Non-linear Conservation Equations

Suppose we solve Burgers' equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

using upwind differences. Then we use

$$\begin{aligned} u\frac{\partial u}{\partial x} &\rightarrow u_i \frac{u_i - u_{i-1}}{\Delta x}, \quad u_i > 0 \\ u\frac{\partial u}{\partial x} &\rightarrow u_i \frac{u_{i+1} - u_i}{\Delta x}, \quad u_i < 0 \end{aligned}$$

Does this work?

Numerical Example

Crossing of Characteristics

- The initial sinusoidal waves steppens into a shock after a finite time.
- The shock propagates at the wrong speed!
- The scheme may still be stable in the presence of the shock, but the numerical solution is not longer an accurate representation of the true weak solution.

The Need for Flux-Conservative Schemes

Since Burgers's equation can be written in flux-conservative form, we can cast it into an integral form that expresses a conservation law:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\int_a^b u\,\mathrm{d}x \left[\frac{1}{2}u(a)^2 - \frac{1}{2}u(b)^2\right]$$

If we have the flux at the boundaries, or if we have periodic boundary conditions, then the integral $\int u\,\mathrm{d}x$ over the entire computation domain should not change with time. However, for our finite-difference scheme, $\sum_i u_i \Delta x$ *does* change with time. So an important physical conservation law is not respected!
The characteristic scheme is *non-conservative*.
As a result, discontinuities can propagate at the wrong speed

The Finite-Volume Method

To construct a conservative scheme (which handles discontinuities correctly), one can use the finite-volume approach. Here, the values of the conserved variables (for the Euler equations ρ_i, $(\rho v)_i$, $(\rho\epsilon + \rho v^2/2)_i$) represent cell-averages of the (true and continuous) solution function, e.g.

$$\rho_i = \frac{1}{\Delta V_i} \int \rho \, \mathrm{d}V,$$

and the rate of change is computed as an integral of the corresponding fluxes over the cell boundary:

$$\frac{\partial}{\partial t}(\rho_i \Delta V_i) = - \oint_{\partial V_i} \rho \mathbf{v} \cdot \mathrm{d}\mathbf{A}.$$

For neighbouring cells that share a common cell face, one must use the same flux through this cell face. Then the scheme is conservative.

Godunov's Method

- How to compute fluxes through cell interfaces?
- As you may guess simple arithmetic averages $F_{i+1/2} = (F_i + F_{i+1})/2$ between neighbouring cells doesn't work: For the advection equation this would lead to the FTCS scheme, and we know that's unstable.
- Upwinding needed, but how?
- In 1959, S. K. Godunov suggested to reconstruct the solution function as a *discontinuous* step function and then to solve the *Riemann problem* at the cell interfaces to get the fluxes.

The Riemann Problem

- Initial conditions: Two different states $\mathbf{U}_L$ and $\mathbf{U}_R$ separated by a discontunity.
- How does this evolve with time?
- We learned that discontinuities must satisfy certain jump conditions.
- If the jump conditions are not satisfied initially, the discontinuity decays into several discontinuities (each satisfying the jump conditions) and waves that connect the initial states $\mathbf{U}_L$ and $\mathbf{U}_R$.

Reminder – Jump Conditions

For any conservative system of equations, the jump conditions with a discontinuity moving at velocity λ read,

$$\mathbf{F}(\mathbf{U}_L) - \mathbf{F}(\mathbf{U}_R) = \lambda(\mathbf{U}_L - \mathbf{U}_R).$$

The Riemann Problem – Burgers' Equation

We can now tackle part of the Riemann Problem for Burgers' equation. Let the left and right states be u_L and u_R with $u_L > u_R$. Then we get a "shock" because the characteristics cross. The shock speed λ will be given by

$$\begin{aligned} \frac{u_L^2}{2} - \frac{u_R^2}{2} &= \lambda(u_L - u_R) \\ \lambda &= \frac{1}{2}\frac{u_L^2 - u_R^2}{u_L - u_R} \\ \lambda &= \frac{u_L + u_R}{2} \end{aligned}$$

The left state and the right state themselves will not change, however, since $\frac{\partial u^2}{\partial x} = 0$ except at the shock itself. The solution will still be a step function that moves to the right or to the left depending on the value of $(u_L + u_R/2)$. So the solution for the flux at the interface (Godunov flux is):

$$F(u_L, u_R) = \begin{cases} \frac{u_L^2}{2}, & u_L > u_R \wedge \frac{u_L+u_R}{2} > 0 \\ \frac{u_R^2}{2} u_R, & u_L > u_R \wedge \frac{u_L+u_R}{2} < 0 \end{cases}$$

What if we have $u_L < u_R$? Then the characteristics *diverge*, and different solutions are conceivable. The correct solution can be obtained by noting that discontinuities *only* arise (and can survive) only when characteristics cross (or at least run parallel to each other). This is called the *Lax entropy condition*. For Burgers' equation, the correct solution joining two states with $u_L < u_R$ must therefore be a smooth solution of Burgers' equation. Since the initial conditions are scale-free, we seek a self-similar solution that depends on x and t only through $\xi = x/t$: $u(t,x) = u(\xi) = u(x/t)$, let's say $u(t,x) = \alpha u x/t$:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\alpha x}{t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\alpha^2 x^2}{t^2}\right) &= 0 \\ -\frac{\alpha x}{t^2} + \frac{\alpha^2 x}{t^2} &= 0 \end{aligned}$$

The solution is $\alpha = 1$, and hence $u(x/t) = x/t$.

The solution $u(x/t) = x/t = \xi$ (called a *rarefaction wave* or *rarefaction fan* for reasons that will become apparent when we discuss the Euler equations) is only valid for a certain range in $\xi = x/t$, however. It must continuously join the left and right states u_L and u_R. Otherwise we would end up with shocks between the left and right states of the rarefaction fan that do not obey the jump conditions. The junctures will be at $\xi_L = U_L$ and $\xi_R = U_R$. Depending on whether $\xi_L > 0$, $\xi_R > 0 > \xi_L$, or $\xi_R < 0$, the Godunov flux therefore comes either from the left state, the rarefaction fan, or the right state:

$$F(u_L, u_R) = \begin{cases} \frac{u_L^2}{2}, & u_L < u_R \wedge u_L > 0 \\ \frac{u_L^2}{2}, & u_L < u_R \wedge u_R < 0 \\ 0, & u_L < u_R \wedge u_R \geq 0 \geq u_L \end{cases}$$

This completes the solution of the Riemann problem for Burgers' equation, and we can now write a Godunov-based solver for it.

The Riemann Problem – Inviscid Euler Equations in 1D

- What are the discontinuities and waves that appear in the solution of the Riemann Problem for the Euler equations?
- The discontinuities are solutions to the jump conditions:

$$\begin{aligned} \rho_L u_L - \rho_R u_R &= \lambda(\rho_L - \rho_R) \\ \rho_L u_L^2 - \rho_R u_R^2 + P_L - P_R &= \lambda(\rho_L u_L - \rho_R u_R) \\ (e_L + P_L)u_L - (e_R + P_R)u_R &= \lambda(e_L - e_R) \end{aligned}$$

- In the second lecture, we discussed *shocks* as a non-trivial solution for $u_L \neq u_R$ (but specialising to shock speed $\lambda = 0$, which does not change the physics.)
- For $u_L = u_R$, there is a separate solution with $\lambda = u_L = u_R$ (from the first jump condition) and $P_L = P_R$ (second jump condition). Arbitrary values for ρ_L and ρ_R are allowed as long as the pressure on both sides is the same (because of different temperature and internal energy density). This is called a *contact discontinuity*. It propagates with the local fluid velocity, and is called a *tangential discontinuity*.
- If velocity components parallel to the interface are present, these may also be discontinuous. They do not affect the structure of the solution.
- The are also self-similar solutions for rarefaction waves with $d\rho/dt = -\partial u/\partial x < 0$. In these waves, the density along a pathline decreases monotonically (hence the name).

Rarefaction Waves

To get the rarefaction fan in the Riemann problem, we seek a similiarity solution just as for Burgers' equation, i.e. we assume that all fluid variables are functions of $\xi = x/t$. This means that we can write time and space derivatives as $\partial/\partial x = (1/t)\partial/\partial\xi$ and $\partial\partial t = -\xi/t\partial/\partial\xi$. Since we're looking for a smooth solution, we can use the entropy equation $\mathrm{d}s/\mathrm{d}t = 0$, which becomes

$$(v - \xi)\frac{\partial s}{\partial \xi} = 0,$$

which implies that the entropy is constant across the rarefaction fan. Using similar arguments, we find that the transverse velocity components are also constant across the rarefaction fan.

Rarefaction Waves

Now consider the continuity equation and the momentum equation:

$$\begin{aligned}(v-\xi)\frac{\partial \rho}{\partial \xi}+\rho\frac{\partial v}{\partial \xi} &= 0\\ (v-\xi)\frac{\partial v}{\partial \xi}+\rho^{-1}\frac{\partial P}{\partial \xi} &= 0\end{aligned}$$

Because the rarefaction fan is isentropic, we have $P' = c_s^2 \rho'$ and hence:

$$(v-\xi)\frac{\partial v}{\partial \xi} = -c_s^2 \frac{\partial \ln \rho}{\partial \xi}$$

To find a non-trivial solution ($\rho \neq \text{const.}$), we solve for v' in the first equation and substitute the solution into the second one to obtain:

$$(v-\xi)^2 = c_s^2,$$

which implies $\xi = v \pm c$ (i.e. $v - \xi = -c$). Let us use the plus solution and put it back into the continuity equation:

$$c\frac{\partial \rho}{\partial \xi} = \rho\frac{\partial v}{\partial \xi}.$$

Rarefaction Fan

We can integrate this if we know the sound speed as a function $c_s(\rho)$ of ρ (which we do for an isentropic fluid). We obtain

$$v = \int \frac{c \, \mathrm{d}\rho}{\rho}$$

For a perfect gas, we have $c_s = \sqrt{\gamma P/\rho}$, and hence $c_s = c_{s,0}(\rho/\rho_0)^{(\gamma-1)/2}$ along and adiabat, so we can do the integral:

$$v = \frac{2}{\gamma - 1}(c_s - c_{s,0})$$

The Riemann Problem – Inviscid Euler Equations in 1D

- There are four different classes of solutions, see, e.g., Landau & Lifshitz, *Fluid Mechanics*, Fig. 62.
- The solution for the interface state cannot be written down explicitly in closed form, it has to be solved for numerically, e.g, using Newton-Raphson iteration.
- This is possible, but it is not always worth the numerical effort:
  - The solution is only exact for an ideal gas equation of state anyway.
  - In multiple dimensions, the Riemann problem becomes much more complicated if we still want to solve it exactly.

Figure: Types of Riemann problems in 1D.

$$\frac{\rho_L}{\rho_R}\frac{P_R}{P_L}\frac{(1-p)^2}{\gamma(1+p)-1+P} = \frac{2\gamma}{(\gamma-1)}\left[1-\left(\frac{p}{P_L/P_R}\right)^{\frac{\gamma-1}{2\gamma}}\right],$$

$$p = P_C/P_R$$

Approximate Riemann Solvers – Kurganov-Tadmor Central Scheme

- Approximate Riemann solvers often provide a cheaper and simpler (i.e. less prone to coding errors) alternative.
- The performance of approximate Riemann solvers for different problems (accuracy, robustness) varies greatly. There is not one single perfect Riemann solver – making a good choice requires experience, good judgement, and an understanding of the physical problem one wants to simulate!
- Approximate Riemann solvers require some (limited) knowledge of the characteristic structure of the equations (e.g. wave speeds).
- The simplest, but also also most diffusive approximate solution to the Riemann problem is provided by the *Kurganov-Tadmor central scheme* (also *Rusanov flux* or *local Lax-Friedrichs flux*), which uses just the maximum signal speed $\lambda_{\max}$ on either side of the interface:

$$\mathbf{F}(\mathbf{U}_L, \mathbf{U}_R) = \frac{\mathbf{F}(\mathbf{U}_L) + \mathbf{F}(\mathbf{U}_R) - \lambda_{\max}(\mathbf{U}_R - \mathbf{U}_L)}{2}.$$

For the Euler equations, $\lambda_{\max} = \max(|u_L| + |c_{s,L}|, |u_R| + |c_{s,R}|)$.

Approximate Riemann Solvers – HLLE

One approach consists in approximating the full Riemann fan with a constant "average" state (Harten, Lax, van Leer, SIAM Rev.,1983). The speeds λ_- and λ_+ of the waves seperating this "star" state $\mathbf{U}^\star$ from the left and right states $\mathbf{U}_L$ and $\mathbf{U}_R$ needs to be specified; no further characteristic information is needed.
The jump conditions between the left state and the star state are:

$$\begin{aligned} \mathbf{F}_L - \mathbf{F}^\star &= \lambda_-(\mathbf{U}_L - \mathbf{U}^\star) \\ \mathbf{F}_R - \mathbf{F}^\star &= \lambda_+(\mathbf{U}_R - \mathbf{U}^\star) \end{aligned}$$

We can eliminate $\mathbf{U}^\star$ to obtain the flux for the intermediate state:

$$\begin{aligned} \lambda_+\mathbf{F}_L - \lambda_+\mathbf{F}^\star &= \lambda_-\lambda_+(\mathbf{U}_L - \mathbf{U}^\star) \\ \lambda_-\mathbf{F}_R - \lambda_-\mathbf{F}^\star &= \lambda_-\lambda_+(\mathbf{U}_R - \mathbf{U}^\star) \\ (\lambda_+ - \lambda_-)\mathbf{F}^\star + \lambda_-\mathbf{F}_R - \lambda_+\mathbf{F}_L &= \lambda_-\lambda_+(\mathbf{U}_R - \mathbf{U}_L) \\ \mathbf{F}^\star &= \frac{\lambda_+\mathbf{F}_L - \lambda_-\mathbf{F}_R + \lambda_-\lambda_+(\mathbf{U}_R - \mathbf{U}_L)}{\lambda_+ - \lambda_-} \end{aligned}$$

Approximate Riemann Solvers – HLLE

We still need to estimate the speeds of the fastest right- and left-bound waves. Different choices are possible, e.g. one chooses the fastest left- and right-going sound waves on both sides of the interface,

$$\lambda_- = \min\left(u_L - c_{s,L}, u_R - c_{s,R}\right), \quad \lambda_+ = \max\left(u_L + c_{s,L}, u_R + c_{s,R}\right).$$

Note that the intermediate state $\mathbf{U}^\star$ (see homework problem) is in general not consistent with the flux $\mathbf{F}^\star$: $\mathbf{F}^\star \neq \mathbf{F}(\mathbf{U}^\star)$.
The interface flux will be

$$\mathbf{F}(\mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{F}^\star = \frac{\lambda_+ \mathbf{F}_L - \lambda_- \mathbf{F}_R + \lambda_- \lambda_+ (\mathbf{U}_R - \mathbf{U}_L)}{\lambda_+ - \lambda_-}, & \lambda_- \leq 0 \leq \lambda_+ \\ \mathbf{F}_L, & \lambda_- > 0 \\ \mathbf{F}_R, & \lambda_+ < 0 \end{cases}$$

While the HLLE solver is quite robust, it smears out contact discontinuities because we approximate the Riemann fan by a single, averaged state.

Approximate Riemann Solvers – HLLC

To achieve a sharper resolution of contact discontinuities and a better performance for slow, subsonic flow, one can restore the contact discontinuity in the HLL Riemann fan structure.
Now the star region consists of two states $\mathbf{U}_L^\star$ and $\mathbf{U}_R^\star$ that are seperated by a contact discontinuity propagating with speed $\lambda^\star$. In addition to the jump conditions for the left- and right-going shocks,

$$\mathbf{F}_L - \mathbf{F}_L^\star = \lambda_-(\mathbf{U}_L - \mathbf{U}_L^\star), \quad \mathbf{F}_R - \mathbf{F}_R^\star = \lambda_+(\mathbf{U}_R - \mathbf{U}_R^\star),$$

we now demand

$$P_L^\star = P_R^\star = P^\star, \quad \mathbf{U}_L^\star = \mathbf{U}_R^\star = \lambda_\star,$$

so that the intermediate wave is indeed a contact discontinuity.

From the jump conditions we obtain:

$$P^\star = P_L + \rho_L u_L^2 - \rho_L^\star \lambda_\star^2 + \lambda_-(\rho_L^\star \lambda_\star - \rho_L u_L)$$

We can eliminate $\rho_L^\star$ by using the jump condition for the density,

$$\rho_L^\star \lambda_\star - \rho_L^\star \lambda_- = \rho_L u_L - \rho_L \lambda_-,$$

to obtain

$$\begin{aligned} P^\star &= P_L + \rho_L u_L^2 + \rho_L \lambda_\star(\lambda_- - u_L) - \rho_L \lambda_- u_L \\ &= P_L + \rho_L u_L(u_L - \lambda_-) + \rho_L \lambda_\star(\lambda_- - u_L) \\ &= P_L + \rho_L(\lambda_\star - u_L)(\lambda_- - u_L), \end{aligned}$$

and similarly,

$$P^\star = P_R + \rho_R(\lambda_\star - u_R)(\lambda_+ - u_R).$$

Approximate Riemann Solvers – HLLC

Now for $P_L^\star = P_R^\star = P^\star$ to hold, we must have

$$\begin{aligned} P_L + \rho_L(\lambda_\star - u_L)(\lambda_- - u_L) &= P_R + \rho_R(\lambda_\star - u_R)(\lambda_- - u_R) \\ \lambda_\star &= \left[\rho_L(\lambda_- - u_L) - \rho_R(\lambda_+ - u_R)\right]\lambda_\star \\ &= P_R - P_L + \rho_L u_L(\lambda_- - u_L) - \rho_R u_R(\lambda_+ - u_R) \\ \lambda_\star &= \frac{P_R - P_L + \rho_L u_L(\lambda_- - u_L) - \rho_R u_R(\lambda_+ - u_R)}{\rho_L(\lambda_- - u_L) - \rho_R(\lambda_+ - u_R)}. \end{aligned}$$

Once we have $\lambda_\star$ all the other state variables follow from the jump conditions (analogous expression hold for the $R^\star$ state, note that $e = \epsilon + (u^2 + v^2 + w^2)/2$ here):

$$\begin{aligned} \rho_L^\star &= \rho_L \frac{\lambda_- - u_L}{\lambda_- - \lambda_\star} \\ (\rho u)_L^\star &= \frac{\rho_L u_L(\lambda_- - u_L) + P^\star - P_L}{\lambda_- - \lambda_\star} \\ (\rho v)_L^\star &= \rho_L v_L \frac{\lambda_- - u_L}{\lambda_- - \lambda_\star} \\ (\rho w)_L^\star &= \rho_L w_L \frac{\lambda_- - u_L}{\lambda_- - \lambda_\star} \\ (\rho e)_L^\star &= \frac{\rho_L e_L(\lambda_- - u_L) + P^\star \lambda_\star - P_L u_L}{\lambda_- - \lambda_\star} \end{aligned}$$

Approximate Riemann Solvers – HLLC

Depending on the signs of λ_-, λ_+, and $\lambda_\star$, the HLLC flux is then given by

$$\mathbf{F}(\mathbf{U}_L, \mathbf{U}_R) = \begin{cases} \mathbf{F}_L, & \lambda_- > 0 \\ \mathbf{F}(\mathbf{U}_L^\star), & \lambda_- < 0 < \lambda_\star \\ \mathbf{F}(\mathbf{U}_R^\star), & \lambda_\star < 0 < \lambda_+ \\ \mathbf{F}_R, & \lambda_+ < 0 \end{cases}$$

Note that in the process of constructing the star states, we have obtained the primitive variables $P_L^\star = P_R^\star = P^\star$ and $u_L^\star = u_R^\star = \lambda_\star$ that we need to compute the fluxes; e.g., the momentum flux in the $L^\star$ region reads

$$(\rho u)_L^\star \lambda_\star + P_\star.$$

Approximate Riemann Solvers – HLLC

Advantages of the HLLC solver:

- Quite robust and often a great deal more accurate than HLLE
- Still cheaper than the Roe or Marquina solver, let alone the exact solver, especially for relativistic hydrodynamics.

Disadvantages:

- Generalisation to GR and MHD is more complicated, whereas HLLE remains straightforward (which is also dangerous!)
- Rotational discontinuities not included in MHD case, more states need to be included for better accuracy (HLLD).
- Prone to some of the instabilities that affect the Roe solver and the exact solver (carbuncle phenomenon, odd-even decoupling, problems at sonic points – more on this in another lecture).

Approximate Riemann Solvers – Linear Solvers

Very accurate solvers can be constructed by a local linearisation of the equations of hydrodynamics. We know how to solve a linear problem: Transform to characteristic variables by projecting on the eigenvectors of the flux Jacobian $\mathbf{J}$), and solve an advection problem using the eigenvalues as wave speeds. This is the underlying idea of the Roe solver.
But around what state should we linearise? The resulting quasi-linear equation

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U}_L, \mathbf{U}_R)\frac{\partial \mathbf{U}}{\partial x} = 0,$$

needs to be "consistent" with original equation in a certain sense. The conditions for the Roe matrix $\mathbf{A}$ are:

- $\mathbf{A}$ should have a complete set of linearly-independent real eigenvalues.
- For identical left and right states, $\mathbf{A}(\mathbf{U}_L, \mathbf{U}_L) = \mathbf{J}(\mathbf{U}_L)$.
- $\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) = \mathbf{A} \cdot (\mathbf{U}_R - \mathbf{U}_L)$.

These conditions are fulfilled if $\mathbf{A}$ is computed as the flux Jacobian of the Roe-averaged state:

$$\bar{\rho} = \sqrt{\rho_L \rho_R}, \quad \bar{\mathbf{v}} = \frac{\sqrt{\rho_L}\mathbf{v}_L + \sqrt{\rho_R}\mathbf{v}_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}, \quad \bar{h} = \frac{\sqrt{\rho_L}h + \sqrt{\rho_R}h}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

From this, we can also obtain $\bar{\epsilon}$ and $\bar{P}$, and then project the jump between the states $\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L$ onto the right eigenvectors $\mathbf{R}_i$ of the Roe matrix $\mathbf{A} = \mathbf{J}(\bar{\mathbf{U}})$:

$$\Delta\mathbf{U} = \sum_{i=1}^{m} \alpha_i \mathbf{R}_i.$$

The flux for the linearised system is then given by:

$$\mathbf{F}_{\mathrm{Roe}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\sum_{i=1}^{m} \alpha_i |\lambda_i| \mathbf{R}_i,$$

where λ_i is the characteristic velocity of the i-th wave for the Roe-averaged state.

Roe Solver – Burgers Equation

Again, the scalar case provides a nice example for the solution structure and the limits of the Roe solver. For Burgers equation, the condition $\mathbf{F}(u_R) - \mathbf{F}(u_L) = A \cdot (u_R - u_L)$ immediately leads to $A = (u_L + u_R)/2$ for the Roe matrix (or Roe scalar in this case). For depending on whether $(u_L + u_R)/2 > 0$ or $(u_L + u_R)/2 < 0$, the Roe solver uses the left or right state as upwind state, and the flux is:

$$\mathbf{F}(u_L, u_R) = \begin{cases} \frac{u_L^2}{2}, & u_L + u_R > 0 \\ \frac{u_L^2}{2}, & u_L + u_R < 0 \\ 0, & u_L + u_R = 0 \end{cases} .$$

Note that the solution for the rarefaction fan (zero flux for $u_L < u_R$ and $u_L < 0 < u_R$) in the full Riemann problem is missing! This leads to artificial expansion shocks in rarefactions in which u changes sign (see animation). This also occurs for the full Euler equations for rarefactions waves containing a *sonic point* $u = \pm c_s$, (also called transsonic rarefaction). This can be cured by a so-called entropy fix.

Approximate Riemann Solvers – Roe Solver with Entropy Fix

One of the simplest entropy fixes for the Roe solver consists in replacing $|\lambda_i|$ in the flux formula with a slightly different signal velocity if it is close to zero:

$$\mathbf{F}_{\mathrm{Roe}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\sum_{i=1}^{m} \alpha_i |\lambda_i| \mathbf{R}_i.$$

If $|\lambda_i| < \epsilon$ (where ϵ is a pre-defined parameter, we set:

$$|\lambda_i| \to \frac{1}{2}\left(\frac{\lambda_i^2}{\epsilon} + \epsilon\right).$$

This needs to be done **only** for the plus and minus characteristics! As you see in our simple numerical example using Burgers' equation, this entropy fix restores the transonic rarefaction fan, but small artifacts remain. There is a trade-off between the additional smearing needed at the sonic point (where we need it) and additional diffusion elsewhere (where we don't want it).

Approximate Riemann Solvers – Marquina Solver

Marquina's flux formula does not require an average state for linearisation, and it also avoids the unphysical expansion shocks that can occur for the Roe soolver. Instead one projects both the fluxes and the state onto the right eigenvectors on either side of the interface:

$$\mathbf{F} = \sum \phi_i \mathbf{R}_i, \quad \mathbf{U} = \sum \alpha_i \mathbf{R}_i,$$

The flux is then computed as

$$\mathbf{F}_{\text{Marquina}} = \sum_i \xi_i \Phi_{i,L} \mathbf{R}_{i,L} + (1-\xi_i)\Phi_{i,R}\mathbf{R}_{i,R} + \delta \frac{1}{2}|\lambda_i|(\alpha_{i,L}\mathbf{W}_{i,L} - \alpha_{i,R}\mathbf{W}_{i,R}),$$

where $\mathbf{W}_{i,L/R}$ are the left eigenvectors for either state, λ_i is the maximum speed of the i-th wave on either side, and ξ is

$$(\xi, \delta) = \begin{cases} (1,0), & \lambda_{i,L} > 0 \wedge \lambda_{i,R} > 0 \\ (0,0), & \lambda_{i,L} < 0 \wedge \lambda_{i,R} < 0 \\ (1/2,1), & \lambda_{i,L} < 0 \wedge \lambda_{i,R} < 0 \end{cases}.$$

Thus, we select the upwind flux for a given wave if the wave speed does not change sign. If the wave speed changes sign, a diffusive (Lax-Friedrichs type) flux is added.