

DAM (SEM III)

MODULE 3

RELATIONS AND FUNCTIONS

UNIT NO :3.2

Relations

Cartesian product

Consider two Non-empty sets X and Y .

The set of all ordered pairs (x,y) where $x \in X$ and $y \in Y$ is called the **Cartesian product**, of X and Y .

it is denoted by $X \times Y$, which is read “ X cross Y .”

Definition

$$X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$$

EXAMPLE

Let $X = \{1, 2\}$ and $Y = \{10, 15, 20\}$. Then write
 $X \times Y, Y \times X, X \times X$

$$X \times Y \\ = \{(1, 10), (1, 15), (1, 20), (2, 10), (2, 15), (2, 20)\}$$

$$Y \times X \\ = \{(10, 1), (15, 1), (20, 1), (10, 2), (15, 2), (20, 2)\}$$

$$\text{Also, } X \times X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

Definition : Relation

A **relation** from a set X to a set Y is any subset of the Cartesian product $X \times Y$, those pair (x,y) which are related with each other.

A relation can be stated as a rule (Infinite sets) or can be given as set of ordered pairs (Finite)

EXAMPLE

Let $X = \{1, 2\}$ and $Y = \{10, 15, 20\}$.

And we can define $R = \{(1,10) (2,20)\}$

Terminologies

The set of first components in the ordered pairs is called the **domain** of the relation and the set of second components is called the **range** of the relation.

For $X = \{1, 2\}$, $Y = \{10, 15, 20\}$ *and*

$R = \{(1, 10) (2, 20)\}$

Domain of $R = \{1, 2\}$

Range of $R = \{10, 20\}$

Terminologies

Suppose R is a relation from X to Y . Then R is a set of ordered pairs where each first element comes from X and each second element comes from Y . That is, for each pair $x \in X$ and $y \in Y$, exactly one of the following is true:

- i. $(x, y) \in R$; we then say “ x is R – related to y ”, written xRy .
- ii. $(x, y) \notin R$; we then say “ x is not R – related to y ”, written $\neg xRy$

Examples : Relation

1. $A = \{1, 2, 3, 4\}$ Then write R as ordered pairs if relation R “is less than” i.e. aRb if $a < b$.

$$R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

2. Let A is \mathbb{Z}^+ (set of positive integers) and R defined as “divides” i.e. aRb if a divides b .

e.g. $3 R 15, 7 R 35$ etc.

3. $A = \{2, 3, 4, 5, 6\}$ relation defined by aRb if $|a - b|$ is divisible by 3, write R as set

$$R = \{(2,5), (5,2), (3,6), (6,3)\}$$

(For finite – ordered pair, For infinite – rule)

Definition : Inverse of R

Let R be any relation from a set A to set B .

The **inverse** of R , denoted by R^{-1} , is the relation from B to A which consists of those ordered pairs, when reversed, belong to R .

That is: $R^{-1} = \{(b, a) : (a, b) \in R\}$

REPRESENTATION OF RELATIONS:

Matrix of a Relation (M_R)

Matrices can be easily used to represent relation

EXAMPLE: For $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$

If $R = \{(1,x), (2,x), (3,y), (3,z)\}$ then matrix of R , M_R is

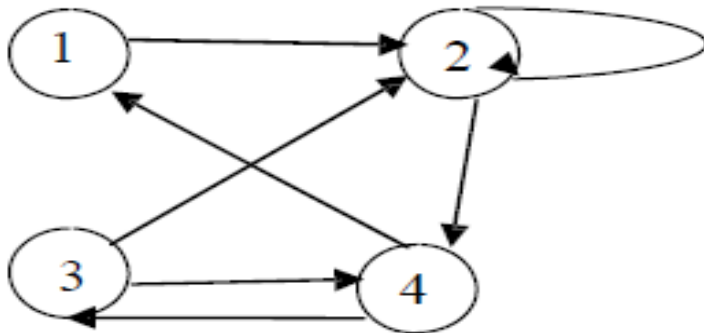
	x	y	z
1	1	0	0
2	1	0	0
3	0	1	1
4	0	0	0

REPRESENTATION OF RELATIONS:

Digraph: Another way of pictorial representation is **digraph**. i.e. Directed Graph

For $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 2), (2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$ Then, the digraph of R is drawn as follows:



The directed graphs are very important data structures that have applications in Computer Science (in the area of networking).

Composite Relation

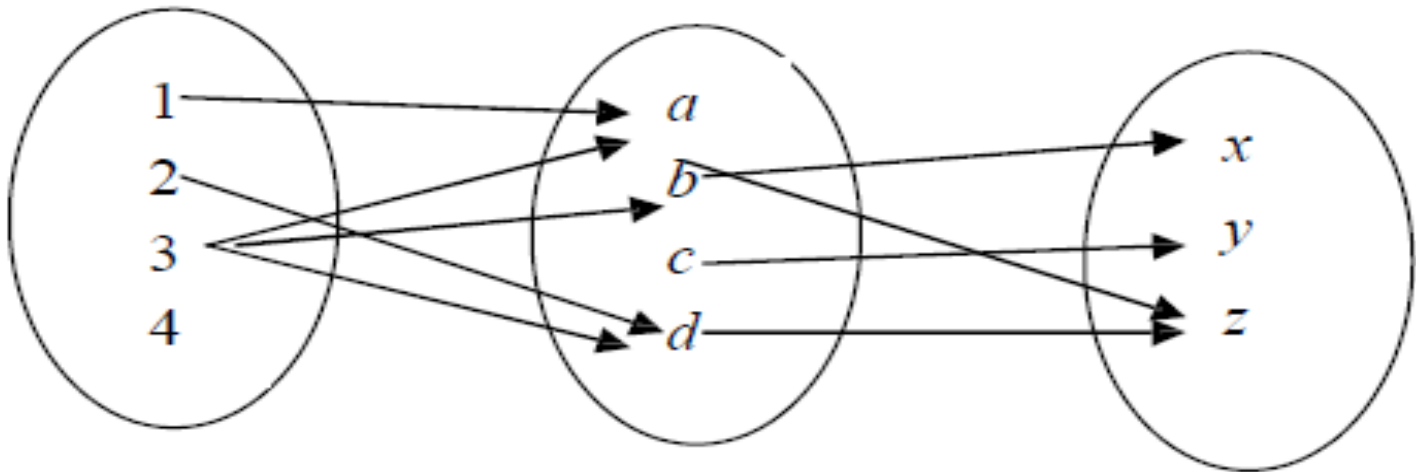
Let A , B and C be three sets.

Let R be a relation from A to B and S be a relation from B to C .

Then, composite relation **$S \circ R$** is a relation from A to C defined by,

$a(S \circ R) c$, if there is some $b \in B$, such that $a R b$ and $b S c$.

Example : Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$. Write SoR



SoR will be given as below.

SoR = $\{(2, z), (3, x), (3, z)\}$.

Properties of Relation

Reflexive Relation

Let A be a nonempty set, a relation R on A is said to be reflexive if for each $a \in A$, $(a, a) \in R$.

Example

Let $A = \{a, b, c, d\}$ and R be defined as follows:

$R = \{(a, a), (a, c), (b, a), (b, b), (c, c), (d, c), (d, d)\}$.

Is R a reflexive relation ?

YES

Properties of Relation

Symmetric Relation

Let A be a nonempty set, a relation R on A is said to be symmetric if for each pair of elements $a, b \in A$,
 $(a, b) \in R$ implies $(b, a) \in R$.

Example

Let $A = \{1, 2, 3, 4\}$ and R be defined as:

$R = \{(1, 2), (2, 3), (2, 1), (3, 2), (3, 3)\}$,

Is R a symmetric relation ?

YES

Observations

If we draw a diagraph of a reflexive relation, then all the vertices will have a loop.

Also if we represent reflexive relation using a matrix, then all its diagonal entries will be 1.

Also if we represent symmetric relation using a matrix then the matrix will be symmetric matrix

Properties of Relation

Anti-Symmetric Relation

Let A be a nonempty set,

A relation R on A is said to be anti-symmetric,
if $a R b$ and $b R a$, then $a = b$, for every $a, b \in A$

Thus, R is not anti-symmetric if there exists $a, b \in A$
such that $a R b$ and $b R a$ but $a \neq b$.

If R is not symmetric or Anti-symmetric then it is called
asymmetric

Example

Example1: Let $A = \{a, b, c, d\}$

R be defined as: $R = \{(a, b), (b, a), (a, c), (c, d), (d, b)\}$.

Check whether R is symmetric or anti-symmetric ?

R is not symmetric, as $a R c$ but $c \not R a$.

R is not anti-symmetric, because $a R b$ and $b R a$, but $a \neq b$,
Hence R is asymmetric

Example2: The relation “less than or equal to (\leq)”, on set of real is an anti-symmetric relation

Because If $a \leq b$ and $b \leq a$ then $a = b$

Check relation “is subset of (\subseteq)”, on set of all subsets of A is an anti-symmetric relation

Properties of Relation

Transitive Relation

Let A be a nonempty set, a relation R on A is said to be transitive if for each triplet of element

$a, b, c \in A$, If $(a, b), (b, c) \in R \implies (a, c) \in R$.

Example

Relation “ a divides b ”, on the set of integers, is a transitive relation.

If $a \mid b$ and $b \mid c$ then $a \mid c$

The relation “less than or equal to (\leq) Or (\geq)”, on set of real numbers is a transitive relation.

If $a \leq b$ and $b \leq c$ then $a \leq c$

Properties of Relation

Partial order Relation

A relation R on the set A is said to be ***partial order relation***, if it is reflexive, anti-symmetric and transitive.

Example : Let $A = \{a, b, c, d, e\}$.

Relation R , represented using following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Is R partial order relation ?

ANS :Yes

Example : Let A be a set of natural numbers and relation R be “less than or equal to relation (\leq)”. Then R is a partial order relation on A .

Answer :

For any $m, n, k \in N$,

$n \leq n$ (reflexive);

if $m \leq n$ and $n \leq m$, then $m = n$ (anti-symmetric);

lastly, if $m \leq n$ and $n \leq k$, then $m \leq k$ (transitive)

Properties of Relation

Equivalence Relation

Let A be a nonempty set.

A relation R on set A is said to be equivalence relation if R is reflexive, symmetric and transitive

Example : Consider the set L of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other.

Is this relation an equivalence relation?

Yes .

$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in R$.

Now,

Let $(L_1, L_2) \in R$.

$\Rightarrow L_1$ is parallel to L_2 .

$\Rightarrow L_2$ is parallel to L_1 .

$\Rightarrow (L_2, L_1) \in R$

$\therefore R$ is symmetric.

Now,

Let $(L_1, L_2), (L_2, L_3) \in R$.

$\Rightarrow L_1$ is parallel to L_2 . Also, L_2 is parallel to L_3 .

$\Rightarrow L_1$ is parallel to L_3 .

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

Example: Determine whether the relation R on a set A is reflexive, symmetric, antisymmetric, transitive, equivalence or partial order.

A = set of all positive integers, $a R b$ iff $|a - b| \leq 2$

R is reflexive because $|a - a| = 0 < 2, \forall a \in A$

R is symmetric because $|a - b| \leq 2 \Rightarrow |b - a| \leq 2 \therefore a R b \Rightarrow b R a$

R is not antisymmetric because $1 R 2$ & $2 R 1$ $1 R 2 \Rightarrow |1 - 2| \leq 2$ & $2 R 1 \Rightarrow |2 - 1| \leq 2$. But $1 \neq 2$

R is not transitive because $5 R 4, 4 R 2$ but $5 \not R 2$

Since it is Not transitive it can not be Partial order or equivalence relation

Terminologies

Congruence

Let m be a fixed positive integer.

Two integers, a, b are said to be congruent modulo m , if m divides $a - b$ (i.e. $a - b = km$ where k is an integer) written as: $a \equiv b \pmod{m}$.

The congruence relation is an equivalence relation (Check!!)

Divides

a is said to be divisible by b (or b divides a) if $a = b.k$ where k is an integer

divides is a Partial order relation (Check!!)

Let R be a relation defined on a set of integers as $a R b$ if $a \equiv b \pmod{5}$ prove that R is an equivalence relation

1. Reflexive: for every integer x , $x - x = 0$ is divisible by 5
so $x \equiv x \pmod{5}$.

2. Symmetric: if $x \equiv y \pmod{5}$ then $x - y$ is divisible by 5

$\Rightarrow x - y = 5k$ where k is an integer

$\Rightarrow y - x = -5k$

$\Rightarrow y - x$ is also divisible by 5

hence $y \equiv x \pmod{5}$.

3. Transitive: assume $x \equiv y \pmod{5}$ and $y \equiv z \pmod{5}$.

Then $x - y = 5m$ and $y - z = 5n$ where m, n are integers

From here, $x - z = (x - y) + (y - z) = 5m + 5n = 5(m + n)$

$\Rightarrow x - z$ is divisible by 5

$\Rightarrow x \equiv z \pmod{5}$.

Example: Check whether relation R on a set of real numbers is reflexive, symmetric, or transitive. $a R b$ if $a \leq b^2$

$$R = \{(a, b) / a \leq b^2\}$$

$$\text{Since } (1/2) > (1/2)^2$$

$$\Rightarrow (1/2, 1/2) \notin R$$

$\therefore R$ is not reflexive.

$$\text{Now, } (1, 4) \in R \text{ as } 1 < 4^2$$

But, 4 is not less than 1^2 .

$$\therefore (4, 1) \notin R$$

$\therefore R$ is not symmetric.

$$(3, 2), (2, 1.5) \in R$$

$$\text{(as } 3 < 2^2 = 4 \text{ and } 2 < (1.5)^2 = 2.25)$$

$$\text{But, } 3 > (1.5)^2 = 2.25$$

$$\therefore (3, 1.5) \notin R$$

$\therefore R$ is not transitive.

Partition

A partition of a set A is a collection of non-empty subsets A_1, A_2, A_3, \dots of A which are pairwise disjoint and whose union equals A

$$1. A_i \cap A_j = \Phi \quad \text{for } i \neq j$$

$$2. \bigcup_n A_n = A$$

Example: Is $P = \{\{1,2\}, \{3,5\}, \{4,5,6\}\}$ partition of $A = \{1,2,3,4,5,6\}$?

Let $A = \{1, 2, 3, 4, 5, 6\}$.

$A_1 = \{1, 2\}; A_2 = \{3, 5\}; A_3 = \{4, 5, 6\}$.

$A = A_1 \cup A_2 \cup A_3$ but $A_2 \cap A_3 \neq \phi$.

P is not partition of A

Example:

Is $P = \{\{1,2\}, \{3,5\}, \{4\}\}$ partition of $A = \{1,2,3,4,5\}$?

$$A_1 = \{1, 2\}; A_2 = \{3, 5\}; A_3 = \{4\}.$$

$$A_1 \cap A_2 = \emptyset, A_1 \cap A_3 = \emptyset, \text{ and } A_2 \cap A_3 = \emptyset.$$

$$A = A_1 \cup A_2 \cup A_3$$

P is partition of A

Equivalence Class

Let R be an equivalence relation on a set A

Let $x \in A$

the set of elements of A related to x is called the **equivalence class of x** , represented by $[x]$

$$[x] = \{y \in A \mid yRx\}.$$

The collection of equivalence classes, represented A/R

$A/R = \{[x] \mid x \in A\}$, is called **Quotient set of A by R**

If R is an equivalence relation on A , then collection of sets $[a]$ or $R(a)$ is called as **equivalence classes of R** .

Theorem

Let R be an equivalence relation on a set A .
Then A/R is a partition of A .

Specifically:

- (i) For each a in A , we have $a \in [a]$. (So every element is covered, nothing is left)
- (ii) $[a] = [b]$ if and only if $(a, b) \in R$ or $b \in [a]$
- (iii) If $[a] \neq [b]$ or $b \notin [a]$,
then $[a]$ and $[b]$ are disjoint.

Example

Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$. Show that R is an equivalence relation on A hence find partition of A induced by R

Consider $M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix}$

It is Reflexive since all diagonal elements are 1.

It is symmetric since Matrix is symmetric.

Due to block structure of Matrix it is transitive. (Check)

Hence R is Equivalence relation on the set A .

We observe that $R(1) = [1] = \{1, 2\} = [2] = R(2)$

and $R(3) = [3] = \{3, 4\} = [4] = R(4)$

hence $A/R = P = \{ \{1, 2\}, \{3, 4\} \}$ which is partition of set A .

Example let $A = \{1, 2, \dots, 8\}$. Let R be the relation defined by $x \equiv y \pmod{4}$. Write R as a set of ordered pairs, Check that it is equivalence relation. Find the partition of A induced by R .

$$R = \{(1,1), (1,5), (2,2), (2,6), (3,3), (3,7), (4,4), (4,8), \\ (5,1), (5,5), (6,2), (6,6), (7,3), (7,7), (8,4), (8,8)\}$$

(Prove equivalence same as previous examples)

Then Equivalence classes of R are

$$[1] = \{1, 5\} = [5], [2] = \{2, 6\} = [6],$$

$$[3] = \{3, 7\} = [7] \text{ and } [4] = \{4, 8\} = [8]$$

So, the partition of A induced by R is

$$A/R = \{[1], [2], [3], [4]\} \text{ *or* } \{[1], [2], [7], [8]\} \text{ etc.}$$

Example : Consider the set L of lines in the Euclidean plane. Two lines in the plane are said to be related, if they are parallel to each other. Show that R is an equivalence relation on A hence find an equivalence class of $y=2x+4$

We have already proved the First Part. Now

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in \mathbf{R}$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in \mathbf{R}$.

Construction of Z_5

Let $A = Z$ (set of integers) and define R as

$R = \{(a, b) \in A \times A : a \equiv b \pmod{5}\}$. Then, we have,

$$R(1) = \{\dots, -14, -9, -4, 1, 6, 11, \dots\}$$

$$R(2) = \{\dots, -13, -8, -3, 2, 7, 12, \dots\}$$

$$R(3) = \{\dots, -12, -7, -2, 3, 8, 13, \dots\}$$

$$R(4) = \{\dots, -11, -6, -1, 4, 9, 14, \dots\}$$

$$R(5) = \{\dots, -10, -5, 0, 5, 10, 15, \dots\}.$$

$R(1), R(2), R(3), R(4)$ and $R(5)$ form partition on Z with respect to given equivalence relation.

$$Z/R = \{R(1), R(2), R(3), R(4), R(5)\}$$

$$Z_5 = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$$

Similarly we can construct any set Z_n using equivalence classes of modulo n

Example 7.47 : Let R and S are equivalence relation on $A = \{1, 2, 3, 4\}$ given by

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$$

$$S = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1)\}$$

Determine partition of A induced by

(i) R^{-1}

(ii) $R \cap S$

Solution : (i)

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$$

$$R^{-1} = \{(1, 1), (2, 1), (1, 2), (2, 2), (4, 3), (3, 4), (3, 3), (4, 4)\}$$

$$\therefore [1]_{R^{-1}} = \{1, 2\}$$

$$[2]_{R^{-1}} = \{1, 2\}$$

$$[3]_{R^{-1}} = \{3, 4\}$$

$$[4]_{R^{-1}} = \{3, 4\}$$

Here,

$$[1]_{R^{-1}} = [2]_{R^{-1}} \text{ and } [3]_{R^{-1}} = [4]_{R^{-1}}$$

$$\therefore \text{Partition of } A \text{ induced by } R^{-1} = [\{1, 2\}, \{3, 4\}]$$

(ii)

$$R \cap S = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$[1]_{R \cap S} = \{1\}$$

$$[2]_{R \cap S} = \{2\}$$

$$[3]_{R \cap S} = \{3\}$$

$$[4]_{R \cap S} = \{4\}$$

$$\therefore \text{Partition of } A \text{ induced by } R \cap S = [\{1\}, \{2\}, \{3\}, \{4\}]$$