SOMAIYA VIDYAVIHAR UNIVERSITY
K J Somaiya College of Engineering

Somaiya TRUST

Module 3
Integration :
Review And Some New Techniques

$\int x^3 dx$

$\int \frac{e^{ax}}{x} dx$

$\int \frac{\sin ax}{x} dx$

Sub-Module 3.2

DUIS

(Differentiation Under Integral Sign)

Syllabus

3	**Integration : Review And Some New Techniques**		7	CO 3
	3.1	Beta and Gamma functions with properties		
	3.2	Differentiation under integral sign with constant limits of integration.(without proof)		
		# Self-learning topic: Differentiation under integral sign with variable limits of integration.		

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DUIS

If $f(x,\alpha)$ is a continuous function of $x, and\ \alpha$ is a parameter and if $\frac{\partial f}{\partial \alpha}$ is a continuous function of $x\ and\ \alpha$ together throughout the interval $[a,b]$ where a, b are constants and independent of α, and if

$$I(\alpha) = \int_a^b f(x,\alpha)dx$$

then,

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)dx$$

$f(x,\alpha)$

$\int \sin ax$

$f(x,a)$

$= \int_0^1 e^{ax}\,dx$

$= \left[\frac{e^{ax}}{a}\right]_0^1$

$I(a) = \left[\frac{e^a}{a} - \frac{1}{a}\right]$

$I(\alpha) = \int_a^b f(x,\alpha)\,dx$

using DUIS

$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha}(f(x,\alpha))\,dx$

$\int_a^b \phi(x,\alpha)\,dx$

$\frac{dI}{d\alpha} = h(\alpha)$

$\int dI = \int h(\alpha)\,d\alpha + c$

$I = \int h(\alpha)\,d\alpha + c$

❖ **Ex.1 Prove that** $\int_0^{\infty} \frac{e^{-\alpha x}\sin x}{x}\,dx = \cot^{-1}\alpha$ **Deduce that,**
$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

$$I(\alpha) = \int_0^{\infty} \frac{e^{-\alpha x}\sin x}{x}\,dx$$

Using DUIS

$$\frac{dI}{d\alpha} = \int_0^{\infty} \frac{\partial}{\partial\alpha}\left(e^{-\alpha x}\frac{\sin x}{x}\right)dx$$

$$= \int_0^{\infty} \frac{\sin x}{\cancel{x}}\left(e^{-\alpha x}\right)(-\cancel{x})\,dx$$

$$= -\int_0^{\infty} e^{-\alpha x}\sin x\,dx \quad //$$

$a = -\alpha \quad b = 1$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\frac{dI}{d\alpha} = -\left[\frac{e^{-\alpha x}}{\alpha^2+1}(-\alpha\sin x - \cos x)\right]_0^{\infty}$$

$$= -\left[0 - \frac{1}{\alpha^2+1}(-\alpha\sin 0 - \cos 0)\right]$$

$e^{-\infty} = 0$

$$= -\left[-\frac{1}{\alpha^2+1}(0-1)\right] = -\frac{1}{\alpha^2+1}$$

$$\frac{dI}{d\alpha} = -\frac{1}{\alpha^2+1}$$

$$dI = -\frac{1}{\alpha^2+1}\,d\alpha$$

On integrating

$$\int dI = \int -\frac{1}{\alpha^2+1}\,d\alpha + c \Rightarrow I = \cot^{-1}(\alpha) + C$$

$$\Rightarrow I(\alpha) = \cot^{-1}(\alpha) + C \quad -(1)$$

$$I(\alpha) = \int_0^{\infty} e^{-\alpha x}\frac{\sin x}{x}\,dx$$

put $\alpha = \infty$ in (1)

$$I(\infty) = \cot^{-1}(\infty) + C$$

$$0 = 0 + C$$

as $e^{-\infty} = 0$
$\therefore I(\infty) = 0$

$$\Rightarrow \boxed{C = 0}$$

$$\therefore I(\alpha) = \cot^{-1}(\alpha)$$

$$\int_0^{\infty} e^{-\alpha x}\frac{\sin x}{x}\,dx = \cot^{-1}(\alpha)$$

put $\alpha = 0$

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \cot^{-1}(0) = \frac{\pi}{2}$$

❖ **Evaluate,** $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx, \quad a > -1$

$$I(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$$

Using DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}(1-e^{-ax})\right)dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(0 - e^{-ax}(-x)\right)dx$$

$$= \int_0^\infty \frac{e^{-x}}{\cancel{x}}(\cancel{x}\, e^{-ax})\,dx$$

$$= \int_0^\infty e^{-x-ax}\,dx$$

$$= \int_0^\infty e^{-(1+a)x}\,dx$$

$$= \left[\frac{e^{-(1+a)x}}{-(1+a)}\right]_0^\infty$$

$$= 0 - \frac{1}{-(1+a)} = \frac{1}{1+a}$$

$$\frac{dI}{da} = \frac{1}{1+a} \Rightarrow dI = \frac{1}{1+a}\,da$$

Integrating $\int dI = \int \frac{1}{1+a}\,da + C$

$$I = \log(1+a) + c$$

$$I(a) = \log(1+a) + c \quad \text{——— (1)}$$

$$I(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$$

put $a=0$ in (1)

$$I(0) = \log(1) + C$$

$$0 = 0 + C \Rightarrow C = 0$$

$$I(a) = \log(1+a)$$

❖ **Prove that** $\int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}, \quad a \geqslant 0$

$$I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\log(1+ax^2)}{x^2}\right]dx$$

$$= \int_0^\infty \frac{1}{\cancel{x^2}}\,\frac{1}{(1+ax^2)}\,(0+\cancel{x^2})\,dx$$

$$= \int_0^\infty \frac{1}{1+(\sqrt{a}\,x)^2}\,dx$$

$$= \left[\frac{1}{1}\,\frac{\tan^{-1}(\sqrt{a}\,x)}{\sqrt{a}}\right]_0^\infty$$

$$= \int_0^\infty \frac{1}{(\sqrt{a})^2\left(\left(\frac{1}{\sqrt{a}}\right)^2+x^2\right)}\,dx$$

$$= \frac{1}{a}\,\frac{1}{\frac{1}{\sqrt{a}}}\tan^{-1}\left(\frac{x}{1/\sqrt{a}}\right)$$

$$= \frac{\sqrt{a}}{a}\times\frac{\sqrt{a}}{\sqrt{a}}\tan^{-1}(x\sqrt{a}).$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int e^{ax}\,dx \qquad \frac{e^{ax}}{a}$$

$$\int (ax+b)^n\,dx \qquad \frac{(ax+b)^{n+1}}{(n+1)\cdot a}$$

$$= \frac{1}{\sqrt{a}}\left(\frac{\pi}{2}-0\right)$$

$$\frac{dI}{da} = \frac{\pi}{2\sqrt{a}}$$

$$dI = \frac{\pi}{2\sqrt{a}}\,da$$

$$\int dI = \frac{\pi}{2}\int a^{-1/2}\,da + C$$

$$I = \frac{\pi}{\cancel{2}}\,\frac{a^{1/2}}{1/\cancel{2}} + C$$

$$I(a) = \sqrt{a}\,\pi + C$$

$$I(a) = \int_0^\infty \frac{\log(1+ax^2)}{x^2}\,dx$$

put $a = 0$

$$I(0) = 0 + C \Rightarrow \boxed{0 = C}$$

$$I(a) = \sqrt{a}\,\pi$$

HW. Prove that $\int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}dx = \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$

$$I(a) = \int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}\,dx \Rightarrow \frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}-e^{-ax}}{x\sec x}\right)dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{1}{x\sec x}\left(0 - e^{-ax}(-x)\right)dx$$

$$\int_0^\infty e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$= \int_0^\infty \frac{1}{\cancel{x}}e^{-ax}\cancel{x}\cos x\,dx$$

$a = (-a) \qquad b = 1$

$$= \left[\frac{e^{-ax}}{a^2+1}\left(-a\cos x + \sin x\right)\right]_0^\infty$$

$$\int \frac{f'(x)}{f(x)} = \log(f(x))$$

$$= \left[0 - \frac{1}{a^2+1}\left(-a(1) + 0\right)\right]$$

$$= \frac{a}{a^2+1}$$

$$\frac{dI}{da} = \frac{a}{a^2+1} \Rightarrow \int dI = \int \frac{a}{a^2+1}\,da + C$$

$$\Rightarrow I = \frac{1}{2}\int \frac{2a}{a^2+1}\,da + C$$

$$\Rightarrow I(a) = \frac{1}{2}\log(a^2+1) + C$$

$$I(a) = \int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}\,dx$$

put $a = 1 \qquad I(1) = 0$

$$I(1) = \frac{1}{2}\log(2) + C$$

$$0 = \frac{1}{2}\log 2 + C \Rightarrow C = -\frac{1}{2}\log 2$$

$$\int_0^\infty \frac{e^{-x}-e^{-ax}}{x\sec x}\,dx = \frac{1}{2}\log(a^2+1) - \frac{1}{2}\log 2$$

$$= \frac{1}{2}\log\left(\frac{a^2+1}{2}\right)$$

❖ **Prove that,** $\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$

$$I(a) = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$$

DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)\right)dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(1 - 0 + \frac{1}{\cancel{x}}e^{-ax}(-\cancel{x})\right)dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$$

Again apply DUIS.

$$\frac{d}{da}\left(\frac{dI}{da}\right) = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}(1-e^{-ax})\right)dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(0 - e^{-ax}(-x)\right)dx$$

$$= \int_0^\infty \frac{e^{-x}}{\cancel{x}}\cancel{x}e^{-ax}\,dx$$

$$= \int_0^\infty e^{-x(1+a)}\,dx$$

$$= \left[\frac{e^{-x(1+a)}}{-(1+a)}\right]_0^\infty$$

$$= \left[0 - \frac{1}{-(1+a)}\right]$$

$m = \frac{dI}{da}$

$$\frac{d}{da}\left(\underbrace{\frac{dI}{da}}_{m}\right) = \frac{1}{1+a}$$

$$\frac{dm}{da} = \frac{1}{1+a} \Rightarrow \int dm = \int \frac{1}{1+a}da + C$$

$$\Rightarrow m = \log(1+a) + C$$

$$\Rightarrow \frac{dI(a)}{da} = \log(1+a) + C \quad \text{—— (1)}$$

$$m = \frac{dI}{da} = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\,dx$$

put $a=0$ in (1) as $m(0) = \frac{dI}{da}(0) = 0$

$$\frac{dI}{da}(0) = \log(1) + C$$

$$0 = 0 + C \Rightarrow C = 0$$

$$\frac{dI}{da} = \log(1+a)$$

$$\int dI = \int \underset{u}{\log(1+a)}\,\underset{v}{(1)}\,da + C$$

$$I = \log(1+a)(a) - \int \frac{(a)}{1+a}da + C$$

$$= a\log(1+a) - \int \frac{a+1-1}{1+a}da + C$$

$$= a\log(1+a) - \int\left(1 - \frac{1}{1+a}\right)da + C$$

$$= a\log(1+a) - (a - \log(1+a)) + C$$

$$= a\log(1+a) - a + \log(1+a) + C$$

$$I(a) = (a+1)\log(1+a) - a + C \quad \text{—— (1)}$$

$$I(0) = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right)dx$$

put $a=0$

$\therefore I(0) = 0$

from (1)

$I(0) = \log 1 + C$

$0 = 0 + C \Rightarrow C = 0$

❖ **Show that,** $\int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$

❖ **Solution:** Let, $I(a) = \int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx \ldots\ldots\ldots\ldots(1)$

❖ Using Differentiation under integral sign w. r. t. 'a', we get

❖ $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{\tan^{-1}ax}{x(1+x^2)}dx$

$= \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{1}{1+a^2x^2} \cdot x \quad dx$

$= \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)} \quad dx$

Let $x^2 = t$ we have,

$$\frac{1}{(1+t)(1+a^2t)} = \frac{A}{1+t} + \frac{B}{1+a^2t} \qquad \text{(Partial Fraction)}$$

$$1 = A(1+a^2t) + B(1+t)$$

For $t = -1$ $\qquad 1 = A(1-a^2) \qquad \therefore A = \frac{1}{1-a^2}$

For $t = -\frac{1}{a^2}$ $\qquad 1 = B\left(1-\frac{1}{a^2}\right) \qquad \therefore B = \frac{a^2}{a^2-1} = \frac{-a^2}{1-a^2}$

❖ $\frac{dI}{da} = \frac{1}{1-a^2}\int_0^\infty \left(\frac{1}{(1+x^2)} - \frac{a^2}{1+a^2x^2}\right)dx$

$$= \frac{1}{1-a^2}\left[\tan^{-1}x - a^2 \cdot \frac{\tan^{-1}ax}{a}\right]\Bigg|_0^\infty$$

$$= \frac{1}{1-a^2}[\tan^{-1}x - a\tan^{-1}ax]\Bigg|_0^\infty$$

$$= \frac{1}{1-a^2}\{[\tan^{-1}\infty - a\tan^{-1}\infty] - [0]\}$$

$$= \frac{1}{1-a^2}\left\{\frac{\pi}{2} - a\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{2}\ \frac{1}{(1-a)(1+a)}(1-a)$$

❖ $\frac{dI}{da} = \frac{\pi}{2(1+a)}$

❖ Integrating both sides w.r.to a ,

$$I(a) = \frac{\pi}{2}\log(1+a) + c \ldots\ldots\ldots\ldots(2)$$

❖ To find c put $a = 0$, $I(0) = c$

But from (1), $I(0) = 0 \qquad \therefore c = 0$

$\therefore$ From (2)

$$I(a) = \int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$$

❖ **Prove That** $\int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx = \pi\left[\sqrt{a+1}-1\right]$

❖ **Solution:** Let $I(a) = \int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx \ldots\ldots\ldots\ldots\ldots.(1)$

❖ Using differentiation under integral sign w. r. t. 'a' we get,

❖ $\frac{dI}{da} = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial a} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx$

❖ $= \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x} \frac{1}{1+a\sin^2 x} \sin^2 x \ dx$

❖ $= \int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin^2 x} \ dx$

❖ Divide N and D by $cos^2 x$

❖ $= \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{sec^2 x + a\tan^2 x} \ dx = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{1+tan^2 x + a\tan^2 x} \ dx$

❖ $= \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{1+(1+a)\tan^2 x} \ dx \ = \int_0^{\infty} \frac{dt}{1+(1+a)t^2} \ dx$

❖ $\frac{dI}{da} = \left. \frac{tan^{-1}\sqrt{1+a}\, t}{\sqrt{1+a}} \right|_0^{\infty}$

❖ $= \frac{1}{\sqrt{1+a}}\left[tan^{-1}\infty - tan^{-1}0\right]$

❖ $\frac{dI}{da} = \frac{1}{\sqrt{1+a}} \ \frac{\pi}{2}$

❖ Integrating both sides w.r.to a , $I(a) = \frac{\pi}{2} \ 2\sqrt{1+a} + c \ldots\ldots\ldots\ldots(2)$

❖ To find c put $a = 0$

❖ $I(0) = \pi + c$

❖ But from (1) $I(0) = 0 \therefore c = -\pi$

❖ $\therefore$ From (2)

❖ $I(a) = \int_0^{\frac{\pi}{2}} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx = \pi\left[\sqrt{a+1}-1\right]$

$\int_0^1 x^2$
$\parallel$
$\int_0^1 t^2$

Prove $\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \cdot dx = \frac{\sqrt{\pi}}{2} \quad e^{-2a}$

$$I(a) = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx$$

DUIS

$$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a} e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx$$

$$= \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \left[-\left(0+\frac{1}{x^2}(2a)\right)\right] dx$$

$$= \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \left(\frac{-2a}{x^2}\right) dx$$

put $\frac{a}{x} = y \Rightarrow x = \frac{a}{y}$

$\frac{-a}{x^2} dx = dy$

as $x: 0 \to \infty$
$y: \infty \to 0$

$$= \int_\infty^0 e^{-\left(\left(\frac{a}{y}\right)^2 + y^2\right)} (2)\, dy$$

$$= -2\left[\int_0^\infty e^{-\left(y^2+\frac{a^2}{y^2}\right)} dy\right] \rightarrow I(a) = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx$$

$\int_0^1 x^2 \equiv \int_0^1 y^2$

$$\frac{dI}{da} = -2\, I(a)$$

$$\frac{dI}{da} = -2I$$

$$\frac{dI}{I} = -2\, da$$

$$\int \frac{dI}{I} = -2\int da + \log C$$

$$\Rightarrow \log(I) = -2a + \log C$$

$$\Rightarrow I = e^{-2a+\log C} = e^{-2a} \cdot e^{\log C}$$

$$I(a) = e^{-2a} C \quad \text{— (1)}$$

$$I(a) = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx$$

put $a = 0$ — in (1)

$$I(0) = C$$

$$C = \int_0^\infty e^{-x^2} dx \qquad \int_0^\infty e^{-x} x^n dx = \Gamma(n+1)$$

put $x^2 = t$

$x = t^{1/2} \Rightarrow dx = \frac{1}{2} t^{-1/2} dt$

$x: 0 \to \infty \quad t: 0 \to \infty$

$$= \int_0^\infty e^{-t} \frac{1}{2} t^{-1/2} dt = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2}\Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$I(a) = e^{-2a} \frac{\sqrt{\pi}}{2}$$

Practice Problems

- **Assuming the validity of differentiation under the integral sign, prove that**

- **1.** $\int_0^{\infty} \frac{1-\cos}{x^2}\, dx = \frac{\pi a}{2}$ $\left(\text{Assume That } \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}\right)$

- **2.** $\int_0^{\infty} e^{-x^2} \cos 2\, ax\;\, dx = \frac{\sqrt{\pi}}{2}\, e^{-a^2}$

  $\left(\text{Assume that, } \int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}\right)$

- **3.** $\int_0^{\pi} \frac{dx}{(a-\cos\;\;)} = \frac{\pi}{\sqrt{a^2-1}}$ $(a > 0)$Deduce that,

i. $\int_0^{\pi} \frac{dx}{(a-\cos x)^2} = \frac{\pi a}{(a^2-1)^{\frac{3}{2}}}$ **ii.** $\int_0^{\pi} \frac{dx}{(2-\cos\;\;)^2} = \frac{2\pi}{3\sqrt{3}}$

- **4.** $\int_0^{\infty} \frac{1-\cos mx}{x} e^{-x} dx = \frac{1}{2}\log(m^2+1)$

- **5.** $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\, dx = \frac{\pi^2}{8} - \frac{\alpha^2}{2}$