Positive Definite

$$\Omega(\vec{x}, \vec{x}) = \vec{x}^T \vec{x} = \sum_{i=1}^{n} x_i^2 \begin{cases} > 0 & \text{if } \vec{x} = \vec{0} \\ = 0 & \text{if } \vec{x} = \vec{0} \end{cases}$$

Hence Ω is positive Definite.

~~Lecture-15(B)~~

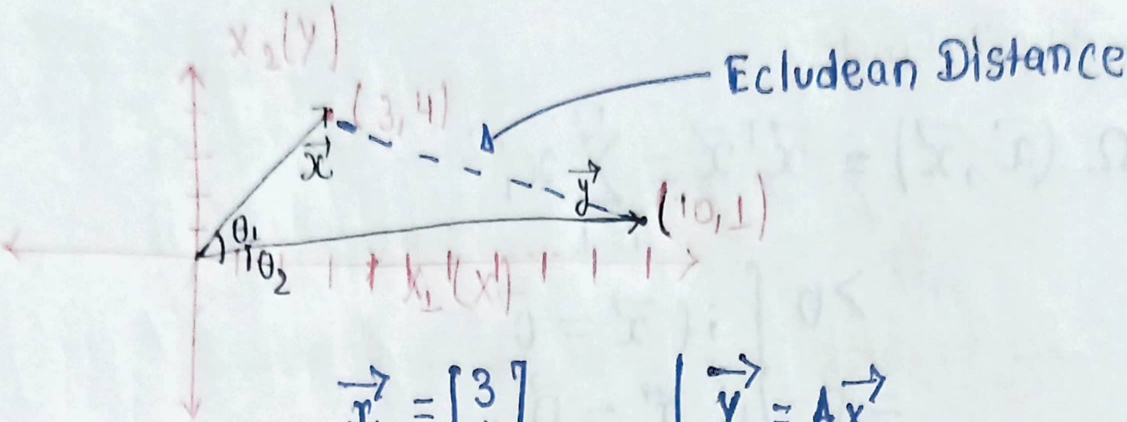
~~X X X X X X X X X~~

~~Convex Optimizations~~

Lecture - 18 (07/10/24)

Lecture-19

- Distance function
- Orthogonal vectors
- Orthonormal vectors
- Orthogonal matrix



$$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$$

$$\vec{y}_1 = A_1 \vec{x} = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -13 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\vec{y} = A\vec{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

① $d : \vec{V} \to \mathbb{R}$ is a distance function

iff $\forall \vec{x}, \vec{y} \in V$ and $\alpha \in \mathbb{R}$

(i) $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$

(ii) $d(\vec{x}, \vec{y}) = \begin{cases} 0, & \text{if } \vec{x} = \vec{y} \\ >0, & \text{if } \vec{x} \neq \vec{y} \end{cases}$

(iii) $d(\alpha\vec{x}, \alpha\vec{y}) = |\alpha| d(\vec{x}, \vec{y})$

② Orthogonal vector:

$\vec{x}, \vec{y} \in \vec{V}$ are orthogonal
if $\langle \vec{x}, \vec{y} \rangle = 0$ and $\vec{x}, \vec{y} \neq \vec{0}$

Inner product

In addition,

③ $\vec{x}, \vec{y}$ are orthonormal if $||x|| = 1$
$||y|| = 1$

④ Orthogonal Matrix

A matrix $A \in \mathbb{R}^n$ is orthogonal if

(i) the column vectors of A are orthonormal

(ii) $AA^T = I$

(iii) $A^T = A^{-1}$

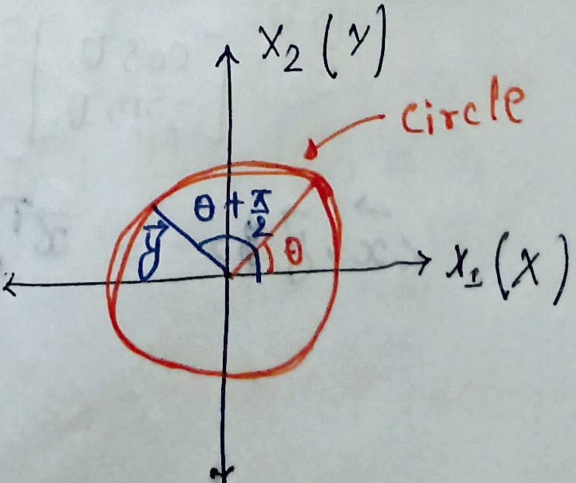
$\Rightarrow AA^T = I = A^TA$

Example:

$V = \mathbb{R}^2$

$\vec{x} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$

$||\vec{x}|| = \sin^2\theta + \cos^2\theta$
$= 1$



$$\vec{y} = \begin{bmatrix} \sin\left(\theta + \frac{\pi}{2}\right) \\ \cos\left(\theta + \frac{\pi}{2}\right) \end{bmatrix}$$

$$\|\vec{y}\| = 1$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\vec{y} = \begin{bmatrix} \sin\left(\theta + \frac{\pi}{2}\right) \\ \cos\left(\theta + \frac{\pi}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta \underbrace{\cos\frac{\pi}{2}}_{0} + \cos\theta \underbrace{\sin\frac{\pi}{2}}_{1} \\ \cos\theta \underbrace{\cos\frac{\pi}{2}}_{0} - \sin\theta \underbrace{\sin\frac{\pi}{2}}_{1} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \qquad \underbrace{\begin{bmatrix} \sin\theta & \cos\theta \end{bmatrix}}_{\vec{x}^T} \cdot \underbrace{\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}}_{\vec{y}}$$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \sin\theta\cos\theta - \cos\theta\sin\theta$$

$$= 0$$

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = [\vec{b_1}, \vec{b_2}]$$

$$\langle \vec{b_1}, \vec{b_2} \rangle = 0$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

A is orthogonal, B is orthogonal

$$A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$B\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 7/\sqrt{2} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \longrightarrow \text{Length} = \sqrt{\vec{x}^T \vec{x}} = 5$$

$$\vec{y} = B\vec{x} = \begin{bmatrix} -1/\sqrt{2} \\ 7/\sqrt{2} \end{bmatrix} \longrightarrow \text{Length} = \sqrt{\vec{y}^T \vec{y}}$$

$$= \sqrt{\frac{1}{2} + \frac{49}{2}} = \sqrt{25} = 5$$

orthogonal matrix multiply by a ~~matrix~~ f vector preserves the length !!

$A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix.

$\forall\ \vec{x} \in \mathbb{R}^n$; let $\vec{y} = A\vec{x}$

$$\|\vec{y}\| = \sqrt{\vec{y}^T\vec{y}}$$

$$= \sqrt{(A\vec{x})^T(A\vec{x})}$$

$$= \sqrt{(\vec{x}^T A^T)(A\vec{x})}$$

$$= \sqrt{\vec{x}^T(A^T A)\vec{x}}$$

$$= \sqrt{\vec{x}^T I \vec{x}} = \|\vec{x}\|$$

Orthonormal basis

If $B = \{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$ is a basis for $V = \mathbb{R}^n$

than B is an orthonormal basis if.

B is an orthogonal matrix

$$\begin{cases} \langle \vec{b_i}, \vec{b_j} \rangle = 0 \quad i \neq j \\ \qquad\qquad = 1 \quad i = j \\ B^T = B^{-1} \end{cases}$$

▷ This limit, if it exists, is called the derivative of f at a and is denoted by $f'(a), f^{(1)}(a), \frac{d(f(a))}{dx}$

Lecture 16 - Saturday Class (Missed)

Lecture 20 (A)

Problem: Optimization (Minimize or Maximize)

$f(x), f'(x), f''(x)$

Oracle access ← Some Subroutine exists (we don't go in details)

Goal: Find a $x \in \Omega$ optimising $f(x)$

Algorithm

1. Start with an initial guess 'x'
2. While 'x' is not an optimal solution do
3. Determine a search direction p
4. $x \leftarrow x + p$.
5. end while
6. Return x

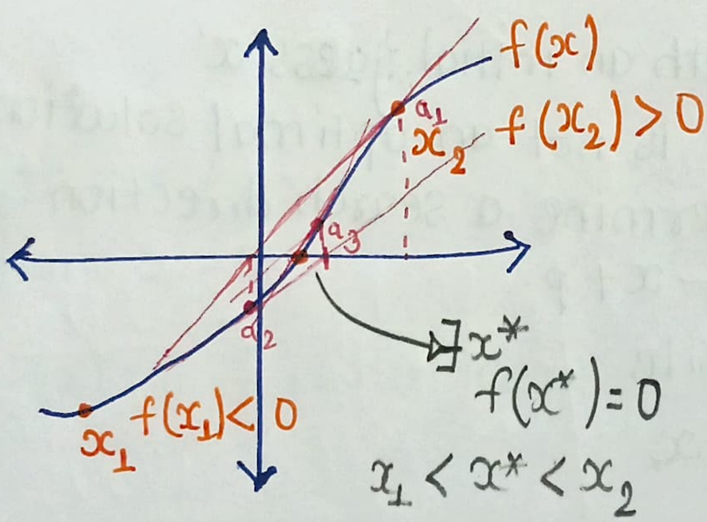
Ex.

$$f(x) = x^2 - 2$$

$$f(x) = 0$$

- Repeatedly check for local optimality
- Check if $f'(x) = 0$. and if $f''(x) \neq 0$
- Calls for finding zeroes of $f'(x)$
- Search direction p (depends on optimality check)

- Global optaimality is hard to find
- There exist efficient Solution for linear and semi-definite case



Binary search between (x_1, x_2)

Netwon's method $\longrightarrow$ Binary search

▷ Given oracle for f and f'

Goal: compute a x^* ~~saty~~ $\exists$ satisfying $f(x^*) = 0$

Algorithm:

1. Start with an initial guess 'x'
2. while $f(x) \neq 0$ and $f'(x) \neq 0$ do
3. $p \leftarrow -\frac{f(x)}{f'(x)}$; $x \leftarrow x + p$; end while
   3a 3b 3c
4. Return x

$f(x) \neq 0$

$$\frac{x_2 - x_1}{2^k} < \epsilon \qquad |f(x)| > \epsilon$$

$$\Rightarrow 2^k > \frac{x_2 - x_1}{\epsilon}$$

$$\Rightarrow k > \log_2\left(\frac{x_2 - x_1}{\epsilon}\right)$$

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Might possible if I written it wrong

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton method Analysis

⚠ Note: (study by yourself)

x_0 (initial guess)

x_k (guess after k iteration)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$e_k = x_k - x^*$$

If the initial difference,

$$e_1 \leq 1$$

$$\vdots$$

$$e_k \leq \frac{1}{2^{2^k}}$$

Lecture 20(B)

Determinent
Geometric Intrepretation
Laplace Expansion
Determinant and orthogonal matrix

For $A \in \mathbb{R}^2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\neq 0}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Determinant of A

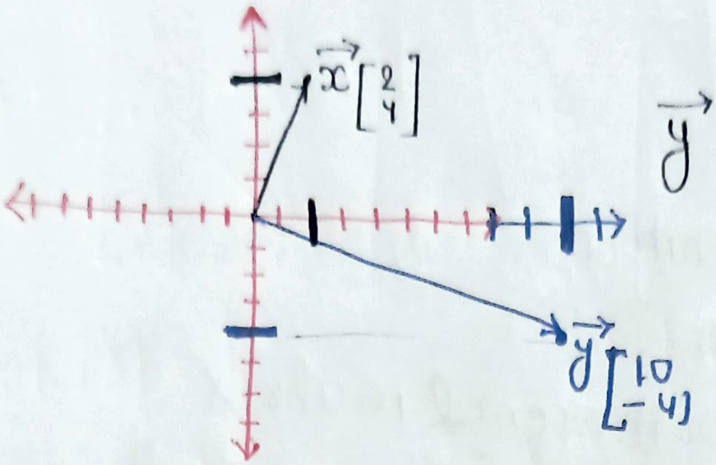
$\Delta(A) \rightarrow$ det(A) or $|A|$

Ex.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{-11}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$$

$\det(A) = -3 - 8 = -11$

$\vec{x} \in \mathbb{R}^2, \vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$



$$\vec{y} = A\vec{x}$$

$$\begin{bmatrix}1 & 2\\4 & -3\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}$$

$$= 2\begin{bmatrix}1\\4\end{bmatrix} + 4\begin{bmatrix}2\\-3\end{bmatrix}$$

$$= \begin{bmatrix}2+8\\8-12\end{bmatrix}$$

$$= \begin{bmatrix}10\\-4\end{bmatrix}$$

length of $\vec{x}$

$$\|x\|_2 = \sqrt{2^2+4^2}$$

$$= \sqrt{20}$$

length of $\vec{y}$

$$\|y\|_2 = \sqrt{10^2+(-4)^2}$$

$$= \sqrt{116}$$

$$\cancel{(\|x\|_2) \times (\|y\|_2) = |\det A|}$$

Laplace transformation of a determinant

$A \in \mathbb{R}^{n \times n}$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \text{ where}$$

'i' is any specific row of A
and 'j' varies from column 1 to n

$a_{ij} = (i, j)^{th}$ element of A

A_{ij} is a matrix of size $(n-1) \times (n-1)$ obtained by eliminating the 'ith' row and 'jth' column

A_{ij} is called minor of A

Ex.

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{bmatrix}$$

$$\det(A) = 3 \begin{bmatrix} 2 & 7 \\ 3 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 7 \\ 1 & 9 \end{bmatrix} + 5 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

11 12 13

$i = 1$

$$= 3(18-21) - 4(9-7) + 5(3-2)$$

$$= -9 - 8 + 5$$

$$= \boxed{-12}$$

$i = 2$

$$\det(A) = -1 \times 1 \times \begin{vmatrix} 4 & 5 \\ 3 & 9 \end{vmatrix} + 1 \times 2 \begin{vmatrix} 3 & 5 \\ 1 & 9 \end{vmatrix} - 1 \times 7 \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -(36-15) + 2(27-5) - 7(9-4)$$

$$= -21 + 44 - 35$$

$$= -56 + 44$$

$$= -12$$

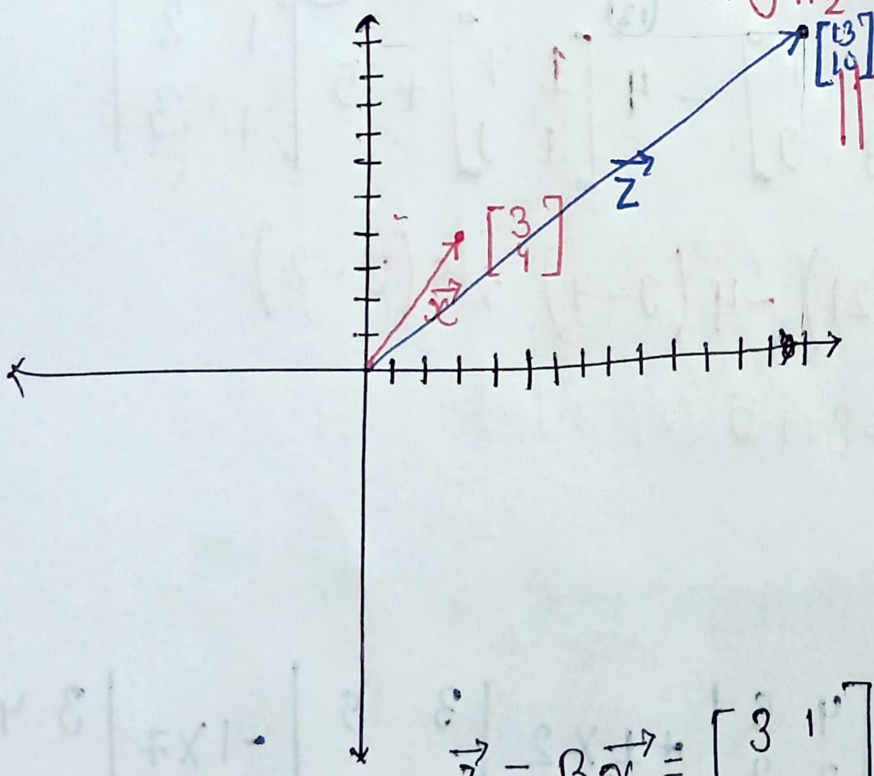
$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$\det(A) = 1$ $\qquad$ $\det(B) = 1$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \vec{y} = A\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ y \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$$

$$\text{length}(\vec{x}) = \|\vec{x}\|_2 = \sqrt{3^2 + 4^2} = 5$$

$$\|\vec{y}\|_2 = \sqrt{\frac{1+49}{2}} = 5$$



$$\|\vec{z}\|_2 = \sqrt{(13)^2 + 10^2} = \sqrt{269}$$

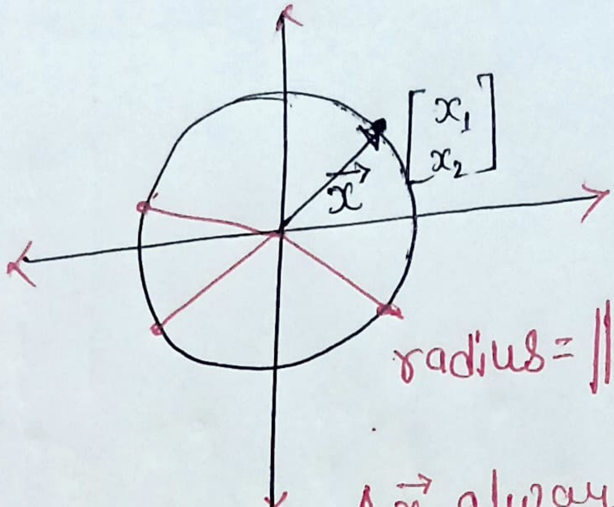
16.something

$$\vec{z} = B\vec{x} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 10 \end{bmatrix}$$

- If A is an orthogonal matrix ($A \in \mathbb{R}^{n \times n}$)

$\det(A) = 1$

$A\vec{x}$ and $\vec{x}$ have the same length



radius $= \|x\|_2$

$A\vec{x}$ always lie on the circumference of the circle if A is orthogonal matrix