0.1 Method of variation of parameters

Again we concentrate on 2nd order equation but it can be applied to higher order ODE. This has much more applicability than the method of undetermined coefficeints. First, the ODE need not be with constant coefficeints. Second, the nonhomogeneos part $r(x)$ can be a much more general function.

Theorem 1. *A particular solution y_p to the linear ODE*

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(y_1,y_2)}dx + y_2(x)\int \frac{y_1(x)r(x)}{W(y_1,y_2)}dx, \tag{2}$$

where y_1 and y_2 are basis solutions for the homogeneous counterpart

$$y'' + p(x)y' + q(x)y = 0.$$

Important note: *The (leading) coefficeint of y'' in (1) must be unity. If it is not unity, then make it unity by dividing the ODE by the leading coefficeint.*

Proof: We try for y_p of the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x),$$

where $u(x)$ and $v(x)$ are unknown functins. Now y_p should satisfy (1). First, we find $y_p'(x) = u'(x)y_1(x) + v'(x)y_2(x) + u(x)y_1'(x) + v(x)y_2'(x)$. Now to make calculations easier (!), we take

$$u'(x)y_1(x) + v'(x)y_2(x) = 0. \tag{3}$$

Next we find $y_p''(x) = u'(x)y_1'(x) + v'(x)y_2'(x) + u(x)y_1''(x) + v(x)y_2''(x)$. Subtituting $y_p(x), y_p'(x)$ and $y_p''(x)$ into (1) (and using the fact that y_1 and y_2 are solution of the homogeneos part), we get

$$u'(x)y_1'(x) + v'(x)y_2'(x) = r(x). \tag{4}$$

We solve u', v' from (3) and (4) as follows (Cramer's rule):

$$u' = -\frac{r(x)y_2(x)}{W(y_1,y_2)}, \quad v' = \frac{r(x)y_1(x)}{W(y_1,y_2)}$$

Integrating we find

$$u = -\int \frac{y_2(x)r(x)}{W(y_1,y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1,y_2)}dx.$$

Substituting u and v in $y_p(x) = y_1(x)u(x) + y_2(x)v(x)$, we find the required form of y_p given in (2).

Note: *We don't write constant of integration in the expression of u and v, since these can be absorbed with the constants of the genral solution of the homogeneous part.*

Example 1. *Consider*

$$y'' - 2y' - 3y = xe^{-x}.$$

(This has been solved before by the method of undetermined coefficeints.) The LI solutions of the homogenous part are $y_1(x) = e^{-x}$ and $y_2(x) = e^{3x}$. Hence,

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Now $W(y_1, y_2) = 4e^{2x}$. Hence,

$$u(x) = -\int \frac{x}{4}dx = -\frac{x^2}{8}$$

$$v(x) = \int \frac{xe^{-4x}}{4}dx = -\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}$$

Thus,

$$y_p(x) = -\frac{x^2}{8}e^{-x} + e^{3x}\left(-\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}\right)$$

Hence, the general solution is

$$y(x) = C_1e^{-x} + C_2e^{3x} + y_p(x)$$

Since, the last term of y_p can be absorbed with the constant C_1, we get the same solution as obatined before.

Example 2. *Consider*

$$y'' + y = \tan x.$$

Solution: (This can not be solved by the method of undetermined coefficeint.) The LI solutions of the homogenous part are $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Hence,

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where

$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx.$$

Now $W(y_1, y_2) = 1$. Hence,

$$u(x) = -\int \sin x \tan x \, dx = -\ln|\sec x + \tan x| + \sin x$$

$$v(x) = \int \sin x \, dx = -\cos x$$

Thus,

$$y_p(x) = -\cos x \ln|\sec x + \tan x|$$

Hence, the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x + y_p(x)$$

Example 3. *Consider*

$$y'' + y = |x|, \qquad x \in (-1, 1)$$

Solution: You can find the general solution using either the method of undetermined coefficients (tricky!) OR method of variation of parameters. Try yourself.

0.2 Method of variation of parameters: extension to higher order

We illustrate the method for the third order ODE

$$y''' + a(x)y'' + b(x)y' + c(x)y = r(x). \tag{5}$$

Note that the leading coefficeint is again unity. Suppose the three LI solutions to (5) are y_1, y_2 and y_3. As before let

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) + w(x)y_3(x). \tag{6}$$

We find

$$y_p'(x) = u'(x)y_1(x) + v'(x)y_2(x) + w'(x)y_3(x) + u(x)y_1'(x) + v(x)y_2'(x) + w(x)y_3'(x)$$

As before for the ease of computation (!) we set

$$u'(x)y_1(x) + v'(x)y_2(x) + w'(x)y_3(x) = 0 \tag{7}$$

Now

$$y_p''(x) = u'(x)y_1'(x) + v'(x)y_2'(x) + w'(x)y_3'(x) + u(x)y_1''(x) + v(x)y_2''(x) + w(x)y_3''(x)$$

Again for the ease of computation (!!), we set

$$u'(x)y_1'(x) + v'(x)y_2'(x) + w'(x)y_3'(x) = 0 \tag{8}$$

Further

$$y_p''' = u'(x)y_1''(x) + v'(x)y_2''(x) + w'(x)y_3''(x) + u(x)y_1'''(x) + v(x)y_2'''(x) + w(x)y_3'''(x)$$

Subtituting $y_p(x), y_p'(x), y_p''(x)$ and $y_p'''(x)$ into (5) (and using the fact that y_1, y_2 and y_3 are solutions of the homogeneos part), we get

$$u'(x)y_1''(x) + v'(x)y_2''(x) + w'(x)y_3'(x) = r(x). \tag{9}$$

Now we find u', v', w' from (7),(8) and (7) by Cramer's rule. Let

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

be the Wronskian of three LI solutions. (Wronskian can similarly defined for n LI solutions). Then

$$u' = \frac{W_1}{W(y_1, y_2, y_3)},\ v' = \frac{W_2}{W(y_1, y_2, y_3)}\ w' = \frac{W_3}{W(y_1, y_2, y_3)}.$$

Here W_i is the determinate obtained from $W(y_1, y_2, y_3)$ by replacing the i-th column by the column vector $(0, 0, r(x))^T$. Hence,

$$u = \int \frac{W_1}{W(y_1, y_2, y_3)}dx,\ v = \int \frac{W_2}{W(y_1, y_2, y_3)}dx,\ w = \int \frac{W_3}{W(y_1, y_2, y_3)}dx.$$

These u, v, w are then substituted into (6) to get y_p.