Mostly Post mid sem

Greedy
Matroid
Amortized } not be included

→ But in the mail it is not mentioned the Greedy will be excluded
So I am not sure.

MFAI

Lecture - 32 (15/11/24)

P1: Min $f(x)$ ← Constrained Minimization Problem

subject to $\boxed{A\vec{x} = \vec{b}}$, $\vec{x} \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

Want to convert to Unconstrained minimize

Positive definite ← ? Hassian

join using a line segment then — take an two points and Atmost value of the function told by the line segment

Assume loss of generality, that $\text{rank}(A) = m$

→ Rows are linearly independents.

$$\left(\overbrace{A_1 \; A_2 \; \cdots \; A_m}^{C} \; \overbrace{A_{m+1} \; \cdots \; A_n}^{D} \right)$$

Columns

$A\vec{x} = \vec{b} \Leftrightarrow (C \; D') \, x' \, x'') = \vec{b}$

$\Leftrightarrow Cx' + Dx'' = \vec{b}$

$\Leftrightarrow Cx' = \vec{b} - Dx''$

$C = \{\vec{x} \mid A\vec{x} = \vec{b}\}$; If $\vec{y} \in C$

Then for any z, $z \in C \Leftrightarrow z = \vec{y} + \vec{\omega}$, $\boxed{\vec{\omega} \in N(A)}$

Set of solutions

Null space

$\boxed{N(A)}$ is a linear subspace of $\mathbb{R}^n$ with dimension $r = n-m$

$$\boxed{r = \text{dimension}(N(A))}$$

$\exists$ exists a Basis of $n-m$ linearly independent vectors in $n(A)$ which spans $N(A)$

$\longrightarrow \{v_1 \dots v_{n-m}\}$

matric

$$Z = [v_1 \; v_2 \dots v_{n-m}]$$

$n \times (n-m)$

Given $\omega \in N(A)$, $\exists\, \vec{v} \in \mathbb{R}^{n-m}$ such that $\boxed{\omega = Z\vec{v}}$

~~$f(x) = f(y$~~

Class of feasible solutions

$$C = \{\vec{y} + z\vec{v} \mid \vec{y} \in \mathbb{R}^n, \vec{v} \in \mathbb{R}^{n-m}\}$$

$$\boxed{\begin{array}{l} \min f(x) \\ \vec{x} \in \mathbb{R}^n \text{ subject to } A\vec{x} = \vec{b} \end{array}} \xrightarrow{\text{Reduced to}} \boxed{\begin{array}{l} \min f(\vec{y} + z\vec{v}) \\ \vec{v} \in \mathbb{R}^{n-m} \end{array}}$$

$u(\vec{v})$

$\mathbb{R}^n$

$\mathbb{R}^{n \times n-m}$

unconstrained Minimization Problem

$$\nabla f(\vec{x}) = \nabla u(\vec{v}) = Z^T \nabla f(\vec{y} + z\vec{v})$$

$$\nabla^2 u(\vec{v}) = Z^T \underbrace{\nabla^2 f(\vec{y} + z\vec{v})}\, Z$$

$$= \underset{(n-m \times n)}{Z^T} \underset{(n \times n)}{\boxed{\nabla^2 f(\vec{x})}} \underset{(n \times n-m)}{Z}$$

Vector Calculus

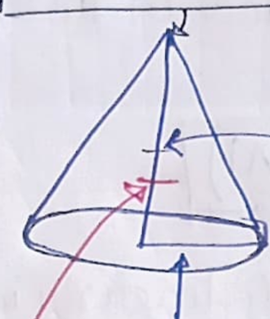
Optimization
- Newton's Method
- Gradient Descent
- Langrange's Method
- Gradient, Hassian?

Q. 8

$$\boxed{V = \frac{1}{3}\pi R^2 H}$$

Rate of flow of water: $5\ ft^3/min$

Given $\frac{dV}{dt} = 5\ ft^3/min$



20 feet (H)

5 feet

8 feet (R)

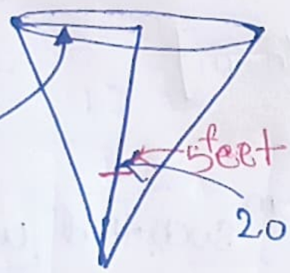
To find: $\frac{dh}{dt}$

height = h

time = t

Water rising in depth will increase by time increase

$\boxed{\text{Speed of depth rising} \propto \text{time}}$



8 feet (R)

5 feet

20 feet (H)

Speed of depth rising $\propto \frac{1}{\text{time}}$

$V = v(t); v(h);$

$h = h(t)$

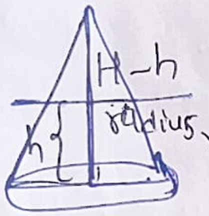
$\boxed{V = v(h)} = v(h(t))$

first this then this

Required to compute

$$\frac{dV}{dt} = \boxed{\frac{dV}{dh}} \cdot \frac{dh}{dt}$$

Splitting the cylinder into two parts to get 'V' as the function of 'h'



$= r(h)$

$$\cancel{\frac{1}{3}\pi r^2 (H-h)} \quad \frac{1}{3}\pi\, r(h)\cdot(H-h)$$

$$\frac{R}{H} = \frac{r(h)}{H-h}$$

$$\Rightarrow r(h) = \frac{R(H-h)}{H}$$

V of top cylinder $= \dfrac{1}{3}\pi\left(\dfrac{R(H-h)}{H}\right)^2 (H-h)$

V of bottom remaining part $= \dfrac{1}{3}\pi R^2 H - \dfrac{1}{3}\pi\left(\dfrac{R\cdot(H-h)}{H}\right)^2 (H-h)$

$$V = \boxed{\frac{1}{3}\pi R^2}\left(H - \left(\frac{H-h}{H}\right)^2\cdot(H-h)\right)$$

constant

$$\frac{dV}{dh} = \frac{1}{3}\pi R^2\cdot\frac{3(H-h)^2}{H^2}$$

$$= \frac{\pi R^2 (H-h)^2}{H^2}$$

$$\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{.5 H^2}{\pi R^2 (H-h)^2}$$

(a) At $h = 5ft$, $H = 20ft$

$$\frac{dh}{dt} = \frac{5 \times 20^2}{\pi \times 8^2 \times (20-5)^2}$$

$$= \frac{.5 \times \cancel{20}^{4} \times \cancel{20}^{4}}{\pi \times \cancel{8}_{2} \times \cancel{8}_{2} \times \cancel{15}_{3} \times \cancel{15}_{3}}$$

$$= \boxed{\frac{5}{36}\pi} \approx 0.4363 \text{ Ans}$$

Q.5 $f(x) = 1$ if $|x| \leq 2$

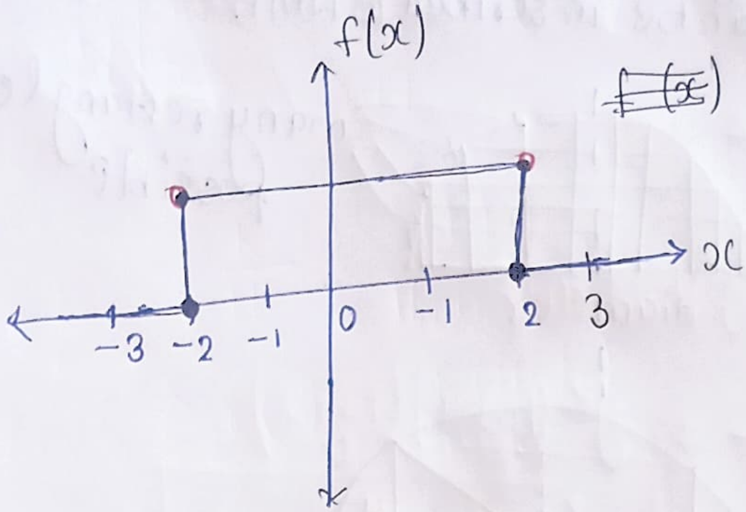
$= 0$ if $|x| > 2$

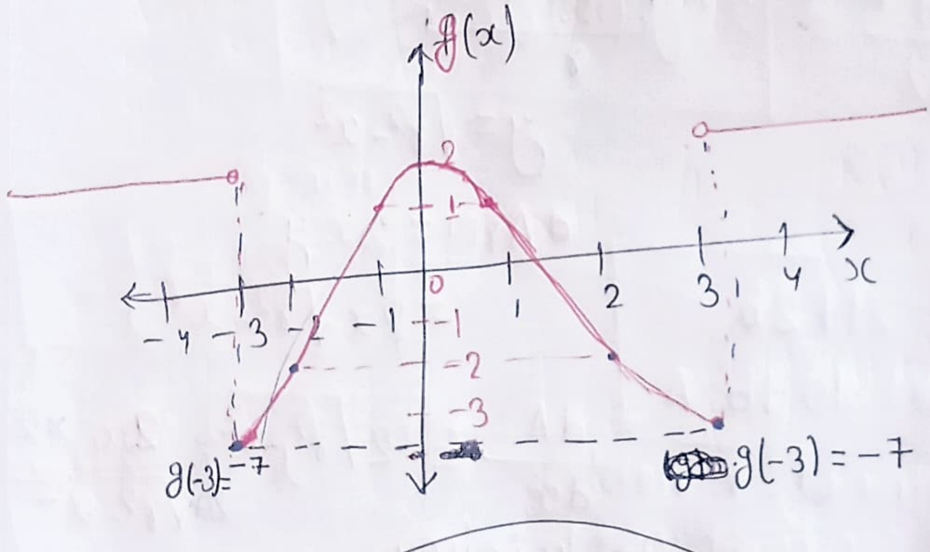
$g(x) = 2 - x^2$ if $|x| \leq 3$

$= 2$ if $|x| > 3$

$h(x) = f(g(x))$

For what values of x, h is continous ?





$h(x) = f(g(x))$

$h(x) = f(x) = f(g(x))$

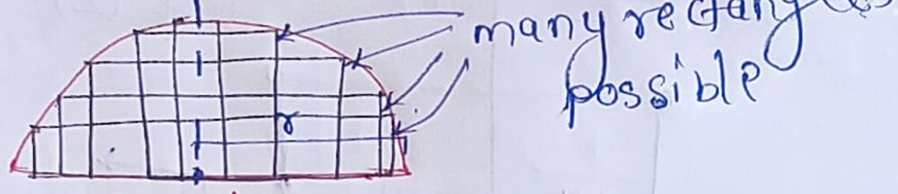
$= \begin{cases} 1 \text{ if } |g(x)| \leq 2 \\ 0 \text{ if } |g(x)| > 2 \end{cases}$

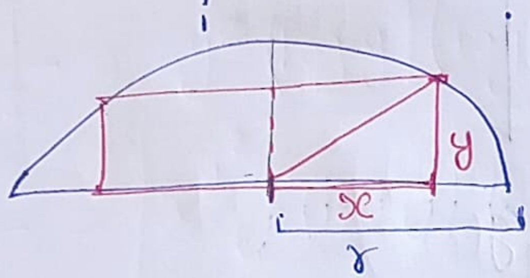
$= \begin{cases} 0 \text{ if } -3 \leq x < -2 \text{ or } 2 < x \leq 3 \\ 1 \text{ if } |x| \leq 2 \text{ or } |x| > 3 \end{cases}$

$(-3, -2, 2, 3) \leftarrow$ Discontinuity points

Q.1)

To find: largest area of rectangle that can be inscribed within





$$y = \sqrt{r^2 - x^2}$$

$$\text{Area} = 2xy$$
$$= 2x\sqrt{r^2 - x^2}$$

$$\frac{dA}{dx} = 2\sqrt{r^2 - x^2} - \frac{2x^2 \times \cancel{2}}{\cancel{2}\times\sqrt{r^2 - x^2}}$$

Checking for minima or maxima

$$\frac{dA}{dx} = 0 \Rightarrow 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = 0$$

$$\Rightarrow 2(r^2 - x^2) = 2x^2$$

$$\Rightarrow r^2 = 2x^2$$

$$\Rightarrow \boxed{x = \frac{r}{\sqrt{2}}} \quad \text{—(i)}$$

$$\frac{d^2A}{dx^2} = \frac{-2x}{\sqrt{r^2-x^2}} - \frac{\left(\sqrt{r^2-x^2}\cdot 4x + \frac{2x^3}{\sqrt{r^2-x^2}}\right)}{r^2-x^2} < 0$$

A is maximized: $\frac{2r}{\sqrt{2}} \times \sqrt{r^2 - \frac{r^2}{2}}$

$$= \frac{2r^2}{2}$$

$$\boxed{A_{max} = r^2}$$

$$\underbrace{\sqrt{2}r}_{L} \times \underbrace{\frac{r}{\sqrt{2}}}_{W}$$

r^2

Q.18(b)

$f(x, y, z) = x^{y^z}$ Need to find $\nabla f(x,y,z)$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

(1) $$\frac{\partial (x^{y^z})}{\partial x} = \boxed{y^z \cdot x^{(y^z-1)}}$$ Ans

(2) $\frac{\partial(x^{y^z})}{\partial y} = \boxed{x^{y^z}\log(x)\cdot z y^{z-1}}$ Ans.

$a^x = a^x \cdot \log_e a$

$\Rightarrow e^{x\log a}$

$f = x^{y^z}$

$\log f = y^z(\log x)$

$\Rightarrow \frac{1}{f}\cdot\frac{\partial f}{\partial z} = y^z(\log y)(\log x)$

$\Rightarrow \frac{\partial f}{\partial z} = \boxed{x^{y^z} y^z(\log y)(\log x)}$ Ans

Q.9

min p

$\boxed{\begin{array}{l}\text{min } p = 2l + 2b \\ \text{subject to } lb = A\end{array}}$

$\boxed{p = 2l + \frac{2A}{l}}$

$\boxed{b = \frac{A}{l}}$; $l \geq 0$

function of 2 variables but can be changed to function of 1 variable.

checking for min or max

$\frac{dP}{dl} = 2 - \frac{2A}{l^2} = 0$

Do samething as Q.11

$\Rightarrow A = l^2 \Rightarrow l = \sqrt{A}$

$\Rightarrow b = \sqrt{A}$

$\frac{d^2P}{dA^2} = \frac{4A}{l^3} > 0$

$l = \sqrt{A}$, $b = \sqrt{A}$

min value of $p = lb$

$= \sqrt{A}\cdot\sqrt{A} = A.$

Reverse question of Q.9

$P = 2l + 2b$

$$\boxed{\begin{aligned}&\max\ A = lb \\ &\text{Subject to } p = 2l + 2b\end{aligned}}$$

$$b = \frac{p - 2l}{2}$$

$$A = \frac{l(p - 2l)}{2}$$

$$\frac{dA}{dl} = \frac{p - 2l}{2} - l = 0$$

$$\Rightarrow \frac{p}{2} = 2l \Rightarrow l = p/4 \qquad b = p/4$$

$$\frac{d^2A}{dl^2} = -2 - 1 = -3 < 0 \ (\text{max})$$

$$l = \frac{p}{4}, \quad b = \frac{p}{4}$$

Deep