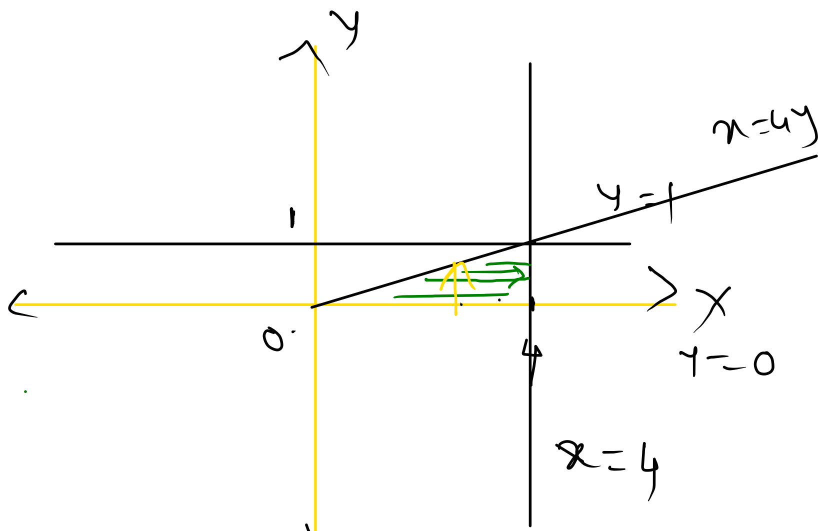
Change the order of following integrals and evaluate (if possible).

1. $\int_0^1 \int_{4y}^4 e^{x^2}\,dx\,dy$

$$\int_{y=0}^{y=1} \int_{x=4y}^{x=4} e^{x^2}\,dx\,dy$$




$y=0$, $y=1$

$x=4y$ $\quad x=4$

$\downarrow$

| | | |
|---|---|---|
| $x$ | 0 | 4 |
| $y$ | 0 | 1 |

$$\int_{x=0}^{4} \int_{y=0}^{x/4} e^{x^2}\,dy\,dx$$

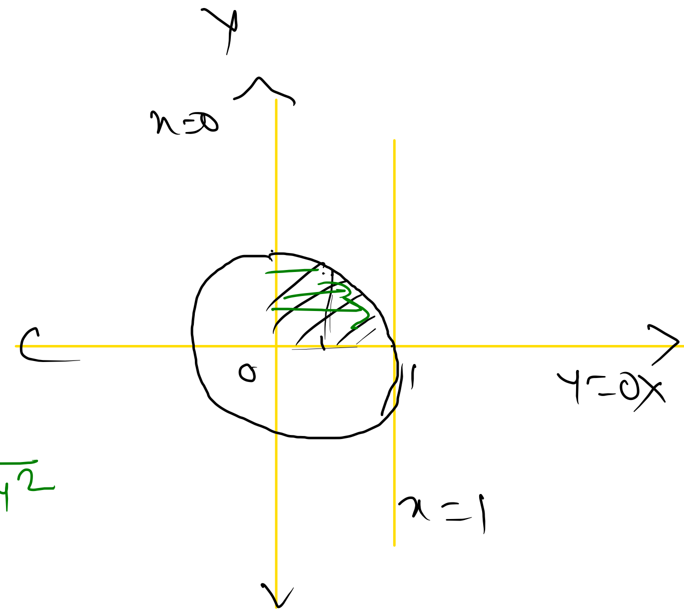
$\frac{1}{8}\left(e^{16}-1\right)$

$\int e^{x^2}\,x$

$x^2 = t$

2. $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\, dy\, dx$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\, dy\, dx$$




$x=0$ , $x=1$

$y=0$ $\quad y=\sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2=1$

$x = \pm\sqrt{1-y^2}$

$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{e^y}{(e^y-1)\sqrt{1-x^2-y^2}}\, dx \quad dy$$

$$= \int_0^1 \frac{e^y}{e^y-1} \int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(\sqrt{1-y^2})^2 - x^2}}\, dx\, dy$$

$$\frac{\pi}{2} \log\left(\frac{e+1}{2}\right).$$

$$\frac{1}{\sqrt{\underbrace{(1-y^2)}_{a} - x^2}} = \frac{1}{\sqrt{(\sqrt{1-y^2})^2 - x^2}}$$

$a - x^2$

$$\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right).$$

**4.** $\displaystyle\int_{y=0}^{2}\int_{x=2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\, dy$

$y=0 \qquad y=2$

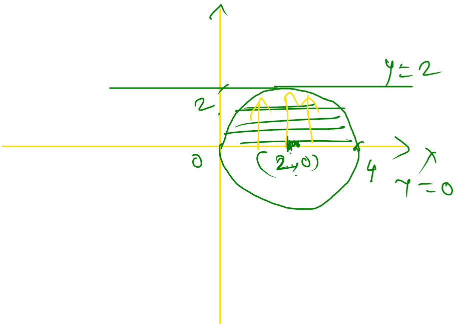
$x = 2 \pm \sqrt{4-y^2}$

$x-2 = \pm\sqrt{4-y^2}$

$(x-2)^2 = 4-y^2$

$(x-2)^2 + y^2 = 4$

center $(2,0)$ rad $2$




$y^2 = 4-(x-2)^2$

$y = \pm\sqrt{4-(x-2)^2}$

$$\int_{x=0}^{4}\int_{y=0}^{\sqrt{4-(x-2)^2}} dy\, dx$$

$$= \int_0^4 (y)_0^{\sqrt{4-(x-2)^2}}\, dx$$

$$= \int_0^4 \sqrt{4-(x-2)^2}\, dx$$

$$= \int_0^4 \sqrt{2^2-\underline{(x-2)}^2}\, dx$$

$$\int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \left[\frac{(x-2)}{2}\sqrt{2^2-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^4$$

$$= \frac{2}{2}\sqrt{\cancel{4}-\cancel{4}} + 2\sin^{-1}(1)$$

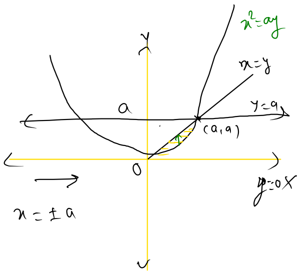
$$- \frac{(-2)}{2}\sqrt{\cancel{4-4}} - 2\sin^{-1}(-1)\Big]$$

$\sin(-\pi/2) = -1$

$$= 2\frac{\pi}{2} + 2\frac{\pi}{2} = 4\frac{\pi}{2} = 2\pi \;//$$

5. $\int_0^a \int_y^{\sqrt{ay}} \frac{x}{x^2+y^2}\,dx\,dy$

$$\int_{y=0}^{a}\int_{x=y}^{\sqrt{ay}} \frac{x}{x^2+y^2}\,dx\,dy$$




Plot $y=0$, $x=y$ $\qquad$ $y=a$, $x=\sqrt{ay}$

$x^2=ay$

$x=\pm\sqrt{ay}$ $\qquad$ $y=a$ $\quad$ $x=\pm a$

$$\int_{x=0}^{a}\int_{y=x^2/a}^{x} \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^a x\int_{x^2/a}^{x} \frac{1}{y^2+x^2}\,dy\,dx$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^a \cancel{x}\left(\frac{1}{\cancel{x}}\tan^{-1}\frac{y}{x}\right)_{x^2/a}^{x} dx$$

$$= \int_0^a \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{x^{\cancel{2}}}{a\cancel{x}}\right)\right] dx$$

$$= \int_0^a \left(\frac{\pi}{4} - \tan^{-1}\left(\frac{x}{a}\right)\right) dx$$

$\int uv = u\int v - \int(\int v)u'$

LIATE

$$= \left(\frac{\pi}{4}x\right)_0^a - \int_0^a \underset{u}{\tan^{-1}\left(\frac{x}{a}\right)}\ \underset{v}{1}\,dx$$

$$= a\frac{\pi}{4} - \left[\tan^{-1}\left(\frac{x}{a}\right)x - \int x\,\frac{1}{1+\left(\frac{x}{a}\right)^2}\,\frac{1}{a}\,dx\right]_0^a$$

$$= \frac{a\pi}{4} - \left[x\tan^{-1}\left(\frac{x}{a}\right) - \int x\,\frac{a^{\cancel{2}}}{(a^2+x^2)}\,\frac{1}{\cancel{a}}\,dx\right]_0^a$$

$$= \frac{a\pi}{4} - \left[x\tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\int \frac{2x}{a^2+x^2}\,dx\right]_0^a$$

$$= \frac{a\pi}{4} - \left[x\tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\log(a^2+x^2)\right]_0^a$$

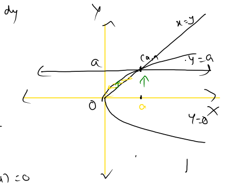
$$= \frac{a\pi}{4} - \left[a\tan^{-1}(1) - \frac{a}{2}\log(2a^2) - 0 + \frac{a}{2}\log(a^2)\right]$$

$$= \cancel{\frac{a\pi}{4}} - \left(\cancel{\frac{a\pi}{4}} - \frac{a}{2}\left(\log(2a^2) - \log(a^2)\right)\right)$$

$$= +\frac{a}{2}\log\left(\frac{2\cancel{a^2}}{\cancel{a^2}}\right) = \frac{a}{2}\log 2$$

6. $\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy$

$$\int_{y=0}^{y=a}\int_{x=y^2/a}^{x=y} \frac{y}{(a-x)\left(\sqrt{ax-y^2}\right)}\, dx\, dy$$




$y=0 \qquad y=a$

$x=y \qquad x=y^2/a \Rightarrow y^2 = xa$

$y = \pm\sqrt{xa}$

$x=y \qquad y^2 = xa \Rightarrow y^2 = ya$

$\Rightarrow y(y-a)=0$

$y=0, \; y=a \qquad x=y \Rightarrow x=0, x=a$

$$\int_{x=0}^{a}\int_{y=x}^{\sqrt{xa}} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dy\, dx$$

$$= \int_0^a \frac{1}{a-x}\int_x^{\sqrt{xa}} \frac{y}{\sqrt{ax-y^2}}\, dy\, dx$$

$ax - y^2 = t$

$-2y\,dy = dt \Rightarrow y\,dy = -\frac{dt}{2}$

$y: x \to \sqrt{xa}$

$t: ax - x^2 \to 0$

$$= \int_0^a \frac{1}{a-x}\int_{ax-x^2}^{0} \frac{-dt}{2\sqrt{t}}\, dx$$

$$= \frac{1}{2}\int_0^a \frac{1}{a-x}\int_0^{ax-x^2} t^{-1/2}\, dt\, dx$$

$$= \frac{1}{2}\int_0^a \frac{1}{(a-x)}\left[\frac{t^{1/2}}{1/2}\right]_0^{ax-x^2} dx$$

$$= \frac{1}{2}\times 2\int_0^a \frac{1}{a-x}(ax-x^2)^{1/2}\, dx$$

$$= \int_0^a (a-x)^{-1}\, x^{1/2}(a-x)^{1/2}\, dx$$

$$= \int_0^a x^{1/2}(a-x)^{-1/2}\, dx \; //$$

$$= \int_0^a x^{1/2} a^{-1/2}\left(1-\frac{x}{a}\right)^{-1/2} dx$$

$\int_0^1 x^m (1-x)^n\, dx = \beta(m+1, n+1)$

$\frac{x}{a} = t$

$x = at$

$dx = a\,dt$

$x: 0 \to a \qquad t: 0 \to 1$

$$= a^{-1/2}\int_0^1 (at)^{1/2}(1-t)^{-1/2}\, a\,dt$$

$$= a^{-1/2} a^{1/2}\, a\int_0^1 t^{1/2}(1-t)^{-1/2}\, dt$$

$$= a\,\beta\left(\tfrac{1}{2}+1, -\tfrac{1}{2}+1\right) = a\,\beta\left(\tfrac{3}{2}, \tfrac{1}{2}\right)$$

$$= a\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2}$$

$$= a\frac{\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{1!}$$

$$= \frac{a}{2}\pi$$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma n = (n-1)!$ ↳ +ve int

$\Gamma n = (n-1)\Gamma n-1$

**14.** $\int_{x=0}^{1} dx \int_{y=1}^{\infty} e^{-y} y^{x} \log y \, dy$

$$\int_{x=0}^{x=1} \int_{y=1}^{\infty} e^{-y} y^{x} \log y \, dy \, dx$$

As all limits are consts we can interchange them directly

$$\int_{1}^{\infty} \int_{0}^{1} e^{-y} \underline{y^{x}} \log y \, dx \, dy$$

$$\int_{1}^{\infty} e^{-y} \cancel{\log y} \left( \frac{y^{x}}{\cancel{\log y}} \right)_{0}^{1} dy$$

$$\int a^{x} = \frac{a^{x}}{\log a}$$

$$= \int_{1}^{\infty} e^{-y} (y - 1) \, dy$$

$$= \int_{1}^{\infty} \underline{(y-1)} \, e^{-y} \, dy$$

LIATE

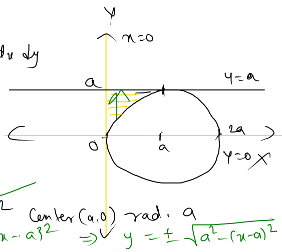
$$= \left[ (y-1) \left( \frac{e^{-y}}{\underline{-1}} \right) - (1) \left( \frac{e^{-y}}{(-1)^{2}} \right) \right]_{1}^{\infty}$$

$$= \left[ 0 - 0 - 0 + e^{-1} \right] = \frac{1}{e}$$

$$e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

**7.** $\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}dx$

$$\int_{y=0}^{a}\int_{x=0}^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy$$




$y=0 \qquad y=a$

$x=0 \qquad x=a-\sqrt{a^2-y^2}$

$(x-a) = -\sqrt{a^2-y^2}$

$(x-a)^2 = a^2-y^2$

$\Rightarrow (x-a)^2+y^2=a^2$ Center $(a,0)$ rad. $a$

$y^2 = a^2-(x-a)^2 \Rightarrow y = \pm\sqrt{a^2-(x-a)^2}$

$$\int_{x=0}^{a}\int_{y=\sqrt{a^2-(x-a)^2}}^{a} \frac{xy\log(x+a)}{(x-a)^2}\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\int_{\sqrt{a^2-(x-a)^2}}^{a} y\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{y^2}{2}\right)_{\sqrt{a^2-(x-a)^2}}^{a} dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{a^2}{2} - \frac{(a^2-(x-a)^2)}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a \frac{x\log(x+a)}{\cancel{(x-a)^2}}\left(\cancel{(x-a)^2}\right)dx$$

$$= \frac{1}{2}\int_0^a \underset{u}{\log(x+a)}\cdot \underset{v}{x}\,dx \qquad \int uv = u\int v - \int\left(\int v\,u'\right)$$

$$= \frac{1}{2}\left[\log(x+a)\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{(x+a)}dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int\frac{x^2}{(x+a)}dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int\frac{x^2-a^2+a^2}{(x+a)}dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int\left[\frac{(x-a)\cancel{(x+a)}}{\cancel{(x+a)}} + \frac{a^2}{x+a}\right]dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\left(\frac{x^2}{2} - ax + a^2\log(x+a)\right)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log(2a) - \frac{1}{2}\left(\frac{a^2}{2} - a^2 + a^2\log(2a)\right) - 0 + \frac{1}{2}\left(0-0+a^2\log(a)\right)\right]$$

$$= \frac{1}{2}\left[\cancel{\frac{a^2}{2}\log(2a)} - \frac{1}{2}\left(-\frac{a^2}{2}\right) - \cancel{\frac{a^2}{2}\log(2a)} + \frac{a^2}{2}\log a\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{4} + \frac{a^2}{2}\log a\right]$$

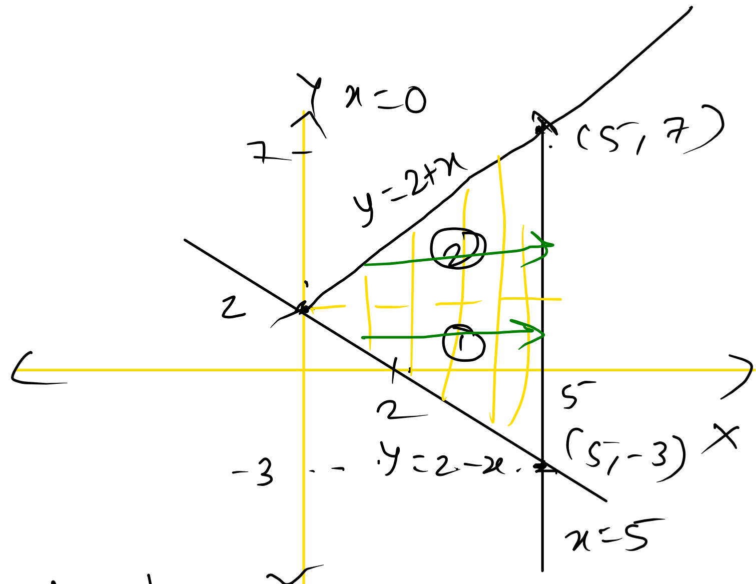
$$\int_0^5 \int_{2-x}^{2+x} dy\, dx$$

$$\int_{x=0}^{5} \int_{y=2-x}^{2+x} dy\, dx$$

To plot $x=0$, $x=5$, $y=2-x$, $y=2+x$

| $x$ | 5 | 0 | 2 |
|---|---|---|---|
| $y$ | $-3$ | 2 | 0 |

| $x$ | 5 | 0 | $-2$ |
|---|---|---|---|
| $y$ | 7 | 2 | 0 |




$$\int_{y=-3}^{2} \int_{x=2-y}^{5} dx\, dy + \int_{y=2}^{7} \int_{x=y-2}^{5} dx\, dy$$

$= 25$

**17.** $\int_{x=0}^{a} \int_{y=x^2/a}^{2a-x} xy \, dy \, dx$

To plot. $x=0$, $x=a$

$y = x^2/a \qquad y = 2a - x$

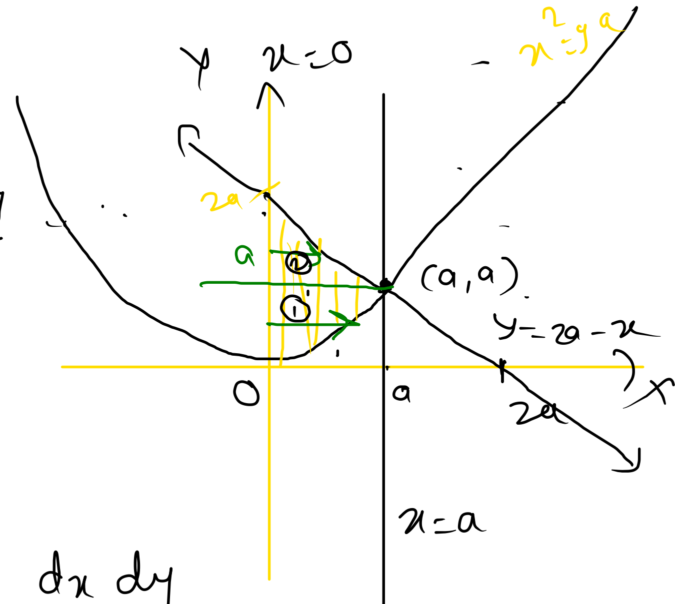
| $x$ | 0 | 2a | a |
|---|---|---|---|
| $y$ | 2a | 0 | a |

$x^2 = ya$

$x = \pm\sqrt{ya}$

$y = a$

$x = \pm a$




$$\int_{y=0}^{a} \int_{x=0}^{\sqrt{ya}} xy \, dx \, dy + \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy \, dx \, dy$$

$$= \frac{3}{8} a^4$$

$$\int_{y=0}^{1} \int_{x=-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{y=1}^{3} dy \int_{x=-1}^{1} dx$$

$y = 0 \quad y = 1 \, , \quad y = 3 \quad x = -1 , \, x = 1$

$x = \pm\sqrt{y} \Rightarrow x^2 = y$

$y = 1 \Rightarrow x = \pm 1$

$$\int_{x=-1} \int_{y=x^2}^{3} dy\,dx$$

$$= 16/3$$