Push-Relabel Algo $\longrightarrow O(n^2 m)$

Goldberg & Tarjan $\longrightarrow O(n^3)$

J.B Orlin — STOC 2013 — $O(mn)$

Chen et al. — FOCS 2022 — $O(m^{1+o(1)})$

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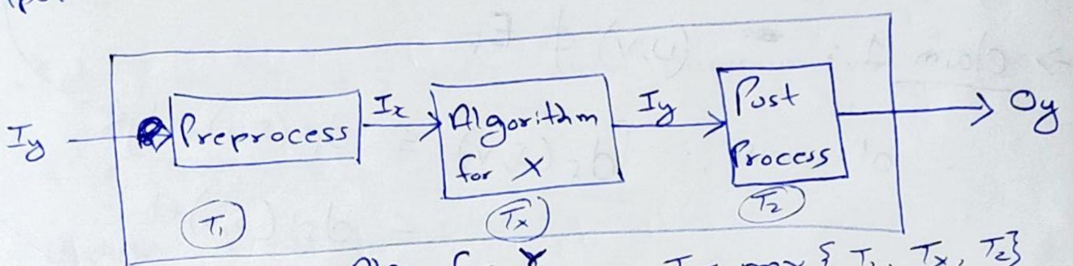
* polynomial time Reductions :

Problem X and Problem Y

input I_x input I_Y

output O_x Output O_y



Algo for Y $T = \max\{T_1, T_x, T_2\}$

Y reduces to X

Q:- Can arbitrary instances of problem Y be solved using a polynomial number of standard computational steps and a polynomial no. of calls to a black-box that solves problem X?

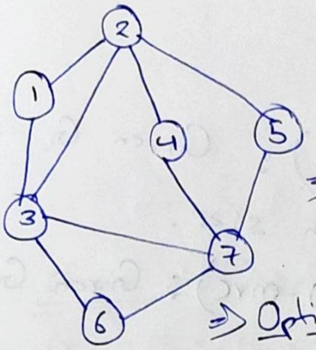
YES $\rightarrow$ $Y \leq_p X$ → polynomial time reduction to X.

eg: → $\boxed{MBMP \leq_p MF}$

Fact 1: If X is solvable in polynomial time then Y is also solvable in polynomial time.

Fact 2: If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

* **Independent Set:** Given a graph $G(V, E)$ an independent set $S \subseteq V$ such that no two vertices in S are joined by an edge.



$S_1 = \{3, 4, 5\}$

$S_2 = \{1, 4, 5, 6\}$

=> Maximum Independent Set (MIS): Independent set of max^n size. Given G, find an independent set of max^n size.

=> Optimization of MIS (O-MIS): Given G, what is the size of the MIS?

=> Decision version of MIS (D-MIS): Given G & $K > 1$, Does G contains an independent set of size at least K.

Claim 1: O-MIS $\leq_p$ D-MIS

Proof:

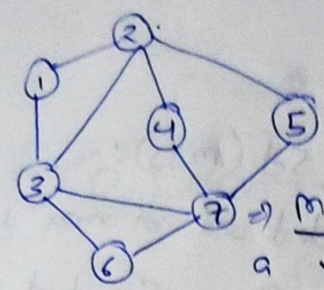
```
O-MIS(G) {
    K = 1
    while (D-MIS(G, K)) {
        K = K+1
    }
    return K-1
}
```

Claim 2: D-MIS $\leq_p$ O-MIS

Proof:

```
D-MIS(G, K) {
    S <- O-MIS(G)
    if S >= K
        return YES
    else
        return NO
}
```

* **Vertex Cover:** Given a graph $G(V, E)$, a set $C \subseteq V$ is a vertex cover if for every edge $e = (u, v)$ in G, either $u \in C$ or $v \in C$.



$C_1 = \{1, 2, 6, 7\}$

$C_2 = \{2, 3, 7\}$

⇒ Minimum Vertex Cover (MVC): Given G, find a vertex cover of minimum size.

⇒ Optimization Version of MVC (O-MVC): Given G, what is the size of min^m vertex cover.

⇒ Decision version of MVC (D-MVC): Given G & K, does G contains a vertex cover of size at most K?

Theorem: Let $G(V, E)$ be a graph then, S is an independent if & only if $V \setminus S$ is a vertex cover.

Proof: ① Suppose S is an independent S

Let $e = (u, v)$ be any arbitrary edge. Then either $u \notin S$ or $v \notin S$. This implies atleast one of u or v is in $V \setminus S$. Since e is any arbitrary edge, $V \setminus S$ is a vertex cover.

② Suppose $V \setminus S$ is a vertex cover.

Lemma 1: $D\text{-}MIS \leq_p D\text{-}MVC$

```
D-MIS(G, K) {
        If D-MVC(G, n-K) is YES
              Output YES
        Else
              Output NO
}
```

Lemma 2: D-MVC $\leq_p$ D-MIS

proof:

```
D-mvc(G,k){
    If D-MIS(G, n-k) is YES
        output YES
    else
        output No
}
```

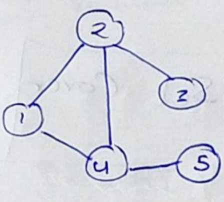
* **Set Cover**: Given a set U of 'n' elements & a collection $S_1, S_2 \ldots S_m$ of subsets of U, & a no. K. Does there exists a collection of at most K sets whose union is equal to U.

Theorem: $\underbrace{\text{Vertex Cover}}$ $\leq_p$ Set Cover

(G,K). $\longrightarrow$ $U = E$

(V,E) $\quad S_i = \{e \mid e \text{ is incident on vertex } i \in V\}$

eg:-



$S_1 = \{\{1,2\}, \{1,4\}\}, K$

$\Downarrow$

Does the reduced instance of set cover contains a set cover of size at most K.

Lemma: U can be covered with at most K of the ~~solution~~ subsets $S_1, S_2 \ldots S_n$ if & only if G contains a vertex cover of size at most K.

Proof: Suppose G has a vertex cover of size at most K.

Then $\exists$ set cover of size at most K.

Suppose we have a set cover of size at most k.

$\exists\ S_{i_1}, S_{i_2}, \ldots S_{i_k}$ such that the union of those sets covers $U = E$.

* <u>Packing Problem</u> (<u>Set Packing</u>):

Given a set U of n elements, a collection $S_1, S_2, \ldots S_m$ of subsets of U & a number k. Does there exists a collection of atleast k of these sets with the property that no two of them intersects.

<u>Theorem</u>: Independent set $\leq_p$ Set Packing

<u>Theorem</u>: If $Z \leq_p Y$ & $Y \leq_p X$ then $\boxed{Z \leq_p X}$

Independent Set $\leq_p$ Vertex Cover — ①

Vertex Cover $\leq_p$ Set Cover — ②

① & ② $\Rightarrow$ $\boxed{\text{Independent set} \leq_p \text{Set Cover}}$