APPENDIX
Optimal Control for Remote Patient Monitoring with Multidimensional Health Representations

APPENDIX I

A. Transition Probabilities and Dynamic Programming Equation

We first discuss the transition probabilities and the cost function for our model -

1. At critical health states $\boldsymbol{h} \in \mathcal{H}_C$ —

No action is taken with the the service ceasing operation. A cost of C_c is incurred.

2. When $\boldsymbol{h} \notin \mathcal{H}_C$ and $1 \leq h^{(x)}, h^{(y)} \leq H-1$ —

(a) *Ordinary Monitoring (m=o), no Switching (a=o):* Does not induce a monitoring change, Starting at state $(o, h^{(x)}, h^{(y)})$, the next state with their respective transition probabilities are:

i) $(o, \min\{h^{(x)}+1, H\}, h^{(y)})$ w.p. $\lambda_{o,x}$

ii) $(o, h^{(x)}, \min\{h^{(y)}+1, H\})$ w.p. $\lambda_{o,y}$

iii) $(o, h^{(x)}-1, h^{(y)})$ w.p. $\mu_{o,x}\mathbb{1}_{\{h^{(x)}\neq 0\}} + \mu_{o,y}\mathbb{1}_{\{h^{(y)}=0\}}$

iv) $(o, h^{(x)}, h^{(y)}-1)$ w.p. $\mu_{o,y}\mathbb{1}_{\{h^{(y)}\neq 0\}} + \mu_{o,x}\mathbb{1}_{\{h^{(x)}=0\}}$

and a cost C_o is incurred. The $\min\{h^{(x)}+1, H\}$ above is used to account for the boundary case of $h^{(x)} = H$ since H is the highest health state. Similarly, the $\mathbb{1}_{\{h^{(x)}\neq 0\}}$ is used to account for the boundary case of $h^{(x)} = 0$ (see Figure 1).

(b) *Intensive Monitoring (m=i), no Switching (a=o):* Does not induce a monitoring change. Starting at state $(o, h^{(x)}, h^{(y)})$, the next state with their respective transition probabilities are:

i) $(o, \min\{h^{(x)}+1, H\}, h^{(y)})$ w.p. $\lambda_{i,x}$

ii) $(o, h^{(x)}, \min\{h^{(y)}+1, H\})$ w.p. $\lambda_{i,y}$

iii) $(i, h^{(x)}-1, h^{(y)})$ w.p. $\mu_{i,x}\mathbb{1}_{\{h^{(x)}\neq 0\}} + \mu_{i,y}\mathbb{1}_{\{h^{(y)}=0\}}$

iv) $(i, h^{(x)}, h^{(y)}-1)$ w.p. $\mu_{i,y}\mathbb{1}_{\{h^{(y)}\neq 0\}} + \mu_{i,x}\mathbb{1}_{\{h^{(x)}=0\}}$

and a cost C_i is incurred.

(c) *Intensive Monitoring (m=i), with switching (a=o):* Induces a switch to ordinary monitoring. The next state, respective transition probabilities, and the cost incurred is same as part (a): ordinary monitoring ($m = o$) with no switching ($a = o$).

(d) *Ordinary Monitoring (m=o), with switching (a=i):* Induces a switch to intensive monitoring. The next state, respective transition probabilities, and the cost incurred is same as part (b): intensive monitoring ($m = i$) with no switching ($a = i$).

Next, we give the dynamic programming equations satisfied by the optimal control $V^*(\cdot,\cdot)$.

1. At critical health states $\boldsymbol{h} \in \mathcal{H}_C$ —

$$V^*(i, \boldsymbol{h}) = V^*(o, \boldsymbol{h}) = C_c.$$

2. When $\boldsymbol{h} \notin \mathcal{H}_C$ and $1 \leq h^{(x)}, h^{(y)} \leq H-1$ —

$$
\begin{aligned}
&V^*(i, \boldsymbol{h}) = V^*(o, \boldsymbol{h}) \\
&= \min\Bigg\{ C_i + \gamma\Big[\lambda_{i,x} V^*\left(i, \min\{h^{(x)}+1, H\}, h^{(y)}\right) \\
&+ \lambda_{i,y} V^*\left(i, h^{(x)}, \min\{h^{(y)}+1, H\}\right) \\
&+ \left(\mu_{i,x}\mathbb{1}_{\{h^{(x)}\neq 0\}} + \mu_{i,y}\mathbb{1}_{\{h^{(y)}=0\}}\right) V^*\left(i, h^{(x)}-1, h^{(y)}\right) \\
&+ \left(\mu_{i,y}\mathbb{1}_{\{h^{(y)}\neq 0\}} + \mu_{i,x}\mathbb{1}_{\{h^{(x)}=0\}}\right) V^*\left(i, h^{(x)}, h^{(y)}-1\right)\Big], \\
&C_o + \gamma\Big[\lambda_{o,x} V^*\left(o, \min\{h^{(x)}+1, H\}, h^{(y)}\right) \\
&+ \lambda_{o,y} V^*\left(o, h^{(x)}, \min\{h^{(y)}+1, H\}\right) \\
&+ \left(\mu_{o,x}\mathbb{1}_{\{h^{(x)}\neq 0\}} + \mu_{o,y}\mathbb{1}_{\{h^{(y)}=0\}}\right) V^*\left(o, h^{(x)}-1, h^{(y)}\right) \\
&+ \left(\mu_{o,y}\mathbb{1}_{\{h^{(y)}\neq 0\}} + \mu_{o,x}\mathbb{1}_{\{h^{(x)}=0\}}\right) V^*\left(o, h^{(x)}, h^{(y)}-1\right)\Big]\Bigg\}
\end{aligned}
$$

B. Proof for Theorem 1

Proof. We work under the asymptotic condition of $H \uparrow \infty$ for this proof. Recall that the one-dimensional model considered in [8] considered health state $h = 0$ as the critical set and defined the parameters γ, λ_o and λ_i. These are the discount factor, probability of health improving under ordinary monitoring, and health improving under intensive monitoring, respectively. Theorem 1 and 2 from [8] together show that the optimal control is always a threshold policy, i.e., there exists $\bar{h}$, such that $\pi^*(h) = i$ for $h \leq \bar{h}$ and $\pi^*(h) = o$ for $h > \bar{h}$. Note that the control where the optimal control at all states is ordinary monitoring is a special case of the threshold policy with $\bar{h} = 0$.

Now in our two-dimensional model, consider the sets $A^{(k)} = \{\boldsymbol{h} \mid h^{(x)} + h^{(y)} = k\}$. Then $\mathbb{P}(\boldsymbol{h}_{t+1} \in A^{(k+1)} \mid \boldsymbol{h}_t \in S^{(k)}, m_t = o) = \lambda_{o,x} + \lambda_{o,y}$. Similarly, $\mathbb{P}(\boldsymbol{h}_{t+1} \in A^{(k-1)} \mid \boldsymbol{h}_t \in S^{(k)}, m_t = o) = \mu_{o,x} + \mu_{o,y}$. Similar transitions are defined for intensive monitoring with analogous probabilities.

Now define $\lambda'_i = \lambda_{i,x} + \lambda_{i,y}$ and $\lambda'_o = \lambda_{o,x} + \lambda_{o,y}$. Suppose the set of health sets $A^{(c)} = \{\boldsymbol{h} \mid h^{(x)} + h^{(y)} = c\}$ is defined as the health set $h' = 0$. Then sets of health states $A^{(k)}$ are given by $h' = k - c$ for $k \geq c$. Then our two-dimensional model can be represented using the one-dimensional model with parameters $\gamma, \lambda_o', \lambda_i'$ and with health states given by h'. Then applying Theorems 1 and 2 from [8] gives us the result that the optimal control in the one-dimensional case can be represented using a threshold. Let that threshold in the one-dimensional case be $\bar{h}'$, then the optimal control in the two-dimensional case is $\pi_{t,f}$ where $f(\boldsymbol{h}) = h^{(x)} + h^{(y)} - (\bar{h}' + c)$. This completes the proof for Theorem 1. $\square$

[8] S. Chandak, I. Thapa, N. Bambos, and D. Scheinker, "Tiered service architecture for remote patient monitoring," arXiv preprint arXiv:2406.18000, 2024