Module 3
Integration :
Review And Some New Techniques

Sub-Module 3.1

Beta functions with properties

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Syllabus

3	**Integration : Review And Some New Techniques**		**7**	**CO 3**
	3.1	Beta and Gamma functions with properties		
	3.2	Differentiation under integral sign with constant limits of integration.(without proof)		
		# Self-learning topic: Differentiation under integral sign with variable limits of integration.		

Beta Functions

- **Beta Functions:** The definite integral

$\int_0^1 x^{m-1}\,(1-x)^{n-1}dx,\ \ (Re(m), Re(n) > 0)$ is called Beta function denoted by $\beta(m,n)$. It is a function of parameters m and n.

- Thus,

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$$

- $\beta(m,n)$ is defined for all n except negative integers (i.e. except $n = -1, -2, -3 \ldots.)$

Imp

$$\boxed{\int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1)}$$

Properties of Beta Function

1. $\beta(m,n) = \beta(n,m)$

2. Relation between Beta & Gamma function :

$$\beta(m,n) = \frac{\lfloor m \lfloor n}{\lfloor m+n} = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

3. $\beta(m,n) = \int_0^\infty \frac{x^{(m-1)}}{(1+x)^{m+n}} dx$ (Definition 2)

4) $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Types. $\int_0^1 x^m (1-x)^n \, dx = \beta(m+1, n+1)$

① $\int_0^a x^m (a-x)^n \, dx$

$\int_0^a x^m a^n \left(1 - \frac{x}{a}\right)^n dx$

$\frac{x}{a} = t \Rightarrow x = at$

② $\int_0^1 x^m (1-x^n)^p \, dx$

$x^n = t, \Rightarrow x = t^{1/n}$

③ $\int_a^b (x-a)^m (b-x)^n \, dx$

put $(x-a) = (b-a)t$

④ $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} \, dx$

$bx = \frac{at}{1-t}$ //

$a + bx = \frac{a}{1-t}$

+

problems related to trigonometric fun^n //

- $\left\lceil\frac{1}{2}\right. = \sqrt{\pi}$

- **Proof:** $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

put p=0 & q=0

$$\int_0^{\frac{\pi}{2}} \sin^0\theta \cos^0\theta d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\frac{\left\lceil\frac{1}{2}\right.\left\lceil\frac{1}{2}\right.}{\left\lceil\frac{1}{2}+\frac{1}{2}\right.}$$

$$[\theta]_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\left\lceil\frac{1}{2}\right.\right)^2$$

$$\frac{\pi}{\cancel{2}} = \frac{1}{\cancel{2}}\left(\left\lceil\frac{1}{2}\right.\right)^2$$

$$\left\lceil\frac{1}{2}\right. = \sqrt{\pi}$$

- **Duplication Formula of Gamma Function:**
- $2^{2m-1}\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2m)$

$\int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1).$

- **Evaluate** $\int_0^1 \sqrt{1-\sqrt{x}}\, dx \cdot \int_0^{\frac{1}{2}} \sqrt{2y-4y^2}\; dy$

$I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\, dx$

$= \int_0^1 (1-x^{1/2})^{1/2} dx$

put $x^{1/2} = t$

$x = t^2$

$dx = 2t\,dt$

$x: 0 \to 1 \quad t: 0 \to 1$

$= \int_0^1 (1-t)^{1/2}\, 2t\,dt$

$= 2\int_0^1 t\,(1-t)^{1/2} dt$

$= 2\,\beta(1+1, \frac{1}{2}+1)$

$= 2\beta(2, 3/2)$

$I_2 = \int_0^{1/2} \sqrt{2y-4y^2}\, dy$

$= \int_0^{1/2} (2y-4y^2)^{1/2} dy$

$= \int_0^{1/2} (2y)^{1/2}(1-2y)^{1/2} dy$

$2y = t$

$\Rightarrow y = t/2 \Rightarrow dy = \frac{dt}{2}$

$y: 0 \to 1/2 \quad t: 0 \to 1$

$= \int_0^1 t^{1/2}(1-t)^{1/2} \frac{dt}{2}$

$= \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2} dt$

$= \frac{1}{2}\,\beta(\frac{1}{2}+1, \frac{1}{2}+1)$

$= \frac{1}{2}\,\beta(3/2, 3/2)$

$I = \int_0^1 \sqrt{1-\sqrt{x}}\, dx \cdot \int_0^{1/2} \sqrt{2y-4y^2}\, dy$

$= \cancel{2}\beta(2, \frac{3}{2}) \cdot \frac{1}{\cancel{2}}\beta(\frac{3}{2}, \frac{3}{2})$

$= \frac{\Gamma 2\,\Gamma\frac{3}{2}}{\Gamma\left(2+\frac{3}{2}\right)} \quad \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}$

$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$= \frac{1!\left(\Gamma\frac{3}{2}\right)^3}{\Gamma\frac{7}{2}\;\Gamma 3}$

$\Gamma n = (n-1)!$ if n is +ve int.

$\Gamma n = (n-1)\Gamma n-1$

$\Gamma\frac{7}{2} = \left(\frac{7}{2}-1\right)\Gamma\frac{7}{2}-1$

$= \frac{5}{2}\Gamma\frac{5}{2}$

$= \frac{5}{2}\,\frac{3}{2}\,\Gamma\frac{3}{2}$ //

$= \frac{\left(\Gamma\frac{3}{2}\right)^3}{\frac{5}{2}\cdot\frac{3}{2}\,\cancel{\Gamma\frac{3}{2}}\cdot 2!}$

$= \frac{\left(\Gamma\frac{3}{2}\right)^2}{\frac{15}{2^2}\,\cancel{2}} = \frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^2}{15/2}$

$= \frac{\frac{1}{4}(\sqrt{\pi})^2}{15/2} = \frac{1}{\cancel{4}}\frac{\pi}{2} \times \frac{\cancel{2}}{15} = \frac{\pi}{30}$ //.

$$\int_0^\infty e^{-x} x^n \, dx = \Gamma(n+1)$$

$$\int_0^1 x^m (1-x)^n \, dx = \beta(m+1, n+1)$$

$$\beta(m,n) = \beta(n,m)$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma (m+n)}$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \beta(m,n)$$

$$\int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1)$$

❖ **Evaluate** $\int_0^{2a} x^2\sqrt{2ax - x^2}\, dx$

$$= \int_0^{2a} x^2 (2ax - x^2)^{1/2} dx$$

$$= \int_0^{2a} x^2 (2ax)^{1/2} \left(1 - \frac{x^2}{2ax}\right)^{1/2} dx$$

$$= (2a)^{1/2} \int_0^{2a} x^{2+\frac{1}{2}} \left(1 - \frac{x^2}{2ax}\right)^{1/2} dx$$

$$= (2a)^{1/2} \int_0^{2a} x^{5/2} \left(1 - \frac{x}{2a}\right)^{1/2} dx$$

put $\frac{x}{2a} = t \Rightarrow x = 2at$

$dx = 2a\,dt$

$x : 0 \to 2a \quad t : 0 \to 1$

$$= (2a)^{1/2} \int_0^1 (2at)^{5/2} (1-t)^{1/2}\, 2a\,dt$$

$$= (2a)^{1/2} (2a)^{5/2} (2a) \int_0^1 t^{5/2} (1-t)^{1/2} dt$$

$$= (2a)^3 (2a)\ \beta\left(\frac{5}{2}+1, \frac{1}{2}+1\right)$$

$$\Gamma n = (n-1)\Gamma(n-1)$$

$$\Gamma \tfrac{7}{2} = \tfrac{5}{2}\Gamma \tfrac{5}{2}$$

$$= (2a)^4\ \beta\left(7/2, \frac{3}{2}\right)$$

$$= 16a^4 \frac{\Gamma 7/2\ \Gamma \frac{3}{2}}{\Gamma\left(\frac{7}{2}+\frac{3}{2}\right)} = \frac{16a^4\ \frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\Gamma\frac{1}{2}\ \frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 5}$$

$$= \cancel{16}a^4 \frac{15}{\cancel{16}} \frac{\sqrt{\pi}\sqrt{\pi}}{4!} = \frac{a^4\, 15\pi}{24} = \frac{5\pi a^4}{8}$$

$1-\cancel{x^n}$

❖ **Prove that** $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2^{\frac{2}{n}-1}\left(\Gamma\frac{1}{n}\right)^2}{n\,\Gamma\frac{2}{n}}$

$\int_0^1 x^m (1-x)^n \, dx$

$I = \int_0^1 (1-x^n)^{-1/2} \, dx$

$x^n = t$

$x = t^{1/n} \Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$x: 0 \to 1 \quad t: 0 \to 1$

$= \int_0^1 (1-t)^{-1/2} \frac{1}{n} t^{\frac{1}{n}-1} dt$

$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-1/2} dt = \frac{1}{n} \beta\left(\frac{1}{n}\cancel{-1}\cancel{+1}, -\frac{1}{2}+1\right)$

$= \frac{1}{n} \beta\left(\frac{1}{n}, \frac{1}{2}\right) = \frac{1}{n} \frac{\Gamma\frac{1}{n}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$ — $*$

By duplication formula.

$2^{2m-1}\,\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma 2m$

put $m = \frac{1}{n}$

$2^{\frac{2}{n}-1}\,\Gamma\frac{1}{n}\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma\frac{2}{n}$

$\Gamma\left(\frac{1}{n}+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\frac{2}{n}}{2^{2/n-1}\,\Gamma\frac{1}{n}}$

from $*$ $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\frac{1}{n}\Gamma\frac{1}{n}\cancel{\sqrt{\pi}}}{\frac{\cancel{\sqrt{\pi}}\,\Gamma\frac{2}{n}}{2^{2/n-1}\,\Gamma\frac{1}{n}}}$

$= \frac{\frac{1}{n}\Gamma\frac{1}{n}}{\Gamma\frac{2}{n}} \times 2^{2/n-1}\,\Gamma\frac{1}{n}$

$= \frac{2^{2/n-1}\left(\Gamma\frac{1}{n}\right)^2}{n\,\Gamma\frac{2}{n}}$

- **Prove That $\int_0^a x^4 (a^2 - x^2)^{\frac{1}{2}} dx = \frac{a^6 \pi}{32}$**

$$\int_0^1 x^m (1-x)^n dx$$

$$I = \int_0^a x^4 (a^2)^{1/2} \left(1 - \frac{x^2}{a^2}\right)^{1/2} dx$$

$$\frac{x^2}{a^2} = t \Rightarrow x^2 = a^2 t$$

$$x = a t^{1/2}$$

$$dx = a \frac{1}{2} t^{-1/2} dt$$

$$x : 0 \to a$$

$$t : 0 \to 1$$

$$I = a \int_0^1 (a t^{1/2})^4 (1-t)^{1/2} a \frac{1}{2} t^{-1/2} dt$$

$$= \frac{a^2 a^4}{2} \int_0^1 t^{2-1/2} (1-t)^{1/2} dt$$

$$= \frac{a^6}{2} \int_0^1 t^{3/2} (1-t)^{1/2} dt = \frac{a^6}{2} \beta\left(\frac{3}{2}+1, \frac{1}{2}+1\right)$$

Evaluate $\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\,dx$ $= \frac{16}{35}$ //.

$$= \int_0^1 x^7 (1-x^2)^{-1/2}\,dx$$

$x^2 = t$

$x = t^{1/2}$

- **Evaluate $\int_0^1 \frac{x^2(4-x^4)}{\sqrt{1-x^2}}\,dx$**

$x = \sin\theta \qquad x : 0 \to 1$

$dx = \cos\theta\, d\theta \qquad \theta : 0 \to \pi/2$

$$I = \int_0^{\pi/2} \frac{\sin^2\theta\,(4-\sin^4\theta)}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{(4\sin^2\theta - \sin^6\theta)}{\cancel{\cos\theta}}\, \cancel{\cos\theta}\, d\theta$$

$$= 4\int_0^{\pi/2} \sin^2\theta\, d\theta - \int_0^{\pi/2} \sin^6\theta\, d\theta$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= 4\int_0^{\pi/2} \sin^2\theta \cos^0\theta\, d\theta - \int_0^{\pi/2} \sin^6\theta \cos^0\theta\, d\theta$$

$$= 4\cdot\frac{1}{2}\beta\left(\frac{3}{2}, \frac{1}{2}\right) - \frac{1}{2}\beta\left(\frac{7}{2}, \frac{1}{2}\right)$$

$$= 2\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} - \frac{1}{2}\,\frac{\Gamma\frac{7}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)}$$

$$\Gamma n = (n-1)\,\Gamma(n-1)$$

$$= \cancel{2}\,\frac{\frac{1}{\cancel{2}}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} - \frac{\frac{1}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 4}$$

$$= \frac{(\sqrt{\pi})^2}{1!} - \frac{15\,(\sqrt{\pi})^2}{16\,(3!)} \qquad \Gamma n = (n-1)! \text{ (+ve int)}$$

$$= \pi - \frac{\overset{5}{\cancel{15}}\,\pi}{16\times\cancel{6}_2} = \frac{27\pi}{32}$$

Evaluate $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\, d\theta$

$$I = \int_0^{\pi} \frac{\sin^4\theta}{(2\cos^2\theta/2)^2}\, d\theta$$

$$= \int_0^{\pi} \frac{(2\sin\theta/2 \cos\theta/2)^4}{4\cos^4\theta/2}\, d\theta$$

$$= \int_0^{\pi} \frac{2^4 \sin^4\theta/2\, \cancel{\cos^4\theta/2}}{4\,\cancel{\cos^4\theta/2}}\, d\theta$$

$$= 4\int_0^{\pi} \sin^4\theta/2\, d\theta$$

put $\theta/2 = t \Rightarrow \theta = 2t$

$d\theta = 2dt$

$\theta : 0 \to \pi$

$t : 0 \to \pi/2$

$$= 4\int_0^{\pi/2} \sin^4 t\; 2\,dt$$

$$= 8\int_0^{\pi/2} \sin^4 t \cos^0 t\, dt$$

$$= 8\, \frac{1}{2}\beta\left(\frac{4+1}{2}, \frac{1}{2}\right) = 4\beta\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= 4\,\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)} = \frac{4\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{3\,(\sqrt{\pi})^2}{2!} = \frac{3\pi}{2}$$

Evaluate $\int_0^1 x^4 \cos^{-1} x \, dx$

$$\int u \cdot v = u \int v - \int \int v \, u'$$

L I A T E

$$\int_0^1 \underset{u}{\cos^{-1}x} \; \underset{v}{x^4} \, dx$$

$$= \left[\cos^{-1}x \int x^4 dx - \int \int x^4 dx \left(-\frac{1}{\sqrt{1-x^2}} \right) dx \right]_0^1$$

$$= \left[\cos^{-1}x \, \frac{x^5}{5} \right]_0^1 + \int_0^1 \frac{x^5}{5} \frac{1}{\sqrt{1-x^2}} dx$$

$$= [0 - 0] + \frac{1}{5} \int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx$$

$x^2 = t$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$x : 0 \to 1 \qquad \theta : 0 \to \pi/2$

$$I = \frac{1}{5} \int_0^{\pi/2} \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

$$= \frac{1}{5} \int_0^{\pi/2} \frac{\sin^5\theta}{\cancel{\cos\theta}} \cancel{\cos\theta} \, d\theta$$

$$= \frac{1}{5} \int_0^{\pi/2} \sin^5\theta \cos^0\theta \, d\theta = \frac{1}{5} \frac{1}{2} \beta\left(\frac{5+1}{2}, \frac{1}{2} \right)$$

$$= \frac{1}{10} \beta\left(3, \frac{1}{2} \right)$$

$$= \frac{1}{10} \frac{\Gamma 3 \, \Gamma \frac{1}{2}}{\Gamma\left(3+\frac{1}{2}\right)}$$

$$= \frac{1}{10} \frac{2! \sqrt{\pi}}{\Gamma \frac{7}{2}}$$

$$\int_0^1 x^m (1-x)^n \, dx = \beta(m+1, n+1)$$

$(1-3x)$

t

$(1-x^2)$

t

t

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \beta(m, n)$$

$\Gamma n = (n-1)\Gamma n-1$

$$\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$2^{2m-1} \Gamma m \, \Gamma m+\tfrac{1}{2} = \sqrt{\pi} \, \Gamma 2m$$

❖ **Evaluate** $\int_0^\infty \frac{x^2}{(1+x^6)^3}\,dx$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

put $x^6 = t$

$x = t^{1/6}$

$dx = \frac{1}{6} t^{-5/6}\,dt$

$x: 0 \to \infty \qquad t: 0 \to \infty$

$$I = \int_0^\infty \frac{(t^{1/6})^2}{(1+t)^3} \frac{1}{6} t^{-5/6}\,dt$$

$$= \frac{1}{6}\int_0^\infty \frac{t^{1/3 - 5/6}}{(1+t)^3}\,dt$$

$$= \frac{1}{6}\int_0^\infty \frac{t^{-1/2}}{(1+t)^3}\,dt$$

$$\int_0^\infty \frac{x^{m-1}}{(1+t)^{m+n}}\,dx = \beta(m,n)$$

$m-1 = -1/2 \qquad m = 1/2$

$m+n = 3 \qquad n = 5/2$

$$= \frac{1}{6}\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$\Gamma n = (n-1)\Gamma n-1$

$$= \frac{1}{6}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)} = \frac{1}{6}\sqrt{\pi}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{1}{\cancel{6}_2}\frac{\cancel{3}^1}{4}\frac{\sqrt{\pi}\sqrt{\pi}}{2!}$$

$$= \frac{\pi}{16}$$

❖ Evaluate $\int_0^{\frac{\pi}{6}} \cos^3 3\theta \sin^2 6\theta \, d\theta$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$I = \int_0^{\pi/6} \cos^3 3\theta \left(2\sin 3\theta \cos 3\theta\right)^2 d\theta$$

$$= 2^2 \int_0^{\pi/6} \sin^2 3\theta \cos^5 3\theta \, d\theta$$

$3\theta = t \Rightarrow \theta = t/3 \quad d\theta = \frac{dt}{3}$

as $\theta : 0 \to \pi/6 \quad t : 0 \to \pi/2$

$$= 4 \int_0^{\pi/2} \sin^2 t \cos^5 t \frac{dt}{3}$$

$$= \frac{4}{3} \cdot \frac{1}{2} \beta\left(\frac{2+1}{2}, \frac{5+1}{2}\right) = \frac{2}{3} \beta\left(\frac{3}{2}, \cancel{\frac{6}{2}}^{3}\right)$$

$$= \frac{2}{3} \frac{\Gamma 3/2 \, \Gamma 3}{\Gamma \frac{3}{2}+3}$$

$\Gamma n = (n-1)\Gamma n-1$

$\Gamma n = (n-1)!$

+ve int.

$$= 2/3 \frac{\frac{1}{2}\Gamma\frac{1}{2} \; 2!}{\Gamma\frac{9}{2}}$$

$$= \frac{2\cancel{\sqrt{\pi}}}{3 \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \cancel{\Gamma\frac{1}{2}}}$$

$$= \frac{2 \times 2^4 \cancel{\sqrt{\pi}}}{3 \times 7 \times 5 \times 3 \cancel{\sqrt{\pi}}}$$

$$= \frac{2^5}{9 \times 7 \times 5}$$

❖ **Evaluate** $\int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^4\, d\theta$

$1+\cos 2\theta = 2\cos^2\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$I = \int_0^{2\pi} (2\sin\theta/_2 \cos\theta/_2)^2 (2\cos^2\theta/_2)^4 d\theta$$

$$= 2^2 \cdot 2^4 \int_0^{2\pi} \sin^2\theta/_2 \cos^{10}\theta/_2 \, d\theta$$

put $\theta/_2 = t \Rightarrow \theta = 2t$ $\quad \theta: 0 \to 2\pi$

$\Rightarrow d\theta = 2\,dt.$ $\quad t: 0 \to \pi$

$$= 2^6 \int_0^{\pi} \sin^2 t \cos^{10} t \; 2\,dt$$

$$= 2^7 \int_0^{\pi} \sin^2 t \cos^{10} t \, dt$$

If $f(a-t) = f(t)$

then $\int_0^a f(t)\,dt = 2\int_0^{a/2} f(t)\,dt.$

$f(\pi - t) = f(t)$

$\int_0^{\pi} f(t)\,dt = 2\int_0^{\pi/2} f(t)\,dt$

$f(t) = \sin^2 t \cos^{10} t$

$f(\pi-t) = [\sin(\pi-t)]^2 [\cos(\pi-t)]^{10}$

$= [0-(-1)\sin t]^2 \, [(-1)\cos t]^{10}$

$= \sin^2 t \cos^{10} t = f(t)$

$$I = 2^7 \int_0^{\pi} \sin^2 t \cos^{10} t \, dt$$

$$= 2^7 \, 2 \int_0^{\pi/2} \sin^2 t \cos^{10} t \, dt$$

$$= 2^7 \cdot \cancel{2}\, \frac{1}{\cancel{2}} \beta\left(\frac{3}{2}, \frac{11}{2}\right).$$

$$= 2^7 \frac{\Gamma\frac{3}{2}\,\Gamma\frac{11}{2}}{\Gamma 7}$$

$\Gamma n = (n-1)\Gamma n-1$

❖ **Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$ and prove that the result**
$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$

$$\int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{-1/2+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$\Gamma n = (n-1)\Gamma n-1$

$$= \frac{1}{2}\frac{\Gamma 3/4\, \Gamma 1/4}{\Gamma 1}$$

$\Gamma p\, \Gamma 1-p = \frac{\pi}{\sin p\pi}$ $0<p<1$

$$= \frac{1}{2}\frac{\Gamma 1/4\, \Gamma 1-1/4}{}$$

$p = 1/4$

$$= \frac{1}{2}\frac{\pi}{\sin \pi/4} = \frac{\pi}{2\times\frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}\pi}{2}$$

$$\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1/2}{2}, \frac{3/2}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{\sqrt{2}\pi}{2}$$

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \cdot \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\sqrt{2}\pi}{2} \times \frac{\sqrt{2}\pi}{2}$$

$$= \frac{\pi^2}{2}$$

❖ **Express $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin\theta + \cos\theta)^{\frac{1}{3}}\, d\theta$ as a Gamma function.**

$$= \int_{-\pi/4}^{\pi/4} \left(\sqrt{2} \left(\sin\theta \frac{1}{\sqrt{2}} + \cos\theta \frac{1}{\sqrt{2}} \right) \right)^{1/3} d\theta$$

$$= (\sqrt{2})^{1/3} \int_{-\pi/4}^{\pi/4} \left(\sin\theta \cos\frac{\pi}{4} + \cos\theta \sin\frac{\pi}{4} \right)^{1/3} d\theta$$

$$= (2)^{1/6} \int_{-\pi/4}^{\pi/4} \sin^{1/3}\left(\theta + \frac{\pi}{4}\right) d\theta$$

$$\theta + \frac{\pi}{4} = t \Rightarrow \theta = t - \pi/4$$
$$d\theta = dt$$
$$\theta : -\frac{\pi}{4} \to \pi/4 \qquad t : 0 \to \pi/2$$

$$= 2^{1/6} \int_0^{\pi/2} \sin^{1/3} t \, \cos^0 t \, dt$$

$$= 2^{1/6} \frac{1}{2} \beta\left(\frac{1/3 + 1}{2}, \frac{1}{2} \right)$$

$$\Gamma n = (n-1)\Gamma n-1$$

$$= 2^{1/6} \frac{1}{2} \beta\left(2/3, \frac{1}{2} \right)$$

$$= \frac{2^{1/6}}{2} \frac{\Gamma 2/3 \, \Gamma 1/2}{\Gamma 2/3 + 1/2} = 2^{-5/6} \frac{\Gamma 2/3 \, \sqrt{\pi}}{\Gamma 7/6}$$

❖ Evaluate $\int_3^7 \sqrt[4]{(x-3)(7-x)}\,dx$

$$\int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1)$$

put $x - 3 = (7-3)t$

$x - 3 = 4t$

$x = 3 + 4t$

$dx = 4\,dt$.

$x: 3 \to 7 \qquad t: 0 \to 1$

$$I = \int_0^1 \sqrt{4t(7-(3+4t))}\; 4dt$$

$$= 2 \times 4 \int_0^1 t^{1/2} (4-4t)^{1/2} dt$$

$$= 2 \times 4 \times (4)^{1/2} \int_0^1 t^{1/2} (1-t)^{1/2} dt$$

$$= 8 \times 2\; \beta\left(\frac{1}{2}+1, \frac{1}{2}+1\right)$$

$$= 16\; \beta(3/2, 3/2) = 16 \frac{\Gamma 3/2\, \Gamma 3/2}{\Gamma 3}$$

$$= 16 \frac{\left(1/2\, \Gamma 1/2\right)^2}{2!}$$

$$= \frac{16}{4 \times 2} \pi = 2\pi$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)$$

$\tan^2\theta$

❖ **Evaluate** $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

$$\int_0^\infty \frac{x^4+x^9}{(1+x)^{15}}\,dx$$

$$= \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$$

$m-1 = 4$
$m = 5$
$m+n = 15$
$n = 10$

$m-1 = 9$ $\quad m = 10$
$m+n = 15$
$n = 5$

$$= \beta(5,10) + \beta(10,5)$$

$\beta(10,5) = \beta(5,10)$

$$= 2\beta(5,10)$$

$$= 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15} = 2\times\frac{4!\,9!}{14!}$$

❖ Evaluate $\int_0^\infty \frac{\sqrt{x}}{1+2x+x^2}\,dx$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}$$

$$\int_0^\infty \frac{x^{1/2}}{(1+x)^2}\,dx$$

$m-1 = 1/2 \qquad m = 3/2$

$m+n = 2 \qquad n = 1/2$

$$= \beta(3/2, 1/2) = \frac{\Gamma 3/2 \,\Gamma 1/2}{\Gamma 2} = \frac{\frac{1}{2}\Gamma \frac{1}{2}\Gamma\frac{1}{2}}{1!}$$

$$= \pi/2$$

$\Gamma n = (n-1)\,\Gamma n-1$

❖ **Prove that $\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$**

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \beta(m,n)$$

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \beta(m,n) \quad \text{— (1)}$$

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2}dy$$

$y = \frac{1}{x}$

$x: 1 \to \infty \qquad y: 1 \to 0$

$$= \int_1^0 \frac{(1/y)^{m-1}}{(1+1/y)^{m+n}}\left(-\frac{1}{y^2}\right)dy$$

$$= \int_0^1 \frac{1}{y^{m-1+2}\left(\frac{y+1}{y}\right)^{m+n}}dy$$

$$= \int_0^1 \frac{y^{\cancel{m}+n-(\cancel{m}+1)}}{(1+y)^{m+n}}dy$$

$\int_0^1 x^2$

$\int_0^1 y^2$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$y = x$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

from (1).

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx = \beta(m,n)$$

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \beta(m,n)$$

❖ **Prove that $\beta(m,m)\cdot\beta\left(m+\frac{1}{2},m+\frac{1}{2}\right)=\frac{\pi}{m}2^{1-4m}$**

$$LHS = \beta(m,m)\ \beta\left(m+\tfrac{1}{2},\ m+\tfrac{1}{2}\right)$$

$$= \frac{\Gamma m\ \Gamma m}{\Gamma 2m}\ \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m+1)}$$

$$= \frac{\left(\Gamma m\ \Gamma\left(m+\frac{1}{2}\right)\right)^2}{\Gamma 2m\ \ 2m\,\Gamma 2m} \qquad \Gamma n = (n-1)\,\Gamma (n-1)$$

$$= \frac{1}{2m}\left(\frac{\Gamma m\ \Gamma\left(m+\frac{1}{2}\right)}{\Gamma 2m}\right)^2 \quad \text{—— (1)}$$

By Duplication formula

$$2^{2m-1}\ \Gamma m\ \Gamma\left(m+\tfrac{1}{2}\right) = \sqrt{\pi}\ \Gamma 2m$$

$$\frac{\Gamma m\ \Gamma\left(m+\frac{1}{2}\right)}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1}}$$

From (1)

$$LHS = \frac{1}{2m}\left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)^2 = \frac{1}{2m}\ \frac{\pi}{2^{4m-2}}$$

$$= \frac{\pi}{2^{4m-1}\,m}$$

$$= 2^{1-4m}\ \frac{\pi}{m}$$

P.T. $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}\,\beta(m,n)$

& hence deduce $\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}\,dx$ & $\int_0^\infty \frac{\sqrt{x}}{(1+2x+x^2)}\,dx$

$$I = \int_0^\infty \frac{x^{m-1}}{a^{m+n}\left(1+\frac{b}{a}x\right)^{m+n}}\,dx$$

$$\frac{b}{a}x = \tan^2\theta$$

$$x = \frac{a}{b}\tan^2\theta$$

$$dx = \frac{a}{b}\,2\tan\theta\sec^2\theta\,d\theta$$

$$x: 0 \to \infty \qquad \theta: 0 \to \pi/2$$

$$= \frac{1}{a^{m+n}}\int_0^{\pi/2} \frac{\left(\frac{a}{b}\tan^2\theta\right)^{m-1}}{(1+\tan^2\theta)^{m+n}}\,\frac{a}{b}\,2\tan\theta\sec^2\theta\,d\theta$$

$$= \frac{1}{a^{m+n}}\,\frac{a^{m-1}}{b^{m-1}}\,\frac{2a}{b}\int_0^{\pi/2} \frac{\tan^{2m-2+1}\theta\,\sec^2\theta}{(\sec^2\theta)^{m+n}}\,d\theta$$

$$= \frac{a^m}{a^{m+n}}\,\frac{2}{b^m}\int_0^{\pi/2} \frac{\tan^{2m-1}\theta}{(\sec\theta)^{2m+2n-2}}\,d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta}\,(\cos\theta)^{2m+2n-2}\,d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} \sin^{2m-1}\theta\,(\cos\theta)^{\cancel{2m}+2n-2-\cancel{2m}+1}\,d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{\cancel{2}}{a^n b^m}\,\frac{1}{\cancel{2}}\,\beta\left(\frac{\cancel{2}m}{\cancel{2}}, \frac{\cancel{2}n}{\cancel{2}}\right)$$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}\,dx = \int_0^\infty \frac{x^{1/2}\,dx}{(2+x)^2}$$

$a=2,\ b=1$

$m-1 = 1/2 \qquad m+n=2$

$m = 3/2 \qquad n = 1/2$

$$= \frac{1}{a^n b^m}\,\beta(m,n)$$

$$= \frac{1}{2^{1/2}(1)^{3/2}}\,\beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\,\frac{\Gamma 3/2\,\Gamma 1/2}{\Gamma 2}$$

$$= \frac{1}{\sqrt{2}}\,\frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1!}$$

$$= \frac{\pi}{2\sqrt{2}}$$