HYPERBOLIC FUNCTIONS

SOME SOLVED EXAMPLES:

1. If $\tanh x = \frac{1}{2}, find \sinh 2x$ $and \cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1}{2}$ $\therefore \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2}$ $\therefore 2\,e^{2x} - 2 = e^{2x} + 1$ $\therefore\ e^{2x} = 3$

Now, $sin\ h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3-(1/3)}{2} = \frac{4}{3}$

Now, $\cos h2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3+(1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Solution: $7cosh\ x + 8sinh\ x = 1$

Putting the values of $coshx$ and $sin\ hx$, we get

$\therefore 7\left(\frac{e^x+e^{-x}}{2}\right) + 8\left(\frac{e^x-e^{-x}}{2}\right) = 1$

$\therefore 7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$

$\therefore 15e^x - e^{-x} = 2$

$\therefore 15e^{2x} - 2e^x - 1 = 0$ Solving it as a quadratic equation in e^x,

$e^x = \frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)} = \frac{2\pm 8}{30} = \frac{1}{3}\ \ or - \frac{1}{5}$

$\therefore x = \log\left(\frac{1}{3}\right)\ \ or\ x = \log\left(-\frac{1}{5}\right)$

Since x is real, $x = log\left(\frac{1}{3}\right) = -\log 3$

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Solution: Let $\sin h^{-1}\ a = \alpha,\ \sin h^{-1} b = \beta\ \ and\ \ \sin h^{-1}\ x = \gamma$

We are given $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ $\therefore \alpha + \beta = \gamma$

$\therefore \sinh(\alpha+\beta) = \sinh\gamma$

$\therefore \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta = \sinh\gamma$ ....(A)

But $\sinh\alpha$ = a, $\sinh\beta = b$, $\sinh\gamma = x$

$\therefore \cos h\ \alpha = \sqrt{1+\sin h^2\alpha} = \sqrt{1+a^2}$ and $\cos h\ \beta = \sqrt{1+\sin h^2\beta} = \sqrt{1+b^2}$

Putting this values in (A), we get $a\sqrt{1+a^2} + b\sqrt{1+b^2} = x$

4. Prove that $16\ sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

Solution: LHS $= 16\sinh^5 x$

$$= 16\left(\frac{e^x-e^{-x}}{2}\right)^5$$
$$= \frac{16}{32}(e^{5x} - 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} - 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} - e^{-5x})$$
$$= \frac{1}{2}(e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x})$$
$$= \left(\frac{e^{5x}-e^{-5x}}{2}\right) - 5\left(\frac{e^{3x}-e^{-3x}}{2}\right) + 10\left(\frac{e^{x}-e^{-x}}{2}\right)$$
$$= \sinh 5x - 5\sinh 3x + 10\sinh x$$
$= \text{RHS}$

5. Prove that $16cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$

Solution: $l.h.s = 16cosh^5 x$

$$= 16\left(\frac{e^x+e^{-x}}{2}\right)^5 \qquad \text{[By Binomial Theorem]}$$
$$= \frac{16}{32}[e^{5x} + 5e^{4x}.e^{-x} + 10e^{3x}.e^{-2x} + 10e^{2x}.e^{-3x} + 5e^{x}.e^{-4x} + e^{-5x}]$$
$$= \frac{(e^{5x}+e^{-5x})}{2} + 5\frac{(e^{3x}+e^{-3x})}{2} + 10\frac{(e^{x}+e^{-x})}{2}$$
$$= \cos h\, 5\, x + 5\cos h\, 3\, x + 10\cos h\, x = r\,.h.s$$

6. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1-cosh^2 x}}} = cosh^2 x$

Solution: $l.h.s = \frac{1}{1-\frac{1}{1-\frac{1}{-sinh^2 x}}} = \frac{1}{1-\frac{1}{1+cosec\, h^2 x}} = \frac{1}{1-\frac{1}{\cot h^2 x}} = \frac{1}{1-\tan h^2 x} = \frac{1}{1-\frac{sinh^2 x}{\cos h^2 x}} = \frac{\cos h^2 x}{\cos h^2 x - \sin h^2 x} = \cos h^2 x$

7. If $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$

(iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Solution: **(i)** $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$\therefore e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$
$$\therefore e^{-u} = \frac{1-\tan\theta/2}{1+\tan\theta/2}$$
$$\therefore \cosh u = \frac{e^u+e^{-u}}{2}$$
$$= \frac{1}{2}\left[\frac{(1+2\tan\theta/2+\tan^2\theta/2)+(1-2\tan\theta/2+\tan^2\theta/2)}{1-\tan^2\theta/2}\right]$$
$$= \frac{1}{2}\left(\frac{2+2\tan^2\theta/2}{1-\tan^2\theta/2}\right)$$

$$= \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1}{\cos\theta} = \sec\theta$$

(ii) $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

(iii) $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

(iv) $\tan h\left(\frac{u}{2}\right) = \frac{\sin h(u/2)}{\cos h(u/2)} = \frac{2\sin h(u/2).\cos h(u/2)}{2\cos h(u/2)\cos h(u/2)} = \frac{\sin hu}{1+\cos hu} = \frac{\tan\theta}{1+\sec\theta}$ (By (i) and (ii))

$\therefore \tan h\left(\frac{u}{2}\right) = \frac{sin\theta/cos\theta}{(\cos\theta+1)/\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)} = \frac{sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Solution: **(i)** $\cosh x = \sec\theta$

$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta$ By definition $\cos hx = \frac{e^x+e^{-x}}{2}$

$\therefore e^x - 2\sec\theta + e^{-x} = 0$

$\therefore (e^x)^2 - 2e^x sec\theta + 1 = 0$

Solving the quadratic in e^x,

$e^x = sec\theta \pm \sqrt{sec^2\theta - 1} = \sec\theta \pm \tan\theta$

$\therefore x = \log(\sec\theta \pm \tan\theta) = \pm log(\sec\theta + \tan\theta)$

(we can prove that $log(\sec\theta - \tan\theta) = -log(\sec\theta + \tan\theta)$)

(ii) Let $tan^{-1}e^{-x} = \alpha$ $\therefore e^{-x} = \tan\alpha$ $\therefore e^x = \cot\alpha$

Now, by data $\sec\theta = \cos hx = \frac{e^x+e^{-x}}{2} = \frac{cot\alpha+tan\alpha}{2}$

$$2\,sec\theta = cot\alpha + tan\alpha = \frac{cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{cos\alpha} = \frac{2}{\sin 2\alpha}$$

$$\therefore cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$$

(iii) $\tan h\,\frac{x}{2} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{e^x-1}{e^x+1} = \frac{sec\theta+tan\theta-1}{sec\theta+tan\theta+1} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$

$$= \frac{(1-cos\theta)+\sin\theta}{(1+cos\theta)+\sin\theta} = \frac{2sin^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)} = \frac{\sin(\theta/2)}{cos(\theta/2)} = tan\frac{\theta}{2}$$

SEPARATION OF REAL AND IMAGINARY PARTS:

Many a time we are required to separate real and imaginary parts of a given complex function. For this, we have to use identities of circular and hyperbolic functions.

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$\therefore$ $\tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\ \tan(\alpha-i\beta)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$

$\therefore 1-x^2-y^2 = 2x\cot 2\alpha \qquad \therefore x^2+y^2+2x\cot 2\alpha - 1 = 0$

Further, $\tan(2i\beta) = \tan[(\alpha+i\beta)-(\alpha-i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$

$$i\tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$

$$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$\therefore 1+x^2+y^2 = 2y\coth 2\beta \qquad$ i. e., $x^2+y^2-2y\coth 2\beta + 1 = 0$

SOME SOLVED EXAMPLES:

1. Separate into real and imaginary parts $tan^{-1}\left(e^{i\,\theta}\right)$

Solution: Let $tan^{-1}e^{i\theta} = x+iy \quad \therefore e^{i\theta} = \tan(x+iy) \quad \therefore cos\theta + i\sin\theta = \tan(x+iy)$

Similarly, $cos\theta - i\sin\theta = \tan(x-iy)$

Now, $tan\ 2x = \ tan\ [\ (x+iy)+\ (x-iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)+(cos\theta-i\sin\theta)}{1-(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ cos\theta}{1-(cos^2\theta+sin^2\theta)} = \frac{2\ cos\theta}{1-1} = \frac{2\ cos\theta}{0} = \infty$$

$\therefore 2x = \frac{\pi}{2} \quad \therefore x = \frac{\pi}{4}$

Also $\tan 2\,iy = \tan[(x+iy)-(x-iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)-(cos\theta-i\sin\theta)}{1+(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ isin\theta}{1+(cos^2\theta+sin^2\theta)} = \frac{2isin\theta}{2}$$

$\therefore i\tan h\ 2y = i\sin\theta \qquad \therefore \tan h\ 2y = \sin\theta$

$\therefore 2y = tanh^{-1}\sin\theta \qquad \therefore y = \frac{1}{2}\tan h^{-1}\sin\theta$

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Solution: $\sin(\alpha - i\,\beta) = x + i\,y$

$\therefore \sin\alpha\cos h\,\beta + i\cos\alpha\sin h\,\beta = x+iy$

Equating real and imaginary parts, we get, $\sin\alpha\cos h\,\beta = x\ \ and \cos\alpha\sin h\,\beta = y$

$\therefore \frac{x^2}{\cos h^2\beta} + \frac{y^2}{\sin h^2\beta} = sin^2\alpha + cos^2\alpha = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = \cos h^2\beta - \sin h^2\beta = 1$

3. If $cos(x + i\,y) = \cos\alpha + i\sin\alpha\,, prove\ that$

(i) $\sin\alpha = \pm sin^2 x = \pm \sin h^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Solution: $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$\cos x\cos(iy) - \sin x\sin(iy) = \cos\alpha + i\sin\alpha$

$\cos x\cosh y - i\sin x\sinh y = \cos\alpha + i\sin\alpha$

Equating real and imaginary parts, we get,

$\cos x\cosh y = \cos\alpha$ and $-\sin x\sinh y = \sin\alpha$

(i) Since $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2 x\sinh^2 y + \cos^2 x\cosh^2 y = 1$

$\sin^2 x\sinh^2 y + (1-\sin^2 x)(1+\sinh^2 y) = 1$

$\sin^2 x\sinh^2 y + 1 + \sinh^2 y - \sin^2 x - \sin^2 x\sinh^2 y = 1$

$1 + \sinh^2 y - \sin^2 x = 1$

$\sinh^2 y - \sin^2 x = 0$

$\therefore \sinh^2 y = \sin^2 x$ ....(i)

$\therefore \sinh y = \pm\sin x$

$\therefore \sin\alpha = -\sin x\sinh y = -\sin x(\pm\sin x) = \pm\sin^2 x$

(ii) $\cos 2x + \cosh 2y = 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sin^2 x$ .... from (i)

$= 2$

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Solution: We have to separate real part $\pi/6$ and imaginary part α

$\because tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy$ $\therefore tan\left(\frac{\pi}{6} - i\alpha\right) = x - iy$

$\therefore tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{tan\left(\frac{\pi}{6}+i\alpha\right)+\tan\left(\frac{\pi}{6}-i\alpha\right)}{1-\tan\left(\frac{\pi}{6}+i\alpha\right).\tan\left(\frac{\pi}{6}-i\alpha\right)}$

$\therefore tan\frac{\pi}{3} = \frac{(x+iy)+(x-iy)}{1-(x+iy).(x-iy)}$

$\therefore \sqrt{3} = \frac{2x}{1-x^2-y^2}$

$\therefore 1 - x^2 - y^2 = \frac{2x}{\sqrt{3}}$

$\therefore x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1.$

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$.

Solution: We have $x + iy = c\,\cot(u+iv)$ $\therefore\ x - iy = c\,\cot(u-iv)$

$\therefore 2x = c[\cot(u+iv) + \cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\,\frac{[\cos(u+iv)\sin(u-iv)+\sin(u+iv)\cos(u-iv)]}{\sin(u+iv)\sin(u-iv)}$$

$$\therefore 2x = \frac{c\sin[(u-iv)+(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u-iv-u+iv)]/2}$$

$$\therefore x = \frac{c\sin 2u}{-[\cos 2u-\cos 2\,iv]} = \frac{c\sin 2u}{\cos h\,2v-\cos 2u} \quad \text{...........(1)}$$

Now, $2iy = c[\cot(u+iv) - \cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} - \frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\left[\frac{\cos(u+iv)\sin(u-iv)-\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$\therefore 2iy = \frac{c\sin[(u-iv)-(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u+iv-u+iv)]/2}$$

$$\therefore\ iy = \frac{c\sin(-2iv)}{-[\cos 2u-\cos 2iv]} = -\frac{i\,c\sin h\,2v}{\cos h\,2v-\cos 2u}$$

$$\therefore\ y = \frac{-\,c\sin h\,2v}{\cos h\,2v-\cos 2u} \quad \text{..........(2)}$$

From (1) & (2) $\quad \frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v-\cos 2u}$

6. If $u + i\,v = cosec\left(\frac{\pi}{4} + i\,x\right), prove\ that\ (u^2+v^2)^2 = 2(u^2-v^2)$

Solution: We have $\frac{1}{\sin[(\pi/4)+ix]} = u + iv$

$$\therefore \sin\left(\frac{\pi}{4}+ix\right) = \frac{1}{u+iv} = \frac{1}{u+iv}\cdot\frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore \sin\frac{\pi}{4}\cos i\,x + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$$

$$\frac{1}{\sqrt{2}}\cos h\,x + i\frac{1}{\sqrt{2}}\sin hx = \frac{u-iv}{u^2+v^2}$$

Equating real and imaginary parts $\quad \cos hx = \sqrt{2}.\left(\frac{u}{u^2+v^2}\right)\ ;\ \sin hx = -\sqrt{2}.\left(\frac{v}{u^2+v^2}\right)$

But $cosh^2x - sinh^2x = 1$

$$\therefore 2\left(\frac{u^2}{(u^2+v^2)^2}\right) - 2\left(\frac{v^2}{(u^2+v^2)^2}\right) = 1$$

$$\therefore 2(u^2-v^2) = (u^2+v^2)^2$$

7. If $x + iy = \cos(\alpha + i\,\beta)$ or if $\cos^{-1}(x+iy) = \alpha + i\,\beta$ express x and y in terms of α and β.

Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2+y^2+1)\lambda + x^2 = 0$

Solution: We have $\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = x + iy$

$\therefore \cos\alpha\cos h\,\beta - i\sin\alpha\sin h\beta = x + iy$

Equating real and imaginary parts $\cos\alpha\cos h\,\beta = x\quad and\quad \sin\alpha\sin h\beta = -y$

We know that, in terms of the roots, the quadratic equation is given by

$\lambda^2 - (sum\ of\ the\ roots)\lambda + (product\ of\ the\ roots) = 0$

Hence the equation whose roots are $\cos^2\alpha$ and $\cosh^2\beta$ is

$\lambda^2 - (cos^2\alpha + cos^2\beta)\lambda + (cos^2\alpha . cos^2\beta) = 0$

This means we have to prove that $x^2 + y^2 + 1 = cos^2\alpha + cos^2\beta$ and $x^2 = cos^2\alpha + cos^2\beta$

Now, $x^2 + y^2 + 1 = cos^2\alpha \cos h^2\beta + sin^2\alpha \sin h^2\beta + 1$

$= cos^2\alpha \cos h^2\beta + (1 - cos^2\alpha)(\cos h^2\beta - 1) + 1$

$= cos^2\alpha \cos h^2\beta + \cos h^2\beta - 1 - cos^2\alpha \cos h^2\beta + cos^2\alpha + 1$

$= cos^2\alpha + \cos h^2\beta = sum\ of\ the\ roots$

And $x^2 = cos^2\alpha \cos h^2\beta$ = Product of the roots

Hence the equation whose roots are $cos^2\alpha$, $\cos h^2\beta$ is $\lambda^2 - (\ x^2 + y^2 + 1)\lambda + x^2 = 0$