Higher Order Linear D.Eq^n with const. coefficients.

A.D.Eq^n is of the form

$$P_0 \frac{d^ny}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_n y = X \quad \text{—— (1)}$$

Here, $P_0, P_1, \ldots P_n$ are const.
X is fun^n of x.

$\frac{d}{dx} = D \qquad \left(\frac{d}{dx} y = Dy \right.$

$$P_0 D^n y + P_1 D^{n-1} y + \cdots + P_n y = X$$

$$(P_0 D^n + P_1 D^{n-1} + \text{——} + P_n)\, y = X$$

$$f(D)\, y = X \quad \text{—— (2)}$$

general sol^n of (2) is given by

y = complementary fun^n + Particular Integral
CF PI

$$y = y_c + y_p$$

* C.F. y_c

Consider $X = 0$ in (2)

$$f(D)y = 0$$

Auxiliary eq^n is. $f(D) = 0$

Then find roots of A.eq^n.
& C.F is depends on nature of the roots.

(1) Roots are real & distinct.

Consider $m_1, m_2, \ldots m_n$ are all real & distinct roots
then $y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \text{——} + C_n e^{m_n x}$

eg. suppose $2, -2, 3$ are roots

$$y_c = C_1 e^{2x} + C_2 e^{-2x} + C_3 e^{3x}$$

(2) Roots are real & repeated.

Consider m_1 is repeated thrice & remaining roots are m_2, m_3, m_4

$$y_c = C_1 e^{m_1 x} + C_2 x e^{m_1 x} + C_3 x^2 e^{m_1 x} + C_4 e^{m_2 x} + C_5 e^{m_3 x} + C_6 e^{m_4 x}$$

eg. consider roots are $2, 2, 2, 2, 3$

$$y_c = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + C_4 x^3 e^{2x} + C_5 e^{3x}$$

(3) Complex & distinct roots: say $a \pm ib$, $(m_1 = a+ib,\ m_2 = a-ib)$

$$y_c = e^{ax}(C_1 \cos bx + C_2 \sin bx)$$

$e^{m_1 x}$

$e^{(a+ib)x}$

$e^{ax} \cdot e^{ibx}$

$e^{ax}(C_1 \cos bx + C_2 \sin bx)$

$+ e^{ax}(C_3 \cos bx + C_4 \sin bx)$

$e^{ax}((C_1 + C_3)\cos bx + (C_2 + C_4)\sin bx)$

(4) Roots are complex & repeated.

Consider $m_1 = m_2 = a + ib$
$m_3 = m_4 = a - ib$

$$y_c = e^{ax}(C_1 \cos bx + C_2 \sin bx) + x e^{ax}(C_3 \cos bx + C_4 \sin bx)$$

eg. suppose roots are $2 \pm 3i$, $2 \pm 3i$

$a = 2,\ b = 3$

$$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + e^{2x} x (C_3 \cos 3x + C_4 \sin 3x).$$

① $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

$D = \frac{d}{dx}$

$D^3y - 6D^2y + 11Dy - 6y = 0.$

$(D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary eq^n · $D^3 - 6D^2 + 11D - 6 = 0$

$D = 1, 3, 2$

$y_c = c_1e^{x} + c_2e^{3x} + c_3e^{2x}$

② $(D^4 - 18D^2 + 81)y = 0$

A. eq^n $= D^4 - 18D^2 + 81 = 0$

$D^2 = m$

$m^2 - 18m + 81 = 0$

$m = 9, 9$

$D^2 = 9$

$D = \pm\sqrt{9}$

$D^2 = 9, 9$

$D = \pm 3, \pm 3$

$D = 3, 3, -3, -3$

$y_c = c_1e^{3x} + c_2xe^{3x} + c_3e^{-3x} + c_4xe^{-3x}$

(3) $$\frac{d^3y}{dx^3} + y = 0.$$

$$D^3 y + y = 0$$

$$(D^3+1)y = 0$$

A. eqⁿ $D^3 + 1 = 0$

$$D = -1, \quad \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

a b

$$y_c = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

(4) $$(D^4+4)y = 0.$$

A eqⁿ. $D^4 + 4 = 0.$

$$(D^2)^2 + 2^2 = 0$$

$$\underbrace{(D^2)^2 + 2D^2 \cdot 2 + 2^2} - 2D^2 \cdot 2 = 0$$

$$(D^2+2)^2 - (2D)^2 = 0$$

$$(D^2+2-2D)(D^2+2+2D) = 0$$

$$(D^2-2D+2)(D^2+2D+2) = 0$$

a=1 b=1

$$D = 1 \pm i, \quad -1 \pm i$$

$$1 + i\,1$$

$$-1 + i$$

$$y_c = e^{x}(c_1 \cos x + c_2 \sin x)$$
$$+ e^{-x}(c_3 \cos x + c_4 \sin x)$$

$$(D^2+1)^3 (D^2+D+1)^2 y = 0.$$

A. eqn $(D^2+1)^3 (D^2+D+1)^2 = 0.$

$D^2+1=0$ or $D^2+D+1=0$

$D^2=-1$ $\quad D = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$D=\pm i$

$$D = \pm i, \pm i, \pm i, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\overset{0}{i}$

$a=-1/2 \quad b=\frac{\sqrt{3}}{2}$

$a=0$

$b=1$

$-\frac{a}{2} \pm i\frac{\sqrt{3}}{2}$

$$y_c = e^{0x}(c_1\cos x + c_2\sin x)$$
$$+ x(c_3\cos x + c_4\sin x)$$
$$+ x^2(c_5\cos x + c_6\sin x)$$
$$+ e^{-1/2 x}\left(c_7\cos\frac{\sqrt{3}}{2}x + c_8\sin\frac{\sqrt{3}}{2}x\right)$$
$$+ x e^{-1/2 x}\left(c_9\cos\frac{\sqrt{3}}{2}x + c_{10}\sin\frac{\sqrt{3}}{2}x\right)$$

$$(D-1)^4 (D^2+2D+2)^2 y = 0$$

$$(D-1)^4 \cdot (D^2+2D+2)^2 = 0$$

$$(D-1)^4 = 0 \quad \text{or} \quad (D^2+2D+2)^2 = 0$$

$$D-1 = 0 \qquad D^2+2D+2 = 0$$

$$D = 1 \qquad D = -1 \pm i$$

$$\therefore D = 1, 1, 1, 1, -1 \pm i, -1 \pm i$$

$-1+i$

$a = -1, b = 1$

$$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x$$
$$+ e^{-x}(c_5 \cos x + c_6 \sin x)$$
$$+ e^{-x} x (c_7 \cos x + c_8 \sin x)$$

★ particular Integral (y_p) / P. I.

$$D = \frac{d}{dx}$$

$$Dy \quad \cancel{\frac{y}{D}} \qquad \cancel{D \quad \frac{1}{D}}$$

$$f(D)y = X$$

$$y_p = \frac{1}{f(D)} X$$

Note $\quad D = \frac{d}{dx} \qquad Dy = \frac{d}{dx} y$

$$\frac{1}{D} X = \int X \, dx$$

① If $X = 0$ $\qquad y_p = \frac{1}{f(D)} X = 0$

② If $X = e^{ax}$

$$y_p = \frac{1}{f(D)} X = \frac{1}{f(D)} e^{ax}$$

$$= \begin{cases} \frac{1}{f(a)} e^{ax} & \text{— } f(a) \neq 0 \\ x \frac{1}{f'(a)} e^{ax} & \text{— } f(a) = 0, \; f'(a) \neq 0. \end{cases}$$

eg. $\quad f(D) = D^2 - 4$

$$y_p = \frac{1}{D^2-4} e^{3x} = \frac{1}{9-4} e^{3x} = \frac{1}{5} e^{3x}$$

$$y_p = \frac{1}{D^2-4} e^{2x} = x \frac{1}{2D} e^{2x} = \frac{x}{4} e^{2x}$$

1. $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-3x/2} + 2^x + 3.$

$\frac{d}{dx} = D$

$6D^2y + 17Dy + 12y = e^{-3x/2} + e^{\log 2^x} + 3e^{0x}$

$(6D^2 + 17D + 12)\,y = e^{-3x/2} + e^{x(\log 2)} + 3e^{0x}$

$\Rightarrow f(D)y = X$

Consider, $(6D^2 + 17D + 12)y = 0$

A eqn $6D^2 + 17D + 12 = 0$

$D = -4/3, -3/2$

$y_c = c_1 e^{-4/3x} + c_2 e^{-3/2x}$

$y_p = \frac{1}{f(D)} X = \frac{1}{6D^2+17D+12}\left(e^{-3x/2} + e^{(\log 2)x} + 3e^{0x}\right)$

$y_p = \frac{1}{6D^2+17D+12} e^{-3x/2} + \frac{1}{6D^2+17D+12} e^{(\log 2)x} + \frac{1}{6D^2+17D+12} 3e^{0x}$

as $f(-3/2) = 0$

$= x \frac{1}{12D+17} e^{-3x/2} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} e^{(\log 2)x} + 3 \frac{1}{12} e^{0x}$

$= x \frac{1}{12(-3/2)+17} e^{-3x/2} + \frac{e^{\log 2^x}}{6(\log 2)^2 + 17\log 2 + 12} + \frac{1}{4}$

$= \frac{x}{(-1)} e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12} + \frac{1}{4}$

$y = y_c + y_p$

$= c_1 e^{-4/3x} + c_2 e^{-3/2x} - x e^{-3x/2} + \frac{2^x}{6(\log 2)^2 + 17\log 2 + 12} + \frac{1}{4}$

(2)

$$\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh^2 2x$$

$$D^3y - 4Dy = 2\left(\frac{e^{2x}+e^{-2x}}{2}\right)^2$$

$$(D^3-4D)y = 2\left(\frac{e^{4x}+2e^{2x}.e^{-2x}+e^{-4x}}{4}\right)$$

$$(D^3-4D)y = \frac{1}{2}\left[e^{4x}+2+e^{-4x}\right]$$

$$f(D)y = X$$

Consider $(D^3-4D)y=0$

A. eqn $D^3-4D=0$

$$D(D^2-4)=0$$

$$D=0 \quad D=\pm 2$$

$$y_c = c_1e^{0x} + c_2e^{2x} + c_3e^{-2x}$$

$$y_p = \frac{1}{f(D)}X$$

$$= \frac{1}{D^3-4D}\frac{1}{2}(e^{4x}+2+e^{-4x})$$

$$= \frac{1}{2}\left[\frac{1}{D^3-4D}e^{4x} + \frac{1}{D^3-4D}2e^{0x} + \frac{1}{D^3-4D}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4^3-4^2}e^{4x} + 2.x\frac{1}{3D^2-4}e^{0x} + \frac{1}{(-4)^3+4^2}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{48} + 2x\frac{1}{(-4)}e^{0x} + \frac{1}{(-48)}e^{-4x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{48} - \frac{x}{2} - \frac{e^{-4x}}{48}\right]$$

$$y = y_c + y_p$$

$$= c_1 + c_2e^{2x} + c_3e^{-2x} + \frac{1}{2}\left[\frac{e^{4x}}{48} - \frac{x}{2} - \frac{e^{-4x}}{48}\right]$$ ✓

(2) P.I if $X = \sin(ax+b)$ / $\cos(ax+b)$

$$y_p = \frac{1}{f(D)} X$$

$$= \frac{1}{f(D)} \sin(ax+b).$$

write $f(D) = \phi(D^2)$

replace every D^2 by $-a^2$

$$= \frac{1}{\phi(-a^2)} \sin(ax+b) \quad \text{———} \quad \phi(-a^2) \neq 0$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax+b) \quad \text{If } \phi(-a^2) = 0.$$

$\hookrightarrow$ replace D^2 by $-a^2$

$(D^2-5D+6)y = \sin 3x \Rightarrow f(D)y = X$

Consider $(D^2-5D+6)y=0$

A. eqns is $D^2-5D+6=0$

$D = 2, 3$

$y_c = C_1 e^{2x} + C_2 e^{3x}$

$y_p = \frac{1}{f(D)} X$

$= \frac{1}{D^2-5D+6} \sin 3x$

replace D^2 by -9

$= \frac{1}{-9-5D+6} \sin 3x$

$= \frac{1}{-5D-3} \sin 3x.$

$= -\frac{1}{5D+3} \sin 3x.$

$= -\frac{1}{5D+3} \frac{(5D-3)}{(5D-3)} \sin 3x$

$= -\frac{(5D-3)}{25D^2-9} \sin 3x.$

D^2 by -9

$= -\frac{(5D-3)}{25(-9)-9} \sin 3x$

$= -\frac{(5D-3)}{-234} \sin 3x$

$= \frac{(5D-3)}{234} \sin 3x.$

$= \frac{5D(\sin 3x) - 3\sin 3x}{234}$ $\quad D = \frac{d}{dx}$

$= \frac{5(\cos 3x)3 - 3\sin 3x}{234}$

$= \frac{\cancel{3}(5\cos 3x - \sin 3x)}{\cancel{234}\ 78}$

$y = y_c + y_p$

$= C_1 e^{2x} + C_2 e^{3x} + \frac{5\cos 3x - \sin 3x}{78}$

$(D^3+D^2+D+1)\,y = \sin^2 x.$ $\qquad f(D)\,y = X$

Consider, $(D^3+D^2+D+1)\cdot y = 0$

A eqn is $\quad D^3+D^2+D+1 = 0$

$D = -1, \pm i \qquad i \Rightarrow a=0,\ b=1$

$y_c = C_1 e^{-x} + e^{0x}(C_2\cos x + C_3 \sin x)$

$1+\cos 2\theta = 2\cos^2\theta$
$1-\cos 2\theta = 2\sin^2\theta$

$$y_p = \frac{1}{f(D)} X$$
$$= \frac{1}{D^3+D^2+D+1} \sin^2 x$$
$$= \frac{1}{D^3+D^2+D+1}\left(\frac{1-\cos 2x}{2}\right)$$
$$= \frac{1}{2}\frac{1}{D^3+D^2+D+1}\left(e^{0x} - \cos 2x\right)$$
$$= \frac{1}{2}\left[\frac{1}{D^3+D^2+D+1}e^{0x} - \frac{1}{D^3+D^2+D+1}\cos 2x\right]$$
$$= \frac{1}{2}\left[\underbrace{\frac{1}{0+1}}_{\text{replace } D \text{ by } 0}e^{0x} - \underbrace{\frac{1}{D(-4)+(-4)+D+1}}_{\text{replace } D^2 \text{ by } -4}\cos 2x\right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{-3D-3}\cos 2x\right]$$
$$= \frac{1}{2}\left[1 + \frac{1}{3(D+1)}\cos 2x\right]$$
$$= \frac{1}{2}\left[1 + \frac{1}{3}\frac{(D-1)}{(D+1)(D-1)}\cos 2x\right]$$
$$= \frac{1}{2}\left[1 + \frac{1}{3}\frac{(D-1)}{D^2-1}\cos 2x\right]$$

$\longrightarrow D^2$ by -4

$$= \frac{1}{2}\left[1 + \frac{1}{3}\frac{(D-1)}{(-4-1)}\cos 2x\right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{15}\left(D(\cos 2x) - \cos 2x\right)\right]$$
$$= \frac{1}{2} - \frac{1}{30}\left(-\sin 2x\,(2) - \cos 2x\right)$$
$$= \frac{1}{2} + \frac{1}{30}\left(+2\sin 2x + \cos 2x\right)$$

$$y = y_c + y_p$$
$$= C_1 e^{-x} + (C_2\cos x + C_3 \sin x) + \frac{1}{2} + \frac{1}{30}(2\sin 2x + \cos 2x)$$

$(D-1)^2(D^2+1)y = e^x + \sin^2 x/2$

$f(D)y = X.$

Consider $(D-1)^2(D^2+1)y = 0.$

A.$\cdot$ $(D-1)^2(D^2+1) = 0$.

$\cancel{D^2+1=0}$.

$(D-1)^2 = 0$ or $D^2+1=0$

$D = 1, 1$ $D^2 = -1$

$\Rightarrow D = \pm i$

$i \rightarrow a=0, b=1$

$y_c = C_1e^x + C_2xe^x + e^{0x}(C_3\cos x + C_4\sin x)$

$$y_p = \frac{1}{f(D)}X$$

$$= \frac{1}{(D-1)^2(D^2+1)}(e^x + \sin^2 x/2)$$

$$= \frac{1}{(D-1)^2(D^2+1)}e^x + \frac{1}{(D-1)^2(D^2+1)}\sin^2 x/2$$

$$\frac{1}{(D-1)^2\underbrace{(D^2+1)}}e^x = \frac{1}{(D-1)^2(1+1)}e^x$$

replacing D by 1

$$= \frac{1}{2(D-1)^2}e^x$$

$$= x\frac{1}{2\times 2(D-1)}e^x$$

$$= \frac{x\,x}{4}\frac{1}{(1)}e^x = \frac{x^2}{4}e^x. \quad \text{——} (1)$$

$$\frac{1}{(D-1)^2(D^2+1)}\sin^2 x/2 = \frac{1}{(D-1)^2(D^2+1)}\left(\frac{1-\cos x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)^2(D^2+1)}e^{0x} - \frac{1}{(D-1)^2(D^2+1)}\cos x\right]$$

Replacing D by 0

$$= \frac{1}{2}\left[\frac{1}{1(1)}e^{0x} - \frac{1}{(D-1)^2}\left(\frac{1}{D^2+1}\cos x\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{(D^2-2D+1)}\frac{1}{(D^2+1)}\cos x\right]$$

D^2 by -1

$$= \frac{1}{2}\left[1 - \frac{1}{(D^2+1)}\frac{1}{((-\cancel{1})-2D+\cancel{1})}\cos x\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{D^2+1}\frac{1}{(-2D)}\cos x\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\frac{1}{(D^2+1)}\int \cos x\,dx\right] \qquad \frac{1}{D} = \int$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\frac{1}{D^2+1}\sin x\right]$$

$D^2 = -1$ give $D^2 = 0$

$$= \frac{1}{2}\left[1 + \frac{1}{2}x\frac{1}{2D}\sin x\right]$$

$$= \frac{1}{2}\left[1 + \frac{x}{4}\int \sin x\,dx\right] = \frac{1}{2}\left[1 + \frac{x}{4}(-\cos x)\right].$$

$$y_p = \frac{x^2}{4}e^x + \frac{1}{2} - \frac{x}{8}\cos x$$

$f(D)y = X$ $D = \frac{d}{dx}$

$f(D)y = 0$ $\longrightarrow$ $y = y_c + y_p$

A eqn $f(D) = 0$

$y_p = \frac{1}{f(D)} X$

$y_p = \frac{1}{f(D)} e^{ax}$

$\longrightarrow$ D by a

$f(a) = 0$

$= x \frac{1}{f'(D)} e^{ax}$

$\longrightarrow$ D by a $f'(a) \neq 0$

$y_p = \frac{1}{f(D)} \cos(ax+b) / \sin(ax+b)$

$\longrightarrow$ D^2 by $-a^2$

$= \frac{1}{\phi(D^2)} \cos(ax+b)$

$\longrightarrow$ D^2 by $-a^2$ $\longrightarrow$ $\phi(-a^2) \neq 0$

If $\phi(-a^2) = 0$

$= x \frac{1}{\phi'(D^2)} \cos(ax+b)$

To find P.I (y_p) when $X = x^m$ where m is +ve int

$$y_p = \frac{1}{f(D)} X = \frac{1}{f(D)} x^m$$

① write f(D) in ascending powers of D

② write f(D) as $1 + \phi(D)$

$$y_p = \frac{1}{1+\phi(D)} x^m$$

Apply following formulae on $\frac{1}{1+\phi(D)}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \text{ ———}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \text{ ———}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 \text{ ———}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 \text{ ———}$$

$$\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2-5x+2$$

$D = \frac{d}{dx}$

$$D^3y - 2Dy + 4y = 3x^2-5x+2$$

$$(D^3-2D+4)y = 3x^2-5x+2$$

$2e^{0x}$

$$f(D)y = X$$

$\sin a x + 2e^{0x}$ $\cancel{2x^0}$

consider $(D^3-2D+4)y = 0$

A. eqn $D^3-2D+4=0$

$$D = -2, \quad 1 \pm i$$

$1+i \to a=1$, $\to b=1$

$$y_c = C_1e^{-2x} + e^{x}(C_2\cos x + C_3 \sin x)$$

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^3-2D+4}(3x^2-5x+2)$$

$$= \frac{1}{4-2D+D^3}(3x^2-5x+2)$$

$$= \frac{1}{4\left(1-\frac{2D}{4}+\frac{D^3}{4}\right)}(3x^2-5x+2)$$

$$= \frac{1}{4}\frac{1}{\left[1-\left(\frac{D}{2}-\frac{D^3}{4}\right)\right]}(3x^2-5x+2)$$

$D(3x^2-5x+2) = 6x-5$

$D^2(3x^2-5x+2) = 6$

$D^3(3x^2-5x+2) = 0$

$$= \frac{1}{4}\left[1+\left(\frac{D}{2}-\frac{D^3}{4}\right)+\left(\frac{D}{2}-\frac{D^3}{4}\right)^2 - \right](3x^2-5x+2)$$

$$= \frac{1}{4}\left[1+\frac{D}{2}+\frac{D^2}{4}\right](3x^2-5x+2)$$

$$= \frac{1}{4}\left[3x^2-5x+2+\frac{D}{2}(3x^2-5x+2)+\frac{D^2}{4}(3x^2-5x+2)\right]$$

$$= \frac{1}{4}\left[3x^2-5x+2+\frac{1}{2}(6x-5)+\frac{1}{4}(6)\right]$$

$$= \frac{1}{4}\left[3x^2-5x+2+3x-\frac{5}{2}+\frac{3}{2}\right]$$

$$= \frac{1}{4}\left[3x^2-2x+1\right]$$

$$y = y_c + y_p = C_1e^{-2x} + e^{x}(C_2\cos x + C_3\sin x) + \frac{1}{4}(3x^2-2x+1)$$

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = \cos t + t^2 + 3 \qquad D = \frac{d}{dt}$$

$$D^3y + Dy = \cos t + t^2 + 3$$

$$(D^3 + D)y = \cos t + t^2 + 3$$

$$f(D)y = X$$

Consider $(D^3+D)Y = 0$ A eq^n $D^3 + D = 0$

$D = 0, \pm i$ $\quad i \begin{cases} a = 0 \\ b = 1 \end{cases}$

$$y_c = C_1 e^{0t} + e^{0t}(C_2 \cos t + C_3 \sin t)$$

$$y_p = \frac{1}{f(D)} X = \frac{1}{D^3+D}(\cos t + t^2 + 3)$$

$$= \frac{1}{D^3+D}\cos t + \frac{1}{D^3+D}(t^2+3)$$

$$= I + II$$

$$I = \frac{1}{D^3+D}\cos t$$

D^2 by $-1^2 = -1$

$D^3 + D = D^2 \cdot D + D$
$= -D + D = 0$

$$= t\frac{1}{3D^2+1}\cos t$$

$\xrightarrow{\quad}$ D^2 by $-1^2 = -1$

$$= t\frac{1}{3(-1)+1}\cos t = \frac{t}{-2}\cos t$$

$$II = \frac{1}{D^3+D}(t^2+3) = \frac{1}{D+D^3}(t^2+3)$$

$$= \frac{1}{D}\frac{1}{(1+D^2)}(t^2+3)$$

$D(t^2+3)$
$= 2t$
$D^2(t^2+3)$
$= 2$

$$= \frac{1}{D}(1 - D^2 + (D^2)^2 \_\_\_\_)(t^2+3)$$

$$= \frac{1}{D}(1 - D^2)(t^2+3)$$

$$= \frac{1}{D}\left[(t^2+3) - D^2(t^2+3)\right]$$

$$= \frac{1}{D}\left[t^2 + 3 - 2\right] = \frac{1}{D}(t^2+1) = \int (t^2+1)\,dt$$

$$= \frac{t^3}{3} + t$$

$$y_p = I + II = -\frac{t}{2}\cos t + \frac{t^3}{3} + t$$

$$y = y_c + y_p = C_1 + (C_2\cos t + C_3 \sin t) - \frac{t}{2}\cos t + \frac{t^3}{3} + t$$

HW $\quad (D^3 - D^2 - 6D)Y = x^2 + 1$

4) To find P.I when $X = e^{ax} V$

where V is funⁿ of x.

$$y_p = \frac{1}{f(D)} X$$

$$= \frac{1}{f(D)} e^{ax} V$$

$$= e^{ax} \frac{1}{f(D+a)} V$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = (x^2e^x)^2$$

$$D = \frac{d}{dx}.$$

$$D^2y - 4Dy + 3y = x^4e^{2x}$$
$$(D^2-4D+3)y = x^4e^{2x}$$
$$f(D)y = X$$

$$(D^2-4D+3)y = 0$$

A.eqn $\quad D^2-4D+3=0$

$$D = 3, 1$$
$$y_c = C_1e^{3x} + C_2e^x$$

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^2-4D+3}e^{2x}\cdot x^4$$

$a=2$

$$= e^{2x}\frac{1}{(D+2)^2-4(D+2)+3}\,x^4$$

$$\frac{1}{1-x} = 1+x+x^2 \text{——}$$

$$= e^{2x}\frac{1}{D^2+\cancel{4D}+4-\cancel{4D}-8+3}\,x^4$$

$$= e^{2x}\frac{1}{(D^2-1)}\,x^4$$

$$= e^{2x}\frac{1}{-1+D^2}\,x^4$$

$$= -e^{2x}\frac{1}{1-D^2}\,x^4$$

$$= -e^{2x}\left(1+D^2+(D^2)^2 \text{——}\right)x^4$$

$$D(x^4) = 4x^3$$
$$D^2(x^4) = 12x^2$$
$$D^3(x^4) = 24x$$
$$D^4(x^4) = 24$$
$$D^5(x^4) = 0$$

$$= -e^{2x}\left(x^4 + D^2(x^4) + D^4(x^4)\right)$$

$$y_p = -e^{2x}(x^4 + 12x^2 + 24)$$

$$y = y_c + y_p$$
$$= C_1e^{3x} + C_2e^x - e^{2x}(x^4+12x^2+24)$$

$(D^3+1)y = e^{x/2}\sin\left(\frac{\sqrt3}{2}x\right)$

$f(D)y = X$

$(D^3+1)y = 0$

A. eqn $\quad D^3+1=0$

$D = -1,\ \frac{1}{2} \pm i\frac{\sqrt3}{2}$ $\qquad \frac{1}{2}+i\frac{\sqrt3}{2} \to a = \frac{1}{2},\ b = \frac{\sqrt3}{2}$

$y_c = C_1 e^{-x} + e^{\frac{1}{2}x}\left(C_2\cos\frac{\sqrt3}{2}x + C_3\sin\frac{\sqrt3}{2}x\right).$

$y_p = \frac{1}{f(D)}X = \frac{1}{D^3+1}e^{x/2}\sin\left(\frac{\sqrt3}{2}x\right) \quad a=\frac{1}{2}$

$= e^{x/2}\frac{1}{(D+\frac{1}{2})^3+1}\sin\left(\frac{\sqrt3}{2}x\right)$

$= e^{x/2}\frac{1}{D^3+3\frac{D^2}{2}+\frac{3D}{4}+\frac{1}{8}+1}\sin\left(\frac{\sqrt3}{2}x\right)$

$= e^{x/2}\frac{1}{D^3+\frac{3D^2}{2}+\frac{3D}{4}+\frac{9}{8}}\sin\left(\frac{\sqrt3}{2}x\right)$

D^2 by $-\left(\frac{\sqrt3}{2}\right)^2 = \frac{-3}{4}$

$-\frac{3}{4}D+\frac{3}{2}\left(-\frac{3}{4}\right)+\cancel{\frac{3D}{4}}+\frac{9}{8}$

$= 0$

$= e^{x/2}x\frac{1}{3D^2+\frac{6D}{2}+\frac{3}{4}}\sin\left(\frac{\sqrt3}{2}x\right)$

D^2 by $-\frac{3}{4}$

$= e^{x/2}x\frac{1}{3\left(-\frac{3}{4}\right)+3D+\frac{3}{4}}\sin\left(\frac{\sqrt3}{2}x\right)$

$= xe^{x/2}\frac{1}{3D-\frac{6}{4}\frac{3}{2}}\sin\left(\frac{\sqrt3}{2}x\right)$

$= \frac{xe^{x/2}}{3}\frac{1}{(D-\frac{1}{2})}\times\frac{(D+\frac{1}{2})}{(D+\frac{1}{2})}\sin\frac{\sqrt3}{2}x$

$= \frac{xe^{x/2}}{3}\frac{(D+\frac{1}{2})}{(D^2-\frac{1}{4})}\sin\frac{\sqrt3}{2}x$

$D^2 \longrightarrow -\frac{3}{4}$

$= \frac{xe^{x/2}}{3}\frac{\left(D\sin\frac{\sqrt3}{2}x + \frac{1}{2}\sin\frac{\sqrt3}{2}x\right)}{-\frac{3}{4}-\frac{1}{4}}$

$= \frac{xe^{x/2}}{3}\frac{(\cos\frac{\sqrt3}{2}x)\frac{\sqrt3}{2}+\frac{1}{2}\sin\frac{\sqrt3}{2}x}{(-1)}$

$= \frac{-xe^{x/2}}{3\times2}\left(\sqrt3\cos\left(\frac{\sqrt3}{2}x\right)+\sin\left(\frac{\sqrt3}{2}\right)x\right)$

$y = y_c + y_p$

$= C_1e^{-x}+e^{x/2}\left(C_2\cos\frac{\sqrt3}{2}x + C_3\sin\frac{\sqrt3}{2}x\right) - \frac{xe^{x/2}}{6}\left(\sqrt3\cos\frac{\sqrt3}{2}x+\sin\frac{\sqrt3}{2}x\right)$

$$(D^2+2)y = e^x\cos x + x^2e^{3x}.$$

$$f(D)y = X$$

$$(D^2+2)y = 0$$

A eqn $D^2+2=0 \Rightarrow D^2=-2$

$$\Rightarrow D = \pm i\sqrt{2}$$

$$y_c = e^{0x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$$

$i\sqrt{2}$ → $a=0$, $b=\sqrt{2}$

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^2+2}(e^x\cos x + e^{3x}x^2)$$

$$= \frac{1}{D^2+2}e^x\cos x + \frac{1}{D^2+2}e^{3x}x^2$$

$$= \mathrm{I} + \mathrm{II}$$

$$\mathrm{I} = \frac{1}{D^2+2}e^x\cos x \quad \to a=1$$

$$= e^x\frac{1}{(D+1)^2+2}\cos x$$

$$= e^x\frac{1}{D^2+2D+1+2}\cos x$$

$$= e^x\frac{1}{D^2+2D+3}\cos x \longrightarrow D^2 \text{ by } -1^2=-1$$

$$= e^x\frac{1}{-1+2D+3}\cos x$$

$$= e^x\frac{1}{2D+2}\cos x = \frac{e^x}{2}\frac{1}{(D+1)}\frac{(D-1)}{(D-1)}\cos x$$

$$= \frac{e^x}{2}\frac{(D-1)}{D^2-1}\cos x = \frac{e^x}{2}\frac{(D\cos x - \cos x)}{(-1-1)}$$

$$D^2=-1$$

$$= -\frac{e^x}{4}(-\sin x - \cos x) = \frac{e^x}{4}(\sin x + \cos x)$$

$$\mathrm{II} = \frac{1}{D^2+2}e^{3x}x^2 = e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$= e^{3x}\frac{1}{D^2+6D+9+2}x^2$$

$$= e^{3x}\frac{1}{11+6D+D^2}x^2$$

$$= \frac{e^{3x}}{11}\frac{1}{1+\left(\frac{6D}{11}+\frac{D^2}{11}\right)}x^2$$

x^2

$D(x^2)=2x$

$D^2(x^2)=2$

$$= \frac{e^{3x}}{11}\left[1-\left(\frac{6D}{11}+\frac{D^2}{11}\right)+\left(\frac{6D}{11}+\frac{D^2}{11}\right)^2 \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}+\frac{25D^2}{121}\right]x^2$$

$$= \frac{e^{3x}}{11}\left[x^2-\frac{6}{11}D(x^2)+\frac{25}{121}D^2(x^2)\right]$$

$$= \frac{e^{3x}}{11}\left(x^2-\frac{6}{11}(2x)+\frac{25}{121}(2)\right)$$

$$y_p = \mathrm{I}+\mathrm{II} = \frac{e^x}{4}(\sin x+\cos x) + \frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

$$y = y_c+y_p = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^x}{4}(\sin x+\cos x) + \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$(D^2-1)\,y = x^2 \sin 3x$

$f(D)\,y = X$

Consider, $(D^2-1)\,y = 0$

A eq^n $D^2-1=0$

$D^2 = 1 \Rightarrow D = \pm 1$

$y_c = c_1 e^{x} + c_2 e^{-x}$

$y_p = \frac{1}{f(D)} X = \frac{1}{D^2-1} x^2 \sin 3x$

$e^{i3x} = \cos 3x + i \sin 3x$

$x^2 e^{i3x} = x^2 \cos 3x + i\, x^2 \sin 3x$

$\frac{1}{D^2-1} x^2 e^{i3x} = \frac{1}{D^2-1} x^2 \cos 3x + i \frac{1}{D^2-1} x^2 \sin 3x$

$\Downarrow$

$A + iB$

$y_p = \frac{1}{D^2-1} x^2 \sin 3x$

$= IP\left[\frac{1}{D^2-1} x^2 e^{i3x}\right] \qquad a = i3$

$\frac{1}{f(D)} e^{ax} V$

$= e^{ax} \frac{1}{f(D+a)} V$

$\frac{1}{1-x} = 1 + x + x^2 \text{---}$

$D(x^2) = 2x$

$D^2(x^2) = 2$

$= IP\left[e^{i3x} \frac{1}{(D+3i)^2-1} x^2\right]$

$= IP\left[e^{i3x} \frac{1}{D^2+6Di-9-1} x^2\right]$

$= IP\left[e^{i3x} \frac{1}{-10+6Di+D^2} x^2\right]$

$= IP\left[\frac{e^{i3x}}{-10} \frac{1}{1-\left(\frac{6Di}{10}+\frac{D^2}{10}\right)} x^2\right]$

$= IP\left[\frac{e^{i3x}}{-10}\left(1+\frac{6Di}{10}+\frac{D^2}{10}+\left(\frac{6Di}{10}+\frac{D^2}{10}\right)^2 \text{---}\right) x^2\right]$

$= IP\left[\frac{e^{i3x}}{-10}\left(1+\frac{6Di}{10}+\frac{D^2}{10}-\frac{36D^2}{100}\right) x^2\right]$

$= IP\left[\frac{e^{i3x}}{-10}\left(x^2+\frac{3i}{5} D(x^2) - \frac{26}{100} D^2(x^2)\right)\right]$

$= IP\left[\frac{e^{i3x}}{-10}\left(x^2+\frac{3i}{5}(2x) - \frac{13}{50}\times 2\right)\right]$ ($\frac{13}{25}$)

$= IP\left[\frac{e^{i3x}}{-10}\left(\left(x^2-\frac{13}{25}\right)+i\frac{6x}{5}\right)\right]$

$= \frac{-1}{10} IP\left[(\cos 3x + i \sin 3x)\left(\left(x^2-\frac{13}{25}\right)+i\frac{6x}{5}\right)\right]$

$= -\frac{1}{10} IP\left[\cos 3x\left(x^2-\frac{13}{25}\right) + i\cos 3x \frac{6x}{5} + i \sin 3x\left(x^2-\frac{13}{25}\right) - \sin 3x \frac{6x}{5}\right]$

$= \frac{-1}{10}\left[\cos 3x \frac{6x}{5} + \sin 3x\left(x^2-\frac{13}{25}\right)\right]$

$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} - \frac{1}{10}\left(\frac{6x}{5}\cos 3x + \left(x^2-\frac{13}{25}\right)\sin 3x\right).$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = xe^{-x}\cos x.$$

$$D = \frac{d}{dx}$$

$$D^2y + 2Dy + y = xe^{-x}\cos x$$

$$(D^2 + 2D + 1)\,y = xe^{-x}\cos x$$

$$(D^2 + 2D + 1)\,y = 0$$

A. eqn : $D^2 + 2D + 1 = 0$

$$(D+1)^2 = 0$$

$$D = -1, -1$$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^2 + 2D + 1}\, x e^{-x}\cos x$$

$a = -1.$

$$= e^{-x}\frac{1}{(D-1)^2 + 2(D-1) + 1}\, x\cos x$$

$$= e^{-x}\frac{1}{D^2 - \cancel{2D} + 1 + \cancel{2D} - 2 + 1}\, x\cos x$$

$$= e^{-x}\frac{1}{D^2}\, x\cos x$$

$$\int u \cdot v = u\int \underbrace{v}_{v_1} - u'\int v_1$$

$$= e^{-x}\frac{1}{D}\int x\cos x \, dx.$$

$$= e^{-x}\frac{1}{D}\left[x\sin x - (1)(-\cos x)\right]$$

$$= e^{-x}\int (x\sin x + \cos x)\, dx.$$

$$= e^{-x}\left[x(-\cos x) - (1)(-\sin x) + \sin x\right]$$

$$= e^{-x}\left[-x\cos x + 2\sin x\right]$$

$$y = y_c + y_p = c_1 e^{-x} + c_2 x e^{-x} + e^{-x}(-x\cos x + 2\sin x)$$

If X does not belong to any of the earlier method.

$$\frac{1}{D} X = \int X \, dx$$

$$\frac{1}{D-a} X = e^{ax} \int e^{-ax} X \, dx$$

$$\frac{1}{D+a} X = e^{-ax} \int e^{ax} X \, dx$$

$(D^2+a^2)\, y = \sec ax$.

$F(D)\, y = X$

Consider $(D^2+a^2)\, y = 0$

A eqn $D^2+a^2=0 \Rightarrow D = \pm ai$

$D^2 = -a^2$

$y_c = e^{0x}(c_1 \cos ax + c_2 \sin ax)$.

ai — RP 0, IP a

$$y_p = \frac{1}{f(D)} X = \frac{1}{D^2+a^2} \sec ax$$

$$= \frac{1}{(D-ai)(D+ai)} \sec ax$$

$$\frac{1}{\cancel{(D-ai)}(D+ai)} = \frac{A}{D-ai} + \frac{B}{D+ai}$$

$$= \frac{A(D+ai) + B(D-ai)}{\cancel{(D-ai)(D+ai)}}$$

$$1 = A(D+ai) + B(D-ai)$$

$D = ai \Rightarrow 1 = A\, 2ai \Rightarrow A = 1/2ai$

$D = -ai \Rightarrow 1 = -2ai\, B \Rightarrow B = -1/2ai$

$$\frac{1}{(D-ai)(D+ai)} = \frac{1/2ai}{(D-ai)} - \frac{1/2ai}{(D+ai)}$$

$$\frac{1}{D-a} X = e^{ax}\int e^{-ax} X\, dx$$

$$y_p = \frac{1}{(D-ai)(D+ai)} \sec ax.$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai}\sec ax - \frac{1}{D+ai}\sec ax\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int e^{-aix}\sec ax\, dx - e^{-aix}\int e^{aix}\sec ax\, dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int (\cos ax - i\sin ax)\frac{1}{\cos ax}\, dx - e^{-iax}\int (\cos ax + i\sin ax)\frac{1}{\cos ax}\, dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\int (1 - i\tan ax)\, dx - e^{-iax}\int (1 + i\tan ax)\, dx\right]$$

$$= \frac{1}{2ai}\left[e^{iax}\left(x - i\frac{\log(\sec ax)}{a}\right) - e^{-iax}\left(x + i\log\left(\frac{\sec ax}{a}\right)\right)\right]$$

$$y = y_c + y_p$$

$$(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x).$$
$$f(D)\,y = X$$

A.E. $D^2+5D+6=0$
$D=-3,-2$
$y_c = c_1 e^{-3x} + c_2 e^{-2x}$

$$y_p = \frac{1}{f(D)}X = \frac{1}{D^2+5D+6}\,e^{-2x}\sec^2 x(1+2\tan x).$$
$$= \frac{1}{(D+3)\underbrace{(D+2)}_{a=2}}\left(e^{-2x}\sec^2 x(1+2\tan x)\right)$$

$$\frac{1}{D+a}X = e^{-ax}\int e^{ax}X\,dx$$

$$= \frac{1}{D+3}\left[e^{-2x}\int e^{2x}\left(e^{-2x}\sec^2 x(1+2\tan x)\right)dx.\right]$$
$$= \frac{1}{D+3}\left[e^{-2x}\int\left(\sec^2 x + 2\tan x\sec^2 x\right)dx\right]$$
$$= \frac{1}{D+3}e^{-2x}\left[\tan x + \int 2\tan x\sec^2 x\,dx\right]$$

put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$y_p = \frac{1}{D+3}e^{-2x}\left[\tan x + 2\int t\,dt\right] = \frac{1}{D+3}e^{-2x}\left[\tan x + \cancel{2}\frac{t^2}{\cancel{2}}\right]$$

$$y_p = \frac{1}{D+3}\underbrace{e^{-2x}(\tan x + \tan^2 x)}_{a=3}$$
$$= e^{-3x}\int e^{3x}e^{-2x}(\tan x + \sec^2 x - 1)\,dx$$
$$= e^{-3x}\int e^{x}(\underline{\tan x + \sec^2 x} - 1)\,dx$$
$$= e^{-3x}\int\left[e^{x}(\underline{\tan x + \sec^2 x}) - e^{x}\right]dx$$

$$\int e^{ax}(a f(x) + f'(x))\,dx = e^{ax}f(x).$$
$$\int e^{-ax}(a f(x) - f'(x))\,dx = -e^{-ax}f(x)$$

$$y_p = e^{-3x}(e^{x}\tan x - e^{x}) = e^{-3x}e^{x}(\tan x - 1)$$
$$= e^{-2x}(\tan x - 1)$$

$$y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}(\tan x - 1).$$

$$(D^2+D)y = \frac{1}{1+e^x}.$$

$f(D)y = X$

A eqn is $D^2+D=0$

$D = 0, -1$

$$y_c = c_1 e^{0x} + c_2 e^{-x}.$$

$$= c_1 + c_2 e^{-x}$$

$$y_p = \frac{1}{f(D)} X = \frac{1}{D^2+D}\left(\frac{1}{1+e^x}\right)$$

$$= \frac{1}{D(D+1)} \frac{1}{1+e^x}$$

$$= \frac{1}{D+1} \frac{1}{D}\left(\frac{1}{1+e^x}\right) = \frac{1}{D+1}\int \frac{1}{1+e^x}\,dx$$

$$= \frac{1}{D+1}\int \frac{e^{-x}}{(1+e^x)e^{-x}}\,dx = \frac{1}{D+1}\int \frac{e^{-x}}{(e^{-x}+1)}\,dx$$

$$= -\frac{1}{D+1}\int \frac{-e^{-x}}{e^{-x}+1}\,dx = -\frac{1}{D+1}\log(e^{-x}+1)$$

$$\frac{1}{D+a} X = e^{-ax}\int e^{+ax} X\,dx$$

$$= -e^{-x}\int e^x \log(e^{-x}+1)\,dx.$$

$$= -e^{-x}\int \underset{u}{\log(e^{-x}+1)}\ \underset{v}{e^x}\,dx$$

$$\int uv = u\int v - \int \underline{\int v}\, u'$$

$$= -e^{-x}\left[\log(e^{-x}+1)e^x - \int e^x \frac{1}{e^{-x}+1}(-e^{-x})\,dx\right]$$

$$= -e^{-x}\left[e^x \log(e^{-x}+1) + \int \frac{1}{e^{-x}+1}\,dx\right]$$

$$= -e^{-x}\left[e^x \log(e^{-x}+1) + \int \frac{e^x}{(e^{-x}+1)e^x}\,dx\right]$$

$$= -e^{-x}\left[e^x \log(e^{-x}+1) + \int \frac{e^x}{1+e^x}\,dx\right]$$

$$= -e^{-x}\left[e^x \log(e^{-x}+1) + \log(1+e^x)\right]$$

$$y = y_c + y_p = c_1 + c_2 e^{-x} - e^{-x}\left[e^x \log(e^{-x}+1) + \log(1+e^x)\right].$$

$$(D^2-D-2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$$

$$f(D)\,y = X$$

A.eqn $D^2 - D - 2 = 0$

$$D = 2, -1.$$

$$y_c = c_1 e^{2x} + c_2 e^{-x}$$

$$y_p = \frac{1}{f(D)} X = \frac{1}{D^2-D-2}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{(D-2)(D+1)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right).$$

$$\frac{1}{D+a} = e^{-ax}\int e^{ax} X\,dx$$

$$= \frac{1}{D-2}\, e^{-x}\int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{D-2}\, e^{-x}\int e^{x}\left(2\log x + \frac{2}{x} - \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{D-2}\, e^{-x}\int\left[e^{x}\left(\underline{2\log x} + \frac{2}{x}\right) + e^{x}\left(-\frac{1}{x} + \frac{1}{x^2}\right)\right]dx$$

$$= \frac{1}{D-2}\, e^{-x}\left(e^{x}(2\log x) + e^{x}\left(-\frac{1}{x}\right)\right)$$

$$= \frac{1}{D-2}\, e^{-x} e^{x}\left(2\log x - \frac{1}{x}\right).$$

$$= \frac{1}{D-2}\left(2\log x - \frac{1}{x}\right)$$

$$\frac{1}{D-a} = e^{ax}\int e^{-ax} X\,dx. \Big/ \int e^{-ax}\left(af(x) - f'(x)\right)dx = -e^{-ax}f(x)$$

$$y_p = e^{2x}\int e^{-2x}\left(2\log x - \frac{1}{x}\right)dx.$$

$$= e^{2x}\left(-e^{-2x}\log x\right) = -\log x$$

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{-x} - \log x.$$

Method of Variation of Parameters
(only for 2nd order D.E.q^n).

$$D^2y + P_1 Dy + P_2 y = X$$

$P_1, P_2 \rightarrow$ const.

1. Consider $y_c = c_1 y_1 + c_2 y_2$

2. find Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$y_p = u y_1 + v y_2$$

$$u = -\int \frac{y_2 X}{W}\, dx \qquad v = \int \frac{y_1 X}{W}\, dx$$

$$\frac{d^2y}{dx^2} + a^2y = \sec ax.$$

$$D^2y + a^2y = \sec ax$$

A-eqn $D^2 + a^2 = 0$

$D = \pm ai$

$ai \rightarrow$ R.P = 0

$ai \rightarrow$ IP = a

$$y_c = e^{0x}(c_1\cos ax + c_2\sin ax)$$

$y_1 = \cos ax \qquad y_2 = \sin ax.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ (-\sin ax)a & (\cos ax)a \end{vmatrix}$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a(\cos^2 ax + \sin^2 ax) = a$$

$$y_p = uy_1 + vy_2$$

$$u = -\int \frac{y_2 X}{W} = -\int \frac{\sin ax \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\int \frac{\sin ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \tan ax\,dx = -\frac{1}{a}\frac{\log(\sec ax)}{a}$$

$$= -\frac{1}{a^2}\log(\sec ax)$$

$$v = \int \frac{y_1 X}{W} = \int \frac{\cos ax \sec ax}{a}\,dx$$

$$= \frac{1}{a}\int \cancel{\cos ax}\,\frac{1}{\cancel{\cos ax}}\,dx$$

$$= \frac{x}{a}$$

$$y_p = uy_1 + vy_2 = -\frac{1}{a^2}\log(\sec ax)\cos ax + \frac{x}{a}\sin ax.$$

$$y = y_c + y_p = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\log(\sec ax) + \frac{x}{a}\sin ax.$$

$(D^2-2D+2)y = e^x \tan x$.

A. eqn $\quad D^2-2D+2=0$
$\quad D = 1 \pm i$

$1+i \nearrow a:1$, $\searrow b:1$

$y_c = e^x(c_1 \cos x + c_2 \sin x)$
$= c_1 e^x \cos x + c_2 e^x \sin x$.

$y_1 = e^x \cos x \qquad y_2 = e^x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(-\sin x) + \cos x e^x & e^x \cos x + \sin x e^x \end{vmatrix}$$

$= e^{2x}\cos^2 x + e^{2x} \cancel{\sin x \cos x} - (-e^{2x}\sin^2 x + e^{2x} \cancel{\sin x \cos x})$
$= e^{2x}(\cos^2 x + \sin^2 x) = e^{2x}$

$y_p = uy_1 + vy_2$

$u = -\int \frac{y_2 X}{W} = -\int \frac{e^x \sin x \, \cancel{e^x} \tan x}{\cancel{e^{2x}}} dx$

$= -\int \sin x \frac{\sin x}{\cos x} dx$.

$= -\int \frac{\sin^2 x}{\cos x} dx$.

$= -\int \frac{(1-\cos^2 x)}{\cos x} dx$

$= -\int (\sec x - \cos x) dx$

$= -[\log(\sec x + \tan x) - \sin x]$

$v = \int \frac{y_1 X}{W} = \int \frac{\cancel{e^x} \cos x \, \cancel{e^x} \tan x}{\cancel{e^{2x}}} dx$

$= \int \cancel{\cos x} \frac{\sin x}{\cancel{\cos x}} dx$

$= -\cos x$

$y_p = uy_1 + vy_2$
$= (-\log(\sec x + \tan x) + \sin x) e^x \cos x$
$+ -\cos x \, e^x \sin x$
$= -e^x \cos x \log(\sec x + \tan x) + \cancel{e^x \cos x \sec x - e^x \sin x \cos x}$

$y = y_c + y_p = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x} \qquad D = \frac{d}{dx}$$

$$D^2y + 3Dy + 2y = e^{e^x}$$

$$(D^2 + 3D + 2)y = e^{e^x}$$

A eq^n $\quad D^2 + 3D + 2 = 0$

$$D = -1, -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_1 = e^{-x} \qquad y_2 = e^{-2x}.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

$$y_p = uy_1 + vy_2$$

$$u = -\int \frac{y_2 X}{W} dx = +\int \frac{e^{-2x} e^{e^x}}{+e^{-3x}} = \int e^x e^{e^x} dx$$

$$e^x = t \Rightarrow e^x dx = dt$$

$$= \int e^t dt = e^t = e^{e^x}$$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} = -\int e^{2x} e^{e^x} dx$$

$$= -\int e^x \cdot \underline{e^x} \, e^{e^x} \underline{dx}$$

$$e^x = t$$

$$e^x dx = dt$$

$$= -\int te^t dt$$

$$= -[t e^t - (1) e^t] = -[e^x e^{e^x} - e^{e^x}]$$

$$y_p = uy_1 + vy_2$$

$$= e^{e^x} e^{-x} + (e^{e^x} - e^x e^{e^x}) e^{-2x}$$

$$= e^{e^x} \cancel{e^{-x}} + e^{-2x} e^{e^x} - \cancel{e^{-x} e^{e^x}}$$

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$