

GAMMA FUNCTION

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DEFINITION

The function of n ($n > 0$) defined by the integral

$\int_0^\infty e^{-x} x^{n-1} dx$ is called Gamma Function and is denoted by $\Gamma(n)$.

$$\text{Thus } \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Sometimes we may write it as $\int_0^\infty e^{-x} x^n dx = \Gamma(n+1)$

PROPERTIES OF GAMMA FUNCTION:

$$\Gamma(n+1) = n\Gamma(n) \quad \text{or} \quad \Gamma(n) = (n-1)\Gamma(n-1)$$

$$\text{Proof:- } \Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

Integrate by parts

$$\Gamma(n+1) = \left[x^n (-e^{-x}) \right]_0^\infty - \int_0^\infty n x^{n-1} (-e^{-x}) dx$$

$$\text{but } \lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{n x^{n-1}}{e^x} = \dots = \lim_{x \rightarrow \infty} \frac{n!}{e^x} = 0$$

(By L'Hospital's Rule)

$$\therefore \Gamma(n+1) = (0-0) + n \int_0^\infty e^{-x} x^{n-1} dx$$

$$\boxed{\Gamma(n+1) = n\Gamma(n)}$$

Values of Γ_n :

i) If n is a positive integer, by successive application of above property

$$\begin{aligned}\overline{\Gamma_{n+1}} &= n\overline{\Gamma_n} = n(n-1)\overline{\Gamma_{n-1}} = n(n-1)(n-2)\overline{\Gamma_{n-2}} \\ &= n(n-1)(n-2) \cdots \cdot 1\overline{\Gamma_1}\end{aligned}$$

$$\begin{aligned}\text{Now } \overline{\Gamma_1} &= \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x} dx = (-e^{-x})_0^\infty \\ &= -(e^{-\infty} - e^0) = 1\end{aligned}$$

$$\overline{\Gamma_{n+1}} = n(n-1)(n-2) \cdots \cdot 1 = n!$$

$$\overline{\Gamma_{n+1}} = n! \text{ if } n \text{ is a positive integer}$$

For this reason, Gamma Function is often referred to as generalised factorial.

$$\text{For eg. } \overline{\Gamma_6} = 5! = 120$$

$$\overline{\Gamma_9} = 8! =$$

$$\overline{\Gamma_2} = 1! = 1$$

$$(ii) \overline{\Gamma_0} = \frac{\overline{\Gamma_1}}{0} = \frac{1}{0} = \infty$$

$$\begin{aligned}\overline{\Gamma_{n+1}} &= n\overline{\Gamma_n} \\ \Rightarrow \overline{\Gamma_n} &= \frac{\overline{\Gamma_{n+1}}}{n}\end{aligned}$$

(iii) If n is a negative integer.

By making use of $\sqrt{n} = \frac{\sqrt{n+1}}{n}$ repeatedly, we will get

$$\begin{aligned}\sqrt{-5} &= \frac{\sqrt{-4}}{(-5)} = \frac{\sqrt{-3}}{(-5)(-4)} = \frac{\sqrt{-2}}{(-5)(-4)(-3)} = \frac{\sqrt{-1}}{(-5)(-4)(-3)(-2)} \\ &= \frac{\sqrt{0}}{(-5)(-4)(-3)(-2)(-1)} = \infty\end{aligned}$$

(iv) If n is a positive fraction, ($n > 1$)

we can use $\sqrt{n+1} = n\sqrt{n}$ repeatedly and find \sqrt{n} in terms of \sqrt{x} where $0 < x < 1$

$$\text{For ex : } \sqrt{\frac{5}{2}} = \frac{3}{2}\sqrt{\frac{3}{2}} = \frac{3}{2}\left(\frac{1}{2}\right)\sqrt{\frac{1}{2}} \quad \sqrt{n+1} = n\sqrt{n}$$

(v) If n is negative fraction,

use $\sqrt{n} = \frac{\sqrt{|n+1|}}{|n|}$ repeatedly and find \sqrt{n} in terms of \sqrt{x} where $0 < x < 1$

$$\begin{aligned}\sqrt{-\frac{5}{3}} &= \frac{\sqrt{-\frac{5}{3}+1}}{\left(-\frac{5}{3}\right)} = \frac{\sqrt{-\frac{2}{3}}}{\left(-\frac{5}{3}\right)} = \frac{\sqrt{-\frac{2}{3}+1}}{\left(-\frac{5}{3}\right)\left(-\frac{2}{3}\right)} = \frac{\sqrt{\frac{1}{3}}}{\left(-\frac{5}{3}\right)\left(-\frac{2}{3}\right)}\end{aligned}$$

(vi) If $0 < n < 1$, we find \sqrt{n} by numerical Integration

Thus \sqrt{n} is defined for all n , except $n = 0, -1, -2, -3, \dots$

Second Form of Gamma Function:-

$$\Gamma(n) = 2 \int_0^\infty e^{-x^2} \cdot x^{2n-1} dx$$

Proof :- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\text{put } x = t^2 \quad dx = 2t dt$$

$$\Gamma(n) = \int_0^\infty e^{-t^2} (t^2)^{n-1} \cdot 2t dt$$

$$= 2 \int_0^\infty e^{-t^2} t^{2n-2} \cdot t dt$$

$$\Gamma(n) = 2 \int_0^\infty e^{-t^2} \cdot t^{2n-1} dt$$

$$\Gamma(n) = 2 \int_0^\infty e^{-x^2} \cdot x^{2n-1} dx \quad \text{using dummy variable property}$$

Types of examples on Gamma Functions:-

Type-I :- Evaluate $\int_0^\infty e^{-ax^n} dx$

Method :- $ax^n = t$

Type-II :- Evaluate $\int_0^\infty x^m \cdot e^{-ax^n} dx$

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$$\text{Method :- } ax^n = t$$

Type-III :- Evaluate $\int_0^1 x^m (\log x)^n dx$

$$\text{Method :- put } \log x = -t$$

Type-IV :- Evaluate $\int_0^\infty \frac{x^a}{e^x} dx$

$$\text{Method :- put } a^x = e^t$$

Ex-1 :- Given $\sqrt{1.8} = 0.9314$, find the value of $\sqrt{-2.2}$

Solⁿ :- we have $\sqrt{n+1} = n\sqrt{n}$

$$\therefore \sqrt{n} = \frac{\sqrt{n+1}}{n}$$

$$\therefore \sqrt{-2.2} = \frac{\sqrt{-2.2+1}}{(-2.2)} = \frac{\sqrt{-1.2}}{(-2.2)} = \frac{\sqrt{-1.2+1}}{(-2.2)(-1.2)} = \frac{\sqrt{-0.2}}{(-2.2)(1.2)}$$

$$= \frac{\sqrt{-0.2+1}}{(-2.2)(-1.2)(-0.2)} = \frac{\sqrt{0.8}}{(-2.2)(-1.2)(-0.2)} = \frac{\sqrt{1.8}}{(-2.2)(-1.2)(-0.2)(0.8)}$$

$$\sqrt{-2.2} = \frac{0.9314}{-2.21}$$

$$\overline{-2.2} = \frac{0.9314}{(-2^2)(-1.02)(-0.2)(0.8)} = -2.21$$

Ex 2:- Prove that $\sqrt{n+\frac{1}{2}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\pi}$

Hence or otherwise prove that $\sqrt{n+\frac{1}{2}} = \frac{(2n)! \sqrt{\pi}}{n! 4^n}$

Soln :- clearly n must be a positive integer

$$\therefore \sqrt{n+\frac{1}{2}} = (n-\frac{1}{2}) \sqrt{n-\frac{1}{2}} \quad (\sqrt{n} = (n-1)/\sqrt{n-1})$$

$$= (n-\frac{1}{2})(n-\frac{3}{2}) \sqrt{n-\frac{3}{2}}$$

$$= (n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2}) \sqrt{n-\frac{5}{2}} \text{ and so on}$$

$$\sqrt{n+\frac{1}{2}} = (n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2}) \cdots \frac{3}{2}, \frac{1}{2} \sqrt{\frac{1}{2}}$$

$$= \left(\frac{2n-1}{2}\right) \left(\frac{2n-3}{2}\right) \left(\frac{2n-5}{2}\right) \cdots \frac{3}{2}, \frac{1}{2} \sqrt{\pi}$$

$$\sqrt{n+\frac{1}{2}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \sqrt{\pi}}{2^n}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-2)(2n-1)(2n) \sqrt{\pi}}{2^n}$$

$$\begin{aligned}
 & 2 \cdot 4 \cdot 6 \cdots (2n-2) (2n) 2^n \\
 = & \frac{(2n)! \sqrt{\pi}}{2^n (1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) 2^n} \\
 \overline{|n+1|} = & \frac{(2n)! \sqrt{\pi}}{n! 4^n}
 \end{aligned}$$

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Revision :- $\Gamma_n = \int_0^\infty e^{-x} x^{n-1} dx$

also $\Gamma_n = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx$

Property :- $\Gamma_{n+1} = n \Gamma_n$

n is +ve integer $\rightarrow \Gamma_n = (n-1)!$

n is zero $\Gamma_0 = \infty$

n is -ve integer $\Gamma_n = \infty$

n is fraction $\rightarrow \Gamma_n$ can be evaluated.

$0 < n < 1 \rightarrow \Gamma_n$ can be evaluated using numerical integration

$$\Gamma_{\frac{1}{2}} = \sqrt{\pi}$$

Ex-3 If $I_n = \frac{\sqrt{\pi}}{2} \frac{\Gamma_{\frac{n+1}{2}}}{\Gamma_n}$, show that $I_{n+2} = \frac{n+1}{n+2} I_n$

$$\frac{\frac{1}{2} \cdot \frac{1}{2}}{\sqrt{\frac{n}{2} + 1}}, \text{ show that } I_{n+2} = \frac{n+1}{n+2} I_n$$

and hence find I_5

$$\underline{\text{Soln:}} \quad I_n = \frac{\frac{\sqrt{\pi}}{2} \sqrt{\frac{n+1}{2}}}{\sqrt{\frac{n}{2} + 1}}$$

$$\Rightarrow I_{n+2} = \frac{\frac{\sqrt{\pi}}{2} \sqrt{\frac{n+2+1}{2}}}{\sqrt{\frac{n+2}{2} + 1}} = \frac{\frac{\sqrt{\pi}}{2} \sqrt{\frac{n+3}{2}}}{\sqrt{\frac{n+2}{2} + 1}}$$

$$\sqrt{n+1} = n \sqrt{n} \quad \text{using this } \sqrt{\frac{n+3}{2}} = \sqrt{\frac{n+1}{2} + 1} = \left(\frac{n+1}{2}\right) \sqrt{\frac{n+1}{2}}$$

$$\sqrt{\frac{n+2}{2} + 1} = \left(\frac{n+2}{2}\right) \sqrt{\frac{n+2}{2}}$$

$$\therefore I_{n+2} = \frac{\frac{\sqrt{\pi}}{2} \cdot \left(\frac{n+1}{2}\right) \sqrt{\frac{n+1}{2}}}{\left(\frac{n+2}{2}\right) \sqrt{\frac{n+2}{2}}} = \frac{(n+1)}{n+2} \cdot \frac{\frac{\sqrt{\pi}}{2} \sqrt{\frac{n+1}{2}}}{\sqrt{\frac{n}{2} + 1}}$$

$$\therefore I_{n+2} = \frac{n+1}{n+2} I_n$$

To find I_5 , put $n=3$

$$I_5 = \frac{3+1}{3+2} I_3 = \frac{4}{5} I_3 \quad \text{--- } ①$$

using the relation again for $n=1$

$$I_3 = \frac{1+1}{1+2} I_1 = \frac{2}{3} I_1$$

$$\therefore I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \quad (\text{using } ①)$$

$$\text{Now, } I_1 = \frac{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\frac{1+1}{2}}}{\sqrt{\frac{1}{2} + 1}} = \frac{\frac{\sqrt{\pi}}{2} \sqrt{\frac{1}{2}}}{\frac{1}{2} \sqrt{\frac{1}{2}}} = \frac{\frac{\sqrt{\pi}}{2}}{\frac{1}{2} \cdot \sqrt{\pi}} = 1$$

$$\therefore I_5 = \frac{8}{15} \cdot (1) = \frac{8}{15}$$

$$\underline{\text{Ex4}} : - \int_0^\infty e^{-h^2 n^2} dn$$

$$\underline{\text{Soln}} : - \quad \text{Put } h^2 n^2 = t \quad \therefore n = \frac{\sqrt{t}}{h} \quad \therefore dn = \frac{1}{h} \cdot \frac{1}{2\sqrt{t}} dt$$

$$\therefore I = \int_0^\infty e^{-h^2 n^2} dn = \int_0^\infty e^{-t} \cdot \frac{1}{h} \cdot \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{2h} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2h} \int_0^\infty e^{-t} t^{\left(\frac{1}{2}-1\right)} dt$$

$$\left[\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \right]$$

$$= \frac{1}{2h} \sqrt{\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{2h}$$

$$\underline{\text{Ex-5}} \quad \int_0^\infty x^{1/4} e^{-\sqrt{x}} dx$$

$$\underline{\text{Soln:}} \quad \text{Let } I = \int_0^\infty x^{1/4} e^{-\sqrt{x}} dx$$

$$\text{put } \sqrt{x} = t \rightarrow x = t^2 \rightarrow dx = 2t dt$$

$$\therefore I = \int_0^\infty (t^2)^{1/4} e^{-t} 2t dt = 2 \int_0^\infty t^{1/2} e^{-t} \cdot t dt$$

$$= 2 \int_0^\infty e^{-t} t^{3/2} dt$$

$$= 2 \int \frac{5}{2}$$

$$\sqrt{n+1} = n \sqrt{n}$$

$$= 2 \left(\frac{3}{2}\right) \sqrt{\frac{3}{2}}$$

$$= 2 \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \sqrt{\frac{1}{2}}$$

$$I = \frac{3\sqrt{\pi}}{2}$$

$$\underline{\text{Ex-6}} \quad \text{Prove that } \int_0^\infty x e^{-x^8} dx \cdot \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$$

$$\underline{\text{Soln:}} \quad \text{Let } I_1 = \int_0^\infty x e^{-x^8} dx \quad \text{and} \quad I_2 = \int_0^\infty x^2 e^{-x^4} dx$$

put $x^8 = t$ in I_1 ,

$$x = t^{1/8} \rightarrow dx = \frac{1}{8} t^{-7/8} dt$$

$$I_1 = \int_0^\infty t^{1/8} e^{-t} \cdot \frac{1}{8} t^{-7/8} dt = \frac{1}{8} \int_0^\infty e^{-t} t^{-3/4} dt$$

$$\therefore I_1 = \frac{1}{8} \Gamma \frac{1}{4}$$

In I_2 , put $x^4 = t$

$$x = t^{1/4} \rightarrow dx = \frac{1}{4} t^{-3/4} dt$$

$$\therefore I_2 = \int_0^\infty t^{1/2} e^{-t} \frac{1}{4} t^{-3/4} dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} t^{-1/4} dt$$

$$I_2 = \frac{1}{4} \Gamma \frac{3}{4}$$

$$\therefore LHS = I_1 \cdot I_2 = \frac{1}{8} \Gamma \frac{1}{4} \cdot \frac{1}{4} \Gamma \frac{3}{4} = \frac{1}{32} \Gamma \frac{1}{4} \Gamma \frac{3}{4}$$

$$\Gamma \frac{1}{4} \Gamma \frac{3}{4} = \frac{\pi}{\sin \frac{\pi}{4}} = \sqrt{2} \pi \quad \left(\Gamma_P \Gamma_{-P} = \frac{\pi}{\sin P \pi} \right)$$

$$\text{LHS} = \frac{1}{32} \cdot \sqrt{2} \pi = \frac{\pi}{16\sqrt{2}} = \text{RHS.}$$

$$\underline{\text{Ex-7}} : - \int_0^1 x^m \left(\log \frac{1}{x} \right)^n dx$$

$$\begin{aligned} \underline{\text{Soln.}} : - I &= \int_0^1 x^m \left(\log \frac{1}{x} \right)^n dx = \int_0^1 x^m (-\log x)^n dx \\ &= (-1)^n \int_0^1 x^m (\log x)^n dx \\ \text{put } \log x &= -t \quad \rightarrow x = e^{-t} \\ dt &= -e^{-t} dt \end{aligned}$$

$$\begin{aligned} x &= 0, \quad t = \infty \\ x &= 1 \quad t = 0 \end{aligned}$$

$$I = (-1)^n \int_{\infty}^0 (-e^{-t})^m (-t)^n (-e^{-t}) dt$$

$$I = (-1)^n \int_{\infty}^0 (-1) (-e^{-t})^{m+1} t^n dt$$

$$= \int_0^{\infty} (-e^{-t})^{m+1} t^n dt$$

$$\text{put } (m+1)t = u \rightarrow t = \frac{u}{m+1} \rightarrow dt = \frac{du}{m+1}$$

$$I = \int_0^{\infty} e^{-u} / \frac{u}{m+1}^n du$$

$$I = \int_0^\infty e^{-u} \left(\frac{u}{m+1}\right)^n \frac{du}{m+1}$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-u} u^n du$$

$$I = \frac{1}{(m+1)^{n+1}} \sqrt{n+1}$$

$$\text{Ex 8} :- \int_0^1 (x \log n)^3 dx$$

$$\text{Soln, } I = \int_0^1 x^3 (\log n)^3 dx$$

$$\text{put } \log n = -t \rightarrow n = e^{-t} \rightarrow dn = -e^{-t} dt$$

$$\text{when } n=0, t=\infty, n=1, t=0$$

$$I = \int_{\infty}^0 (-e^{-t})^3 (-t)^3 (-e^{-t}) dt$$

$$= (-1)^3 \int_0^\infty e^{-4t} t^3 dt$$

$$\text{put } 4t = u \rightarrow t = \frac{u}{4} \quad dt = \frac{du}{4}$$

$$I = - \int_0^\infty e^{-u} \left(\frac{u}{4}\right)^3 \frac{du}{4}$$

$$= -\frac{1}{4^4} \int_0^\infty e^{-u} u^3 du$$

$$= -\frac{1}{4^4} \frac{1}{14} = \frac{-3}{256} = -\frac{3}{128}$$

Ex-9 $\int_0^\infty \frac{x^7}{7^x} dx$

Soln:- $I = \int_0^\infty \frac{x^7}{7^x} dx$

put $7^x = e^t$

$$\Rightarrow t = x \log 7 \Rightarrow dt = (\log 7)dx \Rightarrow dx = \frac{dt}{\log 7}$$

when $x=0, t=0 \log 7 = 0$

when $x=\infty t=(\infty) \log 7 = \infty$

$$\therefore I = \int_0^\infty \left(\frac{t}{\log 7}\right)^7 \cdot e^t \cdot \frac{dt}{\log 7}$$

$$\therefore I = \frac{1}{(\log 7)^8} \int_0^\infty e^t t^7 dt = \frac{1}{(\log 7)^8} = \frac{7!}{(\log 7)^8}$$

Ex-10: $\int_0^\infty 7^{-4x^2} dx$

Ex-10 :- } + dx

$$\underline{\text{Soln}} : - I = \int_0^\infty 7^{-4x^2} dx$$

$$\text{put } 7^{-4x^2} = e^{-t}$$

$$-4x^2 (\log 7) = -t \Rightarrow 4x^2 \log 7 = t$$

$$x^2 = \frac{t}{4 \log 7} \Rightarrow x = \sqrt{\frac{t}{4 \log 7}}$$

$$dx = \frac{1}{2\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} dt$$

when $x=0, t=0$ and when $x=\infty, t=\infty$

$$\therefore I = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{4\sqrt{\log 7}} \cdot \sqrt{\frac{1}{2}} =$$

$$I = \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$$

Ex-11 :- Show that

$$(i) \int_0^\infty x^{m-1} \cos mx dx = \frac{T_m}{am} \cos\left(\frac{m\pi}{2}\right)$$

$$(i) \int_0^\infty x^{m-1} \cos ax dx = \frac{1}{am} \cos\left(\frac{m\pi}{2}\right)$$

$$(ii) \int_0^\infty x^{m-1} \sin ax dx = \frac{1}{am} \sin\left(\frac{m\pi}{2}\right)$$

$[e^{-iam} = \cos am - i \sin am]$

Sol: - $I = \int_0^\infty x^{m-1} e^{-iam} dx$

put $iam = t \rightarrow a = \frac{t}{ia} \rightarrow dx = \frac{dt}{ia}$

$$I = \int_0^\infty \left(\frac{t}{ia}\right)^{m-1} e^{-t} \frac{dt}{ia}$$

$$= \frac{1}{(ia)^m} \int_0^\infty e^{-t} t^{m-1} dt$$

$$I = \frac{1}{(ia)^m} \left(\frac{1}{i^m} \right)$$

$$\text{Now } i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$i^m = \cos \frac{m\pi}{2} + i \sin \frac{m\pi}{2}$$

$$\frac{1}{i^m} = \cos \frac{m\pi}{2} - i \sin \frac{m\pi}{2}$$

$$I = \frac{1}{(ia)^m} \left[\cos\left(\frac{m\pi}{2}\right) - i \sin\left(\frac{m\pi}{2}\right) \right]$$

$$\int_0^\infty e^{-iam} x^{m-1} dx = \frac{1}{(ia)^m} \left[\cos\left(\frac{m\pi}{2}\right) - i \sin\left(\frac{m\pi}{2}\right) \right]$$

$$\int_0^\infty e^{-ixn} x^{m-1} dx = \frac{i}{\alpha^m} \left(\cos\left(\frac{m\pi}{2}\right) - i \sin\left(\frac{m\pi}{2}\right) \right)$$

$$\int_0^\infty x^{m-1} \cos \alpha x dx - i \int_0^\infty x^{m-1} \sin \alpha x dx = \frac{\overline{m}}{\alpha^m} \cos\left(\frac{m\pi}{2}\right) - i \frac{\overline{m}}{\alpha^m} \sin\left(\frac{m\pi}{2}\right)$$

Comparing real and imaginary parts we get the required answers.

Ex-12:- prove that $\int_0^\infty \cos(\alpha x^n) dx = \frac{\overline{n+1}}{\alpha^n} \cos\left(\frac{n\pi}{2}\right)$

Soln:- $I = \int_0^\infty \cos(\alpha x^n) dx$

put $\alpha x^n = t$

$$x^n = \frac{t}{\alpha} \rightarrow x = \frac{t^n}{\alpha^n} \rightarrow dx = \frac{n t^{n-1}}{\alpha^n} dt$$

$$I = \int_0^\infty \cos t \frac{n t^{n-1}}{\alpha^n} dt$$

$$= \frac{n}{\alpha^n} \int_0^\infty t^{n-1} \cos t dt$$

$$= R.P of \underline{n} \int_0^\infty t^{n-1} e^{-it} dt$$

$$= R.P of \int_0^{\infty} t^n e^{-t} dt$$

complete as the previous sum.

Ex-13 prove that

$$(i) \int_0^{\infty} x e^{-ax} \cos bx dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}$$

$$(ii) \int_0^{\infty} x e^{-ax} \sin bx dx = \frac{2ab}{(a^2 + b^2)^2}$$

Sol:-

$$I = \int_0^{\infty} x e^{-ax} \cdot e^{ibx} dx$$

$$= \int_0^{\infty} x e^{-(a-ib)x} dx$$

$$\text{put } (a-ib)x = t \quad x = \frac{t}{a-ib} \rightarrow dx = \frac{dt}{a-ib}$$

$$I = \int_0^{\infty} e^{-t} \cdot \frac{t}{a-ib} \cdot \frac{dt}{a-ib}$$

$$= \frac{1}{(a-ib)^2} \int_0^{\infty} e^{-t} t dt$$

$$I = \frac{1}{(a-ib)^2} \overline{t^2} = \frac{1}{(a-ib)^2} = \frac{1}{(a-ib)^2}$$

$$= \frac{1}{(a^2 - b^2) - 2iab} \cdot \frac{(a^2 - b^2) + 2iab}{(a^2 - b^2) + 2iab}$$

$$= \frac{(a^2 - b^2) + i2ab}{(a^2 - b^2) + i2ab} = \frac{(a^2 - b^2) + i2ab}{\therefore 1^2}$$

$$= \frac{(a^2 - b^2) + i2ab}{(a^2 - b^2)^2 + 4a^2b^2} = \frac{(a^2 - b^2) + 12ab}{(a^2 + b^2)^2}$$

$$\int_0^\infty e^{-(a-ib)x} dx = \frac{(a^2 - b^2)}{(a^2 + b^2)^2} + i \frac{2ab}{(a^2 + b^2)^2}$$

$$\int_0^\infty e^{-ax} \cos bx dx + i \int_0^\infty e^{-ax} \sin bx dx = \frac{a^2 - b^2}{(a^2 + b^2)^2} + i \frac{2ab}{(a^2 + b^2)^2}$$

Comparing real and imaginary parts,
we get the answer.