Name: Chandana Ramesh Galgali Batch: B-1 Roll No. 16010422234

Fourier Series

let $f(x)$ be a periodic function of peroid $2L$ define in the interval $(C, C+2L)$ can be represented in the form fourier series as

$f(x) = \frac{a0}{2} + \sum{n=1}^{\infty} [

```
a_n\cos(\frac{n\pi x}{L})      +
b_n\sin(\frac{n\pi x}{L})]$<br>
```

where $a_0 = \frac{1}{L}\int_C^{C+2L} f(x)dx$

$a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cos(\frac{n\pi x}{L})dx$

$b_n = \frac{1}{L}\int_C^{C+2L} f(x)\sin(\frac{n\pi x}{L})dx$

In [1]:
```
var('x')
var('n')
f(x) = x^2
assume(n,'integer')
f = piecewise([[[-1,1],x^2]])
L = 1
an=(1/L)*integrate((x^2)*cos(n*pi*x),x,-1,1)
a0=(1/L)*integrate(f,x,-1,1)
bn=(1/L)*integrate((x^2)*sin(n*pi*x),x,-1,1)
s =a0/2+sum(an*cos(n*pi*x/L)+bn*sin(n*pi*x/L),n,1,5)
show(an)
show(bn)
show(a0)
show(s)
```

$$\frac{4\,(-1)^n}{\pi^2 n^2}$$

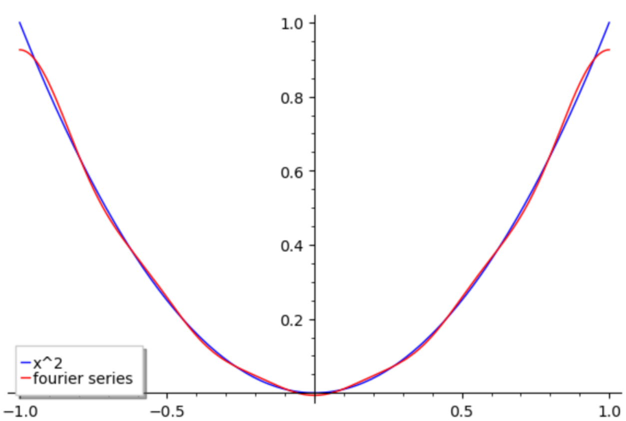
$$0$$

$$\frac{2}{3}$$

$$-\frac{144\,\cos(5\,\pi x) - 225\,\cos(4\,\pi x) + 400\,\cos(3\,\pi x) - 900\,\cos(2\,\pi x) + 3600\,\cos(\pi x)}{900\,\pi^2} + \frac{1}{3}$$

In [2]:
```
plot(f,-1,1,legend_label="x^2") + plot(s,-1,1,color = "red",legend_label="fourier series")
```

Out[2]:



In [3]:
```
g = f.fourier_series_partial_sum(10)
show(g)
```

$$\frac{\cos(10\,\pi x)}{25\,\pi^2} - \frac{4\,\cos(9\,\pi x)}{81\,\pi^2} + \frac{\cos(8\,\pi x)}{16\,\pi^2} - \frac{4\,\cos(7\,\pi x)}{49\,\pi^2} + \frac{\cos(6\,\pi x)}{9\,\pi^2} - \frac{4\,\cos(5\,\pi x)}{25\,\pi^2} + \frac{\cos(4\,\pi x)}{4\,\pi^2} - \frac{4\,\cos(3\,\pi x)}{9\,\pi^2} + \frac{\cos(2\,\pi x)}{\pi^2} - \frac{4\,\cos(\pi x)}{\pi^2} + \frac{1}{3}$$

In [4]:
```
show(f.fourier_series_cosine_coefficient(5))
```

$$-\frac{4}{25\,\pi^2}$$

In [5]:
```
show(f.fourier_series_sine_coefficient(40))
```

$$0$$

In [6]:
```
var('x')
var('n')
f(x) = x^2
assume(n,'integer')
f = piecewise([[[-pi,pi],x^2]])
c = -pi
L = pi
an=(1/pi)*integrate((x^2)*cos(n*pi*x/L),x,-pi,pi)
a0=(1/pi)*integrate(x^2,x,-pi,pi)
bn=(1/pi)*integrate((x^2)*sin(n*pi*x/L),x,-pi,pi)
s =a0/2+sum(an*cos(n*pi*x/L)+bn*sin(n*pi*x/L),n,1,10)
show(an)
show(bn)
show(a0)
show(s)
```

$$\frac{4\,(-1)^n}{n^2}$$

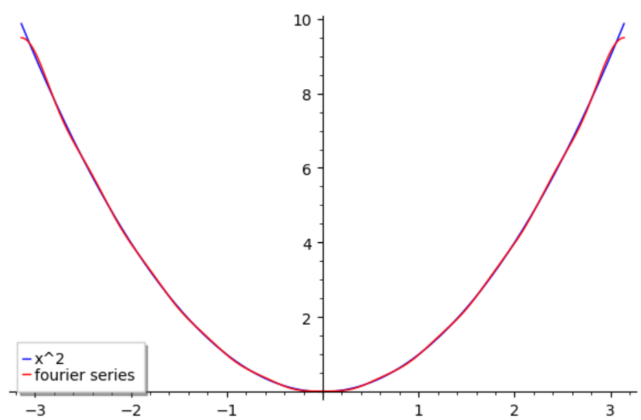
$$0$$

$$\frac{2}{3}\,\pi^2$$

$$\frac{1}{3}\,\pi^2 + \frac{1}{25}\cos(10\,x) - \frac{4}{81}\cos(9\,x) + \frac{1}{16}\cos(8\,x) - \frac{4}{49}\cos(7\,x) + \frac{1}{9}\cos(6\,x) - \frac{4}{25}\cos(5\,x) + \frac{1}{4}\cos(4\,x) - \frac{4}{9}\cos(3\,x) + \cos(2\,x) - 4\cos(x)$$

In [7]:
```
plot(f,-pi,pi,legend_label="x^2") + plot(s,-pi,pi,color = "red",legend_label="fourier series")
```

Out[7]:



In [8]:
```
plot?
```

In [9]:
```
g = f.fourier_series_partial_sum(5)
show(f.fourier_series_partial_sum(5))
```

$$\frac{1}{3}\,\pi^2 - \frac{4}{25}\cos(5\,x) + \frac{1}{4}\cos(4\,x) - \frac{4}{9}\cos(3\,x) + \cos(2\,x) - 4\cos(x)$$

In [10]:
```
show(f.fourier_series_cosine_coefficient(4))
```

$$\frac{1}{4}$$

In [11]:
```
show(f.fourier_series_sine_coefficient(4))
```

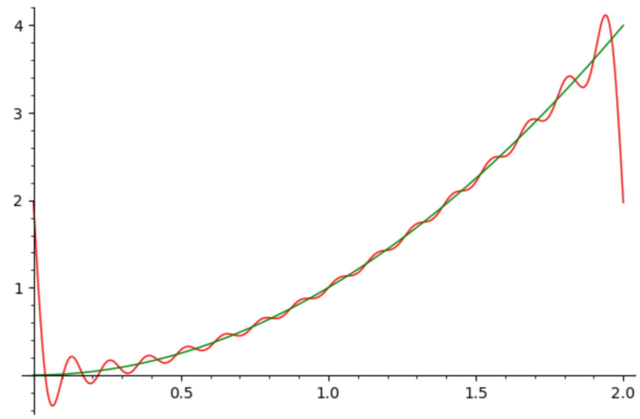
$$0$$

In [12]:
```
x = var('x')
f1(x) = x^2
f = piecewise([[(0,2),f1]])
g = f.fourier_series_partial_sum(15)
show(g)
```

$$-\frac{4\,\sin(15\,\pi x)}{15\,\pi} - \frac{2\,\sin(14\,\pi x)}{7\,\pi} - \frac{4\,\sin(13\,\pi x)}{13\,\pi} - \frac{\sin(12\,\pi x)}{3\,\pi} - \frac{4\,\sin(11\,\pi x)}{11\,\pi} - \frac{2\,\sin(10\,\pi x)}{5\,\pi} - \frac{4\,\sin(9\,\pi x)}{9\,\pi} - \frac{\sin(8\,\pi x)}{2\,\pi} - \frac{4\,\sin(7\,\pi x)}{7\,\pi} - \frac{2\,\sin(6\,\pi x)}{3\,\pi} - \frac{4\,\sin(5\,\pi x)}{5\,\pi} - \frac{\sin(4\,\pi x)}{\pi} - \frac{4\,\sin(3\,\pi x)}{3\,\pi} - \frac{2\,\sin(2\,\pi x)}{\pi} - \frac{4\,\sin(\pi x)}{\pi} + \frac{4\,\cos(15\,\pi x)}{225\,\pi^2} + \frac{\cos(14\,\pi x)}{49\,\pi^2} + \frac{4\,\cos(13\,\pi x)}{169\,\pi^2} + \frac{\cos(12\,\pi x)}{36\,\pi^2} + \frac{4\,\cos(11\,\pi x)}{121\,\pi^2} + \frac{\cos(10\,\pi x)}{25\,\pi^2} + \frac{4\,\cos(9\,\pi x)}{81\,\pi^2} + \frac{\cos(8\,\pi x)}{16\,\pi^2} + \frac{4\,\cos(7\,\pi x)}{49\,\pi^2} + \frac{\cos(6\,\pi x)}{9\,\pi^2} + \frac{4\,\cos(5\,\pi x)}{25\,\pi^2} + \frac{\cos(4\,\pi x)}{4\,\pi^2} + \frac{4\,\cos(3\,\pi x)}{9\,\pi^2} + \frac{\cos(2\,\pi x)}{\pi^2} + \frac{4\,\cos(\pi x)}{\pi^2} + \frac{4}{3}$$

In [13]:
```
plot(g,0,2,color="red")+plot(f1,0,2,color="green")
```

Out[13]:

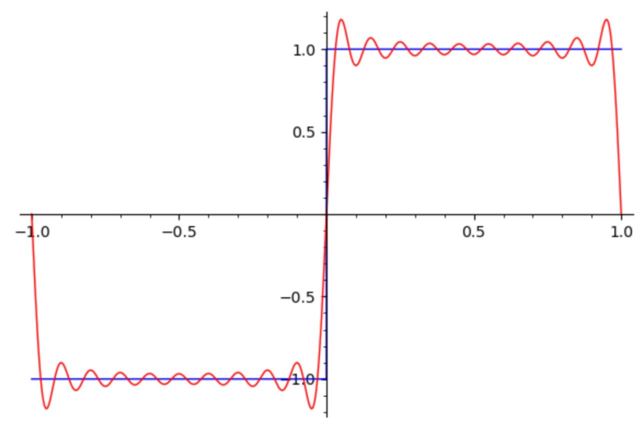


In [14]:
```
f = piecewise([((-1,0), -1), ((0,1), 1)])
g = f.fourier_series_partial_sum(20)
show(g)
```

$$\frac{4\,\sin(19\,\pi x)}{19\,\pi} + \frac{4\,\sin(17\,\pi x)}{17\,\pi} + \frac{4\,\sin(15\,\pi x)}{15\,\pi} + \frac{4\,\sin(13\,\pi x)}{13\,\pi} + \frac{4\,\sin(11\,\pi x)}{11\,\pi} + \frac{4\,\sin(9\,\pi x)}{9\,\pi} + \frac{4\,\sin(7\,\pi x)}{7\,\pi} + \frac{4\,\sin(5\,\pi x)}{5\,\pi} + \frac{4\,\sin(3\,\pi x)}{3\,\pi} + \frac{4\,\sin(\pi x)}{\pi}$$

In [15]:
```
plot(f) + plot(g,-1,1,color ='red')
```

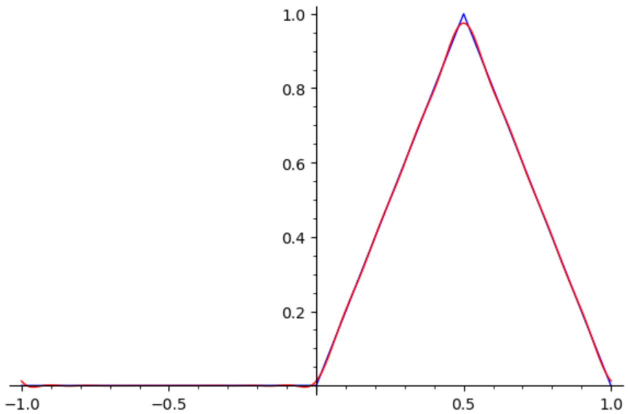
Out[15]:



In [16]:
```
f = piecewise([((-1,0), 0), ((0,1/2), 2*x),((1/2,1), 2*(1-x))])
g = f.fourier_series_partial_sum(15)
show(g)
plot(f)+plot(g,-1,1,color = "red")
```

$$-\frac{2\,\cos(14\,\pi x)}{49\,\pi^2} - \frac{2\,\cos(10\,\pi x)}{25\,\pi^2} - \frac{2\,\cos(6\,\pi x)}{9\,\pi^2} - \frac{2\,\cos(2\,\pi x)}{\pi^2} - \frac{4\,\sin(15\,\pi x)}{225\,\pi^2} + \frac{4\,\sin(13\,\pi x)}{169\,\pi^2} - \frac{4\,\sin(11\,\pi x)}{121\,\pi^2} + \frac{4\,\sin(9\,\pi x)}{81\,\pi^2} - \frac{4\,\sin(7\,\pi x)}{49\,\pi^2} + \frac{4\,\sin(5\,\pi x)}{25\,\pi^2} - \frac{4\,\sin(3\,\pi x)}{9\,\pi^2} + \frac{4\,\sin(\pi x)}{\pi^2} + \frac{1}{4}$$

Out[16]:

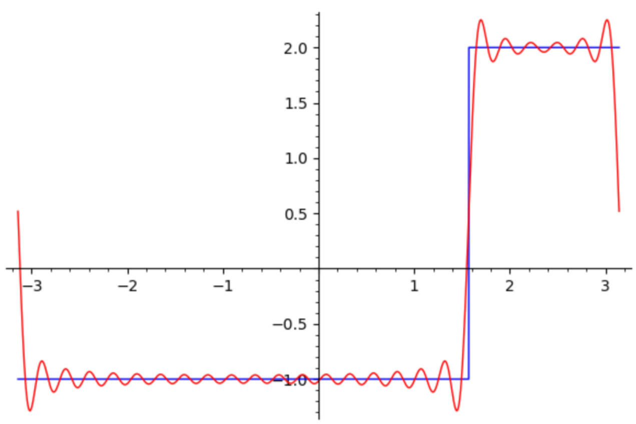


In [18]:

```
f1(x) = -1
f2(x) =  2
f = piecewise([[(-pi,pi/2),f1],[(pi/2,pi),f2]])
g = f.fourier_series_partial_sum(25)
show(g)
plot(f,-pi,pi) + plot(g,-pi,pi, color = "red")
```

$$-\frac{3\,\cos(25\,x)}{25\,\pi}+\frac{3\,\cos(23\,x)}{23\,\pi}-\frac{\cos(21\,x)}{7\,\pi}+\frac{3\,\cos(19\,x)}{19\,\pi}-\frac{3\,\cos(17\,x)}{17\,\pi}+\frac{\cos(15\,x)}{5\,\pi}-\frac{3\,\cos(13\,x)}{13\,\pi}+\frac{3\,\cos(11\,x)}{11\,\pi}-\frac{\cos(9\,x)}{3\,\pi}+\frac{3\,\cos(7\,x)}{7\,\pi}-\frac{3\,\cos(5\,x)}{5\,\pi}+\frac{\cos(3\,x)}{\pi}-\frac{3\,\cos(x)}{\pi}+\frac{3\,\sin(25\,x)}{25\,\pi}+\frac{3\,\sin(23\,x)}{23\,\pi}-\frac{3\,\sin(22\,x)}{11\,\pi}+\frac{\sin(21\,x)}{7\,\pi}+\frac{3\,\sin(19\,x)}{19\,\pi}-\frac{\sin(18\,x)}{3\,\pi}+\frac{3\,\sin(17\,x)}{17\,\pi}+\frac{\sin(15\,x)}{5\,\pi}-\frac{3\,\sin(14\,x)}{7\,\pi}+\frac{3\,\sin(13\,x)}{13\,\pi}+\frac{3\,\sin(11\,x)}{11\,\pi}-\frac{3\,\sin(10\,x)}{5\,\pi}+\frac{\sin(9\,x)}{3\,\pi}+\frac{3\,\sin(7\,x)}{7\,\pi}-\frac{\sin(6\,x)}{\pi}+\frac{3\,\sin(5\,x)}{5\,\pi}+\frac{\sin(3\,x)}{\pi}-\frac{3\,\sin(2\,x)}{\pi}+\frac{3\,\sin(x)}{\pi}-\frac{1}{4}$$

Out[18]:

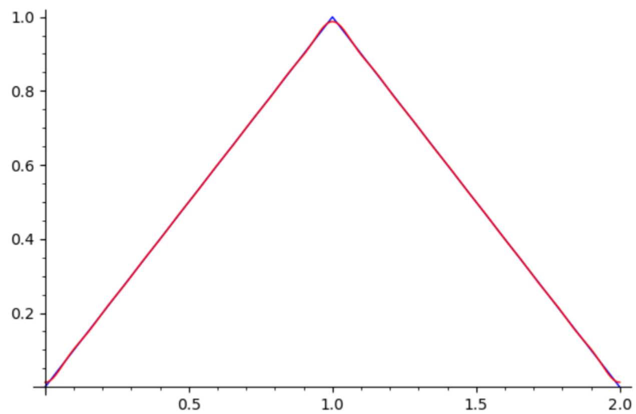


In [19]:

```
f = piecewise([((0,1), x), ((1,2), 2-x)])
g = f.fourier_series_partial_sum(15)
show(g)
plot(f,0,2) + plot(g,0,2,color = 'red')
```

$$-\frac{4\,\cos(15\,\pi x)}{225\,\pi^2}-\frac{4\,\cos(13\,\pi x)}{169\,\pi^2}-\frac{4\,\cos(11\,\pi x)}{121\,\pi^2}-\frac{4\,\cos(9\,\pi x)}{81\,\pi^2}-\frac{4\,\cos(7\,\pi x)}{49\,\pi^2}-\frac{4\,\cos(5\,\pi x)}{25\,\pi^2}-\frac{4\,\cos(3\,\pi x)}{9\,\pi^2}-\frac{4\,\cos(\pi x)}{\pi^2}+\frac{1}{2}$$

Out[19]:



In [20]:

```
var('x')
var('n')
f(x) = x/2
assume(n,'integer')
f = piecewise([[[0,1],x/2]])
c = 0
L = 1/2
an=(1/L)*integrate((x/2)*cos(2*n*pi*x),x,0,1)
a0=(1/L)*integrate(f,x,0,1)
bn=(1/L)*integrate((x/2)*sin(2*n*pi*x),x,0,1)
s =(a0/2)+sum(an*cos(n*pi*x/L)+bn*sin(n*pi*x/L),n,1,5)
show(an)
show(bn)
show(a0)
show(s)
```

$$0$$

$$-\frac{1}{2\,\pi n}$$

$$\frac{1}{2}$$

$$-\frac{12\,\sin(10\,\pi x) + 15\,\sin(8\,\pi x) + 20\,\sin(6\,\pi x) + 30\,\sin(4\,\pi x) + 60\,\sin(2\,\pi x)}{120\,\pi} + \frac{1}{4}$$

In [21]:

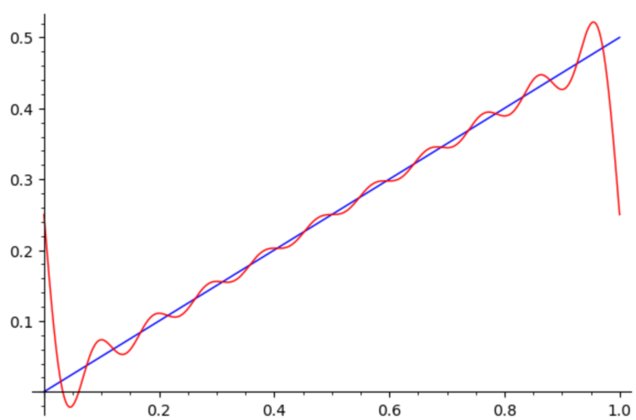
```
g = f.fourier_series_partial_sum(10)
show(g)
```

$$-\frac{\sin(20\,\pi x)}{20\,\pi} - \frac{\sin(18\,\pi x)}{18\,\pi} - \frac{\sin(16\,\pi x)}{16\,\pi} - \frac{\sin(14\,\pi x)}{14\,\pi} - \frac{\sin(12\,\pi x)}{12\,\pi} - \frac{\sin(10\,\pi x)}{10\,\pi} - \frac{\sin(8\,\pi x)}{8\,\pi} - \frac{\sin(6\,\pi x)}{6\,\pi} - \frac{\sin(4\,\pi x)}{4\,\pi} - \frac{\sin(2\,\pi x)}{2\,\pi} + \frac{1}{4}$$

In [22]:

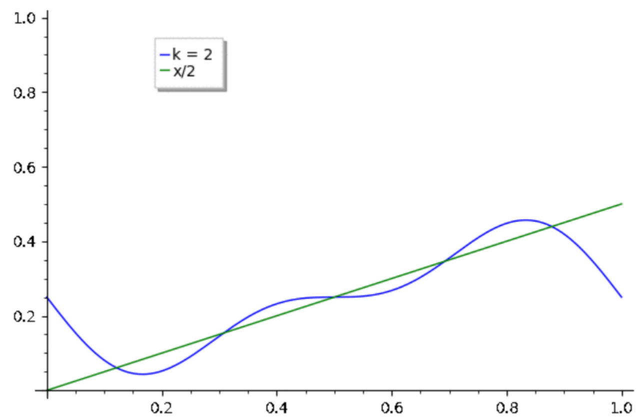
```
plot(f,0,1) + plot(g,0,1,color = 'red')
```

Out[22]:



In [23]:
```
n = var("n")
frames = []
xr = (x, 0, 1)
for k in srange(1,5):
    g = plot(f.fourier_series_partial_sum(k), xr, color="blue", legend_label='k = %d' % k)
    g += plot(x/2, xr, color="green", legend_label="x/2")
    frames.append(g)

a = animate(frames, ymin=0.0, ymax=1.0, legend_loc=(0.2,0.8))
a.show()
```



In [24]:
```
animate??
```

The alternative form to the classical Fourier series is its complex form Fourier series

$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ni\pi x/L}$ *where* $C_n = \frac{1}{2L}\int_C^{C+2L} f(x)\, e^{-ni\pi x/L}\, dx, \quad n = 0, \pm1, \pm2, \ldots;$

In [25]:
```
var('x')
var('n')
assume(n,'integer')
L=1;
f=piecewise([[(0,2),(e)^(2*x)]])
Cn=1/2*integrate((e)^(2*x)*e^(-i*n*pi*x),x,0,2)
show(Cn)
```

$$-\frac{\cosh(4) + \sinh(4)}{2\,(i\,\pi n - 2)} + \frac{1}{2\,(i\,\pi n - 2)}$$

In [26]:
```
s =sum(Cn*e^(i*n*pi*x),n,-1,1)
show(s)
```

$$\frac{\left(2i\,\pi - 2\,(i\,\pi - 2)\cosh(4) + (-2i\,\pi - 2\,(-i\,\pi - 2)\cosh(4) - 2\,(-i\,\pi - 2)\sinh(4) - 4)e^{(2i\,\pi x)}e^{(-i\,\pi x)} - (\pi^2 - (\pi^2 + 4)\cosh(4) - (\pi^2 + 4)\sinh(4) + 4)e^{(i\,\pi x)} - 2\,(i\,\pi - 2)\sinh(4) - 4\right)}{4\,(\pi^2 + 4)}$$

1) Find fourier Series of the following function and also plot graph of function and fourier series.

i) $f(x) = e^{(-x)}$ in $(0, 2\pi)$

In [1]:
```
var('x')
var('n')
f = exp(-x)
assume(n, 'integer')
L = pi
C = 0
a0 = (1/L) * integrate(f, x, 0, 2*pi)
show("a0 = ",a0)
an = (1/L) * integrate((exp(-x))*cos(n*pi*x/L), x, 0, 2*pi)
show("an = ",an)
bn = (1/L) * integrate((exp(-x))*sin(n*pi*x/L), x, 0, 2*pi)
show("bn = ",bn)
s = a0/2 + sum(an*cos(n*pi*x/L) + bn*sin(n*pi*x/L), n, 1, 5)
show("s = ",s)
plot(f, 0, 2*pi, legend_label = "exp(-x)") + plot(s, 0, 2*pi, color = "red", legend_label = "fourier series")
```

a0 $= -\frac{e^{(-2\pi)} - 1}{\pi}$

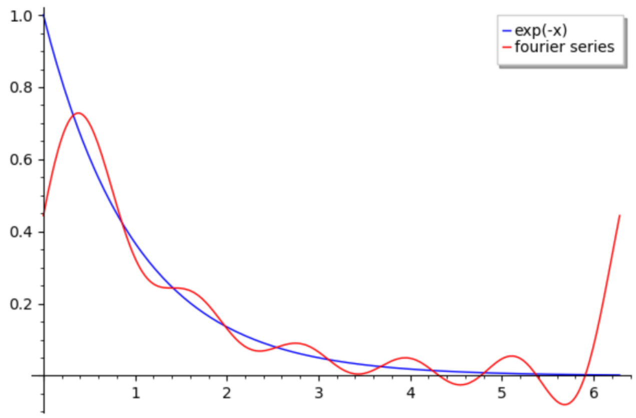
an $= -\frac{\frac{1}{n^2 e^{(2\pi)} + e^{(2\pi)}} - \frac{1}{n^2+1}}{\pi}$

bn $= -\frac{\frac{n}{n^2 e^{(2\pi)} + e^{(2\pi)}} - \frac{n}{n^2+1}}{\pi}$

$$\frac{\left(85\left(e^{(2\pi)} - 1\right)\cos(5x) + 130\left(e^{(2\pi)} - 1\right)\cos(4x) + 221\left(e^{(2\pi)} - 1\right)\cos(3x) + 442\left(e^{(2\pi)} - 1\right)\cos(2x) + 1105\left(e^{(2\pi)} - 1\right)\cos(x) + 425e^{(-2\pi)} \left(e^{(2\pi)} - 1\right)\sin(5x) + 520\left(e^{(2\pi)} - 1\right)\sin(4x) + 663\left(e^{(2\pi)} - 1\right)\sin(3x) + 884\left(e^{(2\pi)} - 1\right)\sin(2x) + 1105\left(e^{(2\pi)} - 1\right)\sin(x)\right)}{2210\,\pi}$$

$$-\frac{e^{(-2\pi)} - 1}{2\pi}$$

Out[1]:



ii) f(x) = xsinx in (0, 2π)

In [2]:

```
var('x')
var('n')
f = x * sin(x)
assume(n, 'integer')
L = pi
C = 0
a0 = (1/L) * integrate(f, x, 0, 2*pi)
show("a0 = ", a0)
an = (1/L) * integrate((x * sin(x))*cos(n*pi*x/L), x, 0, 2*pi)
show("an = ", an)
an_1 = (1/L) * integrate((x * sin(x))*cos(1*pi*x/L), x, 0, 2*pi)
show("an_1 = ", an_1)
bn = (1/L) * integrate((x * sin(x))*sin(n*pi*x/L), x, 0, 2*pi)
show("bn = ", bn)
s = a0/2 + an_1*cos(1*pi*x/L) + sum(an*cos(n*pi*x/L), n, 2, 5) + sum(bn*sin(n*pi*x/L), n, 1, 5)
show("s = ", s)
plot(f, x, 0, 2*pi, legend_label="x * sin(x)") + plot(s, x, 0, 2*pi, color="red", legend_label="Fourier series")
```

a0 $= -2$

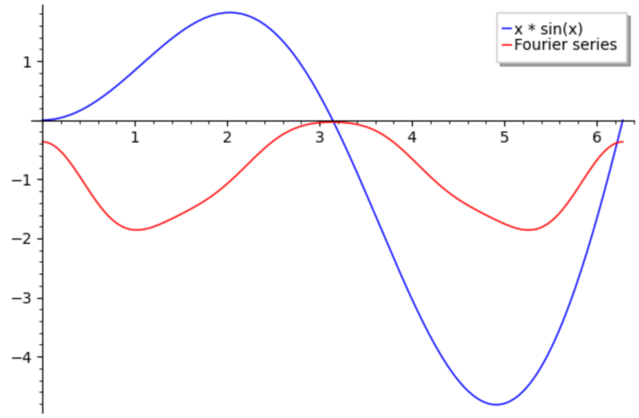
an $= \frac{2}{n^2 - 1}$

an_1 $= -\frac{1}{2}$

bn $= 0$

s $= \frac{1}{12}\cos(5x) + \frac{2}{15}\cos(4x) + \frac{1}{4}\cos(3x) + \frac{2}{3}\cos(2x) - \frac{1}{2}\cos(x) - 1$

Out[2]:



2) obtain half range sine series in (0,π) for cosx

In [3]:
```
var('x')
var('n')
f = cos(x)
assume(n, 'integer')
L = pi
C = 0
a0 = (2/L) * integrate(f, x, 0, pi)
show("a0 = ", a0)
an = 0
show("an = ", an)
bn = (2/L) * integrate(cos(x)*sin(n*pi*x/L), x, 0, pi)
show("bn = ", bn)
bn_1 = (2/L) * integrate(cos(x)*sin(1*pi*x/L), x, 0, pi)
show("bn_1 = ", bn_1)
s = a0/2 + bn_1*sin(1*pi*x/L) + sum(bn*sin(n*pi*x/L), n, 2, 5)
show("s = ", s)
plot(f, 0, pi, legend_label="cos(x)") + plot(s, x, 0, pi, color="red", legend_label="Half range sine series")
```

$\text{a0 } = 0$

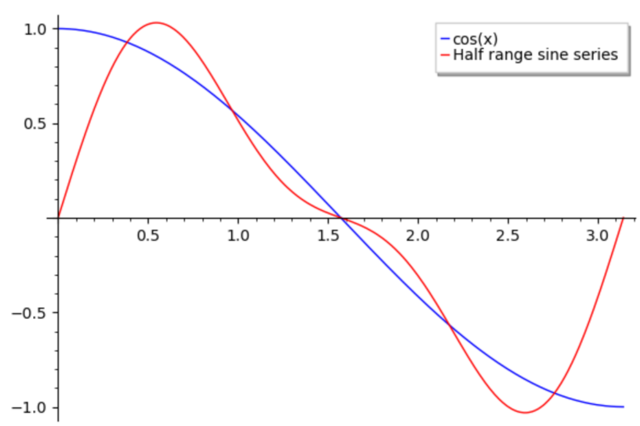
$\text{an } = 0$

$$\text{bn } = \frac{2\left(\frac{(-1)^n n}{n^2-1} + \frac{n}{n^2-1}\right)}{\pi}$$

$\text{bn\_1 } = 0$

$$\text{s } = \frac{8\,(2\,\sin(4\,x) + 5\,\sin(2\,x))}{15\,\pi}$$

Out[3]:



3) obtain half range cosine series in $(0,\pi)$ for $f(x) = x(\pi - x)$

In [4]:
```
var('x')
var('n')
f = x * (pi-x)
assume(n, 'integer')
L = pi
C = 0
a0 = (2/L) * integrate(f, x, 0, pi)
show("a0 = ", a0)
an = (2/L) * integrate((x * (pi-x))*cos(n*pi*x/L), x, 0, pi)
show("an = ", an)
bn = 0
show("bn = ", bn)
s = a0/2 + sum(an*cos(n*pi*x/L), n, 1, 5)
show("s = ", s)
plot(f, 0, pi, legend_label="x * (pi-x)") + plot(s, x, 0, pi, color="red", legend_label="Half range cosine series")
```

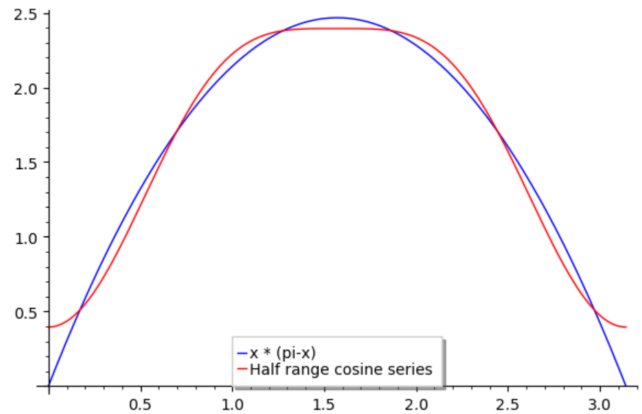
$$\mathbf{a0} = \frac{1}{3}\pi^2$$

$$\mathbf{an} = -\frac{2\left(\frac{\pi(-1)^n}{n^2} + \frac{\pi}{n^2}\right)}{\pi}$$

$$\mathbf{bn} = 0$$

$$\mathbf{s} = \frac{1}{6}\pi^2 - \frac{1}{4}\cos(4x) - \cos(2x)$$

Out[4]:



4) Find the complex form of the Fourier series for $f(x) = 2x$ in $(0,2\pi)$

In [5]:
```
var('x')
var('n')
assume(n,'integer')
C=0
L=pi
f=piecewise([[(0,2*pi),2*x]])
Cn=1/(2*pi)*integrate(2*x*e^(-i*n*pi*x),x,0,2*pi)
Cn_neg1=1/(2*pi)*integrate(2*x*e^(-i*-1*pi*x),x,0,2*pi)
Cn_0=1/(2*pi)*integrate(2*x*e^(-i*0*pi*x),x,0,2*pi)
Cn_1=1/(2*pi)*integrate(2*x*e^(-i*1*pi*x),x,0,2*pi)
show("Cn = ",Cn)
show("Cn_neg1 = ",Cn_neg1)
show("Cn_0 = ",Cn_0)
show("Cn_1 = ",Cn_1)
s =Cn_neg1+Cn_0+Cn_1
show("s = ",s)
plot(f, 0, 2*pi, legend_label="2*x") + plot(s, x, 0, 2*pi, color="red", legend_label="Complex form of Fourier series")
```

$$\mathbf{Cn} = \frac{\frac{(2i\pi^2 n+1)e^{\left(-2i\pi^2 n\right)}}{\pi^2 n^2} - \frac{1}{\pi^2 n^2}}{\pi}$$

$$\mathbf{Cn\_neg1} = -\frac{\frac{(2i\pi^2-1)e^{\left(2i\pi^2\right)}}{\pi^2} + \frac{1}{\pi^2}}{\pi}$$

$$\mathbf{Cn\_0} = 2\pi$$

$$\mathbf{Cn\_1} = \frac{\frac{(2i\pi^2+1)e^{\left(-2i\pi^2\right)}}{\pi^2} - \frac{1}{\pi^2}}{\pi}$$

$$\mathbf{s} = 2\pi - \frac{\frac{(2i\pi^2-1)e^{\left(2i\pi^2\right)}}{\pi^2} + \frac{1}{\pi^2}}{\pi} + \frac{\frac{(2i\pi^2+1)e^{\left(-2i\pi^2\right)}}{\pi^2} - \frac{1}{\pi^2}}{\pi}$$

Out[5]: