Radiation Hydrodynamics

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May 24, 2018

The Role of Radiation

So far we have disregarded heating and cooling processes (except for shock heating). This is only appropriate if we consider length scales L larger than the mean free path λ of radiation and time scales shorter than the diffusion time scale $t_{\mathrm{diff}} = L^2/(c\lambda)$. Very often this is not the case, e.g., for

- Photons in stellar atmospheres
- Photon radiation in star formation problems
- Neutrinos in stellar interiors
- Neutrinos in young neutron stars in a core-collapse supernovae

Radiation Problems

When radiation is involved, we move from a 3-dimensional problem to a 6-dimensional problem. While a fluid is described by a few state variables at every point in space, the radiation field also depends on the direction of the particles (2 dimensions) and their energy or frequency ν (1 dimension). The radiation field is described by the *radiation intensity* $\mathcal{I}(\mathbf{r}, t, \mathbf{n}, \nu)$, which is defined as the energy of particles in the frequency range $[\nu, \nu + \mathrm{d}\nu]$ streaming per unit time through a surface $\mathrm{d}S$ at an angle α to the direction $\mathbf{n}$ of the radiation into a solid angle $\mathrm{d}\omega$.

Number Density and Distribution Function

Sometimes it is easier to consider the number density ψ of particles or their distribution function f in phase space instead of the intensity. These are related to $\mathcal{I}$ as follows:

$$\psi = \frac{\mathcal{I}}{ch\nu}$$

The distribution function is more naturally expressed in terms of the momentum $\mathbf{p} = h\nu\mathbf{n}$ instead of the frequency:

$$f(\mathbf{x}, \mathbf{p}, t) = \frac{c^2}{h^4\nu^3}\mathcal{I}(\mathbf{x}, t, \mathbf{n}, \nu).$$

Boltzmann Equation/Transport Equation

The evolution of the radiation field is governed by the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \dot{p}^i \frac{\partial f}{\partial p^i} = \mathfrak{C}(\text{collision term}),$$

The LHS reflects Liouville's theorem: For non-interacting particles, the distribution function is conserved along particles trajectories. The collision term $\mathfrak{C}$ accounts for the absorption, emission, and scattering of particles. In the Newtonian approximation, the time derivative of the $\dot{p}^i$ momentum can be set to zero if we measure the momentum in a fixed *lab frame*, but this term re-appears if we take gravitational redshift or other relativistic effects into account, or if we measure p^i in a frame comoving with the fluid. The physics behind the LHS is simple: Particles move along curves with $\dot{x} = p/E$, and maybe the momentum of the particles can change along the trajectory (e.g. in curved space time).

The Collision Term

The collision term contains the important physics, but is highly problem-dependent. For example, modern supernova simulations need to take into account processes like these using the detailed cross sections:

$(N,Z)\ e^- \rightleftharpoons (N+1, Z-1)\ \nu_e$	electron capture/absorption on nuclei
$(N,Z)\ \nu \rightleftharpoons (N,Z)\ \nu'$	coherent scattering off nuclei
$\nu_e n \rightleftharpoons e^- p$	absorption on/emission
$\bar{\nu}_e p \rightleftharpoons e^+ n$	by nucleons
$\nu N \rightarrow \nu' N'$	neutrino-nucleon scattering
$\nu e^\pm \rightarrow \nu e^\pm$	neutrino-electron scattering
$e^+ e^- \rightleftharpoons \nu\bar{\nu}$	pair production
$NN \rightleftharpoons NN\nu\bar{\nu}$	bremsstrahlung
$\nu_e\bar{\nu}_e \rightleftharpoons \nu_X\bar{\nu}_X$	purely neutrinic processes

In photon transport, different processes play a role (free-free, bound-free, bound-bound transitions, etc.). Very often one needs to include a large number of lines and finely resolve the frequency spectrum.
A common problem in radiation transport is that the typical time-scales for some of the interactions are short compared to the dynamical time-scale $\lesssim \tau_{dyn}$. This makes the collision term *stiff* and requires expensive implicit solution methods. We shall study this a bit further the labs.

Radiation Hydrodynamics

Unless the collision integral is very simple, one cannot yet solve the full transport equation in 6 dimensions, and needs to use approximations. One strategy consists in considering *angular moments* of the distribution function and the collision integral:

$$J(\epsilon) = \frac{1}{4\pi}\int f \,\mathrm{d}\Omega, H^i(\epsilon) = \frac{1}{4\pi c}\int f v^i \mathrm{d}\Omega, K^{ij}(\epsilon) = \frac{1}{4\pi c^2}\int f v^i v^j \mathrm{d}\Omega, \ldots$$

Note that we keep the dependence on the particle energy ϵ here and solve for different energies separately (multi-group approach). Note that J and H are related to the energy density and flux density in an energy group.
This turns the Boltzmann equation into a hierarchy of equations:

$$\begin{aligned}
\frac{\partial J(\epsilon)}{\partial t} + \frac{\partial H^i(\epsilon)}{\partial x^i} &= \kappa_a (f_{\mathrm{eq}} - J) \\
\frac{\partial H^i(\epsilon)}{\partial t} + \frac{\partial K^{ij}(\epsilon)}{\partial x^j} &= -(\kappa_a + \kappa_s) H^i \\
&\ldots
\end{aligned}$$

Note that we have assumed a special form of the source terms here in terms of the absorption opacity κ_a, the scattering opacity κ_s, and the equilibrium f_{eq} to which neutrino absorption/emission drives the distribution function. The general form is more complicated.

Coupling to Matter

The terms on the RHS side in the first two moment equations need to be fed into the equations of hydrodynamics as source terms. This typically needs to be done in an operator-split approach, and is usually not a major problem.

In photon radiative transfer, however, we often encounter problems where the ionisation state and level populations strongly depart from local thermodynamics equilibrium. Then we also need to solve for the ionisation and the level populations together with the radiation field. This can be extremely demanding.

Radiation Hydrodynamics – Diffusion

Replacing the transport equation with an infinite chain of moment equations,

$$\frac{\partial J(\epsilon)}{\partial t} + \frac{\partial H^i(\epsilon)}{\partial x^i} = \kappa_a(J_{\mathrm{eq}} - J), \frac{\partial H^i(\epsilon)}{\partial t} + \frac{\partial K^{ij}(\epsilon)}{\partial x^j} = -(\kappa_s + \kappa_a)H^i, \ldots,$$

has not simplified the problem yet. To make progress, we can attempt to *truncate* the hierarchy of moment equations. For example, if we assume that scattering quickly drives H^i (the monochromatic flux density) to an equilibrium value, we obtain:

$$\frac{\partial J(\epsilon)}{\partial t} + \frac{\partial H^i(\epsilon)}{\partial x^i} = \kappa_a(J_{\mathrm{eq}} - J), H^i(\epsilon) = -\frac{1}{\kappa_s + \kappa_a}\frac{\partial K^{ij}(\epsilon)}{\partial x^j}.$$

If we further assume that the distribution function is roughly isotropic (i.e. does not depend on direction), we get $K^{ij} = J/3\delta^{ij}$ and hence:

$$H^i = -\frac{1}{3(\kappa_s + \kappa_a)}\frac{\partial J(\epsilon)}{\partial x^i},$$

which we can then put into the equation for J. This is the *diffusion approximation*.

Diffusion Equation – Numerical Solution

How should we discretise the diffusion equation? Consider the prototypical equation $\partial u/\partial t + D\frac{\partial^2 u}{\partial x^2} = 0$. We could finite-difference this using an explicit scheme (FTCS),

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + D\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0.$$

This works but has a stability limit $\Delta t < \Delta x^2/D$. So if we have very fine resolution, we have to take unduly small time steps.
If we evaluate the 2nd derivative at the new time step (backward Euler method), the method is consistent and stable and for arbitrarily large time steps:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + D\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} = 0.$$

Implicit Methods

This comes at the price of having to solve a linear equation that couples variables across the whole grid. In 1D, however, the resulting linear system has a special structure (tridiagonal matrix), so that it can be used in $\mathcal{O}(N)$ operations for a grid of size N. We can even do better at almost no extra effort and make the method 2nd-order in time (Crank-Nicholson method):

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{D}{2}\left(\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\Delta x^2}+\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{\Delta x^2}\right)=0.$$

Pitfalls

One can also construct explicit schemes for the diffusion equation that are unconditionally stable. For example, the Du Fort-Frankel method (reminiscent of Leapfrog)

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}+D\frac{u_{i+1}^n-u_i^{n+1}-u_i^{n-1}+u_{i-1}^n}{\Delta x^2}=0,$$

is unconditionally stable.
But we can't have our cake and eat it: The Du Fort-Frankel scheme is only consistent if $\Delta t/Deltax \to 0$ as $\Delta x \to 0$. So we don't gain anything over the the FTCS scheme.

Stiffness of the Source Terms

Very often, one must also treat the source terms implicitly because they are stiff. This is the case whenever the matter is optically thick, so that $\kappa c \Delta t \sim \kappa \Delta x > 1$, i.e. any particle is absorbed/scattered at least once per time step on average.

Diffusion in Multi-D

The backward Euler and Crank-Nicholson scheme are only cheap in 1D. In multi-D, we still have a sparse-matrix problem, but it is much more difficult to solve it efficiently and parallelisation is not straightforward.

One possibility for solving

$$\frac{\partial u}{\partial t} + D\Delta u = 0$$

in multi-D is directional splitting using unconditionally stable schemes for the 1D diffusion equation (alternating direction implicit or ADI scheme). But one has to be very careful about this: Splitting can help achieve stability, consistency and higher-order accuracy in time if $\Delta t \to 0$. But if we use large time steps, the solution can converge to the wrong asymptotic solution (e.g. not $\Delta u = 0$)! Thus the advantage of unconditional stability is destroyed, and we still have to take small time steps.

Radiation Hydrodynamics – Other Problems with Diffusion

The diffusion approximation,

$$\frac{\partial J(\epsilon)}{\partial t} + \frac{\partial H^i(\epsilon)}{\partial x^i} = \kappa_a(J_{\mathrm{eq}} - J), H^i(\epsilon) = -\frac{1}{3(\kappa_s + \kappa_a)} \frac{\partial J(\epsilon)}{\partial x^i},$$

has other disadvantages. It is obviously not accurate if κ_a and κ_s are small, indeed the energy flux will diverge: $H^i \to \infty$. In this *free-streaming* regime, we still need to satisfy the physical constraints $|H| < J$, which comes from the finite propagation speed of neutrinos. The problem of "infinite signal velocities" in the diffusion equation also leads to severe time-step constraints $\Delta t < \kappa \Delta x^2/(2c)$ for explicit solution methods. Thus, we again have a stiffness problem here that calls for implicit solution methods as we shall discuss in the labs.

The acausal flux $|H| > J$ can be fixed by introducing a *flux limiter* $\lambda(|\nabla E|/\kappa E)$,

$$\mathbf{H} = -\lambda \frac{1}{3(\kappa_s + \kappa_a)} \nabla J,$$

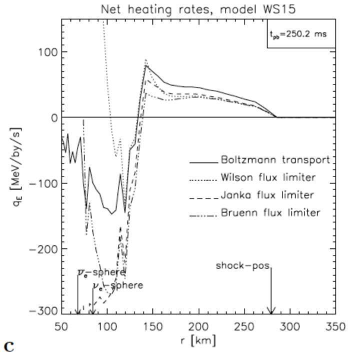
as used, e.g., by the supernova codes CHIMERA (Oakridge) and CASTRO (Princeton).

Impact of Flux Limiters – Example from Supernova Simulations

Most core-collapse supernovae are thought to be powered by neutrinos from the proto-neutron star that deposit energy behind the supernova shock. Whether this works or not has to be determined by simulations. Since the heating depends directly on the flux factor $f = H^r/J$,

$$\dot{q}_{\rm heat} \propto \frac{L}{4\pi r^2 f}.$$

prescribing f by a flux limiter can considerably alter the net neutrino heating compared to a rigorous solutions of the Boltzmann equation.



M. Rampp, PhD thesis

Radiation Hydrodynamics – Two-Moment Approximation

To go beyond the diffusion equation (and, incidentally, to circumvent the need for fully implicit solution methods) one can evolve both J and H^i and make certain (plausible) assumptions for K^{ij}. We have already encountered the simplest one, the Eddington approximation,

$$K^{ij} = J\frac{\delta^{ij}}{3}.$$

Better *closures* account for the fact that the radiation field is forward-peaked far away from the sources of radiation. This can be modelled by having K^{ij} depend both on J and H^i. However, it is not possible to rigorously derive such closures; the choice of a closure always remains somewhat *ad hoc*.

Two-Moment Approximation: Numerical Solution

The left-hand side of the two-moment system looks similar to the hydro equations (system of hyperbolic conservation laws):

$$\begin{aligned}\frac{\partial J}{\partial t}+\frac{\partial H^i}{\partial x^i} &= \kappa_a(f_{\mathrm{eq}}-J),\\ \frac{\partial H^i}{\partial t}+\frac{\partial K^{ij}(J,H^i)}{\partial x^i} &= -(\kappa_a+\kappa_s)H^j.\end{aligned}$$

We can therefore use similar numerical methods to treat the terms on the left-hand side as for the hydro equations. The time step constraint is then only $\Delta t < c\Delta x$ (or better), i.e. it does not decrease with Δx^2. In supernova simulations, this is not much worse than the Courant condition $\Delta t < c_s\Delta x$ for the hydro equations because the sound speed is not too much smaller than the speed of light in the neutron star. However, coupling the left-hand side and the source terms can be tricky as we shall see in the labs. If done incorrectly, the diffusion limit cannot be reproduced.

Closures

In spherical symmetry, the closure is completely determined by the ratio $k = K^{rr}/J$ of the r, r-component of K^{ij} and J. Various closures (ad hoc fits and heuristically derived closures) for k in terms of $h = H^r/J$ include:

$$\begin{aligned}
k &= \frac{1}{3}(1 - h + 3h^2), \quad \text{Wilson} \\
k &= \frac{1}{3}(1 + 0.5h^{1.31} + 1.5h^{4.13}), \quad \text{Janka} \\
k &= 1 - 2h/q(h), \quad \text{Minerbo} \\
k &= h \coth q(h), \quad \text{Levermore-Pomraning.}
\end{aligned}$$

Here q is defined such that $h = \coth q - 1/q$.

In a more rigorous approach, the closure can be com pouted from the solution of a simplified Boltzmann equation (variable Eddington factor method). This is done in the VERTEX supernova code, for example.

Limits of the Two-Moment Approximations

The two-moment approximation also fails for certain problems. For example, if we have two intersecting beams of radiation, the two-moment approximation will give one beam bisecting the angle of the two incident beams. In reality, the two beams would just cross each other.

Full Transport Equation – Formal Solution

If the source term only includes absorption and emission,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \eta - \kappa_a - f,$$

the solution of the transport equation along different rays (direction of radiation) decouples, and we can formally integrate the equation,

$$f = \int \kappa_a[\eta/\kappa_a - f(\mathbf{x} - \mathbf{v}(t - t'), t', \mathbf{p})]\mathrm{d}t',$$

or, if we assume that the radiation field is stationary,

$$f = \int \kappa_a[\eta/\kappa_a - f(\mathbf{x} - \mathbf{n}\lambda, \mathbf{p})]\mathrm{d}\lambda.$$

Although the integral on the right still depends on f, one can write down the solution of these differential/integral equations in terms of definite integrals over the opacity and emissivity.

Formal Solution

If the RHS of the transfer equation does not only include true absorption and emission, but also includes scattering,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = j - \kappa_a f - \kappa_s f,$$

where j includes scattering into the beam, we can still write this as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \kappa_{\mathrm{tot}}(S - f).$$

Here $S = j/\kappa_{\mathrm{tot}}$ is called the *source function*. The problem is that S does not only depend on material properties but on the radiation field itself. But since the transfer equation in this form can be solved easily by formal integration, we can just do this iteratively and update the source function until convergence is reached. This is called Λ-iteration. The problem is that Λ-iteration converges very slowly when scattering dominates.

Alternatives to Λ-iteration

When Λ-iteration converges too slowly one can resort to several strategies for solving the full transport equation:

- Accelerated Λ-iteration
- Combine moment equations and formal solution using variable Eddington factor technique
- Newton-Raphson iteration for the full transport equation (the most expensive approach)

Monte Carlo techniques can also be an alternative, but also require special tricks at high optical depth.

Further Complications – Choice of Frame

When the material velocity is not small compared to the speed of light (or if the optical depth is high), the choice of the frame in which we measure the particle energy and momentum is non-trivial. The opacity and emissivity (material properties) are most readily expressed in the frame comoving with the fluid. The LHS of the transfer equation is simplest in the lab frame. So one has three choices

- Rigorously transform the opacity and emissivity to the lab frame (lab frame approach). This can be dangerous in the optically thick regime.
- Transform the opacity κ and emissivity η to the lab frame using a Taylor expansion for κ and η in terms of comoving-frame energy (mixed frame approach). This is especially dangerous when κ and η vary rapidly (lines!).
- Transform the LHS of the transfer equation. Then we get additional derivatives in phase space, and the conservation of particle number/energy on the LHS is no longer immediately apparent:

$$\frac{\partial f}{\partial t} + A^i \frac{\partial f}{\partial x^i} + B^i \frac{\partial f}{\partial p^i} = \eta - \kappa f.$$