Acey Deucey

- Have a standard deck of 52 cards
  - Ranks of cards: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
  - Three cards drawn (without replacement)
  - What is probability that rank of third card drawn is between the ranks of the first two cards, exclusive?
    - E.g., if ranks of first two cards drawn are 4 and 9, then want probability that third card is a 5, 6, 7 or 8
- Solution set-up
  - Let X = difference between rank of 1st and 2nd card
  - $P(X = 0) = 3/51$
    - After picking first card, there are 3 others with same rank
    - This is not really relevant. Just a warm-up to get you thinking!

Acey Deucey Solution

- Solution
  - P(X = *i*) = $(13-i)\cdot\frac{2}{13}\cdot\frac{4}{51}$, where 1 ≤ *i* ≤ 12
    - (13 – *i*) ways to choose two ranks that differ by *i*
    - First card has 2/13 chance of being one of those 2 ranks
    - Second card is one of 4 cards (out of 51) that differ in rank by *i*
  - Want: $\sum_{i=1}^{12} P(X=i)P(\text{3rd card between first two} \mid X=i)$

$$= \sum_{i=1}^{12} \frac{8(13-i)}{(13)(51)} P(\text{3rd card between first two} \mid X=i)$$

    - Of remaining 50 cards, there are 4 cards of each (*i* – 1) ranks

$$= \sum_{i=1}^{12} \frac{8(13-i)}{(13)(51)} \cdot \frac{4(i-1)}{50} \approx 0.2761$$

Acey Deucey Solution Using Symmetry

- Solution
  - In order for there to be a feasible solution, the ranks of the three cards drawn must all be different
    - If first two cards are same rank, there is no rank between them
    - If third card has same rank as either of first two, it's not between
  - First, compute: P(ranks of all three cards are different)

$$= \frac{48}{51} \cdot \frac{44}{50} \quad \begin{cases} \text{second card has different rank than first} \\ \text{third card has different rank than first two} \end{cases}$$

  - Then, need third card to be in middle of first two

$$= \frac{2!}{3!} = \frac{1}{3} \quad \begin{cases} \text{3! permutations of 3 cards} \\ \text{2! permutations where middle card is third} \end{cases}$$

  - Combining, yields: $\frac{48}{51} \cdot \frac{44}{50} \cdot \frac{1}{3} \approx 0.2761$ (same as before)

Birthdays Tres Compadres

- Have a group of 100 people
  - Let X = number of days of year that are birthdays of *exactly* 3 people in group
- What is E[X]?
  - First, compute probability *p* that a particular day is the birthday of exactly 3 people in the group
    - Let A_i = number of people that have birthday on day *i*
    - $A_i \sim \text{Bin}(100, 1/365)$
    - $p = P(A_i = 3) = \binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97}$
  - Let $X_i = 1$ if $A_i = 3$, and 0 otherwise
  - $E[X] = E[\sum_{i=1}^{365} X_i] = \sum_{i=1}^{365} E[X_i] = \sum_{i=1}^{365} P(A_i = 3) = \sum_{i=1}^{365} p = 365p$

More Birthdays, More Fun

- Have a group of 100 people
  - Let Y = number of distinct days that are birthdays of at least one person
  - What is E[Y]?
- Solution
  - Let Y_i = 1 if day *i* is the birthday of at least 1 person, and 0 otherwise
  - $E[Y_i] = P(Y_i) = 1 - P(Y_i^C) = 1-\left(\frac{364}{365}\right)^{100}$
  - $E[Y] = E[\sum_{i=1}^{365} Y_i] = \sum_{i=1}^{365} E[Y_i] = 365\left[1-\left(\frac{364}{365}\right)^{100}\right]$

MOM Loves the Geometric

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Geo}(p)$
- Estimate p using Method of Moments
- Solution
  - Recall, for $X_i \sim \text{Geo}(p)$, we know $E[X_i] = 1/p$
  - Rewrite as $p = 1/E[X_i]$
  - Using Method of Moments:

$$p = \frac{1}{E[X_i]} \approx \frac{1}{\hat{m}_1} = \frac{1}{\overline{X}} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i} = \hat{p}$$