5. Linear Differential Equations with Constant Coefficients

Linear Differential Equation

A linear differential equation with constant coefficient of the n^{th} order is given by:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots \dots \dots \dots \dots \dots . + a_n y = X$$

Where $a_0, a_1, a_2, \dots \dots \dots . a_n$ are constant and X is a function of x alone.

For solution of differential equation, the operator $\frac{d}{dx}$ is denoted by D.

$$\therefore \frac{d}{dx} = D$$

$\therefore$ The linear differential equation becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots \dots \dots + a_n y = X$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots \dots \dots + a_n) y = X$$

$$\therefore f(D)y = X$$

Where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots \dots \dots + a_n$

General Solution of Differential Equation

The relation between dependant variables and independent variables which satisfy the given differential equation is known as general solution of given differential equation.

The general solution consists of two parts namely, Complementary function (C. F.) and Particular Integral (P. I.)

5.1. Complementary Function

A Part of General Solution which contains arbitrary constants (i.e. $c_1, c_2, c_3, \dots \dots \dots$) is known as Complementary function.

The general solution (G. S.) is given by $y = C.F. + P.I.$ or $y = y_c + y_p$

5.1.1. Methods of Finding Complementary functions:

The given differential equation can be written as

$$f(D)y = X$$

For Complementary function (C. F.) $f(D) = 0$.

This equation is known as Auxiliary equation.

Let $m_1, m_2, m_3, \dots \dots \dots m_n$ be the roots of Auxiliary equation. Depending upon the nature of roots, the Complementary function (C. F.) is determined as follows:

Case – I: Roots are real and non-repeated

If $m_1, m_2, m_3, \dots \dots \dots m_n$ are the real roots, then

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots \dots \dots + c_n e^{m_n x}$$

Case – II: Roots are real and repeated:

If m_1 is repeated 4 times, m_2 is repeated twice and m_3 is not repeated then

$$C.F. = (c_1x^3 + c_2x^2 + c_3x + c_4)e^{m_1x} + (c_5x + c_6)e^{m_2x} + c_7e^{m_3x}$$

Or

$$C.F. = (c_1 + c_2x + c_3x^2 + c_4x^3)e^{m_1x} + (c_5 + c_6x)e^{m_2x} + c_7e^{m_3x}$$

Case – III: Roots are complex and non-repeated:

If $m_1 = \alpha + i\beta$, then for every complex root, there will exist its conjugate pair $m_2 = \alpha - i\beta$

$$C.F. = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]$$

Case – IV: Roots are complex and repeated:

If $m_1 = \alpha \pm i\beta$ is repeated twice and $m_2 = \alpha_1 \pm i\beta_1$ is not repeated then,

$$C.F. = e^{\alpha x}[(c_1x + c_2)\cos(\beta x) + (c_3x + c_4)\sin(\beta x)] + e^{\alpha_1 x}[c_5\cos(\beta_1 x) + c_6\sin(\beta_1 x)]$$

Problems Based on Complementary Functions

1. $(D^2 + 5D + 6)y = 0$

Solution: $f(D) = D^2 + 5D + 6 = 0$

$f(m) = m^2 + 5m + 6 = 0$

This is an auxiliary equation

$\therefore m^2 + 5m + 6 = 0$

$m^2 + 3m + 2m + 6 = 0$

$(m + 3)(m + 2) = 0$

$m_1 = -3 \text{ and } m_2 = -2$

$\therefore C.F. = c_1e^{m_1x} + c_2e^{m_2x}$

$C.F. = c_1e^{-3x} + c_2e^{-2x}$

2. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$

Solution: The Auxiliary equation is given by $D^3 + 2D^2 - 5D - 6 = 0$

By Synthetic Division:

2	1	2	−5	−6
	0	2	8	6
	1	4	3	0

$(D - 2)(D^2 + 4D + 3) = 0$

$(D - 2)(D^2 + 3D + D + 3) = 0$

$(D - 2)[D(D + 3) + 1(D + 3)] = 0$

$(D + 1)(D + 3)(D - 2) = 0$

$\therefore D = -1, -3, 2$

C.F. is given by $y_c = c_1e^{-x} + c_2e^{-3x} + c_3e^{-2x}$