Experiment No. : 5

Title: Floyd-Warshall Algorithm using Dynamic programming approach

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Experiment No.: 05

Aim: To Implement All pair shortest path Floyd-Warshall Algorithm using Dynamic programming approach and analyse its time Complexity.

Algorithm of Floyd-Warshall Algorithm:

```
FLOYD-WARSHALL(W)
n = W.rows
D^(0) = W
for k = 1 to n
    let D^(k) = (d_ij^(k)) be a new n × n matrix
    for i = 1 to n
        for j = 1 to n
            d_ij^(k) = min(d_ij^(k-1), d_ik^(k-1) + d_kj^(k-1))
return D^(n)
```

Constructing Shortest Path:

We can give a recursive formulation of $\pi_{ij}^{(k)}$. When $k = 0$, a shortest path from i to j has no intermediate vertices at all. Thus,

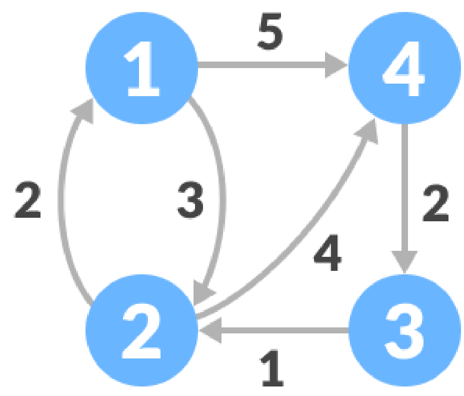
$$\pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty , \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty . \end{cases} \tag{25.6}$$

For $k \geq 1$, if we take the path $i \leadsto k \leadsto j$, where $k \neq j$, then the predecessor of j we choose is the same as the predecessor of j we chose on a shortest path from k with all intermediate vertices in the set $\{1, 2, \ldots, k-1\}$. Otherwise, we choose the same predecessor of j that we chose on a shortest path from i with all intermediate vertices in the set $\{1, 2, \ldots, k-1\}$. Formally, for $k \geq 1$,

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} , \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} . \end{cases} \tag{25.7}$$

Working of Floyd-Warshall Algorithm:

Let the given graph be:



Initial graph

Follow the steps below to find the shortest path between all the pairs of vertices.

1. Create a matrix A0 of dimension n*n where n is the number of vertices. The row and the column are indexed as $\boxed{i}$ and $\boxed{j}$ respectively. $\boxed{i}$ and $\boxed{j}$ are the vertices of the graph. Each cell $\boxed{A[i][j]}$ is filled with the distance from the ith vertex to the jth vertex. If there is no path from ith vertex to the jth vertex, the cell is left as infinity.

$$A^0 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & \infty & 4 \\ \infty & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Fill each cell with the distance between ith and jth vertex

2. Now, create a matrix A1 using matrix A0. The elements in the first column and the first row are left as they are. The remaining cells are filled in the following way.
Let $\boxed{k}$ be the intermediate vertex in the shortest path from source to destination. In this step, $\boxed{k}$ is the first vertex. A[i][j] is filled with (A[i][k] + A[k][j]) if (A[i][j] > A[i][k] + A[k][j]).
That is, if the direct distance from the source to the destination is greater than the path through the vertex $\boxed{k}$, then the cell is filled with A[i][k] + A[k][j].
In this step, k is vertex 1.

We calculate the distance from source vertex to destination vertex through this vertex k.

$$A^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & & \\ \infty & & 0 & \\ \infty & & & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & 9 & 4 \\ \infty & 1 & 0 & 8 \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex k

For example: For A1[2, 4], the direct distance from vertex 2 to 4 is 4 and the sum of the distance from vertex 2 to 4 through vertex (ie. from vertex 2 to 1 and from vertex 1 to 4) is 7. Since 4 < 7, A0[2, 4] is filled with 4.

3. Similarly, A2 is created using A1. The elements in the second column and the second row are left as they are.
   In this step, $\boxed{\text{k}}$ is the second vertex (i.e. vertex 2). The remaining steps are the same as in step 2.

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & & \\ 2 & 0 & 9 & 4 \\ & 1 & 0 & \\ & \infty & & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex 2

4. Similarly, A3 and A4 is also created.

$$A^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & \infty & \\ & 0 & 9 & \\ \infty & 1 & 0 & 8 \\ & & 2 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex 3

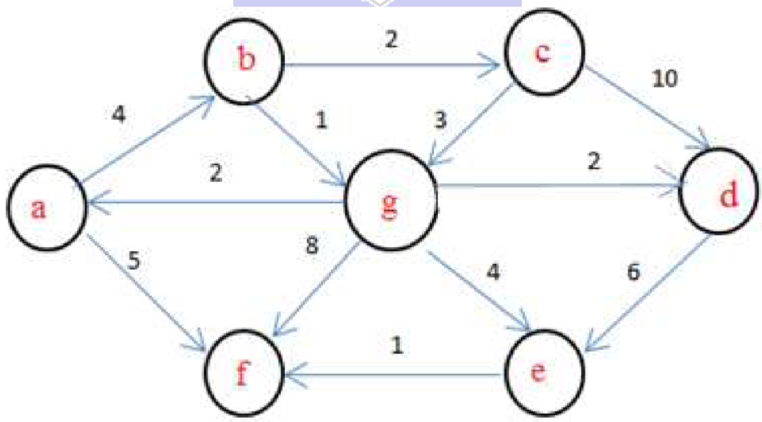
$$A^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & & 5 \\ & 0 & & 4 \\ & & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 7 & 5 \\ 2 & 0 & 6 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex 4

5. A4 gives the shortest path between each pair of vertices.

Problem Statement

Find Shortest Path for each source to all destinations using Floyd-Warshall Algorithm for the following graph.



Solution:

[[0, 4, 6, 11, 6, 5, 9],
 ['INF', 0, 2, 7, 2, 1, 5],
 ['INF', 6, 0, 5, 7, 7, 3],
 ['INF', 18, 17, 0, 6, 7, 15],
 ['INF', 12, 11, 11, 0, 1, 9],

['INF', 11, 10, 10, 1, 0, 8],
['INF', 3, 2, 2, 4, 4, 0]]

Derivation of Floyd-Warshall Algorithm:

Time complexity Analysis

The time complexity is derived from the three nested loops used in the algorithm, each going through all vertices:

For a graph with n vertices, the first loop runs n times for choosing an intermediate vertex k.

The second loop runs n times for picking the starting vertex i.

The third loop runs n times for picking the ending vertex j.

The overall time complexity is $O(n^3)$ since the total number of operations is proportional to $n \times n \times n$.

Program(s) of Floyd-Warshall Algorithm:

```
#include <stdio.h>
#include <stdlib.h>
#include <string.h>

#define INF 99999
#define MAX_INPUT_LEN 100

void printSolution(int **dist, int V);

void floydWarshall(int **graph, int V);

int main() {
    int V, **graph;
    char input[MAX_INPUT_LEN];
    printf("Enter the number of vertices in the graph: ");
    scanf("%d", &V);
    graph = (int **)malloc(V * sizeof(int *));
    for (int i = 0; i < V; i++) {
        graph[i] = (int *)malloc(V * sizeof(int));
    }
    printf("Enter the adjacency matrix (use 'INF' for infinity):\n");
    for (int i = 1; i <= V; i++) {
        for (int j = 1; j <= V; j++) {
            printf("Enter the weight of the edge from vertex %d to vertex %d (or 'INF' for no direct edge): ", i, j);
            scanf("%s", input);
            if (strcmp(input, "INF") == 0) {
                graph[i-1][j-1] = INF;
            } else {
                graph[i-1][j-1] = atoi(input);
            }
        }
```

```
    }
    floydWarshall(graph, V);
    for (int i = 0; i < V; i++) {
        free(graph[i]);
    }
    free(graph);
    return 0;
}

void floydWarshall(int **graph, int V) {
    int **dist = (int **)malloc(V * sizeof(int *));
    for (int i = 0; i < V; i++) {
        dist[i] = (int *)malloc(V * sizeof(int));
        for (int j = 0; j < V; j++) {
            dist[i][j] = graph[i][j];
        }
    }
    for (int k = 0; k < V; k++) {
        for (int i = 0; i < V; i++) {
            for (int j = 0; j < V; j++) {
                if (dist[i][k] != INF && dist[k][j] != INF && dist[i][k] + dist[k][j] < dist[i][j]) {
                    dist[i][j] = dist[i][k] + dist[k][j];
                }
            }
        }
    }
    printSolution(dist, V);
    for (int i = 0; i < V; i++) {
        free(dist[i]);
    }
    free(dist);
}

void printSolution(int **dist, int V) {
    printf("The shortest distances between every pair of vertices:\n");
    for (int i = 1; i <= V; i++) {
        for (int j = 1; j <= V; j++) {
            if (dist[i-1][j-1] == INF) {
                printf("%7s", "INF");
            } else {
                printf("%7d", dist[i-1][j-1]);
            }
        }
        printf("\n");
    }
}
```

Output(o) of Floyd-Warshall Algorithm:

```
"C:\Users\chand\Downloads\I
Enter the number of vertices in the graph: 4
Enter the adjacency matrix (use 'INF' for infinity):
Enter the weight of the edge from vertex 1 to vertex 1 (or 'INF' for no direct edge): 0
Enter the weight of the edge from vertex 1 to vertex 2 (or 'INF' for no direct edge): 3
Enter the weight of the edge from vertex 1 to vertex 3 (or 'INF' for no direct edge): INF
Enter the weight of the edge from vertex 1 to vertex 4 (or 'INF' for no direct edge): 5
Enter the weight of the edge from vertex 2 to vertex 1 (or 'INF' for no direct edge): 2
Enter the weight of the edge from vertex 2 to vertex 2 (or 'INF' for no direct edge): 0
Enter the weight of the edge from vertex 2 to vertex 3 (or 'INF' for no direct edge): INF
Enter the weight of the edge from vertex 2 to vertex 4 (or 'INF' for no direct edge): 4
Enter the weight of the edge from vertex 3 to vertex 1 (or 'INF' for no direct edge): INF
Enter the weight of the edge from vertex 3 to vertex 2 (or 'INF' for no direct edge): 1
Enter the weight of the edge from vertex 3 to vertex 3 (or 'INF' for no direct edge): 0
Enter the weight of the edge from vertex 3 to vertex 4 (or 'INF' for no direct edge): INF
Enter the weight of the edge from vertex 4 to vertex 1 (or 'INF' for no direct edge): INF
Enter the weight of the edge from vertex 4 to vertex 2 (or 'INF' for no direct edge): INF
Enter the weight of the edge from vertex 4 to vertex 3 (or 'INF' for no direct edge): 2
Enter the weight of the edge from vertex 4 to vertex 4 (or 'INF' for no direct edge): 0
The shortest distances between every pair of vertices:
      0      3      7      5
      2      0      6      4
      3      1      0      5
      5      3      2      0

Process returned 0 (0x0)   execution time : 170.453 s
Press any key to continue.
```

Post Lab Questions:

Explain a dynamic programming approach for the Floyd-Warshall algorithm and write the various applications of it.

Ans: The Floyd-Warshall algorithm is a classic example of dynamic programming, used to find the shortest paths in a weighted graph with positive or negative edge weights (but with no negative cycles). The beauty of the Floyd-Warshall algorithm lies in its simplicity and efficiency in computing the shortest paths between all pairs of vertices in a graph.

Dynamic Programming Approach:

The dynamic programming approach of the Floyd-Warshall algorithm iteratively improves the solution by considering all possible paths between each pair of vertices and efficiently updating the shortest paths using a bottom-up approach. The key idea is to incrementally consider intermediate vertices through which a shortest path might pass.

Algorithm Steps:

1. Initialization: Start with a matrix of distances between each pair of vertices. Initially, this matrix is just the adjacency matrix of the graph, where the entry at `dist[i][j]` is the direct distance from `i` to `j` (if there is an edge), or infinity (if there is no direct edge).

2. Iterative Update: For each vertex `k`, consider it as an intermediate vertex in the paths between all pairs of vertices `(i, j)`. For every pair of vertices `(i, j)`, check if a path from `i` to `j` passing through `k` is shorter than the current known shortest path. If so, update the shortest path to this new value. This step is repeated for all vertices `k`.

The key relation for the update is:

```
dist[i][j] = min(dist[i][j], dist[i][k] + dist[k][j])
```

This formula is applied for each pair `(i, j)` for each intermediate vertex `k`.
3. Completion: After considering all vertices as intermediate points, the matrix `dist` will contain the lengths of the shortest paths between all pairs of vertices.

Applications of the Floyd-Warshall Algorithm:

The Floyd-Warshall algorithm's ability to compute shortest paths between all pairs of vertices in a weighted graph makes it versatile for various applications, including:

1. Network Routing: In telecommunications, finding the shortest paths can optimize the routing of messages through a network, minimizing latency or cost.
2. Geographical Mapping: In geographic information systems (GIS), calculating the shortest routes between various locations on a map.
3. Social Network Analysis: Determining the shortest paths can help analyze the degrees of separation between individuals in a social network.
4. Game Development: In pathfinding algorithms for NPCs (non-player characters), to find the shortest path through a game's map.
5. Robotics and Path Planning: In robotics, for calculating the shortest route a robot should take to navigate between points in a space filled with obstacles.
6. Internet Routing Protocols: Algorithms like Floyd-Warshall influence the design of protocols for routing internet traffic to ensure efficient data transmission.
7. Arbitrage Opportunities in Finance: Finding negative cycles in the graph representing currency exchange rates to exploit arbitrage opportunities.

The Floyd-Warshall algorithm, through its dynamic programming approach, offers a robust method for solving a wide range of real-world problems that require efficient computation of shortest paths in various domains.

Conclusion: (Based on the observations)

The experiment with the Floyd-Warshall algorithm underscores its significance in the realm of graph algorithms, offering a powerful tool for solving all-pair shortest path problems with a dynamic programming approach.

Outcome: Implement Greedy and Dynamic Programming algorithms

References:

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