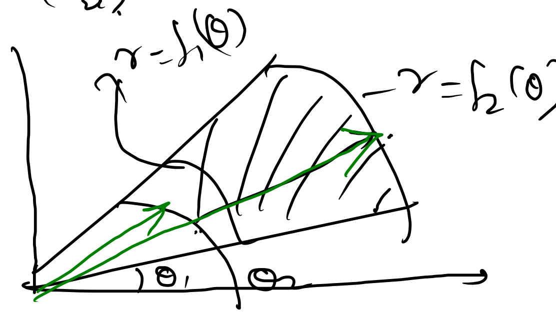
Polar co-ordinates.

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$



$$dx\,dy = r\,dr\,d\theta$$

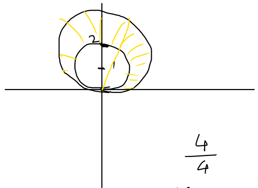
$$\int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\; r\,dr\,d\theta$$

$$\theta_1 \leq \theta \leq \theta_2$$

1. $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

$x = r\cos\theta$
$y = r\sin\theta$

$r = 2\sin\theta$
$r^2 = 2r\sin\theta$
$x^2+y^2 = 2y$
$x^2+y^2-2y = 0$
$x^2+y^2-2y+1 = 1$
$x^2+(y-1)^2 = 1$
Center $(0,1)$ rad 1



$\frac{4}{4}$

$\frac{16}{4} = 4$

$r = 4\sin\theta$
$r^2 = 4r\sin\theta$
$x^2+y^2 = 4y$
$x^2+y^2-4y+4 = 4$
$x^2+(y-2)^2 = 2^2$ — center $(0,2)$ rad 2

$$\int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_0^{\pi}\left(\frac{4^4\sin^4\theta}{\cancel{4}} - \frac{2^4\sin^4\theta}{4}\right) d\theta$$

$$= \int_0^{\pi}\left(64\sin^4\theta - 4\sin^4\theta\right) d\theta$$

$$= \int_0^{\pi} 60(\sin\theta)^4\,d\theta$$

$$= 60\int_0^{\pi}(\sin\theta)^4\,d\theta \qquad \int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$$

$f(\theta) = (\sin\theta)^4 \qquad f(a-x) = f(x)$

$f(\pi-\theta) = [\sin(\pi-\theta)]^4$
$= [0 - (-1)\sin\theta]^4 = \sin^4\theta = f(\theta)$

$$= 60\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta \qquad \int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$= 60\times\cancel{2}\times\frac{1}{\cancel{2}}\,\beta\left(\frac{5}{2},\frac{1}{2}\right)$$

$$= 60\,\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} = 60\,\frac{\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2!}$$

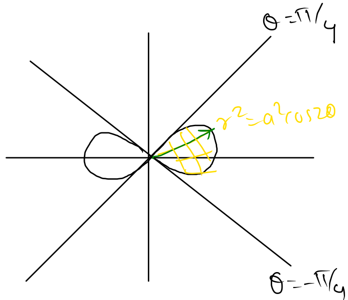
$$= \frac{\overset{15}{\cancel{60}}\times 3}{1\,\cancel{4}\times 2}\pi$$

$$= \frac{45\pi}{2}$$

$\Gamma n = (n-1)\Gamma n-1$
$\Gamma n = (n-1)!$
↳ +ve int
$\Gamma\frac{1}{2} = \sqrt{\pi}$

2. Evaluate $\iint \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$ over one loop of the lemniscate $r^2 = \cancel{a^2\cos\theta}$ $a^2\cos 2\theta$

$r = a\sqrt{\cos 2\theta}$



$$\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{r^2+a^2}}\,dr\,d\theta$$

put $r^2+a^2 = t$

$\Rightarrow 2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$r: 0 \to a\sqrt{\cos 2\theta}$

$t: a^2 \to a^2(1+\cos 2\theta)$

$$= \int_{-\pi/4}^{\pi/4}\int_{a^2}^{a^2(1+\cos 2\theta)} \frac{dt}{2\sqrt{t}}\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\int_{a^2}^{a^2(1+\cos 2\theta)} t^{-1/2}\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\left(\frac{t^{1/2}}{1/2}\right)_{a^2}^{a^2(1+\cos 2\theta)} d\theta$$

$$= \frac{1}{\cancel{2}}\int_{-\pi/4}^{\pi/4}\cancel{2}\left(a\sqrt{1+\cos 2\theta} - a\right) d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left(a\sqrt{2}\cos\theta - a\right) d\theta \qquad 1+\cos 2\theta = 2\cos^2\theta$$

$$= \left[\sqrt{2}a\sin\theta - a\theta\right]_{-\pi/4}^{\pi/4}$$

$\sin(-\pi/4) = -\sin\pi/4 = -\frac{1}{\sqrt{2}}$

$$= \cancel{\sqrt{2}}a\frac{1}{\cancel{\sqrt{2}}} - a\frac{\pi}{4} - \left(\sqrt{2}a\left(-\frac{1}{\sqrt{2}}\right) + a\frac{\pi}{4}\right)$$

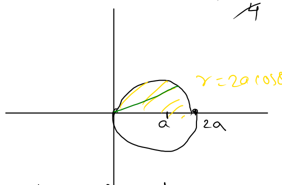
$$= a - a\frac{\pi}{4} + a - a\frac{\pi}{4} = 2a - \frac{\cancel{2}a\pi}{\cancel{4}\,2}$$

3. Evaluate $\iint re^{-r^2/a^2}\cos\theta\sin\theta\, d\theta dr$ over the upper half of the circle $r = 2a\cos\theta$

$$\frac{4a^2}{4}$$

$x = r\cos\theta$
$y = r\sin\theta$

$r = 2a\cos\theta$
$r^2 = 2ar\cos\theta$
$x^2+y^2 = 2ax$
$x^2 - 2ax + a^2 + y^2 = a^2$
$(x-a)^2 + y^2 = a^2$ Center $(a,0)$ rad a



$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r e^{-r^2/a^2}\cos\theta\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2}\cos\theta\sin\theta\int_0^{2a\cos\theta} re^{-r^2/a^2}\, dr\, d\theta$$

$\frac{r^2}{a^2} = t \Rightarrow r^2 = a^2 t$
$2r\,dr = a^2 dt$
$\Rightarrow r\,dr = \frac{a^2}{2}dt$
$r: 0 \to 2a\cos\theta$
$t: 0 \to 4\cos^2\theta$

$$= \int_0^{\pi/2}\cos\theta\sin\theta\int_0^{4\cos^2\theta}\frac{a^2}{2}e^{-t}\,dt\; d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\cos\theta\sin\theta\left(\frac{e^{-t}}{-1}\right)_0^{4\cos^2\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\cos\theta\sin\theta\left(\frac{e^{-4\cos^2\theta}}{-1} + \frac{1}{+1}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(1 - e^{-4\cos^2\theta}\right)\underline{\cos\theta\sin\theta\, d\theta}$$

$\cos^2\theta = t \Rightarrow 2\cos\theta(-\sin\theta)d\theta = dt$
$\Rightarrow \cos\theta\sin\theta\, d\theta = -\frac{dt}{2}$

$\theta: 0 \to \pi/2$ $\quad t: 1 \to 0$

$$= \frac{a^2}{2}\int_1^0 \left(1 - e^{-4t}\right)\left(-\frac{dt}{2}\right)$$

$$= \frac{a^2}{4}\int_0^1 \left(1 - e^{-4t}\right) dt = \frac{a^2}{4}\left(t + \frac{e^{-4t}}{+4}\right)_0^1$$

$$= \frac{a^2}{4}\left(1 + \frac{e^{-4}}{4} - 0 - \frac{1}{4}\right)$$

$$= \frac{a^2}{4}\left(\frac{3}{4} + \frac{e^{-4}}{4}\right).$$

4. Evaluate $\iint_R \sin\theta \, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$

$x = r\cos\theta$

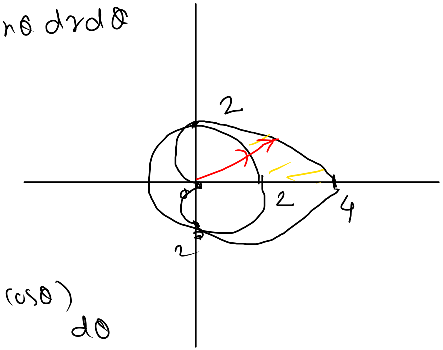
$y = r\sin\theta$

$$\iint_R \sin\theta \, dr \, d\theta$$

$r = 2$

$r^2 = 2^2$

$x^2 + y^2 = 2^2$



$$\int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \, (r)\Big|_2^{2(1+\cos\theta)} \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(2(1+\cos\theta) - 2\right) d\theta$$

$\cancel{2} + 2\cos\theta - \cancel{2}$

$$= \int_0^{\pi/2} 2\sin\theta\cos\theta \, d\theta = \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= \left(\frac{-\cos 2\theta}{2}\right)_0^{\pi/2}$$

$$= -\frac{(-1)}{2} + \frac{1}{2}$$

$$= 1$$

$dA = r\,dr\,d\theta$

$dx\,dy = r\,dr\,d\theta$

$$\int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \sin\theta \; r \, dr \, d\theta$$