Moment Generating Functions

- Moment Generating Function (MGF) of a random variable X, where $-\infty < t < \infty$:

$$M(t) = E[e^{tX}]$$

  - When X is discrete: $M(t) = \sum_x e^{tx} p(x)$
  - When X is continuous: $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

- Oh, that's nice. Um... why should I care?

Bring on the Moments!

- Start with: $M(t) = E[e^{tX}]$
  - Now differentiate *M*(*t*) with respect to *t*, evaluate at *t* = 0

$$M'(t) = \frac{d}{dt} M(t) = \frac{d}{dt} E[e^{tX}] = E[\frac{d}{dt}(e^{tX})] = E[Xe^{tX}]$$

$$M'(0) = E[Xe^{0}] = E[X]$$

  - That's pretty neat, let's do it again:

$$M''(t) = \frac{d}{dt} M'(t) = \frac{d}{dt} E[Xe^{tX}] = E[\frac{d}{dt}(Xe^{tX})] = E[X^2 e^{tX}]$$

$$M''(0) = E[X^2 e^{0}] = E[X^2]$$

  - Do it as often as you like:

$$M^n(t) = \left(\frac{d}{dt}\right)^n M(t) = E[X^n e^{tX}]$$

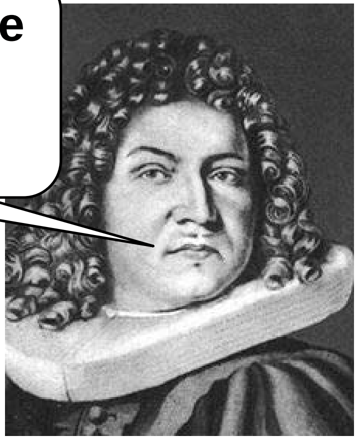
$$M^n(0) = E[X^n]$$

Let's Take It Out For a Spin

- X ~ Ber(p)

$$M(t) = E[e^{tX}] = \sum_{x=0}^{1} e^{tx} p(x)$$

$$= e^{0}(1-p) + e^{t}p = e^{t}p + 1 - p$$



Sadly, no...

$$M'(t) = e^{t}p \quad \Rightarrow \quad M'(0) = E[X] = e^{0}p = p$$

$$M''(t) = e^{t}p \quad \Rightarrow \quad M''(0) = E[X^{2}] = e^{0}p = p$$

$$\mathrm{Var}(X) = E[X^{2}] - (E[X])^{2} = p - p^{2} = p(1-p)$$

Generating Some Binomial Moments

- X ~ Bin(n, p)

$$M(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(pe^t\right)^k (1-p)^{n-k} = (pe^t + 1 - p)^n$$

  - Binomial theorem: $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n \qquad x = pe^t, \ y = (1-p)$

$$M'(t) = n(pe^t + 1 - p)^{n-1} pe^t \Rightarrow M'(0) = E[X] = n(pe^0 + 1 - p)^{n-1} pe^0 = np$$

$$M''(t) = n(n-1)(pe^t + 1 - p)^{n-2} \left(pe^t\right)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

$$M''(0) = E[X^2] = n(n-1)(1)^{n-2} \left(p\right)^2 + n(1)^{n-1} p = n(n-1)p^2 + np$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

Properties of MGFs

- X and Y are independent random variables

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

  - Also, if joint MGF factors, then X and Y independent
- MGF uniquely determines distribution
  - Example: $M_X(t) = (0.3e^t + 0.7)^6$
  - Recall MGF for Binomial: $M_X(t) = (pe^t + 1 - p)^n$
  - So: X ~ Bin(6, 0.3)
- Distributions with same MGF are the same!

$$M_X(t) = M_Y(t) \quad \text{iff} \quad X \sim \mathrm{Y}$$

Joint Moment Generating Functions

- Consider any *n* random variables $X_1, X_2, \ldots X_n$
  - Joint moment generating function:

$$M(t_1, t_2, \ldots, t_n) = E[e^{t_1 X_1 + t_2 X_2 + \ldots + t_n X_n}]$$

  - Individual moment generating functions obtained:

$$M_{X_i}(t) = E[e^{tX_i}] = M(0, \ldots, 0, t, 0, \ldots, 0)$$ where t at i th place

  - $X_1, X_2, \ldots X_n$ independent if and only if:

$$M(t_1, t_2, \ldots, t_n) = M_{X_1}(t_1) M_{X_2}(t_2) \ldots M_{X_n}(t_n)$$

  - Proof:

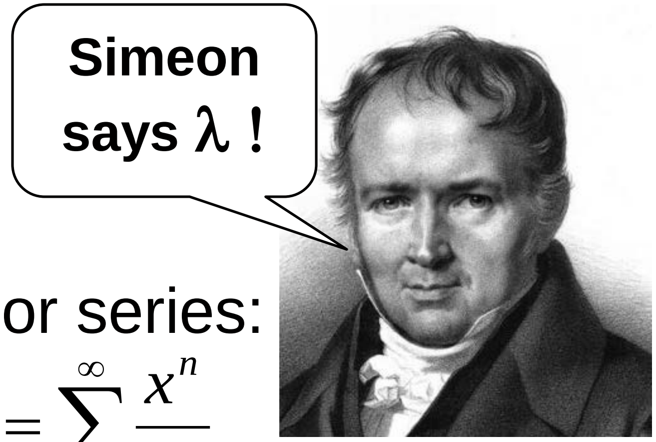
$$M(t_1, t_2, \ldots, t_n) = E[e^{t_1 X_1 + t_2 X_2 + \ldots + t_n X_n}] = E[e^{t_1 X_1} e^{t_2 X_2} \ldots e^{t_n X_n}]$$

By independence:

$$= E[e^{t_1 X_1}] E[e^{t_2 X_2}] \ldots E[e^{t_n X_n}] = M_{X_1}(t_1) M_{X_2}(t_2) \ldots M_{X_n}(t_n)$$

Poisson, May I Have a Moment?

- X ~ Poi(λ)



$$M(t) = E[e^{tX}] = \sum_{n=0}^{\infty} e^{tn} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$M'(t) = (\lambda e^t) e^{\lambda(e^t - 1)} \quad \Rightarrow \quad M'(0) = E[X] = (\lambda e^0) e^{\lambda(e^0 - 1)} = \lambda$$

$$M''(t) = (\lambda e^t)^2 e^{\lambda(e^t - 1)} + (\lambda e^t) e^{\lambda(e^t - 1)} \Rightarrow M''(0) = E[X^2] = \lambda^2 e^0 + \lambda e^0 = \lambda^2 + \lambda$$

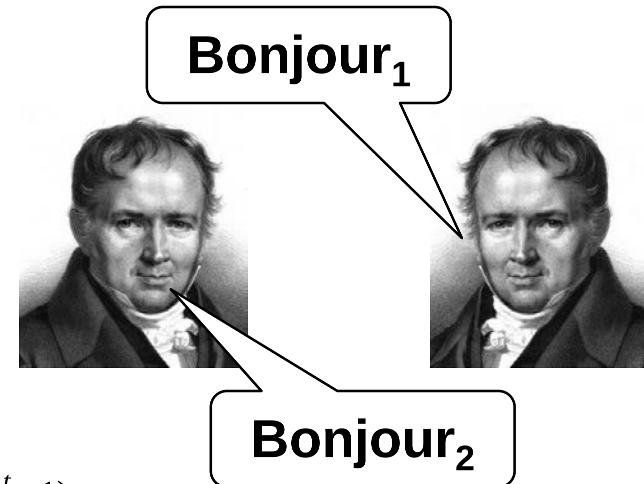
$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

A Tale of Two Poissons

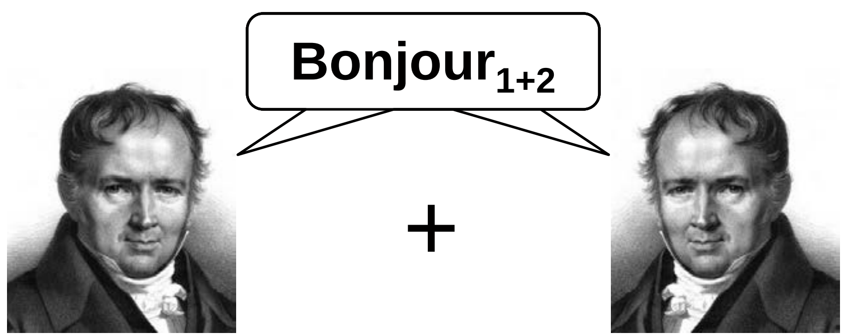
- X ~ Poi(λ_1)   Y ~ Poi(λ_2)
  - X and Y independent

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$= e^{\lambda_1(e^t - 1)} e^{\lambda_2(e^t - 1)} = e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$



  - So, X + Y ~ Poi(λ_1 + λ_2)



MGF of Normal Distribution

- $X \sim N(\mu_1, \sigma_1^2)$

$$M_X(t) = E[e^{tX}] = e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)}$$

- Yes, it's that important...

You Call That Normal?

- X ~ N(μ_1, σ_1^2) $\qquad M_X(t) = E[e^{tX}] = e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)}$

$$M'_X(t) = (\mu_1 + t\sigma_1^2)e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)} \Rightarrow M'_X(0) = E[X] = \mu_1$$

$$M''_X(t) = (\mu_1 + t\sigma_1^2)^2 e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)} + \sigma_1^2 e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)} \Rightarrow M''_X(0) = E[X^2] = \mu_1^2 + \sigma_1^2$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \mu_1^2 + \sigma_1^2 - \mu_1^2 = \sigma_1^2$$

- Now, Y ~ N(μ_2, σ_2^2) where X and Y independent

$$M_Y(t) = E[e^{tY}] = e^{\left(\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right)}$$

$$M_X(t)M_Y(t) = e^{\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right)} e^{\left(\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right)} = e^{\left(\frac{(\sigma_1^2 + \sigma_2^2)t^2}{2} + (\mu_1 + \mu_2)t\right)} = M_{X+Y}(t)$$

  - Uniquely determines: X + Y ~ N($\mu_1 + \mu_2$, $\sigma_1^2 + \sigma_2^2$)