Concepts of Optimization

General Mathematical Formulation for Linear Programming:

The general LPP can be expressed as follows:

Find the values of decision variables $x_1, x_2, x_3, \ldots, x_n$, which satisfy the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \ (\le, =, \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \ (\le, =, \ge)\ b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \ (\le, =, \ge)\ b_m \end{array}\right\} \ldots\ldots\ldots \text{I}$$

and $x_j \ge 0$, where $j = 1, 2, 3, \ldots, n$ II

and maximize or minimize the objective function which is a linear function of x_j, such as

$Z = c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$ III

The above formulation may be put in the following compact form by using the summation sign :

maximize (or minimize) $Z = \sum_{j=1}^{n} c_j x_j$ IV

$$\left.\begin{array}{l} \text{subject to } \sum_{j=1}^{n} a_{ij}x_j (\le, =, \ge) b_i, \quad i = 1, 2, 3, \ldots, m \\ \text{and } x_j \ge 0, \quad j = 1, 2, 3, \ldots, n \end{array}\right\} \ldots\ldots\ldots \text{V}$$

The constant $c_j (j = 1, 2, 3, \ldots, n)$ in equation IV are called cost coefficients;

the constants $b_i (i = 1, 2, 3, \ldots, m)$ in the constraint conditions are called stipulations and

the constants $a_{ij} (i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n)$ are called structural coefficients.

Note : If there are m equality constraints and $m + n$ is the number of variables $(m \le n)$, a start for the optimal solution is made by putting n unknowns [out of $(m + n)$ unknowns] equal to zero and then solving for the m equations in remaining m unknowns, provided that the solution exists and is unique. The n zero variables are called nonbasic variables and the remaining m variables are called basic variables which form a basic solution. If the solution yields all non-negative basic variables, it is called basic feasible solution; otherwise it is infeasible. This step reduces the number of alternatives for the optimal solution from infinite to a finite number, whose maximum limit can be

$${}^{m+n}C_m = \frac{(m+n)!}{m!\,n!}.$$

Solution : x_j $(j = 1, 2, \ldots, n)$ is a solution of the general LPP if it satisfies the constraints I.

Feasible Solution : x_j $(j = 1, 2, \ldots, n)$ is a feasible solution of the LPP if it satisfies conditions I and II.

Basic Solution : The solution of m basic variables when each of the n non-basic variables is set equal to zero is called basic solution.

Basic feasible solution : A feasible solution is called a basic feasible solution if it has no more than m positive x_j. In other words, it is a basic solution which also satisfies the non-negativity condition II.

Non-degenerate basic feasible solution : A basic feasible solution is said to be non-degenerate if it has exactly m positive (non-zero) x_j. The solution, on the other hand, is degenerate if one or more of the m basic variables vanish.

Optimal Solution : A basic feasible solution is said to be optimal or optimum if it also optimizes the objective function [equation (III)] while satisfying conditions I and II.

Canonical and Standard Forms of LPP :

After formulating the LPP , solution can be obtained after putting the problem in a particular form i.e. Canonical Form or Standard Form.

The Canonical Form :

$$\text{maximize } Z = \sum_{j=1}^{n} c_j x_j ,$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \le b_i , \quad i = 1, 2, ..., m ,$$

$$x_j \ge 0 , \quad j = 1, 2, ..., n$$

The characteristics of this form are

a) All decision variables are non-negative,

b) All constraints are of the ($\le$) type, and

c) Objective function is of maximization type

Note : Any LPP can put in the canonical form By the use of some elementary transformations.

1) The minimization of a function is equivalent to the maximization of the negative expression of this function.
2) An inequality in one direction ($\le$ or $\ge$) can be changed to an inequality in the opposite direction ($\ge$ or $\le$) by multiplying both sides of the inequality by -1.
3) An equation may be replaced by two weak inequalities in opposite directions.
4) If a variable is unconstrained, it is expressed as the difference between two non-negative variables.

The Standard Form :

The characteristics of the standard form are

1) All the constraints are expressed in the form of equations, except the non-negativity constraints which remain inequalities (≥ 0).
2) The right hand side of each constraint equation is non-negative.
3) All the decision variables are non-negative.
4) The objective function is of the maximization or minimization type.

The inequality constraints are changed to equality constraints by adding or subtracting a non-negative variable from the left-hand sides of such constraints. These new variables are called **slack variables** or simply slacks. They are added if the constraints are ($\le$) and subtracted if the constraints are ($\ge$). Since in the case of ($\ge$) constraints the subtracted variable represents the surplus of left-hand side over right-hand side, it is commonly known as **surplus variable** and is, in fact, a negative slack. However, we shall always use the name "slack" variable.

Note : Add non-negative variables to the left hand side of all the constraints of ($=$ or $\ge$) type. These variables are called **artificial variables.** The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. It's co-efficient +M for min and –M for max are called as penalties. The high value M ensures that the artificial variable does not remain in the basis.

Let us consider the general LPP.

$$\text{maximize } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \le b_i , (b_i \ge 0), \quad i = 1, 2, 3, ..., m$$

$$x_j \ge 0, \quad j = 1, 2, 3, ..., n$$

This is expressed in the standard form as

$$\text{maximize } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j + s_i = b_i , \quad i = 1, 2, 3, ..., m.$$

$$x_j \ge 0, \quad j = 1, 2, 3, ..., n.$$

$$s_i \ge 0, \quad i = 1, 2, 3, ..., m.$$