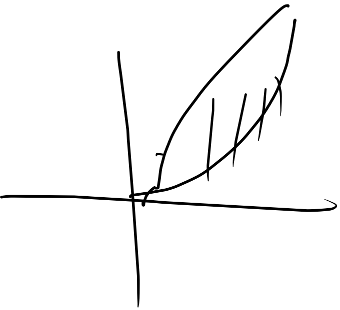
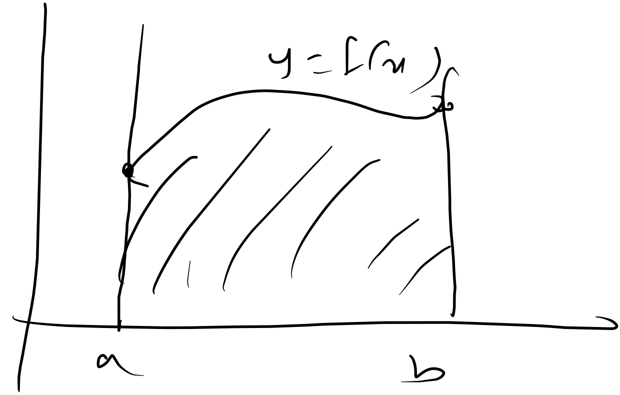
Double Integration.

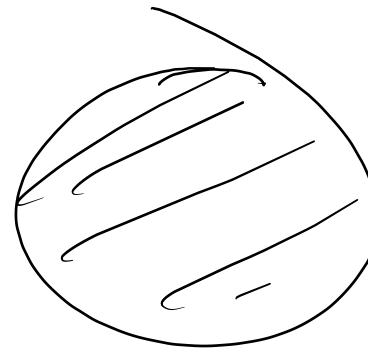


$$\int_{x=a}^{x=b} f(x)\,dx$$



$$\int\int$$

$f(x,y)$



$$\int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx$$

$$\int_{y=c}^{d}\left[\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\right]dy$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$dx\,dy = \underline{r}\,dr\,d\theta$$

$$\int_{\theta=} \int_{r=} f(r,\theta)\, r\,dr\,d\theta \quad //$$

$$\int_{y=0}^{1}\int_{x=0}^{y} xy\,e^{-x^2}\,dx\,dy$$

$$= \int_0^1 y\left(\int_0^y x\,e^{-x^2}\,dx\right)dy$$

$$x^2 = t \Rightarrow 2x\,dx = dt$$

$$x\,dx = \frac{dt}{2}$$

$$x : 0 \to y$$

$$t : 0 \to y^2$$

$$= \int_0^1 y\left(\int_0^{y^2} e^{-t}\,\frac{dt}{2}\right)dy$$

$$= \frac{1}{2}\int_0^1 y\left(\frac{e^{-t}}{-1}\right)_0^{y^2} dy$$

$$= \frac{1}{2}\int_0^1 y\left(\frac{e^{-y^2}}{-1} - \frac{1}{-1}\right)dy$$

$$= \frac{1}{2}\int_0^1 y\left(-e^{-y^2}+1\right)dy$$

$$= \frac{1}{2}\int_0^1 y\left(1-e^{-y^2}\right)dy$$

$$y^2 = t$$

$$2y\,dy = dt \Rightarrow y\,dy = \frac{dt}{2}$$

$$y : 0 \to 1 \qquad t : 0 \to 1$$

$$= \frac{1}{2}\int_0^1 \left(1-e^{-t}\right)\frac{dt}{2}$$

$$= \frac{1}{4}\left[t - \frac{e^{-t}}{-1}\right]_0^1$$

$$= \frac{1}{4}\left[1 - \frac{e^{-1}}{-1} - 0 + \frac{1}{-1}\right]$$

$$= \frac{1}{4}\left[\cancel{1} + \frac{1}{e} - \cancel{1}\right] = \frac{1}{4e}$$

//

$$\int_0^1 \left[\int_0^{x^2} e^{(y/x)}\, dy\right] dx$$

$\int uv$ $\hookrightarrow$ u has vanishing derivative

$= u\underbrace{\int v}_{v_1} - u'\underbrace{\int v_1}_{v_2} + u''\int v_2$ ——

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx = \int_0^1 \left(\frac{e^x}{1/x} - \frac{1}{1/x}\right) dx$$

$$= \int_0^1 \left[\underline{x e^x} - \underset{\sim}{x}\,(1)\right] dx$$

$$= \left[x e^x - (1)e^x - \frac{x^2}{2}\right]_0^1$$

$$= \cancel{e} - \cancel{e} - \frac{1}{2} - 0 + 1 + 0 = \frac{1}{2}$$

$$I = \int_0^\infty \left[\int_{x=0}^\infty e^{-x^2(1+y^2)} \, x\,dx \right] dy$$

$$x^2(1+y^2) = t$$

$$(1+y^2)\, 2x\, dx = dt$$

$$x\,dx = \frac{dt}{2(1+y^2)}$$

$$x : 0 \to \infty \quad t : 0 \to \infty$$

$$= \int_0^\infty \int_0^\infty e^{-t} \frac{dt}{2(1+y^2)} \, dy$$

$$= \frac{1}{2} \int_0^\infty \frac{1}{1+y^2} \left(\int_0^\infty e^{-t} \, dt \right) dy$$

$$= \frac{1}{2} \int_0^\infty \frac{1}{1+y^2} \left(\frac{e^{-t}}{-1} \right)_0^\infty dy$$

$$= \frac{1}{2} \int_0^\infty \frac{1}{1+y^2} \left(0 - \frac{1}{-1} \right) dy \qquad e^{-\infty} = \frac{1}{e^\infty} = 0$$

$$= \frac{1}{2} \int_0^\infty \frac{1}{1+y^2} \, dy$$

$$= \frac{1}{2} \left[\frac{1}{1} \tan^{-1}\left(\frac{y}{1} \right) \right]_0^\infty = \frac{1}{2} \left[\tan^{-1}\infty - \tan^{-1} 0 \right]$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi}{4} \; //$$

HW

$$\int_0^1 \int_{x^2}^{x} xy(x+y)\, dy\, dx = \frac{1}{24}$$

$$\int_0^1 \int_{x^2}^{x} (x^2 y + xy^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2 \frac{y^2}{2} + x \frac{y^3}{3} \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[x^2 \frac{x^2}{2} + x \frac{x^3}{3} - x^2 \frac{(x^2)^2}{2} - x \frac{(x^2)^3}{3} \right] dx$$

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$$

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\underbrace{(1+x^2)}_{a^2}+y^2}\,dy\,dx$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\underbrace{\sqrt{1+x^2}}_{a})^2+y^2}\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\left(\tan^{-1}(1)\right) - \frac{1}{\sqrt{1+x^2}}\tan^{-1}(0)\right] dx$$

$$= \int_0^1 \left[\frac{\pi}{4}\,\frac{1}{\sqrt{1+x^2}} - 0\right] dx$$

$$= \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log(0+1)\right]$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

$$\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\; dx\, dy$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\; dx \quad dy$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right).$$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2-y^2}\right)^2 - x^2}\; dx\, dy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{(a^2-y^2)}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{\sqrt{a^2-y^2}}{2}\sqrt{\cancel{a^2-y^2-(a^2-y^2)}} + \frac{(a^2-y^2)}{2}\sin^{-1}(1) - 0 - 0\right] dy$$

$$= \int_0^a \left(\frac{a^2-y^2}{2}\right)\frac{\pi}{2}\; dy = \frac{\pi}{4}\int_0^a (a^2-y^2)\, dy$$

$$= \frac{\pi}{4}\left(a^2y - \frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{4}\left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{\pi}{\cancel{4}\,2}\left(\frac{\cancel{2}a^3}{3}\right) = \frac{\pi a^3}{6}$$

$$\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\,dx\,dy$$

$$= 2\int_1^2 y^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} x^2\,dx\,dy$$

$$= 2\int_1^2 y^2 \left(\frac{x^3}{3}\right)_{-\sqrt{2-y}}^{\sqrt{2-y}} dy$$

$$= 2\int_1^2 y^2 \left(\frac{(\sqrt{2-y})^3}{3} - \frac{(-\sqrt{2-y})^3}{3}\right) dy$$

$$= \frac{2}{3}\int_1^2 y^2 \left((2-y)^{3/2} + (2-y)^{3/2}\right) dy$$

$$= \frac{2}{3}\times\int_1^2 y^2\, 2\,(2-y)^{3/2}\, dy$$

$$= \frac{4}{3}\int_1^2 y^2 (2-y)^{3/2}\, dy$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$ if $f(x)$ is even

$$2-y=t$$
$$y=2-t$$
$$dy=-dt$$
$y: 1 \to 2$ $\quad t: 1 \to 0$

$$= \frac{4}{3}\int_1^0 (2-t)^2\, t^{3/2}\,(-dt)$$

$$= \frac{4}{3}\int_0^1 (4-4t+t^2)\, t^{3/2}\, dt$$

$$= \frac{4}{3}\int_0^1 \left(4t^{3/2} - 4t^{5/2} + t^{7/2}\right) dt$$

$$= \frac{4}{3}\left[\frac{4t^{5/2}}{5/2} - 4\,\frac{t^{7/2}}{7/2} + \frac{t^{9/2}}{9/2}\right]_0^1$$

$$= \frac{4}{3}\left[4\times\frac{2}{5} - 4\times\frac{2}{7} + \frac{2}{9} - 0\right]$$

$$\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$$

put $1+r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$r : 0 \longrightarrow \sqrt{\cos 2\theta}$

$t : 1 \longrightarrow 1+\cos 2\theta = 2\cos^2\theta$

$$= \int_0^{\pi/4}\int_1^{2\cos^2\theta} \frac{dt}{2(t)^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\int_1^{2\cos^2\theta} t^{-2}\, dt\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{t^{-1}}{-1}\right)_1^{2\cos^2\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{(2\cos^2\theta)^{-1}}{-1} - \frac{(1)^{-1}}{-1}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[-\frac{1}{2\cos^2\theta} + 1\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[-\frac{\sec^2\theta}{2} + 1\right] d\theta$$

$$= \frac{1}{2}\left[-\frac{\tan\theta}{2} + \theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[-\frac{1}{2} + \frac{\pi}{4} + 0\right] = \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

Hw

$$\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$a^2-r^2=t$$

$$-2r\,dr=dt$$

$$r\,dr=-\frac{dt}{2}$$

$$r: 0 \rightarrow a\cos\theta$$

$$t: a^2 \rightarrow a^2-a^2\cos^2\theta$$

$$= a^2(1-\cos^2\theta)=a^2\sin^2\theta$$

$$\int^{\pi/2} \sin^p\theta\,\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

HW

$$\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta \, dr\, d\theta.$$

Hint — $\int_0^{\pi/2} \cos\theta\,(1-\sin\theta)^3$

put $\sin\theta = t$