Continuous Conditional Distributions (Review)

- Let X and Y be continuous random variables
  - Recall, conditional PDF of X given Y (where $f_Y(y) > 0$):

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Let's Do an Example (Review)

- X and Y are continuous RVs with PDF:

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & \text{where } 0 < x,y < 1 \\ 0 & \text{otherwise} \end{cases}$$

  - Compute conditional density: $f_{X|Y}(x \mid y)$

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int_0^1 f_{X,Y}(x,y)\,dx}$$

$$= \frac{\frac{12}{5}x(2-x-y)}{\int_0^1 \frac{12}{5}x(2-x-y)\,dx} = \frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx} = \frac{x(2-x-y)}{\left[x^2 - \frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^1}$$

$$= \frac{x(2-x-y)}{\frac{2}{3} - \frac{y}{2}} = \frac{6x(2-x-y)}{4-3y}$$

Independence and Conditioning

- If X and Y are independent discrete RVs:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{P(X = x)P(Y = y)}{P(Y = y)} = P(X = x)$$

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x)$$

- Analogously, for independent continuous RVs:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{f_X(x)\, f_Y(y)}{f_Y(y)} = f_X(x)$$

Conditional Independence Revisited

- *n* discrete random variables X_1, X_2, …, X_n are called **<u>conditionally independent</u>** given Y if:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n \mid Y = y) = \prod_{i=1}^{n} P(X_i = x_i \mid Y = y) \quad \text{for all } x_1, x_2, \ldots, x_n, y$$

- Analogously, for continuous random variables:

$$P(X_1 \le a_1, X_2 \le a_2, \ldots, X_n \le a_n \mid Y = y) = \prod_{i=1}^{n} P(X_i \le a_i \mid Y = y) \quad \text{for all } a_1, a_2, \ldots, a_n, y$$

- Note: can turn products into sums using logs:

$$\ln \prod_{i=1}^{n} P(X_i = x_i \mid Y = y) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \ln P(X_i = x_i \mid Y = y) = K$$

$$\prod_{i=1}^{n} P(X_i = x_i \mid Y = y) = e^K$$

Mixing Discrete and Continuous

- Let X be a <u>continuous</u> random variable
- Let N be a <u>discrete</u> random variable
  - Conditional PDF of X given N:

$$f_{X|N}(x \mid n) = \frac{p_{N|X}(n \mid x)\, f_X(x)}{p_N(n)}$$

  - Conditional PMF of N given X:

$$p_{N|X}(n \mid x) = \frac{f_{X|N}(x \mid n)\, p_N(n)}{f_X(x)}$$

  - If X and N are independent, then:

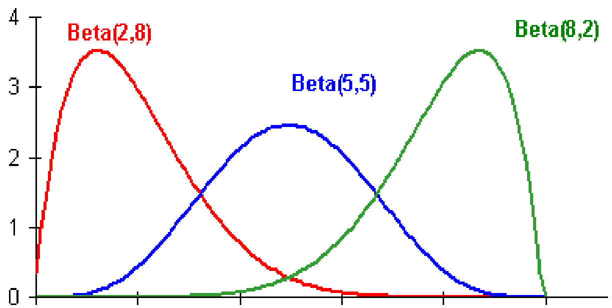
$$f_{X|N}(x \mid n) = f_X(x) \qquad\qquad p_{N|X}(n \mid x) = p_N(n)$$

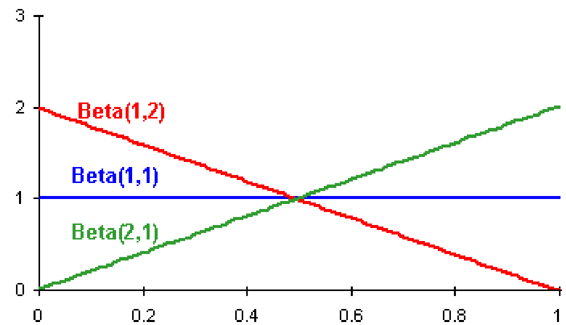
Beta Random Variable

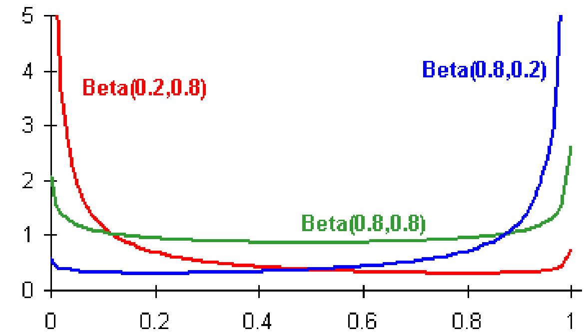
- X is a **Beta Random Variable**: X ~ Beta(a, b)
  - Probability Density Function (PDF): (where a, b > 0)

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$







  - Symmetric when $a = b$
  - $E[X] = \frac{a}{a+b}$ $\qquad Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$

Flipping Coin With Unknown Probability

- Flip a coin (n + m) times, comes up with n heads
  - We don't know probability X that coin comes up heads
  - All we know is that: X ~ Uni(0, 1)
  - What is density of X given n heads in n + m flips?
  - Let N = number of heads
  - Given X = x, coin flips independent: $(N \mid X) \sim \text{Bin}(n+m, x)$
  - Compute conditional density of X given N = n

$$f_{X|N}(x \mid n) = \frac{P(N = n \mid X = x)\overbrace{\boxed{f_X(x)}}^{1}}{P(N = n)} = \frac{\binom{n+m}{n} x^n (1-x)^m}{P(N = n)}$$

$$= \frac{1}{c} \cdot x^n (1-x)^m \quad \text{where} \quad c = \int_0^1 x^n (1-x)^m \, dx$$

Dude, Where's My Beta?!

- Flip a coin (n + m) times, comes up with n heads
  - Conditional density of X given N = n

$$f_{X|N}(x \mid n) = \frac{1}{c} \cdot x^n (1-x)^m \quad \text{where} \quad c = \int_0^1 x^n (1-x)^m \, dx$$

  - Note: $0 < x < 1,$ so $f_{X|N}(x \mid n) = 0$ otherwise
  - Recall Beta distribution:

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

  - Hey, that looks more familiar now...
  - X | (N = n, n + m trials) ~ Beta(n + 1, m + 1)

Understanding Beta

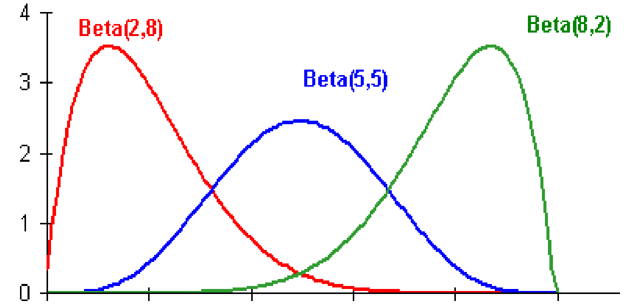
- X | (N = n, m + n trials) ~ Beta(n + 1, m + 1)
  - X ~ Uni(0, 1)
  - Check this out, boss:
    - Beta(1, 1) = Uni(0, 1)
  - So, X ~ Beta(1, 1)
  - "Prior" distribution of X (before seeing any flips) is Beta
  - "Posterior" distribution of X (after seeing flips) is Beta

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} = \frac{1}{B(a,b)} x^0 (1-x)^0$$
$$= \frac{1}{\int_0^1 1\,dx} 1 = 1 \quad \text{where} \quad 0 < x < 1$$

- Beta is a **<u>conjugate</u>** distribution for Beta
  - Prior and posterior parametric forms are the same!
  - Beta is also conjugate for Bernoulli and Binomial
  - Practically, conjugate means easy update:
    - Add number of "heads" and "tails" seen to Beta parameters

Further Understanding Beta

- Can set X ~ Beta(*a*, *b*) as prior to reflect how biased you think coin is apriori
  - This is a subjective probability!
  - Then observe n + m trials, where n of trials are heads



- Update to get posterior probability
  - X | (n heads in n + m trials) ~ Beta(a + n, b + m)
  - Sometimes call *a* and *b* the "equivalent sample size"
  - Prior probability for X based on seeing (*a* + *b* – 2) "imaginary" trials, where (*a* – 1) of them were heads.
  - Beta(1, 1) ~ Uni(0, 1) → we haven't seen any "imaginary trials", so apriori know nothing about coin

Welcome Back Our Friend: Expectation

- Recall expectation for discrete random variable:

$$E[X] = \sum_x x\, p(x)$$

- Analogously for a continuous random variable:

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

- Note: If X always between *a* and *b* then so is E[X]
  - More formally:

if $P(a \le X \le b) = 1$ then $a \le E[X] \le b$

Generalizing Expectation

- Let *g*(*X*, *Y*) be real-valued function of two variables

- Let X and Y be discrete jointly distributed RVs:

$$E[g(X,Y)] = \sum_{y} \sum_{x} g(x,y)\, p_{X,Y}(x,y)$$

- Analogously for continuous random variables:

$$E[g(X,Y)] = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Expected Values of Sums

- Let g(X, Y) = X + Y. Compute E[g(X, Y)] = E[X + Y]

$$E[X+Y] = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} (x+y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dy\, dx + \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{x=-\infty}^{\infty} x\, f_X(x)\, dx + \int_{y=-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E[X] + E[Y]$$

- Generalized: $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$
  - Holds regardless of dependency between X_i's

Tie Me Up! : Bounding Expectation

- If random variable X ≥ *a* then E[X] ≥ *a*

  if $P(a \leq X \leq \infty) = 1$ then $a \leq E[X] \leq \infty$

  - Often useful in cases where *a* = 0
  - But, E[X] ≥ *a* does <u>not</u> imply X ≥ *a* for all X = x
    - E.g., X is equally likely to take on values -1 or 3. E[X] = 1.
- If random variables X ≥ Y then E[X] ≥ E[Y]
  - X ≥ Y $\Rightarrow$ X – Y ≥ 0 $\Rightarrow$ E[X – Y] ≥ 0
  - Note: E[X – Y] = E[X] + E[-Y] = E[X] – E[Y]
  - Substituting: E[X] – E[Y] ≥ 0 $\Rightarrow$ E[X] ≥ E[Y]
  - But, E[X] ≥ E[Y] does <u>not</u> imply X ≥ Y for all X = x, Y = y

Sample Mean

- Consider *n* random variables X_1, X_2, ... X_n
  - X_i are all independently and identically distributed (I.I.D.)
  - Have same distribution function *F* and $E[X_i] = \mu$
  - We call sequence of X_i a **sample** from distribution *F*
  - Sample mean: $\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$