10^{-6}$
  - $X \sim \text{Bin}(10^4, 10^{-6})$ is unwieldy to compute
- Extreme *n* and *p* values arise in many cases
  - # bit errors in file written to disk (# of typos in a book)
  - # of elements in particular bucket of large hash table
  - # of servers crashes in a day in giant data center
  - # Facebook login requests that go to particular server

Binomial in the Limit

- Recall the Binomial distribution

$$P(X=i)=\frac{n!}{i!(n-i)!}p^i(1-p)^{n-i}$$

- Let $\lambda = np$ (equivalently: $p = \lambda/n$)

$$P(X=i)=\frac{n!}{i!(n-i)!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}=\frac{n(n-1)...(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

- When n is large, p is small, and λ is "moderate":

$$\frac{n(n-1)...(n-i+1)}{n^i}\approx 1 \qquad (1-\lambda/n)^n \approx e^{-\lambda} \qquad (1-\lambda/n)^i \approx 1$$

- Yielding: $P(X=i)\approx 1\frac{\lambda^i}{i!}\frac{e^{-\lambda}}{1}=\frac{\lambda^i}{i!}e^{-\lambda}$

Poisson Random Variable

- X is a **<u>Poisson</u>** Random Variable: X ~ Poi(λ)
  - X takes on values 0, 1, 2…
  - and, for a given parameter $\lambda > 0$,
  - has distribution (PMF):

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

- Note Taylor series: $e^{\lambda} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + ... = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$
- So: $\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$

Sending Data on Network Redux

- Recall example of sending bit string over network
  - Send bit string of length $n = 10^4$
  - Probability of (independent) bit corruption $p = 10^{-6}$
  - $X \sim \text{Poi}(\lambda = 10^4 * 10^{-6} = 0.01)$
  - What is probability that message arrives uncorrupted?

$$P(X = 0) = e^{-\lambda} \frac{\lambda^i}{i!} = e^{-0.01} \frac{(0.01)^0}{0!} \approx 0.990049834$$

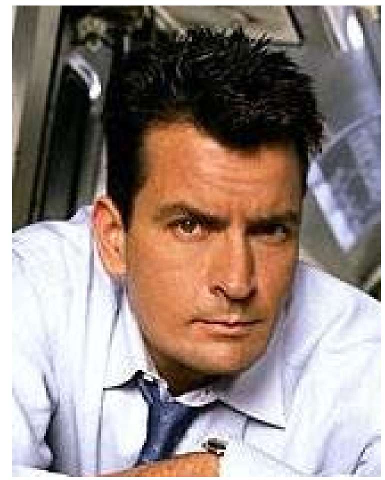
  - Using $Y \sim \text{Bin}(10^4, 10^{-6})$:

$$P(Y = 0) \approx 0.990049829$$

Caveat emptor: Binomial computed with built-in function in R software package, so some approximations may have occurred. Approximations are closer to you than they may appear in some software packages.

Simeon-Denis Poisson

- Simeon-Denis Poisson (1781-1840) was a prolific French mathematician



- Published his first paper at 18, became professor at 21, and published over 300 papers in his life
  - He reportedly said *"Life is good for only two things, discovering mathematics and teaching mathematics."*
- Definitely did not look like Charlie Sheen

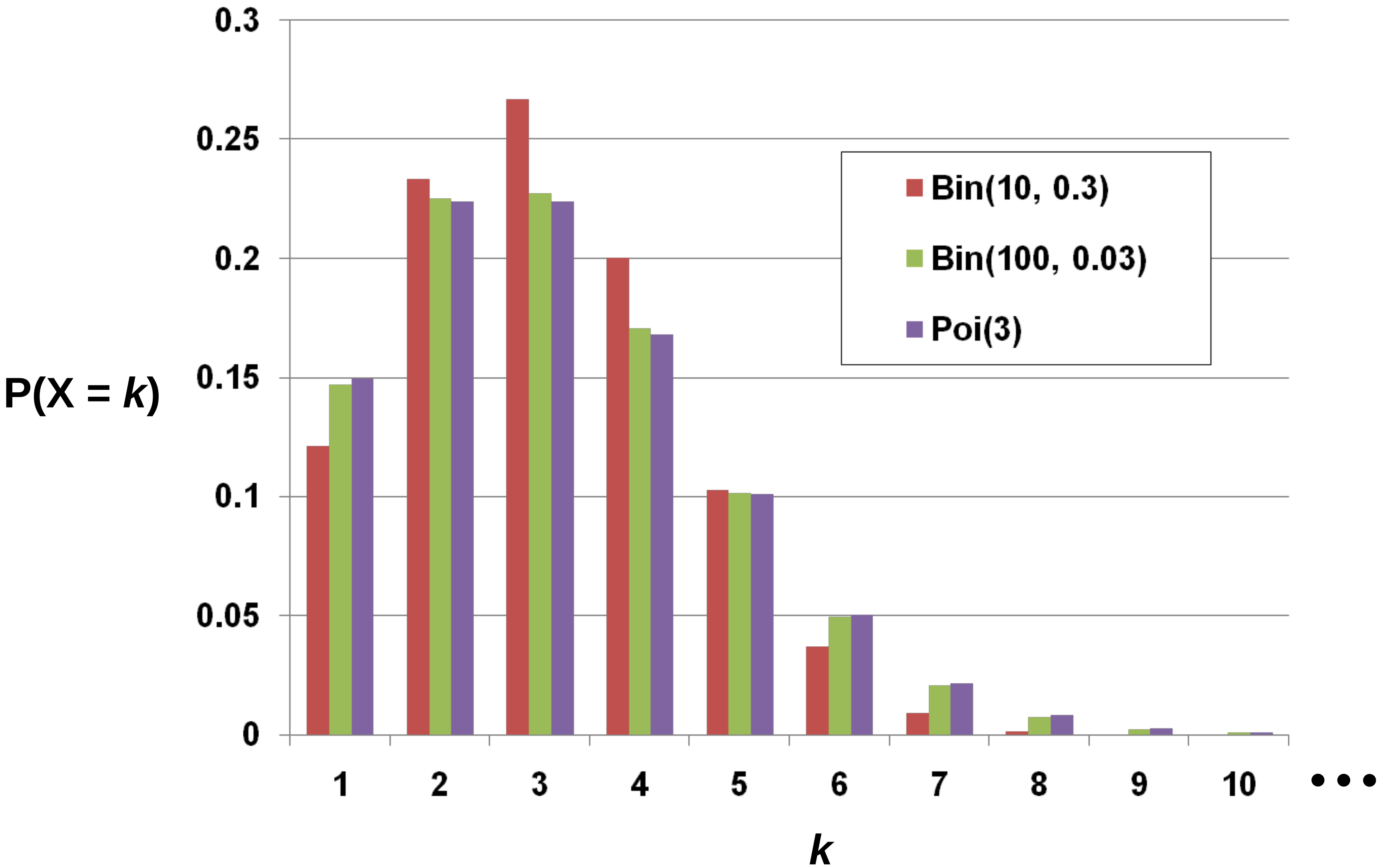
Poisson is Binomial in Limit

- Poisson approximates Binomial where n is large, p is small, and $\lambda = np$ is “moderate”
- Different interpretations of "moderate"
  - $n > 20$ and $p < 0.05$
  - $n > 100$ and $p < 0.1$

- Really, Poisson is Binomial as

$$n \rightarrow \infty \text{ and } p \rightarrow 0, \text{ where } np = \lambda$$

Bin(10, 0.3), Bin(100, 0.03) vs. Poi(3)



A Real License Plate Seen at Stanford

No, it's not mine…
but I kind of wish it was.

Tender (Central) Moments with Poisson

- Recall: Y ~ Bin(n, p)
  - E[Y] = np
  - Var(Y) = $np(1 - p)$

- X ~ Poi(λ) where $\lambda = np$ ($n \to \infty$ and $p \to 0$)
  - E[X] = $np = \lambda$
  - Var(X) = $np(1 - p) = \lambda(1 - 0) = \lambda$
  - Yes, expectation and variance of Poisson are same
    - It brings a tear to my eye…
  - Recall: $\mathrm{Var}(X) = E[X^2] - (E[X])^2$
  - $E[X^2] = \mathrm{Var}(X) + (E[X])^2 = \lambda + \lambda^2 = \lambda(1 + \lambda)$

It’s Really All About Raisin Cake

- Bake a cake using *many* raisins and *lots* of batter
- Cake is enormous (in fact, infinitely so…)
  - Cut slices of “moderate” size (w.r.t. # raisins/slice)
  - Probability p that a particular raisin is in a certain slice is very small (p = 1/# cake slices)
- Let X = number of raisins in a certain cake slice
- X ~ Poi(λ), where $\lambda = \dfrac{\text{total \# raisins}}{\text{\# cake slices}}$

CS = Baking Raisin Cake With Code

- Hash tables
  - strings = raisins
  - buckets = cake slices
- Server crashes in data center
  - servers = raisins
  - list of crashed machines = particular slice of cake
- Facebook login requests (i.e., web server requests)
  - requests = raisins
  - server receiving request = cake slice

Defective Chips

- Computer chips are produced
  - p = 0.1 that a chip is defective (chips are independent)
  - Consider a sample of n = 10 chips
  - What is P(sample contains $\leq$ 1 defective chip)?
  - Let Y = number of defective chips in sample
  - Using Y ~ Bin(10, 0.1). P(Y ≤ 1) = P(Y = 0) + P(Y = 1)

$$P(Y \leq 1) = \binom{10}{0}(0.1)^0(1-0.1)^{10} + \binom{10}{1}(0.1)^1(1-0.1)^9 \approx 0.7361$$

  - Using X ~ Poi(λ = (0.1)(10) = 1)

$$P(X \leq 1) = e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!} = 2e^{-1} \approx 0.7358$$

Approximately Poisson Approximation

- Poisson can still provide a good approximation even when assumptions are “mildly” violated
- “Poisson Paradigm”
- Can apply Poisson approximation when...
  - “Successes” in trials are not entirely independent
    - Example: # entries in each bucket in large hash table
  - Probability of “Success” in each trial varies (slightly)
    - Small relative change in a very small p
    - Example: average # requests to web server/sec. may fluctuate slightly due to load on network

Birthday Problem Redux

- What is the probability that of *m* people, none share the same birthday (regardless of year)?
  - $n = \binom{m}{2}$ trials, one for each pair of people (x, y), $x \neq y$
  - Let $E_{x,y}$ = x and y have same birthday (trial success)
  - $P(E_{x,y}) = p = 1/365$ (note: all $E_{x,y}$ not independent)
  - $X \sim \text{Poi}(\lambda)$ where $\lambda = \binom{m}{2}\frac{1}{365} = \frac{m(m-1)}{730}$

$$P(X = 0) = e^{-m(m-1)/730}\frac{(m(m-1)/730)^0}{0!} = e^{-m(m-1)/730}$$

  - Solve for smallest integer *m*, s.t.: $e^{-m(m-1)/730} \leq 0.5$

$$\ln(e^{-m(m-1)/730}) \leq \ln(0.5) \rightarrow m(m-1) \geq -730\ln(0.5) \rightarrow m \geq 23$$

  - Same as before!

Poisson Processes

- Consider “rare” events that occur over time
  - Earthquakes, radioactive decay, hits to web server, etc.
  - Have time interval for events (1 year, 1 sec, whatever...)
  - Events arrive at rate: λ events per interval of time
- Split time interval into $n \rightarrow \infty$ sub-intervals
  - Assume at most one event per sub-interval
  - Event occurrences in sub-intervals are independent
  - With many sub-intervals, probability of event occurring in any given sub-interval is small
- $N(t)$ = # events in original time interval ~ $\text{Poi}(\lambda)$

Web Server Load

- Consider requests to a web server in 1 second
  - In past, server load averages 2 hits/second
  - X = # hits server receives in a second
  - What is P(X = 5)?
- Model
  - Assume server cannot acknowledge > 1 hit/msec.
  - 1 sec = 1000 msec. (= large *n*)
  - P(hit server in 1 msec) = 2/1000 (= small *p*)
  - X ~ Poi(λ = 2)

$$P(X = 5) = e^{-2}\frac{2^5}{5!} \approx 0.0361$$

Geometric Random Variable

- X is **Geometric** Random Variable: X ~ Geo(*p*)
  - X is number of independent trials until first success
  - *p* is probability of success on each trial
  - X takes on values 1, 2, 3, …, with probability:

$$P(X = n) = (1 - p)^{n-1} p$$

  - E[X] = 1/*p* Var(X) = (1 – *p*)/p^2
- Examples:
  - Flipping a coin (P(heads) = *p*) until first heads appears
  - Urn with N black and M white balls. Draw balls (with replacement, *p* = N/(N + M)) until draw first black ball
  - Generate bits with P(bit = 1) = *p* until first 1 generated

Negative Binomial Random Variable

- X is **Negative Binomial** RV: X ~ NegBin(r, p)
  - X is number of independent trials until r successes
  - p is probability of success on each trial
  - X takes on values r, r + 1, r + 2…, with probability:

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \text{ where } n = r, r+1, \ldots$$

  - E[X] = r/p Var(X) = $r(1 - p)/p^2$
- Note: Geo(p) ~ NegBin(1, p)
- Examples:
  - # of coin flips until r-th "heads" appears
  - # of strings to hash into table until bucket 1 has r entries

Hypergeometric Random Variable

- X is **<u>Hypergeometric</u>** RV: X ~ HypG(n, N, m)
  - Urn with N balls: m white and ($N - m$) black
  - Draw n balls **<u>without</u>** replacement
  - X is number of white balls drawn

$$P(X = i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}, \text{ where } i = 0,1,\dots,n$$

  - E[X] = $n(m/N)$ Var(X) = $[nm(N - n)(N - m)]/[N^2(N - 1)]$
  - Let $p = m/N$ (probability of drawing white on 1st draw)
- Note: HypG(n, N, m) → Bin(n, m/N)
  - As $N \to \infty$ and m/N remains constant

Endangered Species

- Determine *N* = how many of some species remain
  - Randomly tag *m* of species (e.g., with white paint)
  - Allow animals to mix randomly (assuming no breeding)
  - Later, randomly observe another *n* of the species
  - X = number of tagged animals in observed group of *n*
  - X ~ HypG(*n*, *N, m*)
- "Maximum Likelihood" estimate
  - Set *N* to be value that maximizes: $$P(X=i)=\frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

    for the value *i* of X that you observed → $\hat{N}$ = *mn*/*i*
- Similar to assuming: *i* = E[X] = *nm*/*N*