Lecture-24 (14/11/24)

The general Travelling Salesman Problem (TSP)

In the TSP, if we remove the assumption that cost function satisfies triangle property (inequality), then we cannot find any good approximation tour in polynomial time unless $P = NP$.

Theorem: If $P \neq NP$, then for any constant $\rho \geq 1$ there is no polynomial time approximation ratio ρ for general TSP.

Proof: Known result: Hamiltonian Graph (HC) is NP-Complete. Therefore if we can solve Hamiltonian Cycle problem in polynomial time then $P = NP$

Let A be a polynomial time approximation algorithm for solving TSP with approximation ratio ρ, $\rho \geq 1$

Input Instance (for HC): An undirected graph $G(V, E)$

Construction: Given $G(V, E)$ for HC problem we will construct a complete graph $G'(V, E')$ with cost function c. [$E' = \{(u, v) \mid u \in V, v \in V, u \neq v\}$]

$$c(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E \\ \rho|V|+1, & \text{if } (u,v) \in E' \setminus E \text{ or } (u,v) \notin V \end{cases}$$

Claim: G contains HC if and only if optimal general tour cost of G' is $|V|$.

use Algorithm A (ρ-Approximation algorithm) on G'

If G contains a HC, then algorithm A returns cost of TSP tour $\boxed{\leq \rho|V|}$

If G does not contain a Hamiltonian cycle, the optimal tour must use atleast one edge from E'/E. therefore in this case, cost of any optimal tour is

$$\geq (\rho|V|+1) + (|V|-1)$$

$$= (\rho+1)|V|$$

rho

$$\boxed{> \rho|V|}$$

(HC)

→ If G contains a Hamiltonian cycle, then cost returned by A $\leq \rho|V| \Rightarrow$ "YES" instance for H.C

→ If G does not contain Hamiltonian cycle, then cost returned by A $> \rho|V| \Rightarrow$ "No" instance

$\therefore$ using A we can solve Hamiltonian Cycle

If $P \neq NP$, Such an algorithm A must not exist

K-Center Problem

Input: Input is an undirected complete graph $G(V,E)$

- A distance $d(i,j)$ is associated between every pair of vertices $i,j \in V$.
- The distance function satisfies the following
  - (a) $d(i,i) = 0 \quad \forall i \in V$
  - (b) $d(i,j) = d(j,i)$
  - (c) For a triple $i,j,k \in V$

$$d(i,j) + d(j,k) \geq d(i,k) \quad \text{Triangle inequality}$$

- The goal is to find k clusters grouping together vertices that are most similar into some clusters. We select $S \subseteq V$, $|S| = k$ ~~dista~~
- Distance of a vertex i from S is

$$d(i,S) = \min_{j \in S} \{d(i,j)\}$$

- Radius of S is $= \max_{i \in V} \{d(i,S)\}$

The goal of the k-center problem is to find set S of minimum radius.

Greedy-K-Center(G)

1. $S \leftarrow \phi$
2. Pick any arbitrary vertex $i \in V$ and $S \leftarrow \{i\}$
3. While $|S| < K$ do

$$j \leftarrow \arg\left\{\max_{i \in V}\{d(i,S)\}\right\}$$

$$S \leftarrow S \cup \{j\}$$

Theorem: Algo Greedy-K-center is a 2-approximation algorithm for K-center problem

Proof:

Let $S^* = \{j_1, j_2, \dots, j_K\}$ denote the optimal solution.

Let r^* denote the radius of the optimal solution.

Consider the clusters $V_1, V_2 \dots V_K$ created by the optimal solution where

$$V_i = \{j \mid d(i,j) \leq r^*\}$$

Claim: Any pair of vertice j and j' in the same cluster has a distance at most $2r^*$

Proof: suppose j and $j' \in V_i$

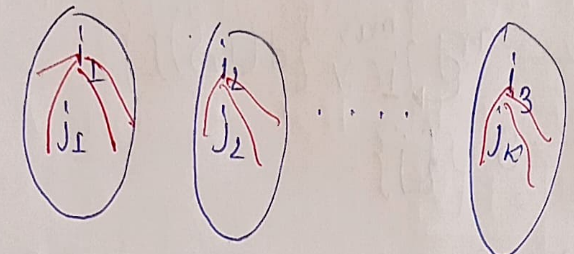
$$d(j,i) \leq r^*$$

$$d(j',i) \leq r^*$$

$$\left.\begin{aligned} d(j,j') &\leq d(j,i) + d(i,j') \\ &= d(j,i) + d(j',i) \\ &\leq r^* + r^* \\ &= 2r^* \end{aligned}\right\} d(j,j') \leq 2r^*$$

Let $S \subseteq V$, $|S| = K$ be the centers selected by the greedy algorithm

<u>case 1</u>: Every center in S in within a single cluster formed by the optimal solution



V_1 V_2 V_K

→ Greedy algorithm finds a solution with radius $\leq 2r^*$

<u>case 2</u>: Suppose two center i_p and i_q selected by greedy in S we within the same cluster V_j

$\because$ Our algorithm is greedy and $d(i_p, i_q) \leq 2r^*$ all points are at a distance of atmost $2r^*$ from some centers in S.

$$R(\text{Greedy}) \leq 2 \cdot R(\text{Optimal})$$

<u>Theorem</u>: If $P \neq NP$, then for any constant ρ, $1 \leq \rho < 2$, there is no polynomial time approximation algorithm with approximation ratio ρ for k-center problem.

<u>Proof</u>:

Dominating set: we are given a graph $G(V, E)$ and integer K. we want to decide if there exists a set $S \subseteq V$ of size $\leq K$. such that each vertex is either in S or adjacent to a vertex in S.

Dominating set is NP Complete.

Suppose, A is a polynomial time algorithm with approximation ratio ρ, $1 \leq \rho < 2$ for K-center problem.

Construction: Given $G(V,E)$ and K for dominating set, we will construct a complete graph $G'(V,E')$ with the following distance function

$$d(i,j) = 1 \quad \text{if } (i,j) \in E,\ i \neq j$$
$$\quad\quad = 2 \quad \text{if } (i,j) \in E'/E,\ i \neq j$$

$$d(i,i) = 0 \quad \forall i \in V$$

Instance for K-center: $G'(V,E'), K, d$

We will use algorithm A on the K-center problem instance.

Claim: G contains a dominating set of size K if and only if K-center instance (G', K, d) then radius ≤ 1.

If G has a dominating set size K, the A on (G', K, d) returns radius $\leq \rho < 2 \Rightarrow$ "YES" instance

If G does not contain a dominating set of size K, any optimal solution for K-center on (G', K, d) for radius $\geq 2 \Rightarrow$ "NO" instance

So if $P \neq NP$ such a algorithm A cannot exists.

Mostly Post mid sem

Greedy
Matroid
Amortized

not be included

But in the mail it is not mentioned the Greedy will be excluded

So I am not sure.