Linear Differential Equations

1. Definitions. 2. Complete solution. 3. Operator D. 4. Rules for finding the Complementary function. 5. Inverse operator. 6. Rules for finding the particular integral. 7. Working procedure. 8. Two other methods of finding P.I. — Method of variation of parameters ; Method of undetermined coefficients. 9. Cauchy's and Legendre's linear equations. 10. Linear dependence of solutions. 11. Simultaneous linear equations with constant coefficients. 12. Objective Type of Questions.

13.1. DEFINITIONS

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^ny}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_n y = X,$$

where $p_1, p_2, \ldots\ldots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + k_n y = X$$

where $k_1, k_2, \ldots\ldots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

13.2. (1) THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + k_n y = 0 \qquad \ldots(1)$$

then $c_1y_1 + c_2y_2 \ (= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \quad \frac{d^ny_1}{dx^n} + k_1 \frac{d^{n-1}y_1}{dx^{n-1}} + k_2 \frac{d^{n-2}y_1}{dx^{n-2}} + \ldots\ldots + k_n y_1 = 0 \qquad \ldots(2)$$

and
$$\frac{d^ny_2}{dx^n} + k_1 \frac{d^{n-1}y_2}{dx^{n-1}} + k_2 \frac{d^{n-2}y_2}{dx^{n-2}} + \ldots\ldots + k_n y_2 = 0 \qquad \ldots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n(c_1y_1 + c_2y_2)}{dx^n} + k_1 \frac{d^{n-1}(c_1y_1 + c_2y_2)}{dx^{n-1}} + \ldots\ldots + k_n(c_1y_1 + c_2y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots\ldots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1\frac{d^{n-1}y_2}{dx^{n-1}} + \ldots\ldots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[by (2) and (3)]}$$

i.e. $$\frac{d^n u}{dx^n} + k_1\frac{d^{n-1}u}{dx^{n-1}} + \ldots\ldots + k_n y = 0 \qquad \ldots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \ldots\ldots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \ldots\ldots + c_ny_n (= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + k_n y = X \qquad \ldots(5)$$

then $$\frac{d^n v}{dx^n} + k_1\frac{d^{n-1}v}{dx^{n-1}} + \ldots\ldots + k_n v = X \qquad \ldots(6)$$

Adding (4) and (6), we have $\dfrac{d^n(u+v)}{dx^n} + k_1\dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \ldots\ldots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ The complete solution (C.S.) of (5) is $\mathbf{y = C.F. + P.I.}$

Thus in order to solve the equation (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

13·3. OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy, \dfrac{d^2y}{dx^2} = D^2y, \dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form*

$(D^n + k_1D^{n-1} + \ldots\ldots + k_n)y = X,$ *i.e.* $f(D)y = X,$

where $f(D) = D^n + k_1D^{n-1} + \ldots\ldots + k_n$, *i.e.* a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e. f(D) can be factorised by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)\,y \text{ or } (D-1)(D+3)\,y.$$

13·4. RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + k_n y = 0 \qquad \ldots(1)$$

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n)\, y = 0 \qquad \ldots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \ldots\ldots, m_n$ be its roots.

Case I. *If all the roots be real and different,* then (2) is equivalent to

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n)\, y = 0 \qquad \ldots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)\, y = 0$, *i.e.* by $\dfrac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

∴ Its solution is $\qquad ye^{-m_n x} = c_n$, *i.e.* $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)\, y = 0$, $(D - m_2)\, y = 0$ etc. *i.e.* by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}$ $\qquad \ldots(4)$

Case II. *If two roots are equal (i.e.* $m_1 = m_2$), then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$y = C\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \qquad [\because c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)\, y = 0$

Putting $(D - m_1)\, y = z$, it becomes $(D - m_1)\, z = 0$ or $\dfrac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. ∴ Its solution is $ze^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\qquad (D - m_1)\, y = z = c_1 e^{m_1 x}$ or $\dfrac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ $\qquad \ldots(5)$

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$ye^{-m_1 x} = \int c_1 e^{m_1 x} dx + c_2 = c_1 x + c_2 \text{ or } y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.* $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.* $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$\begin{aligned} y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\ &= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\ &= e^{\alpha x} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \end{aligned}$$

$[\because$ by *Euler's Theorem*, $e^{i\theta} = \cos\theta + i \sin\theta]$

$= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$ where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two pairs of imaginary roots be equal i.e.* $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}\,[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + \ldots\ldots + c_n e^{m_n x}.$$

Example 13·1. *Solve* $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\dfrac{dx}{dt}(0) = 15$. (V.T.U., 2006)

Sol. Given equation in *symbolic form is* $(D^2 + 5D + 6)\,x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.* $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\dfrac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0$, $x = 0$. $\qquad \therefore\ 0 = c_1 + c_2$ ...(*i*)

When $t = 0$, $dx/dt = 15$ $\qquad \therefore\ 15 = -2c_1 - 3c_2$ ...(*ii*)

Solving (*i*) and (*ii*), $c_1 = 15$, $c_2 = -15$.

Hence the required solution is $x = 15\,(e^{-2t} - e^{-3t})$.

Example 13·2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0$.

Sol. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.* $(D + 3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2 t)\,e^{-3t}$.

Example 13·3. *Solve* $(D^3 + D^2 + 4D + 4) = 0$.

Sol. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.* $(D^2 + 4)(D + 1) = 0$ $\quad \therefore\ D = -1, \pm 2i$.

Hence the C.S. is $\qquad y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e. $\qquad y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$.

Example 13·4. *Solve* (*i*) $(D^2 - 2D + 4)^2\, y = 0$ $\qquad$ (*ii*) $(D^2 + 1)^3\, y = 0$ *where* $D \equiv d/dx$.

Sol. (*i*) The A.E. equation is $(D^2 - 2D + 4)^2 = 0$

$$\therefore\quad D = \frac{2 \pm \sqrt{4 - 16}}{2}\ (twice),\ \therefore\ D = 1 \pm \sqrt{3}i,\ \ 1 \pm \sqrt{3}i$$

Hence the C.S. is $y = e^x\,[(c_1 + c_2 x)\cos\sqrt{3}x + (c_3 + c_4 x)\sin\sqrt{3}x)$ [Roots being repeated complex

(*ii*) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore \qquad D = \pm i,\ \pm i,\ \pm i.$

Hence the C.S. is $y = e^{0x}\,[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$

i.e. $\qquad y = (c_1 + c_2 + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$.

Example 13·5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0$.

Sol. Given equation in *symbolic form is* $(D^4 + 4)\,x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $\qquad (D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $\qquad D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $\qquad D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.* $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos t + c_4 \sin t)$.

Problems 13·1

Solve :

1. $\dfrac{d^2x}{dt^2} + 3a\dfrac{dx}{dt} - 4a^2x = 0.$

2. $y'' - 2y' + 10y = 0, y(0) = 4, y'(0) = 1.$

3. $4y''' + 4y'' + y' = 0.$

4. $\dfrac{d^3y}{dx^3} + y = 0.$ (V.T.U., 2000 S)

5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$

6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$ (J.N.T.U., 2005)

7. $(D^2+1)^2(D-1)y = 0$

8. If $\dfrac{d^4x}{dt^4} = m^4x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt.$

9. Solve $d^4y/dx^4 + a^4y = 0.$

13·5. INVERSE OPERATOR

(1) **Definition.** $\dfrac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e.

$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\dfrac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2) $$\mathbf{\frac{1}{D}X = \int X dx}$$

Let $$\frac{1}{D}X = y \qquad \ldots(i)$$

Operating by D, $D\dfrac{1}{D}X = Dy$ *i.e.* $X = \dfrac{dy}{dx}$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (*i*) does not contain any constant.

Thus $$\frac{1}{D}X = \int X dx.$$

(3) $$\mathbf{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx.}$$

Let $$\frac{1}{D-a}X = y. \qquad \ldots(ii)$$

Operating by $D-a$, $(D-a)\cdot\dfrac{1}{D-a}X = (D-a)y.$

or $$X = \frac{dy}{dx} - ay, \text{ i.e. } \frac{dy}{dx} - ay = X$$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$ye^{-ax} = \int Xe^{-ax}\,dx$, no constant being added as (*ii*) doesn't contain any constant.

Thus $$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}\,dx.$$

13·6. RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $\dfrac{d^n y}{dx^n} + k_1 \dfrac{d^{n-1}y}{dx^{n-1}} + k_2 \dfrac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + k_n y = X$

which in symbolic form is $(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n)\, y = X$.

$$\therefore \quad \text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots\ldots + k_n} X.$$

Case I. When $X = e^{ax}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^n e^{ax} = a^n e^{ax}$$

$\therefore \quad (D^n + k_1 D^{n-1} + \ldots\ldots + k_n)\, e^{ax} = (a^n + k_1 a^{n-1} + \ldots\ldots + k_n)\, e^{ax}$ i.e. $f(D) e^{ax} = f(a) e^{ax}$

Operating on both sides by $\dfrac{1}{f(D)}$, $\dfrac{1}{f(D)} f(D)\, e^{ax} = \dfrac{1}{f(D)} f(a)\, e^{ax}$ or $e^{ax} = f(a) \dfrac{1}{f(D)} e^{ax}$

$\therefore$ Dividing by $f(a)$,

$$\frac{\mathbf{1}}{\mathbf{f(D)}} \mathbf{e}^{ax} = \frac{\mathbf{1}}{\mathbf{f(a)}} \mathbf{e}^{ax} \text{ provided } f(a) \neq 0 \qquad \ldots(1)$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + \ldots\ldots + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\, \phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \qquad \text{[by (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} \cdot e^{-ax}\, dx \qquad \text{[by §13·5 (3)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax}$$

i.e.
$$\frac{\mathbf{1}}{\mathbf{f(D)}} \mathbf{e}^{ax} = \mathbf{x} \frac{\mathbf{1}}{\mathbf{f'(a)}} \mathbf{e}^{ax} \qquad \ldots(2)$$

$$\left[\begin{array}{l} \because f'(D) = (D-a)\, \phi'(D) + 1 \cdot \phi(D) \\ \therefore f'(a) = 0 \times \phi'(a) + \phi(a) \end{array}\right.$$

If $f'(a) = 0$, then applying (2) again, we get $\dfrac{\mathbf{1}}{\mathbf{f(D)}} \mathbf{e}^{ax} = \mathbf{x}^2 \dfrac{\mathbf{1}}{\mathbf{f''(a)}} \mathbf{e}^{ax}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13·6. *Find the P.I. of* $(D^2 + 5D + 6)\, y = e^x$.

Sol. $\quad P.I. = \dfrac{1}{D^2 + 5D + 6} e^x$ [Put $D = 1$] $= \dfrac{1}{1^2 + 5 \cdot 1 + 6} e^x = \dfrac{e^x}{12}$

Example 13·7. *Find the P.I. of* $(D + 2)(D - 1)^2\, y = e^{-2x} + 2 \sinh x$.

Sol. $\quad \text{P.I.} = \dfrac{1}{(D+2)(D-1)^2} [e^{-2x} + 2 \sinh x] = \dfrac{1}{(D+2)(D-1)^2} [e^{-2x} + e^x - e^{-x}]$