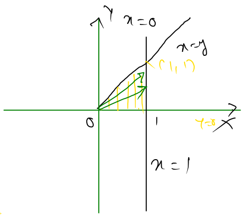
Change to polar coordinates and evaluate.

1. $\int_0^1 \int_0^x (x+y)\,dy\,dx$

$$\int_{x=0}^{1} \int_{y=0}^{x} (x+y)\, dy\, dx$$

To plot $x=0,\ x=1$

$y=0 \quad y=x$



$x = r\cos\theta, \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta : 0 \longrightarrow \pi/4$

$r : 0 \longrightarrow \sec\theta$

$x = 1$

$r\cos\theta = 1$

$r = 1/\cos\theta = \sec\theta$

$$\int_0^1 \int_0^x (x+y)\, dy\, dx$$

$$= \int_0^{\pi/4} \int_0^{\sec\theta} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} \int_0^{\sec\theta} (\cos\theta + \sin\theta)\, r^2\, dr\, d\theta$$

$$= \int_0^{\pi/4} (\cos\theta + \sin\theta) \left(\frac{r^3}{3}\right)_0^{\sec\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi/4} (\cos\theta + \sin\theta)\, \sec^3\theta\, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/4} \left(\cancel{\cos\theta}\, \frac{1}{\cos^{\cancel{3}}\theta} + \frac{\sin\theta}{\cos^3\theta}\right) d\theta$$

$$= \frac{1}{3} \int_0^{\pi/4} \left(\underline{\sec^2\theta} + \sin\theta\, (\cos\theta)^{-3} \right) d\theta$$

$\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$\sin\theta\, d\theta = -dt$

$\theta : 0 \to \pi/4$

$t : 1 \longrightarrow 1/\sqrt{2}$

$$= \frac{1}{3} \left[(\tan\theta)_0^{\pi/4} + \int_1^{1/\sqrt{2}} (t)^{-3}\, -dt \right]$$

$$= \frac{1}{3} \left[(1-0) - \left(\frac{t^{-2}}{-2}\right)_1^{1/\sqrt{2}} \right]$$

$$= \frac{1}{3} \left(1 + \frac{1}{2} \left(\left(\frac{1}{\sqrt{2}}\right)^{-2} - (1)^{-2} \right) \right]$$

$$\left(\frac{1}{\sqrt{2}}\right)^{-2} \qquad \frac{1}{\left(1/\sqrt{2}\right)^2}$$

$$= \frac{1}{3} \left(1 + \frac{1}{2} (2-1) \right)$$

$$= \frac{1}{3} \left(1 + \frac{1}{2} \right) = \frac{1}{3} \times \frac{3}{2} = 1/2 \,//$$

2. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dxdy$

$y=$ $x=$

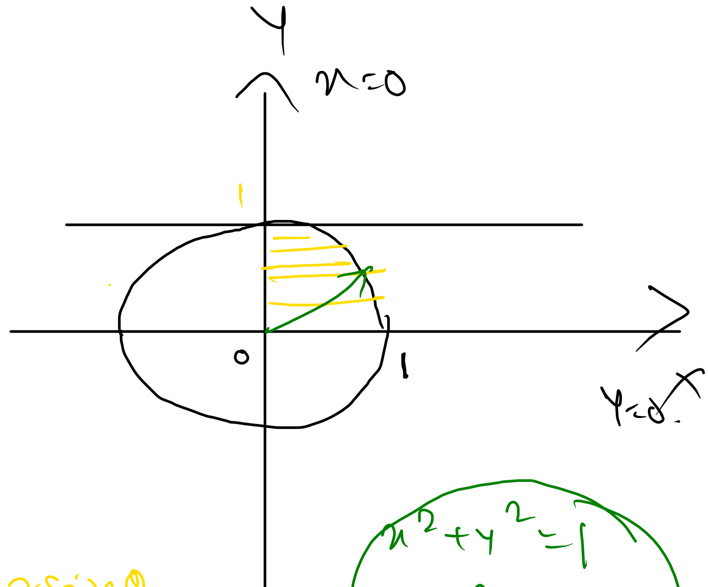
$Y=0 \quad y=1$

$x=0 \quad x=\sqrt{1-y^2}$

$x^2=1-y^2$

$x^2+y^2=1$

Centre $(0,0)$ & rad 1



$x^2+y^2=1$

$r^2=1$

$r=1$

$x=r\cos\theta \quad , \quad y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta: 0 \longrightarrow \pi/2$

$r: 0 \longrightarrow 1$

$$\int_0^{\pi/2}\int_0^1 r^2\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \pi/8$$

$$\frac{M^2}{4F} = \frac{a^2}{4}$$

4. $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-x^2-y^2}}\,dy\,dx$

$x^2 - y^2$

$x = 0 \quad , \quad x = a$

$y = \sqrt{ax - x^2}$

$y^2 = ax - x^2$

$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

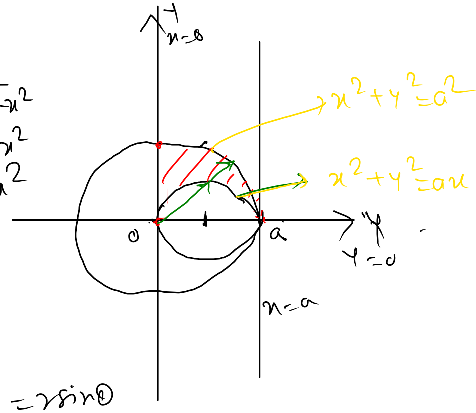
$(x - a/2)^2 + y^2 = \left(\frac{a}{2}\right)^2$

centre $(a/2, 0)$ & rad $a/2$

$y = \sqrt{a^2 - x^2}$

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$



$x^2 + y^2 = ax$

$r^2 = ar\cos\theta$

$x = r\cos\theta \qquad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta : 0 \to \pi/2$

$r : a\cos\theta \to a$

$$\int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{1}{\sqrt{a^2 - r^2}}\, r\,dr\,d\theta$$

$a^2 - r^2 = t$

$-2r\,dr = dt \Rightarrow r\,dr = -dt/2$

$r : a\cos\theta \to a$

$t : a^2 - a^2\cos^2\theta \to 0$

$t : a^2\sin^2\theta \to 0$

$$= \int_0^{\pi/2} \int_{a^2\sin^2\theta}^{0} \frac{-dt}{2\sqrt{t}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \int_0^{a^2\sin^2\theta} t^{-1/2}\,dt\,d\theta = a \,//.$$

$$\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}}\right)^2$$

$$\frac{a^2}{2} + \frac{a^2}{2} = a^2$$

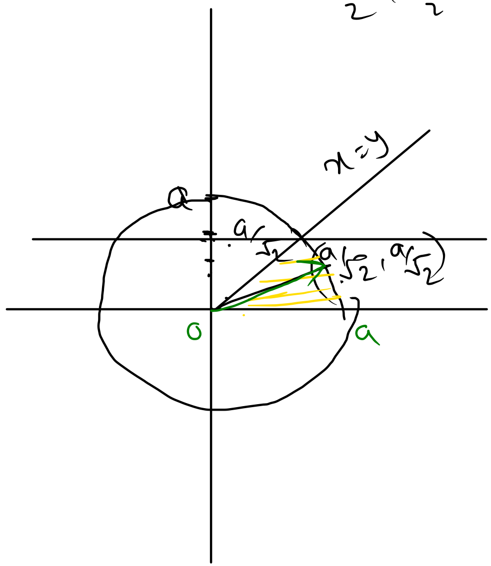
5. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy$

$y =$ $x =$

$y = 0$ $\quad y = a/\sqrt{2}$

$x = y$ $\quad x = \sqrt{a^2-y^2}$

$x^2 + y^2 = a^2$



$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta : 0 \longrightarrow \pi/4$

$r : 0 \longrightarrow a$

$$\int_0^{\pi/4} \int_0^a \log(r^2)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^a 2\log r \; r\,dr\,d\theta$$

$$= 2 \int_0^{\pi/4} \int_0^a \Big[\underbrace{\log r}_{u} \; \underset{v}{r}\Big]\,dr\,d\theta$$

$$= a^2\left(\log a - \frac{1}{2}\right)\frac{\pi}{4}$$

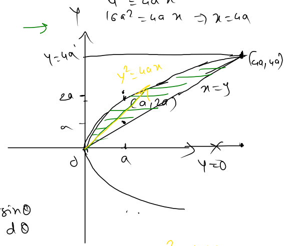
6. $\int_0^{4a}\int_{y^2/4a}^{y}\left(\frac{x^2-y^2}{x^2+y^2}\right)dx\,dy$

$y =$ $x =$

$y = 4a$

$y^2 = 4ax$

$16a^2 = 4ax \Rightarrow x = 4a$



$y = 0 \qquad y = 4a$

$x = y^2/4a \qquad x = y$

$y^2 = 4ax \quad$ vertex $(0,0)$

$y = \pm\sqrt{4ax}$

$x = a \;,\; y = \pm 2a$

$x = r\cos\theta \;,\; y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta = \pi/4 \longrightarrow \pi/2$

$r : 0 \longrightarrow \frac{4a\cos\theta}{\sin^2\theta}$

$y^2 = 4ax$

$r^2\sin^2\theta = 4a\cancel{r}\cos\theta$

$r = 4a\frac{\cos\theta}{\sin^2\theta}$

$$\int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}}\left(\frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}\right)r\,dr\,d\theta$$

$$\int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}}(\cos 2\theta)\,r\,dr\,d\theta$$

$\cos^2\theta + \sin^2\theta = 1$

$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

$$\int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}}d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\left(16a^2\frac{\cos^2\theta}{\sin^4\theta}\right)d\theta$$

$$= \frac{16a^2}{2}\int_{\pi/4}^{\pi/2}(\cot^4\theta-\cot^2\theta)\,d\theta \quad \Big|\; \cot^2\theta(\cot^2\theta-1)$$

$$= \frac{16a^2}{2}\int_{\pi/4}^{\pi/2}\left(\cot^2\theta\cot^2\theta-(\operatorname{cosec}^2\theta-1)\right)d\theta$$

$$= \frac{16a^2}{2}\int_{\pi/4}^{\pi/2}\left[\cot^2\theta(\operatorname{cosec}^2\theta-1)-\operatorname{cosec}^2\theta+1\right)d\theta$$

$$= \frac{16a^2}{2}\int_{\pi/4}^{\pi/2}\left[\cot^2\theta\operatorname{cosec}^2\theta-\cot^2\theta-\operatorname{cosec}^2\theta+1\right]d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left[\cot^2\theta\operatorname{cosec}^2\theta-(\operatorname{cosec}^2\theta-1)-\operatorname{cosec}^2\theta+1\right]d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\left[\underline{\cot^2\theta\operatorname{cosec}^2\theta}-\underline{2\operatorname{cosec}^2\theta+2}\right]d\theta$$

$$= 8a^2\left[\int_{\pi/4}^{\pi/2}\underline{\cot^2\theta\operatorname{cosec}^2\theta\,d\theta}-2(-\cot\theta)_{\pi/4}^{\pi/2}+(2\theta)_{\pi/4}^{\pi/2}\right]$$

put $\cot\theta = t$

$=$ $\quad -\operatorname{cosec}^2\theta\,d\theta = dt$

$\theta : \pi/4 \to \pi/2 \longrightarrow t : 1 \to 0$

$$= 8a^2\left[\int_1^0 t^2(-dt) - 2(0+1) + \left[\pi - \pi/2\right]\right]$$

$$= 8a^2\left[-\left(\frac{t^3}{3}\right)_1^0 - 2 + \pi/2\right] = 8a^2\left[-\left(0-\frac{1}{3}\right)-2+\pi/2\right]$$

$$= 8a^2\left(\frac{1}{3}-2+\pi/2\right]$$

$$= 8a^2\left(\pi/2 - 5/3\right)$$

7. $\int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\,dy\,dx$

$x - y$

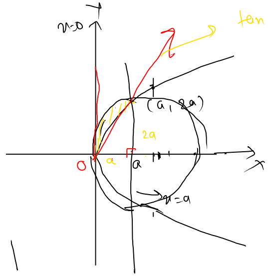
$x=0, \quad x=a$

$y = 2\sqrt{ax} \qquad y = \sqrt{5ax - x^2}$

$y^2 = 4ax$

$y = \pm\sqrt{4ax}$

$x = a \Rightarrow y = \pm 2a$



$\tan\theta = \frac{2a}{a}$

$\theta = \tan^{-1}(2)$

$y^2 = 5ax - x^2$

$x^2 - 5ax + \frac{25a^2}{4} + y^2 = \frac{25a^2}{4}$ $\qquad \frac{25a^2}{4}$

$\left(x - \frac{5}{2}a\right)^2 + y^2 = \left(\frac{5a}{2}\right)^2$ centre $\left(\frac{5}{2}a, 0\right)$ rad $\frac{5}{2}a$

for $x = a$ $\quad y^2 = 5ax - x^2$

$y^2 = 5a^2 - a^2 \Rightarrow y = 4a^2$

$y = \pm 2a$

$\theta: \tan^{-1}(2) \longrightarrow \pi/2$

$r: \frac{4a\cos\theta}{\sin^2\theta} \longrightarrow 5a\cos\theta$

$y^2 = 4ax$

$r^{\cancel{2}}\sin^2\theta = 4a\cancel{r}\cos\theta$

$x^2 + y^2 = 5ax$

$r^{\cancel{2}} = 5a\cancel{r}\cos\theta$

$$I = \int_{\tan^{-1}2}^{\pi/2} \int_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta} \frac{\cancel{r}}{\cancel{r^2}\sin^2\theta}\,\cancel{r}\,dr\,d\theta$$

$$= \int_{\tan^{-1}2}^{\pi/2} \frac{1}{\sin^2\theta}\,(r)_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta}\,d\theta$$

$$= \int_{\tan^{-1}2}^{\pi/2} \frac{1}{\sin^2\theta}\left(5a\cos\theta - \frac{4a\cos\theta}{\sin^2\theta}\right)d\theta.$$

$$= \int_{\tan^{-1}2}^{\pi/2} \left(\frac{5a}{\sin^2\theta} - \frac{4a}{\sin^4\theta}\right)\underline{\cos\theta\,d\theta}$$

$\sin\theta = t$ | $\theta: \tan^{-1}2 \rightarrow \pi/2$

$\cos\theta\,d\theta = dt$ | $t: \frac{2}{\sqrt5} \rightarrow 1$

$\theta = \tan^{-1}2$

$\tan\theta = \frac{2}{1}$

$t = \sin\theta = \frac{2}{\sqrt5}$

$$= \int_{2/\sqrt5}^{1} \left(\frac{5a}{t^2} - \frac{4a}{t^4}\right)dt$$

Evaluate the following integrals over the region stated, by changing to polar coordinates.

1. $\iint y^2 dx\,dy$ over the area outside $x^2+y^2-ax=0$ and inside $x^2+y^2-2ax=0$

$x^2+y^2-ax=0$

$x^2-ax+\frac{a^2}{4}+y^2=\frac{a^2}{4}$

$(x-a/2)^2+y^2=(a/2)^2$

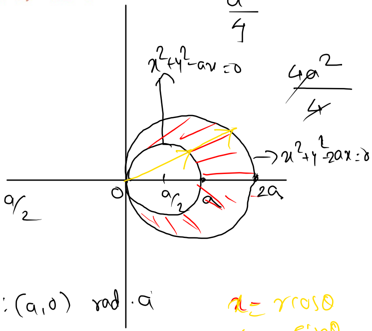
centre $(a/2, 0)$ rad $\cdot\, a/2$

$x^2+y^2-2ax=0$

$x^2-2ax+a^2+y^2=a^2$

$(x-a)^2+y^2=a^2$ $\quad C:(a,0)$ rad $\cdot\, a$

$\frac{a^2}{4}$ $\quad \frac{\cancel{4}a^2}{\cancel{4}}$



$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta: -\pi/2 \to \pi/2$

$r: a\cos\theta \to 2a\cos\theta$

$x^2+y^2-ax=0$

$r^2 - ar\cos\theta = 0$

$r = a\cos\theta$

$x^2+y^2-2ax=0$

$r^2 - 2ar\cos\theta = 0 \Rightarrow r = 2a\cos\theta$

$$\iint y^2\,dy\,dx = \int_{-\pi/2}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta\; r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\sin^2\theta\left(\frac{r^4}{4}\right)_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2}\sin^2\theta\left[(16a^4\cos^4\theta - a^4\cos^4\theta\right] d\theta$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} 15\,\sin^2\theta\, a^4\cos^4\theta\,d\theta$$

$$= \frac{15a^4}{4}\times 2\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta$$

$$= \frac{15a^4}{2}\,\frac{1}{2}\,\beta\left(\frac{3}{2},\frac{5}{2}\right)$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ even} \\ 0 & \text{if } f(x) \text{ odd} \end{cases}$$

$f(-x)=f(x) \to$ even

$f(-x) = -f(x) \to$ odd

eg even

x^2, c, $\cos\theta$

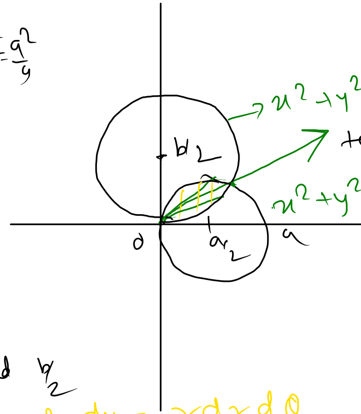
odd

x^3 $\sin x$

2. $\iint \frac{(x^2+y^2)^2}{x^2y^2} dx\, dy$ over the area common to $x^2+y^2=ax$ and $x^2+y^2=by, a, b>0$

$$x^2+y^2=ax \Rightarrow x^2-ax+\frac{a^2}{4}+y^2=\frac{a^2}{4}$$

$$\left(x-\tfrac{a}{2}\right)^2+y^2=\left(\tfrac{a}{2}\right)^2$$



$\rightarrow x^2+y^2=by \Rightarrow r=b\sin\theta.$

$\tan^{-1}(a/b)=\alpha$

$x^2+y^2=ax \Rightarrow r=a\cos\theta$

$$x^2+y^2=by$$

$$x^2+y^2-by+\frac{b^2}{4}=\frac{b^2}{4}$$

$$x^2+\left(y-\tfrac{b}{2}\right)^2=\left(\tfrac{b}{2}\right)^2$$

Centre $(0, b/2)$ rad $b/2$

$x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy=r\,dr\,d\theta$

$$x^2+y^2=by \qquad\qquad x^2+y^2=ax$$

$$r^{\cancel{2}}=b\cancel{r}\sin\theta \qquad\qquad r^2=a\cancel{r}\cos\theta$$

$$r=b\sin\theta \qquad\qquad r=a\cos\theta$$

$$b\sin\theta=a\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta}=\frac{a}{b}$$

$$\Rightarrow \tan\theta=a/b$$

$$\theta=\tan^{-1}(a/b)=\alpha$$

$$I=\iint\frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$$

$$=\int_0^{\alpha}\int_0^{b\sin\theta}\frac{\cancel{(r^2)^2}}{\cancel{r^2}\cos^2\theta\,\cancel{r^2}\sin^2\theta}\,r\,dr\,d\theta+\int_{\alpha}^{\pi/2}\int_0^{a\cos\theta}\frac{r\,dr\,d\theta}{\sin^2\theta\cos^2\theta}$$

$$=ab$$

$\cancel{\tan}(\alpha)$

$\cot\,\cancel{\tan}(\tan^{-1} a/b)$

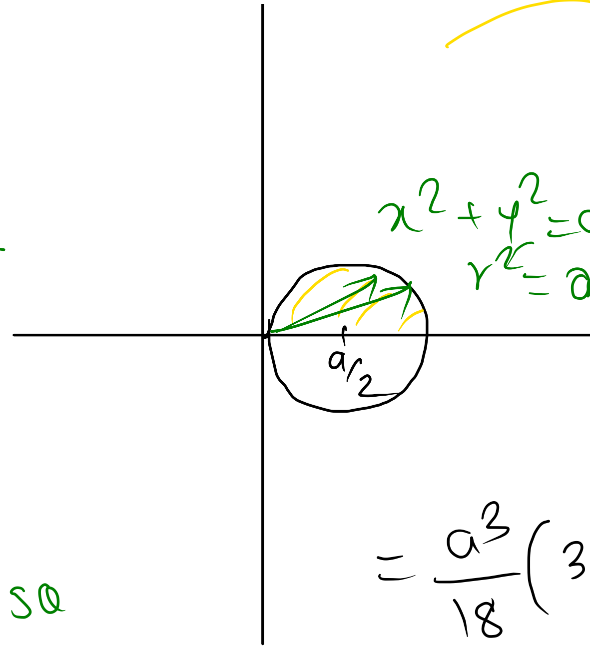
$\frac{1}{\tan(\tan^{-1} a/b)}=\frac{1}{a/b}=b/a$

3. $\iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$ where R is the area of the upper half of the circle $x^2 + y^2 = ax$

$$x^2 + y^2 = ax$$

$$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2 \quad \times$$



$$x^2 + y^2 = ax$$

$$r^2 = ar\cos\theta$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta$$

$$\theta : 0 \to \pi/2$$

$$r : 0 \to a\cos\theta$$

$$= \frac{a^3}{18}(3\pi - 4)$$

$$\int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2 - r^2}\quad r\,dr\,d\theta$$

$$a^2 - r^2 = t$$

$$-2r\,dr = dt \qquad r\,dr = -dt/2$$

$$r : 0 \to a\cos\theta \qquad t : a^2 \to a^2(\sin^2\theta)$$

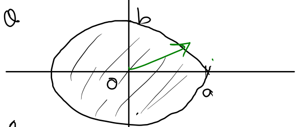
6. $\iint \sqrt{\left[\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}\right]}\,dx\,dy$ where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$a^2b^2-b^2a^2r^2\cos^2\theta-a^2b^2r^2\sin^2\theta$$
$$a^2b^2(1-r^2)$$

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \\ dx\,dy = ab\,r\,dr\,d\theta \end{cases}$$

$\theta: 0 \to 2\pi$

$r: 0 \to 1$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{\cancel{a^2}r^2\cos^2\theta}{\cancel{a^2}}+\frac{\cancel{b^2}r^2\sin^2\theta}{\cancel{b^2}}=1$$
$$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)=1$$
$$\Rightarrow r^2=1$$
$$\Rightarrow r=1$$

$$\iint \sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}\,dx\,dy$$

$$=\int_0^{2\pi}\int_0^1 \sqrt{\frac{\cancel{a^2b^2}(1-r^2)}{\cancel{a^2b^2}(1+r^2)}}\,ab\,r\,dr\,d\theta$$

$$=ab\int_0^{2\pi}\int_0^1 \frac{\sqrt{1-r^2}\,\sqrt{1-r^2}}{\sqrt{1-r^4}}\,r\,dr\,d\theta$$

$$=ab\int_0^{2\pi}\int_0^1 \frac{(1-r^2)}{\sqrt{1-r^4}}\,\underline{r\,dr}\,d\theta$$

$$r^2=\sin t$$
$$2r\,dr=\cos t\,dt$$
$$r\,dr=\frac{\cos t}{2}\,dt$$

$r: 0 \to 1 \qquad t: 0 \to \pi/2$

$$=ab\int_0^{2\pi}\int_0^{\pi/2}\frac{(1-\sin t)}{\cancel{\sqrt{1-\sin^2 t}}}\,\frac{\cancel{\cos t}}{2}\,dt\,d\theta$$

$$=\underline{\underline{\pi ab\left(\pi/2-1\right)}}$$

7. $\iint \frac{1}{(1+x^2+y^2)^2}\,dxdy$ over one loop of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta: -\pi/4 \longrightarrow \pi/4.$

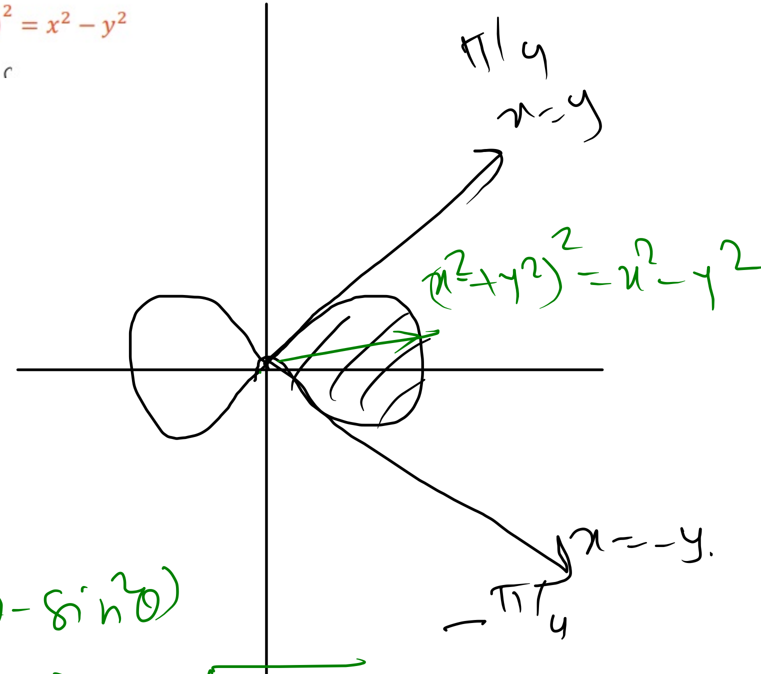
$r: 0 \longrightarrow \sqrt{\cos 2\theta}$

$(x^2+y^2)^2 = x^2 - y^2$

$(r^2)^{\cancel{2}} = \cancel{r^2}(\cos^2\theta - \sin^2\theta)$

$r^2 = \cos 2\theta \Rightarrow r = \sqrt{\cos 2\theta}$

$$\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{1}{1+r^2}\, r\,dr\,d\theta$$



$= \pi/4 - 1/2$

8. Evaluate $\iint_R (3x+4y^2)\,dxdy$ where R is the region in the upper half of the area bounded by the circle $x^2+y^2=1, x^2+y^2=4$

$x = r\cos\theta$
$y = r\sin\theta$

$dxdy = r\,dr\,d\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$\theta : 0 \longrightarrow \pi$
$r : 1 \longrightarrow 2$

$$\int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= {}^{15\pi}/_{2}.$$