Lecture - 12 : (MFAI | 02-09-24)

$\boxed{\text{2nd implied 1st}}$

1st ... Imply ... 2nd

$\boxed{append(cons(A, L_1), L_2, cons(A, L_3))}$:- $\boxed{append(L_1, L_2, L_3)}$

$append(nil, L_1, L_1)$

:- append((cons(a, cons(b, nil))), cons(b, cons(c, nil)), z)

negation.

goal clause

append([b, nil],
[a, b, c, nil]
→ [b, a, b, c, nil]

cons(a, [b, c, nil])
⇒ [a, b, c, nil]

written as

a : [b, c, nil]

simplified

$append(A' : L_1', L_2', A' : L_3')$:- $append(L_1', L_2', L_3')$

∨ ¬

$append(nil, L_1, L_1)$

:- $append(a : b : nil, b : c : nil, z)$

¬

$A \to a$
$L_1 \to b : nil$
$L_2 \to b : c : nil$
$z \to A : L_3 = a : L_3$
} unification

resolution

$\neg append(b : nil, b : c : nil, L_3)$

$A' \to b$
$L_1' \to nil$
$L_2' \to b : c : nil$
$L_3 \to A' : L_3' = b : L_3'$

$\neg append(nil, b : c : nil, L_3')$

{□}

$L_1 \to b : c : nil$
$L_3' \to b : c : nil$

$$Z \rightarrow a : L_3$$
$$\downarrow$$
$$a : b : L_3'$$
$$\downarrow$$
$$a : b : b : c : nil$$
$$Z = [a, b, b, c, nil]$$

append $(A : L_1, L_2, A : L_3)$:- append (L_1, L_2, L_3)
append (nil, L_1, L_1)
:- append (Cons(a, Cons(b, nil)) y, cons(a, b, c, nil))

$$\Downarrow$$

In First order logic

$$\forall L_1, L_2, L_3, A \ \ append(L_1, L_2, L_3) \longrightarrow append(A : L_1, L_2, A : L_3)$$
$$\wedge \forall L_1, L_2. \ append(nil, L_1, L_1)$$
$$\wedge \forall y \ \ \neg append(a : b : nil, y, a : b : c : nil)$$

$$\neg append(x, y, b : c : nil)$$

Natural language statement
↓
1st order logic
↓
Prolog program
↓ Resolution/unification
$\{\square\}$ — null clause
(refutation of computation)
↑
refusal

~~Program~~ Prolog Program : Horn Clause Programming

$(\neg A \vee \neg B \vee C)$ ← positive
$(\neg D \vee \neg E \vee F)$ ← positive

$\boxed{\neg \text{Goal Clause}}$ ← negative to save computational power.

Resolution and Unification

General Linear Group:

$\mathbb{A} = \{A\}$, $A \in \mathbb{R}^{n \times n}$ and A is invertible
then $\mathbb{A}$ is a Group with respect to matrix multiplication
This is not an abelian group.
usually, it is noted by
$GL(n, \mathbb{R})$

Linear Combination, Linear Independence and Linear Dependent:

Consider a set of vectors $\vec{x}_1, \vec{x}_2 \cdots \vec{x}_k \in \mathbb{R}^n$
If any vector $\vec{v}$ which can be expressed as
$\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_k\vec{x}_k$ where $\alpha_i \in \mathbb{R}$
then $\vec{v}$ is a linear combination of $\{\vec{x}_1 \cdots \vec{x}_k\}$

The set of vectors $\{\vec{x}_1, \vec{x}_2 \cdots \vec{x}_k\}$ is called linearly independent
if $\alpha_1\vec{x} + \alpha_2\vec{x}_2 + \cdots \alpha_k\vec{x}_k = \vec{0} \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = \vec{0}$
otherwise the set is called linearly dependent.

Example:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\vec{x}_1, \vec{x}_2, \vec{x}_3 \in \mathbb{R}^4$$

$$2\vec{x}_1 - \vec{x}_2 - \vec{x}_3 = \vec{0}$$

$$\alpha_1 = 2, \alpha_2 = -1, \alpha_3 = -1$$

Therefore $A = \{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$ are linearly dependent.

Principle:

1. Reduce the set of vectors to a row-echelon (or reduced row-echelon form).
- If row-echelon (or reduced row echelon) has pivot columns only then the vectors are linearly independent. otherwise the set of vectors are linearly dependent

Example 1:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \\ 2 & 2 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & -5 & 5 \\ 0 & -2 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -5 & 5 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x_3} = 2\vec{x} + (-1)\vec{x}$$

$$2\vec{x_1} - \vec{x_2} - \vec{x_3} = \vec{0}$$

are linearly dependent

Example 2:

$$\vec{x_1} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{x_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{x_3} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 5 \\ 2 & 1 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ 0 & -3 & -4 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 7/5 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix}$$

this set is linear dependent only if

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 7/5 \\ 4/5 \\ 1 \end{bmatrix} = \vec{0}$$ where atleast one i $\alpha_i \neq 0$

$\alpha_1 + \frac{7}{5}\alpha_3 = \vec{0}$, $\alpha_1 = \vec{0}$

$\alpha_2 + 4/5\,\alpha_3 = 0$, $\alpha_2 = 0$

$\alpha_3 = 0$

$\exists$ no nonzero α_i for which the linear combination gives $\vec{0}$

$\Rightarrow \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ are linearly independent

Generating Set, Span, minimal generating set, ~~Bases~~ Basis, Rank

Consider a vector space.

$V = (V, +, \cdot)$ is a group (scalar $\times$ vector)

and a set $A = \{\vec{x}_1, \vec{x}_2 \dots \vec{x}_k\}$

(a) The A is a generating set for the vector space V if for every vector $\vec{v} \in V$, $\vec{v}$ can be written as a linear combination of the vectors of the set A.

$V = (\mathbb{R}^3, +, \cdot)$

$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ← Generating set

for any $\vec{v}$ = say, $\begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$

$$\vec{v} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\vec{v} = \alpha_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$$

$\alpha_1 + \alpha_2 + \alpha_3 = 5$, $\alpha_1 = -4$

$\alpha_2 + \alpha_3 = 9$, $\alpha_2 = 5$

$\alpha_3 = 4$

$C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ ← This is not a generating set

$$\vec{r} = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix} = \alpha_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\alpha_1 + \alpha_2 = 5$

$\alpha_2 = 9$

$0 = 4$

Span.

A set A is called to span a vector space V if every element of the V can be obtained as a linear combination of the vectors in A.

Notationally, $V = \text{Span}(A)$

$$A = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

$$(\mathbb{R}, +, \cdot) = \text{Span}(A)$$
$$= \text{span}(B)$$

scalar x vector

$$A' = \left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}}_{A}, \begin{bmatrix}2\\3\\1\end{bmatrix} \right\}$$

$\vec{v} = \text{span}(A)$

$\vec{v} =$ linear combination $+ 0 \begin{bmatrix}2\\3\\1\end{bmatrix}$

$$(\mathbb{R}^3, +, \cdot) = \text{span}(A')$$

▷ A set $A = \{\vec{x}_1, \vec{x}_2 \dots \vec{x}_k\}$ is a minimal generating set of V if

1. $V = \text{Span}(A)$
2. No proper subset of A spans V
3. The set of vectors in A are linearly independent

(2 and 3 are equivalent)

Basis

A Basis for a vector Space is a set A which is a minimal generating set.

If A is the matrix representation of the element of the set A then the rank of A is the number of columns in A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Minimal generating set

Basis

RanK = 3

RanK < n | the matrix is not invertible or Singular