Triple Integration

$$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} \int_{z=g_1(x,y)}^{g_2(x,y)} f(x,y,z) \; dz \; dy \; dx$$

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{(1-x^2-y^2-z^2)}}$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

$$1-x^2-y^2 = a^2$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{a^2}} \frac{dz}{\sqrt{a^2-z^2}}\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sin^{-1}\left(\frac{z}{a}\right)\right)_0^a dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sin^{-1}(1) - \sin^{-1}(0)\right) dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2}\right) dy\,dx$$

$$= \pi/2 \int_0^1 (y)_0^{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\,dx$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1^2-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\,0 + \frac{1}{2}\,\frac{\pi}{2} - 0 - 0\right] = \frac{\pi}{2}\left[\frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

2. $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\,dz\,dy\,dx$

$\int e^{ax}\,dx = \frac{e^{ax}}{a}$

$\hookrightarrow \underline{e^x \cdot e^y}\left(e^z\right)$

$\left(e^z\right)_0^{2x+2y} = \left(e^{2x+2y} - 1\right)$

Ans $= \frac{e^{12}}{8} - \frac{e^4}{2} - \frac{e^6}{9} + e^2 - \frac{4}{9}$

$$\int_0^{\pi} 2d\theta \int_0^{a(1+\cos\theta)} r\, dr \int_0^{h} \left|1 - \frac{r}{a(1+\cos\theta)}\right| dz$$

$$= 2\int^{\pi}\int^{a(1+\cos\theta)} r\left(1 - \frac{r}{a(1+\cos\theta)}\right)(z)_0^h \, dr\, d\theta$$

$$= 2\int^{\pi}\int^{a(1+\cos\theta)} \left(r - \frac{r^2}{a(1+\cos\theta)}\right)(h)\, dr\, d\theta$$

$$= 2h\int^{\pi}\left(\frac{r^2}{2} - \frac{r^3}{3a(1+\cos\theta)}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= 2h\int^{\pi}\left[\frac{a^2(1+\cos\theta)^2}{2} - \frac{a^3(1+\cos\theta)^3}{3\cancel{a}\,\cancel{(1+\cos\theta)}}\right] d\theta$$

$$= 2ha^2\int^{\pi}\left[\frac{(1+\cos\theta)^2}{2} - \frac{(1+\cos\theta)^2}{3}\right] d\theta$$

$$= 2ha^2\int^{\pi}\frac{(1+\cos\theta)^2}{6}\, d\theta$$

$1+\cos 2\theta = 2\cos^2\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$$= \frac{\cancel{2}ha^2}{\cancel{6}\,3}\int^{\pi}(1+2\cos\theta+\cos^2\theta)\, d\theta$$

$$= \frac{ha^2}{3}\int^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{ha^2}{3\times 2}\int^{\pi}(2+4\cos\theta+1+\cos 2\theta)\, d\theta$$

$$= \frac{ha^2}{6}\int^{\pi}(3+4\cos\theta+\cos 2\theta)\, d\theta$$

$$= \frac{ha^2}{6}\left(3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$= \frac{ha^2}{6}(3\pi+0+0) = \frac{\cancel{3}\pi ha^2}{\cancel{6}\,2}$$

7. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (z)_{x^2+3y^2}^{8-x^2-y^2}\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (8-x^2-y^2-x^2-3y^2)\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-4y^2)\, dy\, dx.$$

$$= \int_{-2}^{2} \left[8y - 2x^2y - \frac{4y^3}{3}\right]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{4-x^2}/2} dx$$

$$= \int_{-2}^{2} \left[8\frac{\sqrt{4-x^2}}{2} - 2x^2\frac{\sqrt{4-x^2}}{2} - 4\frac{(4-x^2)^{3/2}}{3\times 2^3} + 8\frac{\sqrt{4-x^2}}{2} + 2x^2\left(-\frac{\sqrt{4-x^2}}{2}\right) + 4\frac{(-(4-x^2)^{3/2})}{3\times 2^3}\right] dx$$

$$= \int_{-2}^{2}\left[\frac{\cancel{16}\,8}{\cancel{2}}\sqrt{4-x^2} - \frac{\cancel{4}\,2\,x^2}{\cancel{2}\,1}\sqrt{4-x^2} - \frac{8}{3\times\cancel{8}}(4-x^2)^{3/2}\right] dx.$$

$$= \int_{-2}^{2}\left[2\sqrt{4-x^2}\,(4-x^2) - \frac{1}{3}(4-x^2)^{3/2}\right] dx$$

$$= \int_{-2}^{2}\left[2(4-x^2)^{3/2} - \frac{1}{3}(4-x^2)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \frac{5}{3}(4-x^2)^{3/2}\, dx$$

$x = 2\sin\theta$ $\quad$ $x: -2 \to 2$ $\quad$ $\sin\theta = -1$

$dx = 2\cos\theta\, d\theta$ $\quad$ $\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$ $\quad$ $\sin(-\frac{\pi}{2}) = -1$, $-\sin\frac{\pi}{2}$

$$= \frac{5}{3}\int_{-\pi/2}^{\pi/2} (4-4\sin^2\theta)^{3/2}\, 2\cos\theta\, d\theta$$

$$= \frac{5\times 2}{3}\int_{-\pi/2}^{\pi/2} (4)^{3/2}(\cos^2\theta)^{3/2}\cos\theta\, d\theta$$

$$= \frac{10}{3}\times 2^3 \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \to f(x)$ is even

$$= \frac{10\times 8}{3}\times 2\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{160}{3}\,\frac{1}{2}\,\beta\left(\frac{0+1}{2}, \frac{5}{2}\right)$$

$$= \frac{80}{3}\,\beta\left(\tfrac{1}{2}, \tfrac{5}{2}\right) = \frac{80}{3}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma\frac{6}{2}}$$

$$= \frac{80}{3}\,\frac{\Gamma\frac{1}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{80}{\cancel{3}}\times\frac{\pi\times\cancel{3}}{4\times 2}$$ $\quad$ $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$= 10\pi.$$

8. $\int_0^a \int_0^a \int_0^a (yz + zx + xy)\,dx\,dy\,dz.$

$$\int_0^a \int_0^a \left(xyz + \frac{zx^2}{2} + \frac{yx^2}{2} \right)_0^a dy\,dz$$

$$\frac{3}{4} a^5$$

TYPE II : WHEN THE REGION OF INTEGRATION IS BOUNDED BY PLANES

1. Evaluate $\iiint x^2yz\,dx\,dy\,dz$ throughout the volume bounded by the planes $x = 0, y = 0, z = 0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad , \; x=0 \,,\; y=0 \,,\; z=0$$

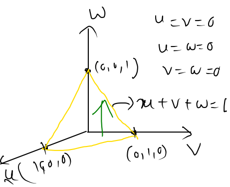
$$\frac{x}{a}=u \quad \frac{y}{b}=v \quad , \frac{z}{c}=w$$

$$\Rightarrow x=au \quad y=bv \quad z=cw$$

$$u+v+w=1 \quad , \; u=0 \,,\; v=0 \,,\; w=0$$

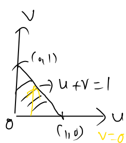
$$\left[\hookrightarrow x=0 \quad au=0 \Rightarrow u=0 \right]$$

$$u=v=0 \quad w=1$$
$$u=w=0 \quad v=1$$
$$v=w=0 \quad u=1$$



$$\iiint x^2yz\;dx\,dy\,dz$$

$$x=au \quad y=bv \quad z=cw$$
$$dx=a\,du \quad dy=b\,dv \quad dz=c\,dw$$



$$I = \int\int\int a^2u^2\, bv\, cw\;\, a\,du\, b\,dv\, c\,dw$$

$$= a^3b^2c^2 \int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} u^2vw\,dw\,dv\,du$$

$$= a^3b^2c^2 \int_0^1\int_0^{1-u} u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v} dv\,du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2v\,(1-u-v)^2\,dv\,du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2v\left[(1-u)^2-2(1-u)v+v^2\right]dv\,du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u}\left[u^2(1-u)^2v-2u^2(1-u)v^2+u^2v^3\right]dv\,du$$

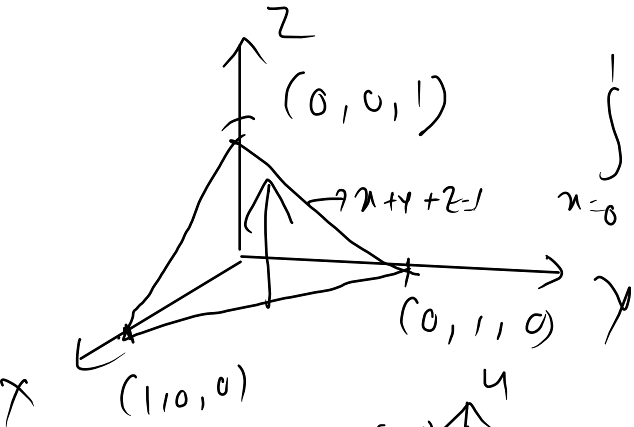
$$= \frac{a^3b^2c^2}{2}\int_0^1\left[u^2(1-u)^2\frac{v^2}{2}-2u^2(1-u)\frac{v^3}{3}+\frac{u^2v^4}{4}\right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\left[\frac{u^2(1-u)^2(1-u)^2}{2}-\frac{2(u^2(1-u)(1-u)^3}{3}+\frac{u^2(1-u)^4}{4}\right]du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2(1-u)^4\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]du$$

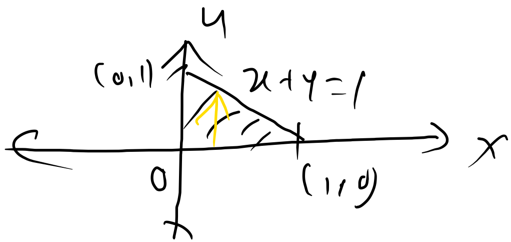
$$= \frac{a^3b^2c^2}{2\times12}\int_0^1 u^2(1-u)^4\,du = \frac{a^3b^2c^2}{24}\,\beta(3,5)$$

3. Evaluate $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$



$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{1}{(1+x+y+z)^3}\,dz\,dy\,dx$$

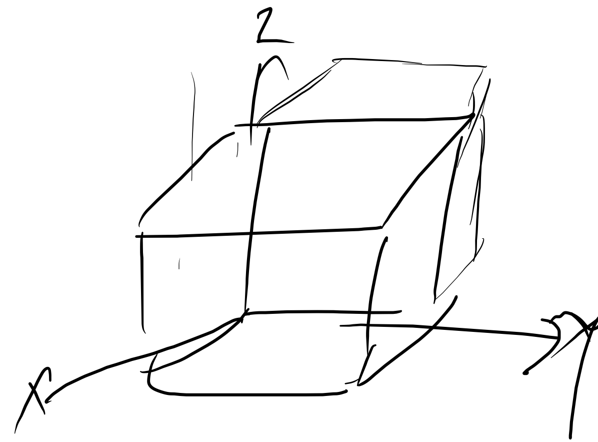
$$(a+z)^{-3}$$



$$\int (bx+c)^n dx = \frac{(bx+c)^{n+1}}{(n+1)b}$$

6. Evaluate the integral $\iiint_v xyz^2\,dv$ over the region bounded by the planes $x = 0, x = 1, y = -1, y = 2,$ $z = 0, z = 3$

$$\int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dy\,dx$$



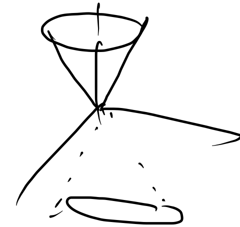
Cylindrical co-ordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$dz\,dy\,dx = r\,dz\,dr\,d\theta$$

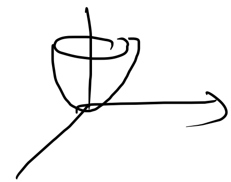
Cylinder
$$x^2+y^2=a^2$$

Cone
$$x^2+y^2=z^2$$

$z>0$



paraboloid
$$x^2+y^2=z$$

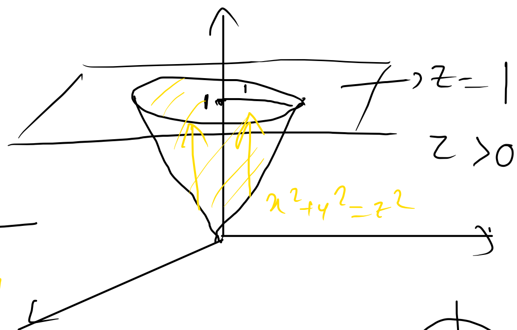


TYPE IV : WHEN THE REGION OF INTEGRATION IS BOUNDED BY A CONE OR A CYLINDER OR A PARABOLOID.

1. $\iiint \sqrt{x^2+y^2}\,dx\,dy\,dz$ over the volume bounded by the right circular cone $x^2+y^2=z^2, z>0$ and the planes $z=0$ and $z=1$.

$x^2+y^2=z^2$

$z=1 \quad x^2+y^2=1^2$



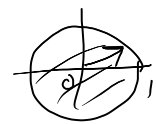
By Cylindrical coordi—

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$



$\theta : 0 \to 2\pi$

$r : 0 \to 1$

as $z^2 = x^2+y^2$

$z^2 = r^2$

$z : r \to 1$

$\iiint \sqrt{x^2+y^2}\,dx\,dy\,dz$

$= \int_0^{2\pi}\int_0^1\int_r^1 r\; r\; dz\,dr\,d\theta = \pi/6$

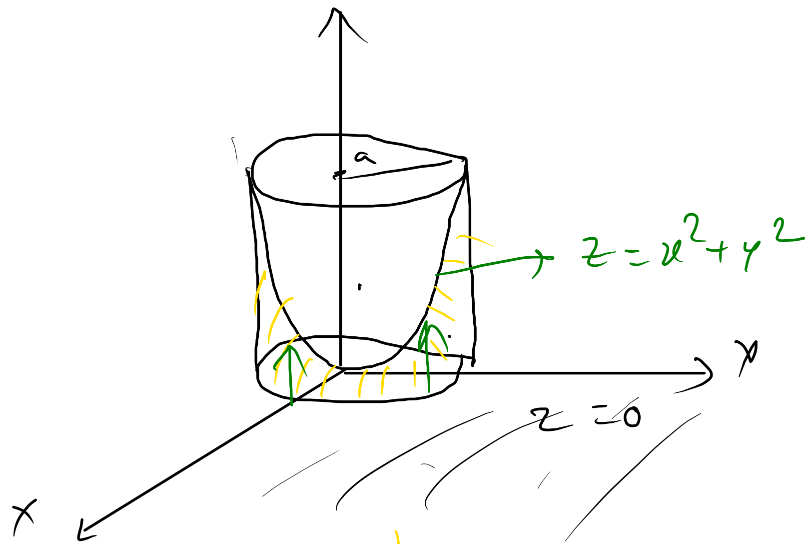
2. $\iiint z^2\,dx\,dy\,dz$ over the volume bounded by the cylinder $x^2+y^2=a^2$ and the paraboloid $x^2+y^2=z$ and the plane $z=0$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

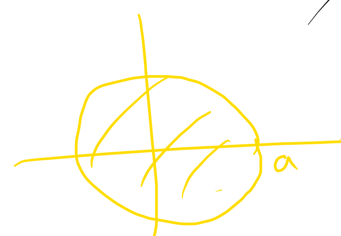
$dz\,dy\,dx = r\,dz\,dr\,d\theta$



$\theta : 0 \to 2\pi$

$r : 0 \to a$

$z : 0 \to x^2+y^2 = r^2$

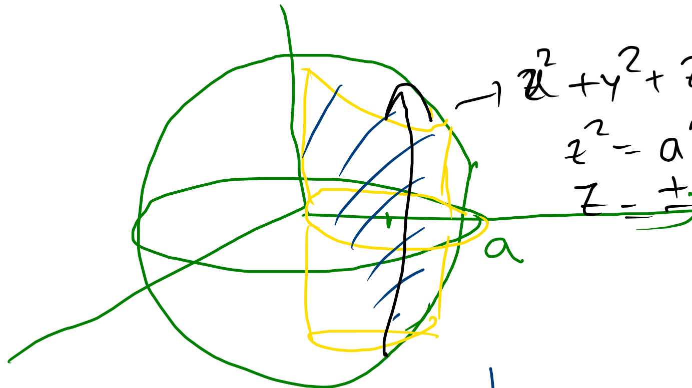


$$\iiint z^2\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{a}\int_0^{r^2} z^2\,r\,dz\,dr\,d\theta$$

$$= \pi a^8$$

3. $\iiint z^2\,dx\,dy\,dz$ over the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2 = ax$.



$\rightarrow x^2+y^2+z^2=a^2$

$z^2=a^2-x^2-y^2$

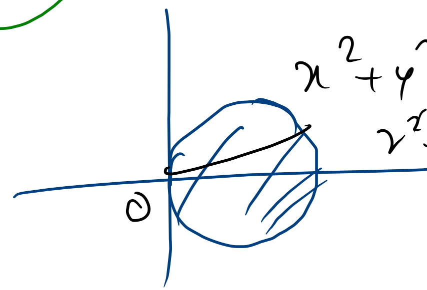
$z=\pm\sqrt{a^2-r^2}$

$x^2+y^2=ax$

$x^2-ax+\frac{a^2}{4}+y^2=\frac{a^2}{4}$

$(x-a/2)^2+y^2=(a/2)^2$

$(a/2, 0)$ & $r=a/2$



$x^2+y^2=ax$

$r^2=ar\cos\theta$

$\theta : -\pi/2 \rightarrow \pi/2$

$r : 0 \rightarrow a\cos\theta$

$z : -\sqrt{a^2-r^2} \rightarrow \sqrt{a^2-r^2}$

$x=r\cos\theta$

$y=r\sin\theta$

$z=z$

HW

4. Evaluate $\iiint_V (x^2+y^2)\,dV$ where V is the solid bounded by the surface $x^2+y^2=z^2$ and the planes $z=0, z=2$

z : cone $\longrightarrow$ plane $z=2$

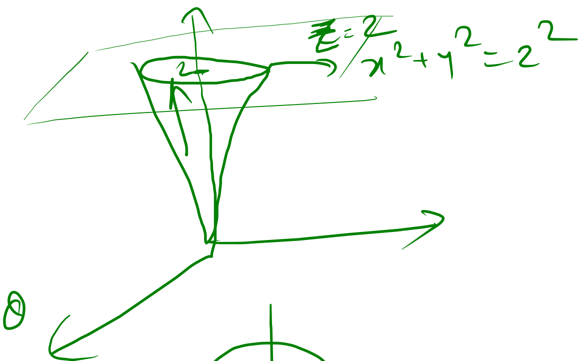
$x = r\cos\theta$

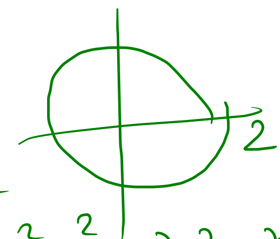
$y = r\sin\theta$

$z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$

$$\int_0^{2\pi}\int_0^{2}\int_r^{2} (r^2)\, r\,dz\,dr\,d\theta$$





$z^2 = x^2+y^2$

$\Rightarrow z^2 = r^2 \Rightarrow z = r$

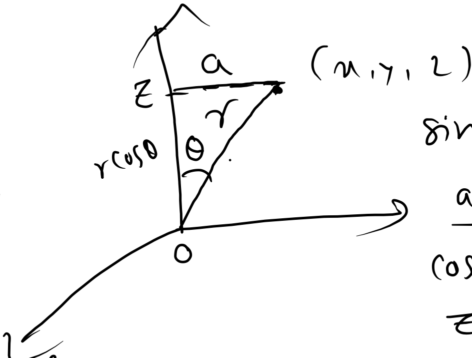
Spherical Co-ordinates

$x^2+y^2+z^2=a^2 \longrightarrow$ Sphere

(r, ϕ, θ) → angle from the z axis.

r ↓ radius in 3 dim.

ϕ ↓ is angle on xy plane how the diagram is rotated around z axis



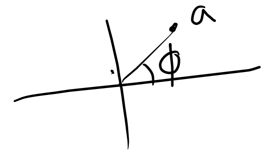
$\sin\theta = \frac{a}{r}$

$\underline{a = r\sin\theta}$

$\cos\theta = \frac{z}{r}$

$\underline{\underline{z = r\cos\theta.}}$

$$\left\{ \begin{aligned} x &= \underline{\underline{r\sin\theta}}\cos\phi \\ y &= \underline{\underline{r\sin\theta}}\sin\phi. \\ z &= r\cos\theta \end{aligned} \right.$$



$$\left| \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \right.$$

$dz\, dy\, dx = r^2 \sin\theta\, dr\, d\theta\, d\phi.$

$x = a\cos\phi$

$y = a\sin\phi$

$$\left. \begin{aligned} 0 &< r < \infty \\ 0 &< \theta < \pi \\ 0 &< \phi < 2\pi \end{aligned} \right\}$$

TYPE III : WHEN THE REGION OF INTEGRATION IS NOT BOUNDED BY PLANES, BUT BY SPHERE, ELLIPSOID ETC.

Evaluate the following integrals.

1. $\iiint_V \frac{dx\,dy\,dz}{\left(1+x^2+y^2+z^2\right)^2}$ where V is the volume in the first octant.

Imagine very big sphere with rad ∞ in 1st oct

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

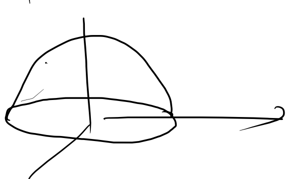
$dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$

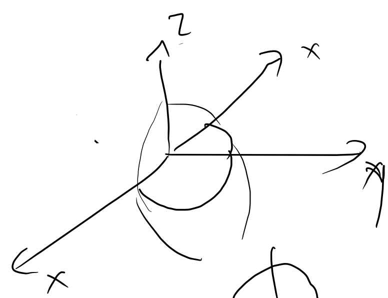
$$\iiint_V \frac{1}{(1+x^2+y^2+z^2)^2}\,dz\,dy\,dx = \pi^2/8$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} \frac{1}{(1+r^2)^2}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$1+r^2 = t$

$\phi \to 0 \to \pi$

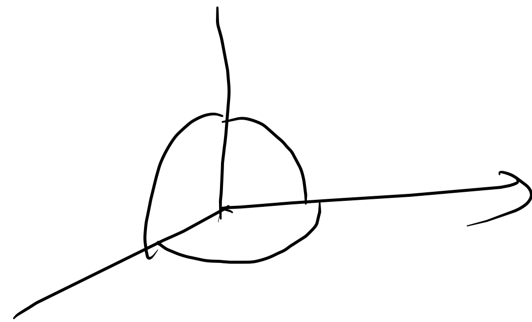




2. $\iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$ over the first octant of the sphere $x^2 + y^2 + z^2 = a^2$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$dz\, dy\, dx = r^2 \sin\theta\, dr\, d\theta\, d\phi$$



$0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/2$

$0 \le r \le a$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\pi a^5}{10}$$

6. $\iiint \frac{z^2\,dx\,dy\,dz}{x^2+y^2+z^2}$ over the volume of the sphere $x^2 + y^2 + z^2 = 2$

rad $= \sqrt{2}$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

$0 \le r \le \sqrt{2}$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}} \frac{r^2\cos^2\theta}{\cancel{r^2}}\,\cancel{r^2}\sin\theta\,dr\,d\theta\,d\phi$$

$$= \boxed{\frac{8\sqrt{2}\,\pi}{9}}$$

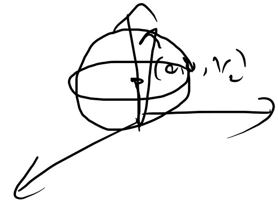
10. $\iiint_v \frac{z^2}{x^2+y^2+z^2}\,dx\,dy\,dz$ where V is the volume bounded by the sphere $x^2 + y^2 + z^2 = z$

$$x^2+y^2+z^2-z=0$$

$$x^2+y^2+z^2-z+\frac{1}{4}=\frac{1}{4}$$

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\left(\tfrac{1}{2}\right)^2$$

Centre $(0,0,\tfrac{1}{2})$ & rad $\tfrac{1}{2}$



$$0 \le \theta \le \pi/2$$

$$0 \le \phi \le 2\pi$$

$$0 \le r \le \cos\theta$$

$$x^2+y^2+z^2=z$$

$$r^{\cancel{2}}=\cancel{r}\cos\theta$$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$$

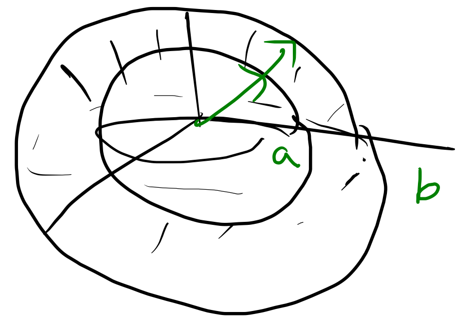
$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\theta} \frac{r^2\cos^2\theta}{\cancel{r^2}}\,\cancel{r^2}\sin\theta\,dr\,d\theta\,d\phi$$

$$= \pi/4.$$

11. $\iiint_v \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$ where V is the volume bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2, (b > a)$

$$\begin{cases} x = \underline{r\sin\theta}\cos\phi \\ y = \underline{r\sin\theta}\sin\phi \\ z = r\cos\theta \end{cases}$$

$$dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$$



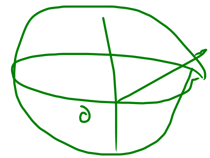
$$\int_0^{\pi}\int_0^{2\pi}\int_a^{b} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(r^2)^{3/2}}$$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$

$a \le r \le b$

12. $\iiint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$ throughout the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



$x = a\, r \sin\theta \cos\phi$

$y = b\, r \sin\theta \sin\phi$

$z = c\, r \cos\theta$

$dz\,dy\,dx = abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

$0 \le \phi \le 2\pi \qquad 0 \le \theta \le \pi$

$0 \le r \le 1 \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$r^2 = 1 \Rightarrow r = 1$

$$\iiint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\, dx\, dy\, dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sqrt{1-r^2}\; abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$r = \sin t$

Volume

$$V = \iiint 1 \, dz \, dy \, dx$$

cylindrical.

$$V = \iiint r \, dz \, dr \, d\theta$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Spherical.

$$\begin{cases} x = r\sin\theta \cos\phi \\ y = r\sin\theta \sin\phi \\ z = r\cos\theta \end{cases}$$

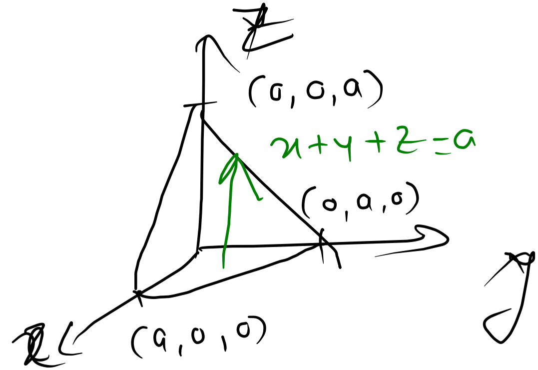
$$dz \, dy \, dx = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

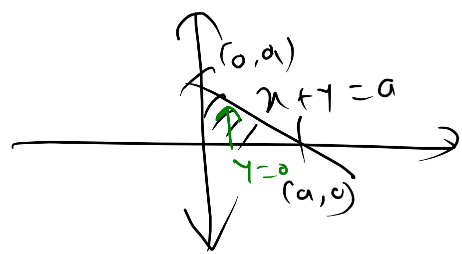
$$V = \iiint r^2 \sin\theta \, dr \, d\theta \, d\phi$$

1. Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a$.

$$V = \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-y-x} dz\, dy\, dx$$

$$= \quad a^3/6.$$





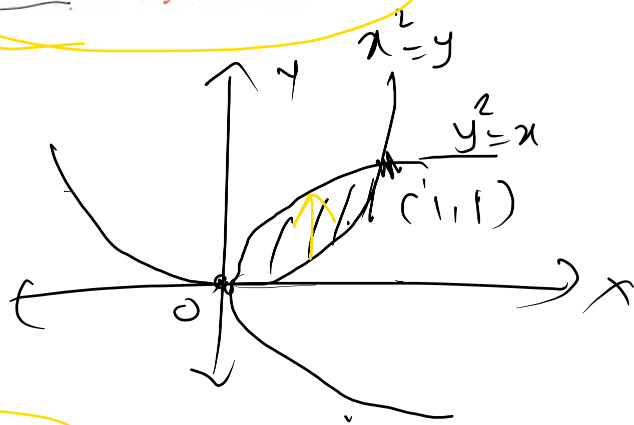
2. Find the volume bounded by $y^2 = x, x^2 = y$ and the planes $z = 0$ and $x + y + z = 1$.

$y^2 = x$

$y = \pm\sqrt{x}$ | $x^2 = y$

$x = \pm\sqrt{y}$

$$V = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{0}^{1-x-y} dz\, dy\, dx$$



$= 1/30$

3. A cylindrical hole of radius b is bored through a sphere of radius a. Find the volume of the remaining solid.

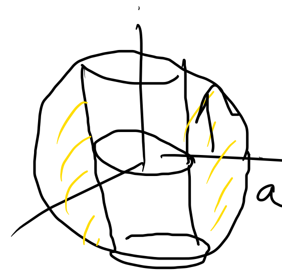
Cylindrical coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$



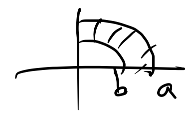
$x^2+y^2+z^2 = a^2$

$r^2 + z^2 = a^2$

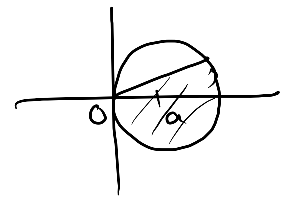
$z = \pm\sqrt{a^2-r^2}$

Total volume = 8 (volume of solid in 1st oct)

$$= 8\int_0^{\pi/2}\int_b^a\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$



4. Show that the volume of the wedge intercepted between the cylinder $x^2 + y^2 = 2ax$ and planes $z = mx, z = nx$ is $\pi(n-m)\,a^3$.



$$x^2 + y^2 = 2ax$$

$$x^2 - 2ax + \frac{4a^2}{4} + y^2 = a^2$$

$$(x-a)^2 + y^2 = a^2$$

$(a, 0)$ $\quad a$

$x^2 + y^2 = 2ax$

$r^{\cancel{2}} = 2a\cancel{r}\cos\theta$

$r : 0 \longrightarrow$

$x = r\cos\theta$

$y = r\sin\theta$

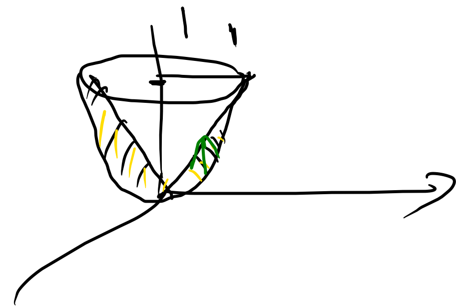
$z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$

$$V = \int_{-\pi/2}^{\pi/2} \int_{0}^{2a\cos\theta} \int_{mx}^{nx} r\,dz\,dr\,d\theta$$

5. Find the volume bounded by the cone $z^2 = x^2 + y^2$ and the paraboloid $z = x^2 + y^2$

$$z^2 = z$$
$$z^2 - z = 0$$
$$z(z-1) = 0$$
$$z = 0,\ z = 1$$



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$dz\,dy\,dx = r\,dz\,dr\,d\theta$$

$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$$

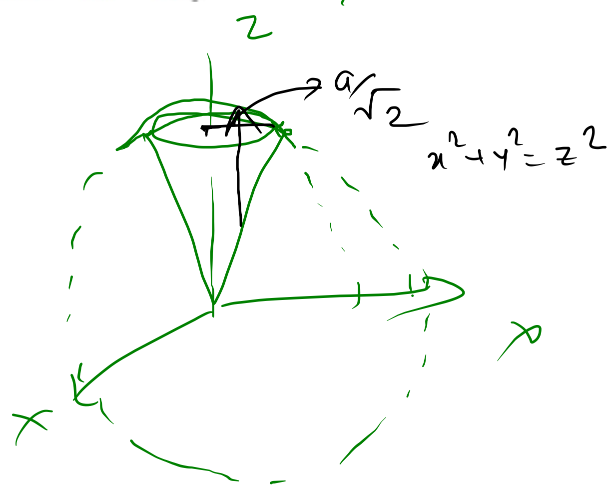
$z = x^2+y^2$ | $z^2 = x^2+y^2$

$z = r^2$ | $z^2 = r^2$

$z = r$

6. Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$

$z > 0$



$$x^2 + y^2 + z^2 = a^2$$

$$x^2 + y^2 = z^2$$

$$z^2 + z^2 = a^2$$

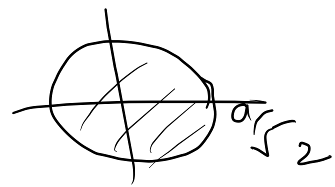
$$2z^2 = a^2$$

$$z = \pm \frac{a}{\sqrt{2}}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$dz\,dy\,dx = r\,dz\,dr\,d\theta$$

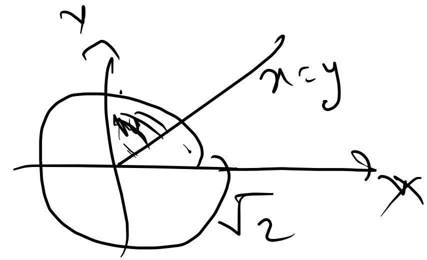
$$z^2 = x^2 + y^2$$
$$z^2 = r^2$$
$$z = r$$

$$V = \int_0^{2\pi} \int_0^{a/\sqrt{2}} \int_r^{\sqrt{a^2 - r^2}} r\,dz\,dr\,d\theta$$



$$x^2 + y^2 + z^2 = a^2$$
$$r^2 + z^2 = a^2$$
$$z = \pm\sqrt{a^2 - r^2}$$

7. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x, z = 0$ and $x = 0$.



$$V = \int_{\pi/4}^{\pi/2} \int_{0}^{\sqrt{2}} \int_{0}^{r(\cos\theta + \sin\theta)} r \, dz \, dr \, d\theta$$

$\pi/4 \le \theta \le \pi/2$

$0 \le r \le \sqrt{2}$

$0 \le z \le x + y$

$= r(\cos\theta + \sin\theta)$

8. Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$.

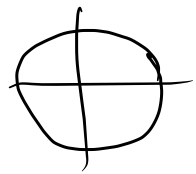
$za = a^2$

$z = a$

$0 \le \theta \le 2\pi$

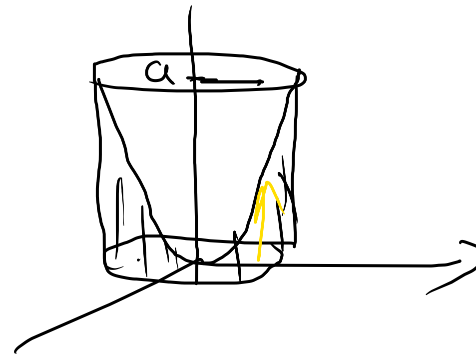
$0 \le r \le a$

$0 \le z \le r^2/a$



$x^2 + y^2 = az$

$r^2 = az$



10. Find the volume cut off from the paraboloid $x^2+\frac{1}{4}y^2+z=1$ by the plane $z=0$.

$$x^2+\frac{1}{4}y^2=(1-z)$$

vertex $(0,0,1)$

$$x^2+\frac{1}{4}y^2=1$$

$$\frac{y^2}{4}=1-x^2$$

$$y=\pm 2\sqrt{1-x^2}$$

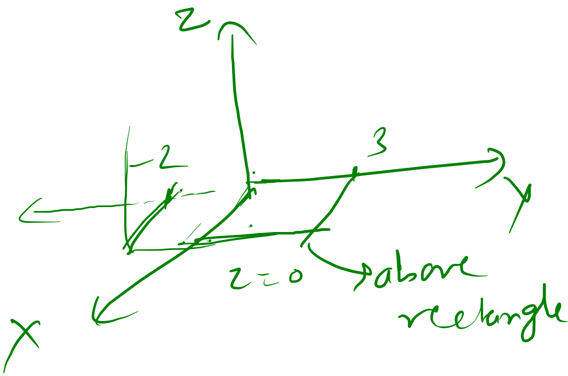
$$V=4\int_{x=0}^{1}\int_{y=0}^{2\sqrt{1-x^2}}\int_{0}^{1-x^2-\frac{y^2}{4}} dz\,dy\,dx$$

11. Find the volume of the triangular prism formed by the planes $ay = bx, y = 0, x = a$ from $z = 0$ to $z = c + xy$

$$V = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}x} \int_{z=0}^{c+xy} dz\, dy\, dx$$

12. Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R\{(x, y) | 0 \le x \le 1, -2 \le y < 3\}$

$$V = \int_{x=0}^{1} \int_{y=-2}^{3} \int_{z=0}^{12-3x-2y} dz \; dy \; dx$$



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$x^2+y^2+z^2=a^2$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin\theta\,dr\,d\theta\,d\phi$$