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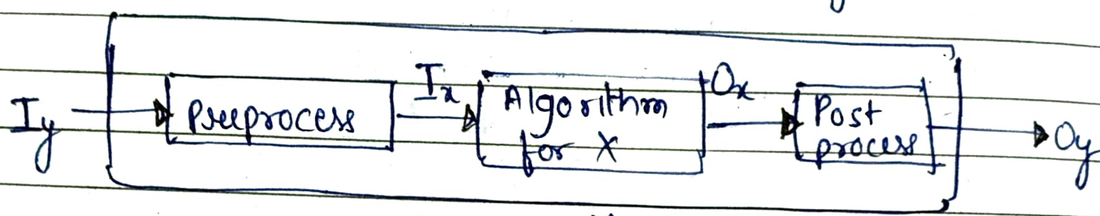
Polynomial time Reductions :-

	Problem X	Problem Y
Input	I_x	input I_y
Output	O_x	Output O_y



Algo for Y

Y reduces to X : $Y \leq_p X$

Question: Can arbitrary instances of Problem Y be solved using a $\boxed{\text{polynomial}}$ (w.r.t input size of Y) number of standard computational steps and polynomial number of calls to a black boss that solves problem X?

Yes → $\boxed{Y \leq_p X}$

Fact 1 : If X is solvable in polynomial time then Y is also solvable in polynomial time

$[MBMP \leq_p MF]^*$

Fact 2 : If Y cannot be solved in polynomial time This implies that X cannot be solved in polynomial time

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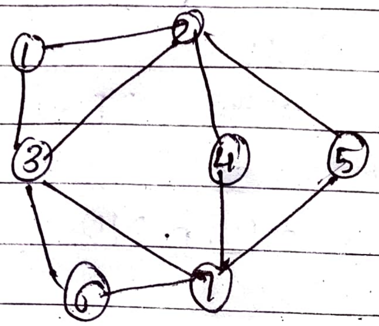
Independent set Given a graph $G(V,E)$, an independent set $S \subseteq V$ and such that no two vertices in S are joined by edge



$S_1 = \{3,4,5\}$

$S_2 = \{1,4,5,6\}$

→ Maximum Independent Set (MIS)

Given G, find an independent set of maximum ~~edg~~ size

→ Optimized Version of MIS (O-MIS)

Given G, what is the size of the maximum independent set.

→ Decision Version of MIS (D-MIS).

Given G and $k>1$, Does G contain an independent set of size $\boxed{\text{at least}}$ k ?

Claim 1: O-MIS $\le_p$ D-MIS

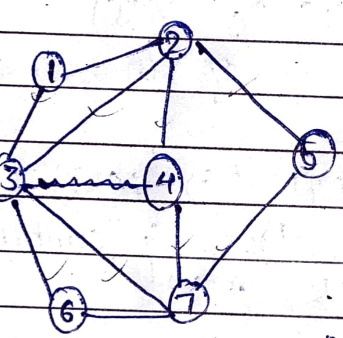
Proof:

```
O-MIS (G) {
    k=1
    while (D-MIS(G,k))
    {
        k=k+1
    }
    return k-1;
}
```

Claim 2: D-MIS $\leq_p$ O-MIS

```
D-MIS (G, K) {
        S ← O-MIS (G)
        if S ≥ K
              return YES
        else
              return NO
        }
```

* **Vertex Cover** : Given a graph $G(V, E)$, a set of $C \subseteq V$ is a vertex cover if for every edge $e = (u, v)$ in G, either $u \in C$ or $v \in C$



$C_1 = \{1, 2, 6, 7\}$
$C_2 = \{2, 3, 7\}$

- Minimum vertex cover (MVC): Given G, find a vertex cover of minimum size.

[O-MVC]

- Optimization Version of MVC : Given G, what is the size of the minimum vertex cover?

- Decision Version of MVC [D-MVC]: Given G and k, Does G consist a vertex cover of size $\boxed{\text{atmost}}$ k

Theorem: let $G(V, E)$ be a graph. Then S is an independent set $\boxed{\text{if and only if}}$ V/S is an vertex cover

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Proof : ① Suppose S is an independent set

let $e = (u, v)$ be any arbitrary edge.
Then, either $u \notin S$ or $v \notin S$. This implies atleast one of u or v is in V/S. Since e is any arbitrary edge, V/S is a vertex cover

② Suppose V/S is a vertex cover.

- lemma 1: $\boxed{\text{D-MIS} \leq_p \text{D-MVC}}$

Proof: D-MIS(G, k)
{
if {D-MVC(G, n-k) is Yes}
Output Yes
Else
Output NO
}

- lemma 2: D-MVC $\leq_p$ DMIS

Proof: DMVC(G,k)
{
if (D-MIC(G, n-k) is Yes)
Output Yes
else
Output NO
}

$\{1,2,3,4\}$ $\{1,2\}$ $\{2,3,4\}$

$\{(1,2)\ (1,4) \ldots\}$

minimization – atmost

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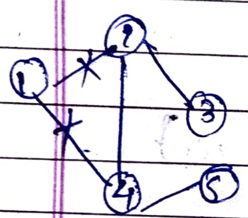
Set Cover Given a set U of n elements and a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and a number K. Does there exists a collection of $\boxed{\text{atmost}}$ k sets whose union is equal to U.

Theorem : Vertex-Cover $\leq_p$ Set Cover

V.C atmost size k $(G, k) \longrightarrow U = E$

(V, E)

$S_i = \{e \mid e \text{ is incident on vertex } i \in V\}$



$S_1 = \{(1,2), (1,4)\}, k$

$\Downarrow$

Does the reduced instance of set cover contains a set cover of size atmost k

* vertex cover containing edge

Lemma : U can be covered with atmost k of the subsets $S_1, S_2, \ldots S_n$ $\boxed{\text{if and only if}}$ G contains vertex cover of size atmost k

Proof : Suppose G has vertex cover of size atmost K Then $\exists$ set cover of size atmost K

Suppose we have a set cover of size at most k $\exists\ S_{i_1}, S_{i_2}, \ldots S_{i_k}$ such that the union of those set cover $U = E$

Packing Problem (Set Packing)

Given a set U of n elements, a collection $S_1, S_2 \ldots S_m$ of subsets of U and a number k. Does there exists a collection of atleast k of these ~~exist~~ sets with the property that no two of them intersects.

Theorem : Independent Set $\leq_p$ Set Packing.

Theorem If $Z \leq Y$ and $Y \leq_p X$ then $Z \leq_p X$

① Independent Set $\leq_p$ Vertex Cover

② Vertex Cover $\leq_p$ Set Cover

① & ② Indepent Set $\leq_p$ Set Cover