Literature

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The Equations of Fluid Dynamics

The first equations of fluid dynamics are conservation equations for mass, momentum and energy. For mass conservation consider the rate of change of mass in a volume, which is given by the fluxe through the surface:

$$\frac{\partial}{\partial t}\int \rho \,\mathrm{d}V + \oint \rho \mathbf{v} \cdot \mathbf{dA} = 0$$

Using Gauss' theorem, we can write this as a partial differential equation if the volume we consider is small:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

This is the continuity equation.

Lagrangian vs. Eulerian Form

We obtained the continuity equation for the change of density at a fixed point in space. Sometimes it is advantageous to have equations for the rate of change of state variables along the trajectory of a fluid element (Lagrangian formulation). We can obtain this as follows,

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho = -\nabla\cdot(\rho\mathbf{v}) + (\mathbf{v}\cdot\nabla)\rho && (1) \\ &= -(\mathbf{v}\cdot\nabla)\rho - \rho\nabla\cdot\mathbf{v} + (\mathbf{v}\cdot\nabla)\rho = -\rho\nabla\cdot\mathbf{v}, && (2) \end{aligned}$$

or in other words

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}t} + \nabla\cdot\mathbf{v} = 0.$$

You can also obtain this directly by noting that $\nabla\cdot\mathbf{v}$ is the rate of change of the volume of an infinitesimal fluid element whose mass is conserved.
The derivative $\mathrm{d}/\mathrm{d}t$ is called an absolute/convective/comoving/substantial derivative.

Conservation Form

An equation of the form

$$\frac{\partial}{\partial t}(\text{conserved quantity}) + \nabla \cdot (\text{flux}) = 0$$

is said to be in conservation form. This form will be advantageous for numerical discretisation. Sometimes one cannot get rid of all source terms on the right-hand side, in which case one can still avoid *derivatives* of the hydrodynamics state variables on the right-hand side.

The Momentum Equation

For the momentum equation, it is advantageous to start from the Lagrangian point of view. If we disregard viscous forces, the force on a fluid element is given by

$$\mathbf{f} = -\oint P\,\mathbf{dA},$$

so that

$$\begin{aligned}\left(\int \rho\,\mathrm{d}V\right)\frac{\mathrm{d}v}{\mathrm{d}t} + \oint P\,\mathbf{dA} &= 0\\ \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nabla P &= 0\end{aligned}$$

If we add gravitational acceleration, we get

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nabla P = \rho\mathbf{g}$$

Momentum Equation – Eulerian Form

Now expand the comoving derivative to get the Eulerian form:

$$
\begin{aligned}
\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]+\nabla P &= \mathbf{g}\\
\frac{\partial \rho\mathbf{v}}{\partial t}-\mathbf{v}\frac{\partial \rho}{\partial t}+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla P &= \mathbf{g}\\
\frac{\partial \rho\mathbf{v}}{\partial t}+\mathbf{v}\nabla\cdot(\rho\mathbf{v})+(\rho\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla P &= \mathbf{g}
\end{aligned}
$$

This can be written using the dyadic product $\rho\mathbf{v}\otimes\mathbf{v}$ or tensor notation as:

$$
\begin{aligned}
\frac{\partial \rho\mathbf{v}}{\partial t}+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})+\nabla P &= \mathbf{g}\\
\frac{\partial \rho v^i}{\partial t}+\frac{\partial}{\partial x^j}(\rho v^i v^j)+\frac{\partial P}{\partial x^i} &= g^i
\end{aligned}
$$

Internal Energy Equation

Now consider the work done by pressure forces on a fluid element. The internal energy ϵ per unit mass will change due to $P\,\mathrm{d}V$ work according to the first law of thermodynamics

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + P\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) = 0.$$

In Eulerian form, this becomes

$$\begin{aligned}
\frac{\mathrm{d}\epsilon}{\mathrm{d}t} - \frac{P}{\rho}\frac{\mathrm{d}\ln\rho}{\mathrm{d}t} &= 0\\
\rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + P\nabla\cdot\mathbf{v} &= 0\\
\frac{\mathrm{d}\rho\epsilon}{\mathrm{d}t} + \rho\epsilon\nabla\cdot\mathbf{v} + P\nabla\cdot\mathbf{v} &= 0\\
\frac{\partial\rho\epsilon}{\partial t} + (\mathbf{v}\cdot\nabla)(\rho\epsilon) + \rho\epsilon\nabla\cdot\mathbf{v} + P\nabla\cdot\mathbf{v} &= 0\\
\frac{\partial\rho\epsilon}{\partial t} + \nabla\cdot(\rho\epsilon\mathbf{v}) + P\nabla\cdot\mathbf{v} &= 0
\end{aligned}$$

Total Energy Equation

Note that we cannot simply integrate the internal energy equation in space to obtain a global conservation law as for mass conservation. This is because of the $P\nabla\cdot\mathbf{v}$ term on the LHS. To obtain a global conservation law, we must add the kinetic energy of the fluid. We can obtain the rate of change of kinetic energy from the continuity equation and the momentum equation:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= \frac{1}{2}\left(\mathbf{v}\frac{\partial \rho\mathbf{v}}{\partial t}+\rho\mathbf{v}\frac{\partial \mathbf{v}}{\partial t}\right) \\
&= \frac{1}{2}\rho\mathbf{v}\cdot\left[-(\mathbf{v}\cdot\nabla)\mathbf{v}-\frac{\nabla P}{\rho}+\mathbf{g}\right] \\
&\quad +\frac{1}{2}\mathbf{v}\cdot\left[-\mathbf{v}\nabla\cdot(\rho\mathbf{v})-(\rho\mathbf{v}\cdot\nabla)\mathbf{v}-\nabla P+\rho\mathbf{g}\right] \\
&= -\frac{1}{2}\left\{2\rho\mathbf{v}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]-v^2\nabla\cdot(\rho\mathbf{v})\right\}-\rho\mathbf{v}\cdot\nabla P+\rho\mathbf{v}\cdot g \\
&= -\frac{1}{2}\left\{\rho\mathbf{v}\cdot\nabla v^2-v^2\nabla\cdot(\rho\mathbf{v})\right\}-\rho\mathbf{v}\cdot\nabla P+\rho\mathbf{v}\cdot g \\
&= -\nabla\cdot\left(\rho\mathbf{v}\frac{v^2}{2}\right)-\rho\mathbf{v}\cdot\nabla P+\rho\mathbf{v}\cdot g
\end{aligned}$$

Total Energy Equation

Now we add this to the internal energy equation and obtain:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho\epsilon+\frac{v^2}{2}\right)+\nabla\cdot(\rho\epsilon\mathbf{v})+\nabla\cdot\left(\rho\mathbf{v}\frac{v^2}{2}\right)+P\nabla\cdot\mathbf{v}+\rho\mathbf{v}\cdot\nabla P &= \rho\mathbf{v}\cdot g\\ \frac{\partial}{\partial t}\left(\rho\epsilon+\frac{v^2}{2}\right)+\nabla\cdot\left(\rho\epsilon\mathbf{v}+\frac{v^2}{2}+P\mathbf{v}\right) &= \rho\mathbf{v}\cdot g.\end{aligned}$$

If the right-hand side vanishes, the total energy is manifestly conserved. Note that the energy flux has a component due to advection, but also an extra term $P\mathbf{v}$ to account for $P\,\mathrm{dV}$ work.

If the gravitational field is time-independent, or generated by the fluid itself, one can still obtain global conservation laws, but we will only discuss this in later lectures.

Note that we need an *equation of state* to close the system of five equations by providing the pressure $P = P(\rho, u)$.

Viscous Fluids

So far we have neglected viscous forces and heat conduction. If these are to be added, one needs to add extra terms to the momentum equation,

$$\frac{\partial \rho v^i}{\partial t} + \frac{\partial}{\partial x^j}(\rho v^i v^j) + \frac{\partial P}{\partial x^i} - \frac{\partial}{\partial x^j}\left[\eta\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial x^j}{\partial x^i} - \frac{2}{3}\delta_{ij}\frac{\partial v^k}{\partial v^k}\right)\right] - \zeta\frac{\partial}{\partial x_i}(\frac{\partial v^j}{\partial x^j}) \quad = \quad g^i$$

where η and ζ are the (dynamics) shear and bulk viscosity. These terms contain second spatial derivatives and will tend to smear out the velocity field. One also gets additional terms for viscous heating and heat conduction in the energy equation.

The *Reynolds number* expresses the relative importance of the advection terms and viscous terms in the momentum equation:

$$\mathrm{Re} = \frac{vl}{\nu} = \frac{vl}{\eta/\rho}.$$

In many astrophysical systems **Re** is extremely large ($10^{10\dots15}$), so we are justified in using the inviscid form of the equations. This does not mean that viscosity is not important at all in astrophysical systems; Viscous dissipation still occurs at small scales, but the dynamics on large scales is typically decoupled from the precise behaviour on small scales.
The small viscosity leads to two important phenomena: Astrophysical flows are often *turbulent*, and the inviscid equations can develop discontinuous solutions with shocks, which require a special numerical treatment.

Special Limits – Isentropic Flow

For the inviscid case, one case also invoke conservation of entropy of fluid elements instead of the energy equation,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0.$$

This, however, breaks down once shocks form in an ideal fluid. Nevertheless, conservation of entropy is important because it allows simplifications of the fluid equations in the absence of shocks: If the we have initial conditions with $s = \mathrm{const.}$, this will be maintained as long as now shocks form (such a fluid with constant entropy is a special case of a *barotropic* fluid). In such a case, one can derive an equation for the velocity that does not contain the pressure at all:

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})].$$

$\nabla \times \mathbf{v}$ is called *vorticity*.

Potential Flow

If $s = \mathrm{const.}$ and $\nabla \times \mathbf{v} = 0$ initially, no vorticity will develop later (unless shocks form). Thus, the flow stays *irrotational*, and this implies that the velocity can be expressed as a gradient,

$$\mathbf{v} = \nabla \phi.$$

This is useful because one is left only with a partial differential for a scalar field, which sometimes even allows analytic solutions for non-trivial problems.

Incompressible Limit

If a flow is very close to hydrostatic, the advection term is much smaller than the pressure term in the momentum equation ($\rho v^i v^j \ll P$). One can show that compressibility effects scale as $\delta\rho/\rho \sim v^2/(P/\rho)$, so the fluid is nearly incompressible $\rho = \mathrm{const.}$ in this case. One can then simplify the equations as follows. The continuity equation becomes

$$\nabla \cdot v = 0,$$

i.e., it is no longer an evolution equation, but a *constraint equation*. The flow must now be *purely* rotational, $\mathbf{v} = \nabla \otimes \omega$. The momentum equation (with viscous terms) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{\nabla P}{\rho} = \mathbf{g} + \frac{\eta}{\rho}\Delta \mathbf{v}.$$

Solving this equation numerically is tricky. The problem is that The right-hand side has to be irrotational as well, so one has another constraint equation that needs to be solved:

$$\frac{\Delta P}{\rho} = \nabla \cdot \mathbf{g} + \nabla \cdot \left(\frac{\eta}{\rho}\Delta \mathbf{v}\right).$$

Thus, we can no longer obtain the pressure from an evolution equation, instead we must obtain it from the solution of a boundary value problem.

Weakly Compressible Flow

The range of validity of the incompressible approximation is very limited in astrophysics. Although we often have weakly compressible flow (e.g. convection in stellar interiors), density constrants typically drive the flow via buoyancy forces in these cases. There are approximations that take buoyancy forces into account, while still accounting for the weak compressiblity by imposing a constraint equation for the velocity field (Boussinesq approximation, anelastic approximation).

Wave Solutions

We can gain some insights into the equations of hydrodynamics by considering linearized solutions. Let us consider a homogenous medium with constant pressure and density and vanishing velocity, and let primed quantities denote small perturbations of the background flow. Then the linearized continuity and momentum equation read:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho \nabla \cdot \mathbf{v}' &= 0 \\ \rho \frac{\partial \mathbf{v}'}{\partial t} + \nabla P' &= 0 \end{aligned}$$

The energy equation is tantamount to entropy conservation, so we know that P' and ρ' are related by the adiabatic condition

$$P' = \left(\frac{\partial P}{\partial \rho}\right)_s .$$

Wave Solutions

So our linearized equations become

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho \nabla \cdot \mathbf{v}' &= 0, \\ \rho \frac{\partial \mathbf{v}'}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \rho' &= 0. \end{aligned}$$

Let us now assume irrotational flow so that we can express $\mathbf{v}' = \nabla \phi$ and solve the monetum equation:

$$\begin{aligned} \rho \nabla \frac{\partial \phi}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \rho' &= 0 \\ \left(\frac{\partial P}{\partial \rho}\right)_s \rho' &= -\rho \frac{\partial \phi}{\partial t} \\ P' &= -\rho \frac{\partial \phi}{\partial t} \end{aligned}$$

Plugging this into the continuity equation yields

$$-\left(\frac{\partial P}{\partial \rho}\right)_s^{-1} \frac{\partial^2 \Phi}{\partial t^2} + \Delta \phi = 0$$

Wave Equation

The result is a wave equation,

$$\frac{\partial^2 \Phi}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_s \Delta\phi = 0,$$

with wave speed $c_\mathrm{s} = (\partial P/\partial \rho)^{1/2}$. This is the speed of sound. If we specalize to one dimension, the solutions are given by $\phi = f_1(x - c_s t) + f_2(x + c_s t)$ (left- and right-going sound wave). Specifically, we have periodic solutions like

$$\begin{aligned}
\phi &= \mathrm{Ma}\, c_s k^{-1} \sin[k(x - c_s t)] \\
v' &= \mathrm{Ma}\, c_s \cos[k(x - c_s t)] \\
P' &= -\rho \mathrm{Ma}\, c_s^2 \sin[k(x - c_s t)] \\
\rho' &= -\rho \mathrm{Ma} \sin[k(x - c_s t)]
\end{aligned}$$

where Ma is the Mach number of the sound wave.

Other waves and characteristics

Invoking Galilean invariance, we can see that sound waves will propagate with velocity $v \pm c_s$ if the background flow has constant non-vanishing velocity. We can also obtain two other waves characterized by $P' = 0$ and $v' = 0$. These are entropy waves (with $\delta\rho' \neq 0$) and vorticity waves (with transverse velocity pertubations $v'_T \neq 0$. They propagete with the velocity of v of the background flow.

This remains true in the non-linear regime as well, where the wave speeds $\pm c_s + v$ and v describe how information propagates through the fluid. We call these characteristic speeds. We will go through a more rigorous treatment of characteristics in a later lecture.

Effects of Non-linearity

For perturbations of finite amplitude, the characteristic velocity will no longer be constant. The wave crests (maximum v') will propagate faster than the troughts (minimum v'). One can estimate that the crests would overtake the troughs after a time λ/v'. Since we can't have a multivalued solution something else must happpen: The wave will steepen into a discontinuity.

Jump Conditions

Because the inviscid equations contain non-linear terms but no dissipative terms to smear out the solution, the solutions can develop discontinuities, where the differential form of the equations break down. To correctly solve the equations, we must then look for *weak solution* that satisfy the integral form of the equations.

This is done as follows: Consider integrals of the conserved quantities over a small neighborhood $A = [x_D(t) - \epsilon, x_D(t) + \epsilon]$ around the disconinuity at $x_D(t)$. For the continuinity equation, we get

$$\frac{\partial}{\partial t} \int_{x_D-\epsilon}^{x_D+\epsilon} \rho \,\mathrm{d}x = [(\rho_L v_L - \rho_L \lambda_L) - (\rho_R v_R \rho_R \lambda)],$$

where $\lambda = \dot{x}_D(t)$ is the speed of the discontinuity. If the solution is to remain well-behaved even for $\epsilon \to 0$, we must have

$$\rho_L v_L - \rho_L \lambda_L = \rho_R v_R - \rho_R \lambda,$$

Jump Conditions

Similarly, we have for the one-dimensional momentum equation and energy equation:

$$\begin{aligned}
\rho_L v_L^2 + P_L - \rho_L v_L \lambda &= \rho_R v_R^2 + P_R - \rho_R v_R \lambda \\
\rho_L v_L (h_L + v_L^2/2) - \rho_L(\epsilon_L + v_L^2/2)\lambda &= \rho_R v_R (h_R + v_R^2/2) - \rho_R(\epsilon_R + v_R^2/2)\lambda
\end{aligned}$$

Let us consider the possible cases of discontinuities. The simplest solution is to have $v_L = v_R$ and $P_L = P_R$, in which case $\lambda = v_L = v_R$ is a solution, and ρ_L and ρ_R can take on any combination of values. This is called a *contact discontinuity*. Note that there is no mass flux *across* the discontinuity.