Inequality, Probability, and Joviality

- In many cases, we don’t know the true form of a probability distribution
  - E.g., Midterm scores
  - But, we know the mean
  - May also have other measures/properties
    - Variance
    - Non-negativity
    - Etc.
  - Inequalities and bounds still allow us to say something about the probability distribution in such cases
    - May be imprecise compared to knowing true distribution!

Markov's Inequality

- Say X is a non-negative random variable

$$P(X \geq a) \leq \frac{E[X]}{a}, \quad \text{for all } a > 0$$

- Proof:
  - I = 1 if X ≥ *a*, 0 otherwise
  - Since $X \geq 0, \quad I \leq \frac{X}{a}$
  - Taking expectations:

$$E[I] = P(X \geq a) \leq E\left[\frac{X}{a}\right] = \frac{E[X]}{a}$$

Andrey Andreyevich Markov

- Andrey Andreyevich Markov (1856-1922) was a Russian mathematician

  - Markov’s Inequality is named after him
  - He also invented Markov Chains…
    - …which are the basis for Google’s PageRank algorithm

Markov and the Midterm

- Statistics from a prior quarter's CS109 midterm
  - X = midterm score
  - Using sample mean $\overline{X} = 86.7 \approx E[X]$
  - What is P(X ≥ 100)?

$$P(X \geq 100) \leq \frac{E[X]}{100} = \frac{86.7}{100} \approx 0.867$$

  - Markov bound: ≤ 86.7% of class scored 100 or greater
  - In fact, 30.1% of class scored 100 or greater
    - Markov inequality can be a very loose bound
    - But, it made <u>no</u> assumption at all about form of distribution!

Chebyshev's Inequality

- X is a random variable with $E[X] = \mu$, $\mathrm{Var}(X) = \sigma^2$

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}, \quad \text{for all } k > 0$$

- Proof:
  - Since $(X - \mu)^2$ is non-negative random variable, apply Markov's Inequality with $a = k^2$

$$P((X-\mu)^2 \geq k^2) \leq \frac{E[(X-\mu)^2]}{k^2} = \frac{\sigma^2}{k^2}$$

  - Note that: $(X - \mu)^2 \geq k^2 \Leftrightarrow |X - \mu| \geq k$, yielding:

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Pafnuty Chebyshev

- Pafnuty Lvovich Chebyshev (1821-1894) was also a Russian mathematician

  - Chebyshev’s Inequality is named after him
    - But actually formulated by his colleague Irénée-Jules Bienaymé
  - He was Markov’s doctoral advisor
    - And sometimes credited with first deriving Markov’s Inequality
  - There is a crater on the moon named in his honor

Of the Midterm What Say You Chebyshev?

- Statistics from a prior quarter's CS109 midterm
  - X = midterm score
  - Using sample mean $\overline{X} = 86.7 \approx E[X]$
  - Using sample variance $S^2 = (20.14)^2 \approx 405.62 \approx \sigma^2$
  - What is $P(|X - 86.7| \geq 25)$?

$$P(|X - E[X]| \geq 25) \leq \frac{\sigma^2}{(25)^2} = \frac{405.62}{625} \approx 0.6490$$

$$P(|X - E[X]| < 25) = 1 - P(|X - E[X]| \geq 25) \geq 1 - 0.6490 = 0.3510$$

  - Chebyshev bound: $\leq$ 64.90% scored $\geq$ 111.7 or $\leq$ 61.7
  - In fact, 20.61% of class scored $\geq$ 111.7 or $\leq$ 61.7
    - Chebyshev's inequality is really a theoretical tool

One-Sided Chebyshev's Inequality

- X is a random variable with E[X] = 0, Var(X) = σ^2

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}, \quad \text{for any } a > 0$$

  - Equivalently, when E[Y] = μ and Var(Y) = σ^2:

$$P(Y \geq E[Y] + a) \leq \frac{\sigma^2}{\sigma^2 + a^2}, \quad \text{for any } a > 0$$

$$P(Y \leq E[Y] - a) \leq \frac{\sigma^2}{\sigma^2 + a^2}, \quad \text{for any } a > 0$$

  - Follows directly by setting X = Y – E[Y], noting E[X] = 0

Comments on Midterm, One-Sided One?

- Statistics from a prior quarter's CS109 midterm
  - X = midterm score
  - Using sample mean $\overline{X} = 86.7 \approx E[X]$
  - Using sample variance $S^2 = (20.14)^2 \approx 405.62 \approx \sigma^2$
  - What is $P(X \geq 96.7)$?

$$P(X \geq 86.7 + 10) \leq \frac{405.62}{405.62 + (10)^2} \approx 0.8022$$

  - One-sided Chebyshev bound: $\leq$ 80.22% scored ≥ 96.7
  - In fact, 36.82% of class scored ≥ 96.7
  - Using Markov's inequality: $P(X \geq 96.7) \leq \dfrac{86.7}{96.7} \approx 0.8966$

Chernoff Bound

- Say we have MGF, M(*t*), for a random variable X
  - Chernoff bounds:

$$P(X \geq a) \leq e^{-ta} M(t), \quad \text{for all } t > 0$$
$$P(X \leq a) \leq e^{-ta} M(t), \quad \text{for all } t < 0$$

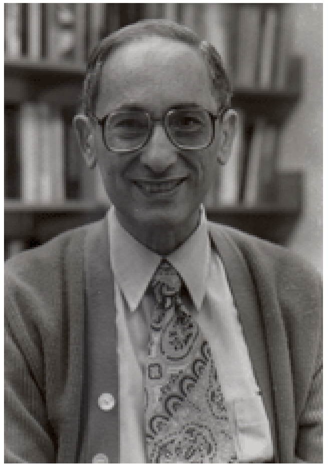
  - Bounds hold for $t \neq 0$, so use *t* that minimizes $e^{-ta}M(t)$
- Proof:
  - X has MGF: $M(t) = E[e^{tX}]$
  - Note $P(X \geq a) = P(e^{tX} \geq e^{ta})$, use Markov's inequality:

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-ta} E[e^{tX}] = e^{-ta} M(t), \quad \text{for all } t > 0$$

  - Similarity for $P(X \leq a)$ when $t < 0$

Herman Chernoff

- Herman Chernoff (1923-) is an American mathematician and statistician



  - Chernoff Bound is named after him
    - And it actually was derived by him!
  - He is Professor Emeritus of Applied Mathematics at MIT and of Statistics at Harvard University
    - I did not know if he was a fan of Charlie Sheen… until 2013.

No, You Didn’t…

From: Trey Deitch
Sent: Thursday, February 21, 2013 9:18 PM
To: Chernoff, Herman
Subject: Light-hearted question

Prof. Chernoff,

I'm in an introductory statistics class at Stanford University. As part of our lectures, our professor often links mathematicians to the actor Charlie Sheen in interesting ways. These comparisons keep students engaged and are usually pretty amusing. Your name came up in association with Chernoff bounds and I was wondering if you had any thoughts on Charlie Sheen. This could be either a way you are like him or just what you think of him.

Thank you,
Trey Deitch

Yes, You Did

From: "Chernoff, Herman" <chernoff@stat.harvard.edu>
Date: February 22, 2013, 8:25:05 AM PST
To: Trey Deitch <tdeitch@stanford.edu>
Subject: RE: Light-hearted question

I think Sheen is a good actor, almost as good as his father. I enjoyed the character he played in two and a half men. I think he was in real life somewhat deluded, as was the people who paid money to see him after he was fired. In personality, consider an equilateral triangle with Sheen, his character, and myself at the three vertices.

Sincerely,
Herman Chernoff

PS. Who is your instructor? HC

Chernoff's Feeling (Unit) Normal

- Z is standard normal random variable: Z ~ N(0, 1)
  - Moment generating function: $M_Z(t) = e^{t^2/2}$
  - Chernoff bounds for P(Z ≥ *a*)

$$P(Z \geq a) \leq e^{-ta} e^{t^2/2} = e^{t^2/2 - ta}, \quad \text{for all } t > 0$$

  - To minimize bound, minimize: $t^2/2 - ta$
    - Differentiate w.r.t. *t*, and set to 0: $t - a = 0 \Rightarrow t = a$

$$P(Z \geq a) \leq e^{-a^2/2}, \quad \text{for all } \mathrm{t} = a > 0$$

  - Can proceed similarly for $t = a < 0$ to obtain:

$$P(Z \leq a) \leq e^{-a^2/2}, \quad \text{for all } t = a < 0$$

  - Compare to: $P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

Chernoff's Poisson Pill

- X is Poisson random variable: X ~ Poi(λ)
  - Moment generating function: $M_X(t) = e^{\lambda(e^t - 1)}$
  - Chernoff bounds for P(X ≥ *i*)

$$P(X \geq i) \leq e^{\lambda(e^t - 1)} e^{-it} = e^{\lambda(e^t - 1) - it}, \quad \text{for all } t > 0$$

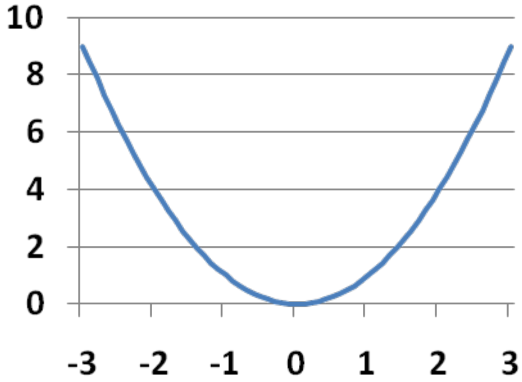
  - To minimize bound, minimize: $\lambda(e^t - 1) - it$
    - Differentiate w.r.t. *t*, and set to 0: $\lambda e^t - i = 0 \Rightarrow e^t = i/\lambda$

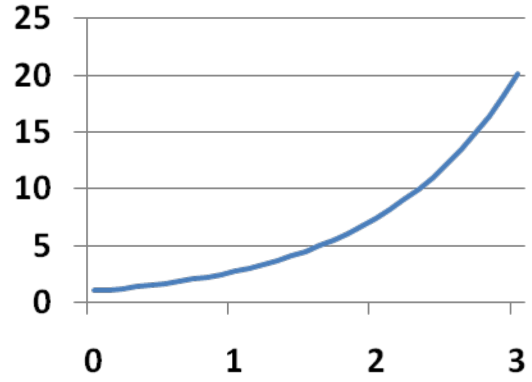
$$P(X \geq i) \leq e^{\lambda(i/\lambda - 1)} \left(\frac{i}{\lambda}\right)^{-i} = e^i e^{-\lambda} \left(\frac{\lambda}{i}\right)^i = \left(\frac{e\lambda}{i}\right)^i e^{-\lambda}, \quad \text{for all } i/\lambda > 1$$

  - Compare to: $P(X = i) = e^{-\lambda} \dfrac{\lambda^i}{i!}$

Jensen's Inequality

- If $f(x)$ is a *convex* function then $E[f(X)] \geq f(E[X])$
  - $f(x)$ is **convex** if $f''(x) \geq 0$ for all x
  - Intuition: Convex = "bowl". E.g.: $f(x) = x^2$, $f(x) = e^x$





  - if $g(x) = -f(x)$ is convex, then $f(x)$ is **concave**
  - Proof outline: Taylor series of $f(x)$ about μ. Be happy.
  - Note: $E[f(X)] = f(E[X])$ only holds when $f(x)$ is a line
    - That is when: $f''(x) = 0$ for all x

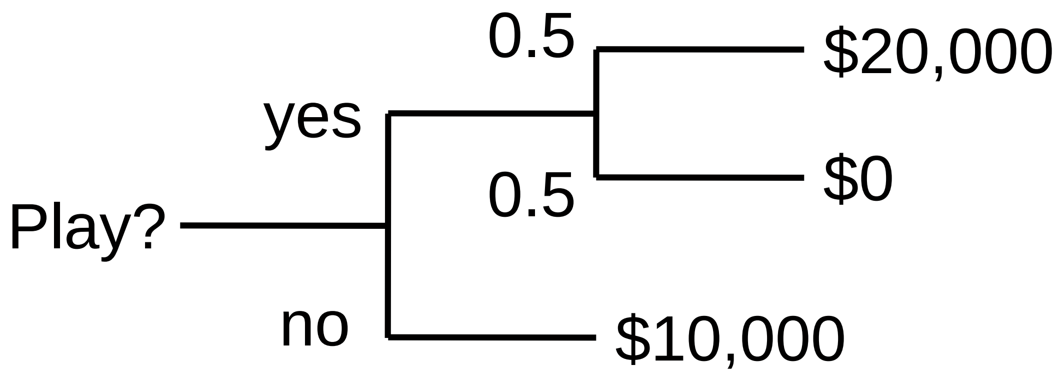
Johan Jensen

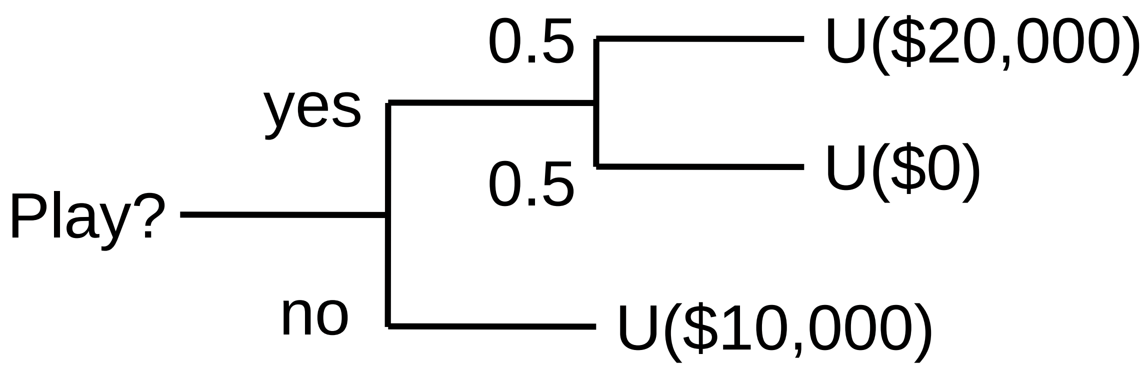
- Johan Ludwig William Valdemar Jensen (1859-1925) was a Danish mathematician

  - He derived Jensen’s inequality
  - He was president of the Danish Mathematical Society from 1892 to 1903

A Brief Digression on Utility Theory

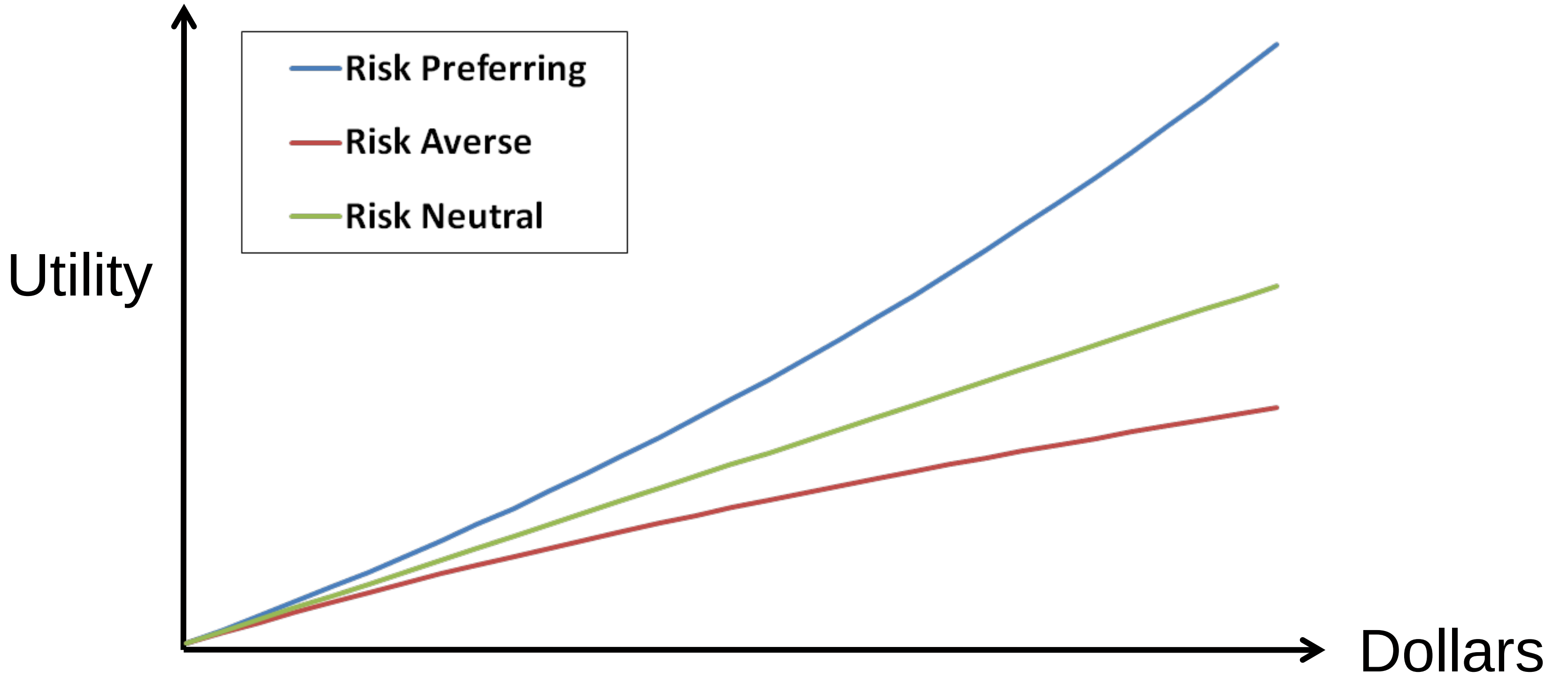
- Utility U(x) is "value" you derive from x





  - Can be monetary, but often includes intangibles
    - E.g., quality of life, life expectancy, personal beliefs, etc.

Utility Curves



- Utility curve determines your “risk preference”
  - Can be different in different parts of the curve
  - We’ll talk more about this near the end of the quarter

Jensen's Investment Advice

- Example: risk-taking investor, with two choices:
  - Choice 1: Invest money to get return X where E[X] = μ
  - Choice 2: Invest money to get return μ (probability 1)
- Want to maximize utility: *u*(R), where R is return
  - Convex *u* $\Rightarrow$ "risk preferring", concave *u* $\Rightarrow$ "risk averse"

  - if *u*(X) <u>convex</u> then E[*u*(X)] $\geq$ *u*(μ), so choice 1 better
  - If *u*(X) <u>concave</u> then E[*u*(X)] $\leq$ *u*(μ) so choice 2 better