Dynamic Programming

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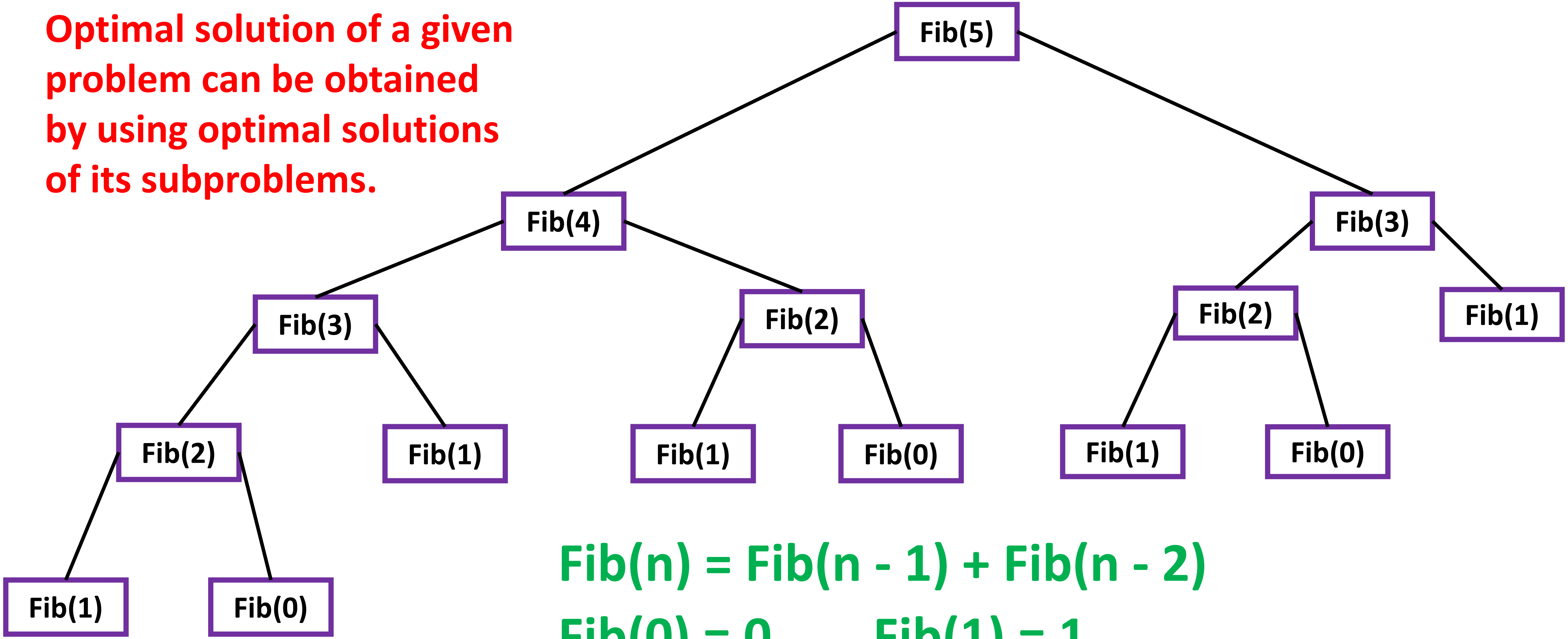
n-th Fibonacci Number: Recursion

0 1 1 2 3 5 8 13 21 34

```
int fib(int n)
{
	if (n <= 1)
		return n;
	return fib(n-1) + fib(n-2);
}
```

Optimal Substructure

Optimal solution of a given problem can be obtained by using optimal solutions of its subproblems.

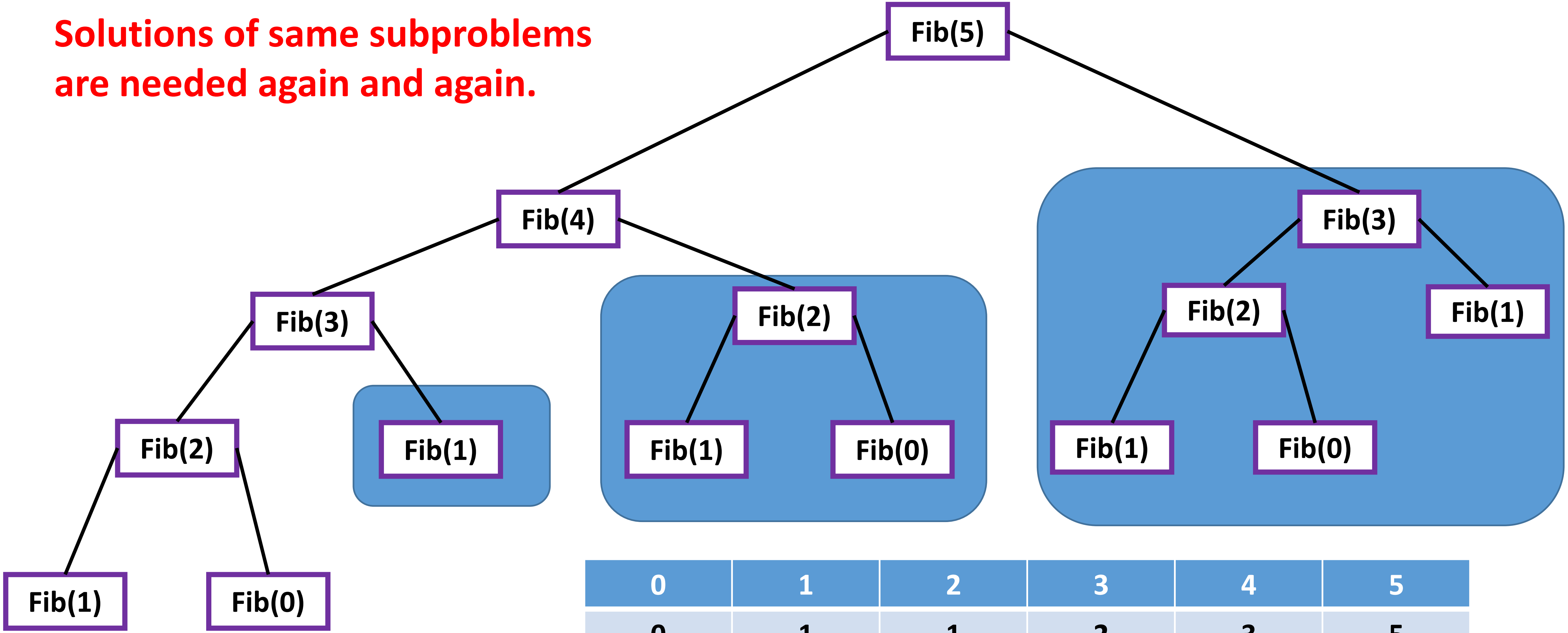


$$Fib(n) = Fib(n - 1) + Fib(n - 2)$$

$$Fib(0) = 0 \qquad Fib(1) = 1$$

Overlapping Subproblems

Solutions of same subproblems are needed again and again.



0	1	2	3	4	5
0	1	1	2	3	5

Dynamic Programming

- Break a problem into subproblems:
  - Find a recurrence relation with base cases (Optimal Substructure Property)
- Stores the results of subproblems in a table
  - Avoid computing the same subproblem again. (Overlapping Subproblem Property)
  - Size of the table/array can be determined from the recursion tree.
- How to populate the table?
  - **Top-down approach**: Memoization (Recursive)
  - **Bottom-up approach**: Tabulation (Iterative)

Memoization: A Top-Down Approach

1. Recursion with lookup table.
2. Initialize a lookup table with

   all initial values as -1
3. If (lookup = -1)
   a. Calculate the value and

      put the result in the

      lookup table
4. Else
   a. Return precomputed value

```
int fib(int n)
{
    if (T[n] == -1) {
      if (n <= 1)
        T[n] = n;
      else
        T[n] = fib(n-1) + fib(n-2);
    }
    return T[n];
}
```

Tabulation: A Bottom-Up Approach

1. Iteration with lookup table.
2. Compute and store the result of the subproblems in the table in a bottom-up fashion (start from the base case of the recurrence relation).
3. Return last entry of the table

```
int fib(int n)
{
	int i, f[n+1];
	f[0] = 0; f[1] = 1;
	for (i = 2; i <= n; i++)
		f[i] = f[i-1] + f[i-2];
	return f[n];
}
```

Dynamic Programming

1. Recursion
2. Memoization: Recursion + Table
    1. It is better than Tabulation only when a subset of the subproblems are needed to solve the original problem.
3. Tabulation: Table (typically used methodology to solve a DP)
    - It is better than Memoization only when all of the subproblems are needed to solve the original problem.

Substring vs Subsequence

- A subsequence of a string is a sequence of characters in the string that maintains the same relative order as in the string with one or more characters left out.
- Example: “CBCDB” is a subsequence in “ACBCCDAB”
- A substring of a string is a sequence of contiguous characters in the string .
- Example: “CBCCD” is a substring in “ACBCCDAB”
- A substring is a subsequence, but a subsequence may not be a substring.

Longest Common Subsequence (LCS)

- **Input:**

  X = “ACBCCDAB”   Y = “DABCADB”

- **Output: Length of LCS**

  5 (Z = “ABCDB” or Z = “ABCAB”)

Length of LCS: Optimal Substructure

Input: **x[] = “ACBCCDAB”** **y[] = “DABCADB”**

Since x[7] = y[6], then

LCS (**“ACBCCDAB” , “DABCADB”**) = 1 + LCS(**“ACBCCDA” , “DABCAD”**)

Input: **x[] = “ACBCCDAB”** **y[] = “DABCADA”**

Since x[7] ≠ y[6], then

LCS (**“ACBCCDAB” , “DABCADA”**) = MAX{

LCS(**“ACBCCDAB” , “DABCAD”**), LCS(**“ACBCCDA” , “DABCADA”**)}

Length of LCS: Optimal Substructure

Input: $X_m = x_0, x_1, \ldots, x_{m-1}$

$Y_n = y_0, y_1, \ldots, y_{n-1}$

1. If $x[m-1] = y[n-1]$, then

   $LCS(X_m, Y_n) = 1 + LCS(X_{m-1}, Y_{n-1})$

2. If $x[m-1] \neq y[n-1]$,

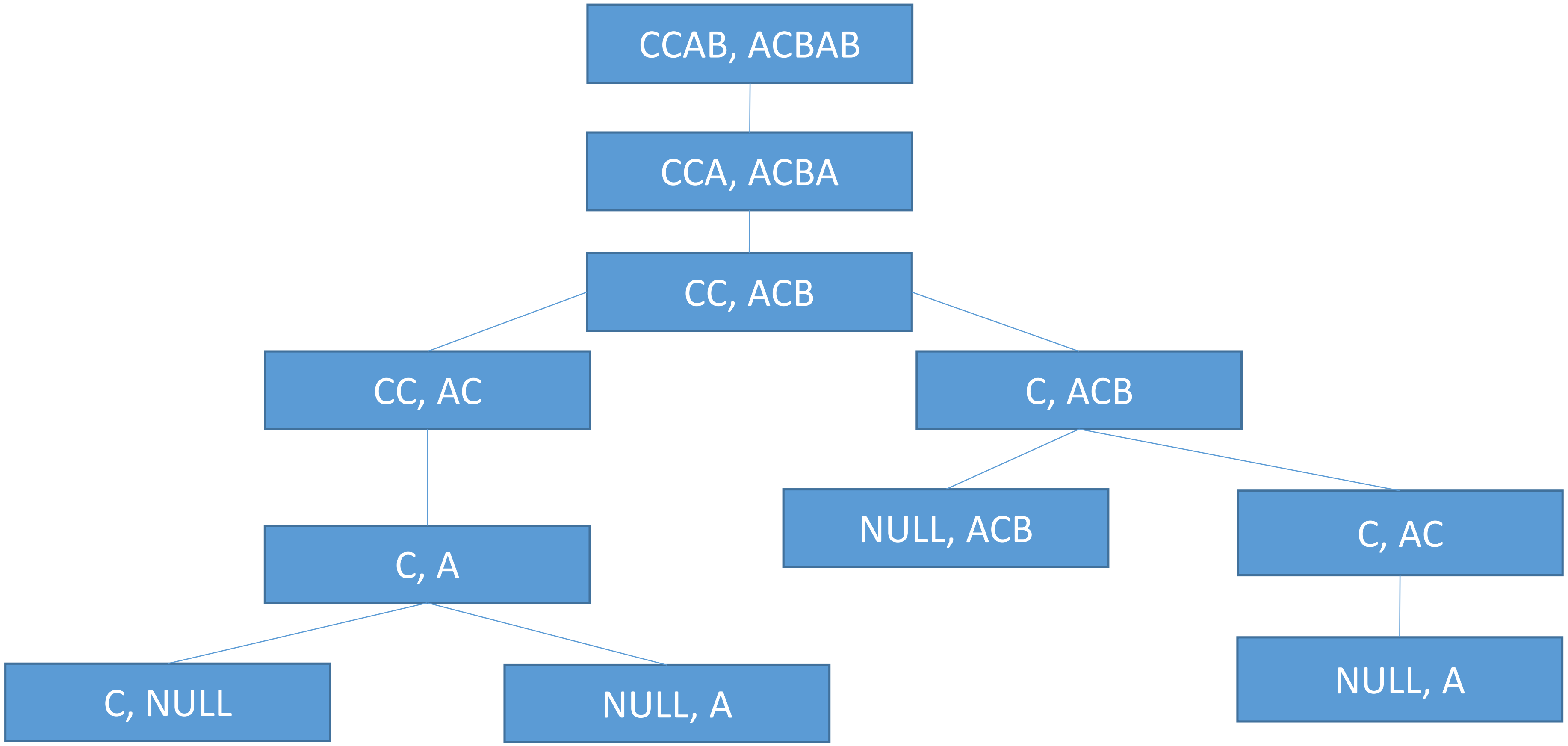
$$LCS(X_m, Y_n) = MAX\{ LCS(X_{m-1}, Y_n), LCS(X_m, Y_{n-1}) \}$$

Important Observation:

$LCS(X_m, Y_n)$ depends on

Either $LCS(X_{m-1}, Y_{n-1})$

Or $LCS(X_{m-1}, Y_n)$ and $LCS(X_m, Y_{n-1})$


CCAB, ACBAB
CCA, ACBA
CC, ACB
CC, AC
C, ACB
C, A
NULL, ACB
C, AC
C, NULL
NULL, A
NULL, A

Length of LCS (X[m], Y[n])

L	Y=	A	C	B	A	B
X=		0	0	0	0	0
C	0	0	1	1	1	1
C	0	0	1	1	1	1
A	0	1	1	1	2	2
B	0	1	1	2	2	3

int L[m+1][n+1];
L[1][2] = Length of LCS(“C”, “AC”);
L[4][5] = Length of LCS(“CCAB”, “ACBAB”)=1 +LCS(“CCA”, “ACBA”)=L[3][4];

L[i][j] = Length of LCS of string X[0..i-1] and string Y[0..j-1]

Length of LCS (X[m], Y[n])

	Y=	A	C	B	A	B
X=	0	0	0	0	0	0
C	0	0	1	1	1	1
C	0	0	1	1	1	(2,5)1
A	0	1	1	1	(3,4)2	(3,5)2
B	0	1	1	(4,3)2	(4,4)2	(4,5)3

Input: $Xm = x0, x1, \ldots, xm-1$

$Yn = y0, y1, \ldots, yn-1$

1. If $xm-1 = yn-1$, then

   $LCS (Xm, Yn) = 1 + LCS (Xm-1, Yn-1)$

2. If $xm-1 \neq yn-1$,

$LCS (Xm, Yn) = MAX\{ LCS (Xm-1, Yn),$

$LCS (Xm, Yn-1)\}$

L[i][j] = LCS(X, i, Y, j) = Length of LCS of string X[0..i] and string Y[0..j]

LCS: Assignment

1. Write a Recursive program for longest common subsequence (LCS).
2. Write a Memoized DP for LCS.
3. Write a Tabulated DP for LCS.
4. Print an LCS.

Length of LCS (X[m], Y[n]) (Tabulation)

```
int lcs(char *X, char *Y)
{
    int i, j, m = strlen(X), n = strlen(Y), L[m+1][n+1];
    /* L[i][j] is length of LCS of string X[0..i] and string Y[0..j] */
    for (i = 0; i <= m; i++) {
        for (j = 0; j <= n; j++) {
            if (i == 0 || j == 0)             L[i][j] = 0;
            else if (X[i-1] == Y[j-1])        L[i][j] = L[i-1][j-1] + 1;
            else if (L[i-1][j] > L[i][j-1])   L[i][j] = L[i-1][j];
            else                              L[i][j] = L[i][j-1];
        }
    }
    return L[m][n];
}
```

Longest Palindromic Subsequence (LPS)

LPS(“ACBCCDA”)
= 2 + LPS(“CBCCD”)
= 2 + max(LPS(“CBCC”), LPS(“BCCD”))
= 2 + max(2+LPS(“BC”), max(LPS(“BCC”), LPS(“CCD”)))
= 2 + max(2 + 1, max(max(LPS(“BC”), LPS(“CC”)) , max(LPS(“CC”), LPS(“CD”))))
= 2 + max(3, 2) = 5

- **Output:** Length of LPS

    5 (Z = “ACBCA” or Z = “ACCCA”)

Length of LPS: Optimal Substructure

Input: x[8] = "**ACBCCDBA**"

Since x[0] = x[7], then

LPS(x, 0, 7) = 2 + LPS (x, 1, 6)

LPS("**ACBCCDBA**") = 2 + LPS("**CBCCDB**")

Input: x[8] = "**ACBCCDAB**"

Since x[0] ≠ x[7], then

LPS(x, 0, 7) = MAX{ LPS (x, 0, 6), LPS (X, 1, 7)}

LPS("**ACBCCDAB**") = MAX{ LPS("**ACBCCDA**"), LPS("**CBCCDAB**")}

Length of LPS: Optimal Substructure

Input: $X_m = x_0, x_1, \ldots, x_{m-1}$

Output: $Z, 0, k-1 = LPS(X, 0, m-1)$

1. If $x_0 = x_{m-1}$, then $z_0 = x_0$ and $z_{k-1} = x_{m-1}$

   $Z, 1, k-2 = LPS(X, 1, m-2)$

1. If $x_0 \neq x_{m-1}$, then

   $Z, 0, k = MAX\{ LPS(X, 0, m-2),$

   $LPS(X, 1, m-1)\}$

Important Observation:

LPS (X, i, j) = Length of LPS X[i..j]

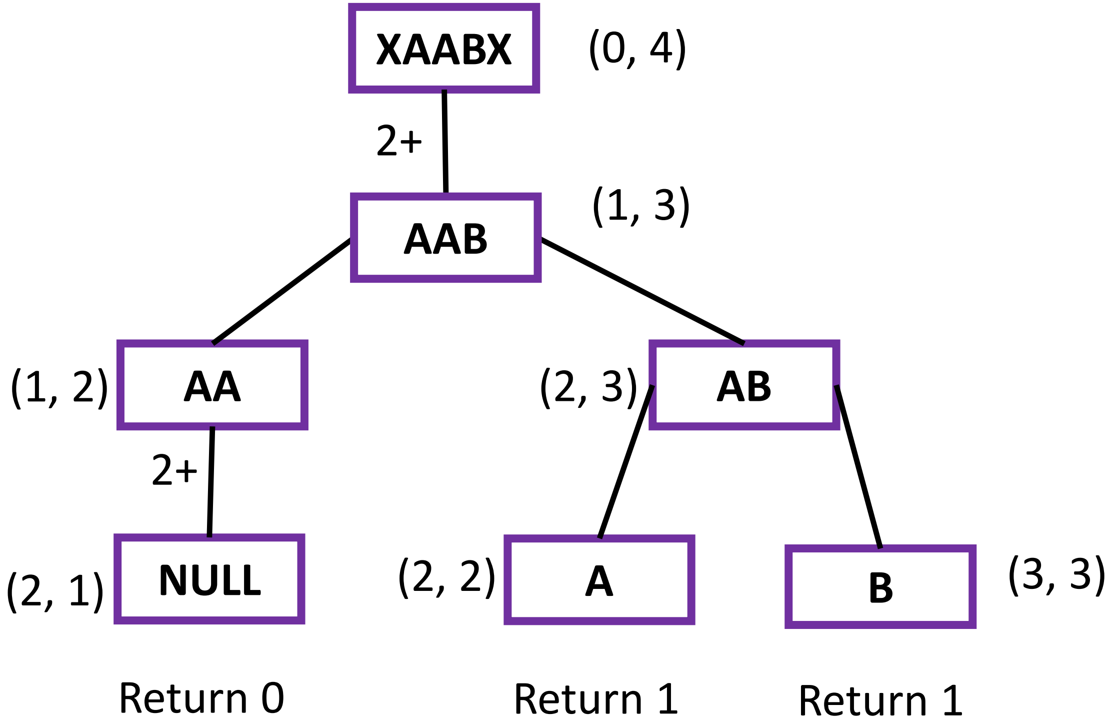
LPS (X, i, j) depends on

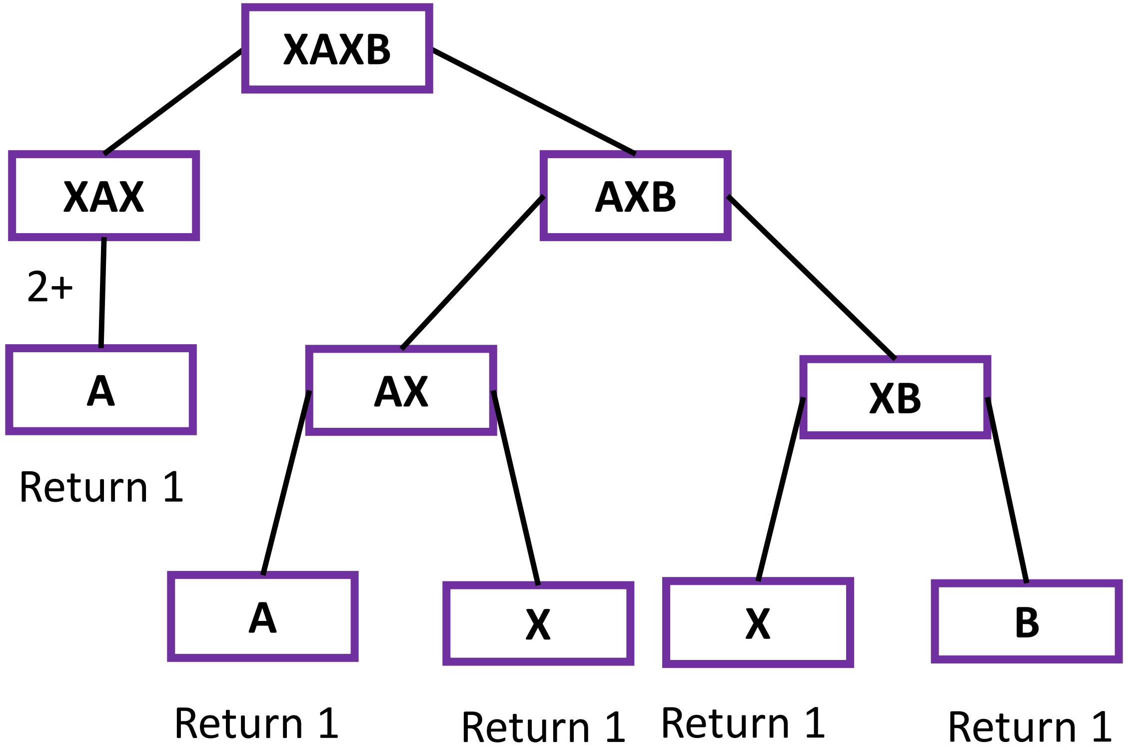
Either LPS (X, i+1, j−1) if X[i]=X[j]

Or Maximum of

{LPS (X, i, j-1) and LPS (X, i+1, j)}

Optimal Substructure: LPS





Length of LPS (X[m])

	A	C	B	B	A
A	0, 0	0, 1	0, 2	**0, 3**	**0, 4**
C		1, 1	1, 2	(1, 3)= max{(1,2),(2,3)}	**1, 4**
B			2, 2	2, 3	**2, 4**
B				3, 3	3, 4
A					4, 4

	A	C	B	B	A
A	1	1	1	2	4
C	0	1	1	2	2
B		0	1	2	2
B			0	1	1
A				0	1

A	C	B	B	A
0	1	2	3	4

0	1	2	3
A	B	B	A

L[i][j] = LPS(X, i, j) = Length of LPS X[i..j]

Reduction: Solving LPS using LCS

$LPS(X) = LCS(X, X_R) = LCS(ACBBA, ABBCA) = ABBA$

LPS: Assignment

1. Write a Recursive program for longest palindromic subsequence (LPS).
2. Write a Memoized DP for LPS.
3. Write a Tabulated DP for LPS.
4. Print an LPS.

Length of LPS (X[m]) (Tabulation)

```
int lps(char X[], int m) {
    int i, j, k, l, L[m][m];
    for(i = 0; i < m; i++) L[i][i] = 1;
    for(l = 1; l < m; l++) {
        for(i = 0; i < m-l; i++) {
        if(X[i] == X[i+l]) {
            if(i+1 <= i+l-1)   L[i][i+l] = 2 + L[i+1][i+l-1];
            else               L[i][i+l] = 2;
        } else {
            int a = L[i][i+l-1];
            int b = L[i+1][i+l];
            L[i][i+l] = (a > b)? a : b;
        }
    }
    return L[0][m-1];
}
```

Longest Increasing Subsequence (LIS)

- **Input:**

  X[10] = {3, 2, 1, 3, 5, 4, 4, 5, 6, 3}

  X[10] = {3, 2, 1, 3, 5, 4, 4, 5, 6, 3}

- **Output: Length of LIS**

  5 {1, 3, 4, 5, 6} or {2, 3, 4, 5, 6}

Length of LIS (A[n])

A	3	2	1	3	5	4	4	5	6	3
i	0	1	2	3	4	5	6	7	8	9

Length[i] = Length of LIS that ends with A[i]

Previous[i] = Previous index of LIS that ends with A[i]

Optimal Substructure:

Length[i] = MAX (0, Length[k]) + 1, where i > k AND A[i] > A[k]

Objective:

Max(Length(i)) for all i = 0 to n-1

Initialization

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

Initialization

Length	1	1	1	1	1	1	1	1	1	1
Previous	0	1	2	3	4	5	6	7	8	9

Execution: LIS

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

$i = 1$

No update: A[1] < A[0]

Length	1	1	1	1	1	1	1	1	1	1
Previous	0	1	2	3	4	5	6	7	8	9

Execution: LIS

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

i = 2

No update: A[2] < A[1] AND A[2] < A[0]

Length	1	1	1	1	1	1	1	1	1	1
Previous	0	1	2	3	4	5	6	7	8	9

Execution: LIS

A									
3	2	1	3	5	4	4	5	6	3
0	1	2	3	4	5	6	7	8	9

i = 3

Update: A[2] < A[3]

Length	1	1	1	2	1	1	1	1	1	1
Previous	0	1	2	2	4	5	6	7	8	9

Execution: LIS

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

i = 4

Update: A[3] < A[4]

Length	1	1	1	2	3	1	1	1	1	1
Previous	0	1	2	2	3	5	6	7	8	9

Execution: LIS

A									
3	2	1	3	5	4	4	5	6	3
0	1	2	3	4	5	6	7	8	9

i = 5

No Update: A[4] > A[5] j = 4

Update: A[3] < A[5] j = 3 (If get a longer subsequence)

	0	1	2	3	4	5	6	7	8	9
Length	1	1	1	2	3	3	1	1	1	1
Previous	0	1	2	2	3	3	6	7	8	9

Since i = 5, the variable j iterates from j= 4 to j = 0, and check whether A[j] < A[i] and Length[j]+1 > Length[i]. If so then update Length[i] = Length[j] + 1 and Previous [i] = j

Execution: LIS

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

i = 6

Update: A[3] < A[6]

Length	1	1	1	2	3	3	3	1	1	1
Previous	0	1	2	2	3	3	3	7	8	9

Execution: LIS

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

i = 7

Update: A[6] < A[7]

Length	1	1	1	2	3	3	3	4	1	1
Previous	0	1	2	2	3	3	3	6	8	9

Since i = 7, the variable j iterates from j= 6 to j = 0, and check whether A[j] < A[i] and Length[j]+1 > Length[i]. If so then update Length[i] = Length[j] + 1 and Previous [i] = j

Execution: LIS

A									
3	2	1	3	5	4	4	5	6	3
0	1	2	3	4	5	6	7	8	9

i = 8

Update: A[7] < A[8]

Length	1	1	1	2	3	3	3	4	5	1
Previous	0	1	2	2	3	3	3	6	7	9

Execution: LIS

A										
	3	2	1	3	5	4	4	5	6	3
	0	1	2	3	4	5	6	7	8	9

i = 9

Update: A[2] < A[9]

Length	1	1	1	2	3	3	3	4	5	2
Previous	0	1	2	2	3	3	3	6	7	2

LIS

A

3	2	1	3	5	4	4	5	6	3
0	1	2	3	4	5	6	7	8	9

Length	1	1	1	2	3	3	3	4	5	2
Previous	0	1	2	2	3	3	3	6	7	2

LIS

1	3	4	5	6
0	1	2	3	4

Algorithm: LIS (Tabulation)

```
For each i = 1 to n-1 {
    For each j = i-1 to 0 {
        if( A[i]  >  A[j]   AND
          length[j] + 1  >  length[i]) {
            length[i]     =  length[j] + 1;
            previous[i]  =   j;
            if(length[i]   >   maximum) {
                maximum      =   length[i];
                LastIndexLIS  =   i;
            }
        }
    }
}
```

```
printLIS(int A[], int previous[], int i)
{
    if(previous[ i ]  !=  i)
        printLIS(A, previous, previous[ i ]);
    print(A[ i ]);
}
```