Gamma functions.

The funⁿ of n defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \text{ is called}$$

Gamma funⁿ (where $n > 0$)

$$\boxed{\int_0^\infty e^{-x} x^n dx = \Gamma n+1}$$

Properties of Gamma funⁿ

① $\Gamma 1 = 1$ $\Gamma 0 = \infty$, $\Gamma n = \infty$ if n is $-ve$

② $\Gamma \frac{1}{2} = \sqrt{\pi}$

③ $\Gamma n = (n-1)\,\Gamma n-1$

eg. $\Gamma \frac{5}{2} = \left(\frac{5}{2}-1\right)\Gamma \frac{5}{2}-1 = \frac{3}{2}\Gamma \frac{3}{2}$

$= \frac{3}{2}\left(\frac{3}{2}-1\right)\Gamma \frac{3}{2}-1$

$= \frac{3}{2}\cdot\frac{1}{2}\,\Gamma \frac{1}{2} = \frac{3}{4}\sqrt{\pi}$

$\Gamma 5 = 4\Gamma 4 = 4\cdot 3\Gamma 3 = 4\cdot 3\cdot 2\,\Gamma 2 = 4\cdot 3\cdot 2\cdot 1$
$= 4!$

④ $\Gamma n = (n-1)!$ if n is +ve int.

⑤ $\Gamma p\,\Gamma 1-p = \frac{\pi}{\sin p\pi}$ $0 < p < 1$

eg $\Gamma \frac{1}{4}\,\Gamma \frac{3}{4} = \Gamma \frac{1}{4}\,\Gamma 1-\frac{1}{4} = \frac{\pi}{\sin \pi/4} = \frac{\pi}{1/\sqrt{2}}$

$p = \frac{1}{4}$ $= \sqrt{2}\,\pi$

$$I = \int_0^\infty e^{-a^2x^2}\,dx$$

$$\int_0^\infty e^{-x} x^n\,dx = \Gamma\,n+1$$

put $a^2x^2 = t$

$$x^2 = \frac{t}{a^2} \Rightarrow x = \frac{t^{1/2}}{a}$$

$$dx = \frac{1}{a}\,\frac{1}{2}\,t^{-1/2}\,dt$$

$x : 0 \to \infty \qquad t : 0 \to \infty$

$$I = \int_0^\infty e^{-t}\,\frac{1}{a}\,\frac{1}{2}\,t^{-1/2}\,dt = \frac{1}{2a}\int_0^\infty e^{-t}\,t^{-1/2}\,dt$$

$$= \frac{1}{2a}\,\Gamma\,{-\tfrac{1}{2}+1} = \frac{1}{2a}\,\Gamma\,\tfrac{1}{2}$$

$$= \frac{\sqrt{\pi}}{2a}$$

$$I = \int_0^\infty x^{1/4} e^{-\sqrt{x}} \, dx$$

put $\sqrt{x} = t$ $\qquad$ $x : 0 \to \infty$

$x = t^2$ $\qquad$ $t : 0 \to \infty$

$dx = 2t \, dt$

$$I = \int_0^\infty (t^2)^{1/4} e^{-t} \, 2t \, dt$$

$$= 2\int_0^\infty e^{-t} t^{1/2} t \, dt = 2\int_0^\infty e^{-t} t^{3/2} \, dt$$

$$= 2 \Gamma\left(\frac{3}{2}+1\right) = 2\Gamma\left(\frac{5}{2}\right)$$

$$= 2 \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right) \quad \text{—— } \Gamma(n) = (n-1)\Gamma(n-1)$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

P.T. $I = \int_0^\infty x e^{-x^8} dx \cdot \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$

$I_1 = \int_0^\infty x e^{-x^8} dx$

$x^8 = t$

$x = t^{1/8}$

$dx = \frac{1}{8} t^{-7/8} dt$

$x: 0 \to \infty \qquad t: 0 \to \infty$

$I_1 = \int_0^\infty (t)^{1/8} e^{-t} \frac{1}{8} t^{-7/8} dt$

$= \frac{1}{8} \int_0^\infty e^{-t} t^{-\frac{6^3}{8_4}} dt$

$= \frac{1}{8} \Gamma\left(-\frac{3}{4}+1\right) = \frac{1}{8} \Gamma\left(\frac{1}{4}\right)$

$I_2 = \int_0^\infty x^2 e^{-x^4} dx$

$x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

$x: 0 \to \infty \qquad t: 0 \to \infty$

$I_2 = \int_0^\infty (t^{1/4})^2 e^{-t} \frac{1}{4} t^{-3/4} dt$

$= \frac{1}{4} \int_0^\infty e^{-t} t^{2/4 - 3/4} dt$

$= \frac{1}{4} \int_0^\infty e^{-t} t^{-1/4} dt$

$= \frac{1}{4} \Gamma\left(-\frac{1}{4}+1\right) = \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$

$I = I_1 \cdot I_2 = \frac{1}{8} \Gamma\left(\frac{1}{4}\right) \frac{1}{4} \Gamma\left(\frac{3}{4}\right) = \frac{1}{32} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1-\frac{1}{4}\right)$

$= \frac{1}{32} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{32} \frac{\pi}{1/\sqrt{2}}$

— $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

$0 < \frac{1}{4} < 1$

$= \frac{\sqrt{2}\pi}{32} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\pi}{16\sqrt{2}}$

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$$

$$= \int_0^1 x^m (\log 1 - \log x)^n dx$$

$$= \int_0^1 x^m (-\log x)^n dx$$

$-\log x = t$

$\Rightarrow \log x = -t \Rightarrow x = e^{-t}$

$dx = e^{-t}(-1)\,dt$

$x : 0 \to 1$ $\quad t : \infty \to 0$

$$= \int_\infty^0 (e^{-t})^m (t)^n e^{-t}(-1)\,dt$$

$$= \int_0^\infty e^{-tm-t}\, t^n\, dt$$

$$= \int_0^\infty e^{-t(m+1)}\, t^n\, dt$$

$t(m+1) = u$

$t = \frac{u}{m+1} \Rightarrow dt = \frac{du}{m+1}$

$t : 0 \to \infty \quad u : 0 \to \infty$

$$= \int_0^\infty e^{-u} \left(\frac{u}{m+1}\right)^n \frac{du}{(m+1)}$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-u} u^n du = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

HW

$$\int_0^1 (x \log x)^3 dx$$

$$\int_0^\infty \frac{x^7}{7^x}\,dx.$$

$$I = \int_0^\infty x^7\, 7^{-x}\,dx$$

$$7^{-x} = e^{-t}$$

$$\log 7^{-x} = \log e^{-t}$$

$$+x\log 7 = +t\log e$$

$$x = \frac{t}{\log 7} \Rightarrow dx = \frac{dt}{\log 7}$$

$x: 0 \to \infty \quad t: 0 \to \infty$

$$I = \int_0^\infty \left(\frac{t}{\log 7}\right)^7 (e^{-t})\,\frac{dt}{\log 7}$$

$$= \frac{1}{(\log 7)^8}\int_0^\infty e^{-t}\,t^7\,dt$$

$$= \frac{1}{(\log 7)^8}\Gamma 8 = \frac{7!}{(\log 7)^8}$$

$\Gamma n = (n-1)!$ if n +ve int

$$\int_0^\infty e^{-x} x^n \, dx = \Gamma(n+1)$$

$\Gamma(n)$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)$

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi} \qquad 0<p<1$$

$\Gamma 0 = \Gamma\text{-ve} = \infty$

$$\int_0^\infty 7^{-4x^2} dx$$

put $7^{-4x^2} = e^{-t}$

$$\log 7^{-4x^2} = \log e^{-t}$$

$$\cancel{+}4x^2 \log 7 = \cancel{+}t$$

$$x^2 = \frac{t}{4\log 7} \Rightarrow x = \frac{t^{1/2}}{2(\log 7)^{1/2}}$$

$$dx = \frac{1}{2\sqrt{\log 7}} \frac{1}{2} t^{-1/2} dt$$

$x : 0 \to \infty$

$t : 0 \to \infty$

$$I = \int_0^\infty e^{-t} \frac{1}{2\sqrt{\log 7}} \frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{4\sqrt{\log 7}} \Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \frac{1}{4\sqrt{\log 7}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$$

p.7 $\displaystyle\int_0^\infty x^{m-1} \cos ax \, dx = \frac{\Gamma m}{a^m} \cos\left(\frac{m\pi}{2}\right)$

$$\int_0^\infty x^{m-1} \sin ax \, dx = \frac{\Gamma m}{a^m} \sin\left(\frac{m\pi}{2}\right)$$

$$\begin{cases} e^{\pm i\theta} = \cos\theta \pm i\sin\theta \\ e^{-iax} = \cos ax - i \sin ax \end{cases}$$

Consider $\displaystyle\int_0^\infty x^{m-1} e^{-iax} \, dx$

put, $iax = t$

$x = \dfrac{t}{ia} \Rightarrow dx = \dfrac{dt}{ia}$

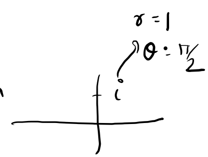
$x : 0 \to \infty \qquad t : 0 \to \infty$

$$\int_0^\infty \left(\frac{t}{ia}\right)^{m-1} e^{-t} \frac{dt}{ia}.$$

$$= \frac{1}{(ia)^m} \int_0^\infty e^{-t} t^{m-1} dt = \frac{1}{(ia)^m} \Gamma m$$

$$\therefore \int_0^\infty x^{m-1} e^{-iax} dx = \frac{\Gamma m}{a^m} (i)^{-m}$$

$$= \frac{\Gamma m}{a^m} \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-m}$$



$$\int_0^\infty x^{m-1} (\cos ax - i \sin ax) \, dx = \frac{\Gamma m}{a^m}\left(\cos\left(-\frac{m\pi}{2}\right) + i\sin\left(-\frac{m\pi}{2}\right)\right)$$

By DeMoivre's thm

$$\Rightarrow \int_0^\infty x^{m-1} \cos ax \, dx - i\int_0^\infty x^{m-1} \sin ax \, dx = \frac{\Gamma m}{a^m}\left(\cos\left(\frac{m\pi}{2}\right) - i\sin\left(\frac{m\pi}{2}\right)\right)$$

By comparing real & img parts

$$\int_0^\infty x^{m-1} \cos ax \, dx = \frac{\Gamma m}{a^m} \cos\left(\frac{m\pi}{2}\right), \quad \int_0^\infty x^{m-1} \sin ax \, dx = \frac{\Gamma m}{a^m} \sin\left(\frac{m\pi}{2}\right)$$

$$\int_0^\infty \cos(a x^{1/n})\, dx = \frac{\Gamma(n+1)}{a^n} \cos\left(\frac{n\pi}{2}\right).$$

put $a x^{1/n} = t$

$$x^{1/n} = \frac{t}{a}$$

$$x = \frac{t^n}{a^n} \Rightarrow dx = \frac{1}{a^n} n t^{n-1} dt$$

$x: 0 \to \infty \quad t: 0 \to \infty$

$$\int_0^\infty \cos(t)\, \frac{1}{a^n} n t^{n-1} dt$$

$$= \frac{n}{a^n} \int_0^\infty t^{n-1} \cos t \, dt.$$

$$e^{-it} = \cos t - i \sin t$$

Consider $\frac{n}{a^n} \int_0^\infty t^{n-1} e^{-it} \, dt$

put $it = u$

HW

$$\int_0^\infty x\, e^{-ax} \cos bx\, dx = \frac{a^2-b^2}{(a^2+b^2)^2}$$

$$\int_0^\infty x\, e^{-ax} \sin bx\, dx = \frac{2ab}{(a^2+b^2)^2}$$

$$e^{-ibx} = \cos bx - i \sin bx$$

Consider

$$\int_0^\infty x\, e^{-ax}\, e^{-ibx}\, dx$$

$$= \int_0^\infty x\, e^{-x(a+ib)}\, dx$$

$$x(a+ib) = t \Rightarrow \quad x = \frac{t}{a+ib}$$

$$dx = \frac{dt}{a+ib}$$

$$x : 0 \to \infty \qquad t : 0 \to \infty$$

$$= \int_0^\infty \left(\frac{t}{a+ib}\right) e^{-t}\, \frac{dt}{a+ib}$$

$$= \frac{1}{(a+ib)^2} \int_0^\infty e^{-t}\, t\, dt = \frac{1}{a^2+i2ab-b^2}\, \Gamma 2$$

$$\int_0^\infty x\, e^{-ax}\, e^{-ibx}\, dx = \frac{1}{(a^2-b^2)+i2ab} \quad (1!)$$

$$= \frac{(a^2-b^2)-i2ab}{((a^2-b^2)+i2ab)((a^2-b^2)-i2ab)}$$

$$= \frac{(a^2-b^2)-i2ab}{(a^2-b^2)^2+4a^2b^2}$$

$$= \frac{(a^2-b^2)-i2ab}{a^4-2a^2b^2+b^4+4a^2b^2}$$

$$\int_0^\infty x\, e^{-ax}(\cos bx - i \sin bx)\, dx = \frac{(a^2-b^2)-i2ab}{a^4+2a^2b^2+b^4}$$

$$\int_0^\infty x\, e^{-ax} \cos bx\, dx - i \int_0^\infty x\, e^{-ax} \sin bx\, dx = \frac{(a^2-b^2)-i2ab}{(a^2+b^2)^2}$$

Given $\Gamma(1.8) = 0.9314$ find $\Gamma(-2.2)$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(-2.2) = \frac{\Gamma(-2.2+1)}{(-2.2)} = \frac{\Gamma(-1.2)}{(-2.2)} = \frac{1}{(-2.2)}\frac{\Gamma(-1.2+1)}{(-1.2)}$$

$$= \frac{1}{(2.2)(1.2)}\Gamma(-0.2)$$

$$= \frac{1}{(2.2)(1.2)}\frac{\Gamma(-0.2+1)}{(-0.2)}$$

$$= \frac{1}{(2.2)(1.2)(-0.2)}\Gamma(0.8)$$

$$= -\frac{1}{(2.2)(1.2)(0.2)}\frac{\Gamma(0.8+1)}{(0.8)}$$

$$= -\frac{1}{(2.2)(1.2)(0.2)}\frac{\Gamma(1.8)}{(0.8)}$$

$$= -\frac{0.9314}{(2.2)(1.2)(0.2)(0.8)}$$

$$= -2.21$$

prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{2^n}\sqrt{\pi}$

$\rightarrow$ $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\Gamma\left(n-\tfrac{1}{2}\right)$$

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\Gamma\left(n-\tfrac{3}{2}\right)$$

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)\Gamma\left(n-\tfrac{5}{2}\right)$$

& so on

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)\cdots\tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \left(\tfrac{2n-1}{2}\right)\left(\tfrac{2n-3}{2}\right)\left(\tfrac{2n-5}{2}\right)\cdots\tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\sqrt{\pi}$$

$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}.$$

Hence or o.w p.T. $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{n!\,4^n}$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-2)(2n-1)(2n)}{2\cdot 4\cdots(2n-2)(2n)\,2^n}\sqrt{\pi}$$

$$= \frac{(2n)!}{2^n(1\cdot 2\cdots n)\,2^n}\sqrt{\pi} = \frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi}$$

$$= \frac{(2n)!\,\sqrt{\pi}}{(2^2)^n\,n!}$$

If $I_n = \dfrac{\dfrac{\sqrt{\pi}}{2}\,\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2}+1\right)}$ Show that $I_{n+2} = \dfrac{n+1}{n+2} I_n$.

$$I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{(n+2)+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)} = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)}$$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$\Gamma\left(\frac{n+3}{2}\right) = \left(\frac{n+3}{2}-1\right)\Gamma\left(\frac{n+3}{2}-1\right) = \left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma\left(\frac{n+2}{2}+1\right) = \left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+2}{2}\right) = \left(\frac{n+2}{2}\right)\Gamma\left(\frac{n}{2}+1\right)$$

$$I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\left(\frac{n+1}{\cancel{2}}\right)\Gamma\left(\frac{n+1}{2}\right)}{\left(\frac{n+2}{\cancel{2}}\right)\Gamma\left(\frac{n}{2}+1\right)} = \left(\frac{n+1}{n+2}\right)\frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$= \frac{n+1}{n+2}\, I_n$$