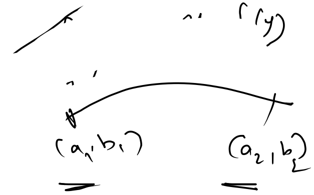
Rectification

(To find length of curve) $x = f(y)$

length of curve $= S$



① If $y = f(x)$

$$S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

② If $x = f(y)$

$$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

③ If $x = f_1(t)$ & $y = f_2(t)$ → Parametric eq^{n}s

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

polar

④ If $r = f(\theta)$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

⑤ If $\theta = f(r)$

$$S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

Find the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ → Astroid

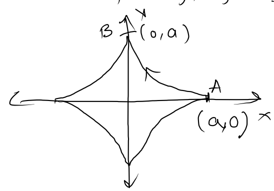
4-cusped hypocycloid.

S = Total length of astroid

$= 4\, l(AB)$

$= 4 \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = f(x)$



$x^{2/3} + y^{2/3} = a^{2/3}$

Diff. w.r.t. x

$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$

$y^{-1/3} \frac{dy}{dx} = -x^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$

$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$

$\left(\frac{dy}{dx}\right)^2 = +\frac{y^{2/3}}{x^{2/3}}$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{2/3}}{x^{2/3}}$

$= \frac{x^{2/3} + y^{2/3}}{x^{2/3}}$

$1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^{2/3}}{x^{2/3}}$

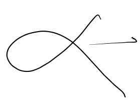
$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\left(\frac{a^{1/3}}{x^{1/3}}\right)^2} = \frac{a^{1/3}}{x^{1/3}}$

$S = 4 \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 4 \int_a^0 \frac{a^{1/3}}{x^{1/3}}\, dx$

$= 4a^{1/3} \int_a^0 x^{-1/3}\, dx$

$= 4a^{1/3} \left[\frac{x^{-1/3+1}}{-1/3+1}\right]_a^0$

$= 4a^{1/3} \left[\frac{x^{2/3}}{2/3}\right]_a^0$



$= 4a^{1/3} \left[0 - \frac{a^{2/3}}{2/3}\right]$

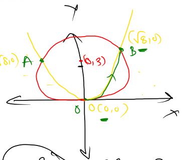
$= -\cancel{4}^{\,2} a^{1/3} \frac{3}{\cancel{2}} a^{2/3} = -6a$

$\therefore$ total length of Astroid (curve) $= 6a$

Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$.

$x^2 = 4y$ $\qquad x = \pm\sqrt{4y}$

Vertex $(0,0)$



$x^2 + y^2 = 6y$

$x^2 + \underline{y^2 - 6y} = 0$

$x^2 + \underline{y^2 - 6y + 9} = 9$

$x^2 + (y-3)^2 = 3^2$

Center $(0,3)$ $\qquad$ rad. 3

$\dfrac{M^2}{4F} = \dfrac{36}{4(1)} = 9$

$ax^2 + bx + c$

$-\dfrac{b^2}{4a}$

$x^2 = 4y$ $\qquad x^2 + y^2 = 6y$

$x^2 = 6y - y^2$

$4y = 6y - y^2 \Rightarrow y^2 - 2y = 0$

$y(y-2) = 0$

$y = 0$ or $y = 2$

as $x^2 = 4y$

$y = 0 \Rightarrow x = 0$ $\qquad (0,0)$

$y = 2 \Rightarrow x^2 = 8$ $\qquad \Rightarrow x = \pm\sqrt{8}$

$(\sqrt{8}, 0)$ $\quad (-\sqrt{8}, 0)$

$x^2 = 4y$

$y = x^2/4$ $\qquad y = f(x)$

$S = l(AOB)$

$= 2\,l(OB)$

$= 2\displaystyle\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\displaystyle\int_0^{\sqrt{8}} \sqrt{1 + \left(\frac{2x}{4}\right)^2}\, dx$

$= 2\displaystyle\int_0^{\sqrt{8}} \sqrt{1 + \frac{x^2}{2^2}}\, dx$

$= \dfrac{\cancel{2}}{\cancel{2}}\displaystyle\int_0^{\sqrt{8}} \sqrt{2^2 + x^2}\, dx$

$\displaystyle\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 + a^2}\right)$

$a = 2$

$= \left[\dfrac{x}{2}\sqrt{x^2 + 2^2} + \dfrac{2^{\cancel{2}}}{\cancel{2}}\log\left(x + \sqrt{x^2 + 2^2}\right)\right]_0^{\sqrt{8}}$

$= \left[\dfrac{\sqrt{8}}{2}\sqrt{8+4} + 2\log\left(\sqrt{8} + \sqrt{8+4}\right) - 2\log 2\right]$

$= \dfrac{\cancel{2}\sqrt{2}}{\cancel{2}}\sqrt{12} + 2\log\left(2\sqrt{2} + \sqrt{12}\right) - 2\log 2$

$= \sqrt{24} + 2\log\left(\dfrac{2\sqrt{2} + 2\sqrt{3}}{2}\right)$

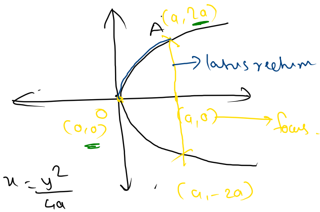
$= \sqrt{24} + 2\log\left(\sqrt{2} + \sqrt{3}\right)$

Hw Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is $a[\sqrt{2} + log\,(1+\sqrt{2})]$.
Find the length of the arc cut off by the line $3y = 8x$.

$y^2 = 4ax$ $\quad y = \pm\sqrt{4ax}$

$x = a$

vertex $(0,0)$ $\quad y = \pm 2a$



$S = l(OA)$

Σ $\quad$ as $y^2 = 4ax \Rightarrow x = \frac{y^2}{4a}$

$x = f(y)$

$$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

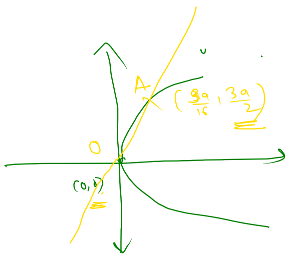
$$= \int_0^{2a} \sqrt{1 + \left(\frac{2y}{4a}\right)^2}\, dy$$

Hw

$y^2 = 4ax$

$3y = 8x$

$x = \frac{3y}{8}$



$y^2 = 4a\,\frac{3y}{8_2}$

$y^2 - \frac{3a}{2}y = 0$ $\quad y\left(y - \frac{3a}{2}\right) = 0$

$y = 0, \quad y = 3a/2$

$x = \frac{3y}{8} \Rightarrow$ for $y = 0 \Rightarrow x = 0$

for $y = 3a/2 \Rightarrow x = \frac{3}{8} \times \frac{3a}{2}$

$x = \frac{9a}{16}$

$\left(\frac{9a}{16}, \frac{3a}{2}\right)$

$S = l(O, A)$

$$= \int_0^{3a/2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

❖ **Find the total length of the loop of the curve**
$9y^2 = (x+7)(x+4)^2$

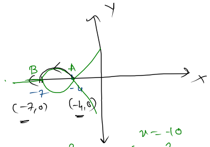
LHS = 0 for $y = 0$

$\therefore\ 0 = (x+7)(x+4)^2$

$x = -7,\ x = -4$

$(-7, 0)\quad (-4, 0)$

$x < -7$



$x = -10$

$9y^2 = (-3)(\ \ )^2$

$9y^2 = -ve$ ✗

S = Total length of loop

$= 2\, l(AB)$

$= 2\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$9y^2 = (x+7)(x+4)^2 \qquad y = f(x)$

$18y\frac{dy}{dx} = (x+7)\,2(x+4) + (x+4)^2$

$18y\frac{dy}{dx} = (x+4)(2x+14+x+4)$

$\frac{dy}{dx} = \frac{(x+4)(3x+18)}{18y}$

$= \frac{(x+4)\,\cancel{3}(x+6)}{\cancel{18}_{6}\,y}$

$\left(\frac{dy}{dx}\right)^2 = \frac{(x+4)^2(x+6)^2}{36y^2}$

$= \frac{(x+4)^2(x+6)^2}{4(9y^2)}$

$= \frac{\cancel{(x+4)^2}(x+6)^2}{4(x+7)\cancel{(x+4)^2}}$

$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x+6)^2}{4(x+7)}$

$= \frac{4x+28+x^2+12x+36}{4(x+7)}$

$= \frac{x^2+16x+64}{4(x+7)} = \frac{(x+8)^2}{4(x+7)}$

$S = 2\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\int_{-4}^{-7} \sqrt{\frac{(x+8)^2}{4(x+7)}}\, dx = \cancel{2}\int_{-4}^{-7} \frac{x+8}{\cancel{2}\sqrt{x+7}}\, dx$

put $x+7 = t$

$x = t-7$

$dx = dt$

$x: -4 \to -7 \qquad t: 3 \to 0$

$= \int_3^0 \frac{t-7+8}{\sqrt{t}}\, dt = \int_3^0 \left(\frac{t+1}{\sqrt{t}}\right) dt$

$= \int_3^0 \left(t^{1/2} + t^{-1/2}\right) dt$

$= \left[\frac{t^{3/2}}{3/2} + \frac{t^{1/2}}{1/2}\right]_3^0$

$= 0 - \left[\frac{3^{3/2}}{3}\times 2 + 2(3)^{1/2}\right]$

$= -\left[2(3)^{1/2} + 2(3)^{1/2}\right]$

$= -4\sqrt{3}$

$\therefore$ length of loop $4\sqrt{3}$

- HW Find the length of the loop $3ay^2 = x(x-a)^2$

LHS $= 0$ $\quad y = 0$

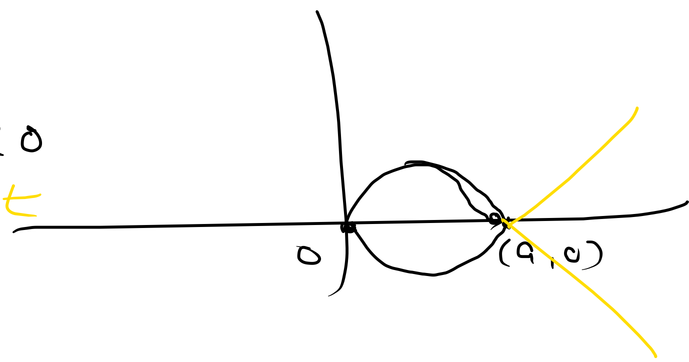
$x(x-a)^2 = 0$

$x = 0, \ x = a$

$(0, 0) \quad (a, 0)$

curve does not exist for $x < 0$

$\rightarrow t$



Prove that the length of the arc of the curve $y = log\left(\frac{e^x-1}{e^x+1}\right)$ from $x = 1, x = 2$ is $log\left(e+\frac{1}{e}\right)$

$y = f(x)$

$$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y = \log\left(\frac{e^x-1}{e^x+1}\right) = \log(e^x-1) - \log(e^x+1)$$

$$\frac{dy}{dx} = \frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}$$

$$= \frac{\cancel{e^{2x}}+e^x - (\cancel{e^{2x}} - e^x)}{(e^x)^2-1}$$

$$= \frac{2e^x}{e^{2x}-1}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{4e^{2x}}{(e^{2x}-1)^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{(e^{2x})^2 - 2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{(e^{2x})^2+2e^{2x}+1}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$$

$$S = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \int_1^2 \left(\frac{e^{2x}+1}{e^{2x}-1}\right)dx$$

e^x+e^{-x}

$e^x+\frac{1}{e^x}$

$$= \int_1^2 \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)dx$$

$$= \left[\log(e^x-e^{-x})\right]_1^2 \quad \text{as } (e^x-e^{-x})' = e^x+e^{-x}$$

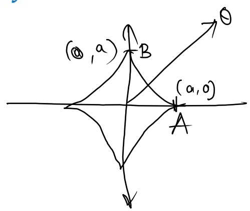
$$= \log(e^2-e^{-2}) - \log(e-e^{-1}) \quad //$$

$$= \log\left[\frac{\left(e^2-\frac{1}{e^2}\right)}{\left(e-\frac{1}{e}\right)}\right] = \log\left[\frac{\cancel{(e-\frac{1}{e})}(e+\frac{1}{e})}{\cancel{(e-\frac{1}{e})}}\right]$$

$x^{2/3}+y^{2/3}=a^{2/3}$

Find the length of the asteroid $x = a\cos^3 t\ ,\ y = a\sin^3 t.$

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$



$$\left[\begin{aligned} x^{2/3} &= a^{2/3}\cos^2 t \\ y^{2/3} &= a^{2/3}\sin^2 t \end{aligned}\right]$$

S = Total length of astroid

$= 4\ l(AB)$

$$= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{—— } (1)$$

$x = a\cos^3 t$ $\qquad\qquad$ $y = a\sin^3 t$

$\frac{dx}{dt} = a\,3\cos^2 t\,(-\sin t)$ $\qquad$ $\frac{dy}{dt} = a\,3\sin^2 t\,(\cos t)$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t$$

$$= 9a^2\cos^2 t\sin^2 t\,(\cos^2 t + \sin^2 t)$$

$$= 9a^2\sin^2 t\cos^2 t$$

$$S = 4\int_0^{\pi/2} \sqrt{9a^2\sin^2 t\cos^2 t}\,dt \quad \text{—— from } (1)$$

$$= 4\int_0^{\pi/2} 3a\sin t\cos t\,dt$$

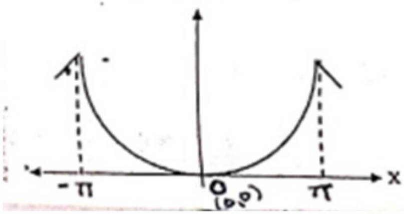
$$= 12a\int_0^{\pi/2} \sin t\cos t\,dt$$

$$= 12a\,\frac{1}{2}\beta\left(\frac{2}{2},\frac{2}{2}\right) = \frac{\cancel{12}^{6}}{\cancel{2}_{1}}a\,\beta(1,1)$$

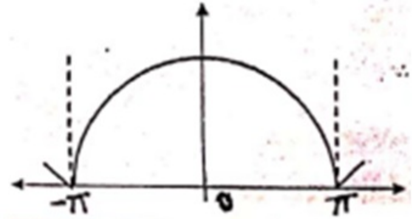
$$\left[\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)\right]$$

$$S = 6a\,\frac{\Gamma 1\,\Gamma 1}{\Gamma 2} = 6a$$

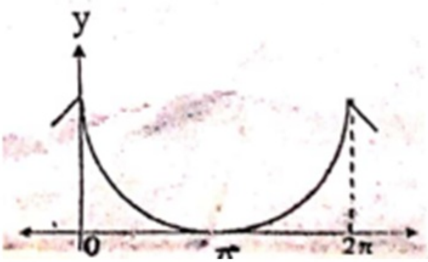
(1) $x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$



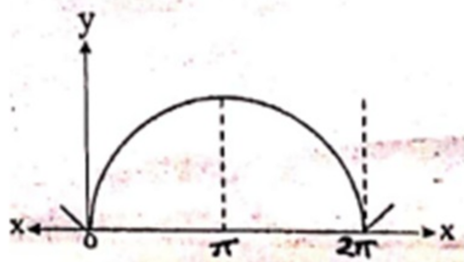
(2) $x = a(\theta + \sin\theta)$
$y = a(1 + \cos\theta)$



4) $x = a(\theta - \sin\theta)$
$y = a(1 + \cos\theta)$



(3) $x = a(\theta - \sin\theta)$
$y = a(1 - \cos\theta)$



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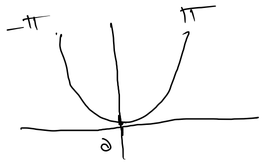
1. Find the length of the cycloid from one cusp to the next cusp $x = a(\theta + sin\theta)$, $y = a(1 - cos\,\theta)$.

2. If S is the length of the arc from the origin to a point $p(x, y)$ show that $s^2 = 8ay$. angle : $0 \to \theta$



at $\theta = 0$

$x = 0$, $y = 0$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{—— (1)}$$

$x = a(\theta + \sin\theta)$ $\qquad$ $y = a(1 - \cos\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = a(0 + \sin\theta)$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\left[1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}\right]$$

$$= a^2\left[2 + 2\cos\theta\right]$$

$$= 2a^2(1 + \cos\theta) = 2a^2\, 2\cos^2\theta/2$$

$$= (2a\cos\theta/2)^2$$

$$S = \int_{-\pi}^{\pi} \sqrt{(2a\cos\theta/2)^2}\, d\theta \quad \text{— from (1)}$$

$$= \int_{-\pi}^{\pi} 2a\cos\theta/2\, d\theta$$

$$= 2a\left[\frac{\sin\theta/2}{1/2}\right]_{-\pi}^{\pi}$$

$$= 4a\left[\sin\pi/2 - \sin(-\pi/2)\right]$$

$$= 4a\left(\sin\pi/2 + \sin\pi/2\right)$$

$$= 4a(2) = 8a$$

$4a\left(\sin\frac{\theta}{2}\right)$

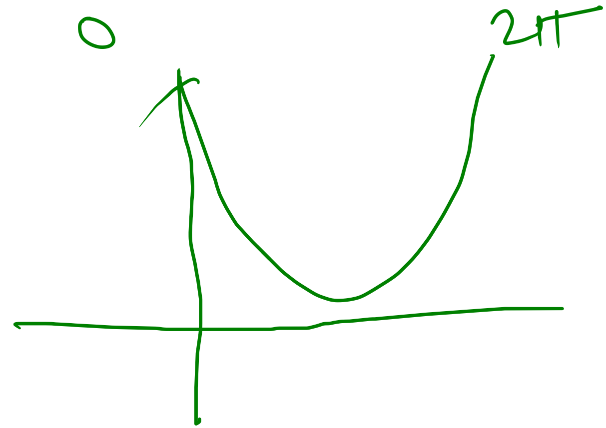
$\sqrt{\frac{1-\cos\theta}{2}}$

Hw Find the length of the one arc of the Cycloid $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$

$\theta = 0$

$x = 0$

$y = 2a$



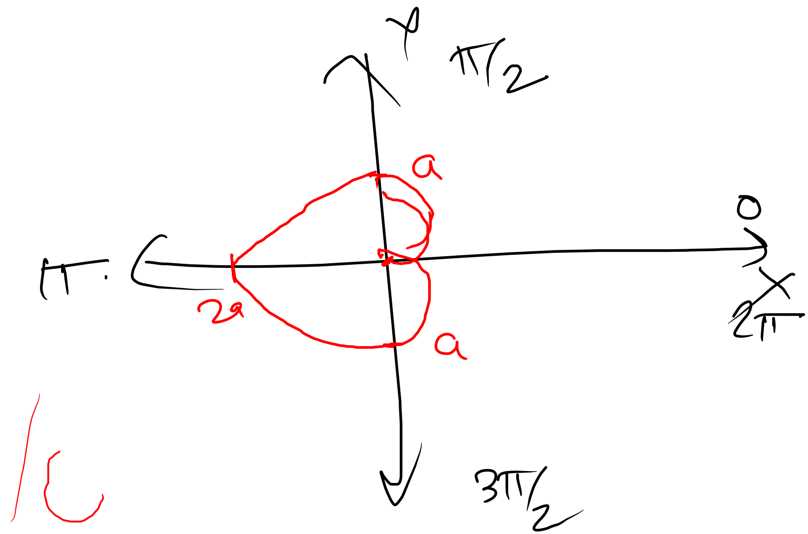
- HW Prove that the length of the arc of the curve

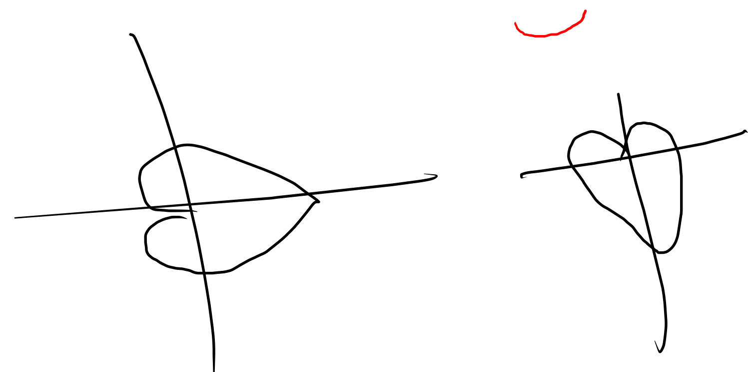
$x=a \sin 2\theta\,(1+\cos 2\theta)$, $y=a \cos 2\theta\,(1-\cos 2\theta)$ measured from the origin to (x,y) is $4/3\, a \sin 3\theta$.

$r = a(1+\cos\theta)$

$$\left\{ r = a(1-\cos\theta) \right]$$

θ	$\cos\theta$	r
0	1	0
$\pi/2$	0	a
π	-1	$2a$
$3\pi/2$	0	a
2π	1	0

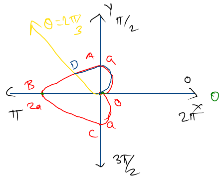




❖ **Find the perimeter of the cardioide $r = a(1-\cos\theta)$ and prove that the line $\theta = \frac{2\pi}{3}$ bisects the upper half of the cardioide.**

$r = a(1-\cos\theta)$

θ	$\cos\theta$	r
0	1	0
$\pi/2$	0	a
π	-1	$2a$
$3\pi/2$	0	a
2π	1	0



S = Perimeter of cardioide $= l(OABCO)$

$= 2\, l(OAB)$

$$= 2\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r = a(1-\cos\theta) \Rightarrow \frac{dr}{d\theta} = a(+\sin\theta)$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1 - 2\cos\theta + \underline{\cos^2\theta + \sin^2\theta})$$

$$= a^2(1 - 2\cos\theta + 1)$$

$$= a^2(2 - 2\cos\theta)$$

$$= 2a^2(1-\cos\theta)$$

$$= 2a^2\, 2\sin^2\theta/2 = (2a\sin\theta/2)^2$$

$$S = 2\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{(2a\sin\theta/2)^2}\, d\theta$$

$$= 2\int_0^{\pi} 2a\sin\theta/2\, d\theta$$

$$= 4a\left(\frac{-\cos\theta/2}{1/2}\right)_0^{\pi}$$

$$= 4a \times 2\,(-\cos\pi/2 + \cos 0)$$

$$= 8a\,(0+1) = 8a$$

perimeter of cardioide $= 8a$

$\therefore$ length of upper half of cardioide

$= l(OAB) = 4a$

To p. that $\theta = 2\pi/3$ bisects upper half of cardioide

i.e. T.p.T. $l(OD) = 2a$

$$\Rightarrow \int_0^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 2a$$

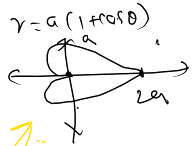
$$\int_0^{2\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi/3} 2a\sin\theta/2\, d\theta$$

$$= 2a\left[\frac{-\cos\theta/2}{1/2}\right]_0^{2\pi/3}$$

$$= 4a\left[-\cos\pi/3 + \cos 0\right]$$

$$= 4a\left[-1/2 + 1\right] = \frac{4a}{2} = 2a$$

❖ Find the length of the cardioide $r = a(1-\cos\theta)$ lying inside the circle $r = a\cos\theta$.



$r = a(1-\cos\theta)$

$r = a\cos\theta$ —— ①

$x = r\cos\theta \qquad y = r\sin\theta$

① × r

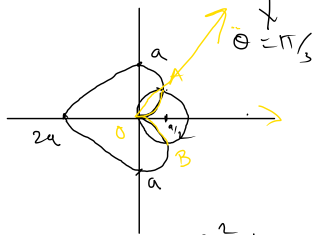
$r^2 = a\,r\cos\theta$

$x^2+y^2 = ax$

$x^2 - ax + y^2 = 0$

$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$ center $(a/2, 0)$ rad $a/2$



ax^2+bx+c

$\frac{M^2}{4F} = \frac{b^2}{2a}$

S = Required length of cardioide

$= 2\, l(OA)$

$= 2\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$r = a(1-\cos\theta) \qquad r = a\cos\theta$

$\cancel{a}(1-\cos\theta) = \cancel{a}\cos\theta \Rightarrow 2\cos\theta = 1$

$\Rightarrow \cos\theta = 1/2$

$\theta = \pi/3$

$S = 2\int_0^{\pi/3} \sqrt{(2a\sin\theta/2)^2}\, d\theta$

$= 2\int_0^{\pi/3} (2a\sin\theta/2)\, d\theta$

$= 4a\left(\frac{-\cos\theta/2}{1/2}\right)_0^{\pi/3} = 4a\times 2\left[-\cos\pi/6 + \cos 0\right]$

$= 8a\left(-\frac{\sqrt{3}}{2}+1\right)$

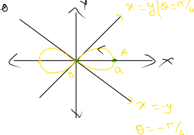
❖ **Show that the perimeter of $r^2 = a^2\cos 2\theta$ is $\frac{a}{\sqrt{2\pi}}\left(\Gamma\frac{1}{4}\right)^2$**

$r^2 = a^2\cos 2\theta$ lemniscate

$\theta = 0 \quad r^2 = a^2 \Rightarrow r = a$

$\theta = \pi/4 \quad r^2 = 0$

$\theta = \pi \quad r^2 = a^2$



S = perimeter of lemniscate

$= 4\, l(AO)$

$= 4\int_0^{a} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{—— (1)}$

$\theta : 0 \to \pi/4$

$r^2 = a^2\cos 2\theta$

$r = a(\cos 2\theta)^{1/2}$

$\frac{dr}{d\theta} = a\frac{1}{\cancel{2}}(\cos 2\theta)^{-1/2}(-\sin 2\theta)\cancel{2}$

$= \frac{-a\sin 2\theta}{\sqrt{\cos 2\theta}}$

$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^2\sin^2 2\theta}{\cos 2\theta}$

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}$

$= \frac{a^2\cos^2 2\theta + a^2\sin^2 2\theta}{\cos 2\theta}$

$= \frac{a^2}{\cos 2\theta}$

$S = 4\int_0^{\pi/4}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= 4\int_0^{\pi/4}\frac{a}{(\cos 2\theta)^{1/2}}\, d\theta$

$= 4a\int_0^{\pi/4}(\cos 2\theta)^{-1/2}\, d\theta$

$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$ ✓

put $2\theta = t$

$\theta = t/2 \Rightarrow d\theta = \frac{dt}{2}$

$\theta : 0 \to \pi/4 \quad t : 0 \to \pi/2$

$S = \overset{2}{\cancel{4}}a\int_0^{\pi/2}(\cos t)^{-1/2}\frac{dt}{\cancel{2}}$

$= 2a\int_0^{\pi/2}(\sin t)^0(\cos t)^{-1/2}dt$

$= 2a\frac{1}{\cancel{2}}B\left(\frac{1}{2}, \frac{1/2}{2}\right) = aB\left(\frac{1}{2}, \frac{1}{4}\right)$

$= a\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

$= a\frac{\sqrt{\pi}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

$\Gamma p\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

$\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \frac{\pi}{\sin\pi/4} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\pi$

$\Gamma\frac{3}{4} = \frac{\sqrt{2}\pi}{\Gamma\frac{1}{4}}$

$S = a\frac{\sqrt{\pi}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}} = a\frac{\sqrt{\pi}\,\Gamma\frac{1}{4}}{\sqrt{2}\pi/\Gamma\frac{1}{4}} = \frac{a}{\sqrt{2}\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2$