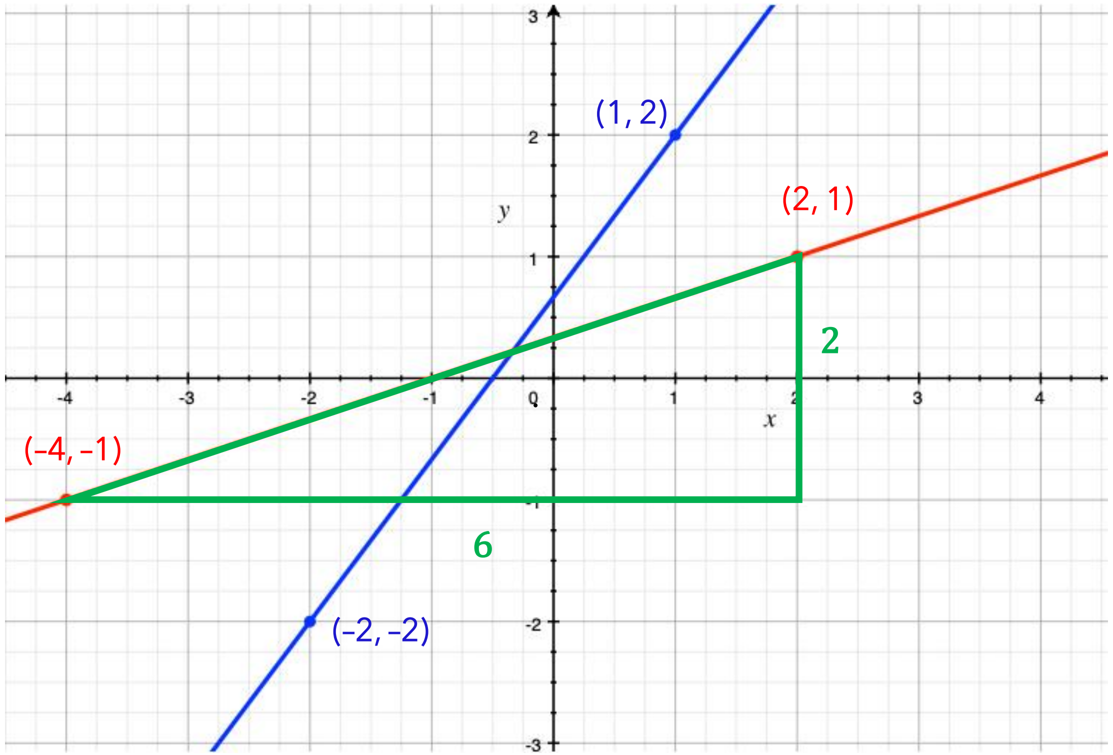
SLOPE

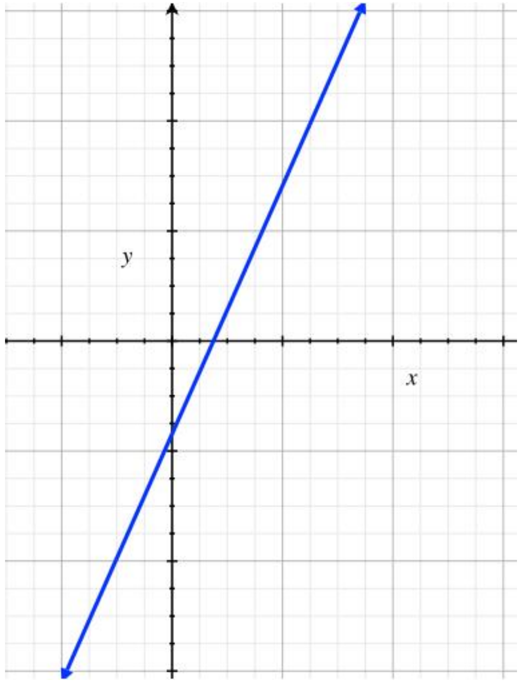
$$\text{Slope} = \frac{\text{Vertical change}}{\text{Horizontal change}}$$

$$m = \frac{2-(-2)}{1-(-2)} = \frac{4}{3}$$

$$m = \frac{1-(-1)}{2-(-4)} = \frac{2}{6} = \frac{1}{3}$$

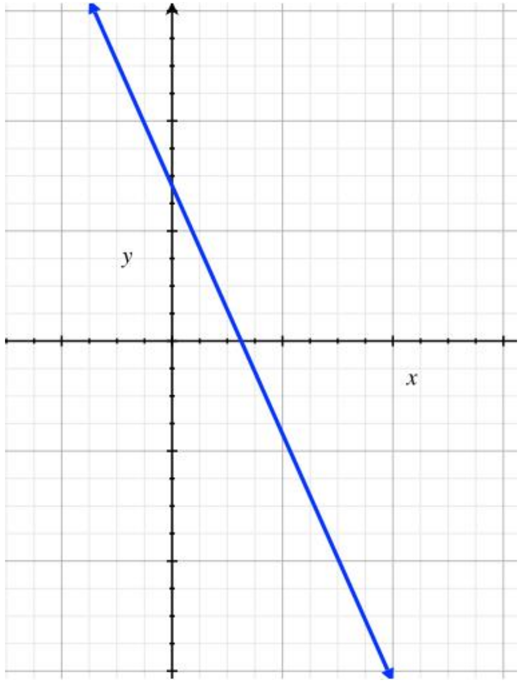


Positive slope



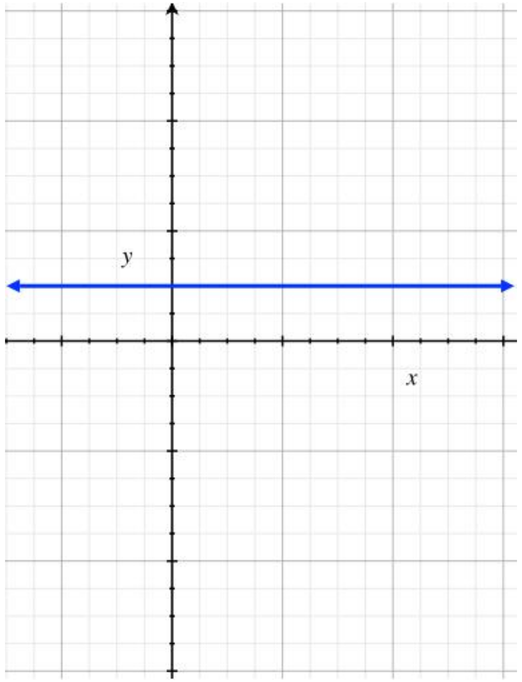
$y = mx + b$
$m > 0$

Negative slope



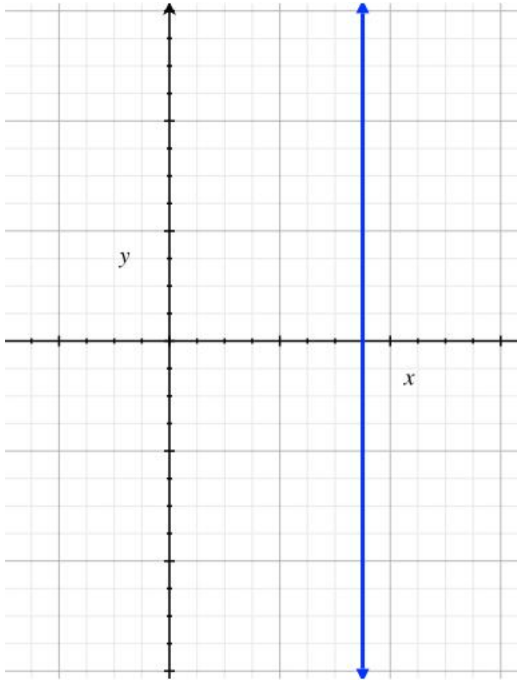
$y = mx + b$
$m < 0$

Zero slope



$y = b$

Undefined slope



$x = a$

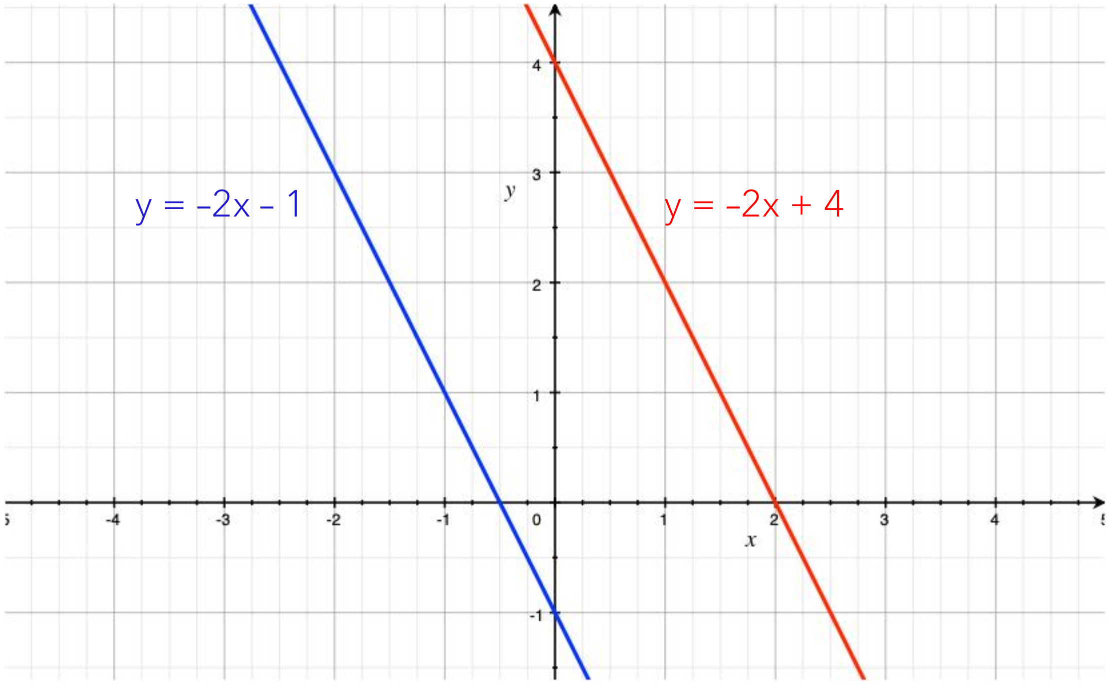
LINEAR EQUATIONS

- Slope-intercept form: $y = mx + b$
- Standard form: $Ax + By = C$, where A and B are not both 0.

PARALLEL LINES

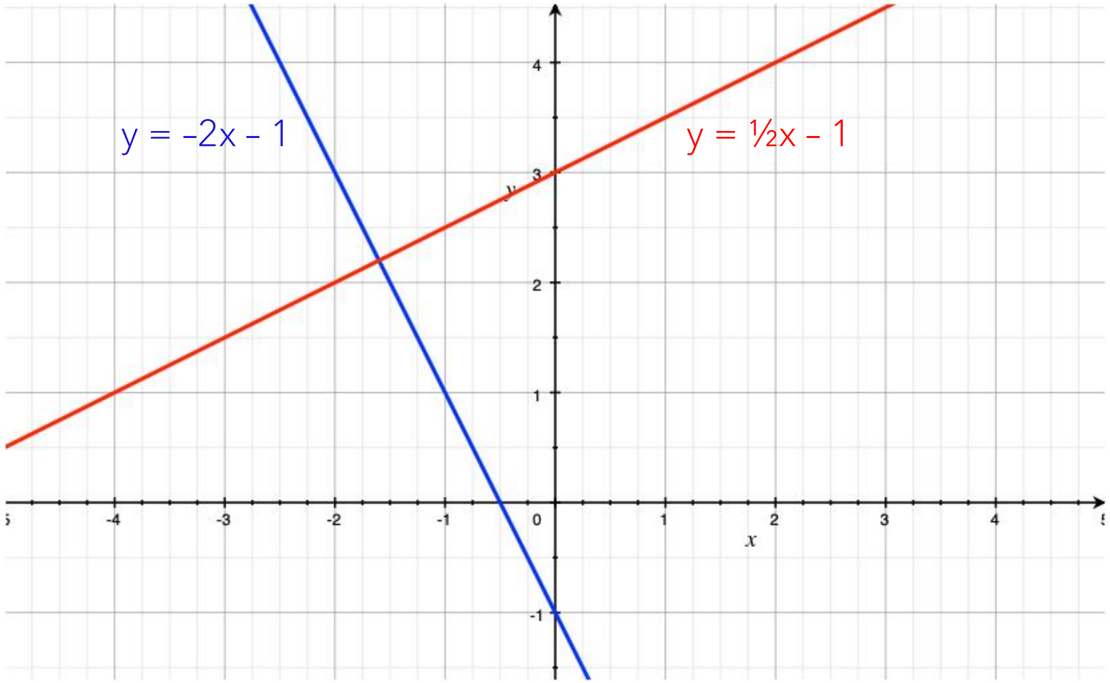
Same slope (positive, negative, zero)

or both vertical



PERPENDICULAR LINES

- Product of slopes is –1 or one is vertical and the other horizontal



EXAMPLE

Passes through (–2, 6) and parallel to

$y = \frac{2}{3}x - \frac{5}{3}$

$$m = \frac{2}{3}$$

$$y = \frac{2}{3}x + b$$

$$\left(\frac{2}{3}\right)(-2) + b = 6$$

$$-\frac{4}{3} + b = 6$$

$$b = \frac{4}{3} + \frac{18}{3} = \frac{22}{3}$$

$$\boldsymbol{y = \frac{2}{3}x + \frac{22}{3}}$$

Passes through (–2, 6) and perpendicular to

$y = \frac{2}{3}x - \frac{5}{3}$

$$m = -\frac{3}{2}$$

$$y = -\frac{3}{2}x + b$$

$$\left(-\frac{3}{2}\right)(-2) + b = 6$$

$$3 + b = 6$$

$$b = 6 - 3 = 3$$

$$\boldsymbol{y = -\frac{3}{2}x + 3}$$

BREAK-EVEN ANALYSIS

- Linear cost function, $C(x) = mx + b$

m is the marginal cost, b is the fixed cost, x is the number of items produced

- Revenue function, $R(x) = px$

p is the price per unit and x is the number of units sold

- Profit function, $P(x) = R(x) – C(x)$

- Break-even point: The point where $R(x) = C(x)$

Occurs where the two lines intersect

EXAMPLE

The cost to produce x widgets is given by $C(x) = 105x + 6000$ and each widget sells for \$250. Determine the break-even quantity.

<u>Solution:</u>

$R(x) = 250x$

$250x = 105x + 6000$

$145x = 6000$

$x \sim 41.38$

$R(41) = 250(41) = 10{,}250$

and

$C(41) = 105(41) + 6000 = 10{,}305$

$R(42) = 250(42) = 10{,}500$

and

$C(42) = 105(42) + 6000 = 10{,}410$

<u>Note:</u> Selling 41 widgets is not enough.

<u>The breakeven quantity is 42 widgets.</u>

LEAST SQUARES LINE

Minimize the sum of the squares of the vertical distances from the data points to the line

$$y = mx + b$$

Data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

and

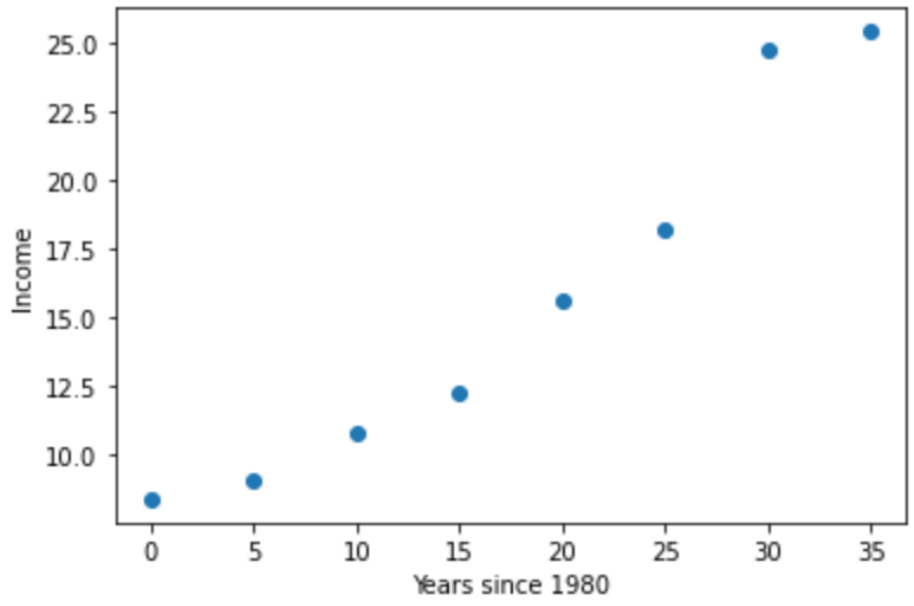
$$b = \frac{\sum y - m(\sum x)}{n}$$

SCATTERPLOT

Income from side business

Year	Income
1980	8,414
1985	9,124
1990	10,806
1995	12,321
2000	15,638
2005	18,242
2010	24,792
2015	25,436

Let x represent the number of years since 1980 and y represent the income in thousands of dollars



LEAST SQUARES CALCULATIONS

Year	Income
1980	8,414
1985	9,124
1990	10,806
1995	12,321
2000	15,638
2005	18,242
2010	24,792
2015	25,436

Least Squares Calculations				
x	y	xy	x^2	y^2
0	8.414	0	0	70.795396
5	9.124	45.62	25	83.247376
10	10.806	108.06	100	116.769636
15	12.321	184.815	225	151.807041
20	15.638	312.76	400	244.547044
25	18.242	456.05	625	332.770564
30	24.792	743.76	900	614.643264
35	25.436	890.26	1225	646.990096
140	**124.773**	**2741.325**	**3500**	**2261.57042**

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{8(2741.325) - (140)(124.773)}{8(3500) - (140)^2}$$

$$= 0.5312$$

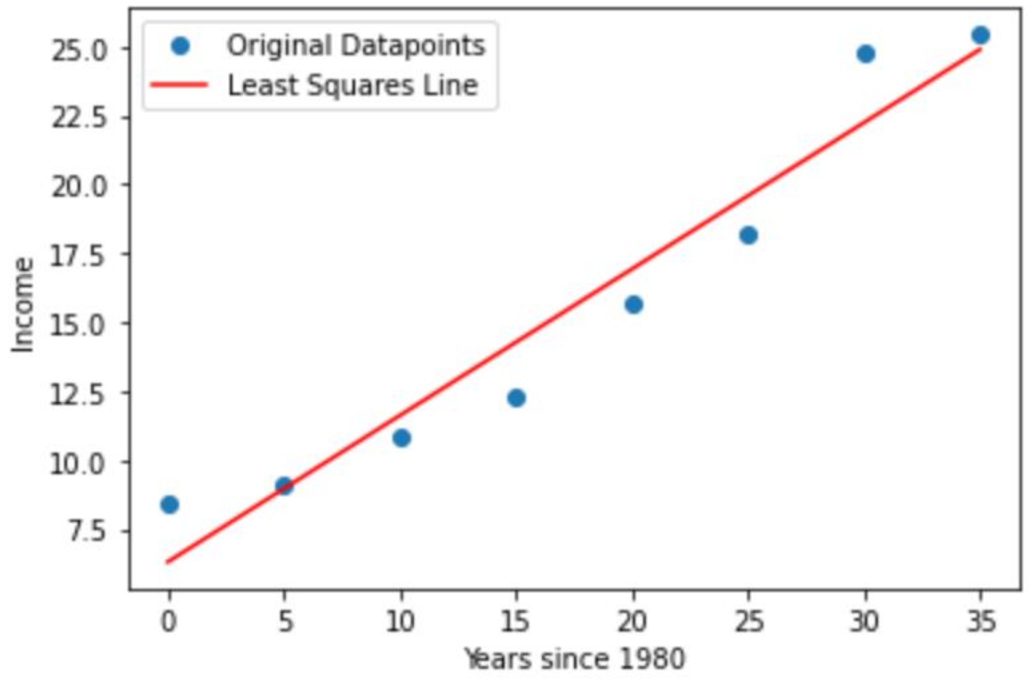
$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{124.773 - (0.5312)(140)}{8}$$

$$= 6.3$$

$$\boldsymbol{y = 0.5312x + 6.3}$$

GRAPH OF LEAST SQUARES LINE



LEAST SQUARES LINE PREDICTION

Use the least squares line $y = 0.5312x + 6.3$ to predict income in 2025

Recall, x is the number of years since 1980, so $x = 45$ corresponds to 2025

$$y = (0.5312)(45) + 6.3 = 30.204$$

Since y is in thousands of dollars, the predicted income in 2025 is $30,204

CORRELATION COEFFICIENT

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$= \frac{8(2741.325)(140)(124.773)}{\sqrt{8(3500) - (140)^2} \cdot \sqrt{8(2261.57042) - (124.773)^2}}$$

$$\mathbf{= 0.9691}$$

Least Squares Calculations				
x	y	xy	x^2	y^2
0	8.414	0	0	70.795396
5	9.124	45.62	25	83.247376
10	10.806	108.06	100	116.769636
15	12.321	184.815	225	151.807041
20	15.638	312.76	400	244.547044
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35	25.436	890.26	1225	646.990096
140	**124.773**	**2741.325**	**3500**	**2261.57042**

Prediction | Forecast

Time

PYTHON

Gradient Descent (GD)

1. Basic GD
2. Momentum GD
3. Nesterov Momentum GD
4. AdaGrad
5. RMS Prop
6. AdaDelta
7. Adam
8. Hybrid Gradient

```python
from scipy import stats
x = [0,5,10,15,20,25,30,35]
y = [8.414,9.124,10.806,12.321,15.638,18.242,24.792,25.436]
slope, intercept, r_value, p_value, std_err = stats.linregress(x, y)
print("slope = ", slope)
print("intercept = ", intercept)
print("correlation coefficient = ", r_value)
```

```
slope =  0.5312357142857143
intercept =  6.300000000000001
correlation coefficient =  0.9690801754643459
```

AVERAGE RATE OF CHANGE

The average rate of change of $f(x)$ with respect to x as x changes from a to b is

$$\frac{f(b)-f(a)}{b-a}$$

Based on population projections for 2000 to 2050, the projected Hispanic population (in millions) for a certain country can be modeled by the exponential function

$$H(t) = 37.791(1.021)^t$$

where $t = 0$ corresponds to 2000 and $0 \leq t \leq 50$. Use H to estimate the average rate of change in the Hispanic population from 2000 to 2010.

The years 2000 and 2010 correspond to $t = 0$ and $t = 10$, respectively

Tip: Use technology

```
(37.791*1.021**10-37.791*1.021**0)/10
0.8729653294860398
```

$$\frac{H(10)-H(0)}{10-0} = \frac{37.791(1.021)^{10} - 37.791(1.021)^{0}}{10}$$

$$\approx \frac{8.73}{10} = 0.873$$

Never round until the last step

Based on this model, the Hispanic population increased at an average rate of approximately 873,000 people per year between 2000 and 2010

INSTANTANEOUS RATE OF CHANGE

Suppose a car is stopped at a traffic light. When the light turns green, the car begins to move along a straight road. Assume that the distance traveled by the car is given by $s(t) = 3t^2$, for $0 \le t \le 15$ where t is time in seconds and $s(t)$ is distance traveled in feet.

How do we find the exact velocity of the car at say, $t = 10$?

Interval	Average velocity
$t = 10$ to $t = 10.1$	$\frac{s(10.1) - s(10)}{10.1 - 10} = \frac{306.03 - 300}{0.1} = 60.3$
$t = 10$ to $t = 10.01$	$\frac{s(10.01) - s(10)}{10.01 - 10} = \frac{300.6003 - 300}{0.01} = 60.03$
$t = 10$ to $t = 10.001$	$\frac{s(10.001) - s(10)}{10.001 - 10} = \frac{300.060003 - 300}{0.001} = 60.003$

Table suggests that the velocity at $t = 10$ is 60 ft/sec.

Consider the following where h is small but not 0

$$\frac{s(10+h) - s(10)}{(10+h) - 10} = \frac{s(10+h) - s(10)}{h}$$

Velocity represents both how fast something is moving and its direction, so ***<u>velocity can be negative.</u>***

$$\frac{s(10+h) - s(10)}{h} = \frac{3(10+h)^2 - 3(10)^2}{h}$$

$$= \frac{3(100 + 20h + h^2) - 300}{h}$$

$$= \frac{300 + 60h + 3h^2 - 300}{h}$$

$$= \frac{60h + 3h^2}{h} = \frac{h(60 + 3h)}{h} = 60 + 3h$$

$$\lim_{h \to 0} \frac{s(10+h) - s(10)}{h} = \lim_{h \to 0} (60 + 3h) = 60 \text{ ft/sec}$$

INSTANTANEOUS RATE OF CHANGE

The instantaneous rate of change for a function f when $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists

<u>Difference Quotient</u>

$$\frac{f(a+h) - f(a)}{h}$$

Alternate Form

The instantaneous rate of change for a function f when $x = a$ can be written as

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

provided this limit exists

EXAMPLE

Suppose the total profit in hundreds of dollars from selling x items is given by $P(x) = 2x^2 - 5x + 6$. Find and interpret the following:

(a) The average rate of change of profit from $x = 2$ to $x = 4$
(b) The average rate of change of profit from $x = 2$ to $x = 3$
(c) The instantaneous rate of change of profit with respect to the number produced when $x = 2$

$$\frac{P(4) - P(2)}{4 - 2} = \frac{(2(4)^2 - 5(4) + 6) - (2(2)^2 - 5(2) + 6)}{2}$$

$$= \frac{18 - 4}{2} = 7$$

The average rate of change of profit from $x = 2$ to $x = 4$ is $\$700$ per item

$$\frac{P(3) - P(2)}{3 - 2} = \frac{(2(3)^2 - 5(3) + 6) - (2(2)^2 - 5(2) + 6)}{1}$$

$$= 9 - 4 = 5$$

The average rate of change of profit from $x = 2$ to $x = 3$ is $\$500$ per item

$$\lim_{h \to 0} \frac{P(2 + h) - P(2)}{h} = \lim_{h \to 0} \frac{(2(2 + h)^2 - 5(2 + h) + 6) - 4}{h}$$

$$= \lim_{h \to 0} \frac{(8 + 8h + 2h^2 - 10 - 5h + 6) - 4}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 3h}{h}$$

$$= \lim_{h \to 0} (2h + 3) = 3$$

The instantaneous rate of change of profit with respect to the number of items produced when $x = 2$ is $\$300$ per item

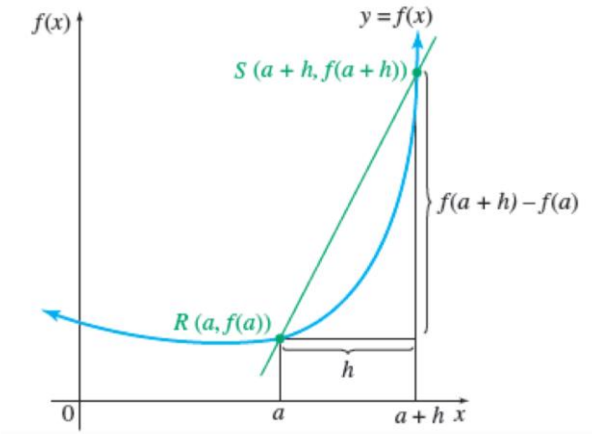
DERIVATIVES

SECANT AND TANGENT LINES

The slope of the secant line of the graph of $y = f(x)$ containing the points $(a, f(a))$ and $(a + h, f(a + h))$ is given by

$$\frac{f(a+h) - f(a)}{h}$$

Slope of secant line = average rate of change

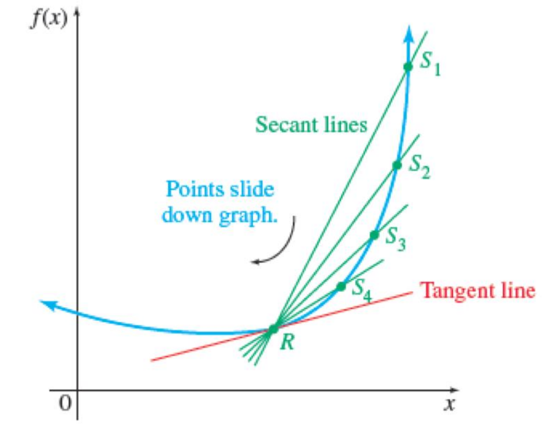


The slope of the tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

Slope of tangent line = instantaneous rate of change



DEFINITION OF THE DERIVATIVE

The derivative of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function $f'(x)$ represents the instantaneous rate of change of $y = f(x)$ with respect to x

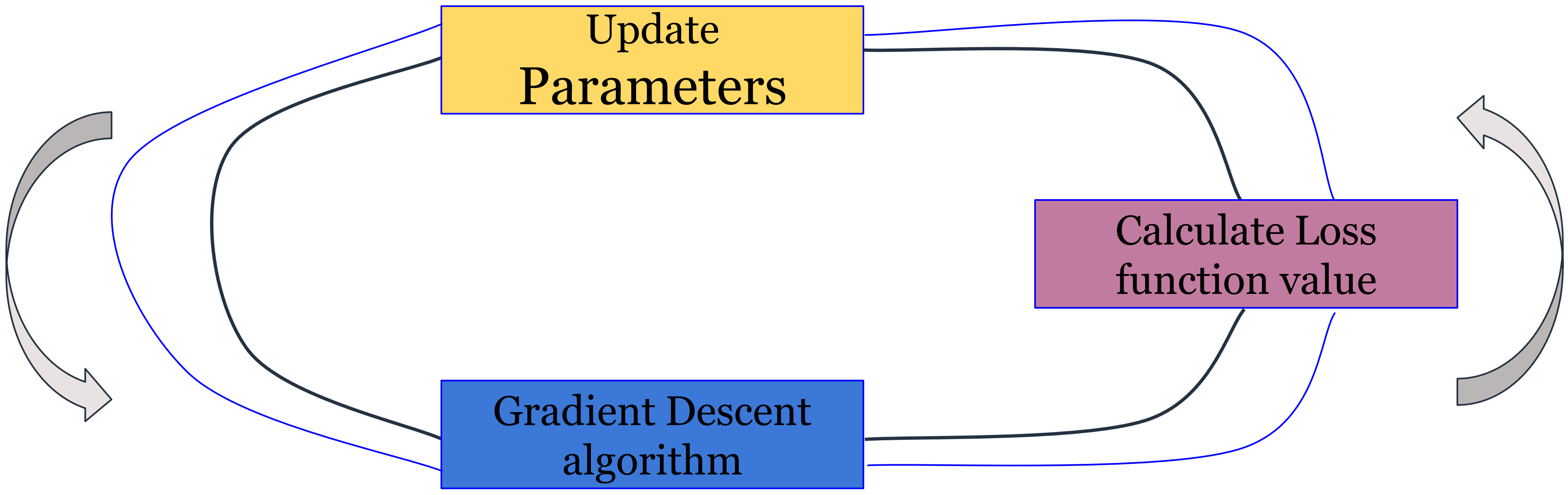
The function $f'(x)$ represents the slope of the graph at any point x

If $f'(x)$ is evaluated at the point $x = a$, then it represents the slope of the curve, or the slope of the tangent line at that point

APPLICATIONS OF DERIVATIVES

- Rate of Change of Quantities
- Increasing and Decreasing Functions
- Maxima and Minima

OVERVIEW



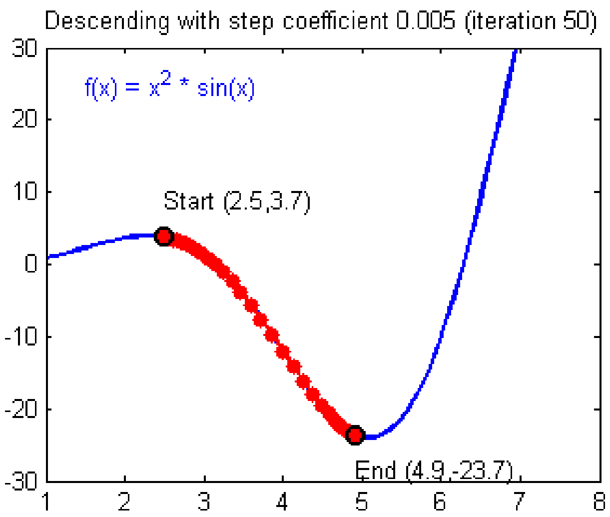
Do this multiple times...

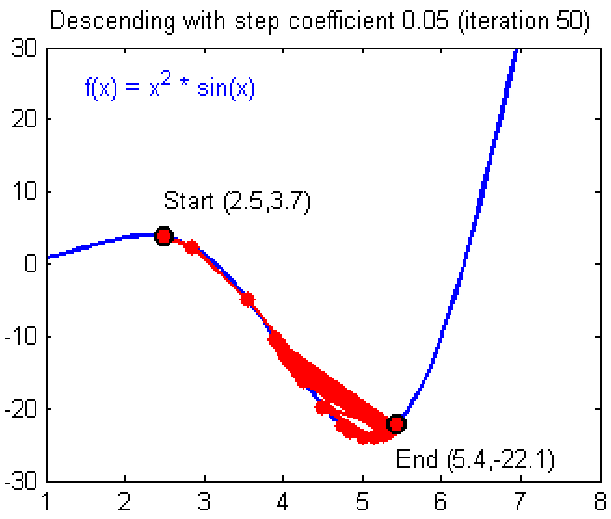
VISUALIZING GD : LEARNING RATE AND LOSS FUNCTION

$$Loss = \frac{\sum_{i=1}^{m} (Prediction_i - Actual_i)^2}{2 \times m}$$

Target: Find optimal model parameters to minimize the Loss

$$W_{new} = W_{old} - \eta \frac{d}{dW} Loss(W_{old})$$





QUESTIONS?