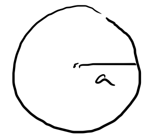
Area.



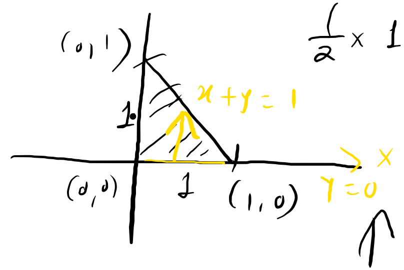

$\pi a^2$

$$\text{Area} = \iint 1 \, dy \, dx \; / \; = \iint 1 \, dx \, dy$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx \, dy = r \, dr \, d\theta$$

$$\text{Area} = \iint r \, dr \, d\theta$$

$$\int_{x=0}^{1} \int_{y=0}^{y=1-x} dy \, dx$$




$\frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$$= \int_0^1 (y)_0^{1-x} \, dx = \int_0^1 (1-x) \, dx$$

$$= \left[ x - \frac{x^2}{2} \right]_0^1 = \left[ 1 - \frac{1}{2} \right] = \frac{1}{2}$$

**1.** Find by double integration the area enclosed $y^2 = x^3$ and $y = x$

$y^2 = x^3$ $\quad$ $y^2 = x$

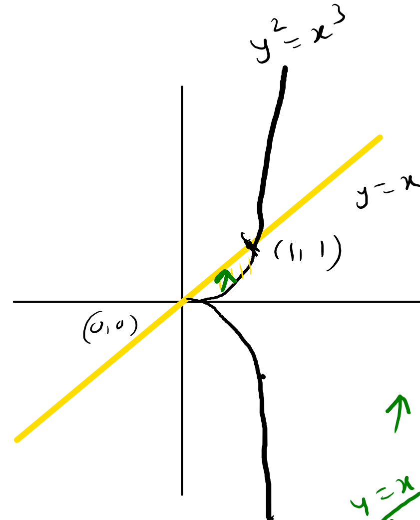
$x(-ve) \Rightarrow x^3(-ve)$

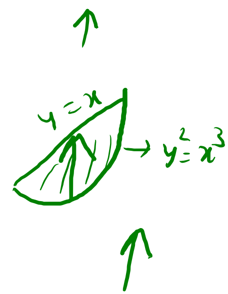
$y^2 = -ve \rightarrow\leftarrow$

$y^2 = x^3 \Rightarrow y = \pm\sqrt{x^3}$

$x = 1 \Rightarrow y = \pm 1$

$(1,1) \quad (1,-1)$




$$\text{Area} = \int_{x=0}^{1}\int_{y=x^{3/2}}^{x} dy\,dx$$

$$= \int_0^1 (y)_{x^{3/2}}^{x}\,dx = \int_0^1 \left(x - x^{3/2}\right) dx$$

$$= \left(\frac{x^2}{2} - \frac{x^{5/2}}{5/2}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{5}\right) = \frac{1}{10}$$

$$\int x^n = \frac{x^{n+1}}{n+1}$$

**2.** Find the area between parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$

$(y-a) = (x-b)^2$

vertex$(b, a)$

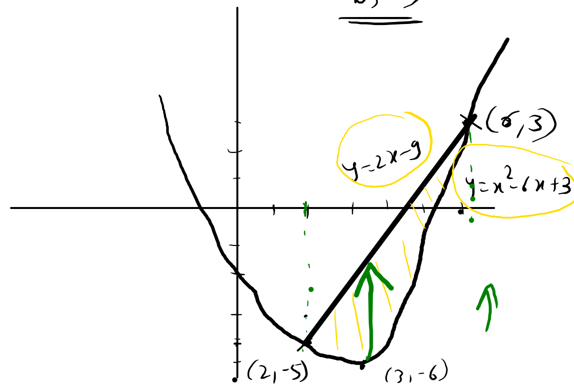
$y = x^2 - 6x + 3$

$y = x^2 - 6x + \frac{\overset{9}{\cancel{36}}}{\cancel{4}_1} - 9 + 3$

$y = (x-3)^2 - 6$

$(y+6) = (x-3)^2$

vertex$(3, -6)$




$ax^2 + bx + c$

$\frac{m^2}{4c} = \frac{b^2}{4a}$

$A = \int_2^6 \int_{x^2-6x+3}^{2x-9} dy\, dx$

$= 32/3$

$y = x^2 - 6x + 3$ $\qquad$ $y = 2x - 9$

$x^2 - 6x + 3 = 2x - 9$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

$x = 6, 2$

$y = 2x - 9 \Rightarrow$

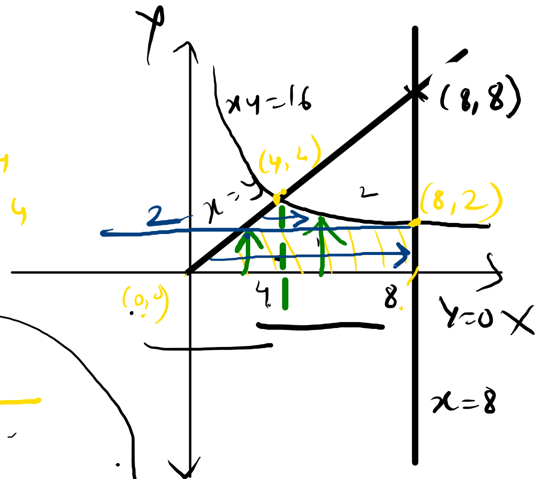
| $x$ | 6 | 2 |
|---|---|---|
| $y$ | 3 | -5 |

3. Sketch the region bounded by the curves $xy = 16, y = x, x = 8$ and $y = 0$.
Express the area of this region as a double integral in two ways

$xy = 16$
$y = x$
$x = 8$
$y = 0$

$xy = 16$
$x = y$
$y^2 = 16 \Rightarrow y = \pm 4$
as $x = y \Rightarrow x = \pm 4$

for $x = 8$
$xy = 16$
$\Rightarrow 8y = 16 \Rightarrow y = 2$




$$A = \iint dy\, dx$$

$$= \int_{x=0}^{4}\int_{y=0}^{x} dy\, dx + \int_{x=4}^{8}\int_{y=0}^{16/x} dy\, dx$$

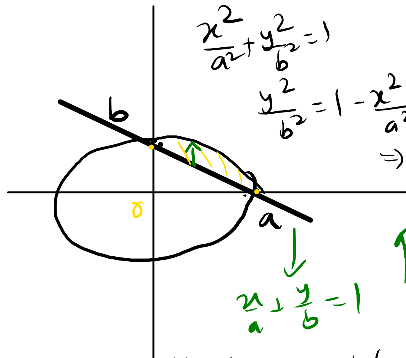
$xy = 16$

$$A = \int_{y=0}^{2}\int_{x=y}^{8} dx\, dy. + \int_{y=2}^{4}\int_{x=y}^{16/y} dx\, dy$$

Find by double integration the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$

$$\frac{x}{a}+\frac{y}{b}=1$$

| | | |
|---|---|---|
| $x$ | $0$ | $a$ |
| $y$ | $b$ | $0$ |




$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \Rightarrow y^2=b^2\frac{(a^2-x^2)}{a^2}$$

$$\Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{y}{b}=1-\frac{x}{a} \Rightarrow y=b\left(1-\frac{x}{a}\right)$$

$$=b\frac{(a-x)}{a}$$

$$A=\int_{x=0}^{a}\int_{y=\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$=\int_0^a (y)_{b/a\,(a-x)}^{b/a\,\sqrt{a^2-x^2}}\,dx$$

$$=\int_0^a\left[\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right]dx$$

$$=\frac{b}{a}\int_0^a\left[\sqrt{a^2-x^2}-(a-x)\right]dx \quad = \quad \frac{ba}{4}(\pi-2) \rightarrow \text{Ans}$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \quad //$$

**5.** Using double integration find the area bounded by the parabolas $x = y^2, x = 2y - y^2$

$x = y^2$ vertex $(0,0)$

$y = \pm\sqrt{x}$ //

$x = 1$ $y = \pm 1$

$x = 2y - y^2$

$-x = y^2 - 2y + \frac{4}{4} - 1$

$(1-x) = (y-1)^2$ vertex $(1,1)$

$x = y^2$ $x = 2y - y^2$

$y^2 = 2y - y^2 \Rightarrow 2y^2 = 2y \Rightarrow y^2 - y = 0$

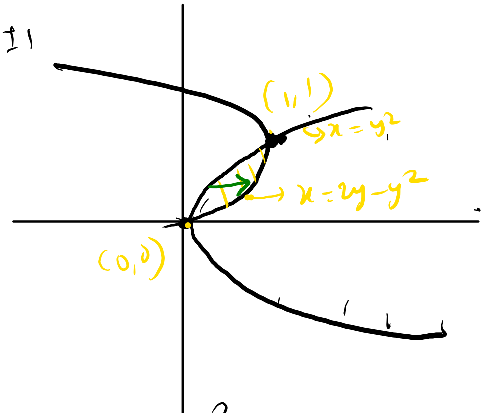
$y(y-1) = 0$

$\Rightarrow y = 0, \ y = 1$

Intersection Points

$x = y^2$

| | | |
|---|---|---|
| $x$ | 0 | 1 |
| $y$ | 0 | 1 |




$$A = \int_{y=0}^{1} \int_{x=y^2}^{2y-y^2} dx\, dy$$

$$= 1/3$$

6. Find by double integration the area included between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$

$y = 3x^2 - x - 3$

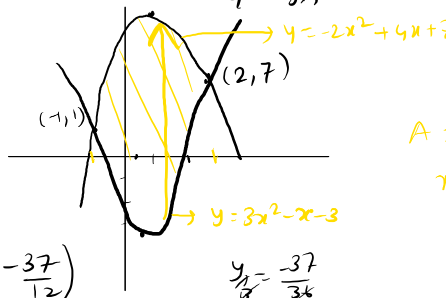
$= 3\left(x^2 - \frac{x}{3} - 1\right)$

$\frac{y}{3} = \left(x^2 - \frac{x}{3} + \frac{1}{36} - \frac{1}{36} - 1\right)$

$\frac{y}{3} = \left(x - \frac{1}{6}\right)^2 - \frac{37}{36}$

$\left(\frac{y}{3} + \frac{37}{36}\right) = \left(x - \frac{1}{6}\right)^2$ vertex $\left(\frac{1}{6}, -\frac{37}{12}\right)$

$\frac{1/9}{4} = \frac{1}{9 \times 4}$




$\frac{y}{\cancel{3}} = \frac{-37}{\cancel{36}}$ 12

$\frac{4}{4(1)} = 1$

$A = \int_{x:-1}^{2} \int_{y:3x^2-x-3}^{-2x^2+4x+7} dy\, dx$

$= \frac{45}{2}$

$y = -2x^2 + 4x + 7$

$= -2\left(x^2 - 2x - \frac{7}{2}\right)$

$-\frac{y}{2} = \left(x^2 - 2x + 1 - 1 - \frac{7}{2}\right)$

$-\frac{y}{2} = (x-1)^2 - \frac{9}{2} \Rightarrow \left(\frac{9}{2} - \frac{y}{2}\right) = (x-1)^2$

vertex $(1, 9)$

$\frac{y}{\cancel{2}} = \frac{9}{\cancel{2}}$

$y = 3x^2 - x - 3 \qquad y = -2x^2 + 4x + 7$

$3x^2 - x - 3 = -2x^2 + 4x + 7$

$5x^2 - 5x - 10 = 0$

$x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$x = 2, -1$

$y = 3x^2 - x - 3$

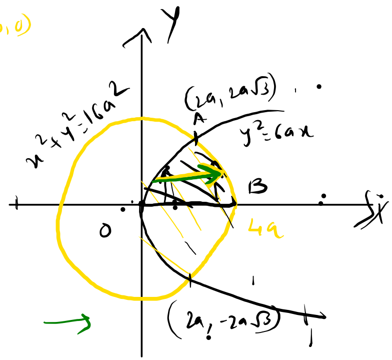
| x | 2 | -1 |
|---|---|---|
| y | 7 | 1 |

$x = 2 \quad y = 12 - 2 - 3$

$x = -1 \quad y = \cancel{3} + 1 \cancel{-3}$

**7.** Find the larger of the two areas into which the circle $x^2 + y^2 = 16a^2$ is divided by the parabola $y^2 = 6ax$

circle → centre $(0,0)$
& rad $4a$




$y^2 = 6ax$

$y = \pm\sqrt{6ax}$ vertex $(0,0)$

$x^2+y^2 = 16a^2$ $\quad y^2 = 6ax$

$x^2 + 6ax - 16a^2 = 0$

$(x+8a)(x-2a) = 0$

$x = -8a$ $\quad x = 2a$

$y^2 = 6ax$

$x = 2a$ $\quad y^2 = 12a^2$

$y = \pm\sqrt{12a^2}$

| | | |
|---|---|---|
| $x$ | $2a$ | $2a$ |
| $y$ | $2a\sqrt{3}$ | $-2a\sqrt{3}$ |

Area $= 2$ (Area of $OAB$)

$$= 2\int_{y=0}^{2a\sqrt{3}}\int_{x=y^2/6a}^{\sqrt{16a^2-y^2}} dx\, dy = \frac{4}{3}(4\pi+\sqrt{3})a^2$$

$x^2+y^2 = 16a^2$

$x^2 = 16a^2 - y^2$

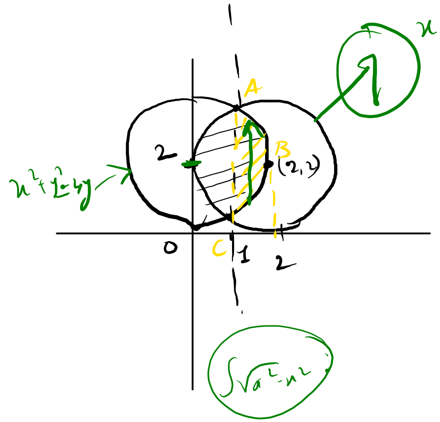
$x = \pm\sqrt{16a^2-y^2}$

$\int \sqrt{a^2-x^2}$ ...

**8.** Find by double integration the area common to the circles $x^2+y^2-4y=0$ and $x^2+y^2-4x-4y+4=0$

$$x^2+y^2-4y=0$$
$$x^2+y^2-4y+\frac{16}{4}=4$$
$$x^2+(y-2)^2=2^2$$
centre $(0,2)$ & rad 2.

$$x^2+y^2-4x-4y+4=0$$
$$x^2-4x+4+y^2-4y+4=4$$
$$(x-2)^2+(y-2)^2=2^2$$
centre $(2,2)$ & rad 2




$x^2+y^2-4x-4y+4=0$ ✓

$$A=2(\text{Area } ABC)$$
$$=2\int_{x=1}^{2}\int_{y=2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dy\,dx \;=4\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$$

$$x^2+y^2-4y=0$$
$$x^2+(y-2)^2=2^2$$
$$(y-2)^2=4-x^2$$
$$(y-2)=\pm\sqrt{4-x^2}$$
$$y=2\pm\sqrt{4-x^2}$$

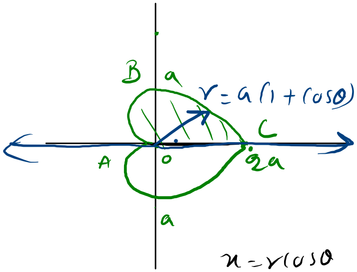
$$x^2+y^2-4y=0$$
$$x^2+y^2=4y$$

$$x^2+y^2-4x-4y+4=0$$
$$\cancel{4y}-4x-\cancel{4y}+4=0$$
$$4x=4\Rightarrow x=1$$

**9.** Find the area of the cardioide $r = a(1+\cos\theta)$

**Solution:**

| $\theta$ | $\cos\theta$ | $r$ |
|---|---|---|
| $0$ | $1$ | $2a$ |
| $\pi/2$ | $0$ | $a$ |
| $\pi$ | $-1$ | $0$ |
| $3\pi/2$ | $0$ | $a$ |
| $2\pi$ | $1$ | $2a$ |




$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$1+\cos 2\theta = 2\cos^2\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$$\text{Area} = 2\,(\text{Area } ABC)$$

$$= 2\left(\int_0^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta\right)$$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}\left(\cos^2\theta/_2\right)^2 d\theta = a^2\int_0^{\pi}\cos^4(\theta/_2)\,d\theta$$

$=$ put $\theta/_2 = t \Rightarrow d\theta = 2dt$

$\theta : 0 \to \pi \quad t : 0 \to \pi/_2$

$$= a^2\int_0^{\pi/2}\cos^4 t\;2dt$$

$$= 2a^2\,\frac{1}{2}\,\beta\left(\frac{0+1}{2}, \frac{4+1}{2}\right)$$

$$= \frac{3}{2}\pi a^2$$

Polar //.

**10.** Find the total area enclosed by the lemniscate of Bernoulli $(x^2 + y^2)^2 = a^2(x^2 - y^2)$

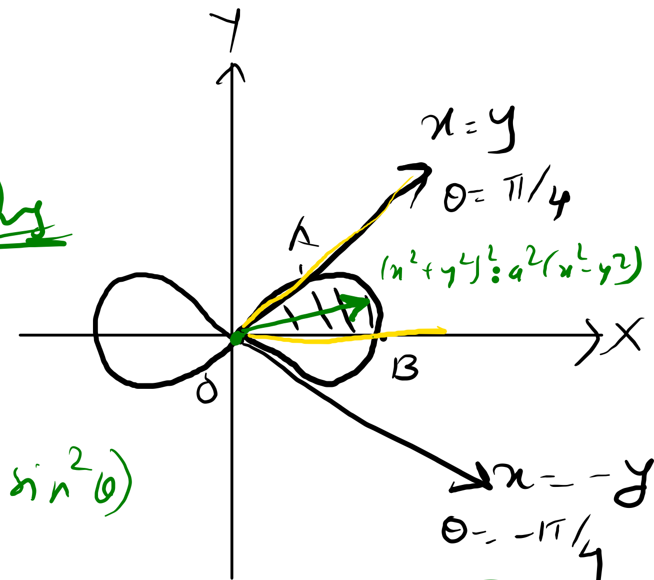
$x = r\cos\theta$ , $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$A = 4$ (Area of OAB)

$$= 4\int_{0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$= a^2$ ~~Ans~~




$(x^2+y^2)^2 = a^2(x^2-y^2)$

$(r^2)^{\cancel{2}} = a^2\cancel{r^2}(\cos^2\theta - \sin^2\theta)$

$r^2 = a^2\cos 2\theta$

$r = a\sqrt{\cos 2\theta}$

$\int \cos ax\,dx = \frac{\sin ax}{a}$.

**11.** Find the area inside the circle $r = a\sin\theta$ and outside the cardioide $r = a(1 - \cos\theta)$.

| $\theta$ | $\cos\theta$ | $r$ |
|---|---|---|
| 0 | 1 | 0 |
| $\pi/2$ | 0 | $a$ |
| $\pi$ | $-1$ | $2a$ |
| $3\pi/2$ | 0 | $a$ |
| $2\pi$ | 1 | 0 |

$x = r\cos\theta$

$y = r\sin\theta$

$r = a\sin\theta$

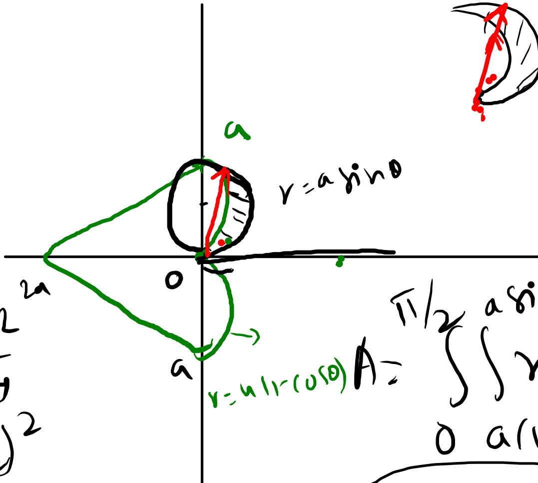
$r^2 = ar\sin\theta$

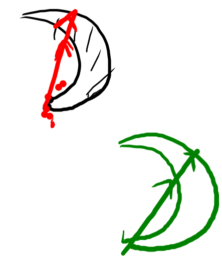
$x^2 + y^2 = ay$

$x^2 + y^2 - ay + \frac{a^2}{4} = \frac{a^2}{4}$

$x^2 + (y - a/2)^2 = (a/2)^2$

Centre $(0, a/2)$ rad $a/2$

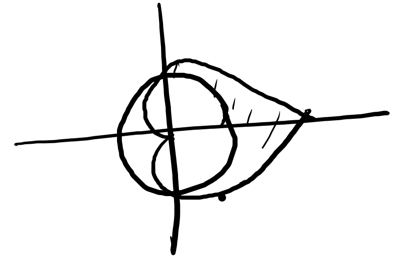


$$A = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \frac{a^2}{4}(\pi + 8)$$

**12.** Find the area outside the circle $r = a$ and inside the cardioide $r = a\,(1 + cos\theta\,)$.



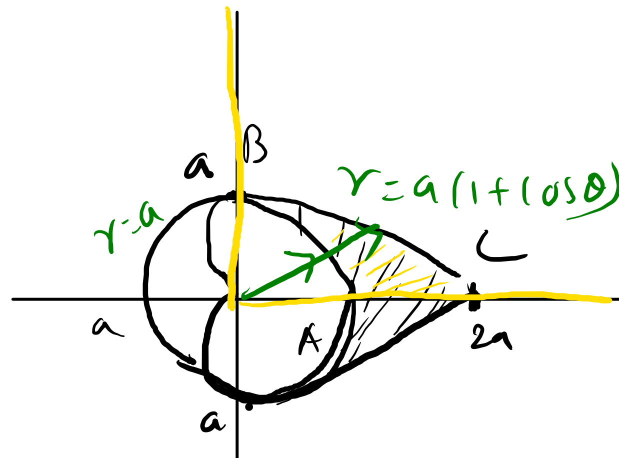
$x = r\cos\theta$

$y = r\sin\theta$

$r = a$

$r^2 = a^2$

$x^2 + y^2 = a^2$




$$A = 2\,(\text{Area } ABC)$$

$$= 2 \int_0^{\pi/2} \int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \frac{a^2}{4}(\pi + 8)$$

**13.** Find the area outside the circle $r = a\sqrt{2}$ and inside circle $r = 2a\cos\theta$.

$r = a\sqrt{2}$

centre $(0,0)$

& rad - $a\sqrt{2}$

$r = 2a\cos\theta$

$r^2 = 2a\, \underline{r\cos\theta}$

$x = r\cos\theta$

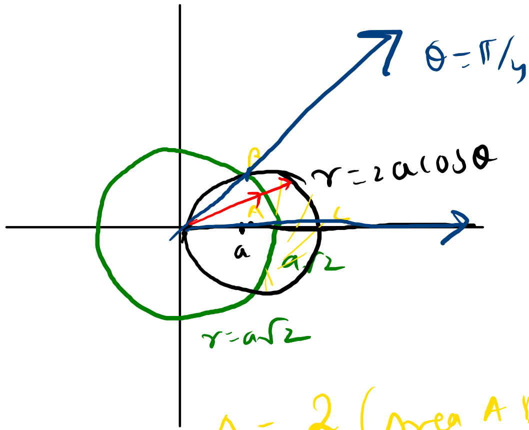
$y = r\sin\theta$

$x^2 + y^2 = 2ax$

$x^2 - 2ax + \frac{4a^2}{4} + y^2 = a^2$

$(x-a)^2 + y^2 = a^2$

centre $(a,0)$ & rad $a$




$A = 2\,(\text{Area } ABC)$

$$= 2\int_0^{\pi/4}\int_{r=a\sqrt{2}}^{2a\cos\theta} r\,dr\,d\theta = a^2$$

$r = a\sqrt{2}$ $\qquad$ $r = 2a\cos\theta$

$a\sqrt{2} = 2a\cos\theta \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4$