LAPLACE TRANSFORM

I. FIND THE LAPLACE TRANSFORM OF FOLLOWING FUNCTIONS:

1. $f(t) = (t-1)^4$, $t > 4$; $f(t) = 0$, $0 < t < 4$

2. $f(t) = t$, $0 < t < 1/2$; $f(t) = t-1$, $1/2 < t < 1$; $f(t) = 0$, $t > 1$ [Ans: $\dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2} - \dfrac{e^{-s/2}}{s}$]

3. $f(t) = 0$, $0 < t < \pi$; $f(t) = \sin^2(t - \pi)$, $t > \pi$ [Ans: $\dfrac{e^{-\pi s}}{2s} - \dfrac{s.e^{-\pi s}}{s^2 + 4}$]

4. $\cos t \cdot \cos 2t \cdot \cos 3t$ [Ans: $\dfrac{1}{4}\left(\dfrac{1}{s} + \dfrac{s}{s^2 + 2^2} + \dfrac{s}{s^2 + 4^2} + \dfrac{s}{s^2 + 6^2}\right)$]

5. $\left(\sqrt{t} - 1\right)^2$ [Ans: $\dfrac{1}{s^2} - \dfrac{\sqrt{\pi}}{s^{3/2}} + \dfrac{1}{s}$]

6. $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ [Ans: $\sqrt{\dfrac{\pi}{s}} \cdot e^{-1/4s}$]

7. If $L\left[\sin\sqrt{t}\right] = \dfrac{\sqrt{\pi}}{2s\sqrt{s}} \cdot e^{-1/4s}$, find $L\left[\sin 2\sqrt{t}\right]$ [Ans: $\dfrac{\sqrt{\pi}}{s\sqrt{s}} \cdot e^{-1/s}$]

8. $\sinh(t/2) \cdot \sin(\sqrt{3}\,t/2)$ [Ans: $\dfrac{\sqrt{3}}{2} \cdot \dfrac{s}{(s^4 + s^2 + 1)}$]

9. $e^{4t}\sin^3 t$ [Ans: $\dfrac{6}{(s^2 - 8s + 17)(s^2 - 8s + 25)}$]

10. $\dfrac{\cos 2t \cdot \sin t}{e^t}$ [Ans: $\dfrac{s^2 + 2s - 2}{(s^2 + 2s + 10)(s^2 + 2s + 2)}$]

11. $e^{-4t}\sinh t \cdot \sin t$ [Ans: $\dfrac{2(s+4)}{(s^2 + 6s + 10)(s^2 + 10s + 26)}$]

12. $e^{2t}(1+t)^2$ [Ans: $\dfrac{1}{(s-2)} + \dfrac{2}{(s-2)^2} + \dfrac{2}{(s-2)^3}$]

13. If $L[f(t)] = \dfrac{s}{s^2 + s + 4}$, find $L[e^{-3t} f(2t)]$ [Ans: $\dfrac{s+3}{s^2 + 8s + 10}$]

14. $\left(1 + te^{-t}\right)^3$ [Ans: $\dfrac{1}{s} - \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^3}$]

15. $t\sin^3 t$ [Ans: $24 \cdot \dfrac{s(s+5)}{(s^2 + 1)^2 (s^2 + 9)^2}$]

16. $t^5 \cosh t$

[Ans: $60\left(\dfrac{1}{(s-1)^6} + \dfrac{1}{(s+1)^6}\right)$]

17. $t\sqrt{1+\sin t}$

[Ans: $4\dfrac{(4s^2+4s-1)}{(4s^2+1)^2}$]

18. $t\left(\dfrac{\sin t}{e^t}\right)^2$

[Ans: $\dfrac{1}{2}\left(-\dfrac{1}{(s+2)^2} + \dfrac{s^2+4s}{(s^2+4s+8)^2}\right)$]

19. If $L[f(t)] = \dfrac{s+3}{s^2+s+1}$, find $L[t\,f(2t)]$ [Ans: $\dfrac{s^2+12s+8}{(s^2+2s+4)^2}$]

20. $t\,e^{-2t}\sinh 4t$

[Ans: $\dfrac{8(s+2)}{(s^2+4s-12)^2}$]

21. $t\cos(\omega t - \alpha)$

[Ans: $\dfrac{(s^2-\omega^2)\cos\alpha - 2\omega s\sin\alpha}{(\omega^2+s^2)^2}$]

22. $(t\sinh 2t)^2$

[Ans: $\dfrac{1}{2}\left(\dfrac{1}{(s-4)^3} + \dfrac{1}{(s+4)^3}\right)$]

23. $(t+\sin 2t)^2$

[Ans: $\dfrac{2}{s^3} + \dfrac{8s}{(s^2+4)^2} + \dfrac{1}{2s} - \dfrac{s}{2(s^2+4^2)}$]

24. $\dfrac{1}{t}(1-\cos t)$

[Ans: $\dfrac{1}{2}\log\left(\dfrac{s^2+1}{s^2}\right)$]

25. $\dfrac{1}{t}e^{-t}\sin t$

[Ans: $\cot^{-1}(s+1)$]

26. $\dfrac{\sin^2 2t}{t}$

[Ans: $\dfrac{1}{4}\log\left(\dfrac{s^2+16}{s^2}\right)$]

27. $\dfrac{1-\cos t}{t^2}$

[Ans: $\dfrac{\pi}{2} - \dfrac{s}{2}\log\left(\dfrac{s^2+1}{s^2}\right) - \tan^{-1}s$]

28. Find the Laplace transform of $\dfrac{\sin at}{t}$. Does Laplace transform of $\dfrac{\cos at}{t}$ exist?

   [Ans: $\cot^{-1}\dfrac{s}{a}$, does not exist]

29. $\dfrac{\cosh 2t \sin 2t}{t}$

[Ans: $\pi + \tan^{-1}\left(\dfrac{s-2}{2}\right) + \tan^{-1}\left(\dfrac{s+2}{2}\right)$]

30. $\dfrac{e^{-at} - \cos at}{t}$

[Ans: $\log\left(\dfrac{\sqrt{s^2+a^2}}{s+a}\right)$]

31. Given that $f(t) = t+1, \; 0 \le t \le 2$, & $f(t) = 3, \; t > 2$ find $L[f(t)], \; L[f'(t)] \,\&\, L[f''(t)]$

   [Ans: $\dfrac{1}{s} + \dfrac{1}{s^2}(1-e^{-2s}), \; \dfrac{1}{s}(1-e^{-2s}), \; s^2\left[\dfrac{1}{s} + \dfrac{1}{s^2}(1-e^{-2s})\right] - s - 1$]

32. Find the Laplace transform of $\dfrac{d}{dt}\left(\dfrac{\sin 3t}{t}\right)$ [Ans: $s\cot^{-1}(s/3) - 3$]

33. $\mathrm{erf}\sqrt{t}$

[Ans: $\dfrac{1}{s\sqrt{s+1}}$]

34. $\mathrm{erf}\,2\sqrt{t}$

[Ans: $\dfrac{2}{s\sqrt{s+4}}$]

35. $e^{3t} t \, erf \sqrt{t}$ [Ans: $\dfrac{3s-7}{2(s-3)^2(s-2)^{3/2}}$]

36. $\displaystyle\int_0^t\int_0^t\int_0^t t \sin t \, (dt)^3$ [Ans: $\dfrac{2}{s^2(s^2+1)^2}$]

37. $\displaystyle\int_0^t ue^{-3u}\cos^2 2u \, du$ [Ans: $\dfrac{1}{2s(s+3)^2}+\dfrac{s^2+6s-7}{2s(s^2+6s+25)^2}$]

38. $\displaystyle\int_0^t \dfrac{1-e^{-au}}{u}\, du$ [Ans: $\dfrac{1}{s}\log\left(\dfrac{s-a}{s}\right)$]

39. $t^{-1}\displaystyle\int_0^t e^{-u}\sin u \, du$ [Ans: $\dfrac{1}{4}\log\left(\dfrac{s^2+2s+2}{s^2}\right)-\dfrac{1}{2}\cot^{-1}(s+1)$]

40. $e^{-4t}\displaystyle\int_0^t u \sin 3u \, du$ [Ans: $\dfrac{6}{(s^2+8s+25)^2}$]

41. $\cosh t \displaystyle\int_0^t e^u \cosh u \, du$ [Ans: $\dfrac{1}{2}\left[\dfrac{s-2}{(s-1)^2(s-3)}+\dfrac{s}{(s+1)^2(s-1)}\right]$]

42. $\displaystyle\int_0^t ue^{-3u}\sin^2 u \, du$ [Ans: $\dfrac{1}{2s}\left[\dfrac{1}{(s+3)^2}+\dfrac{s^2+6s+5}{(s^2+6s+13)^2}\right]$]

43. $\dfrac{1}{t}(\cos at - \cos bt)$ [Ans: $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$]

44. Find $L\{\cosh 2t \cdot erf 3\sqrt{t}\}$ if $L\{erf \sqrt{t}\}=\dfrac{1}{s\sqrt{s+1}}$ [Ans:

$\dfrac{1}{2}\left[\dfrac{3}{(s+2)\sqrt{s+7}}+\dfrac{3}{(s-2)\sqrt{s+11}}\right]$]

45. If $L\left(2\sqrt{\dfrac{t}{\pi}}\right)=\dfrac{1}{s^{3/2}}$, show that $L\left(\dfrac{1}{\sqrt{\pi t}}\right)=\dfrac{1}{\sqrt{s}}$

46. A function $f(t)$ obeys the equation $f(t)+2\displaystyle\int_0^t f(t)\, dt = \cosh 2t$ find the Laplace

transform of $f(t)$ [Ans: $\dfrac{s^2}{(s^2-4)(s+2)}$]

II. EVALUATE THE FOLLOWING INTEGRALS USING LAPLACE TRANSFORM:

1. $\displaystyle\int_0^\infty e^{-2t}\sin^3 t \, dt$ [Ans: 6/65]

2. If $\displaystyle\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\, dt = 3/8$ then find α.   [Ans: $\pi/4$]

3. $\displaystyle\int_0^\infty e^{-3t}\, t \sin t \, dt$ [Ans: 3/50]

4. If $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$, prove that $\displaystyle\int_0^\infty e^{-3t} t \, J_0(4t)\, dt = 3/125$

5. $\displaystyle\int_0^\infty \dfrac{t^2 \sin 3t}{e^{2t}}\, dt$ [Ans: 18/2197]

6. $\displaystyle\int_0^\infty \dfrac{\cos at - \cos bt}{t}\, dt$ [Ans: $\log\dfrac{b}{a}$]

7. $\displaystyle\int_0^\infty e^{-st}\, \dfrac{\sin^2(at/2)}{t}\, dt$ [Ans: $\dfrac{1}{2}\log\left(\dfrac{s^2+a^2}{s^2}\right)$]

8. Prove that $\displaystyle\int_0^\infty e^{-st}\, \dfrac{\sin t \sinh t}{t}\, dt = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2a}{1+s^2-a^2}\right)$

9. $\displaystyle\int_0^\infty \dfrac{e^{-t} - \cos t}{t\, e^{4t}}\, dt$ [Ans: $\log\dfrac{\sqrt{17}}{5}$]

10. Prove that $\displaystyle\int_0^\infty \dfrac{\sin 2t + \sin 3t}{t\, e^t}\, dt = \dfrac{3\pi}{4}$

11. $\displaystyle\int_0^\infty e^{-2t} \sinh t\, \dfrac{\sin t}{t}\, dt$ [Ans: $\dfrac{1}{2}\tan^{-1}\dfrac{1}{2}$]

12. $\displaystyle\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u \, du\right) dt$ [Ans: $-\dfrac{2}{125}$]

13. $\displaystyle\int_0^\infty e^{-4t}\left(\cosh t \int_0^t e^u \cosh u \, du\right) dt$ [Ans: 31/225]

14. Prove that $\displaystyle\int_0^\infty e^{-st}\, \dfrac{\sin bt + \sin at}{t}\, dt = \pi - \tan^{-1}\left(\dfrac{s(a+b)}{ab - s^2}\right)$

15. $\displaystyle\int_0^\infty e^{-t} \sin^5 t \, dt$ [Ans: $\dfrac{3}{8}$]

16. $\displaystyle\int_0^\infty \dfrac{\cos 4t - \cos 3t}{t}\, dt$ [Ans: $\log\dfrac{3}{4}$]

17. $\displaystyle\int_0^\infty e^{-t} t^3 \sin t \, dt$ [Ans: 0]

18. $\displaystyle\int_{t=0}^\infty \int_{u=0}^t \dfrac{e^{-t} \sin u}{u}\, du\, dt$ [Ans: $\dfrac{\pi}{4s}$]