Mathematical Foundations of AI (Aug'24-Dec'24)

Problem Sheet

1. Prove or disprove the following.

   (a) $Lt_{x\to 0}\frac{\tan 2x}{\sin x} = 2$. (Hint : $Lt_{x\to 0}\frac{\sin x}{x} = 1$.)

   (b) $Lt_{x\to 0}\frac{\sin 5x - \sin 3x}{x} = 2$.

   (c) $Lt_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2} = \frac{1}{2}$.

2. Let $f(x)$ be defined as $f(x) = 2\cos x$ if $x \le c$ and $f(x) = ax^2 + b$ if $x > c$. a, b, c are constants. Given b and c, find all values of a for which $f(x)$ is continuous at $x = c$.

3. Consider $f(x) = \sin(x^{-1})$ for $x \neq 0$ and $f(0) = a$. Determine (if any) the value of a such that $f(x) \to A$ (for some A) as $x \to 0$. Justify your answer.

4. Consider $f(x) = [x^{-1}]$ for $x \neq 0$. $[t]$ denotes the greatest integer less than or equal to t. For example, $[2.5] = 2$ and $[-3.5] = -4$. Can you define $f(0)$ so that f is continuous at 0.

5. Let f and g be functions defined as $f(x) = 1$ if $|x| \le 2$ and $f(x) = 0$ if $|x| > 2$, and $g(x) = 2 - x^2$ if $|x| \le 3$ and $g(x) = 2$ if $|x| > 3$. Determine a formula for computing $h(x) = f(g(x))$. For what values of x is h continuous ?

6. Let $f(x)$ be defined by $f(x) = x^2$ if $x \le c$ and $f(x) = ax^3 + b$ if $x > c$ where a, b, c are constants. Find values of a and b (in terms of c) such that $f'(c)$ exists.

7. Solve Exercise (6.) for $f(x)$ where $f(x) = |x|^{-1}$ if $|x| > c$ and $f(x) = a + bx^2$ if $|x| \le c$.

8. A reservoir has the shape of a right-circular cone. The altitude is 20 feet, and the radius is 8 feet. Water is poured into the reservoir at a constant rate of 5 cubic feet per minute. How fast is the water level rising when the depth of the water is 5 feet if (a) the vertex of the cone is up ? and (b) the vertex of the cone is down ?

9. Prove that among all rectangles of a given area, square has the smallest perimeter.

10. A farmer has L feet of fencing to enclose a rectangular pasture adjacent to a long stone wall. What dimensions give the maximum area of the pasture ?

11. Find the rectangle of the largest area that can be inscribed in a semi-circle, the lower base being on the diameter.

12. Given n real numbers $x_1, \ldots, x_n$, prove that the sum $\sum_{i=1}^{n}(x-a_i)^2$ is the smallest when x is the arithmetic mean of $a_1, \ldots, a_n$.

13. Let $0 < a < b$ be given. We want to approximate an unknown $x \in [a, b]$ by another number $t \in [a, b]$ so that the maximum relative error $M(t) = \max_{x\in[a,b]} \frac{|t-x|}{x}$ is as small as possible. Prove that $M(t)$ is achieved at either $x = a$ or $x = b$, for each t. Prove that the smallest value of $M(t)$ is achieved by the harmonic mean of a and b, that is, for t satisfying $\frac{1}{t} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$.

14. Determine the Taylor series expansion for $f(x)$ around $x = 0$ for each of the following definitions of $f(x)$.

    (a) $f(x) = a^x$, $a > 0$, for all x.

    (b) $f(x) = \frac{1}{2-x}$ for $|x| < 2$.

    (c) $f(x) = \log\sqrt{\frac{1+x}{1-x}}$ for $|x| < 1$.

15. For each of the following scalar functions, determine the set of points (x, y) or (x, y, z) at which (i) f is defined and (ii) f is continuous.

    (a) $f(x, y, z) = \log_e(x^2+y^2+z^2)$ for $(x, y, z) \neq (0, 0, 0)$; $f(0, 0, 0) = 1$.

    (b) $f(x, y) = \frac{x+y}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

    (c) $f(x, y) = \frac{x}{\sqrt{x^2+y^2}}$.

16. For $(x, y) \neq (0, 0)$, define $f(x, y) = \frac{x^3-y^3}{x^2+y^2}$. Is it possible to define $f(0, 0)$ so as to make f continuous at $(0, 0)$.

17. Define $v(r,t) = t^n \cdot e^{-r^2/(4t)}$. Find a value of the constant n such that v satisfies the following equation $\frac{\partial v}{\partial t} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right)$.

18. Find the gradient vector at each point at which it exists for the scalar function given by the following definitions.

    (a) $f(x,y,z) = \log_e(x^2 + 2y^2 - 3z^2)$.

    (b) $f(x,y,z) = x^{y^z}$.

19. Find the directional derivatives for each of the following scalar functions for the points and directions given :

    (a) $f(x,y,z) = x^2 + 2y^2 + 3z^2$ at $(1,1,0)$ in the direction of $(1,-1,2)$.

    (b) $f(x,y,z) = \left(\frac{x}{y}\right)^z$ at $(1,1,1)$ in the direction of $(2,1,-1)$.

20. For each of the following definitions of $f(x,y)$, $x = X(t)$ and $y = Y(t)$, define $F(t) = f(X(t), Y(t))$ and compute $F'(t)$ and $F''(t)$.

    (a) $f(x,y) = x^2 + y^2$, $X(t) = t$ and $Y(t) = t^2$.

    (b) $f(x,y) = e^{xy}\cos(xy^2)$, $X(t) = \cos t$, $Y(t) = \sin t$.