Choosing a Random Subset

- From set of n elements, choose a subset of size k such that all $\binom{n}{k}$ possibilities are <u>equally</u> likely
  - Only have **`random()`**, which simulates X ~ Uni(0, 1)
- Brute force:
  - Generate (an ordering of) all subsets of size k
  - Randomly pick one (divide (0, 1) into $\binom{n}{k}$ intervals)
  - Expensive with regard to time and space
  - Bad times!

(Happily) Choosing a Random Subset

- Good times:

```
int indicator(double p) {
   if (random() < p) return 1; else return 0;
}

subset rSubset(k, set of size n) {
   subset_size = 0;
   I[1] = indicator((double)k/n);
   for(i = 1; i < n; i++) {
      subset_size += I[i];
      I[i+1] = indicator((k – subset_size)/(n – i));
   }
   return (subset containing element[i] iff I[i] == 1);
}
```

$P(I[1]=1)=\frac{k}{n}$ and $P(I[i+1]=1 \mid I[1],\ldots,I[i])=\frac{k-\sum_{j=1}^{i} I[j]}{n-i}$ where $1<i<n$

Random Subsets the Happy Way

- Proof (Induction on (k + n)): (i.e., why this algorithm works)
  - Base Case: k = 1, n = 1, Set S = {*a*}, **`rSubset`** returns {*a*} with p=$1 \Big/ \binom{1}{1}$
  - Inductive Hypoth. (IH): for k + x ≤ c, Given set S, |S| = x and k ≤ x, **`rSubset`** returns any subset S' of S, where |S'| = k, with p = $1 \Big/ \binom{x}{k}$
  - Inductive Case 1: (where k + n ≤ c + 1) |S| = n (= x + 1), **`I[1] = 1`**
    - Elem 1 in subset, choose k – 1 elems from remaining n – 1
    - By IH: **`rSubset`** returns subset S' of size k – 1 with p = $1 \Big/ \binom{n-1}{k-1}$
    - P(I[1] = 1, subset S') = $\frac{k}{n} \cdot 1 \Big/ \binom{n-1}{k-1} = 1 \Big/ \binom{n}{k}$
  - Inductive Case 2: (where k + n ≤ c + 1) |S| = n (= x + 1), **`I[1] = 0`**
    - Elem 1 not in subset, choose k elems from remaining n – 1
    - By IH: **`rSubset`** returns subset S' of size k with p = $1 \Big/ \binom{n-1}{k}$
    - P(I[1] = 0, subset S') = $\left(1 - \frac{k}{n}\right) \cdot 1 \Big/ \binom{n-1}{k} = \left(\frac{n-k}{n}\right) \cdot 1 \Big/ \binom{n-1}{k} = 1 \Big/ \binom{n}{k}$

Sum of Independent Binomial RVs

- Let X and Y be independent random variables
  - $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$
  - $X + Y \sim \text{Bin}(n_1 + n_2, p)$
- Intuition:
  - X has n_1 trials and Y has n_2 trials
    - Each trial has same "success" probability p
  - Define Z to be $n_1 + n_2$ trials, each with success prob. p
  - $Z \sim \text{Bin}(n_1 + n_2, p)$, and also $Z = X + Y$
- More generally: $X_i \sim \text{Bin}(n_i, p)$ for $1 \leq i \leq N$

$$\left(\sum_{i=1}^{N} X_i\right) \sim \text{Bin}\left(\sum_{i=1}^{N} n_i,\ p\right)$$

Sum of Independent Poisson RVs

- Let X and Y be independent random variables
  - X ~ Poi(λ_1) and Y ~ Poi(λ_2)
  - X + Y ~ Poi($\lambda_1 + \lambda_2$)
- Proof: (just for reference)
  - Rewrite (X + Y = *n*) as (X = *k*, Y = *n* – *k*) where 0 ≤ *k* ≤ *n*

$$P(X+Y=n)=\sum_{k=0}^{n}P(X=k,Y=n-k)=\sum_{k=0}^{n}P(X=k)P(Y=n-k)$$

$$=\sum_{k=0}^{n}e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}=e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!}=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k}$$

  - Noting Binomial theorem: $(\lambda_1+\lambda_2)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k}$
  - $P(X+Y=n)=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\left(\lambda_1+\lambda_2\right)^n$ so, X + Y = n ~ Poi($\lambda_1 + \lambda_2$)

Reference: Sum of Independent RVs

- Let X and Y be independent Binomial RVs
  - $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$
  - $X + Y \sim \text{Bin}(n_1 + n_2, p)$
  - More generally, let $X_i \sim \text{Bin}(n_i, p)$ for $1 \le i \le N$, then

$$\left(\sum_{i=1}^{N} X_i\right) \sim \text{Bin}\left(\sum_{i=1}^{N} n_i,\ p\right)$$

- Let X and Y be independent Poisson RVs
  - $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$
  - $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$
  - More generally, let $X_i \sim \text{Poi}(\lambda_i)$ for $1 \le i \le N$, then

$$\left(\sum_{i=1}^{N} X_i\right) \sim \text{Poi}\left(\sum_{i=1}^{N} \lambda_i\right)$$

Dance, Dance, Convolution

- Let X and Y be independent random variables
  - Cumulative Distribution Function (CDF) of X + Y:

$$F_{X+Y}(a) = P(X+Y \le a)$$
$$= \iint_{x+y\le a} f_X(x)\, f_Y(y)\, dx\, dy = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{a-y} f_X(x)\, dx\;\; f_Y(y)\, dy$$
$$= \int_{y=-\infty}^{\infty} F_X(a-y)\;\, f_Y(y)\, dy$$

  - F_{X+Y} is called **convolution** of F_X and F_Y
  - Probability Density Function (PDF) of X + Y, analogous:

$$f_{X+Y}(a) = \int_{y=-\infty}^{\infty} f_X(a-y)\;\, f_Y(y)\, dy$$

  - In discrete case, replace $\int_{y=-\infty}^{\infty}$ with $\sum_{y}$, and $f(y)$ with $p(y)$

Sum of Independent Uniform RVs

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \le x \le 1$
  - What is PDF of X + Y?

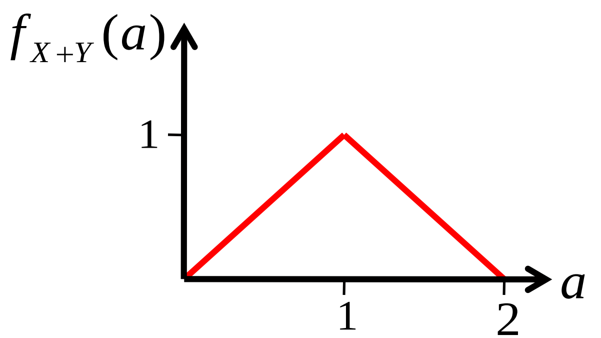
$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y)\, f_Y(y)\, dy = \int_{y=0}^{1} f_X(a-y)\, dy$$

  - When $0 \le a \le 1$ and $0 \le y \le a$, $0 \le a–y \le 1$ → $f_X(a-y) = 1$

$$f_{X+Y}(a) = \int_{y=0}^{a} dy = a$$

  - When $1 \le a \le 2$ and $a–1 \le y \le 1$, $0 \le a–y \le 1$ → $f_X(a-y) = 1$

$$f_{X+Y}(a) = \int_{y=a-1}^{1} dy = 2-a$$



  - Combining: $f_{X+Y}(a) = \begin{cases} a & 0 \le a \le 1 \\ 2-a & 1 < a \le 2 \\ 0 & \text{otherwise} \end{cases}$

Sum of Independent Normal RVs

- Let X and Y be independent random variables
  - $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
  - $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

- Generally, have *n* independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$:

$$\left(\sum_{i=1}^{n} X_i\right) \sim N\left(\sum_{i=1}^{n} \mu_i,\ \sum_{i=1}^{n} \sigma_i^2\right)$$

Virus Infections

- Say your RCC checks dorm machines for viruses
  - 50 Macs, each independently infected with $p = 0.1$
  - 100 PCs, each independently infected with $p = 0.4$
  - A = # infected Macs   A ~ Bin(50, 0.1) ≈ X ~ N(5, 4.5)
  - B = # infected PCs   B ~ Bin(100, 0.4) ≈ Y ~ N(40, 24)
  - What is P(≥ 40 machine infected)?
  - P(A + B ≥ 40) ≈ P(X + Y ≥ 39.5)
  - X + Y = W ~ N(5 + 40 = 45, 4.5 + 24 = 28.5)

$$P(W \geq 39.5) = P\left(\frac{W-45}{\sqrt{28.5}} > \frac{39.5-45}{\sqrt{28.5}}\right) = 1 - \Phi(-1.03) \approx 0.8485$$

Discrete Conditional Distributions

- Recall that for *events* E and F:

$$P(E \mid F) = \frac{P(EF)}{P(F)} \quad \text{where} \quad P(F) > 0$$

- Now, have X and Y as discrete random variables
  - Conditional PMF of X given Y (where $p_Y(y) > 0$):

$$P_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

  - Conditional CDF of X given Y (where $p_Y(y) > 0$):

$$F_{X|Y}(a \mid y) = P(X \le a \mid Y = y) = \frac{P(X \le a, Y = y)}{P(Y = y)}$$

$$= \frac{\sum_{x \le a} p_{X,Y}(x, y)}{p_Y(y)} = \sum_{x \le a} p_{X|Y}(x \mid y)$$

Operating System Loyalty

- Consider person buying 2 computers (over time)
  - X = 1st computer bought is a PC (1 if it is, 0 if it is not)
  - Y = 2nd computer bought is a PC (1 if it is, 0 if it is not)
  - Joint probability mass function (PMF):

Y \ X	0	1	$p_Y(y)$
0	0.2	0.3	0.5
1	0.1	0.4	0.5
$p_X(x)$	0.3	0.7	1.0

  - What is P(Y = 0 | X = 0)?

$$P(Y=0 \mid X=0) = \frac{p_{X,Y}(0,0)}{p_X(0)} = \frac{0.2}{0.3} = \frac{2}{3}$$

  - What is P(Y = 1 | X = 0)?

$$P(Y=1 \mid X=0) = \frac{p_{X,Y}(0,1)}{p_X(0)} = \frac{0.1}{0.3} = \frac{1}{3}$$

  - What is P(X = 0 | Y = 1)?

$$P(X=0 \mid Y=1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{0.1}{0.5} = \frac{1}{5}$$

And It Applies to Books Too…

P(Buy Book Y | Bought Book X)

Web Server Requests Redux

- Requests received at web server in a day
  - X = # requests from humans/day   X ~ Poi(λ_1)
  - Y = # requests from bots/day   Y ~ Poi(λ_2)
  - X and Y are independent → X + Y ~ Poi($\lambda_1 + \lambda_2$)
  - What is P(X = *k* | X + Y = *n*)?

$$P(X=k \mid X+Y=n) = \frac{P(X=k, Y=n-k)}{P(X+Y=n)} = \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)}$$

$$= \frac{e^{-\lambda_1}\lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!} \cdot \frac{n!}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n} = \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

  - X | X + Y $\sim \mathrm{Bin}\left(X+Y, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

Continuous Conditional Distributions

- Let X and Y be continuous random variables
  - Conditional PDF of X given Y (where $f_Y(y) > 0$):

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$f_{X|Y}(x \mid y)\, dx = \frac{f_{X,Y}(x, y)\, dx\, dy}{f_Y(y)\, dy}$$

$$\approx \frac{P(x \le X \le x + dx, y \le Y \le y + dy)}{P(y \le Y \le y + dy)} = P(x \le X \le x + dx \mid y \le Y \le y + dy)$$

  - Conditional CDF of X given Y (where $f_Y(y) > 0$):

$$F_{X|Y}(a \mid y) = P(X \le a \mid Y = y) = \int_{-\infty}^{a} f_{X|Y}(x \mid y)\, dx$$

  - Note: Even though P(Y = *a*) = 0, can condition on Y = *a*
    - Really considering: $P(a - \frac{\varepsilon}{2} \le Y \le a + \frac{\varepsilon}{2}) = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f_Y(y) dy \approx \varepsilon\, f(a)$

Let's Do an Example

- X and Y are continuous RVs with PDF:

$$f(x,y)=\begin{cases}\frac{12}{5}x(2-x-y) & \text{where } 0<x,y<1\\ 0 & \text{otherwise}\end{cases}$$

  - Compute conditional density: $f_{X|Y}(x\,|\,y)$

$$f_{X|Y}(x\,|\,y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{f_{X,Y}(x,y)}{\int_0^1 f_{X,Y}(x,y)\,dx}$$

$$=\frac{\frac{12}{5}x(2-x-y)}{\int_0^1 \frac{12}{5}x(2-x-y)\,dx}=\frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx}=\frac{x(2-x-y)}{\left[x^2-\frac{x^3}{3}-\frac{x^2y}{2}\right]_0^1}$$

$$=\frac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}}=\frac{6x(2-x-y)}{4-3y}$$