1.)

$X_1 - X_2$ ARE iiD GAUSSIAN.

unknown mean θ
Know variance σ^2

(A) $\theta \sim N(\mu, \sigma^2)$,

so for a single observation, probability is,

$$P(x_i|\theta) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x_i-\theta)}{2\sigma_0^2}}$$

as $x_1, x_2 - x_n$ are independent,

For n observations, in a dataset D: -

$$P(D|\theta) = \prod_{i=1}^{n} P(x_i|\theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\theta)^2}{2\sigma^2}}$$

$$P(\theta) = \frac{1}{\sqrt{2\pi\sigma}} \times e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}$$

Maximum posterior,

$$\theta_{MAP} = \arg\max \left\{ P(D|\theta) \cdot P(\theta) \right\}$$

$$= \arg\max \left\{ \prod_{i=1}^{n} P(x_i|\theta) \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \right\}$$

$$= \arg\max \left\{ \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x_i-\theta)^2}{2\sigma_0^2}} \times \frac{1}{\sqrt{2\pi\sigma}} \times e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \right\}$$

$$= \arg\max \left\{ \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\sigma_0^2}} \times \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \right\}$$

$$= \arg\max \left\{ \frac{1}{(2\pi)^{\frac{n+1}{2}} \sigma_0^n \sigma} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma_0^2}} e^{-\frac{(\theta-\mu)^2}{2\sigma^2}} \right\}$$

Taking log (to smothe) & log is monotically increasing :-

$$\log\left(P(D|\theta) \cdot P(\theta) \right) = \log\left(\frac{1}{(2\pi)^{n+1/2} \sigma_0^n \sigma} \right) - \sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\sigma_0^2} - \frac{(\theta-\mu)^2}{2\sigma^2}$$

Maximizing by derivation:

$$\partial/\partial\theta \left[\log(P(D|\theta)\cdot P(\theta)\right] = \frac{\partial}{\partial\theta}\left[\log\left(\frac{1}{(2\pi)^{n+1/2}\sigma_0^n\sigma}\right) - \sum_{i=1}^{n}\frac{(x_i-\theta)^2}{2\sigma_0^2} - \frac{(\theta-\mu)^2}{2\sigma^2}\right)$$

$$= \sum_{i=1}^{n}\frac{(x_i-\theta)}{\sigma_0^2} - \frac{(\theta-\mu)}{\sigma^2}$$

Now equating to zero: -

$$\sum_{i=1}^{n}\frac{(x_i-\theta)}{\sigma_0^2} - \frac{(\theta-\mu)}{\sigma^2} = 0$$

$$\sum_{i=1}^{n}\frac{(x_i-\theta)}{\sigma_0^2} = \frac{\theta-\mu}{\sigma^2}$$

$$\sigma^2\sum_{i=1}^{n}(x_i-\theta) = \sigma_0^2(\theta-\mu)$$

$$\sigma^2\left(\sum_{i=1}^{n}(x_i) - n\theta\right) = \sigma_0^2\theta - \sigma_0^2\mu.$$

$$\sigma^2\sum_{i=1}^{n}(x_i) - \sigma^2 n\theta = \sigma_0^2\theta - \sigma_0^2\mu$$

$$\sum_{i=1}^{n}(x_i)\sigma^2 + \sigma_0^2\mu = \sigma_0^2\theta + \sigma^2 n\theta$$

$$\sum_{i=1}^{n}(x_i)\sigma^2 + \sigma_0^2\mu = \theta(\sigma_0^2 + \sigma^2 n)$$

$$\therefore \theta = \frac{\sigma_0^2 \mu + \sigma^2 \sum_{i=1}^{n} (x_i)}{\sigma_0^2 + \sigma^2 n}$$

$$\therefore \theta_{MAP} = \frac{\sigma_0^2 \mu + \sigma^2 \sum_{i=1}^{n} x_i}{\sigma_0^2 + \sigma^2 n}$$

(b) $P(\theta) = \frac{1}{2b} e^{\frac{-|\theta-\mu|}{b}}$

By MAP as we did previously;

$$\sum_{i=1}^{n} \frac{(x_i - \theta)}{\sigma_0^2} + \frac{(\theta - \mu)}{|\theta - \mu| b} = 0$$

$$\therefore \sum_{i=1}^{n} \frac{(x_i - \theta)}{\sigma_0^2} = \frac{-(\theta - \mu)}{|\theta - \mu| b} \quad \text{--- (1)}$$

since it is assumed that $\mu = 0$ we can rewrite the above equation (1) as:

$$\sum_{i=1}^{n} \frac{(x_i - \theta)}{\sigma_0^2} = \frac{-\theta}{|\theta| b}$$

$$\sum_{i=1}^{n} (x_i) - n\theta = \frac{(\text{sign } \theta)\, \sigma_0^2}{b}$$

$$\sum_{i=1}^{n} (x_i) + \frac{(\text{sign } \theta)\, \sigma_0^2}{b} = n\theta$$

$$\theta = \frac{\sum_{i=1}^{n} (x_i) + \frac{\text{sign}(\theta)}{b} \sigma_0^2}{n}$$

(c)

$$\theta \sim N(0, \Sigma = \sigma^2 I),$$

$$\theta_{MAP} = \arg\max \left\{ \frac{1}{\sqrt[n]{2\pi}\, |\Sigma_0|^{n/2}} e^{\left(-1/2(-\theta+x)^T \Sigma_0^{-1} (x-\theta)\right)} \right.$$

$$\times \frac{1}{(2\pi)^{n/2} \sigma^{-2}} e^{\left(\frac{-1}{2\sigma^2} \theta^T \theta\right)}$$

Applyng log:—

$$\log \left(\frac{1}{(2\pi)^{n/2} (2\pi)^{n/2} |\Sigma_0|^{n/2} \sigma^{-2}} \right)$$

$$- 1/2 (x-\theta)^T \Sigma_0^{-1} (x-\theta))$$

$$- 1/2 \sigma^{-2} \theta^T \theta$$

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So, now apply derivative and equate it to zero: -

$$\partial/\partial\theta \left(1/2\, (x-\theta)^T \Sigma_0^{-1} (x-\theta) \right.$$
$$\left. + \ \partial/\partial\theta \left(\frac{1}{2\sigma^2} \theta^T\theta \right) = 0 \right.$$

$$\partial/\partial\theta \left(x^T x \Sigma_0^{-1} - x\theta^T \Sigma_0^{-1} - x^T \theta \Sigma_0^{-1} \right.$$
$$\left. + \theta\theta^T \Sigma_0^{-1} \right)$$
$$+ \frac{\partial}{\partial\theta} \left(\frac{1 \cdot \theta^T \theta}{\sigma^2} \right) = 0$$

$$-\left(x^T \Sigma_0^{-1} \right)^T - x\Sigma_0^{-1} + \Sigma_0^{-1^T} \theta + \theta\Sigma_0^{-1}$$
$$+ \frac{1}{\sigma^2} \theta = 0$$

$$\frac{-1}{\sigma^2} x \theta = -\left(x^T \Sigma_0^{-1} \right)^T - \Sigma_0^{-1} x$$
$$+ \Sigma_0^{-1^T} \theta + \theta \Sigma_0^{-1}$$

$$\frac{-1}{\sigma^2} \theta = - \Sigma_0^{-1} x - x\Sigma_0^{-1} + \theta\Sigma_0^{-1}$$
$$+ \theta \Sigma_0^{-1^T}$$

$$-2\Sigma_0^{-1} x + 2\Sigma_0^{-1} \theta = \frac{-2\theta}{\sigma^2}$$

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Since $\Sigma_0^{-1} = \Sigma_0^{-1^T}$

$$-2\Sigma_0^{-1}X = +2\Sigma_0^{-1}\theta - \frac{2\theta}{\sigma^{-2}}$$

$$-2\Sigma_0^{-1}X = \left(2\Sigma_0^{-1} \cdot \frac{2}{\sigma^{-2}}\right)\theta$$

$$\therefore \quad \theta = \frac{\Sigma_0^{-1}X}{\Sigma_0^{-1}(-1/\sigma^{-2})}$$