Lecture-01: MFAI (24/07/24)

Syllabus

→ Logic and Deduction
→ Vector calculation
→ Linear Algebra
→ Probabilistic Analysis
→ Markov's Chain/Queuing Theory

Midsem: 30
Endsem: 40
Assignment: 10
Attendence: 10
Surprise Test: 10

} Marking scheme

"Logic And Deduction"

↓

Russel's paradox

In a city, there is a barber, the barber shaved those people who could not shave themselves.

People who shave themselves	People who do not shave themselves
A	B

Now the question is:

Does Barber belong to part A or Part B?

↓

Barber ∈ A ?
or
Barber ∈ B ?

(i) $Br \in A \longrightarrow Br \notin A$

(ii) $Br \in B \longrightarrow Br \notin B$

Milkman and Lady

Lady: On anyday that you leave milk be sure to leave milk next day as well

Anyday leave milk ⟶ Leave milk next day

~~Milkman~~
Lady (Should have said):

Inductive + Base case:
(i) On any day that you leave milk be sure to leave milk next day as well.
(ii) Leave milk today.

↑ Russel's Induction

- Luca cardely asked by a monk: What is truth?
  Luca replied (Logically): A set of facts and some rules of inferences.

- Aristotle (oldest uses of logic)

All men are mortal
Socrates was a man] Premises

Socrates was mortal] Conclusion (Inference)

All bats can fly
Socrates is a bat] Premises

Socrates can fly] Conclusion

A > B
B > C
C > D
∴ A > D

Language of logic

First order logic

Propositional logic

Question: Assume a doctor give a strip of medicine to patient and said to take $1 \rightarrow \frac{1}{2} \rightarrow 1 \ldots$ medicine day after day. How can Patient remember how much to take that day with marking it somewhere?

$1 \rightarrow \frac{1}{2} \rightarrow 1 \rightarrow \frac{1}{2} \rightarrow 1 \rightarrow \frac{1}{2} \ldots$

Solution:

Connectives:

AND (∧) : Conjunction
OR (∨) : Disjunction
NOT (¬) : Negation
⊃ (→) : Implication

Books:

1. A beginner's guide to mathematical logic
- Smullyan

Prolog: Programming Language for Logic

Lecture 2: MFAI

Propositional logic → Rules to reason about $\boxed{\text{propositions}}$ → A statement which takes values (True) or (False)

Ex.

It is raining.
Laxman is lying.
If it rains, grass becomes wet

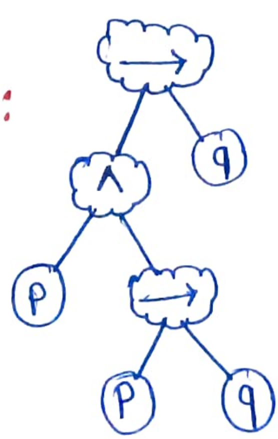
Stop!
Why are you angry?
} Not propositions.

Rules:

1. $\top$ → Always True
   $\bot$ → Always False
   } propositional formula

2. '(' and ')' → Auxiliary symbols
3. Propositional variables p, q, r... are propositional formula.
4. If 'p' and 'q' are propositional formula then so are $\sim p$, $\sim q$, $p \rightarrow q$, $p \vee q$, $p \wedge q$.
5. Bigger propositional formulas are constructed by using rule 3 and rule 4.

Ex. $(p \wedge (p \rightarrow q)) \rightarrow q$:



p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

fig: Truth table: p∨q

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	F

Truth tables

$(p \to q)$	$\wedge$	$(q \to p)$
T		T
F		T
T		F
T		T

Laws:

(i) $p \wedge q = q \wedge p$
(ii) $p \vee q = q \vee p$
} commutative

(iii) $p \wedge p = p$

(iv) $p \vee p = p$

(v) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
(vi) $p \vee (q \vee r) = (p \vee q) \vee r$
} associative

(vii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
(viii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
} distributive

(ix) $\neg\neg p = p$ (Double implication rule)

(x) $\neg(p \wedge q) = \neg p \vee \neg q$
(xi) $\neg(p \vee q) = \neg p \wedge \neg q$
} DeMogan's law

(xii) $p \wedge (p \vee q) = p$
(xiii) $p \vee (p \wedge q) = p$
} Absorption law

① If $n \mid \{(n-1)! + 1\} \longrightarrow n$ is prime

$\mid$ → Divides

$\neg(n \text{ is prime}) \to n \nmid \{(n-1)!+1\}$

$p \to q$

$\Downarrow$

$\boxed{\neg q \to \neg p}$

$p \vee \neg q$

$\neg q \vee p$

$\neg q \to \neg p$

Study: Try to do the proof and take some numerical values for ~~a, b~~ a, b, n

Proof: suppose n is composite

~~Case 1:~~ $n = a * b$

case 1: $a \neq b$; $a, b < n$

$(n-1)! = ((a*b)-1)!$

that means

$a, b \mid (n-1)! \longrightarrow a, b \nmid (n-1)!+1$ (contradiction)

then $n \nmid (n-1)!+1$

case 2: $a = b$, $n = b^2$

$(n-1)! = (b^2-1)!$

$= (b^2-1) \times (b^2-2) \ldots b \ldots \times 1$

$b \mid (n-1)! \longrightarrow b \nmid (n-1)!+1$ (contradiction)

then $n \nmid (n-1)!+1$

It is hard to understand by this but this is just a proof by contradiction.

Let say,

$n = 6$

$a = 2$, $b = 3$

$(n-1)!+1 = 121$

$(n-1)! = 120$

$a, b \mid (n-1)!$

$a, b \nmid (n-1)!+1$

Let say,

$n = 9$

$a = 3, b = 3$ | $a = b$

$(n-1)! = 40320$

$(n-1)!+1 = 40321$

$b \mid (n-1)!$

$b \nmid (n-1)!+1$

Suppose

$n = 5$

$(n-1)! + 1 = 25$

$\{n \mid (n-1)! + 1\} \wedge \{n \text{ is prime}\}$

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$\equiv$

(Equivalent)

Books:

1. Logic and structure by Van Dalen (Springer)
2. Mathematical Logic by smullyan,
3. Logic for computer Science by Jean H. Gallier

Truth table : $\to$ Semantic domain

Laws : $\to$ Syntax Domain

Lecture 3(A): MFAI (31/07/24)

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\equiv$

[God's Rule]

$p \to q \equiv \neg p \vee q$

$\equiv \neg q \to \neg p$

T
F
F
T

Proving in Semantic domain (Model Theory)

Truth Table

$p \to q \equiv \neg p \vee q \equiv q \vee \neg p$

$\equiv \neg(\neg q) \vee \neg p$

$\equiv \neg q \to \neg p$

Proving in Syntatic domain (Proof Theory)

Boolean Algebra

(i) If we prove a property holds in semantic domain then a proof exists in syntax domain.
(Completeness)

(ii) If we prove a property in proof theory then proving using model theory gives same results
(Soundness)

[Propositional Logic is sound and complete]

Operator precedence

(High)

$\neg$
$\wedge$
$\vee$
$\rightarrow$
$\leftrightarrow$

(Low)

[All operators are right associative]

$(\neg x) \rightarrow ((y \vee z) \rightarrow (x \vee (y \wedge z)))$

{$\rightarrow, \neg$: Functionally complete}

① $p \vee q \equiv \neg(\neg p) \vee q$
$\equiv \neg p \rightarrow q$

② $p \wedge q \equiv \neg\neg(p \wedge q)$
$\equiv \neg(\neg p \vee \neg q)$
$\equiv \neg(p \rightarrow \neg q)$

③ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$\equiv \neg\neg((p \rightarrow q) \wedge (q \rightarrow p))$
$\equiv \neg[\neg(p \rightarrow q) \vee \neg(q \rightarrow p)]$
$\equiv \neg[(p \rightarrow q) \rightarrow (\neg(q \rightarrow p))]$

Exclusive-OR
(XOR)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

9

I will not see total solar eclipse if I am not in the path

p : I am in the path

q : I will see the total solar eclipse

$\boxed{\neg p \rightarrow \neg q}$

Lecture-04: Maths Foundations for AI

Valid formula
Satisfiable formula
Unsatisfiable formula

A	B	$A \to B$	$X = A \wedge (A \to B)$	$X \to B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$X \to B$ $[(A \wedge (A \to B)) \to B]$

▷ If all valuations make a formula true then it is called a valid formula or Tautology.

$\neg B$	$X \to \neg B$
F	F
T	T
F	T
T	T

▷ If there exist a valuation which makes formula true we say the formula is Satisfiable.

▷ If no valuation can make a formula true we say formula is invalid / unsatisfiable or Contradiction.

A	$\neg A$	$A \wedge \neg A$
T	F	F
F	T	F

→ Proof by Contrapositive
→ Reduce to Absurdity (contradiction)

Ex.

$n \mid (n-1)! + 1 \longrightarrow n$ prime

$\boxed{p \to q \equiv \neg q \to \neg p}$

$\neg(n \text{ prime}) \longrightarrow \neg(n \mid (n-1)! + 1)$

$\neg(p \to q)$

$\vdots$

$\perp$

$\neg(p \to q) \equiv \neg(\neg p \vee q)$

$\equiv p \wedge \neg q$

composite

$n \mid (n-1)! + 1 \wedge n$ is ~~composition~~

We will prove that 'n' does not divide $(n-1)!+1$.

Study: Proof this as a practice

$A_1, A_2 \ldots A_n$ are propositional formula
$B_1, B_2 \ldots B_n$ are propositional formula

$A_1, A_2 \ldots A_n \Longrightarrow B_1, B_2 \ldots B_n$ is called a SEQUENT.

- A valuation 'v' makes this sequent true of '

$$v \models \boxed{A_1 \wedge A_2 \ldots \wedge A_n} \to \boxed{B_1 \vee B_2 \ldots \vee B_n}$$

(satisfy)

Can be also written as $\supset$ (in some books)

▷ A valuation 'v_1' makes the $\boxed{\text{sequent false}}$ if 'v_1' make $\boxed{A_1 \wedge A_2 \wedge \cdots A_n \text{ True}}$ and $\boxed{B_1 \vee B_2 \vee \cdots \vee B_n \text{ false}}$

Ex.

$$A \supset B \Rightarrow A, B$$

$$W = \{(A \rightarrow B) \rightarrow A \vee B\}$$

$$\underbrace{T \rightarrow F}_{F} \qquad \underbrace{T \vee F}_{T}$$

$$\underbrace{F \longrightarrow T}_{T}$$

$\{A = F, B = F\}$
then W is False

$$\left\{ \begin{array}{l} \bigcirc \Rightarrow A, B \\ A, B \Rightarrow \bigcirc \end{array} \right\} \text{ are also sequent}$$

Gentzen's Table method

Lecture 5-A: MFAI

$$\underbrace{A_1, A_2, A_3}_{\text{True}} \Rightarrow \underbrace{B_1, B_2, B_3}_{\text{False}} \Big\} \text{ To falsify.}$$

$$(A_1 \wedge A_2 \wedge A_3) \wedge (\neg B_1 \wedge \neg B_2 \wedge \neg B_3)$$

(i) $\Gamma, \neg A, \Delta \Rightarrow \Lambda$ (Not in the left)

$\Gamma, \Delta \Rightarrow A, \Lambda$

(ii) $\Gamma \Rightarrow \Delta, \neg A, \Lambda$ (Not in the right)

$\Gamma, A \Rightarrow \Delta, \Lambda$

(iii) AND in left

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Lambda}{\Gamma, A \wedge B, \Delta \Rightarrow \Lambda}$$

(iv) AND in right

$$\frac{\Gamma \Rightarrow \Delta, A, \Lambda \quad \text{or} \quad \Gamma \Rightarrow \Delta, B, \Lambda}{\Gamma \Rightarrow \Delta, A \wedge B, \Lambda}$$

(v) OR in left ($\vee L$)

$$\frac{\Gamma, A, \Delta \Rightarrow \Lambda \quad \text{or} \quad \Gamma, B, \Delta \Rightarrow \Lambda}{\Gamma, A \vee B, \Delta \Rightarrow \Lambda}$$

(vi) OR in the right ($\vee R$)

$$\frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, A \vee B, \Lambda}$$

(vii) $\rightarrow$, R

$$\frac{\Gamma, A \Rightarrow B, \Lambda}{\Gamma \Rightarrow A \supset B, \Lambda}$$

(viii) $\rightarrow$, L

$$\frac{\Gamma \Rightarrow A, \Lambda \quad | \quad \Gamma, B \Rightarrow \Lambda}{\Gamma, A \supset B \Rightarrow \Lambda}$$

Ex.

Known as deduction tree

$$\frac{(P \supset Q) \Rightarrow (\neg Q \supset \neg P)}{\Rightarrow (P \supset Q) \supset (\neg Q \supset \neg P)}$$

$\Rightarrow P, (\neg Q \supset \neg P) \quad | \quad Q \Rightarrow (\neg Q \supset \neg P)$

$\neg Q \Rightarrow P, \neg P$ (Contradiction) $\qquad Q, \neg Q \Rightarrow \neg P$ (contradiction)

Assignment: Take a formula and create a deduction Tree

Contradiction

$$\frac{B \Rightarrow A, \neg A \;\mid\; B, \neg B \Rightarrow \neg A}{}\ L\rightarrow$$

Contradiction

$$\frac{A \supset \neg B, \Rightarrow \neg A, A \;\mid\; B, A \supset \neg B \Rightarrow \neg A}{A \supset B, A \supset \neg B \Rightarrow \neg A}\ L\rightarrow$$

$$\frac{A \supset B, A \supset \neg B \Rightarrow \neg A}{A \supset B \Rightarrow (A \supset \neg B) \supset \neg A}\ R\rightarrow$$

$$\frac{A \supset B \Rightarrow (A \supset \neg B) \supset \neg A}{\Rightarrow (A \supset B) \supset \{(A \supset \neg B) \supset \neg A\}}\ R\rightarrow$$

5-B
Lecture 6: MFAI

(19)

▷ Linear Algebra (17-69)
▷ Analytic Geometry (70-97)
▷ Matrix Decompositions (98-138)

Assignment: Google Page rank

Mathematics for machine learning
[M.P. Deisenroth, A.A. Faisal and C.S. Ong]

Algebra: Intuitive development of concepts of object and rules for manipulation of the object

{For all $a, b \in Z$ where $a^2+b^2=c^2 \mid c \in Z$} → $3^2+4^2=5^2$

$6^2+8^2=10^2$

$15^2+8^2=17^2$

```
for i=1 to n-1:
    for j=i+1 to n:
        S = i^2 + j^2
        K = √S
        if K = int(K):
            Print (i,j,k)
    end for
end for
```

Ex.

m: moutain ranges

$N = (N_1, N_2 \dots N_m)$

$N_j \rightarrow$ number of hilltops in j^{th} mountain range

X : [ ] ↓ (column)

X^T : [→ (Row)]

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

, B is the row echleon for of A

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

① $b_{ij} = 0 \mid i > j$

② b_{ii} can be any value if $b_{ii} \neq 0$ then $b_{i-1}, b_{i-1} \neq 0$

Reduced Row-echelon form

$b_{ii} = 0$ or 1

Ex.

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Find A^{-1}

If A^{-1} exists then we know $AA^{-1} = I$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - R_2$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$R_1 \to R_1 - R_3$

$R_2 \to R_2 + R_3$

$R_4 \to R_4 - \frac{1}{2} R_3$

(Next page)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 + 2R_4$
$R_2 \rightarrow R_3 - 2R_4$
$R_3 \rightarrow R_3 - R_4$
$R_4 \rightarrow 2R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right] \rightarrow A^{-1}$$

Lecture-06 : MFAI (12-08-24)

Symbols and its meanging → Given in the book (Maths for ML - DFO)

▷ **Linear Equation:**

If $\vec{x} \in \mathbb{R}^n$

then $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

where $a_i, i = 1:n$ and b are scalars.

▷ **System of linear equation**

If $\vec{x} \in \mathbb{R}^n$

then
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = b_m \end{array}\right\}$$

can be compacted into matrix form

Compact form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$AX = B$ ← General matrix equation for system of linear equations

Equation-1

(a) R_1 $2x_1 + x_2 - 2x_3 = -3$
R_2 $3x_1 - 2x_2 + x_3 = 11$
R_3 $-x_1 + 3x_2 - 3x_3 = 4$

$R_2 + R_3 = (3x_1 - x_1) + (3x_2 - 2x_2) + (x_3 - 3x_3) = 11 + 4$

$\Rightarrow 2x_1 + x_2 - 2x_3 = 15$ —(i)

~~$R_2 + R_3 \neq R_1$ there is some dependency~~

~~But~~

But equation (i) and R_1 are equal on LHS but not on RHS. So the system of equations are inconsistent.

No Solution (Inconsisent)

(b) R_1 $2x_1 + x_2 - 2x_3 = -3$
R_2 $3x_1 - 2x_2 + x_3 = 11$
R_3 $-x_1 + 3x_2 - 3x_3 = -14$

$R_2 + R_3 = \boxed{R_1}$ (There is dependency)

Can be achieved by other two equations, So there is actually on two independent equations

Infinitely many solutions (consistent)

let $x_3 = 0$

$2 \times (2x_1 + x_2) = (-3) \times 2$
$3x_1 - 2x_2 = 11$

$7x_1 = 5$

$2R_1 + R_2$

$x_1 = 5/7$
$x_2 = \frac{-31}{7}$

~~Equation-2~~

(e)
$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= -3 \\ 3x_1 - 2x_2 + x_3 &= 11 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$
→ Unique Solution (Consistent)

$$X = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

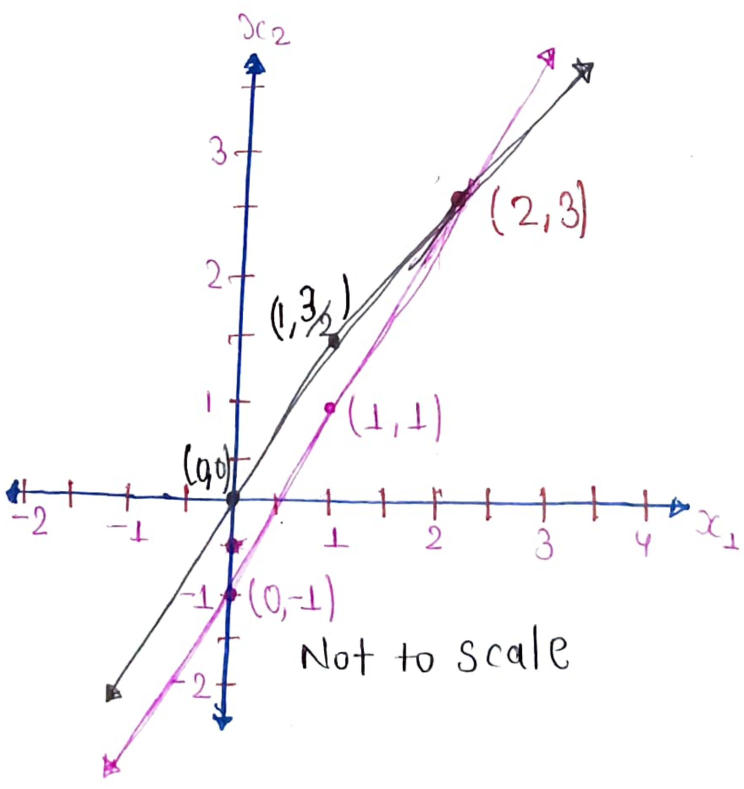
Equation-2

$$2x_1 - x_2 = 1$$
$$3x_1 - 2x_2 = 0$$
$$X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$x_2 = 2x_1 - 1$ — (i)

$x_2 = \frac{3}{2}x_1 + 0$ — (ii)

	(i)		(ii)	
x_1	0	1	0	1
x_2	-1	1	0	$3/2$



Laws/Rules:

Given $A, B \in \mathbb{R}^{m \times n}$

(i) $A + B = B + A \in \mathbb{R}^{m \times n}$

(ii) $\alpha(A) = \alpha(a_{ij})$
$= (\alpha a_{ij})$

Matrix Properties

Given $A, B \in \mathbb{R}^{m \times n}$

1. It is commulative with respect to Addition.

$$A+B = B+A \in \mathbb{R}^{m \times n}$$

2. $\alpha[A] = [\alpha A]$, $\alpha(a_{ij}) = (\alpha a_{ij})$ $i: 1-m$, $j: 1-n$

3. Matrix multiplication is not com~~putation~~mulative

Given A, B

(i) BA and AB may not be defined

(ii) Even if AB or BA or both is defined

$$AB \neq BA$$

$A \in \mathbb{R}^{m \times \underline{k}}$, $B \in \mathbb{R}^{\underline{k} \times n}$ | then AB is defined and $AB \in \mathbb{R}^{m \times n}$

$B \in \mathbb{R}^{k \times n}$, $A \in \mathbb{R}^{m \times k}$ | then BA is defined iff $m = n$ and $BA \in \mathbb{R}^{k \times k}$

If $A, B \in \mathbb{R}^{n \times n}$ | AB and $BA \in \mathbb{R}^{n \times n}$

$C = AB$, $D = BA$

then ij^{th} element of $C = c_{ij}$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

then ij^{th} element of $D = d_{ij}$

$$= \sum_{k=1}^{n} b_{ik} a_{kj}$$

4. Matrix multiplication is distributive over addition

$$(A+B)C = AC + BC$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 10 \\ 5 & 8 \end{bmatrix} \qquad BA = \begin{bmatrix} 5 & 4 \\ 11 & 10 \end{bmatrix}$$

$$AB \neq BA$$

5. Matrix Inversion

Given $A \in \mathbb{R}^{n \times n}$, A is $\boxed{\text{invertable}}$ → regular | non-singular
if and only if (iff) $\exists\, B \in \mathbb{R}^{n \times n}$
where $AB = BA = I$, $I \in \mathbb{R}^{n \times n}$

$$I_{5\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then B is written as A^{-1}

$$\boxed{AA^{-1} = A^{-1}A = I}$$

$A \in \mathbb{R}^{2\times2}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \times \frac{1}{a_{11}a_{22} - a_{12}a_{21}}$$

only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$

Proof

Let $B = A^{-1}$

$$AB = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a_{11}b_{11}+a_{12}b_{21} = 1 \quad \text{—} (1)$$

$$a_{11}b_{12}+a_{12}b_{22} = 0 \quad \text{—} (2)$$

$$a_{21}b_{11}+a_{22}b_{21} = 0 \quad \text{—} (3)$$

$$a_{21}b_{21}+a_{22}b_{22} = 1 \quad \text{—} (4)$$

$$(1)-(3) = b_{11}(a_{11}-a_{21}) + b_{21}(a_{12}-a_{22}) = 1$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} \frac{1}{-10}$$

6. Given

$A, B \in \mathbb{R}^{n \times n}$

and A^{-1}, B^{-1} exist

the $(AB)^{-1} = B^{-1}A^{-1}$

and $(AB)^{-1} \neq A^{-1}B^{-1}$

(not always

Let $(AB)^{-1} = C$

$B^{-1}A^{-1} = D$

$B^{-1}A^{-1} = D$

$\Rightarrow BB^{-1}A^{-1} = BD$

$\Rightarrow BB^{-1}A^{-1} = BD$

$\Rightarrow A A^{-1} = BD$

$\Rightarrow AA^{-1} = A(BD)$

$\Rightarrow I = (AB)D$

$\Rightarrow (AB)^{-1} = D = B^{-1}A^{-1}$

7. If $A \in \mathbb{R}^{m \times n}$ then A^T is a matrix where row and columns reversed

$A = (a_{ij}) \in \mathbb{R}^{m \times n}$

$A^T = (b_{ij}) \in \mathbb{R}^{n \times m}$

$\boxed{b_{ij} = a_{ji}}$

Study: MIMO Technology

{26 April : World Intelltual Property Day}

8. Given $A, B \in \mathbb{R}^{n \times n}$

$(A^T)^T = A$

$(AB)^T = B^T A^T$

9. If $A \in \mathbb{R}^{n \times n}$

and if A is invertible

then $(A^{-1})^T = (A^T)^{-1}$

Proof:

$$I^T = I$$

$$(AA^{-1})^T = I$$

$$\Rightarrow (A^{-1})^T A^T = I$$

$$\Rightarrow (A^{-1})^T A^T (A^T)^{-1} = I (A^T)^{-1}$$

$$\Rightarrow (A^{-1})^T \cancel{I} = \cancel{I} (A^T)^{-1}$$

$$\Rightarrow \boxed{(A^{-1})^T = (A^T)^{-1}}$$

Solving set of Linear Equations

$$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n = b_m$$

- Compact Representation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$A\vec{x} = \vec{b}$

$A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^n$

$y \in \mathbb{R}^m$

Given, $A\vec{x} = \vec{b}$
there is always an associated compact space $A\vec{x} = \vec{0}$

Statement:

If $\vec{\hat{x}}$ is a specific solution to $A\vec{x} = \vec{b}$
and if $\vec{x'}$ is a specific solution to $A\vec{x} = \vec{0}$ $(\vec{x'} = \vec{0})$
then $\forall \alpha \in \mathbb{R}, \alpha\vec{x'} + \vec{\hat{x}}$ is a solution $A\vec{x} = \vec{b}$

Proof :

$$A(\alpha\vec{x'}+\vec{\hat{x}}) = A(\alpha\vec{x'}) + A(\vec{\hat{x}})$$
$$= \alpha(A\vec{x'}) + A(\vec{\hat{x}})$$
$$= \alpha(\vec{0}) + \vec{b} = \vec{b}$$

Ex.

$$x_1 + x_3 - 4x_4 = 42$$
$$x_2 + 2x_3 + 12x_4 = 8$$

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

Associated $A\vec{\alpha} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(\vec{\hat{x}})^T = x_1, x_2, x_3, x_4 = [42, 8, 0, 0]$ — (i)

$(\vec{x'})^T = x_1, x_2, x_3, x_4 = [1, 2, -1, 0]$ — (ii)

$$\alpha\vec{x'} + \vec{\hat{x}} = \alpha\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha+42 \\ 2\alpha+8 \\ -\alpha \\ 0 \end{bmatrix}$$ is a solution

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_2 = \frac{-a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}} \quad \Big| \quad x_2 = \frac{-a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}}$$

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

If $A\vec{x} = \vec{b}$ has one solution in n-dimensional space means they have a comman intersection point

Gallier book
→ 3.4.1
→ 3.4.2
→ 3.4.4

Propositional logic ~~can~~ cannot reason about elements of a set or structure.

First order logic (FOL)

Constants: $c_0, c_1, c_2 \ldots$
Variable: $x, y, z \ldots$
functions: $f_0, f_1, f_2 \ldots f_i$ mean f is of arity i
→ can take i variable
Predicate: $P_0, P_1 \ldots P_n$ are predicate symbols

Term in FOL

$c \in$ TERMS, c is a constant
$x \in$ TERM, x is a variable

If f_n is a function with arity 'n'
then $f_n(t_1, t_2 \ldots t_n) \in$ TERMS

$t_1, t_2, t_3 \ldots \in$ TERMS

Predicate: Any that evaluates to → True / → False

True → $\top, \bot \in$ PRED ← False

(or) PREDICATE formula

P_n is a predicate symbol then $P_n(t_1, t_2 \ldots t_n)$ is a new Predicate where $t_1, t_2 \ldots t_n \in$ TERMS

If $P(\ldots, x, \ldots)$ is a predicate then

$\exists x\, P(x)$ and $\forall\, P(x) \in PRED$

There is atleast one student in the class whose height is more than 6ft.

$$\exists x \left[\text{in-class}(x) \wedge \text{more-than}(\text{height}(x), 6)\right]$$

Predicate

function

There is nobody in MFAI whose height is less than 4ft

$$\sim \forall x \left[\text{in-MFAI}(x) \supset \text{less-than}(\text{height}(x, 4))\right]$$

Domain

Always

$$\forall x (\text{In-MFAI}(x) \supset \text{happy}(x))$$

$$\exists y (\text{In-MFAI}(y) \wedge \text{Know swimming}(x))$$

Always

[$\forall x$: Universal quantification]

[$\exists x$: Existential quantification]

$$\forall x \quad x > 0 \supset Q(x)$$

$\{1, 2, 3\}$

$1 > 0$; $Q(1)$

$2 > 0$, $Q(2)$

$3 > 0$, $Q(3)$

$\{-1, -2, -3\}$

$\forall x\, x > 0 \supset Q(x)$: TRUE

$\exists x\, [x > 0 \wedge Q(x)]$: FALSE

$>(x, 0)$

There are two coins
Sum of their values is 55 paise
one of these is not a 50 paise coin
what are coins?

$\exists x, \exists y \cdot Coin(x) \wedge coin(y) \wedge plus(x, y, 55) \wedge (NE(x, 50) \vee NE(y, 50))$

$x = 50, y = 5$

In MFAI, class every Student has a friend in the Same class.

every Student $\to \forall x$, a $\to \exists y$

$\forall x \, \exists y \; in\text{-}MFAI(x) \supset \{ InMFAI(y) \wedge friend\text{-}of(y\text{-}x) \}$

Lecture-09 (MFAI) | (21-08-24)

Elementary Transformations on $\boxed{SLE}$

SLE: System of Linear Equations

1. Exchange of any two equation keep the Solution ~~Space~~ set Same.
2. Multiplication of a scalar in any of the equation will keep the Solution set same.
3. Addition of any two equation Keeps the solution set Same.
4. Combination of 1, 2, and 3 Keeps the Solution set Same.

$$a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n = b_m$$

$A \in \mathbb{R}^{m \times n}$

$\vec{x} \in \mathbb{R}^n$

$\vec{b} \in \mathbb{R}^m$

$A\vec{x} = \vec{b}$ is the compact form

$A\vec{x} = \vec{b}$ is the compact form

with a series of elementary transformation

obtain $A'\vec{x}' = \vec{b}'$

then $A\vec{x} = \vec{b} \equiv A'\vec{x} = \vec{b}'$ and $A \neq A'$.

$\vec{b} \neq \vec{b}$

- System of linear equations has three possibilites
  - (i) Inconsistent
  - (ii) More than '1' solution
  - (iii) Unique solution.

Properties:

① For $A\vec{x} = \vec{b}$, $A \in \mathbb{R}^{n\times n}$, $\vec{x}, \vec{b} \in \mathbb{R}^n$

if A^{-1} exists the A^{-1} is unique

Proof: Let B and C are the inverse of A

$\Rightarrow AB = I$

$\Rightarrow C(AB) = C(I)$

$\Rightarrow (CA)B = C$

$\Rightarrow IB = C$

$\Rightarrow \boxed{B = C}$

② Given $A\vec{x} = \vec{b}$, A is invertible then $A\vec{x} = \vec{b}$ has a unique solution.

(Proof in the next page)

Proof: Let $\vec{x'}$ and $\vec{x''}$ are two solutions

$\Rightarrow A\vec{x} = \vec{b} \Rightarrow \vec{x'} = A^{-1}\vec{b}$ | $\vec{x'} = A^{-1}\vec{b}$, $\vec{x''} = A^{-1}\vec{b}$

$\Rightarrow \vec{x'} = \vec{x''}$

Solve $A\vec{x} = \vec{b}$ were A is $\boxed{\text{invertible}}$ → Inverse of A exists.

Job will be easy if

(I) find a series of elementary transformations where $A\vec{x} = \vec{b}$ becomes $I\vec{x} = \vec{b'}$

(II) find a series of transformations where $A = LU$ is a lower triangular and U is a upper triangular matrix

Property:

If A is $\boxed{\text{invertible}}$ then → Inverse of A exists [~~Invers~~ L and U exists if A^{-1} exists]

Find L and U where $A = LU$

It means solving $A\vec{x} = \vec{b}$

$(LU)\vec{x} = \vec{b}$

$\Rightarrow L(U\vec{x}) = \vec{b}$ → $\boxed{\begin{array}{l} L\vec{y} = \vec{b} \\ U\vec{x} = \vec{y} \end{array}}$ —(i) —(ii)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$x_1 = 2, x_2 = 3, x_3 = 1$

Not identity matrix

Some elementary transformations

$R_1 \leftrightarrow R_3$ (swap)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

If A is $\boxed{\text{invertible}}$ then always we can get $PA = LU$, where P is a permutation of identity matrix I

Inverse of A exists

Given $A\vec{x} = \vec{b}$ and A is invertible

$\Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{b}$

$\Rightarrow I\vec{x} = A^{-1}\vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \vdots & 0 & 0 & 1 & 0 \\ a_{m1} & a_{m2} & & a_{mn} & 0 & 0 & 0 & 1 \end{array}\right]$$

by some elementary transformation

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & 0 & a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 & a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 & a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right]$$

$A\vec{x} = \vec{b}$

$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b} = \vec{b'}$

$$= \begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_m' \end{bmatrix}$$

Row Echelon form

Given $A\vec{x} = \vec{b}$ is a row-echelon form then

(I) all equation with zero Coefficent is at the bottom

(II) the $\boxed{\text{first non-zero}}$ coefficent of a row from the bottom is to the right if first non-zero coefficient of a row above it

This non zero coefficient is called pivot.

Reduced Row-Echelon form

(i) It is in row echelon form

(ii) the pivot element value is '1' and it is only non zero element in its column

$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lecture - 09 | Part B

You cannot find someone in the class who is happy.

$\neg \exists x \; class(x) \wedge happy(x)$

Everyone in the class is unhappy

$\forall x \; class(x) \rightarrow \neg happy(x)$

$\neg \exists x \; happy(x)$

$\forall x \; \neg happy(x)$

$\forall x \; \neg(class(x) \wedge happy(x))$

$\equiv \forall x \; \neg class(x) \vee \neg happy(x)$

$\equiv \forall x \; class(x) \rightarrow \neg happy(x)$

$\left\{ \begin{matrix} P \rightarrow Q \\ \neg P \vee Q \end{matrix} \right\}$

$$\boxed{\begin{matrix} \neg \exists x \; P(x) \equiv \forall x \; \neg P(x) \\ \neg \forall x \; P(x) \equiv \exists x \; \neg P(x) \end{matrix}}$$

Negation change $\forall$ to $\exists$ and vice versa

- There is someone in the class who is good in all sports activities.

  $\exists x \forall y \; class(x) \wedge [sport(y) \rightarrow good(y, x)]$

  ~~$\exists x \; class(x) \wedge goodinallsports(x)$~~

  $\exists x \; class(x) \wedge \{\forall y \; Sport(y) \rightarrow good(y, x)\}$ $\quad$ x is good in y

- Any student who has taken MFAI has a friend who is not in MFAI calass.

  $\forall x \; \exists y \; MFAI(x) \rightarrow [friend(y, x) \wedge \neg MFAI(y)]$

  $\forall x \; MFAI(x) \rightarrow \{\exists y \; friend(y, x) \wedge \neg MFAI(y)\}$

  $\exists x \cdot class(x) \wedge \{\forall y \cdot spob(y) \rightarrow good(y, x)\}$