Mathematical Foundation of AI

(CS6LXXXX; L-T-P: 3-0-0; Credits: 4; IIT Bhubaneswar)

Syllabus (Part - Linear Algebra):

Advanced Linear Algebra, Eigenvalues and eigenvectors, Singular value decomposition, and Matrix approximation

Faculty: Santosh Mohanty

Textbook Referred: Mathematics for Machine Learning (Cambridge University Press, 2020 publication), Authored by – M. P. Deisenroth, A. A. Faisal, and C. S. Ong.

Lecture and Topics Covered (Lecture Hall, Lalit Giri, 1st Floor)

Mid Term coverage: Lecture 01 to Lecture 09 (Lecture 01 excluded from exam perspective)

Final Term coverage: Lecture 10 to 17 (Lecture 12 excluded from exam perspective)

For MID TERM:

Lecture 01: 2024-07-31/Wednesday/11 AM to 12 Noon (01 Hour)

- Motivation to learn Mathematics/Quantitative Techniques essential for AI/ML
- Occupational Therapy as a use case

Lecture 02: 2024-08-07/Wednesday/10 AM to 11 AM (01 Hour)

- Algebra to Linear Algebra with geometric interpretation
- Concept of Matrix and Matrix Inversion

Lecture 03: 2024-08-12/Monday/2:30 PM to 3:30 PM (01 Hour)

- Linear Equations, System of Linear Equations, Solution Types (inconsistent, many solutions, unique solution), Geometric Interpretation

Lecture 04 & 05: 2024-08-14/Wednesday/10 AM to 12 Noon (02 Hours)

- Matrix, Transpose of a Matrix, Inverse of a Matrix, Algebraic properties of a Matrix, its transpose and inversion

Lecture 06: 2024-08-21/Wednesday/10 AM to 11 AM (01 Hours)

- Elementary transformation associated with SLEs, Row-Echelon Form of a Matrix, Reduced Row-Echelon Form of a Matrix, Gaussian Elimination, Obtaining inverse of a matrix through Gaussian Elimination
- LU factorization, Permutation matrix P, PA = LU for every invertible A

Lecture 07: 2024-08-28/Wednesday/10 AM to 11 AM

- Groups, Vector Space as a Group
- General Linear Group

Lecture 08 & 09: 2024-09-04/Wednesday/10 AM to 12 Noon (02 Hours)

- Linear Combination, Linear Independency, and Linear Dependency
- Generating Set and Span of Vector Space
- Minimal Generating Set, Basis, and Rank

For FINAL TERM:

Lecture 10: 2024-09-23/Monday/2:30 PM to 3:30 PM

- Structure preserving mapping in Vector spaces
- Homomorphism, Linear Mapping
- Linear Mapping properties – Injective, Surjective, and Bijective
- Isomorphism, Endomorphism, and Automorphism

Lecture 11: 2024-09-25/Wednesday/10 AM to 11 AM

- Norms
- Bilinear Mapping
- Inner Product and Inner Product Spaces

Lecture 12: 2024-10-07/Monday/2:30 PM to 3:30 PM

- Data Management Reference Architecture – Data Life Cycle, Data Analysis, and Data Management Platform (*Side Note: Student attendance was 25 – the day before the start of puja holiday break. Did not cover the topic from MFAI subject as the subsequent lectures would have critical dependency on the coverage. Covered an important topic of general interest to any student while dealing with data and data analysis.*)

Lecture 13: 2024-10-14/Monday/2:30 PM to 3:30 PM

- Distance, Angle, and Orthogonality
- Orthogonal and Orthonormal Vectors
- Matrix Interpretation/Representation of Orthogonality and Inner Product
- Properties of Orthogonal Matrix and Orthonormal Basis

Lecture 14: 2024-10-16/Wednesday/11 AM to 12 Noon

- Determinant and Geometrical Interpretation
- Laplace Expansion to compute the value of the determinant
- Determinant of an Orthogonal Matrix
- Properties of Determinant
- Similar Matrices

Lecture 15: 2024-10-21/Monday/2:30 PM to 3:30 PM

- Characteristic Polynomial, Eigenvalues and Eigenvectors, Eigenvalues with Multiplicity,
- Relationship between Determinant and Eigenvalues

Lecture 16 & 17: 2024-10-23/Wednesday/10 AM to 12 Noon (02 Hours)

- Eigenvalues and Eigenvectors of a symmetric matrix
- Diagonalization of a symmetric matrix and properties
- Singular Value Decomposition (SVD)
- Singular Value Decomposition (SVD) of the symmetric matrix A^TA
- SVD as a Tool in Machine Learning – Data Compression, Image Compression and Recognition, Noise Reduction, Signal Filtering, Feature Extraction, Dimensionality Reduction, Constraint Optimization, etc.
- A brief on the concept behind Google Search Engine's Page Rank algorithm

FINAL TERM – Guidance for Exam Preparation

Topics	Exam Preparation
• Structure preserving mapping in Vector spaces • Homomorphism, Linear Mapping • Linear Mapping properties – Injective, Surjective, and Bijective • Isomorphism, Endomorphism, and Automorphism	✓ Define Linear Mapping in a Vector Space ✓ Demonstrate Linear Mapping by citing an example where the mapping is from R^2 to R^3 ✓ State the conditions for a Linear Mapping to be Injective, Surjective, and Bijective ✓ Define Homomorphism, Isomorphism, Endomorphism, and Automorphism. Cite examples in R^3
• Norms • Bilinear Mapping • Inner Product and Inner Product Spaces	✓ Define Norm in a Vector Space ✓ Explain Manhattan Norm, Euclidean Norm, and Max Norm with Geometric Interpretation ✓ Define Bilinear Mapping in a Vector Space ✓ State Bilinear Mapping that is symmetric ✓ State Bilinear Mapping that is positive definite ✓ Define Inner Product and Inner Product Space ✓ Share an example of Inner Product in R^3
• Distance, Angle, and Orthogonality • Orthogonal and Orthonormal Vectors • Matrix Representation of Orthogonality and Inner Product	✓ Define Distance Function in a Vector Space ✓ State when two vectors are orthogonal and orthonormal ✓ Demonstrate two symmetric matrices in R^2 where one of them is positive definite and the other one is not positive definite ✓ Define an orthogonal matrix

• Properties of Orthogonal Matrix and Orthonormal Basis	✓ Prove that an orthogonal matrix preserves the length of a vector ✓ Demonstrate an example of a set of vectors in R^3 (other than the columns of an Identity Matrix) that represents an orthonormal basis for R^3
• Determinant and Geometrical Interpretation • Laplace Expansion to compute the value of the determinant • Determinant of an Orthogonal Matrix • Properties of Determinant • Similar Matrices	✓ Define determinant of a matrix A (det (A)) ✓ If A represents an invertible matrix of size N x N, find the Laplace Expansion of the **det (A).** Also, prove that ○ $\det(A^T) = \det(A)$ ○ $\det(A^{-1}) = 1 / \det(A)$ ○ $\det(\alpha A) = \alpha^N \det(A)$, for any α in R ○ Determinant of an Orthogonal Matrix is ±1 ✓ Define the trace of a square matrix A – **trace (A)** ✓ Prove that trace (αA) = α trace (A), for any α in R ✓ Given two matrices A and B in R^{NxN}, prove that ○ det (AB) = det (A) x det (B) = det (BA) ○ trace (A + B) = trace (A) + trace (B) ○ trace (AB) = trace (BA) ✓ Given two matrices A and B in R^{NxN}, state the condition for A and B to be similar. ✓ Prove that det (A) = det (B), if A and B are similar
• Characteristic Polynomial, Eigenvalues and Eigenvectors, Eigenvalues with Multiplicity, • Relationship between Determinant and Eigenvalues	✓ For a square matrix A in R^{NxN}, define Characteristic Polynomial, Eigenvalues, and Eigenvectors of A. ✓ Define algebraic multiplicity of an eigenvalues and cite examples ✓ Prove that for a square matrix A, both A and A^T have the same eigenvalues but the associated eigenvectors may be different. ✓ For a square matrix A in R^{NxN} with its Characteristic Polynomial $P_N(\lambda) = \det(A - \lambda I)$ as: $C_0 + C_1 \lambda + C_2 \lambda + \dots\dots\dots + C_{N-1} \lambda^{N-1} + (-1)^N \lambda^N$, prove ○ $C_0 = \det(A)$ ○ $C_{N-1} = (-1)^{N-1} \text{trace}(A)$ ○ $\det(A) = \prod \lambda_{i,}\ i = 1 : N$

	○ trace (A) = $\sum \lambda_{i,}$ i = 1 : N
• Eigenvalues and Eigenvectors of a symmetric matrix • Diagonalization of a symmetric matrix and properties • Singular Value Decomposition (SVD) • Singular Value Decomposition (SVD) of the symmetric matrix ATA	✓ If A is symmetric, prove that all the eigenvalues of A are real and there exists a set of associated eigenvectors of A that forms an orthonormal basis of the corresponding vector space ✓ If A is symmetric and positive definite, then all the eigenvalues are real and positive. ✓ Define the condition for a matrix to be diagonalizable ✓ If A is a symmetric matrix, prove that it is diagonalizable and there exists a diagonal matrix D and a matrix P representing an orthonormal basis, where $A = PDP^{-1}$, the diagonal entries of the diagonal matrix D are the eigenvalues of A and P is consisting of the associated eigenvectors ✓ Define the Singular Value Decomposition (SVD) of a matrix A in R^{MxN} ✓ For a matrix A in R^{MxN}, define the matrix $B = A^TA$. Then prove that the set of singular values of B is same as the set of eigenvalues of B.