Lecture-17

[Tardos Book]

Analysis of Ford Fulkerson Algorithm

Termination

→ Assumption: All capacities are integers

Lemma 1: At every intermediate stage, the flow value and the residual capacities are integers.

Proof: (By Induction)

Base Case: Before the first iteration of the while loop the claim true

(IH) Induction Hypotheses: Suppose the claim is true for 'j' iterations

Induction Step: What about $(j+1)^{th}$ iteration

From (IH); all capacities in the residual graph (G_f) are integer and value of the flow obtained so far is also integer.

b = Bottleneck (f, P) is an integer

▷ Addition/Subtraction between two integer is an integer'

Therefore all residual capacities and the augmented flow value are integers at the end of $(j+1)^{th}$ iteration.

Lemma 2: Let f be a flow value in G and P be an s-t path in G_f, then $v(f') = v(f) +$ bottle(P, f)

Proof: The first edge 'e' in the path is an outgoing edge from 's' (source) $\therefore$

▷ By definition of flow network has only outgoing edges. Therefore is a forward edge

$$(s) \xrightarrow{e} \circ$$

forward

$$v(f') = \sum_{e \text{ out of } s} f'(e) = v(f) + \underbrace{\text{bottleneck}(P, f)}_{> 0}$$

$$\therefore v(f') > v(f)$$

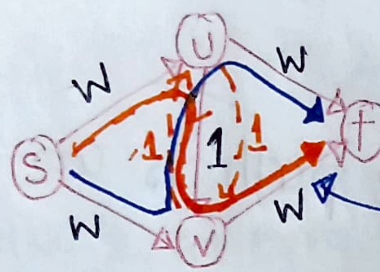
$$v(f) = \sum_{e \text{ out of } \{s\}} f(e) \leq \boxed{\sum_{e \text{ out of } \{s\}} c_e = C}$$

upper bound on flow

flow increase by atleast '1' → Max 'C' times.

$T(G) = O(mC)$

Number of edges in the input flow network



In the worst case it will run upto $2w$ times

$\boxed{W >> 1}$

Optimality of the algorithm

Trivial upper bound: $v(f) \leq \sum_{e \text{ out of 's'}} C_e = C$

$A \cap B = \Phi$
$A \cup B = V$

Cut in a graph.

s-t cuts: An s-t cut is a partition (A, B) of the vertex set such $s \in A$ and $t \in B$.

Capacity of a cut:

→ Capacity of an s-t (A, B) is

$$c(A, B) = \sum_{e \text{ out of } A} c_e = \sum_{e' \text{ into } B} c_{e'}$$

$A = \{s\}$ | $c(A,B) = \sum_{e \text{ out of } A} f(e) = C$

$B = V - A$

We know that for this particular cut (A,B)

$$v(f) \leq c(A,B)$$

Goal: We want to show that the capacity of any s-t cut is an upperbound on the value of s-t flow.

Lemma: Let f be any s-t flow and (A,B) be any s-t cut. Then

$$v(f) = f^{out}(A) - f^{in}(A)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e' \text{ into } A} f(e')$$

Proof:

From definition of flow $v(f) = \sum_{e \text{ out of } \{s\}} f(e)$

and $\sum_{e' \text{ into } \{s\}} f(e') = 0$.

(i) $A = \{s\}$
$B = V - A$ $\longrightarrow$ $v(f) = f^{out}(A) - f^{in}(A)$

Trivial case

(ii) $A \supset \{s\}$

▷ $v \in A$ and $v \neq \{s\}$, such a vertex v is an internal vertex

▷ From conservation condition:

$$\sum_{e \text{ out of } v} f(e) - \sum_{e' \text{ into } v} f(e') = 0$$

$$\Rightarrow f^{out}(v) - f^{in}(v) = 0$$

$$\sum_{v \in A} \left(f^{out}(v) - f^{in}(v)\right) = f^{out}(s) - f^{in}(s) = v(f)$$

$$= \boxed{f^{out}(A) - f^{in}(A)}$$

1. What about the $e = \overset{x \to y}{(x, y)}$ for which $x \in A$ and $y \in A$?

$f(e) \to f^{out}(x)$

$f(e) \to f^{in}(y)$

Does not contribute to ~~c(A,B)~~ ~~or~~ $v(f)$?

2. What about $e = \overset{p \to q}{(p, q)}$ | $p \in A, q \in B$?

$f(e) \to f^{out}(p)$

3. What about $e' = \overset{p' \to q'}{(p', q')}$: $p' \in B$, and $q' \in A$

$f(e) \to f^{in}(q')$

Contributes to ~~c(A,B) or~~ $v(f)$?

$$\boxed{v(f) = f^{in}(B) - f^{out}(B)}$$

Lemma 'B': Let f be any s-t flow and (A, B) any s-t cut. Then

$$v(f) \leq c(A, B)$$

Proof:

$$v(f) = f^{out}(A) - f^{in}(A)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e' \text{ into } A} f(e')$$

$$\leq \sum_{e \text{ out of } A} f(e) \leq \sum_{e \text{ out of } A} c_e$$

Capacity Condition.

$$= c(A, B)$$

Lemma 'C': If f is any s-t flow such that there is no s-t path in G_f (residual network), then there is an s-t cut (A^*, B^*) in G for which

$$v(f) = \boxed{c(A^*, B^*)}$$

Maximumflow / Minimum Capacity

Proof:

A^* = set of vertices for which there is a path in G_f from $\{s\}$

$s \in A^*$

$t \notin A^*$

$B^* = V - A^*$

$A^* \cap B^* = \phi$

$A^* \cup B^* = V$

Let $e = (u, v)$ be an edge in G such that $u \in A^*$ and $v \in B^*$

Claim 1: $f(e) = c_e$

If not, then we will have a forward edge 'e' with capacity $c_e - f(e)$

$$s \rightsquigarrow u \rightarrow v \Rightarrow v \in A^*$$

Let $e' = (p, q)$ be an edge in G such that $p \in B^*$ and $q \in A^*$

Claim 2: $f(e') = 0$

If not, then we will have a backward edge

$$e'' : q' \longrightarrow p'$$

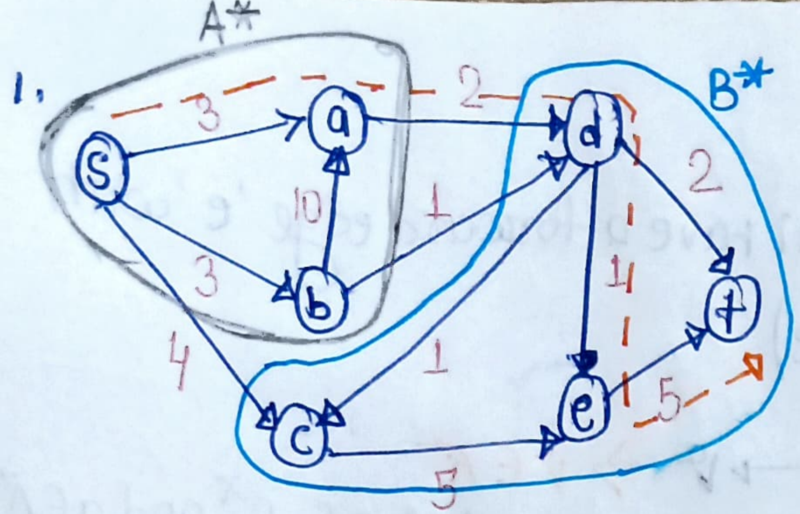
$$v(f) = f^{out}(A^*) - f^{in}(A^*)$$

$$= \sum_{e \text{ out of } A^*} c_e = c(A^*, B^*)$$

(Maxflow | Min cut Theorem)

In Every flow network, the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.

1.



--- Path

$f = 0, b = 1$

P: $s \to a \to d \to e \to t$

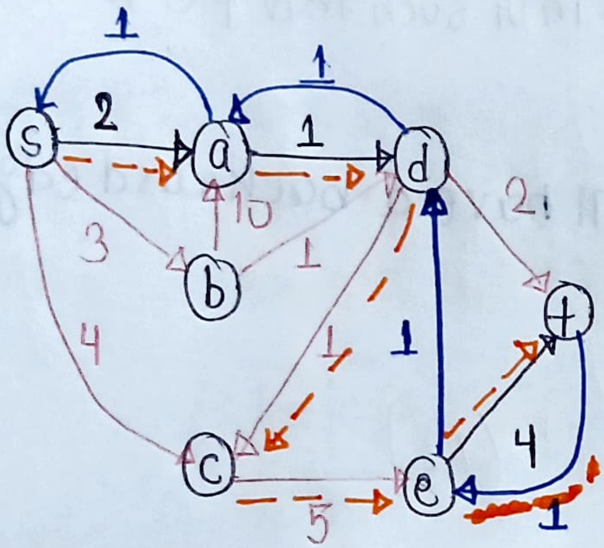
$f' = f + b = 1$

— Forward Edge

— Back edge

— Unchanged

2.

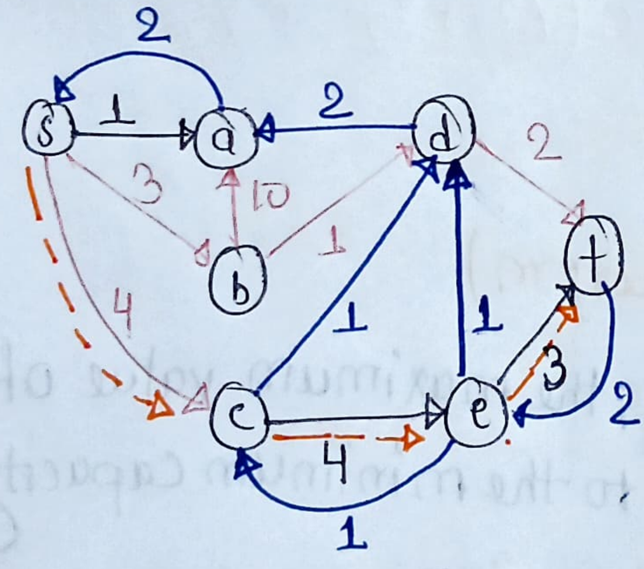


$f = 1, b = 1$

P: $s \to a \to d \to c \to e \to t$

$f' = f + b = 2$

3.

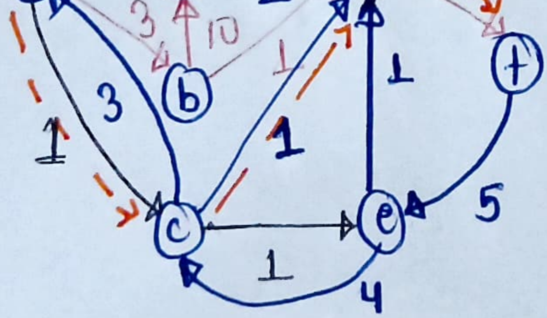


$f = 2, b = 3$

P: $s \to c \to e \to t$

$f' = f + b = 5$

4.

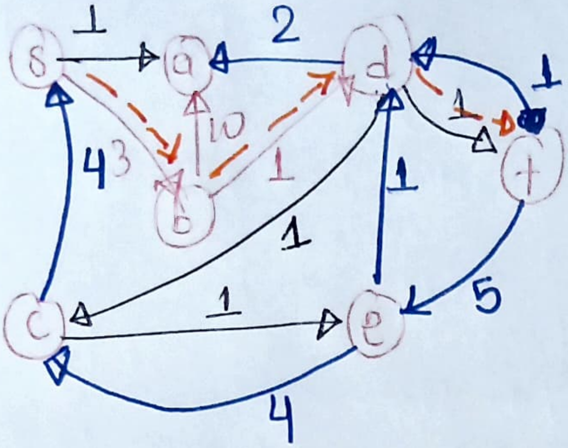


$f = 5, b = 1$

P: $s \to c \to d \to t$

$f' = f + b = 6$

5.

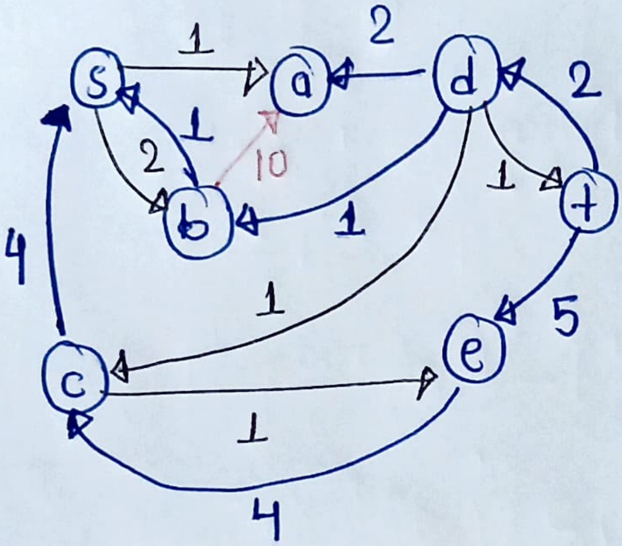


$f = 6, \ b = 1$

$P: s \to b \to d \to t$

$f' = f + b = 6 + 1 = 7$

6.



$f = 7$

$A^* = \{s, a, b\}$

$B^* = \{c, d, e, f\}$

$c(A^*, B^*) = 2 + 1 + 4$

$= 7$

(a-d) (b-d) (s-c)