**Extension:** Given a matroid $M(S, I)$ for $A \in I$, an element $x \notin A$ is an extension of $A$ if $A \cup \{x\} \in I$.

example: For a graphic matroid $M_G$, e is an extension for A if addition 'e' in $G_A(V, A)$ does not create a cycle.

**Maximal:** If A is an independent subset in a matroid M, then we say that A is maximal if it has no extension.

Lemma: All maximal independent subset in a matroid have the same size.

Proof: Suppose A & B are maximal independent subset of a matroid & $|A| < |B|$

By exchange property, $\exists x \in B \setminus A$ and $A \cup \{x\} \in I$

$\Rightarrow$ A has extension, however A is maximal & doesn't have extension

$|A| = |B|$

weighted Graphic Matroid

A matroid $M_G(S_G, I_G)$ is weighted if it is associated with a weight function $\omega$ that assigns strictly positive weight $\omega(e)$ to each element $e \in S_G$ for any $A \subseteq S_G$

$$\omega(A) = \sum_{e \in A} \omega(e)$$

## Minimum Spanning Tree Problem (MST)

Input : $G(V,E)$ weight $w_e$ $\forall e \in E$

Output: Spanning Tree $T(V, E')$ with $E' \subseteq E$ that minimizes $w(T) = \sum_{e \in E'} w(e)$

How to formulate MST on Matroid?

$$G(V,E),\ w_e \longrightarrow M_G(S_G, I_G),\ w'_e$$

For any edge $e_1$, $w'(e) = w_0 - w(e)$ where $w_0 > \max\{w(e) / e \in E\}$

For weighted Graphic matroid

For any weighted graphic matroid an independent subset with maximum possible weight is an optimal solution because the weights are positive, an optimal set is always a maximal independent subset.

Let A be an optimal set for $M_G(S_G, I_G), w'$

$$w'(A) = \sum_{e \in A} w'(e)$$

$$= \sum_{e \in A} (w_0 - w(e))$$

$$= \sum_{e \in A} w_0 - \sum_{e \in A} w(e)$$

Constant

$$w'(A) = \boxed{(|V| - 1)\, w_0} - w(A)$$

~~minimize~~ ~~maximize~~

~~maximize~~ ~~minimize~~ minimize

maximize

## Greedy Algorithm for weighted matroid



Input: weighted matroid $M(S, I)$ and weight function $w$

Algo-GREEDY $(M, w)$

1. $A = \phi$
2. sort $M \cdot S$ into monotonically decreasing order by $w$
3. for each $x \in M \cdot S$ taken in sorted order
4. if $A \cup \{x\} \in M \cdot I$
5. then $A \leftarrow A \cup \{x\}$
6. return $A$

## Correctness for Algo-GREEDY

**Lemma:** Suppose $M(S, I)$ is a weighted matroid with weight function $w$ that $S$ is sorted into monotonically decreasing order by weight. Let $x$ be the first element of $S$ such that $\{x\}$ is independent. If any such $x$ exists, then there exists an optimal subset $A$ of $S$ that contains $x$.

**Proof:** Let $B$ be any optimal solution and $x \notin B$
Let $y$ be any element in $B$. Since $B \in I$ : $\{y\} \in I$ because of Hereditary property.
Because of our Greedy where $\{x\} \in I$ and $w(x) \geq w(y)$

Construction of A using B by repeatedly applying exchange argument

$A \in I$, $B \in I$, $|A| < |B|$

$\exists z \in B/A$ and $A \cup \{z\} \in I$

Apply exchange argument until $|A| = |B|$

After this phase,

$A = B - \{y\} \cup \{x\}$

$w(A) \geq w(B) - w(y) + w(x)$

$\because$ B is an optimal solution $\Rightarrow$ A is also an optimal solution.

Lemma: Let $M(S, I)$ be any matroid. If $x \in S$ that is an extension of some $A \in I$, then $x$ is also an extension of $\phi$.

Proof: $A \cup \{x\} \in I$

then $\{x\} \in I$

(optimal substructural property)

Lemma: Let $x$ be the first element of S selected by Greedy for $M(S, I)$ w. The remaining problem of finding maximum weight independent subset containing $x$ reduces to finding maximum weight subset of the weighted matroid $M' = (S', I)$

when,

$S' = \{y \in S \mid (x, y) \in I\}$

$I' = \{B \subseteq S - \{x\} \mid B \cup \{x\} \in I\}$

and the weight ~~function~~ function for $M'$ in weight function $\omega$ for $M$ restricted to $S'$

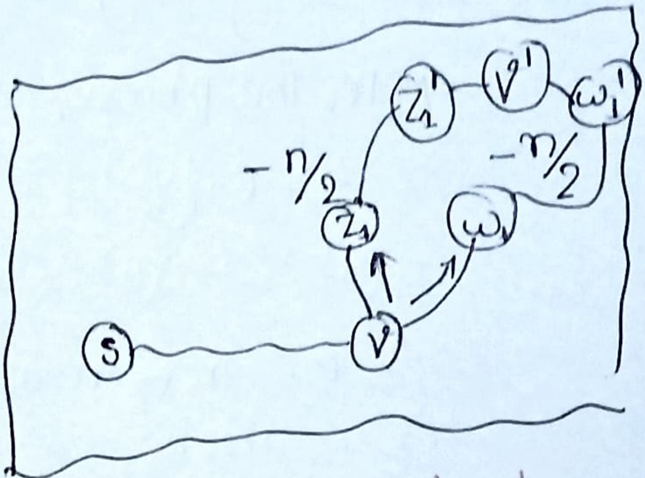
$A = A' \cup \{x\}$

$\omega(A) = \omega(A') + \omega(x)$

$S \subseteq U$

$\boxed{S}$ | $V - S$

partial tree




$s \sim v - z_1 \sim z_1' - v'$

$s \sim v - \omega_1 \sim \omega_1' - v'$