

Module :3

Matrices

Properties:

Eigen Values & Eigen Vectors

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❖ Singular Matrices have Zero Eigenvalues.

Statement: Suppose A is a square matrix. Then A is singular if and only if $\lambda=0$ is an eigenvalue of A .

Proof: A is singular \Leftrightarrow there exists $x \neq 0$,

\Leftrightarrow there exists $x \neq 0$, $Ax=0x$

$\Leftrightarrow \lambda=0$ is an eigenvalue of A

❖ Eigenvalues of the Transpose of a Matrix

Statement: Suppose A is a square matrix and λ is an eigenvalue of A . Then λ is an eigenvalue of the matrix A^T . i.e. A and A^T have eigenvalues.

Proof: Suppose A has order n .

$$\text{Now, } \det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$$

$$\therefore \det(A - \lambda I)^T = 0 \iff \det(A^T - \lambda I) = 0$$

So A and A^T have the same characteristic polynomial and their eigenvalues are identical and have equal algebraic multiplicities.

❖ Eigenvalues of a Scalar Multiple of a Matrix

Statement: Suppose A is a square matrix and λ is an eigenvalue of A . Then $\alpha\lambda$ is an eigenvalue of αA .

Proof: Let $x \neq 0$ be one eigenvector of A for λ . Then
 $(\alpha A)x = \alpha(Ax)$

$$= \alpha(\lambda x)$$

$$= (\alpha\lambda)x$$

So $x \neq 0$ is an eigenvector of αA for the eigenvalue $\alpha\lambda$.

❖ Eigenvalues Of Matrix Powers

Statement: Suppose A is a square matrix of order n and λ is an eigenvalue of A . Then for any integer $s \geq 0$, λ^s is an eigenvalue of A^s .

Proof: Let $x \neq 0$ be one eigenvector of A for λ .

We will prove the theorem by mathematical induction on s .

Step 1: For $s=0$,

$$\begin{aligned} A^s x &= A^0 x = I_n x = x \\ &= 1x = \lambda^0 x = \lambda^s x \end{aligned}$$

\therefore The theorem is true for $s=0$.

Step 2: Assume the theorem is true for s , then we will prove its true for $s+1$.

$$A^{s+1} x = A^s A x = A^s (\lambda x) = \lambda (A^s x) = \lambda (\lambda^s x) = (\lambda \lambda^s) x = \lambda^{s+1} x$$

So, $x \neq 0$ is an eigenvector of A^{s+1} corresponding to eigenvalue λ^{s+1}

By Mathematical induction, Theorem is true for all $s \geq 0$.

❖ Eigenvalues of the Polynomial of a Matrix

Statement: Suppose A is a square matrix and λ is an eigenvalue of A . Let $q(x)$ be a polynomial in the variable x . Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Proof : Let $x \neq 0$ be one eigenvector of A for λ , and write $q(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$

$$\text{Then } q(A)x = (a_0 + a_1A + a_2A^2 + \cdots + a_mA^m)x$$

$$= a_0(A^0x) + a_1(A^1x) + a_2(A^2x) + \cdots + a_m(A^mx)$$

$$= a_0(\lambda^0x) + a_1(\lambda^1x) + a_2(\lambda^2x) + \cdots + a_m(\lambda^mx)$$

$$= (a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_m\lambda^m)x$$

$$= q(\lambda)x$$

So $x \neq 0$ is an eigenvector of $q(A)$ for the eigenvalue $q(\lambda)$.

❖ Eigenvalues of the Inverse of a Matrix

Statement: Suppose A is a square nonsingular matrix and λ is an eigenvalue of A . Then λ^{-1} is an eigenvalue of the matrix A^{-1} .

Proof: since A is assumed nonsingular, A^{-1} exists and $1/\lambda$ does not involve division by zero.

Let $x \neq 0$ be one eigenvector of A for λ . Suppose A has order n . Then

$$\begin{aligned} A^{-1}x &= A^{-1}(\lambda x) = A^{-1}\left(\frac{1}{\lambda} \lambda x\right) \\ &= \frac{1}{\lambda} A^{-1}(\lambda x) \\ &= \frac{1}{\lambda} A^{-1}(Ax) = \frac{1}{\lambda} (A^{-1}Ax) \\ &= \frac{1}{\lambda} (I_n x) = \frac{1}{\lambda} x \end{aligned}$$

So $x \neq 0$ is an eigenvector of A^{-1} for the eigenvalue $\frac{1}{\lambda}$.

❖ Eigenvalues of the Adjoint of a Matrix

Statement: Suppose A is a square nonsingular matrix and λ is an eigenvalue of A . Then $\frac{|A|}{\lambda}$ is an eigenvalue of the matrix $\text{Adj.}(A)$.

Proof: We know that $A \text{ adj.}A = |A|I$

Premultiplying by A^{-1}

$$A^{-1} A \text{ adj.}A = A^{-1} |A|I$$

$$\therefore \text{adj.}A = A^{-1} |A|I$$

Post multiplying by x , x is the eigenvector corresponding to λ .

$$(\text{adj.}A)x = |A| A^{-1} x = |A| \frac{1}{\lambda} x = \frac{|A|}{\lambda} x$$

$\therefore \frac{|A|}{\lambda}$ is an eigenvalue of the matrix $\text{Adj.}(A)$.

❖ The eigenvalues of a unitary matrix are of unit modulus.

Statement: Suppose A is a Unitary matrix and λ is an eigenvalue of A .
Then $|\lambda| = 1$.

Proof: But A is Unitary. $\therefore AA^{\theta} = A^{\theta}A = I$

Let $x \neq 0$ be one eigenvector of A corresponding to eigenvalue λ .

$$\therefore Ax = \lambda x \dots (1)$$

Taking complex conjugate transpose on both sides

$$(Ax)^{\theta} = (\lambda x)^{\theta}$$

$$x^{\theta} A^{\theta} = \bar{\lambda} x^{\theta} \dots (2)$$

Multiplying (1) & (2)

$$(x^{\theta} A^{\theta})(Ax) = (\bar{\lambda} x^{\theta})(\lambda x)$$

$$\therefore x^{\theta} (A^{\theta}A) x = \bar{\lambda} \lambda (x^{\theta}x)$$

$$\therefore x^{\theta} x = \bar{\lambda} \lambda (x^{\theta}x)$$

Since X is an eigen vector, $x^{\theta} x \neq 0$

$$\therefore \lambda \bar{\lambda} = 1 \therefore |\lambda| = 1$$

❖ Hermitian Matrices have Real Eigenvalues

Statement: Suppose that A is a Hermitian matrix and λ is an eigenvalue of A . Then $\lambda \in \mathbb{R}$.

Proof : Let $x \neq 0$ be one eigenvector of A corresponding to eigenvalue λ .

we know $Ax = \lambda x$

Premultiplying by x^θ we get

$$x^\theta Ax = x^\theta \lambda x = \lambda x^\theta x \dots\dots\dots 1$$

Taking complex conjugate transpose on both sides

$$(x^\theta Ax)^\theta = (\lambda x^\theta x)^\theta$$

$$x^\theta A^\theta (x^\theta)^\theta = \bar{\lambda} x^\theta (x^\theta)^\theta \dots\dots\dots 2$$

But A is Hermitian. $\therefore A = A^\theta$

$$\text{By 2, } x^\theta Ax = \bar{\lambda} x^\theta x \dots\dots\dots 3$$

From (1) & (3), we get

$$\lambda x^\theta x = \bar{\lambda} x^\theta x$$

Since X is an eigen vector, $x^\theta x \neq 0$

$$\therefore \lambda = \bar{\lambda}$$

Corollaries

- ❖ Determinant of Hermitian matrix is real
- ❖ Real Symmetric Matrices have Real Eigenvalues
- ❖ Eigenvalues of a Skew-Hermitian Matrix are either purely imaginary or zero.
- ❖ The eigen values of a real skew-symmetric matrix are purely imaginary or zero.

❖ The eigenvalues of an Orthogonal Matrix are of unit modulus.

Statement: *Suppose A is a Orthogonal matrix and λ is an eigenvalue of A . Then $|\lambda| = 1$.*

Hint: If the element of a unitary matrix A are all real, then A becomes the orthogonal Matrix.

Ex. Find the eigenvalues of Adj. A and inv. A Where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

❖ Eigenvalues of A are $\lambda = 1, 2, 4, 6$

By properties,

Eigenvalues of Adj.A = $\frac{|A|}{\lambda}$ & $|A| = \text{Product of eigenvalues}$

\therefore Eigenvalues of Adj.A = $\frac{48}{1}, \frac{48}{2}, \frac{48}{4}, \frac{48}{6}$

\therefore Eigenvalues of Adj.A = 48, 24, 12, 8

Eigenvalues of Inv.A = $\frac{1}{\lambda} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots, \dots, \dots, \dots$

Ex. 1 Find the eigenvalues and eigen vectors of $6A^{-1}+A^3+2I$. Where $A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

Eigenvalues of A are $\lambda = -1, -1$ and corresponding eigenvector is $X = [1, 0]'$ (Can be found by students)

By properties,

Eigenvalues of $6A^{-1}+A^3+2I = 6\frac{1}{\lambda} + \lambda^3 + 2(1)$

\therefore Eigenvalues of $6A^{-1}+A^3+2I$ are $(-6)+(-1)^3+2 = 5, 5$ with corresponding eigenvector is $X = [1, 0]'$.

Ex. Find eigen values of A and prove they are of unit modulus.

$$A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} \frac{1+i}{2} - \lambda & -\frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} - \lambda \end{vmatrix} = 0$$

$$\therefore \left(\frac{1+i}{2} - \lambda \right) \left(\frac{1-i}{2} - \lambda \right) + \left(\frac{1+i}{2} \right) \left(\frac{1-i}{2} \right) = 0$$

$$\therefore \left\{ \left(\frac{1}{2} - \lambda \right)^2 - \left(\frac{i}{2} \right)^2 \right\} + \left\{ \left(\frac{1}{2} \right)^2 - \left(\frac{i}{2} \right)^2 \right\} = 0$$

$$\therefore \lambda^2 - \lambda + 1 = 0$$

Eigen values are: $\lambda = \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$ & their modulus are 1.

Ex. If $A = \begin{bmatrix} \sin\theta & \operatorname{cosec}\theta & 1 \\ \sec\theta & \cos\theta & 1 \\ \tan\theta & \cot\theta & 1 \end{bmatrix}$ then prove that there does not exist a real value of θ for which characteristic roots of A are $-1, 1$ & 4 .

Soln. Sum of A = sum of diagonal elements
 $= \sin\theta + \cos\theta + 1 \dots\dots\dots (1)$

By properties, trace of A = sum of eigen values = $4 \dots\dots (2)$

By (1) & (2),

$$\sin\theta + \cos\theta + 1 = 4 \dots\dots\dots (3)$$

For no real value of θ equation (3) holds true.

Hence the proof.