Non-Linear Programming

Part (I) : Unconstrained Problems

Let f be a function of variables $x_1, x_2, x_3, \cdots\cdots$ then, the **necessary condition** for f to have extreme value at any point is,

$$\nabla f = 0 \Rightarrow \frac{\partial f}{\partial x_1} = 0,\ \frac{\partial f}{\partial x_2} = 0,\ \cdots\cdots\cdots$$

Let if $P = (x_1, x_2, x_3, \cdots\cdots\cdots)$ be one of the points.

Now the **Hessian matrix** is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \cdots\cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \cdots\cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_3^2} & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{bmatrix}$$

Then find $[H]_p$

The **sufficient conditions** are

If $[H]_p$ is **positive definite** then **minima** occurs at point P .

If $[H]_p$ is **negative definite** then **maxima** occurs at point P .

Note :

1) A matrix H is said to be positive definite, if the values of all principle minor determinants of H are positive..
2) A matrix H is said to be negative definite, if the values of all principle minor determinants of H are alternately positive and negative (starting with negative).

Part-II : Constrained Problems

A) Equality Constraints (Lagrange's Multiplier Method)

1) Optimize $z = f(x)$

   subject to $g_i(x) = 0$

   where, $x = x_1, x_2, \cdots\cdots\cdots, x_n \leftarrow$ 'n' No. of variables

   $g = g_1, g_2, \cdots\cdots\cdots, g_m \leftarrow$ 'm' No. of constraints

   $L(x, \lambda) = f(x) - \lambda_i g_i(x)$ is called Lagrange's function.

2) The **necessary condition** for z to have extreme values are

   i) $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0$ and so on

   ii) $\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \frac{\partial L}{\partial \lambda_1} = 0,\ \frac{\partial L}{\partial \lambda_2} = 0$ and so on

3) Solving these equations, we get different sets of values of $x_1, x_2, x_3, \cdots\cdots\cdots$.

4) Let $P_0 = (x_1, x_2, \cdots\cdots\cdots)$ be one of the points. At the point P_0, find the **bordered Hessian matrix**

   given by $H_B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(m+n)\times(m+n)}$

   where,

   $$P = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \\ \nabla g_m(x) \end{bmatrix}_{m\times n}$$

   $P^T =$ Transpose of P

   $Q =$ Hessian matrix

   $O =$ Null matrix

   Also, find the values of $(-1)^m$ and $(-1)^{m+1}$

5) **Sufficient Condition**

   a) If starting with the principle minor determinant of order $(2m+1)$, the last $(n-m)$ principle minor determinants of H_B have the same sign of $(-1)^m$, then, there is minimum value.

   b) If starting with the principle minor determinant of order $(2m+1)$, the last $(n-m)$ principle minor determinants of H_B have an alternate sign pattern starting with a sign of $\left|(-1)^{m+1}\right|$, then, there is maximum value.

B) Inequality Constraints (Kuhn-Tucker's Conditions)

Optimize $z = f(x)$

subject to $g_m(x_n) \le 0$

1) $\nabla f(x) - \lambda_i \nabla g_i(x) = 0$
2) $\lambda_i g_i(x) = 0,\quad i = 1, 2, 3, \cdots\cdots, m$
3) $g_m(x_n) \le 0$
4) All $\lambda_i \ge 0$ for max

   $\lambda_i \le 0$ for min