7.4.2 Runs Tests

1. *Runs up and runs down.* Consider a generator that provided a set of 40 numbers in the following sequence:

0.08	0.09	0.23	0.29	0.42	0.55	0.58	0.72	0.89	0.91
0.11	0.16	0.18	0.31	0.41	0.53	0.71	0.73	0.74	0.84
0.02	0.09	0.30	0.32	0.45	0.47	0.69	0.74	0.91	0.95
0.12	0.13	0.29	0.36	0.38	0.54	0.68	0.86	0.88	0.91

Both the Kolmogorov-Smirnov test and the chi-square test would indicate that the numbers are uniformly distributed. However, a glance at the ordering shows that the numbers are successively larger in blocks of 10 values. If these numbers are rearranged as follows, there is far less reason to doubt their independence:

0.41	0.68	0.89	0.84	0.74	0.91	0.55	0.71	0.36	0.30
0.09	0.72	0.86	0.08	0.54	0.02	0.11	0.29	0.16	0.18
0.88	0.91	0.95	0.69	0.09	0.38	0.23	0.32	0.91	0.53
0.31	0.42	0.73	0.12	0.74	0.45	0.13	0.47	0.58	0.29

The runs test examines the arrangement of numbers in a sequence to test the hypothesis of independence.

Before defining a run, a look at a sequence of coin tosses will help with some terminology. Consider the following sequence generated by tossing a coin 10 times:

$$H \quad T \quad T \quad H \quad H \quad T \quad T \quad T \quad H \quad T$$

There are three mutually exclusive outcomes, or events, with respect to the sequence. Two of the possibilities are rather obvious. That is, the toss can result in a head or a tail. The third possibility is "no event." The first head is preceded by no event and the last tail is succeeded by no event. Every sequence begins and ends with no event.

A run is defined as a succession of similar events preceded and followed by a different event. The length of the run is the number of events that occur in the run. In the coin-flipping example above there are six runs. The first run is of length one, the second and third of length two, the fourth of length three, and the fifth and sixth of length one.

There are two possible concerns in a runs test for a sequence of numbers. The number of runs is the first concern and the length of runs is a second concern. The types of runs counted in the first case might be runs up and runs down. An up run is a sequence of numbers each of which is succeeded by a larger number. Similarly, a down run is a sequence of numbers each of which is succeeded by a smaller number. To illustrate the concept, consider the following sequence of 15 numbers:

$$^{-}0.87 \quad ^{+}0.15 \quad ^{+}0.23 \quad ^{+}0.45 \quad ^{-}0.69 \quad ^{-}0.32 \quad ^{-}0.30 \quad ^{+}0.19 \quad ^{-}0.24$$
$$^{+}0.18 \quad ^{+}0.65 \quad ^{+}0.82 \quad ^{-}0.93 \quad ^{+}0.22 \quad 0.81$$

The numbers are given a "+" or a "−" depending on whether they are followed by a larger number or a smaller number. Since there are 15 numbers, and they are all different, there will be 14 +'s and −'s. The last number is followed by "no event" and hence will get neither a + nor a −. The sequence of 14 +'s and −'s is as follows:

− + + + − − − + − + + + − +

Each succession of +'s and −'s forms a run. There are eight runs. The first run is of length one, the second and third are of length three, and so on. Further, there are four runs up and four runs down.

There can be too few runs or too many runs. Consider the following sequence of numbers:

0.08 0.18 0.23 0.36 0.42 0.55 0.63 0.72 0.89 0.91

This sequence has one run, a run up. It is unlikely that a valid random-number generator would produce such a sequence. Next, consider the following sequence:

0.08 0.93 0.15 0.96 0.26 0.84 0.28 0.79 0.36 0.57

This sequence has nine runs, five up and four down. It is unlikely that a sequence of 10 numbers would have this many runs. What is more likely is that the number of runs will be somewhere between the two extremes. These two extremes can be formalized as follows: if N is the number of numbers in a sequence, the maximum number of runs is $N - 1$ and the minimum number of runs is one.

If a is the total number of runs in a truly random sequence, the mean and variance of a are given by

$$\mu_a = \frac{2N - 1}{3} \tag{7.4}$$

and

$$\sigma_a^2 = \frac{16N - 29}{90} \tag{7.5}$$

For $N > 20$, the distribution of a is reasonably approximated by a normal distribution, $N(\mu_a, \sigma_a^2)$. This approximation can be used to test the independence of numbers from a generator. In that case the standardized normal test statistic is developed by subtracting the mean from the observed number of runs, a, and dividing by the standard deviation. That is, the test statistic is

$$Z_0 = \frac{a - \mu_a}{\sigma_a}$$

Substituting Equation (7.4) for μ_a and the square root of Equation (7.5) for σ_a yields

$$Z_0 = \frac{a - [(2N - 1)/3]}{\sqrt{(16N - 29)/90}}$$

where $Z_0 \sim N(0, 1)$. Failure to reject the hypothesis of independence occurs when $-z_{\alpha/2} \leq Z_0 \leq z_{\alpha/2}$, where α is the level of significance. The critical values and rejection region are shown in Figure 7.3.

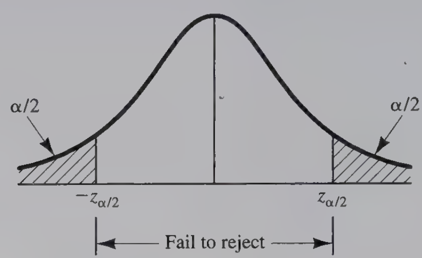

Figure 7.3. Failure to reject hypothesis.

EXAMPLE 7.8

Based on runs up and runs down, determine whether the following sequence of 40 numbers is such that the hypothesis of independence can be rejected where $\alpha = 0.05$.

0.41	0.68	0.89	0.94	0.74	0.91	0.55	0.62	0.36	0.27
0.19	0.72	0.75	0.08	0.54	0.02	0.01	0.36	0.16	0.28
0.18	0.01	0.95	0.69	0.18	0.47	0.23	0.32	0.82	0.53
0.31	0.42	0.73	0.04	0.83	0.45	0.13	0.57	0.63	0.29

The sequence of runs up and down is as follows:

+ + + − + − + − − − + + − + − − + − +
− − + − − + − + + − − + + − + − − + + −

There are 26 runs in this sequence. With $N = 40$ and $a = 26$, Equations (7.4) and (7.5) yield

$$\mu_a = \frac{2(40) - 1}{3} = 26.33$$

and

$$\sigma_a^2 = \frac{16(40) - 29}{90} = 6.79$$

Then,

$$Z_0 = \frac{26 - 26.33}{\sqrt{6.79}} = -0.13$$

Now, the critical value is $z_{0.025} = 1.96$, so the independence of the numbers cannot be rejected on the basis of this test. ◀

2. *Runs above and below the mean.* The test for runs up and runs down is not completely adequate to assess the independence of a group of numbers. Consider the following 40 numbers:

0.63	0.72	0.79	0.81	0.52	0.94	0.83	0.93	0.87	0.67
0.54	0.83	0.89	0.55	0.88	0.77	0.74	0.95	0.82	0.86
0.43	0.32	0.36	0.18	0.08	0.19	0.18	0.27	0.36	0.34
0.31	0.45	0.49	0.43	0.46	0.35	0.25	0.39	0.47	0.41

The sequence of runs up and runs down is as follows:

$$+ + + - + - + - - - + + - + - - + - + - - + - -$$
$$+ - + + - - + + - + - - + + -$$

This sequence is exactly the same as that in Example 7.8. Thus, the numbers would pass the runs-up and runs-down test. However, it can be observed that the first 20 numbers are all above the mean $[(0.99 + 0.00)/2 = 0.495]$ and the last 20 numbers are all below the mean. Such an occurrence is highly unlikely. The previous runs analysis can be used to test for this condition, if the definition of a run is changed. Runs will be described as being above the mean or below the mean. A "+" sign will be used to denote an observation above the mean, and a "−" sign will denote an observation below the mean.

For example, consider the following sequence of 20 two-digit random numbers:

0.40	0.84	0.75	0.18	0.13	0.92	0.57	0.77	0.30	0.71
0.42	0.05	0.78	0.74	0.68	0.03	0.18	0.51	0.10	0.37

The pluses and minuses are as follows:

$$- + + - - + + + - + - - + + + - - + - -$$

In this case, there is a run of length one below the mean followed by a run of length two above the mean, and so on. In all, there are 11 runs, five of which are above the mean and six of which are below the mean. Let n_1 and n_2 be the number of individual observations above and below the mean and let b be the total number of runs. Notice that the maximum number of runs is $N = n_1 + n_2$, and the minimum number of runs is one. Given n_1 and n_2, the mean—with a continuity correction suggested by Swed and Eisenhart [1943]—and the variance of b for a truly independent sequence are given by

$$\mu_b = \frac{2n_1n_2}{N} + \frac{1}{2} \tag{7.6}$$

and

$$\sigma_b^2 = \frac{2n_1n_2(2n_1n_2 - N)}{N^2(N - 1)} \tag{7.7}$$

For either n_1 or n_2 greater than 20, b is approximately normally distributed. The test statistic can be formed by subtracting the mean from the number of runs and dividing by the standard deviation, or

$$Z_0 = \frac{b - (2n_1n_2/N) - 1/2}{\left[\dfrac{2n_1n_2(2n_1n_2 - N)}{N^2(N-1)}\right]^{1/2}}$$

Failure to reject the hypothesis of independence occurs when $-z_{\alpha/2} \leq Z_0 \leq z_{\alpha/2}$, where α is the level of significance. The rejection region is shown in Figure 7.3.

EXAMPLE 7.9

Determine whether there is an excessive number of runs above or below the mean for the sequence of numbers given in Example 7.8. The assignment of $+$'s and $-$'s results in the following:

$$\begin{array}{cccccccccccccccccccc} - & + & + & + & + & + & + & + & - & - & - & + & + & - & + & - & - & - & - & - \\ - & - & + & + & - & - & - & - & + & + & - & - & + & - & + & - & - & + & + & - \end{array}$$

The values of n_1, n_2, and b are as follows:

$$n_1 = 18$$
$$n_2 = 22$$
$$N = n_1 + n_2 = 40$$
$$b = 17$$

Equations (7.6) and (7.7) are used to determine μ_b and σ_b^2 as follows:

$$\mu_b = \frac{2(18)(22)}{40} + \frac{1}{2} = 20.3$$

and

$$\sigma_b^2 = \frac{2(18)(22)[(2)(18)(22) - 40]}{(40)^2(40-1)} = 9.54$$

Since n_2 is greater than 20, the normal approximation is acceptable, resulting in a Z_0 value of

$$Z_0 = \frac{17 - 20.3}{\sqrt{9.54}} = -1.07$$

Since $z_{0.025} = 1.96$, the hypothesis of independence cannot be rejected on the basis of this test. ◀

3. *Runs test: length of runs.* Yet another concern is the length of runs. As an example of what might occur, consider the following sequence of numbers:

0.16, 0.27, 0.58, 0.63, 0.45, 0.21, 0.72, 0.87, 0.27, 0.15, 0.92, 0.85, ...

Assume that this sequence continues in a like fashion: two numbers below the mean followed by two numbers above the mean. A test of runs above and below the mean would detect no departure from independence. However, it is to be expected that runs other than of length two should occur.

Let Y_i be the number of runs of length i in a sequence of N numbers. For an independent sequence, the expected value of Y_i for runs up and down is given by

$$E(Y_i) = \frac{2}{(i+3)!}[N(i^2+3i+1) - (i^3+3i^2-i-4)], \quad i \le N-2 \tag{7.8}$$

$$E(Y_i) = \frac{2}{N!}, \quad i = N-1 \tag{7.9}$$

For runs above and below the mean, the expected value of Y_i is approximately given by

$$E(Y_i) = \frac{Nw_i}{E(I)}, \quad N > 20 \tag{7.10}$$

where w_i, the approximate probability that a run has length i, is given by

$$w_i = \left(\frac{n_1}{N}\right)^i \left(\frac{n_2}{N}\right) + \left(\frac{n_1}{N}\right)\left(\frac{n_2}{N}\right)^i, \quad N > 20 \tag{7.11}$$

and where $E(I)$, the approximate expected length of a run, is given by

$$E(I) = \frac{n_1}{n_2} + \frac{n_2}{n_1}, \quad N > 20 \tag{7.12}$$

The approximate expected total number of runs (of all lengths) in a sequence of length N, $E(A)$, is given by

$$E(A) = \frac{N}{E(I)}, \quad N > 20 \tag{7.13}$$

The appropriate test is the chi-square test with O_i being the observed number of runs of length i. Then the test statistic is

$$\chi_0^2 = \sum_{i=1}^{L} \frac{[O_i - E(Y_i)]^2}{E(Y_i)}$$

where $L = N-1$ for runs up and down and $L = N$ for runs above and below the mean. If the null hypothesis of independence is true, then χ_0^2 is approximately chi-square distributed with $L-1$ degrees of freedom.

EXAMPLE 7.10

Given the following sequence of numbers, can the hypothesis that the numbers are independent be rejected on the basis of the length of runs up and down at $\alpha = 0.05$?

0.30	0.48	0.36	0.01	0.54	0.34	0.96	0.06	0.61	0.85
0.48	0.86	0.14	0.86	0.89	0.37	0.49	0.60	0.04	0.83
0.42	0.83	0.37	0.21	0.90	0.89	0.91	0.79	0.57	0.99
0.95	0.27	0.41	0.81	0.96	0.31	0.09	0.06	0.23	0.77
0.73	0.47	0.13	0.55	0.11	0.75	0.36	0.25	0.23	0.72
0.60	0.84	0.70	0.30	0.26	0.38	0.05	0.19	0.73	0.44

For this sequence the +'s and −'s are as follows:

+ − − + − + − + + − + − + + − + + − +
− + − − + − + − − + − − + + + − − − + +
− − − + − + − − − + − + − − − + − + + −

The length of runs in the sequence is as follows:

1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 2, 1, 1,
1, 2, 1, 2, 3, 3, 2, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 2, 1

The number of observed runs of each length is as follows:

Run Length, i	1	2	3
Observed Runs, O_i	26	9	5

The expected numbers of runs of lengths one, two, and three are computed from Equation (7.8) as

$$
\begin{aligned}
E(Y_1) &= \frac{2}{4!}[60(1 + 3 + 1) - (1 + 3 - 1 - 4)] \\
&= 25.08 \\
E(Y_2) &= \frac{2}{5!}[60(4 + 6 + 1) - (8 + 12 - 2 - 4)] \\
&= 10.77 \\
E(Y_3) &= \frac{2}{6!}[60(9 + 9 + 1) - (27 + 27 - 3 - 4)] \\
&= 3.04
\end{aligned}
$$

The mean total number of runs (up and down) is given by Equation (7.4) as

$$\mu_a = \frac{2(60) - 1}{3} = 39.67$$

Thus far, the $E(Y_i)$ for $i = 1, 2,$ and 3 total 38.89. The expected number of runs of length 4 or more is the difference $\mu_a - \sum_{i=1}^{3} E(Y_i)$, or 0.78.

As observed by Hines and Montgomery [1990], there is no general agreement regarding the minimum value of expected frequencies in applying the chi-square test. Values of 3, 4, and 5 are widely used, and a minimum of 5 was suggested earlier in this chapter. Should an expected frequency be too small,

Table 7.4. Length of Runs Up and Down: χ^2 Test

Run Length, i	*Observed Number of Runs, O_i*	*Expected Number of Runs, $E(Y_i)$*	$\frac{[O_i - E(Y_i)]^2}{E(Y_i)}$
1	26	25.08	0.03
2 ≥ 3	$\left.\begin{array}{r}9\\5\end{array}\right\}14$	$\left.\begin{array}{r}10.77\\3.82\end{array}\right\}14.59$	$\left.\begin{array}{r}\\\end{array}\right\}0.02$
	40	39.67	0.05

it can be combined with the expected frequency in an adjacent class interval. The corresponding observed frequencies would then be combined also, and L would be reduced by one. With the foregoing calculations and procedures in mind, we construct Table 7.4. The critical value $\chi^2_{0.05,1}$ is 3.84. (The degrees of freedom equals the number of class intervals minus one.) Since $\chi^2_0 = 0.05$ is less than the critical value, the hypothesis of independence cannot be rejected on the basis of this test. ◀

Example 7.11

Given the same sequence of numbers in Example 7.10, can the hypothesis that the numbers are independent be rejected on the basis of the length of runs above and below the mean at $\alpha = 0.05$? For this sequence, the $+$'s and $-$'s are as follows:

$$\begin{array}{cccccccccccccccccccc}
- & - & - & - & + & - & + & - & + & + & - & + & - & + & + & - & - & + & - & + \\
- & + & - & - & + & + & + & + & + & + & + & - & - & + & + & - & - & - & - & + \\
+ & - & - & + & - & + & - & - & - & + & + & + & + & - & - & - & - & - & + & -
\end{array}$$

The number of runs of each length is as follows:

Run Length, i	1	2	3	≥ 4
Observed Runs, O_i	17	9	1	5

There are 28 values above the mean ($n_1 = 28$) and 32 values below the mean ($n_2 = 32$). The probabilities of runs of various lengths, w_i, are determined from Equation (7.11) as

$$w_1 = \left(\frac{28}{60}\right)^1 \frac{32}{60} + \frac{28}{60}\left(\frac{32}{60}\right)^1 = 0.498$$

$$w_2 = \left(\frac{28}{60}\right)^2 \frac{32}{60} + \frac{28}{60}\left(\frac{32}{60}\right)^2 = 0.249$$

$$w_3 = \left(\frac{28}{60}\right)^3 \frac{32}{60} + \frac{28}{60}\left(\frac{32}{60}\right)^3 = 0.125$$

$$\vdots$$

The expected length of a run, $E(I)$, is determined from Equation (7.12) as

$$E(I) = \frac{28}{32} + \frac{32}{28} = 2.02$$

Now, Equation (7.10) can be used to determine the expected numbers of runs of various lengths as

$$E(Y_1) = \frac{60(0.498)}{2.02} = 14.79$$

$$E(Y_2) = \frac{60(0.249)}{2.02} = 7.40$$

$$E(Y_3) = \frac{60(0.125)}{2.02} = 3.71$$

The total number of runs expected is given by Equation (7.13) as $E(A) = 60/2.02 = 29.7$. This indicates that approximately 3.8 runs of length four or more can be expected. Proceeding by combining adjacent cells in which $E(Y_i) < 5$ produces Table 7.5.

Table 7.5. Length of Runs Above and Below the Mean: χ^2 Test

Run Length, i	*Observed Number of Runs, O_i*	*Expected Number of Runs, $E(Y_i)$*	$\frac{[O_i - E(Y_i)]^2}{E(Y_i)}$
1	17	14.79	0.33
2	9	7.40	0.35
3	1 } 6	3.71 } 7.51	} 0.30
≥ 4	5	3.80	
	32	29.70	0.98

The critical value $\chi^2_{0.05,2}$ is 5.99. (The degrees of freedom equals the number of class intervals minus one.) Since $\chi_0^2 = 0.98$ is less than the critical value, the hypothesis of independence cannot be rejected on the basis of this test. ◀

7.4.3 *Tests for Autocorrelation*

The tests for autocorrelation are concerned with the dependence between numbers in a sequence. As an example, consider the following sequence of numbers:

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

From a visual inspection, these numbers appear random, and they would probably pass all the tests presented to this point. However, an examination of the 5th, 10th, 15th (every five numbers beginning with the fifth), and so on, indicates a very large number in that position. Now, 30 numbers is a rather small

7.4.4 Gap Test

The gap test is used to determine the significance of the interval between the recurrences of the same digit. A gap of length x occurs between the recurrences of some specified digit. The following example illustrates the length of gaps associated with the digit 3:

$$
\begin{array}{l}
4,\ 1,\ \underline{3},\ 5,\ 1,\ 7,\ 2,\ 8,\ 2,\ 0,\ 7,\ 9,\ 1,\ \underline{3},\ 5,\ 2,\ 7,\ 9,\ 4,\ 1,\ 6,\ \underline{3} \\
\underline{3},\ 9,\ 6,\ \underline{3},\ 4,\ 8,\ 2,\ \underline{3},\ 1,\ 9,\ 4,\ 4,\ 6,\ 8,\ 4,\ 1,\ \underline{3},\ 8,\ 9,\ 5,\ 5,\ 7 \\
\underline{3},\ 9,\ 5,\ 9,\ 8,\ 5,\ \underline{3},\ 2,\ 2,\ \underline{3},\ 7,\ 4,\ 7,\ 0,\ \underline{3},\ 6,\ \underline{3},\ 5,\ 9,\ 9,\ 5,\ 5 \\
5,\ 0,\ 4,\ 6,\ 8,\ 0,\ 4,\ 7,\ 0,\ \underline{3},\ \underline{3},\ 0,\ 9,\ 5,\ 7,\ 9,\ 5,\ 1,\ 6,\ 6,\ \underline{3},\ 8 \\
8,\ 8,\ 9,\ 2,\ 9,\ 1,\ 8,\ 5,\ 4,\ 4,\ 5,\ 0,\ 2,\ \underline{3},\ 9,\ 7,\ 1,\ 2,\ 0,\ \underline{3},\ 6,\ 3
\end{array}
$$

To facilitate the analysis, the digit 3 has been underlined. There are eighteen 3's in the list. Thus, only 17 gaps can occur. The first gap is of length 10, the second gap is of length 7, and so on. The frequency of the gaps is of interest. The probability of the first gap is determined as follows:

$$
\begin{aligned}
P(\text{gap of } 10) &= \overbrace{P(\text{no } 3) \cdots P(\text{no } 3)}^{10 \text{ of these terms}} P(3) \\
&= (0.9)^{10}(0.1)
\end{aligned}
$$

since the probability that any digit is not a 3 is 0.9, and the probability that any digit is a 3 is 0.1. In general,

$$P(t \text{ followed by exactly } x \text{ non-}t \text{ digits}) = (0.9)^x(0.1), \quad x = 0, 1, 2, \ldots$$

In the example above, only the digit 3 was examined. However, to fully analyze a set of numbers for independence using the gap test, every digit, 0, 1, 2, ..., 9, must be analyzed. The observed frequencies of the various gap sizes for all the digits are recorded and compared to the theoretical frequency using the Kolmogorov-Smirnov test for discretized data.

The theoretical frequency distribution for randomly ordered digits is given by

$$P(\text{gap} \le x) = F(x) = 0.1 \sum_{n=0}^{x} (0.9)^n = 1 - 0.9^{x+1} \tag{7.14}$$

The procedure for the test follows the steps below. When applying the test to random numbers, class intervals such as [0, 0.1), [0.1, 0.2), ... play the role of random digits.

Step 1. Specify the cdf for the theoretical frequency distribution given by Equation (7.14) based on the selected class interval width.

Step 2. Arrange the observed sample of gaps in a cumulative distribution with these same classes.

Step 3. Find D, the maximum deviation between $F(x)$ and $S_N(x)$ as in Equation (7.3).

Step 4. Determine the critical value, D_α, from Table A.8 for the specified value of α and the sample size N.

Step 5. If the calculated value of D is greater than the tabulated value of D_α, the null hypothesis of independence is rejected.

It should be noted that using the Kolmogorov-Smirnov test when the underlying distribution is discrete results in a reduction in the Type I error, α, and an increase in the Type II error, β. The exact value of α can be found using the methodology described by Conover [1980].

Example 7.13

Based on the frequency with which gaps occur, analyze the 110 digits above to test whether they are independent. Use $\alpha = 0.05$. The number of gaps is given by the number of data values minus the number of distinct digits, or $110 - 10 = 100$ in the example. The number of gaps associated with the various digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
Number of Gaps	7	8	8	17	10	13	7	8	9	13

The gap test is presented in Table 7.6. The critical value of D is given by

$$D_{0.05} = \frac{1.36}{\sqrt{100}} = 0.136$$

Since $D = \max|F(x) - S_N(x)| = 0.0224$ is less than $D_{0.05}$, do not reject the hypothesis of independence on the basis of this test. ◀

Table 7.6. Gap-Test Example

Gap Length	*Frequency*	*Relative Frequency*	*Cumulative Relative Frequency*	$F(x)$	$\lvert F(x) - S_N(x)\rvert$
0–3	35	0.35	0.35	0.3439	0.0061
4–7	22	0.22	0.57	0.5695	0.0005
8–11	17	0.17	0.74	0.7176	0.0224
12–15	9	0.09	0.83	0.8147	0.0153
16–19	5	0.05	0.88	0.8784	0.0016
20–23	6	0.06	0.94	0.9202	0.0198
24–27	3	0.03	0.97	0.9497	0.0223
28–31	0	0.0	0.97	0.9657	0.0043
32–35	0	0.0	0.97	0.9775	0.0075
36–39	2	0.02	0.99	0.9852	0.0043
40–43	0	0.0	0.99	0.9903	0.0003
44–47	1	0.01	1.00	0.9936	0.0064