APPLICATIONS OF DE MOIVER'S THEOREM:

1) Expansion of $\boldsymbol{sin\ n\theta, cos\ n\theta\ in\ powers\ of\ sin\ \theta, \cos\theta}$:

By De Moivre's theorem $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$$= cos^n\,\theta + {}^nC_1 cos^{n-1}\theta.\,i\sin\theta + {}^nC_2\,cos^{n-2}\theta.\,(i\,sin\,\theta)^2 + {}^nC_3 cos^{n-3}\theta(i\sin\theta)^3 + \ldots\ldots.$$

$$= (cos^n\,\theta - {}^nC_2 cos^{n-2}\theta sin^2\theta + \ \ldots\ldots)$$

$$+\, i(\,{}^nC_1\,cos^{n-1}\,\theta\sin\theta - {}^nC_3 cos^{n-3}\theta sin^3\theta + \ldots\ldots\ldots)$$

Comparing real imaginary part on both sides

$$cos\,n\theta = cos^n\theta - {}^nC_2 cos^{n-2}\theta sin^2\theta + \ldots\ldots\ldots\ldots\ldots$$

$$sin\,n\theta = \ {}^nC_1 cos^{n-1}\theta\sin\theta - {}^nC_3\,cos^{n-2}sin^3\theta + \ldots\ldots\ldots\ldots.$$

SOME SOLVED EXAMPLES:

1. Using De Moivre's Theorem, prove that, Cos $3\theta = cos^3\theta - 3\cos\theta\,sin^2\theta$ and $\sin 3\theta = 3\sin\theta\,cos^2\theta - sin^3\theta$

Solution: By De Moivre's theorem,

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equating real and imaginary parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

2. Using De Moivre's Theorem express $\sin 3\theta, \cos 3\theta$, $\tan 3\theta$ in terms of powers of $\sin\theta$, $\cos\theta$, $\tan\theta$ respty.

Solution: continue as example (1) and obtain

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^2\theta - \sin^3\theta$$

$$= 3\sin\theta - 4\,sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta)$$
$$= \cos^3\theta - 3\cos\theta + 3\cos^2\theta$$
$$= 4\,cos^3\theta - 3\cos\theta$$
$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

Dividing the numerator and denominator by $\cos^3\theta$

$$\tan 3\theta = \frac{(3\tan\theta - tan^3\theta)}{(1-3\,tan^2\theta)}$$

3. Show that, **(i)** $\sin 5\theta = 5\sin\theta - 20\,sin^3\theta + 16\,sin^5\theta$

(ii) $\cos 5\theta = 5\cos\theta - 20cos^3\theta + 16\,cos^5\theta$

Solution: By De Moivre's Theorem, $(cos\ 5\theta + i\ sin\ 5\theta) = (cos\ \theta + i\ sin\ \theta)^5$

$$= cos^5\theta + 5\,cos^4\theta(i\sin\theta) + 10cos^3\theta(i\sin\theta)^2 + 10cos^2\theta(i\sin\theta)^3$$
$$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$ … Using Binomial Theorem

$$= cos^5\theta + i\ 5cos^4\theta\ \sin\theta - 10cos^3\theta sin^2\theta + i.10cos^2\theta sin^3\theta + 5\cos\theta sin^4\theta + i\ sin^5\theta$$
$$= (cos^5\theta - 10\ cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta)$$
$$+ i(5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta)$$

Equating real and imaginary parts

$$\cos 5\,\theta = cos^5\theta - 10\ cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta$$
$$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta$$

We have $\sin 5\theta = 5\,cos^4\,\theta\ \sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$

$$= 5(1-sin^2\theta)^2\sin\theta - 10(1-sin^2\theta)\,sin^3\,\theta + sin^5\theta$$
$$= 5(1-2\,sin^2\theta + sin^4\theta)\sin\theta - 10(1-\ sin^2\theta)sin^3\,\theta + sin^5\theta$$
$$= 5\sin\theta - 20sin^3\,\theta + 16\,sin^5\theta$$

And $\cos 5\theta = cos^5\theta - 10\,cos^3\theta\ \ sin^2\theta + 5\cos\theta\,sin^4\theta$

$$= cos^5\theta - 10cos^3\theta(1-cos^2\theta) + 5\cos\theta(1-cos^2\theta)^2$$
$$= cos^5\theta - 10cos^3\theta(1-cos^2\theta) + 5\cos\theta\,(1-2cos^2\theta + cos^4\theta)$$
$$= 5\,cos\theta - 20cos^3\theta + 16\,cos^5\theta$$

4. Show that, $\frac{\sin 5\theta}{\sin \theta} = 16\,cos^4\theta - 12\,cos^2\theta + 1$

Solution: From above example (3)

$$\sin 5\theta = 5cos^4\theta\ \sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta$$

$$\therefore \frac{\sin 5\theta}{\sin\theta} = 5cos^4\theta - 10cos^2\theta sin^2\theta + sin^4\theta$$

$$= 5cos^4\theta - 10cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$$

$$= 5cos^4\theta - 10cos^2\theta + 10cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$$

$$= 16\,cos^4\theta - 12\,cos^2\theta + 1$$

5. Use De Moiver's Theorem to show that $tan5\theta = \frac{5\tan\theta - 10\,tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta}$ and hence deduce that $5tan^4\frac{\pi}{10} - 10tan^2\frac{\pi}{10} + 1 = 0$

Solution: From above example (3)

$$\cos 5\,\theta = cos^5\theta - 10\ cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta$$

$$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta$$

$$\therefore \tan 5\theta = \frac{\sin 5\,\theta}{\cos 5\,\theta} = \frac{5\,cos^4\theta\sin\theta - 10cos^2\theta sin^3\theta + sin^5\theta}{cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta}$$

Dividing the numerator and denominator by $cos^5\theta$

$$\tan 5\theta = \frac{\tan\theta - 10tan^3\theta + tan^5\theta}{1 - 10tan^2\theta + 5tan^4\theta} \quad \text{................(1)}$$

Now, Put $\theta = \frac{\pi}{10}$.

Then $\tan 5\theta = \tan\frac{\pi}{2} = \infty$ and hence the denominator in (1) must be zero.

$$\therefore 5\,tan^4\frac{\pi}{10} - 10tan^2\frac{\pi}{10} + 1 = 0\ .$$

6. If $\sin 6\theta = a\,cos^5\theta\sin\theta + b\,cos^3\theta\,sin^3\theta + c\cos\theta\ sin^5\theta$, find the values of a, b, c.

Solution: By De Moivre's Theorem $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$

$$= (\cos\theta)^6 + 6(\cos\theta)^5(i\sin\theta) + 15(\cos\theta)^4(i\sin\theta)^2 + 20(\cos\theta)^3(i\sin\theta)^3$$

$$+15(\cos\theta)^2(i\sin\theta)^4+6(\cos\theta)^1(i\sin\theta)^5+(i\sin\theta)^6$$

$$=\cos^6\theta+i6\cos^5\theta\sin\theta-15\cos^4\theta\sin^2\theta-i20\cos^3\theta\sin^3\theta+15\cos^2\theta\sin^4\theta$$
$$+i6\cos\theta\sin^5\theta-\sin^6\theta$$

$$=(\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta)$$
$$+i(6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta)$$

Equating imaginary parts, $\sin 6\theta=6\cos^6\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta$

Comparing with $\sin 6\theta=a\cos^5\theta\sin\theta+b\cos^3\theta\sin^3\theta+c\cos\theta\sin^5\theta$

we get, $a=6, b=-20, c=6$

7. Prove that,

$$\cos 8\theta=cos^8\theta-28\,cos^6\theta\,sin^2\theta+70\,cos^4\theta\,sin^4\theta-28\,cos^2\theta\,sin^6\theta+sin^8\theta$$
$$\sin 8\theta=8\,cos^7\theta\sin\theta-56\,cos^5\theta\,sin^3\theta+56cos^3\theta sin^5\theta-8\cos\theta\,sin^7\theta.$$

Solution: By De Moivre's Theorem $\cos 8\theta+i\sin 8\theta=(\cos\theta+i\sin\theta)^8$

$$=(\cos\theta)^8+8(\cos\theta)^7(i\sin\theta)+28(\cos\theta)^6(i\sin\theta)^2+56(\cos\theta)^5(i\sin\theta)^3$$
$$+70(\cos\theta)^4(i\sin\theta)^4+56(\cos\theta)^3(i\sin\theta)^5+28(\cos\theta)^2(i\sin\theta)^6$$
$$+8(\cos\theta)(i\sin\theta)^7+(i\sin\theta)^8$$

$$=\cos^8\theta+i\cos^7\theta\sin\theta-28\cos^6\theta\sin^2\theta-i56\cos^5\theta\sin^3\theta+28\cos^4\theta\sin^4\theta$$
$$+i56\cos^3\theta\sin^5\theta-28\cos^2\theta\sin^6\theta-i8\cos\theta\sin^7\theta+\sin^8\theta$$

$$=(\cos^8\theta-28\cos^6\theta\sin^2\theta+70\cos^4\theta\sin^4\theta-28\cos^2\theta\sin^6\theta+\sin^8\theta)$$
$$+i(8\cos^7\theta\sin\theta-56\cos^5\theta\sin^3\theta+56\cos^3\theta\sin^5\theta-8\cos\theta\sin^7\theta)$$

Equating real and imaginary parts

$$\cos 8\theta=cos^8\theta-28\,cos^6\theta\,sin^2\theta+70\,cos^4\theta\,sin^4\theta-28\,cos^2\theta\,sin^6\theta+sin^8\theta$$
$$\sin 8\theta=8\,cos^7\theta\sin\theta-56\,cos^5\theta\,sin^3\theta+56cos^3\theta sin^5\theta-8\cos\theta\,sin^7\theta.$$

8. Using De Moivre's theorem prove that,

$$2(1+\cos 8\theta)=(x^4-4x^2+2)^2\ where\ x=2\cos\theta.$$

Solution: $2(1+\cos 8\theta)=2(2cos^2 4\,\theta)=(2\cos 4\theta)^2$ ……………………(1)

To find $\cos 4\theta$ in powers of $\cos\theta$,

Consider $(\cos 4\theta+i\sin 4\theta)=(\cos\theta+i\sin\theta)^4$

$$= cos^4\theta + 4cos^3\theta i \sin\theta + 6cos^2\,\theta i^2 sin^2\theta + 4\,cos\theta i^3 sin^3\theta + i^4 sin^4\theta$$

$$= (cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta) + i(4cos^3\theta\sin\theta - 4\,cos\theta sin^3\theta)$$

Equating real Parts, $\cos 4\theta = cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta$

$$= cos^4\theta - 6cos^2\theta(1 - cos^2\theta) + (1 - cos^2\theta)^2$$

$$= cos^4\theta - 6cos^2\theta + 6cos^4\theta + 1 - 2cos^2\theta + cos^4\theta$$

$$= 8cos^4\theta - 8cos^2\theta + 1$$

$\therefore 2\cos 4\theta = 16cos^4\theta - 16cos^2\theta + 2$ Putting this value in (1)

$$2(1 + \cos 8\theta) = (16cos^4\theta - 16cos^2\theta + 2)^2$$

$$= [(2\cos\theta)^4 - 4(2\cos\theta)^2 + 2]^2$$

$$= (x^4 - 4x^2 + 2)^2 \quad \text{where } x = 2\,cos\,\theta$$

9. Prove that $\frac{1+\cos 9A}{1+\cos A} = [16cos^4 A - 8\,cos^3 A - 12\,cos^2 A + 4\cos A + 1]^2$

Solution: $\frac{1+\cos 9A}{1+\cos A} = \frac{2\cos^2\left(\frac{9A}{2}\right)}{2\cos^2\left(\frac{A}{2}\right)} = \left[\frac{\cos\left(\frac{9A}{2}\right)}{\cos\left(\frac{A}{2}\right)}\right]^2$

$$= \left(\frac{2\cos\left(\frac{9A}{2}\right)\sin\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)}\right)^2 = \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right)-\sin\left(\frac{9A}{2}-\frac{A}{2}\right)}{\sin A}\right]^2$$

$$= \left(\frac{\sin(5A)-\sin(4A)}{\sin A}\right)^2 \qquad \text{............... (1)}$$

By De Moivre's Theorem**,** $(cos\,5\theta + i\,sin\,5\theta) = (cos\,\theta + i\,sin\,\theta)^5$

$$= cos^5\theta + 5\,cos^4\theta(i\sin\theta) + 10cos^3\theta(i\sin\theta)^2 + 10cos^2\theta(i\sin\theta)^3$$

$$+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$ .... Using Binomial Theorem

$$cos^5\theta + i\,5cos^4\theta\,\sin\theta - 10cos^3\theta sin^2\theta - i.10cos^2\theta sin^3\theta + 5\cos\theta sin^4\theta + i\,sin^5\theta$$

$$= (cos^5\theta - 10\,cos^3\theta sin^2\theta + 5\cos\theta sin^4\theta) + i(5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta)$$

Equating imaginary parts

$$\sin 5\,\theta = 5\,cos^4\theta\sin\theta - 10cos^2\theta\,sin^3\theta + sin^5\theta \quad \text{............... (2)}$$

Consider $(\cos 4\theta + i\sin 4\theta) = (\cos\theta + i\sin\theta)^4$

$$= cos^4\theta + 4cos^3\theta i\sin\theta + 6cos^2\,\theta i^2 sin^2\theta + 4\,cos\theta i^3 sin^3\theta + i^4 sin^4\theta$$

$$= (cos^4\theta - 6cos^2\theta sin^2\theta + sin^4\theta) + i(4cos^3\theta\sin\theta - 4\,cos\theta sin^3\theta)$$

Equating imaginary parts

$\sin 4\theta = 4\,cos^3\theta \sin\theta - 4cos\theta\, sin^3\theta$ .... (3)

Put (2) & (3) in (1) we get

$$\frac{1+\cos 9A}{1+\cos A} = \left[\frac{(5\,cos^4 A\sin A - 10cos^2 A sin^3 A + sin^5 A) - (4\,cos^3 A\sin A - 4cosA sin^3 A)}{\sin A}\right]^2$$

$$= (5\cos^2 A - 10\cos^2 A\sin^2 A + \sin^4 A - 4\cos^2 A + 4\cos A\sin^2 A)^2$$

$$= [5\cos^2 A - 10\cos^2 A\,(1-\cos^2 A) + (1-\cos^2 A)^2 - 4\cos^3 A + 4\cos A\,(1-\cos^2 A)]^2$$

$$= [5\cos^2 A - 10\cos^2 A + 10\cos^4 A + 1 - 2\cos^2 A + \cos^4 A - 4\cos^3 A + 4\cos A - 4\cos^3 A]^2$$

$$= (16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1)^2$$

10. Prove that $\frac{1-\cos 9A}{1-\cos A} = [16\,cos^4 A + 8\,cos^3 A - 12\,cos^2 A - 4\cos A + 1]^2$

Solution:

$$\frac{1-\cos 9A}{1-\cos A} = \frac{2\sin^2\left(\frac{9A}{2}\right)}{2\sin^2\left(\frac{A}{2}\right)} = \left(\frac{\sin\left(\frac{9A}{2}\right)}{\sin\left(\frac{A}{2}\right)}\right)^2 = \left(\frac{2\sin\left(\frac{9A}{2}\right)\cos\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}\right)^2 = \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right)+\sin\left(\frac{9A}{2}-\frac{A}{2}\right)}{\sin A}\right]^2$$

$$= \left(\frac{\sin(5A)+\sin(4A)}{\sin A}\right)^2$$ Continue as above example

SOME PRACTICE PROBLEMS

1. Using De Moivre’s Theorem prove that,

$\cos 4\,\theta = cos^4\theta - 6\,cos^2\theta sin^2\theta + sin^4\theta$ and

$\sin 4\theta = 4cos^3\theta sin\theta - 4\cos\theta\, sin^3\theta$

2. Prove that, $\frac{\sin 6\theta}{\sin 2\theta} = 16\,cos^4\theta - 16\,cos^2\theta + 3$

3. If $\cos 6\theta = a\,cos^6\theta + b\,cos^4\theta\,sin^2\theta + c cos^2\theta\,sin^4\theta + d\,sin^6\theta$, find a, b, c, d.

4. Express $\sin 7\theta$ *and* $\cos 7\theta$ in terms of powers of $\sin\theta$ *and* $\cos\theta$.

5. Prove that, $\frac{\sin 7\theta}{\sin\theta} = 7 - 56\,sin^2\theta + 112\,sin^4\theta - 64\,sin^6\theta$

6. Show that $\tan 7\theta = \frac{7\tan\theta - 35tan^3\theta + 21tan^5\theta - tan^7\theta}{1 - 21tan^2\theta + 35tan^4\theta - 7tan^6\theta}$.

7. Express $\tan 7\theta$ in terms of powers of $\tan\theta$

Hence deduce $7\,tan^6\frac{\pi}{14} - 35\,tan^4\frac{\pi}{14} + 21\,tan^2\frac{\pi}{14} - 1 = 0$

8. Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2\ where\ x = 2\cos\theta$

9. Prove that $\frac{1-\cos 7\theta}{1-\cos\theta} = (x^3 + x^2 - 2x - 1)^2\ where\ x = 2\cos\theta$

Expansion of $\boldsymbol{sin^n\theta, cos^n\theta}$ in term of sin $\boldsymbol{n\ \theta, cos\ n\theta}$ (n is a + ve integer):

Let $x = \ cos\ \theta + i\ sin\ \theta = e^{i\theta}$ $\quad \therefore \frac{1}{X} = \ cos\ \theta - i\ sin\ \theta = e^{-i\theta}$

Hence $x + \frac{1}{x} = 2cos\ \theta$ and $x - \frac{1}{x} = 2i\ sin\ \theta$

Again, $x^n = (cos\ \theta + \ i\ sin\ \theta)^n = cos\ n\ \theta + i\ sin\ n\theta = e^{in\theta}$

$$\frac{1}{x^n} = (cos\ \theta - i\ sin\ \theta)^n = cos\ n\theta - i\ sin\ n\theta = e^{-in\theta}$$

$$x^n + \frac{1}{x^n} = 2\cos \text{n}\ \theta \quad \text{and} \quad x^n - \frac{1}{x^n} = 2i\ sin\ n\theta$$

To expand $cos^n\theta$, write $cos^n\theta = \frac{1}{2^n}\left(x + \frac{1}{x}\right)^n$

To expand $sin^n\theta$, write $sin^n\theta = \frac{1}{(2i)^n}\left(x - \frac{1}{x}\right)^n$ and expand R.H.S. using binomial expansion $\quad (x + a)^n = x^n \ + {}^nC_1x^{n-1}a \ + \ {}^nC_2x^{n-2}a^2 \ +$……….+ a^n

SOME SOLVED EXAMPLES:

1. Show that $sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Solution: Let $x = \cos\theta + i\sin\theta, then\ \frac{1}{x} = cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and\ \ x - \frac{1}{x} = 2\,\text{i}\sin\theta \quad \text{…………(1)}$$

$$x^n = \cos n\theta + i\sin n\theta\ \ and\ \ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and\ \ x^n - \frac{1}{x^n} = 2\,\text{i}\sin n\theta \quad \text{…………(2)}$$

$$\therefore (2\,i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5 \quad \text{from (1)}$$

$$= x^5 - 5x^4.\frac{1}{x} + 10x^3.\frac{1}{x^2} - 10x^2.\frac{1}{x^3} + 5x.\frac{1}{x^4} - \frac{1}{x^5}$$

$$= x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$$

$$32\, i^5 sin^5\theta = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$\therefore 32\ i\ sin^5\theta = 2\ i \sin 5\ \theta - 5(2i \sin 3\theta) + 10\ (2i\ sin\theta)$ from (2)

$$\therefore sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

2. Using De Moivre's Theorem prove that, $cos^6\theta + sin^6\theta = \frac{1}{8}(3\cos 4\,\theta + 5)$

Solution: Let $x = \cos\theta + \ i\sin\theta, then\ \frac{1}{x} = \cos\theta - \ i\ sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta \quad and\ \ x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta$ ....(1)

$x^n = \cos n\theta + i\sin n\theta\ \ and\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore x^n + \frac{1}{x^n} = 2\ \cos n\theta \quad and\ \ x^n - \frac{1}{x^n} = 2\ \mathrm{i}\sin n\theta$ ....(2)

$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6$ from (1)

$$= x^6 + 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x.\frac{1}{x^5} + \frac{1}{x^6}$$

$2^6 cos^6\theta = x^6 + 6x^4 + 15x^2 + 20 + 15.\frac{1}{x^2} + 6.\frac{1}{x^4} + \frac{1}{x^6}$ ....(3)

$(2\ i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6$ from (1)

$$= x^6 - 6x^5.\frac{1}{x} + 15x^4.\frac{1}{x^2} - 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} - 6x.\frac{1}{x^5} + \frac{1}{x^6}$$

$$-2^6 sin^6\theta = x^6 - 6x^4 + 15x^2 - 20 + 15.\frac{1}{x^2} - 6.\frac{1}{x^4} + \frac{1}{x^6}$$

$\therefore 2^6 sin^6\theta = -x^6 + 6x^4 - 15x^2 + 20 - 15.\frac{1}{x^2} + 6.\frac{1}{x^4} - \frac{1}{x^6}$ ....(4)

Adding (3) and (4) $2^6(cos^6\theta + sin^6\theta) = 12x^4 + 40 + 12.\frac{1}{x^4}$

$$2^6(cos^6\theta + sin^6\theta) = 4\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$$

$$\therefore cos^6\theta + sin^6\theta = \frac{1}{16}\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$$

$\therefore cos^6\theta + sin^6\theta = \frac{1}{16}[6\cos 4\theta + 10]$ from (2)

$$= \frac{1}{8}[3\ \cos 4\theta + 5]$$

3. Expand $sin^7\theta$ in a series of sines of multiples of θ

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \;\; and \;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{............(2)}$$

$$(2\,i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7 \quad \text{from (1)}$$

$$= x^7 - 7x^6.\frac{1}{x} + 21x^5.\frac{1}{x^2} - 35x^4.\frac{1}{x^3} + 35x^3.\frac{1}{x^4} - 21x^2.\frac{1}{x^5} + 7x.\frac{1}{x^6} - \frac{1}{x^7}$$

$$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$-2^7 i\, sin^7\theta = 2i\sin 7\theta - 7.(2i\sin 5\theta) + 21.(2i\sin 3\theta) - 35.(2i\sin\theta) \qquad \text{from (2)}$$

$$\therefore -2^6 sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$$

$$\therefore sin^7\theta = -\frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$$

4. Expand $cos^7\theta$ in a series of cosines of multiples of θ

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \;\; and \;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{............(2)}$$

$$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7 \qquad \text{............. from (1)}$$

$$= x^7 + 7x^6\frac{1}{x} + 21x^5\frac{1}{x^2} + 35x^4\frac{1}{x^3} + 35x^3\frac{1}{x^4} + 21x^2\frac{1}{x^5} + 7x\frac{1}{x^6} + \frac{1}{x^7}$$

$$= x^7 + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$\therefore 2^7\cos^7\theta = 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(\cos\theta) \qquad \text{From (2)}$$

$$cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta]$$

5. Show that $2^5 sin^4\theta cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2.$

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{............(2)}$$

$$(2i\sin\theta)^4(2\cos\theta)^4 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^2 \quad \text{From (1)}$$

$$\therefore 2^6\sin^4\theta\cos^2\theta = \left(x - \frac{1}{x}\right)^2\left(x - \frac{1}{x}\right)^2\left(x + \frac{1}{x}\right)^2 = \left(x - \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^2$$

$$= \left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$= x^6 - 2x^2 + \frac{1}{x^2} - 2x^4 + 4 - \frac{2}{x^4} + x^2 - \frac{2}{x^2} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - \left(x^2 + \frac{1}{x^2}\right) + 4$$

$$= 2\cos 6\theta - 2(2\cos 4\theta) - (2\cos 2\theta) + 4 \quad \text{From (2)}$$

$$\therefore 2^5\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

6. Prove that $cos^5\theta sin^3\theta = -\frac{1}{2^7}\,[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta]$

Solution: Let $x = \cos\theta + i\sin\theta \quad \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\mathrm{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \quad and \quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\mathrm{i}\sin n\theta \quad \text{............(2)}$$

$$(2\cos\theta)^5(2i\sin\theta)^3 = \left(x + \frac{1}{x}\right)^5\left(x - \frac{1}{x}\right)^3$$

$$2^8 i^3\cos^5\theta\sin^3\theta = \left(x + \frac{1}{x}\right)^2\left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)^3$$

$$-2^8 i\cos^5\theta\sin^3\theta = \left(x + \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= x^8 - 3x^4 + 3 - \frac{1}{x^4} + 2x^6 - 6x^2 + \frac{6}{x^2} - \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) - 6\left(x^2 - \frac{1}{x^2}\right)$$

$$= (2i\sin 8\theta) + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta) \quad \text{From (2)}$$

$$\therefore -2^7\cos^5\theta\sin^3\theta = \sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta$$

$$\therefore \cos^5\theta\sin^3\theta = -\frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right]$$

7. If $sin^4\theta\, cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$,

Prove that $a_1 + 9a_3 + 25a_5 + 49a_7 = 0$.

Solution: Let $x = \cos\theta + i\sin\theta$ $\therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore x + \frac{1}{x} = 2\cos\theta \quad and \quad x - \frac{1}{x} = 2\,\text{i}\sin\theta \quad \text{............(1)}$$

$$x^n = \cos n\theta + i\sin n\theta \;\; and \;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \quad and \quad x^n - \frac{1}{x^n} = 2\,\text{i}\sin n\theta \quad \text{............(2)}$$

Consider $(2\,i\sin\theta)^4(2\,\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$

$$= \left(x - \frac{1}{x}\right)^3\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)^3$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x - \frac{1}{x}\right)$$

$$= \left(x^6 - 3x^2 + 3.\frac{1}{x^2} - \frac{1}{x^6}\right)\left(x - \frac{1}{x}\right)$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$\therefore (2\,i\sin\theta)^4(2\cos\theta)^3 = 2\cos 7\theta - 2\cos 5\theta - 6cos3\theta + 6\cos\theta \quad \text{from (2)}$$

$$\therefore sin^4\theta cos^3\theta = \frac{\cos 7\theta}{2^6} - \frac{\cos 5\theta}{2^6} - \frac{3\cos 3\theta}{2^6} + \frac{3\cos\theta}{2^6}$$

Comparing this with the given equality, $a_1 = \frac{3}{2^6}, a_3 = -\frac{3}{2^6}, a_5 = -\frac{1}{2^6}, a_7 = \frac{1}{2^6}$

$$\therefore\ a_1 + 9a_3 + 25a_5 + 49a_7 = \frac{3}{2^6} - \frac{27}{2^6} - \frac{25}{2^6} + \frac{49}{2^6} = \frac{52-52}{2^6} = 0$$

SOME PRACTICE PROBLEMS:

1. Show that $cos^6\theta = \frac{1}{32}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$

2. Prove that $cos^6\theta - sin^6\theta = \frac{1}{16}[\cos 6\theta + 15\cos 2\theta]$

3. Express $sin^8\theta$ in a series of cosines of multiples of θ.

4. Prove that, $cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$

5. Prove that $cos^8\theta + sin^8\theta = \frac{1}{64}[\cos 8\theta + 28\cos 4\theta + 35]$.

6. Show that $2^6 sin^4\theta cos^3\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$.

7. Prove that $sin^7\theta cos^3\theta = -\frac{1}{512}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta]$

ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.

General values of $\cos\theta = \cos(2k\pi + \theta)$ and $\sin\theta = \sin(2k\pi + \theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\theta + i\sin\theta$, we apply De Moivre's theorem

$$z = (cos\,\theta + i\,sin\,\theta)^{\frac{1}{n}} = cos\frac{\theta}{n} + i\,sin\,\frac{\theta}{n}$$

This shows that $\left(cos\frac{\theta}{n} + i\,sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = cos\,\theta + i\,sin\,\theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}^{\frac{1}{n}} = cos\left(\frac{2k\pi+\theta}{n}\right) + i\,sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,……………,(n – 1). We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Solution: Consider $x^3 = 1 \quad \therefore x = 1^{1/3}$

$$\therefore x = (\cos 0 + i\sin 0)^{1/3} = (\cos 2\,k\,\pi + i\sin 2\,k\,\pi)^{1/3} = cos\frac{2\,k\,\pi}{3} + i\sin\frac{2\,k\,\pi}{3}$$

Putting $k = 0, 1, 2$, the cube roots of unity are

$x_0 = 1,\ x_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$ (say)

And $x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left[cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]^2 = \omega^2$

Now, $1 + \omega + \omega^2 \quad = 1 + \left(cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - 1 = 0$$

$\therefore 1 + \omega^2 = -\omega$

Now, $(1-\omega)^6 = [(1-\omega)^2]^3 = (1 - 2\omega + \omega^2)^3$

$$= (-\omega - 2\omega)^3 = (-3\omega)^3 - 27\omega^3 = -27$$

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

Solution: $\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos\left((8k+1)\frac{\pi}{4}\right) + i\sin\left((8k+1)\frac{\pi}{4}\right)\right]^{1/3}$$

$$\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) + i\sin\left((8k+1)\frac{\pi}{12}\right)$$

Similarly, $\sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) - i\sin\left((8k+1)\frac{\pi}{12}\right)$

$$\therefore \sqrt[3]{\frac{(1+i)}{\sqrt{2}}} + \sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = 2\cos\left((8k+1)\frac{\pi}{12}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as $2\cos\frac{\pi}{12},\ 2\cos\frac{9\pi}{12},\ 2\cos\frac{17\pi}{12}$

i.e., $2\cos\frac{r\pi}{12}$ where $r = 1, 9, 17$

3. Find the cube roots of $(1 - cos\theta - i\ sin\theta)$.

Solution: $$(1-\cos\theta - i\sin\theta)^{1/3} = \left[2\sin^2\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\left(2\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(2k\pi - \left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right) + i\sin\left(2k\pi - \left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{(4k-1)+\theta}{6}\right) + i\sin\left(\frac{(4k-1)+\theta}{6}\right)\right]$$

Putting $k = 0, 1, 2$ we get the three roots

4. Find the continued product of all the value of $(-i)^{2/3}$

Solution: $$(-i)^{2/3} = \left(0 + i(-1)\right)^{2/3} = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{2/3}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{2/3}$$

$$= \cos\left((4k+1)\frac{\pi}{3}\right) - i\sin\left((4k+1)\frac{\pi}{3}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$$\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right), \left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right), \left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

$\therefore$ Continued product

$$= \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)\left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{8\pi}{3} + \frac{9\pi}{3}\right) - i\sin\left(\frac{\pi}{3} + \frac{8\pi}{3} + \frac{9\pi}{3}\right)$$

$$= \cos 6\pi + i\sin 6\pi$$
$$= 1 - i(0)$$
$$= 1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

Solution: $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left\{\left(cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right\}^{1/4}$

$$= (\cos\pi + i\sin\pi)^{1/4}$$
$$= [cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}$$
$$= cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$$

Putting $k = 0,1,2,3$ we get the four roots as,

$$\left(cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \left(cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \left(cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \left(cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$\therefore \left(cos\frac{r\pi}{4} + i\ sin\frac{r\pi}{4}\right)\ where\ r = 1,3,5,7$

The required product $= cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + isin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$

$$= cos4\pi + i\sin 4\pi = 1.$$

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

Solution: $x^7 + x^4 + x^3 + 1 = 0$

$\therefore x^4(x^3+1) + (x^3+1) = 0$

$\therefore (x^3+1)(x^4+1) = 0$

$\therefore x^3 = -1, x^4 = -1$

Consider $x^3 = -1$

$\therefore x = (-1+i0)^{1/3} = (\cos\pi + i\sin\pi)^{1/3} = [\cos(2k+1)\pi - i\sin(2k+1)\pi]^{1/3}$

$$= \cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}$$

Putting $k = 0, 1, 2$ we get the three roots

Similarly from $x^4 = -1$ we get the remaining four roots as

$x = \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$ where $k = 0, 1, 2, 3$

7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

Solution: $x^4 + x^3 + x^2 + x + 1 = 0$

Multiplying the given equation by $x - 1$, we get $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

$\therefore$ We have $x^5 - 1 = 0$ $\therefore x^5 = 1 = \cos 0 + i\sin 0$

$\therefore x^5 = cos(2k\pi) + i\sin(2k\pi)$

$\therefore x = (\cos 2k\pi + i\sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$

Putting $k = 0, 1, 2, 3, 4$, we get the roots of the equation.

$x_0 = \cos 0 + i\sin 0 = 1,$

$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$ $x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5},$ $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

It is clear that 1 is the roots of $x - 1 = 0$

and the remaining roots are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$

i.e., $cos\frac{2\pi}{5} \pm i\ sin\frac{2\pi}{5},$ $cos\frac{4\pi}{5} \pm i\ sin\frac{4\pi}{5}$

8. SOLVE: $x^4 - x^2 + 1 = 0$

Solution: $x^4 - x^2 + 1 = 0$

Multiplying the given equation by $(x^2 + 1)$, we get, $(x^2+1)(x^4 - x^2 + 1) = 0$

$\therefore (x^2)^3 + (1)^3 = 0$ $\therefore x^6 + 1 = 0$ $\therefore x^6 = -1$

$\therefore x = (-1 + 0i)^{1/6} = (\cos\pi + i\sin\pi)^{1/6}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/6}$

$= \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots of equation

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ $x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$ $\qquad$ $x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i$ $\qquad$ $x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

It is clear that i and $-i$ are the roots of $x^2 + 1 = 0$ and the remaining roots x_0, x_2, x_3, x_5 are roots of $x^4 - x^2 + 1 = 0$

9. Find the roots common to $x^4 + 1 = 0$ *and* $x^6 - i = 0$.

Solution: Consider $x^4 + 1 = 0 \quad \therefore x^4 = -1$

$$x = (-1 + i0)^{1/4} = (\cos\pi + i\sin\pi)^{1/4} = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$$

$$x = \cos\left((2k+1)\frac{\pi}{4}\right) + i\sin\left((2k+1)\frac{\pi}{4}\right)$$

Putting $k = 0, 1, 2, 3$ we get the three roots as

$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = 1$ $\qquad$ $x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$ $\qquad$ $x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Now consider, $x^6 - i = 0 \quad \therefore x^6 = i$

$$x = (0 + 1i)^{1/6} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/6} = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$$

$$= \cos\left((4k+1)\frac{\pi}{12}\right) + i\sin\left((4k+1)\frac{\pi}{12}\right)$$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots as

$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ $\qquad$ $x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore$ common roots are $\pm\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

10. If $(1+x)^6 + x^6 = 0$

$show\ that\ x = -\frac{1}{2} - \frac{i}{2} cot\frac{\theta}{2}\ where \quad \theta = (2n+1)\,\pi/6\,, n = 0,1,2,3,4,5.$

Solution: $(1+x)^6 + x^6 = 0 \qquad \therefore \frac{(1+x)^6}{x^6} = -1$

$$\frac{1+x}{x} = (-1)^{1/6} = (\cos\pi + i\sin\pi)^{1/6} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$$

$$= \cos\left((2k+1)\frac{\pi}{6}\right) + i\sin\left((2k+1)\frac{\pi}{6}\right)$$

$$\frac{x+1-x}{x} = \cos\theta + i\sin\theta - 1$$

$$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$$

$$x = \frac{1}{(\cos\theta-1)+i\sin\theta} \times \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta} = \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{-2\sin^2(\theta/2) - i2\sin(\theta/2)\cos(\theta/2)}{2(2\sin^2(\theta/2))}$$

$$= -\frac{1}{2} - \frac{i}{2}\cot\left(\frac{\theta}{2}\right) \qquad \text{where } \theta = (2k+1)\frac{\pi}{6}$$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0\ is\ 1 + i$, find all other roots.

Solution: The given equation is $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

Since one of the root is $1 + i$

$\therefore$ other root must be $1 - i$ (since roots always occurs as complex conjugate pairs)

$\therefore x = 1 \pm i$ are the two roots

$\therefore x - 1 = \pm i$

$\therefore (x-1)^2 = (\pm i)^2$

$\therefore x^2 - 2x + 1 = -1$

$\therefore x^2 - 2x + 2 = 0$

Now we want to find other two remaining roots for that we divide

$x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 4x + 2$ and we obtain

$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = (x^2 - 4x + 2)(x^2 - 4x + 5)$

$\therefore$ the remaining two roots are the roots of equation $x^2 - 4x + 5 = 0$

$$\therefore x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)} = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm 2i}{2} = 2 \pm i$$

$\therefore$ The required remaining roots of given equation are $1 - i,\ 2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5.$

Solution: We have $x^5 = 1 = \cos O + i \sin O$ $\quad\therefore x^5 = cos(2k\pi) + i\sin(2k\pi)$

$$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i \sin\frac{2k\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as

$$x_0 = \cos 0 + i\sin 0 = 1, \qquad x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$$

$$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \quad x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \quad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5},$$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$, we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$, and hence

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \frac{x^5-1}{x-1}$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$$

Putting $x = 1$, we get $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^5) = 5$

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$.

Solution: We have $z^4 = i(z-1)^4$

$$\therefore \left(\frac{z}{z-1}\right)^4 = i = cos\frac{\pi}{2} + i\ sin\frac{\pi}{2} = cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$\therefore \frac{z}{z-1} = \left[cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{1/4}$$

$$= cos(4n+1)\frac{\pi}{8} + i\sin(4n+1)\frac{\pi}{8}$$

$$\therefore \frac{z}{z-1} = \cos\theta + i\sin\theta \qquad \text{where } \theta = (4n+1)\frac{\pi}{8}$$

$\therefore \frac{z}{z-1-z} = \frac{z}{-1} = \frac{\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$ Simplifying as in the above example, we get

$\therefore \frac{z}{-1} = \frac{-\sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)}$

$\therefore -z = -\frac{1}{2} + \frac{i}{2}\, cot\, \frac{\theta}{2}$

$\therefore z = \frac{1}{2} - \frac{i}{2}\, cot\, \frac{\theta}{2}, \quad where\ \theta = (4n+1)\frac{\pi}{8}$

For, $n = 0, 1, 2$, we get three roots, All these roots have the real part $1/2$

14. If ω is a 7[th] root of unity, prove that

$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$

if n is a multiple of 7 and is equal to zero otherwise.

Solution: We have $x = 1^{\frac{1}{7}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{7}}$

$$= \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}, \quad where\ \ n = 0, 1, 2, 3, 4, 5, 6$$

Let $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1 \therefore \omega^{7n} = 1^n = 1$

If n is not a multiple of 7, $\therefore \omega^n \neq 1$

Now, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \ \ldots\ldots\ldots + \omega^{6n} = \frac{1-\omega^{7n}}{1-\omega^n}$ sum of 7 terms of G.P

$$= \frac{1-1}{1-\omega^n} = \frac{0}{1-\omega^n} = 0$$

If n is a multiple of 7, say $n = 7k$

Then, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$

$$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + (\omega^7)^{4k} + (\omega^7)^{5k} + (\omega^7)^{6k}$$

$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$$

15. Prove that $\sqrt{1 + sec(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} + \left(1 + e^{-i\theta}\right)^{-1/2}$

Solution: We have to show that $\sqrt{1 + sec(\theta/2)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

Squaring both sides, we get, $1 + sec\frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

We shall prove this result

Now, $r.h.s = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{-i\theta}+e^{i\theta}+1}}$$

$$= 1 + \frac{2}{\sqrt{2+(e^{i\theta}+e^{-i\theta})}} = 1 + \frac{2}{\sqrt{2+2\cos\theta}}$$

$$= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}} = 1 + \frac{2}{\sqrt{4cos^2(\theta/2)}}$$

$$= 1 + \frac{2}{2cos(\theta/2)} = 1 + sec\frac{\theta}{2} = l.h.s$$

SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1 + \omega + \omega^2 = 0$ **(ii)** $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1 - i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9 + 8x^6 + x^3 + 8 = 0$ **(ii)** $x^4 - x^3 + x^2 - x + 1 = 0$

(iii) $(x+1)^8 + x^8 = 0$

9. If $(x+1)^6 = x^6, show\ that\ x = -\frac{1}{2} - i\ cot\frac{\theta}{2}\ where\ \theta = \frac{2k\pi}{6}, k = 0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7 = (x-1)^7$are given by $\pm i\, cot\frac{r\pi}{7}, r = 1,2,3.$

11. If $\alpha, \alpha^2, \alpha^3, \dots \alpha^6$ are the roots of $x^7 - 1 = 0$, find them and prove that

$(1-\alpha)(1-\alpha^2) \dots \dots \dots \dots (1-\alpha^6) = 7.$

12. Prove that $x^5 - 1 = (x-1)\left(x^2 + 2x\, cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\, cos\frac{3\pi}{5} + 1\right) = 0.$

13. Solve the equation $z^n = (z+1)^n$ and show that the real part of all the roots is $-1/2.$

14. If $a = e^{i\,2\pi/7}$ and $b = a + a^2 + a^4,\ c = a^3 + a^5 + a^6$**.** then prove that b & c are roots of quadratic equation $x^2 + x + 2 = 0.$

15. Prove that **(i)** $\sqrt{1 - cosce(\theta/2)} = \left(1 - e^{i\theta}\right)^{-1/2} - \left(1 - e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1 - sce(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} - \left(1 + e^{-i\theta}\right)^{-1/2}$

16. If $1 + 2i$ is a root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find all the other roots.