COS 226 Programming Assignment

8 Puzzle

Write a program to solve the 8-puzzle problem (and its natural generalizations) using the A* search algorithm.

The problem. The [8-puzzle problem]() is a puzzle invented and popularized by Noyes Palmer Chapman in the 1870s. It is played on a 3-by-3 grid with 8 square tiles labeled 1 through 8 and a blank square. Your goal is to rearrange the tiles so that they are in order, using as few moves as possible. You are permitted to slide tiles horizontally or vertically into the blank square. The following shows a sequence of legal moves from an *initial board* (left) to the *goal board* (right).

```
    1  3        1     3        1  2  3        1  2  3        1  2  3
 4  2  5   =>   4  2  5   =>   4     5   =>   4  5      =>   4  5  6
 7  8  6        7  8  6        7  8  6        7  8  6        7  8

 initial        1 left          2 up          5 left          goal
```

Best-first search. Now, we describe a solution to the problem that illustrates a general artificial intelligence methodology known as the [A* search algorithm](). We define a *search node* of the game to be a board, the number of moves made to reach the board, and the previous search node. First, insert the initial search node (the initial board, 0 moves, and a null previous search node) into a priority queue. Then, delete from the priority queue the search node with the minimum priority, and insert onto the priority queue all neighboring search nodes (those that can be reached in one move from the dequeued search node). Repeat this procedure until the search node dequeued corresponds to a goal board. The success of this approach hinges on the choice of *priority function* for a search node. We consider two priority functions:

- *Hamming priority function.* The number of tiles in the wrong position, plus the number of moves made so far to get to the search node. Intuitively, a search node with a small number of tiles in the wrong position is close to the goal, and we prefer a search node that have been reached using a small number of moves.
- *Manhattan priority function.* The sum of the Manhattan distances (sum of the vertical and horizontal distance) from the tiles to their goal positions, plus the number of moves made so far to get to the search node.

For example, the Hamming and Manhattan priorities of the initial search node below are 5 and 10, respectively.

```
 8  1  3        1  2  3     1  2  3  4  5  6  7  8    1  2  3  4  5  6  7  8
 4     2        4  5  6     ----------------------    ----------------------
 7  6  5        7  8        1  1  0  0  1  1  0  1    1  2  0  0  2  2  0  3

 initial          goal           Hamming = 5 + 0          Manhattan = 10 + 0
```

We make a key observation: To solve the puzzle from a given search node on the priority queue, the total number of moves we need to make (including those already made) is at least its priority, using either the Hamming or Manhattan priority function. (For Hamming priority, this is true because each tile that is out of place must move at least once to reach its goal position. For Manhattan priority, this is true because each tile must move its Manhattan distance from its goal position. Note that we do not count the blank square when computing the Hamming or Manhattan priorities.) Consequently, when the goal board is dequeued, we have discovered not only a sequence of moves from the initial board to the goal board, but one that makes the fewest number of moves. (Challenge for the mathematically inclined: prove this fact.)

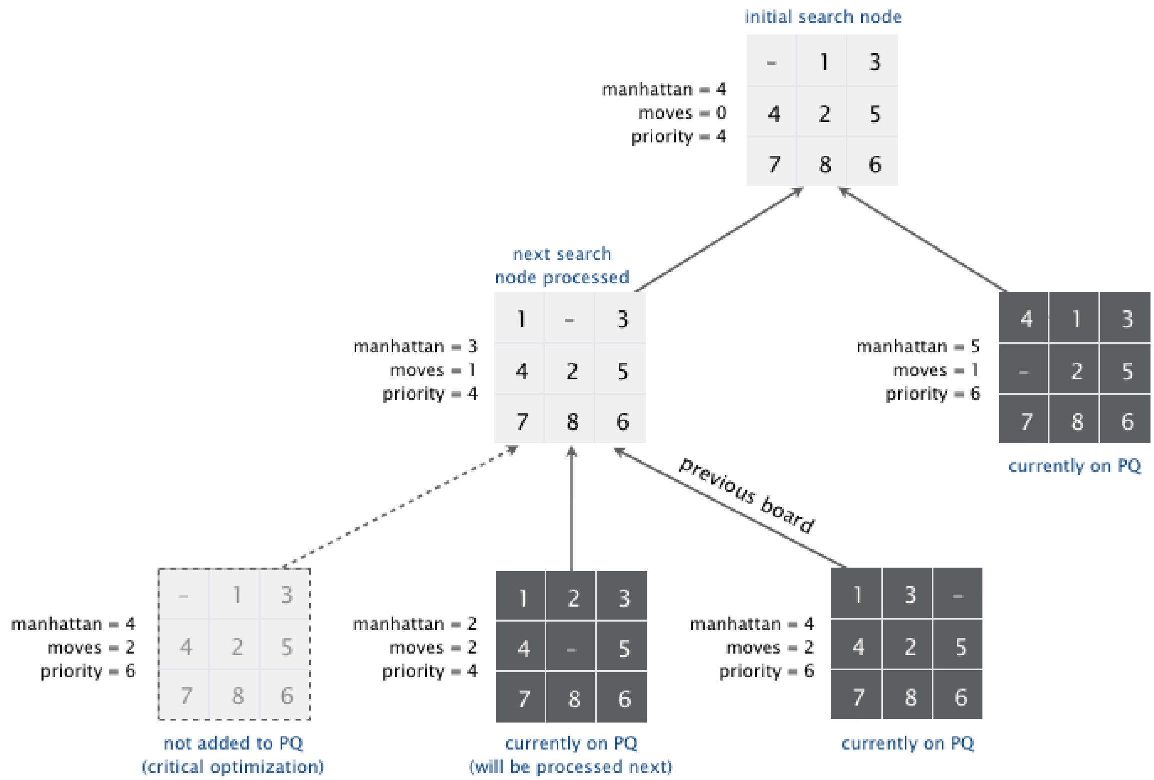
A critical optimization. Best-first search has one annoying feature: search nodes corresponding to the same board are enqueued on the priority queue many times. To reduce unnecessary exploration of useless search nodes, when considering the neighbors of a search node, don't enqueue a neighbor if its board is the same as the board of the previous search node.

```
 8  1  3       8  1  3       8  1          8  1  3       8  1  3
 4     2       4  2          4  2  3       4     2       4  2  5
 7  6  5       7  6  5       7  6  5       7  6  5       7  6

 previous    search node    neighbor     neighbor      neighbor
                                        (disallow)
```

A second optimization. To avoid recomputing the Manhattan distance of a board (or, alternatively, the Manhattan priority of a solver node) from scratch each time during various priority queue operations, compute it at most once per object; save its value in an instance variable; and return the saved value as needed. This *caching technique* is broadly applicable: consider using it in any situation where you are recomputing the same quantity many times *and* for which computing that quantity is a bottleneck operation.

Game tree. One way to view the computation is as a *game tree*, where each search node is a node in the game tree and the children of a node correspond to its neighboring search nodes. The root of the game tree is the initial search node; the internal nodes have already been processed; the leaf nodes are maintained in a *priority queue*; at each step, the A* algorithm removes the node with the smallest priority from the priority queue and processes it (by adding its children to both the game tree and the priority queue).



Detecting unsolvable puzzles. Not all initial boards can lead to the goal board by a sequence of legal moves, including the two below:

```
 1  2  3         1  2  3  4
 4  5  6         5  6  7  8
 8  7            9 10 11 12
                13 15 14
unsolvable
                unsolvable
```

To detect such situations, use the fact that boards are divided into two equivalence classes with respect to reachability: (i) those that lead to the goal board and (ii) those that cannot lead to the goal board. Moreover, we can identify in which equivalence class a board belongs *without* attempting to solve it.

- *Odd board size.* Given a board, an *inversion* is any pair of tiles *i* and *j* where *i* < *j* but *i* appears after *j* when considering the board in row-major order (row 0, followed by row 1, and so forth).

```
         1  2  3              1  2  3              1  2  3              1  2  3              1  2  3
         4  5  6      =>      4  5  6      =>      4     6      =>         4  6      =>      4  6  7
         8  7                 8     7              8  5  7              8  5  7              8  5

 1 2 3 4 5 6 8 7      1 2 3 4 5 6 8 7      1 2 3 4 6 8 5 7      1 2 3 4 6 8 5 7      1 2 3 4 6 7 8 5

  inversions = 1       inversions = 1       inversions = 3       inversions = 3       inversions = 3
          (8-7)                (8-7)         (6-5 8-5 8-7)        (6-5 8-5 8-7)        (6-5 7-5 8-5)
```

If the board size *N* is an odd integer, then each legal move changes the number of inversions by an even number. Thus, if a board has an odd number of inversions, then it *cannot* lead to the goal board by a sequence of legal moves because the goal board has an even number of inversions (zero).

The converse is also true: if a board has an even number of inversions, then it *can* lead to the goal board by a sequence of legal moves.

```
     1  3               1     3              1  2  3              1  2  3              1  2  3
  4  2  5     =>        4  2  5     =>       4     5     =>       4  5        =>       4  5  6
  7  8  6               7  8  6              7  8  6              7  8  6              7  8

1 3 4 2 5 7 8 6     1 3 4 2 5 7 8 6     1 2 3 4 5 7 8 6     1 2 3 4 5 7 8 6     1 2 3 4 5 6 7 8

 inversions = 4      inversions = 4      inversions = 2      inversions = 2      inversions = 0
(3-2 4-2 7-6 8-6)  (3-2 4-2 7-6 8-6)        (7-6 8-6)           (7-6 8-6)
```

- *Even board size*. If the board size *N* is an even integer, then the parity of the number of inversions is not invariant. However, the parity of the number of inversions *plus* the row of the blank square is invariant: each legal move changes this sum by an even number. If this sum is even, then it *cannot* lead to the goal board by a sequence of legal moves; if this sum is odd, then it *can* lead to the goal board by a sequence of legal moves.

```
 1  2  3  4            1  2  3  4            1  2  3  4            1  2  3  4            1  2  3  4
 5     6  8     =>     5  6     8     =>     5  6  7  8     =>     5  6  7  8     =>     5  6  7  8
 9 10  7 11            9 10  7 11            9 10    11            9 10 11               9 10 11 12
13 14 15 12           13 14 15 12           13 14 15 12           13 14 15 12           13 14 15

  blank row = 1       blank row  = 1       blank row  = 2       blank row  = 2       blank row  = 3
 inversions = 6       inversions = 6       inversions = 3       inversions = 3       inversions = 0
 --------------       --------------       --------------       --------------       --------------
        sum = 7              sum = 7              sum = 5              sum = 5              sum = 3
```

.

Board and Solver data types. Organize your program by creating an immutable data type `Board` with the following API:

```
public class Board {
    public Board(int[][] tiles)           // construct a board from an N-by-N array of tiles
                                          // (where tiles[i][j] = tile at row i, column j)
    public int tileAt(int i, int j)       // return tile at row i, column j (or 0 if blank)
    public int size()                     // board size N
    public int hamming()                  // number of tiles out of place
    public int manhattan()                // sum of Manhattan distances between tiles and goal
    public boolean isGoal()               // is this board the goal board?
    public boolean isSolvable()           // is this board solvable?
    public boolean equals(Object y)       // does this board equal y?
    public Iterable<Board> neighbors()    // all neighboring boards
    public String toString()              // string representation of this board (in the output format specified below)

    public static void main(String[] args) // unit testing (required)
}
```

Corner cases. You may assume that the constructor receives an *N*-by-*N* array containing the N^2 integers between 0 and N^2 – 1, where 0 represents the blank square. The `tileAt()` method should throw a `java.lang.IndexOutOfBoundsException` unless both `i` or `j` are between 0 and *N* – 1.

Performance requirements. Your implementation should support all `Board` methods in time proportional to N^2 (or better) in the worst case, with the exception that `isSolvable()` may take up to N^4 in the worst case.

Also, create an immutable data type `Solver` with the following API:

```
public class Solver {
    public Solver(Board initial)           // find a solution to the initial board (using the A* algorithm)
    public int moves()                     // min number of moves to solve initial board
    public Iterable<Board> solution()      // sequence of boards in a shortest solution
    public static void main(String[] args) // unit testing
}
```

To implement the A* algorithm, *you must use the* `MinPQ` *data type from* `algs4.jar` *for the priority queue.*

Corner cases. The constructor should throw a `java.lang.IllegalArgumentException` if the initial board is not solvable and a `java.lang.NullPointerException` if the initial board is `null`.

Input and output formats. The input and output format for a board is the board size *N* followed by the *N*-by-*N* initial board, using 0 to represent the blank square.

```
% more puzzle04.txt
3
 0  1  3
 4  2  5
 7  8  6

% java-algs4 Solver puzzle04.txt
Minimum number of moves = 4
3
 0  1  3
 4  2  5
 7  8  6

3
 1  0  3
 4  2  5
 7  8  6

3
 1  2  3
 4  0  5
 7  8  6

3
 1  2  3
 4  5  0
 7  8  6

3
 1  2  3
 4  5  6
 7  8  0
% more puzzle-unsolvable3x3.txt
3
 1  2  3
 4  5  6
 8  7  0

% java-algs4 Solver puzzle3x3-unsolvable.txt
Unsolvable puzzle
```

Your program should work correctly for arbitrary *N*-by-*N* boards (for any 1 ≤ *N* ≤ 32768), even if it is too slow to solve some of them in a reasonable amount of time.

Challenge for the bored. Implement a better solution which is capable of solving puzzles that the required solution is incapable of solving.

Deliverables. Submit the files `Board.java` and `Solver.java` (with the Manhattan priority). We will supply `algs4.jar`. Your may not call any library functions other than those in `java.lang`, `java.util`, and `algs4.jar`. You must use [`MinPQ`] for the priority queue. Finally, submit a [readme.txt] file and answer the questions.