11

Differential Equations of First Order

11.1. DEFINITIONS

(1) **A differential equation** *is an equation which involves differential coefficients or differentials.*

Thus (*i*) $e^x dx + e^y dy = 0$ (*ii*) $\dfrac{d^2x}{dt^2} + n^2x = 0$

(*iii*) $y = x\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$ (*iv*) $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \Big/ \dfrac{d^2y}{dx^2} = c$

(*v*) $\dfrac{dx}{dt} - wy = a \cos pt,\ \dfrac{dy}{dt} + wx = a \sin pt$ (*vi*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

(*vii*) $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ are all examples of differential equations.

(2) *An* **ordinary differential equation** *is that in which all the differential coefficients have reference to a single independent variable.* Thus the equations (*i*) to (*v*) are all ordinary differential equations.

A **partial differential equation** *is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.* Thus the equations (*vi*) and (*vii*) are partial differential equations.

(3) *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

Thus, from the examples above,

(*i*) is of the first order and first degree ; (*ii*) is of the second order and first degree ;

(*iii*) written as $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (*iv*) written as $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3 = c^2\left(\dfrac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

11·2. PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases :

(i) formulation of differential equation from the given physical situation, called **modelling**

(ii) solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and

(iii) physical interpretation of the solution.

11·3. FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

Example 11·1. *Form the differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$.*

Sol. To eliminate the constants A and α differentiating it twice, we have

$$\frac{dx}{dt} = -nA\sin(nt+\alpha) \text{ and } \frac{d^2x}{dt^2} = -n^2A\cos(nt+\alpha) = -n^2x$$

Thus $$\frac{d^2x}{dt^2} + n^2x = 0$$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

Example 11·2. *Obtain the differential equation of all circles of radius a and centre (h, k).* *(Andhra, 1999)*

Sol. Such a circle is $(x-h)^2 + (y-k)^2 = a^2$ ...(*i*)

where h and k, the co-ordinates of the centre, and a are the constants.

Differentiating it twice, we have

$$x - h + (y-k)\frac{dy}{dx} = 0 \text{ and } 1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then $$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and $$x - h = -(y-k)\,dy/dx = \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2}$$

Substituting these in (*i*) and simplifying, we get $[1 + (dy/dx)^2]^3 = a^2(d^2y/dx^2)^2$ ...(*ii*)

as the required differential equation.

Writing (*ii*) in the form $\dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = a$,

it states that the radius of curvature of a circle at any point is constant.

Example 11.3. *Obtain the differential equation of the coaxial circles of the system $x^2 + y^2 + 2ax + c^2 = 0$ where c is a constant and a is a variable.* (*J.N.T.U., 2003*)

...(*i*)

Sol. We have $x^2 + y^2 + 2ax + c^2 = 0$

Differentiating w.r.t. x, $2x + 2y\, dy/dx + 2a = 0$

or
$$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (*i*), $x^2 + y^2 - 2(x + y\, dy/dx)\, x + c^2 = 0$

or
$$2xy\, dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4. (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example $\quad x = A\cos(nt + \alpha)$ ...(1)

is a solution of $\quad \dfrac{d^2x}{dt^2} + n^2x = 0$ [Example 11.1] ...(2)

The **general** (or **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.* Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example $\quad x = A\cos(nt + \pi/4)$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. *Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation*

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad ...(3)$$

are said to be linearly independent if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent*.

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of (3).

Example 11.4. *Find the differential equation whose set of independent solutions is [e^x, xe^x].*

Sol. Let the general solution of the required differential equation be $y = c_1e^x + c_2xe^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = c_1e^x + c_2(e^x + xe^x)$$

$$\therefore \quad y - y_1 = c_2 e^x \qquad \text{...(ii)}$$

Again differentiating (*ii*) w.r.t. x, we obtain

$$y_1 - y_2 = c_2 e^x \qquad \text{...(iii)}$$

Subtracting (*iii*) from (*ii*), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) **Geometrical meaning of a differential equation.** Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \qquad \text{...(1)}$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx $(= m)$ when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

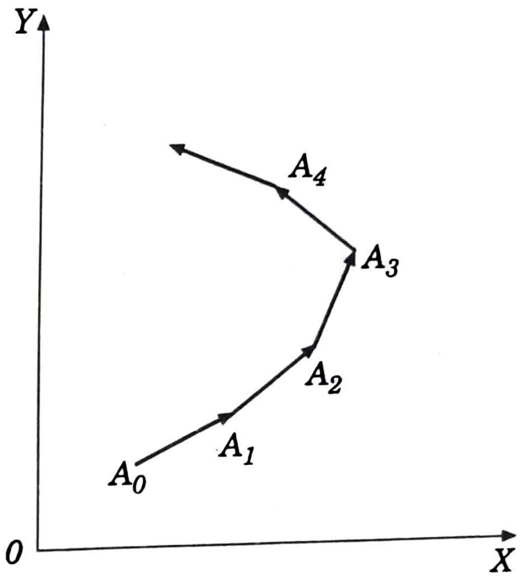

Fig. 11·1.

If the successive points A_0, A_1, A_2, A_3...... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point $A_0(x_0, y_0)$. Clearly the slope of the tangent to C at any point and the co-ordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

11·11. EXACT DIFFERENTIAL EQUATIONS

(1) **Def.** *A differential equation of the form* $\mathbf{M}(x, y)\,\mathbf{dx} + \mathbf{N}(x, y)\,\mathbf{dy} = \mathbf{0}$ *is said to be* **exact** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.* $du \equiv Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) **Theorem.** *The necessary and sufficient condition for the differential equation* $Mdx + Ndy = 0$ *to be exact is*

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \qquad \ldots(1)$$

where u is some function of x and y.

But $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad ...(2)$$

$\therefore$ Equating coefficients of dx and dy in (1) and (2), we get $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{(Assumption)}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e. if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \text{ i.e. } M = \frac{\partial u}{\partial x} \qquad \left\{ \begin{aligned} &\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ (given)} \\ &\text{and } \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \end{aligned} \right. \quad ...(3)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ or } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Integrating both sides w.r.t. x (taking y as constant).

$N = \frac{\partial u}{\partial y} + f(y)$, where $f(y)$ is a function of y alone. ...(4)

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x}dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\}dy \quad \text{[by (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + f(y)\,dy = du + f(y)\,dy = d\left[u + \int f(y)\,dy\right] \quad ...(5)$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) **Method of solution.** By (5), the equation $Mdx + Ndy = 0$ becomes $d\left[u + \int f(y)\,dy\right] = 0$

Integrating $\quad u + \int f(y)\,dy = 0.$

But $\quad u = \int_{y \text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y \text{ cons.})} \mathbf{M}dx + \int (\textbf{terms of N not containing x})\,dy = c$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11·25. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$ (V.T.U., 2006)

Sol. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2} \cdot 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e. $$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3)\,dx + \int (-3y^2)\,dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11·26. *Solve* $\left\{ y\left(1+\frac{1}{x}\right)+\cos y \right\} dx + (x + \log x - x \sin y)\, dy = 0$ (V.T.U., 2006)

Sol. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y\ const)} \left\{ \left(1+\frac{1}{x}\right) y + \cos y \right\} dx = c \quad \text{or} \quad (x + \log x)\, y + x \cos y = c.$$

Example 11·27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Sol. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e., $$\int_{(y\ const)} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \cdot 2x dx - \int 2\,x\,dx \right] = c$$

or $$x + y \sin x^2 - y\, x^2 = c.$$

Example 11·28. *Solve* $\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ (*Kurukshetra, 2005 ; Sambalpur, 2002*)

Sol. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\ const.)} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e. $$\int_{(y\ const.)} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Example 11·29. *Solve* $(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y dy = 0$ (U.P.T.U., 2005)

Sol. Given equation can be written as

$$\frac{ydy}{x\,dx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{y\,dy + xdx}{ydy - xdx} = \frac{5\,(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$ [By componendo & dividendo

or $$\frac{x\,dx + y\,dy}{x^2 + y^2 - 3} = 5 \cdot \frac{x\,dx - y\,dy}{x^2 - y^2 - 1}$$

Integrating both sides oides, we get

$$\int \frac{2x\,dx + 2y\,dy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log(x^2 + y^2 - 3) = 5 \log(x^2 - y^2 - 1) + \log c'$$ [Writing $c = \log c'$

or $$x^2 + y^2 - 3 = c'(x^2 - y^2 - 1)^5$$

which is the required solution.

Problems 11·7

Solve the following equations :

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy$.
2. $(x^2 + y^2 - a^2)\,xdx + (x^2 - y^2 - b^2)\,ydy = 0$. *(Kurukshetra, 2005)*
3. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$.
4. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$.
5. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$.
6. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$.
7. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$.
8. $\dfrac{2x}{y^3}\,dx + \dfrac{y^2 - 3x^2}{y^4}\,dy = 0$.
9. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$.
10. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$.
11. $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy$. *(Osmania, 2000 S)*

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor $(Mx + Ny \neq 0)$.*

Example 11.31. *Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.* *(Osmania, 2003 S)*

Sol. This equation is homogeneous in x and y.

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ The solution is $\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$ or $\frac{x}{y} - 2\log x + 3\log y = c.$

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Example 11.32. *Solve $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$.* *(Raipur, 2005)*

Sol. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y,\; N = (1 - xy)\,x.$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)\,yx - (1 - xy)\,xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

or $$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

or $$\log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $Mdx + Ndy = 0$,

(*a*) *if* $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ *be a function of x only $= f(x)$ say, then $e^{\int f(x)\,dx}$ is an integrating factor.*

(*b*) *if* $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ *be a function of y only $= F(y)$ say, then $e^{\int F(y)\,dy}$ is an integrating factor.*

Example 11.33. *Solve $(xy^2 - e^{1/x^3})\,dx - x^2\,ydy = 0$.* *(Raipur, 2004)*

Sol. Here $\quad M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-4}{x}dx} = e^{-4\log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx - \frac{y}{x^2} dy = 0$

which is an exact equation.

$\therefore$ The solution is $\int\limits_{(y\ const)} M\,dx + \int$ (terms of N not containing x) $dy = c$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\,dx = c \qquad \text{or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Example 11·34. *Solve* $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0.$

Sol. Here $\quad M = xy^3 + y,\ N = 2\,(x^2y^2 + x + y^4)$

$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y\,(xy^2 + 1)} (4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y}$, which is a function of y alone.

$\therefore \quad$ I.F. $= e^{\int 1/y\,dy} = e^{\log y} = y$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\,dx + (2x^2y^3 + 2xy + 2y^5)\,dy = 0$, which is an exact equation.

$\therefore$ Its solution is $\int\limits_{(y\ const)} M\,dx + \int$ (terms of N not containing x) $dy = 0$

or $$\int\limits_{(y\ const)} (x\,y^4 + y^2)\,dx + \int 2y^5\,dy = c \qquad \text{or} \qquad \tfrac{1}{2} x^2y^4 + xy^2 + \tfrac{1}{3} y^6 = c.$$

(5) For the equation of the type

$$x^a y^b\,(my\,dx + nx\,dy) + x^{a'} y^{b'}\,(m'y\,dx + n'x\,dy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a + h + 1}{m} = \frac{b + k + 1}{n},\ \frac{a' + h + 1}{m'} = \frac{b' + k + 1}{n'}.$$

Example 11·35. *Solve* $(y \log y)\,dx + (x - \log y)\,dy = 0$ *(U.P.T.U., 2004)*

Sol. Here $M = y \log y$ and $N = x - \log y$

$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y} (1 - \log y - 1) = -\frac{1}{y}$, which is a function of y alone.

$\therefore \quad$ I.F. $= e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\,dx + \frac{1}{y}(x - \log y)\,dy = 0$$

which is an exact equation $\quad \left[\because \frac{\partial}{\partial y}(\log y) = \frac{\partial}{\partial x}\left(\frac{x - \log y}{y}\right)\right]$

$\therefore$ Its solution is $\int_{(y\ const.)} M\,dx + \int$ (terms of N not containing x) $dy = c$

or $$\log y \int dx + \int \left(-\frac{\log y}{y}\right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

Example 11·36. *Solve* $y\,(xy + 2x^2y^3)\,dx + x\,(xy - x^2y^2)\,dy = 0.$

(*Hissar, 2005 ; Kurukshetra, 2005*)

Sol. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^a y^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1,\ m = n = 1\ ;\ a' = b' = 2,\ m' = 2,\ n' = -1.$

$\therefore$ $\quad$ I.F. $= x^h y^k$.

where $\quad \dfrac{a+h+1}{m} = \dfrac{b+k+1}{n},\ \dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}$

i.e. $\quad \dfrac{1+h+1}{1} = \dfrac{1+k+1}{1},\ \dfrac{2+h+1}{2} = \dfrac{2+k+1}{-1}$

or $\quad h - k = 0,\ h + 2k + 9 = 0$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\displaystyle\int_{(y\ const)} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

or $$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c$$

or $$2\log x - \log y - 1/xy = c.$$