CALCULUS & LINEAR ALGEBRA

MODULE - 3

Integral Calculus** in this Module begins with the review of **Basic Integration.

Multiple Integrals** is introduced with the direct evaluation of **Double and Triple integrals.

*Further, evaluation of double integrals by **changing the order of integration** and by changing **into polars** is presented.*

*Applications is discussed with the computation of **Area, Volume** and **Centre of gravity.***

*Finally, two special functions **Beta and Gamma functions** is presented.*

INTEGRAL CALCULUS

3.0 Review of Elementary Integral Calculus

Basic integration, covering the conceptual content, necessary rules / formulae along with illustrative examples is vividly presented in *Appendix* - 2. (Article 0.64)

3.1 Multiple Integrals

We are already conversant with the indefinite and definite integrals of a function of a single independent variable along with applications.

In this module the concept is discussed for a function of two and three independent variables along with the applications.

Further we also discuss two special functions 'Beta function' and 'Gamma function' defined in the form of definite integrals.

Basic integration, definition and properties of definite integrals along with the following *two* established *Reduction Formulae* is an essential pre-requisite.

(1) $$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \times k$$

where n is a positive integer and $k = 1$ if n is odd and $k = \pi/2$ if n is even.

(2) $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots} \times k$$

where m, n are positive integers and $k = \pi/2$ only when m and n are even. Otherwise $k = 1$.

Ex - (i) : $$\int_0^{\pi/2} \sin^5 x\, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15} = \int_0^{\pi/2} \cos^5 x\, dx$$

Ex - (ii) : $$\int_0^{\pi/2} \cos^6 x\, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \times \frac{\pi}{2} = \frac{5\pi}{32} = \int_0^{\pi/2} \sin^6 x\, dx$$

Ex - (iii) : $$\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{[(4)(2)][(3)(1)]}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \frac{8}{315}$$

Ex - (iv) : $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx = \dfrac{[(6)(4)(2)][(4)(2)]}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} = \dfrac{1}{120}$

Ex - (v) : $\int_0^{\pi/2} \sin^8 x \cos^6 x \, dx = \dfrac{[(7)(5)(3)(1)][(5)(3)]}{14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \times \dfrac{\pi}{2} = \dfrac{5\pi}{4096}$

3.2 Double and Triple integrals

In this topic we discuss a repeated process of integration of a function of two and three variables referred to as *double integrals* : $\iint f(x,y)\,dx\,dy$ and *triple integrals* : $\iiint f(x,y,z)\,dx\,dy\,dz$.

The principle of partial differentiation is adopted here in the process of integration.

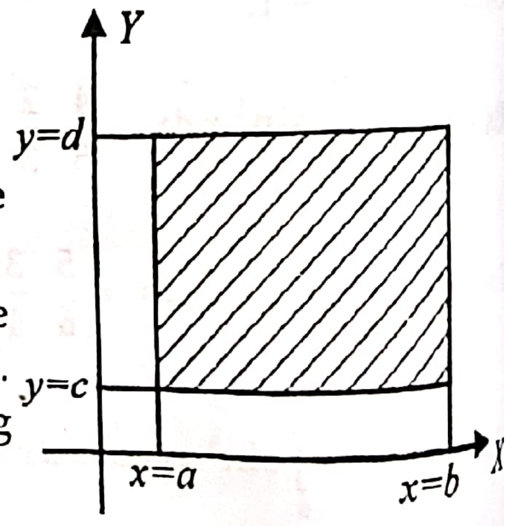
Ex - (i) $\iint (x+y)\,dx\,dy = \int \left(\dfrac{x^2}{2} + y \cdot x\right) dy = \dfrac{x^2}{2} \cdot y + \dfrac{y^2}{2} \cdot x = \dfrac{xy}{2}(x+y)$

Ex - (ii) $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} xyz \, dz\, dy\, dx$

ie., $= \int_{x=0}^{1} \int_{y=0}^{2} xy \left[\dfrac{z^2}{2}\right]_0^3 dy\ dx = \int_{x=0}^{1} \int_{y=0}^{2} xy\left(\dfrac{9}{2} - 0\right) dy dx = \dfrac{9}{2} \int_{x=0}^{1} \int_{y=0}^{2} xy\, dy\, dx$

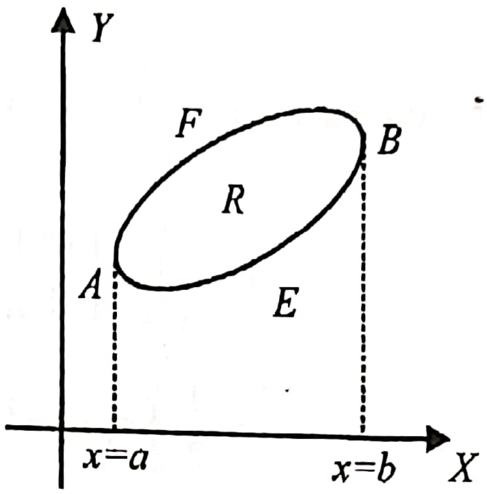
ie., $= \dfrac{9}{2} \int_{x=0}^{1} x \left[\dfrac{y^2}{2}\right]_0^2 dx = \dfrac{9}{2} \int_{x=0}^{1} 2x\, dx = 9\left[\dfrac{x^2}{2}\right]_0^1 = \dfrac{9}{2}$

3.21 Geometrical meaning



$\int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\, dy\, dx$ can be regarded as the integral over the region bounded by the rectangle with sides $x = a$, $x = b$, $y = c$, $y = d$. The integral can also be evaluated by writing in the form $\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$ and the value will be the same.

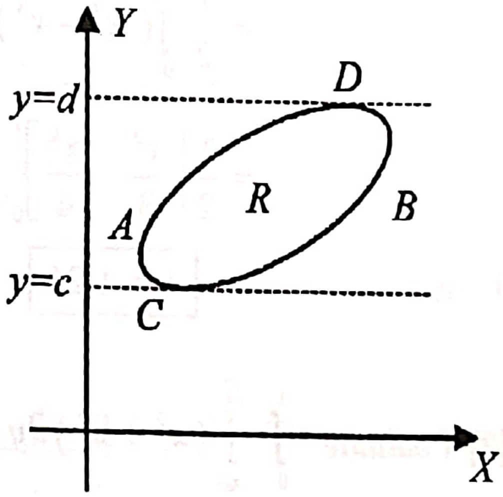
That is to say that when limits are constants the integral can be evaluated in either way.



If R is a region of the $x-y$ plane bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$ we have,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$$

where $y = y_1(x)$ and $y = y_2(x)$ are the equations of the lower and upper part of the boundary curve respectively being AEB and AFB.



Further the integral can also be expressed in the form

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$$

where $x = x_1(y)$, $x = x_2(y)$ are the equations of the left and right part of the boundary curve respectively being CAD and CBD.

It should be observed that if a function of x is involved as a limit in the double integral it corresponds to y in which case the limits for x will be constant. Similar argument holds good for a function of y in the limit and also in the case of triple integral involving three variables x, y, z.
A form of triple integral is as follows.

$$I = \int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)}\int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

WORKED PROBLEMS

Direct evaluation of double and triple integrals

[1] *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

☞ We have, $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy\, dy\, dx$

$$I = \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx = \int_{x=0}^{1} \frac{x}{2}\left[\left(\sqrt{x}\right)^2 - x^2\right] dx$$

ie.,

$$= \frac{1}{2}\int_0^1 x(x - x^2)\, dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - x^3)\, dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = \frac{1}{24}$$

Thus, $\boxed{I = 1/24}$

[2] *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

☞ We have, $I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3}\right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$I = \int_{x=0}^{1} \left[x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\right] dx$$

$$I = \left[\frac{x^{7/2}}{7/2} + \frac{1}{3}\frac{x^{5/2}}{5/2} - \frac{4}{3}\cdot\frac{x^4}{4}\right]_{x=0}^{1}$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{9}{105} = \frac{3}{35}$$

Thus, $\boxed{I = 3/35}$

[3] *Evaluate* $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$

☞ We have, $I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} x^3 y \, dx \, dy$

$$I = \int_{y=0}^{1} y \left[\frac{x^4}{4}\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{4}\int_{y=0}^{1} y(1-y^2)^2 \, dy = \frac{1}{4}\int_{y=0}^{1} y(1-2y^2+y^4)\, dy$$

ie., $$= \frac{1}{4}\int_{y=0}^{1} (y - 2y^3 + y^5)\, dy$$

$$= \frac{1}{4}\left[\frac{y^2}{2} - \frac{y^4}{2} + \frac{y^6}{6}\right]_0^1 = \frac{1}{4}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{24}$$

Thus, $\boxed{I = 1/24}$

[4] *Evaluate* $\int_{-c}^{c}\int_{-b}^{-a}\int_{-a}^{a} (x^2+y^2+z^2)\, dz\, dy\, dx$

☞ $I = \int_{x=-c}^{c} \int_{y=-b}^{b} \int_{z=-a}^{a} (x^2+y^2+z^2)\, dz\, dy\, dx$

$$= \int_{x=-c}^{c} \int_{y=-b}^{b} \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{z=-a}^{a} dy\, dx$$

$$= \int_{x=-c}^{c} \int_{y=-b}^{b} \left[x^2(a+a) + y^2(a+a) + (a^3/3 + a^3/3)\right] dy\, dx$$

Hence,

$$I = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_{x=0}^{1}$$

$$= \frac{\pi}{2}\left[0 + \frac{1}{2}(\sin^{-1}1 - \sin^{-1}0)\right] = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

Thus,

$$\boxed{I = \pi^2/8}$$

Note : *Similar problem*

Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$ **[June 2017]**

☞ Proceeding on the same lines, we can obtain $\boxed{I = \pi^2 a^2/8}$

[8] *Evaluate* $\displaystyle\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

☞ We have, $\displaystyle I = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y}\,e^{z}\,dz\,dy\,dx$

$$I = \int_{x=0}^{a}\int_{y=0}^{x} e^{x+y}\left[e^{z}\right]_{z=0}^{x+y} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{x} e^{x+y}\left[e^{x+y}-1\right] dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{x} (e^{2x}\cdot e^{2y} - e^{x}\cdot e^{y})\,dy\,dx$$

$$= \int_{x=0}^{a}\left\{e^{2x}\left[\frac{e^{2y}}{2}\right]_{y=0}^{x} - e^{x}\left[e^{y}\right]_{y=0}^{x}\right\}dx$$

$$= \int_{x=0}^{a}\left\{\frac{e^{2x}}{2}(e^{2x}-1) - e^{x}(e^{x}-1)\right\}dx$$

$$\int_{x=0}^{a}\left(\frac{e^{4x}}{2} \quad e^{2x} + e^{x}\right)dx.$$

$$I = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{-1}{2(1+x+y+z)^2}\right]_{z=0}^{1-x-y} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[-\frac{1}{8}+\frac{1}{2(1+x+y)^2}\right] dy\,dx$$

$$= \int_{x=0}^{1}\left[-\frac{1}{8}y-\frac{1}{2(x+y+1)}\right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1}\left[-\frac{1}{8}(1-x)-\frac{1}{4}+\frac{1}{2(x+1)}\right] dx$$

$$I = \int_{x=0}^{1}\left[-\frac{3}{8}+\frac{x}{8}+\frac{1}{2(x+1)}\right] dx$$

$$= \left[-\frac{3x}{8}+\frac{x^2}{16}+\frac{1}{2}\log(x+1)\right]_{x=0}^{1}$$

$$= -\frac{3}{8}+\frac{1}{16}+\frac{1}{2}\log 2 = \frac{-5}{16}+\log\sqrt{2}$$

Thus, $\boxed{I = \log\sqrt{2}-(5/16)}$

3.22 Evaluation of $\iint_R f(x,y)\,dx\,dy$ over the specific region R

We need to draw the befitting figure from the given description to identify the specific region R. We have to then express

$$I = \iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx \quad \ldots (1)$$

or

$$I = \iint_R f(x,y)\,dx\,dy = \int_{y=c}^{d}\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy \quad \ldots (2)$$

is obtained by the evaluation of (1) or (2)

Remark : *Carefully take a note of the content in article 3.21.*

3.23 Evaluation of double integral by changing the order of integration

The method is illustrated stepwise.

Step-1: Given the integral in either of the forms as in article 3.22, say (1) we have to identify the region of integration R by writing the figure and express (1) in the form (2).

Step-2 : The evaluation of (2) will be the value of (1) on changing the order of integration. This can be vice versa also.

Step-3 : The advantage of this procedure is that, some times the double integral which is difficult to be evaluated in the existing form becomes easy for evaluation on changing the order of integration.

3.24 Evaluation of double integral by changing into polar form

The method is illustrated stepwise.

Step-1 : Given a double integral with limits we use the well known polar form of substitution $x = r \cos\theta$, $y = r \sin\theta$. This will give u $x^2 + y^2 = r^2$, $y/x = \tan\theta$ and it should be noted that $dx\,dy = J\,dr\,d$ where J is the Jacobian of the transformation given by

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Step-2 : Hence, $dx\,dy = r\,dr\,d\theta$ and we need to change the limits of integrati to r, θ suitably for the purpose of evaluation.

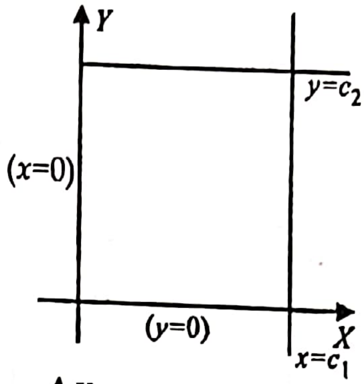
Step-3 : The method might be advantageous if the terms of the form $x^2 +$ are involved in $f(x, y)$ and terms like $\sqrt{a^2 - y^2}$, $\sqrt{a^2 - x^2}$ etc. involved in limits.

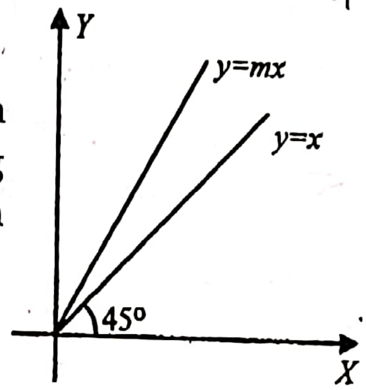
Note : *Some of the important and standard curves along with their equations shape is given below as it will be highly useful for working problems.*

1. ***Straight lines***

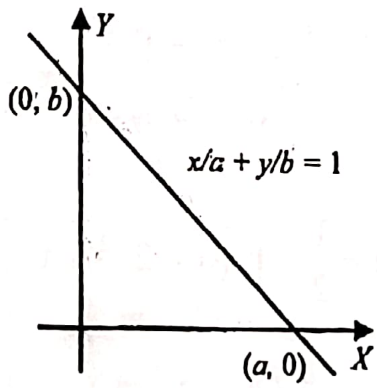
(i) $x = 0$ and $y = 0$ are respectively the equations of y and x - axis.

(ii) $x = c_1$ and $y = c_2$ are respectively the equations of a line parallel to y-axis and a line parallel to x - axis.

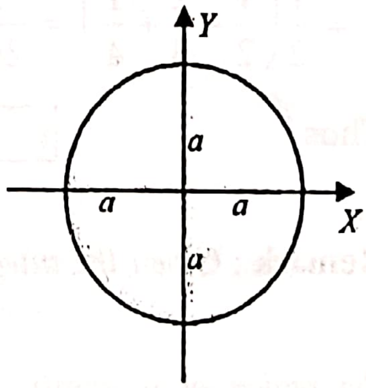




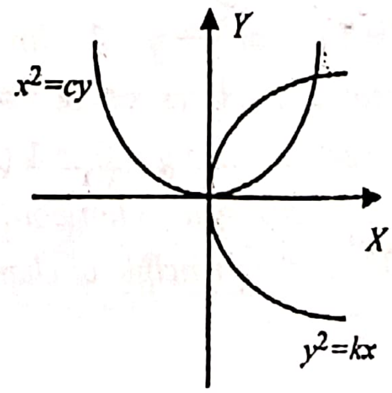
(iii) $y = mx$ is a straight line passing through the origin and in particular $y = x$ is a straight line passing through the origin subtending an angle 45^0 with the x - axis.



(iv) $x/a + y/b = 1$ is a straight line having x intercept a and y intercept b, being a straight line passing through $(a, 0)$ and $(0, b)$



. ***Circle*** $x^2 + y^2 = a^2$ is a circle with centre origin nd radius a.



Parabola $y^2 = kx$ is a parabola symmetrical ɔout the x - *axis.*

$= cy$ is a parabola symmetrical about ıe y - *axis.*

WORKED PROBLEMS

Type-1 : Evaluation over a given region

[13] *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region bounded by the coordinate axes and the line* $x + y = 1$.

☞ R is the region bounded by $x = 0$, $y = 0$ being the coordinate axes and $x + y = 1$ being a straight line through $(1, 0)$ and $(0, 1)$.

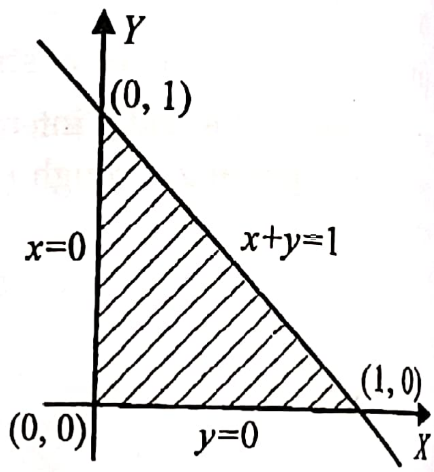
Shaded portion in the figure is the region R.
From the figure we have,

$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dy\,dx \quad \text{or} \quad \int_{y=0}^{1}\int_{x=0}^{1-y} xy\,dx\,dy$$

$$I = \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx = \int_{x=0}^{1} \frac{x}{2}(1-x)^2\,dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}(x - 2x^2 + x^3)\,dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$I = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24}$$



Thus, $\boxed{I = 1/24}$

Remark : *Given the integral* $\int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dy\,dx$ *we can write the figure to identify the region of integration. This being the region bounded by* $y = 0$ *(x - axis),* $y = 1 - x$ *or* $x + y = 1$ *a line passing through the points* $(1, 0)$ *and* $(0, 1)$ *embedded between the lines* $x = 0$, $x = 1$. $x = 0$ *to* 1 *being the horizontal strip can be changed to vertical strip* $y = 0$ *to* 1 *(constant limits)* $y = 0$ *to* $(1 - x)$ *being the vertical strip can be changed to horizontal strip* $x = 0$ *to* $1 - y$ *(variable limits).*
This is the principle of changing the order of integration of a given double integral.

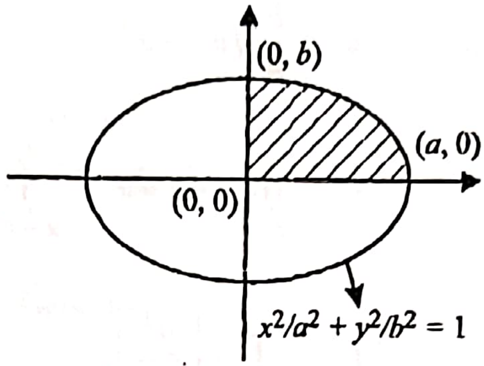
[14] *Evaluate* $\iint y \, dx \, dy$ *over the region bounded by the first quadrant of the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

☞ Shaded portion in the figure is the region (R) of integration. We observe that x varies from 0 to a and we need to express $x^2/a^2 + y^2/b^2 = 1$ in the form $y = f(x)$.

i.e., $y^2/b^2 = 1 - (x^2/a^2) = (a^2 - x^2)/a^2$

or $y = (b/a)\sqrt{a^2 - x^2}$.

Since $y = 0$ is the equation of x axis, we can say that y varies from 0 to $(b/a)\sqrt{a^2 - x^2}$



$$\therefore \quad \iint_R y \, dx \, dy = \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} y \, dy \, dx \qquad \ldots (1)$$

$$= \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{(b/a)\sqrt{a^2-x^2}} dx = \frac{b^2}{2a^2} \int_{x=0}^{a} (a^2 - x^2) \, dx$$

$$= \frac{b^2}{2a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{b^2}{2a^2}\left[\left(a^3 - \frac{a^3}{3}\right) - 0\right] = \frac{a b^2}{3}$$

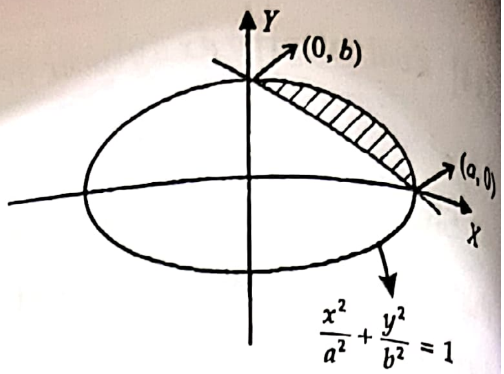
Thus, $\boxed{I = ab^2 / 3}$

Note: *From the figure, on a similar argument we can also have*

$$\iint_R y \, dx \, dy = \int_{y=0}^{b} \int_{x=0}^{(a/b)\sqrt{b^2-y^2}} y \, dx \, dy = \frac{ab^2}{3} \qquad \ldots (2)$$

Remark : *Given the double integral in the form (1) writing the same in the form as in (2) with the help of the figure is the change of order of integration.*

[15] *Evaluate* $\iint xy \, dx \, dy$ *taken over the region bounded by* $x^2/a^2 + y^2/b^2 = 1$ *and* $x/a + y/b = 1$



☞ x varies from 0 to a

$$\frac{x}{a}+\frac{y}{b}=1 \text{ or } y=\frac{b}{a}(a-x)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ or } y^2=\frac{b^2}{a^2}(a^2-x^2)$$

or $$y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$I=\iint\limits_R xy\,dx\,dy=\int_{x=0}^{a}\int_{b(a-x)/a}^{(b/a)\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$I=\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{b(a-x)/a}^{(b/a)\sqrt{a^2-x^2}} dx=\frac{1}{2}\int_{x=0}^{a} x\left\{\frac{b^2}{a^2}(a^2-x^2)-\frac{b^2}{a^2}(a-x)^2\right\}dx$$

$$I=\frac{b^2}{2a^2}\int_0^a (a^2x-x^3-a^2x+2ax^2-x^3)\,dx$$

$$=\frac{b^2}{2a^2}\int_0^a 2(ax^2-x^3)\,dx$$

$$=\frac{b^2}{a^2}\left[a\frac{x^3}{3}-\frac{x^4}{4}\right]_0^a=\frac{b^2}{a^2}\left(\frac{a^4}{3}-\frac{a^4}{4}\right)=\frac{a^2b^2}{12}$$

Thus, $$\boxed{I=a^2b^2/12}$$

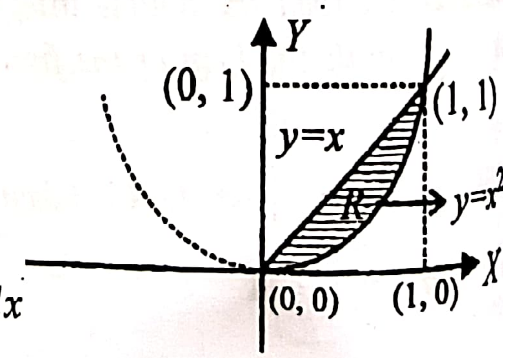
[16] *Evaluate* $\iint xy(x+y)\,dy\,dx$ *taken over the area between* $y=x^2$ *and* $y=x$

[June 2018]

☞ Now $x^2=x$ or $x(x-1)=0 \Rightarrow x=0,\ x=1$. This gives $y=0,\ y=1$ and hence the two curves intersect at the points $(0, 0)$ and $(1, 1)$

$$I=\iint\limits_R xy(x+y)\,dy\,dx$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x}(x^2y+xy^2)\,dy\,dx$$



$$=\int_{x=0}^{1}\left\{x^2\left[\frac{y^2}{2}\right]_{y=x^2}^{x}+x\left[\frac{y^3}{3}\right]_{y=x^2}^{x}\right\}dx$$

$$I = \int_{x=0}^{1} \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3} \right) dx$$

$$= \left[\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right]_{x=0}^{1}$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{3}{56}$$

Thus, $\boxed{I = 3/56}$

Note : *We can also write I in the form*

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} xy(x+y)\, dx\, dy \; ; \; I = \frac{3}{56}$$

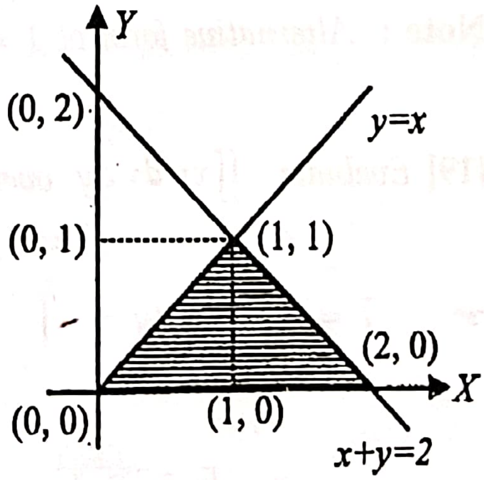
[17] *Evaluate* $\iint_R x^2 y \, dx\, dy$ *where R is the region bounded by the lines* $y = x$, $y + x = 2$ *and* $y = 0$.

☞ The lines $y = x$ and $y + x = 2$ intersect at $(1, 1)$.

$$I = \iint_R x^2 y\, dx\, dy = \int_{y=0}^{1} \int_{x=y}^{2-y} x^2 y\, dx\, dy = \int_{y=0}^{1} y \left[\frac{x^3}{3} \right]_{x=y}^{2-y} dy$$

$$I = \frac{1}{3} \int_{y=0}^{1} y \{ (2-y)^3 - y^3 \} dy$$

$$= \frac{1}{3} \int_{y=0}^{1} y(8 - 12y + 6y^2 - y^3 - y^3)\, dy$$



$$I = \frac{1}{3} \int_{y=0}^{1} (8y - 12y^2 + 6y^3 - 2y^4)\, dy$$

$$I = \frac{1}{3} \left[4y^2 - 4y^3 + \frac{3}{2} y^4 - 2\frac{y^5}{5} \right]_{y=0}^{1}$$

$$= \frac{1}{3}\left(4 - 4 + \frac{3}{2} - \frac{2}{5} \right) = \frac{11}{30}$$

Thus, $\boxed{I = 11/30}$

$$I = \frac{1}{2}\int_{x=0}^{a} x(a^2 - x^2)\,dx = \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{a}$$

$$I = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}$$

Thus, $\boxed{I = a^4/8}$

Note : *Alternative form of* I : $\int_{y=0}^{a}\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\,dy$; $I = \dfrac{a^4}{8}$

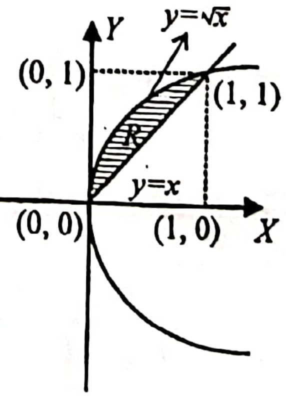
Type-2 : ***Evaluation of a double integral by changing the order of integration***
The procedure is illustrated in the article 3.23 and in every problem (13 to 19) the alternative form of the double integral is exactly the integral by changing the order of integration.
We complete the problem by evaluating the new form of the double integral.

20] *Evaluate* $\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$ *by changing the order of integration.* **[June2016]**

$$I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} xy\,dy\,dx$$

We need to first identify the region of integration R bounded by the curves $y = x$, $y = \sqrt{x}$ between the lines $x = 0$, $x = 1$. We shall find the points of intersection of $y = x$ and $y = \sqrt{x}$ by equating their RHS.

i.e., $x = \sqrt{x} \Rightarrow x^2 = x$ or $x(x-1) = 0$ or $x = 0, 1$.
This will give us $y = 0$, $y = 1$ and hence the points of intersection are (0, 0) and (1, 1). Further we know that $y = x$ is a straight line passing through the origin making an angle 45^0 with the x-axis and $y = \sqrt{x}$ or $y^2 = x$ is a parabola symmetrical about the x - *axis*. The befitting figure indicating R is given.



On changing the order of integration we must have constant limits for y an variable limits for x. From the figure we observe that y varies from 0 to 1 an x varies from $y^2\ (\because y = \sqrt{x})$ to y.

[It should be noted that $y = x$ and $\sqrt{x}$ are the lower and upper parts of the bounda of R and on changing the order, $x = y^2$ and $x = y$ represent the left and right par of the boundary of R]

Hence, we have on changing the order of integration,

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_{x=y^2}^{y} dy = \frac{1}{2}\int_{y=0}^{1} y(y^2 - y^4)\, dy$$

$$I = \frac{1}{2}\int_0^1 (y^3 - y^5)\, dy$$

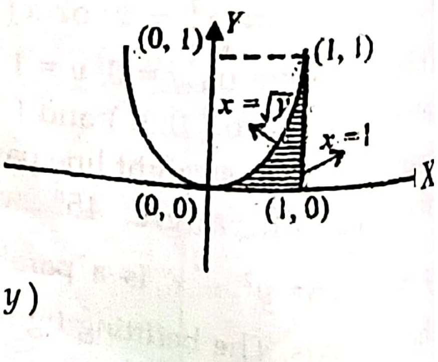
$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{24}$$

Thus, $\boxed{I = 1/24}$

Remark : *Referring to Problem-1 it may be seen that we have obtained the sam answer by direct evaluation.*

[21] *Change the order of the integration and hence evaluate* $\int_0^1 \int_{\sqrt{y}}^1 dx\, dy$

☞ Let $I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} dx\, dy$



On changing the order of integration,

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} dy\, dx \quad (\because x = \sqrt{y} \Rightarrow x^2 = y)$$

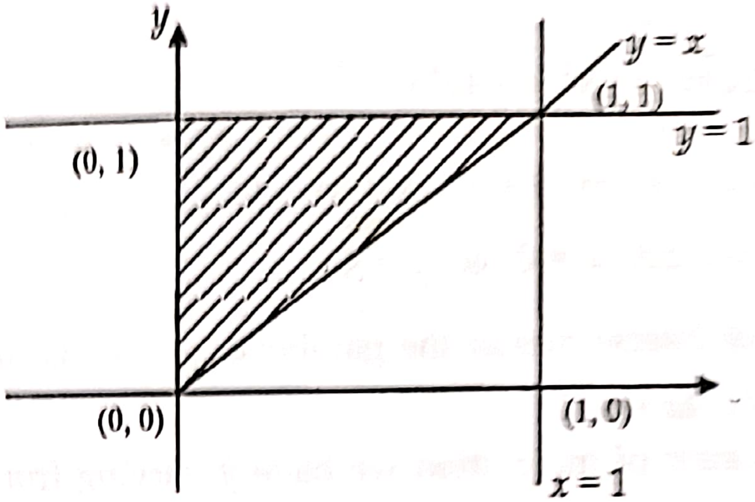
$$I = \int_{x=0}^{1} [y]_0^{x^2} \, dx$$

$$I = \int_{x=0}^{1} x^2 \, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Thus, $\boxed{I = 1/3}$

22] *Evaluate by change of order of integration* $\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$

➤ Let $I = \int_{x=0}^{1} \int_{y=x}^{1} \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx$



On changing the order of integration we have,

$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}} \, dx \, dy$$

Put $x^2 + y^2 = t \therefore 2x \, dx = dt$ or $x \, dx = dt/2$

We have, $\int \frac{x}{\sqrt{x^2+y^2}} \, dx$ reducing to $\frac{1}{2} \int \frac{dt}{\sqrt{t}} = \sqrt{t}$

$$\therefore \quad I = \int_{y=0}^{1} \left[\sqrt{x^2+y^2}\right]_{x=0}^{y} dy$$

$$= \int_{y=0}^{1} (\sqrt{2}\,y - y)\,dy = \int_{0}^{1} (\sqrt{2}-1)\,y\,dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 = \frac{\sqrt{2}-1}{2}$$

Thus, $\boxed{I = (\sqrt{2}-1)/2}$

[23] *Change the order of integration and hence evaluate* $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$ **[Dec 2016]**

☞ We have, $I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{y=2\sqrt{ax}} xy\,dy\,dx$

We have, $\frac{x^2}{4a} = 2\sqrt{ax}$ or $x^4 = 64a^3 x$

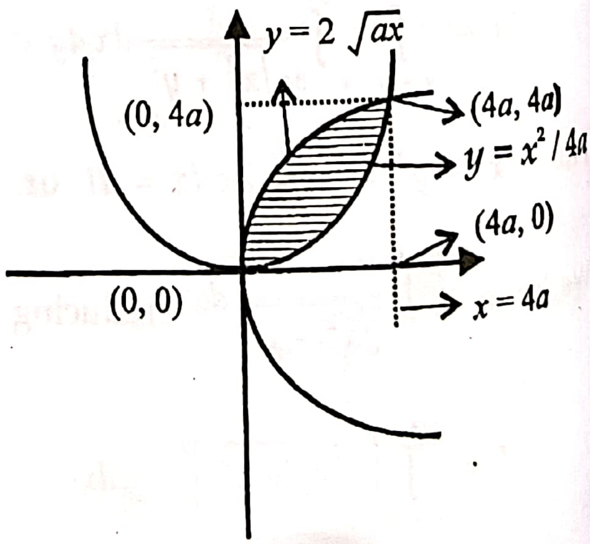
ie., $x(x^3 - 64a^3) = 0 \Rightarrow x = 0$ and $x = 4a$

From $y = x^2/4a$ we get $y = 0$ & $y = 4a$.

Hence the points of intersection of the parabolas $y = x^2/4a$ and $y = 2\sqrt{ax}$ are $(0, 0)$ and $(4a, 4a)$

On changing the order of integration we have y varying from 0 to $4a$ and x varying from $y^2/4a$ ($\because y = 2\sqrt{ax}$) to $2\sqrt{ay}$ ($\because y = x^2/4a$)

Now, $I = \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$



$$= \int_{y=0}^{4a} y\left[\frac{x^2}{2}\right]_{x=y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_{y=0}^{4a} y\left[4ay - \frac{y^4}{16a^2}\right]dy$$

$$= \frac{1}{2}\left[4a\frac{y^3}{3} - \frac{1}{16a^2}\frac{y^6}{6}\right]_{y=0}^{4a}$$

$$I = \frac{1}{2}\left[4a\left(\frac{64\,a^3}{3}\right) - \frac{1}{96\,a^2}(4096\,a^6)\right]$$

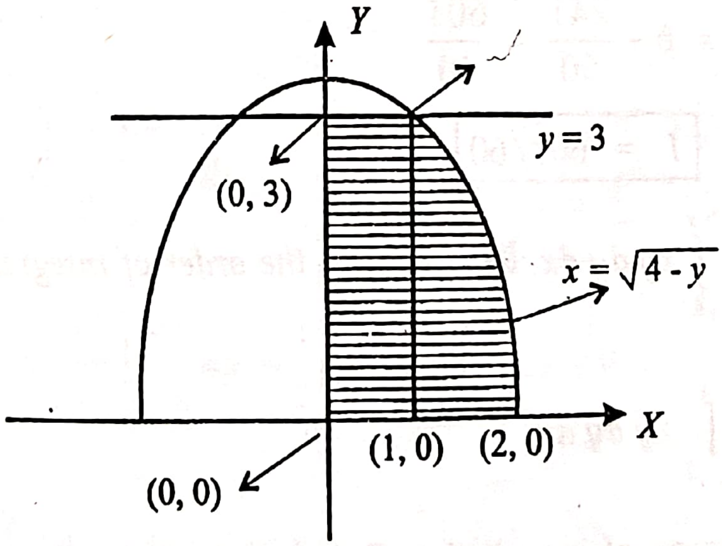
$$= \frac{1}{2}\left[\frac{256\,a^4}{3} - \frac{128\,a^4}{3}\right] = \frac{64\,a^4}{3}$$

Thus, $\boxed{I = 64\,a^4/3}$

[24] *Change the order of integration and evaluate* $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\,dx\,dy$

☞ Let $I = \int_{y=0}^{3} \int_{x=0}^{\sqrt{4-y}} (x+y)\,dx\,dy$

The points of intersection of the parabola $x = \sqrt{4-y}$ or $x^2 = 4-y$ with $y=0$ are $(\pm 2, 0)$ and with $y=3$ are $(\pm 1, 0)$. Since y varies from 0 to 3 the points for consideration are (2, 0) and (1, 0). The region is shown in the figure.



On changing the order we have,

$$I = \int_{x=0}^{1} \int_{y=0}^{3} (x+y)\,dy\,dx + \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\,dy\,dx = I_1 + I_2 \text{ (say)}$$

Now, $I_1 = \int_{x=0}^{1} \left[xy + \frac{y^2}{2}\right]_{y=0}^{3} dx$

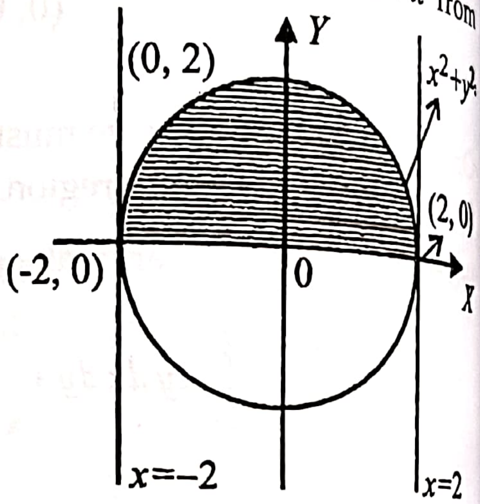
[26] *Evaluate* $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}(2-x)\,dy\,dx$ *by changing the order of integration.*

☞ We have $I = \int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}}(2-x)\,dy\,dx$

Here, $y = \sqrt{4-x^2}$ or $x^2+y^2 = 4$

This is a circle with centre origin and radius 2. $y = 0$ to $\sqrt{4-x^2}$ is the upper half of the circle being bounded by the lines $x = -2$ and 2.

On changing the order we must have y varying from 0 to 2 and x from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$.



$$\therefore \quad I = \int_{y=0}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(2-x)\,dx\,dy$$

$$I = \int_{y=0}^{2}\left[2x-\frac{x^2}{2}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}dy$$

$$I = \int_{y=0}^{2}\left[2\cdot 2\sqrt{4-y^2}-0\right]dy = 4\int_{y=0}^{2}\sqrt{2^2-y^2}\,dy$$

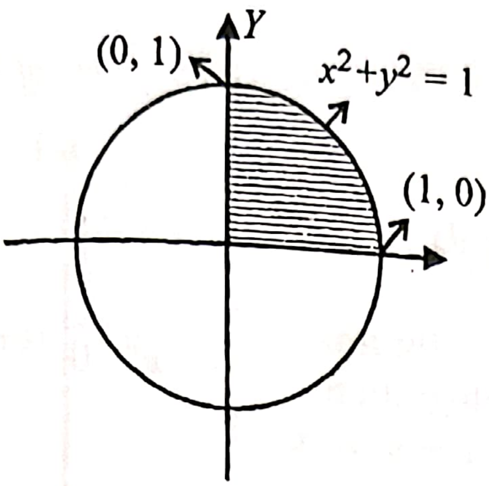
$$I = 4\left[\frac{y\sqrt{4-y^2}}{2}+\frac{2^2}{2}\sin^{-1}\frac{y}{2}\right]_0^2 = 4(0+2\sin^{-1}1) = 8\cdot\frac{\pi}{2} = 4\pi$$

Thus, $\boxed{I = 4\pi}$

[27] *Change the order of integration and hence evaluate* $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} y^2\,dx\,dy$

☞ $I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx$

On changing the order of integration we have from the figure,



$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 \, dx \, dy$$

$$I = \int_{y=0}^{1} y^2 [x]_{x=0}^{\sqrt{1-y^2}} \, dy = \int_{y=0}^{1} y^2 \sqrt{1-y^2} \, dy$$

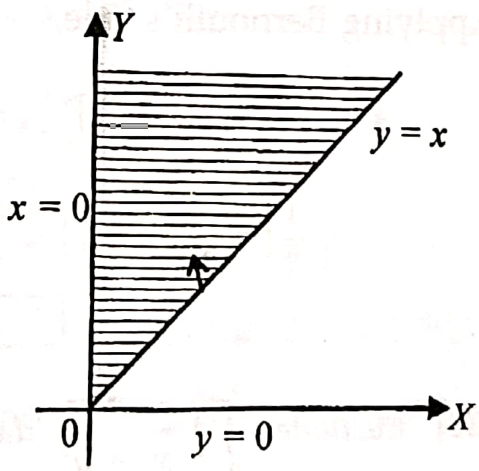
Put, $y = \sin\theta \;\therefore\; dy = \cos\theta \, d\theta$ and θ varies from 0 to $\pi/2$.

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta = \frac{(1)(1)}{(4)(2)} \cdot \frac{\pi}{2} = \frac{\pi}{16}$$ by reduction formula.

Thus, $\boxed{I = \pi/16}$

[28] *Change the order of integration and evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$ **[June 2018]**

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} \, dy \, dx$$



On changing the order we must have $y = 0$ to ∞ and $x = 0$ to y

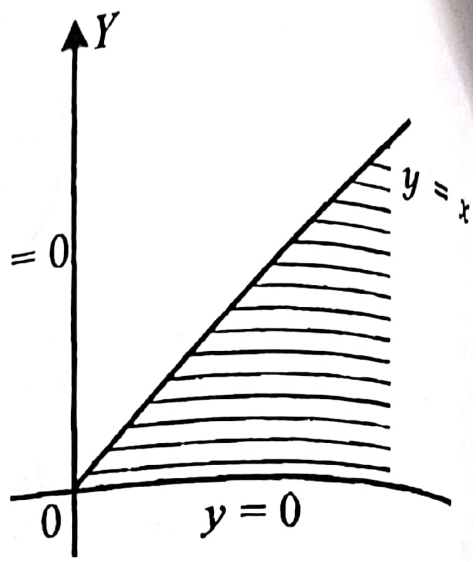
$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y} \, dx \, dy = \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$I = \int_{y=0}^{\infty} \frac{e^{-y}}{y} \cdot y \, dy = \int_{y=0}^{\infty} e^{-y} \, dy = -[e^{-y}]_0^{\infty} = 1$$

Thus, $\boxed{I = 1}$

[29] *Evaluate* $\int_0^\infty \int_0^x x e^{-x^2/y}\, dy\, dx$ *by changing the order of integration.*

☞ $$I = \int_{x=0}^{\infty} \int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$$



The region is as shown in the figure.
On changing the order of integration
we must have $y = 0$ to ∞ ; $x = y$ to ∞

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-x^2/y}\, dx\, dy$$

Put, $\frac{x^2}{y} = t \quad \therefore \quad \frac{2x}{y} dx = dt \quad$ or $\quad x\, dx = y\, dt/2$

Also when $x = y$, $t = y$ and when $x = \infty$, $t = \infty$

$$\therefore \quad I = \int_{y=0}^{\infty} \int_{t=y}^{\infty} e^{-t} \frac{y}{2} dt\, dy$$

$$= \int_{y=0}^{\infty} \frac{y}{2} [-e^{-t}]_{t=y}^{\infty}\, dy$$

$$= \frac{1}{2} \int_{y=0}^{\infty} y e^{-y}\, dy.$$

Applying Bernoulli's rule,

$$I = \frac{1}{2}\left\{ [y(-e^{-y})]_{y=0}^{\infty} - [(1)(e^{-y})]_{y=0}^{\infty} \right\}$$

$$= \frac{1}{2}[0-(0-1)] = \frac{1}{2}$$

Thus, $\boxed{I = 1/2}$

[30] *Evaluate* $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ *by changing the order of integration.*

[June, Dec. 2017]

☞ The region bounded by the curves $x = y$, $x = a$ embedded between the lines $y = 0$, $y = a$ is shown in the figure.

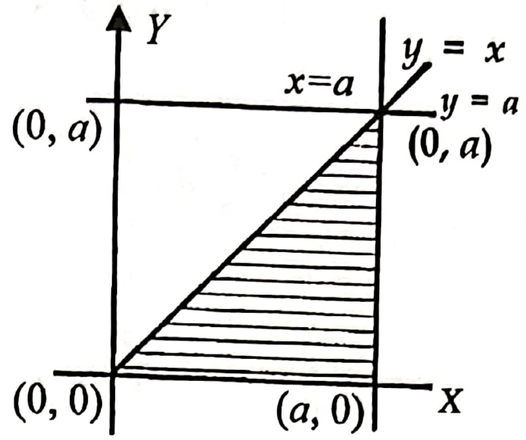
On changing the order x varies from 0 to a and y varies from 0 to x.

$$I = \int_{x=0}^{a}\int_{y=0}^{x} x \cdot \frac{1}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x \cdot \frac{1}{x}\left[\tan^{-1}(y/x)\right]_{y=0}^{x} dx$$

$$\int_{x=0}^{a} (\tan^{-1} 1 - \tan^{-1} 0)\, dx$$

$$\int_{x=0}^{a} \frac{\pi}{4} dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$



Thus, $\boxed{I = \pi a / 4}$

[31] *Change the order of integration in* $\int_0^1 \int_{\sqrt{y}}^{2-y} xy\, dx\, dy$ *and hence evaluate.*

☞ $$I = \int_{y=0}^{1}\int_{x=\sqrt{y}}^{2-y} xy\, dx\, dy$$

First, let us find the points of intersection of the curves, $x = \sqrt{y}$ and $x = 2 - y$

ie., $\sqrt{y} = 2 - y$ or $y = (2-y)^2$ or $y^2 - 5y + 4 = 0$

ie., $(y-1)(y-4) = 0 \Rightarrow y = 1$ and $4 \therefore x = \pm 1, \pm 2$

The points of intersection are $(1,1)(-1,1)(2,4)(-2,4)$

Since y varies from 0 to 1 the point (1, 1) is of consideration. $x = \sqrt{y}$ or $x^2 = y$ is a parabola symmetrical about the y-axis and $x = 2 - y$ or $x + y = 2$ is a line passing through (2, 0) and (0, 2). The region is as shown in the figure. *(Figure in the following page)*

The integral on changing the order consist of two parts.

$$I = \int_{x=0}^{1}\int_{y=0}^{x^2} xy\, dy\, dx + \int_{x=1}^{2}\int_{y=0}^{2-x} xy\, dy\, dx = I_1 + I_2\ (say)$$

[37] *Evaluate* $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ *by changing into polars.* **[Dec. 2017]**

$$I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$$

$x = \sqrt{a^2-y^2}$ or $x^2+y^2 = a^2$ is a circle with centre origin and radius a.

Since y varies from 0 to a, the region of integration is the first quadrant of the circle.

In polars we have, $x = r\cos\theta,\ y = r\sin\theta \ \therefore\ x^2+y^2 = r^2$

ie., $r^2 = a^2 \Rightarrow r = a$.

Also $x = 0, y = 0$ will give $r = 0$ and hence we can say that r varies from 0 to a.

In the first quadrant θ varies from 0 to $\pi/2$.

Further we know that, $dx\, dy = r\, dr\, d\theta$.

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r\sin\theta \cdot r \cdot r\, dr\, d\theta = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta\, dr\, d\theta$$

$$I = \int_{r=0}^{a} r^3 \left[-\cos\theta\right]_0^{\pi/2} dr = \int_{r=0}^{a} -r^3(0-1)\, dr = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

Thus, $\boxed{I = a^4/4}$

ASSIGNMENT

Evaluate the following (1 to 5)

1. $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

2. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{\sqrt{1+x^2+y^2}}$

3. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x^2+y^2+z^2)\, dz\, dy\, dx$

4. $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\, dx\, dy\, dz$

5. $\int_{1}^{e}\int_{1}^{\log y}\int_{0}^{e^x} \log z \, dz \, dy \, dx$

6. Evaluate $\iint_R xy^2 \, dx \, dy$ over the region bounded by $y = x^2$, $y = 0$ and $x = 1$

7. Evaluate $\iint_R xy(x+y) \, dx \, dy$ taken over the region bounded by the parabolas $y^2 = x$ and $y = x^2$

8. Evaluate $\iint_R x^2 y \, dx \, dy$ over the region bounded by the curves $y = x^2$ and $y = x$

9. Evaluate $\iint_R xy \, dx \, dy$ where R is the region in the first quadrant bounded by the line $x + y = 1$.

10. Evaluate $\iint_R x^2 y^2 \, dx \, dy$ taken over the region bounded by the y-axis, x-axis and $x^2 + y^2 = 1$.

Evaluate the following by changing the order of integration (11 to 15)

11. $\int_{0}^{a}\int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) \, dy \, dx$

12. $\int_{0}^{a}\int_{0}^{2\sqrt{ax}} x^2 \, dx \, dy$

13. $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (a - x) \, dy \, dx$

14. $\int_{0}^{a}\int_{\sqrt{ax}}^{a} \dfrac{y^2 \, dy \, dx}{\sqrt{y^4 - a^2x^2}}$

15. $\int_{0}^{a}\int_{x^2/a}^{2a-x} xy \, dy \, dx$

ANSWERS

1. $3/8$	2. $\pi/4 \cdot \log(1+\sqrt{2})$	3. $a^5/20$
4. $\pi a^4/16$	5. $(e^2 - 8e + 13)/2$	6. $1/24$
7. $3/28$	8. $1/35$	9. $1/6$
10. $\pi/96$	11. $a^3/28 + a/20$	12. $4a^4/7$
13. $\pi a^3/2$	14. $\pi a^2/6$	15. $3a^4/8$

3.3 Applications of Double and Triple Integrals

The reader is already familiar with the application of definite integrals of a single variable to find the area of a curve $y = f(x)$ bounded by the x - axis and the extreme ordinates $x = a$ and $x = b$ which is given by $\int_a^b f(x)\,dx$.

In this article, with the knowledge of double and triple integrals, we discuss *Area, Volume* and *Centre of gravity* as *Applications.*

Note the following Applications formulae for Area and Volume.

1. $\iint_R dx\,dy$ = Area of the region R in the cartesian form.

2. $\iint_R r\,dr\,d\theta$ = Area of the region R in the polar form.

3. $\iiint_V dx\,dy\,dz$ = Volume of the solid in the cartesian form.

4. $\iint_A 2\pi r^2 \sin\theta\,dr\,d\theta$ = Volume of a solid obtained by the revolution of a curve enclosing an area A about the initial line in the polar form.

WORKED PROBLEMS

[38] *Find the area of ellipse $x^2/a^2 + y^2/b^2 = 1$ by double integration.*

☞ Area $(A) = \iint_R dx\,dy.$

Referring to the figure in Problem - 14 we can write $A = 4A_1$ where A_1 is the area in the first quadrant and with reference to the same figure we have,

$$A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{a^2-x^2}} dy\,dx = 4\int_{x=0}^{a}\left[\,y\,\right]_0^{(b/a)\sqrt{a^2-x^2}} dx$$

$$= 4\int_{x=0}^{a}\frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{x=0}^{a}$$

i.e.,
$$= \frac{4b}{a}\left[0+\frac{a^2}{2}(\sin^{-1}1-\sin^{-1}0)\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi ab$$

Thus the required area $\boxed{A = \pi ab \text{ sq. units.}}$

Note : *The area of the circle $x^2+y^2 = a^2$ by double integration is πa^2. This is a particular case of this problem when $b = a$.*

[39] (a) *Find by double integration the area enclosed by the curve $r = a(1+\cos\theta)$ between $\theta = 0$ and $\theta = \pi$.* [June 2018]

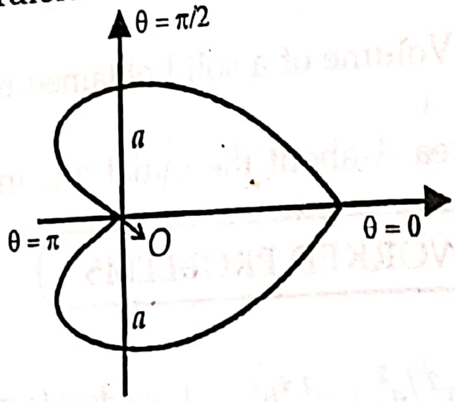
☞ Let, $f(r,\theta) = a(1+\cos\theta) = f(r,-\theta)$

⇒ Curve is symmetrical about the initial line.

We tabulate the value of r for certain angles θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r = a(1+\cos\theta)$	$2a$	$3a/2$	a	$a/2$	0

The shape of the cardioide is as follows.



Area $A = \iint r\,dr\,d\theta$ where r varies from 0 to $a(1+\cos\theta)$ and θ from 0 to π.

$$\therefore \quad A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$A = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$A = \frac{a^2}{2}\int_{\theta=0}^{\pi}\{2\cos^2(\theta/2)\}^2\,d\theta = 2a^2\int_0^{\pi}\cos^4(\theta/2)\,d\theta$$

Put $\theta/2 = \phi$, $d\theta = 2d\phi$ and ϕ varies from 0 to $\pi/2$

$$\therefore \quad A = 2a^2 \int_0^{\pi/2} \cos^4\phi \cdot 2\, d\phi$$

$$= 4a^2 \int_0^{\pi/2} \cos^4\phi \cdot d\phi = 4a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$ by the reduction formula.

Thus the required area $\boxed{A = 3\pi a^2/4 \text{ sq. units.}}$

[39] (b) *Find the volume generated by the revolution of the cardioide* $r = a(1 + \cos\theta)$ *about the initial line.*

☞ Volume of the solid of revolution in polars is given by

$$V = \iint_A 2\pi r^2 \sin\theta\, dr\, d\theta.$$

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} 2\pi \left[\frac{r^3}{3}\right]_{r=0}^{a(1+\cos\theta)} \sin\theta\, d\theta$$

$$= \frac{2\pi}{3} \int_{\theta=0}^{\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$$

Put $1 + \cos\theta = t \quad \therefore \quad -\sin\theta\, d\theta = dt$

If $\theta = 0, t = 2; \ \theta = \pi, \ t = 0$

$$V = \frac{2\pi a^3}{3} \int_2^0 t^3 (-dt) = \frac{2\pi a^3}{3} \int_0^2 t^3\, dt = \frac{2\pi a^3}{3} \left[\frac{t^4}{4}\right]_0^2 = \frac{8\pi a^3}{3}$$

Thus the required volume $\boxed{V = 8\pi a^3/3 \text{ cubic units.}}$

[40] *Find the volume of the tetrahedron bounded by the planes* $x = 0, y = 0, z = 0, x/a + y/b + z/c = 1$.

☞ $V = \iiint dx\, dy\, dz$

$$x/a + y/b + z/c = 1 \quad \therefore \quad z = c(1 - x/a - y/b)$$

If $z = 0$, then $x/a + y/b = 1 \ \therefore \ y = b(1 - x/a)$.

If $z = 0, y = 0$ then $x = a$

$$\therefore \quad V = \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} \int_{z=0}^{c(1-x/a-y/b)} dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx \quad \left(\because \int dz = z\right)$$

$$= c \int_{x=0}^{a} \left[y - \frac{x}{a} y - \frac{y^2}{2b} \right]_0^{b(1-x/a)} dx$$

$$= c \int_{x=0}^{a} \left\{ b\left(1-\frac{x}{a}\right) - \frac{x}{a} b\left(1-\frac{x}{a}\right) - \frac{b}{2}\left(1-\frac{x}{a}\right)^2 \right\} dx$$

$$= c \int_{x=0}^{a} b\left(1-\frac{x}{a}\right)\left\{1-\frac{x}{a}-\frac{1}{2}\left(1-\frac{x}{a}\right)\right\} dx$$

$$= c \int_{x=0}^{a} b\left(1-\frac{x}{a}\right)\frac{1}{2}\left(1-\frac{x}{a}\right) dx$$

$$= \frac{bc}{2} \int_{x=0}^{a} \left(1-\frac{x}{a}\right)^2 dx = \frac{bc}{2}\left[\frac{-a}{3}\left(1-\frac{x}{a}\right)^3\right]_{x=0}^{a}$$

$$V = -\frac{abc}{6}(0-1) = \frac{abc}{6}$$

Thus the required volume $\boxed{V = abc/6 \text{ cubic units.}}$

Note : ***Similar problem***

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0,$ $x + y + z = 1$ [June 2015]

☞ This particular case where, $a = b = c = 1$

We can obtain $\boxed{V = 1/6}$

[41] *A pyramid is bounded by three coordinate planes and the plane, $x + 2y + 3z = 6$. Compute the volume by double integration.*

☞ $V = \iint z\, dx\, dy$ (*This is same as* $V = \iiint dx\, dy\, dz$)

Consider, $x + 2y + 3z = 6$ or $x/6 + y/3 + z/2 = 1$

We have, $z = 2[1-(x/6)-(y/3)]$

If $z = 0$, $(x/6)+(y/3) = 1 \Rightarrow y = 3[1-(x/6)]$

If $z = 0$, $y = 0$, then $x = 6$.

$$V = \int_{x=0}^{6} \int_{y=0}^{3[1-(x/6)]} 2[1-(x/6)-(y/3)]\, dy\, dx = 6, \text{ on evaluation.}$$

Thus the required volume $\boxed{V = 6 \text{ cubic units.}}$

[42] *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$, $z = 0$.*

☞ $V = \iiint dx\, dy\, dz$

$y + z = 4$ gives $z = 4 - y$. Hence z varies from 0 to $(4-y)$.

Also, $x^2 + y^2 = 4$ gives $y^2 = 4 - x^2$.

Hence y varies from $-\sqrt{4-x^2}$ to $+\sqrt{4-x^2}$.

Further when $y = 0$, $x^2 = 4$ and hence x varies from -2 to 2.

$$\therefore \quad V = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{(4-y)} dz\, dy\, dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_{z=0}^{(4-y)}\, dy\, dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx$$

$$= \int_{x=-2}^{2} \left[4y - \frac{y^2}{2}\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$V = \int_{x=-2}^{2} \left[4\left(\sqrt{4-x^2} + \sqrt{4-x^2} \right) - \frac{1}{2}\{(4-x^2)-(4-x^2)\} \right] dx$$

$$= \int_{x=-2}^{2} 8\sqrt{4-x^2}\,dx = 8 \times 2 \int_0^2 \sqrt{4-x^2}\,dx, \text{ by a property.}$$

By using, $\int \sqrt{a^2-x^2}\,dx = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)$,

$$V = 16\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) \right]_0^2 \quad (a = 2)$$

$$= 16\left[0 + 2(\sin^{-1}1 - \sin^{-1}0)\right] = 32\left(\frac{\pi}{2} - 0\right) = 16\pi$$

Thus the required volume is $\boxed{16\pi \text{ cubic units.}}$

Centre of gravity

Let A be the area of a plane lamina in the xy plane having density 'ρ' as a function of two variables x and y. If $P(x, y)$ is a point of the lamina then its total mass (M_A) is given by

$$M_A = \iint_A \rho\,dx\,dy$$

The *Centre of gravity* $(\bar{x}, \bar{y})$ of the plane lamina is given by the following formulae.

$$\bar{x} = \frac{\iint_A x\rho\,dx\,dy}{M_A} \;;\; \bar{y} = \frac{\iint_A y\rho\,dx\,dy}{M_A}$$

Further, let V be the volume of a solid having density 'ρ' as a function of three variables x, y, z. If $P(x, y, z)$ is a point of the solid then its total mass (M_V) is given by

$$M_V = \iiint_V \rho\,dx\,dy\,dz$$

The *Centre of gravity* $(\bar{x}, \bar{y}, \bar{z})$ of the solid is given by the following formulae.

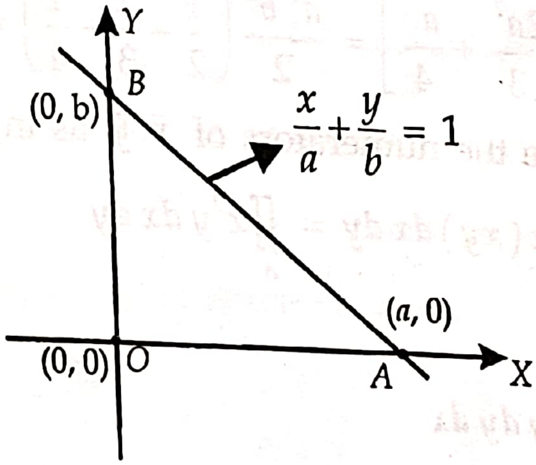
$$\bar{x} = \frac{\iiint_V x\rho\, dx\, dy\, dz}{M_V} \; ; \; \bar{y} = \frac{\iiint_V y\rho\, dx\, dy\, dz}{M_V} \; ; \; \bar{z} = \frac{\iiint_V z\rho\, dx\, dy\, dz}{M_V}$$

Note : $\rho = \lambda xy$ *in two dimensions and* $\rho = \mu xyz$ *in three dimensions.* (λ, μ *being fixed constants). However, while computing the centre of gravity, we take* $\rho = xy$ *or* $\rho = xyz$ *as the fixed constants cancels out.*

WORKED PROBLEMS

[43] *Find the centre of gravity of the triangular lamina bounded by the coordinate axes and the line* $\frac{x}{a} + \frac{y}{b} = 1$.

☞



OAB be the triangular lamina. Centre of gravity $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{\iint_A x\rho\, dx\, dy}{M_A} \; ; \; \bar{y} = \frac{\iint_A y\rho\, dx\, dy}{M_A} \qquad \ldots (1)$$

where, $M_A = \iint_A \rho\, dx\, dy$ and $\rho = xy$.

$\frac{x}{a} + \frac{y}{b} = 1$ gives $y = b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a and y from 0 to $b\left(1 - \frac{x}{a}\right)$

$$\therefore \quad M_A = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} xy\, dy\, dx$$

$$= \int_{x=0}^{a} x\left(\frac{y^2}{2}\right)_{y=0}^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \frac{b^2}{2}\int_{x=0}^{a} x\left(1-\frac{x}{a}\right)^2 dx = \frac{b^2}{2}\int_{x=0}^{a} x\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right) dx$$

ie.,
$$= \frac{b^2}{2}\left[\frac{x^2}{2}-\frac{2x^3}{3a}+\frac{x^4}{4a^2}\right]_0^a$$

$$M_A = \frac{b^2}{2}\left[\frac{a^2}{2}-\frac{2a^2}{3}+\frac{a^2}{4}\right] = \frac{a^2b^2}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{a^2b^2}{24} \quad \ldots(2$$

Next, we shall compute the numerators of $\overline{x}, \overline{y}$ as in (1).

$$\iint_A x\rho\, dx\, dy = \iint_A x(xy)\, dx\, dy = \iint_A x^2y\, dx\, dy$$

That is, $\displaystyle\int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} x^2y\, dy\, dx$

$$= \int_{x=0}^{a} x^2\left[\frac{y^2}{2}\right]_{y=0}^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \frac{b^2}{2}\int_{x=0}^{a}\left[x^2\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{b^2}{2}\int_{x=0}^{a}\left[x^2\left(1-\frac{2x}{a}+\frac{x^2}{a^2}\right)\right] dx$$

$$= \frac{b^2}{2}\left[\frac{x^3}{3}-\frac{2x^4}{4a}+\frac{x^5}{5a^2}\right]_0^a = \frac{b^2a^3}{2}\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right) = \frac{a^3b^2}{60}$$

Hence, $\displaystyle\iint_A x\rho\, dx\, dy = \frac{a^3b^2}{60}$

$$= \frac{1}{2} \int_{x=0}^{a} (a^2x^2 - 3a^{4/3}x^{8/3} + 3a^{2/3}x^{10/3} - x^4)\,dx$$

$$= \frac{1}{2}\left[a^2 \cdot \frac{x^3}{3} - 3a^{4/3} \cdot \frac{3}{11}x^{11/3} + 3a^{2/3} \cdot \frac{3}{13}x^{13/3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{a^5}{2}\left[\frac{1}{3} - \frac{9}{11} + \frac{9}{13} - \frac{1}{5}\right] = \frac{8a^5}{2145} \quad \ldots (3)$$

Using (2) and (3) in (1) we have,

$$\bar{x} = \frac{8a^5/2145}{a^4/80} = \frac{128a}{429}$$

Similarly we can obtain, $\bar{y} = \dfrac{128a}{429}$

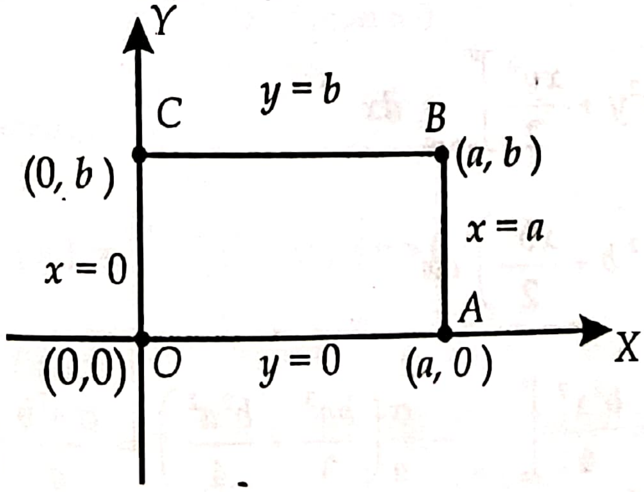
Thus the required centre of gravity is $\boxed{\left(\dfrac{128a}{429}, \dfrac{128a}{429}\right)}$

Note : ***Generalized form of this problem***

If the curve is of the form $(x/a)^{2/3} + (y/b)^{2/3} = 1$ *then the centre of gravity can be obtained as* $\boxed{\left(\dfrac{128a}{429}, \dfrac{128b}{429}\right)}$

[46] *The density at any point* (x, y) *of a lamina is* $\dfrac{\sigma}{a}(x+y)$, *where* λ *and a are constants. The lamina is bounded by the lines* $x = 0, y = 0, x = a, y = b$. *Find the position of its centre of gravity.*

☞



OABC is a rectangular lamina. Centre of gravity is given by

$$\bar{x} = \frac{\iint_A x\rho \, dx\, dy}{M_A} \; ; \; \bar{y} = \frac{\iint_A y\rho \, dx\, dy}{M_A} \qquad \ldots (1)$$

Where $M_A = \iint_A \rho \, dx\, dy$ and $\rho = \frac{\sigma}{a}(x+y)$ by data.

Evidently x varies from 0 to a and y from 0 to b.

$$\therefore \quad M_A = \int_{x=0}^{a} \int_{y=0}^{b} \frac{\sigma}{a}(x+y)\, dy\, dx$$

$$= \frac{\sigma}{a} \int_{x=0}^{a} \left[xy + \frac{y^2}{2} \right]_{y=0}^{b} dx$$

$$= \frac{\sigma}{a} \int_{x=0}^{a} \left(bx + \frac{b^2}{2} \right) dx$$

$$= \frac{\sigma}{a} \left[\frac{bx^2}{2} + \frac{b^2 x}{2} \right]_{x=0}^{a} = \frac{\sigma}{2a}(ba^2 + b^2 a) = \frac{\sigma ab}{2a}(a+b)$$

That is, $M_A = \frac{\sigma b}{2}(a+b) \qquad \ldots (2)$

Next, $\iint_A x\rho \, dx\, dy = \iint_A x \cdot \frac{\sigma}{2a}(x+y)\, dx\, dy$

That is, $\frac{\sigma}{a} \int_{x=0}^{a} \int_{y=0}^{b} (x^2 + xy)\, dy\, dx$

$$= \frac{\sigma}{a} \int_{x=0}^{a} \left[x^2 y + \frac{xy^2}{2} \right]_{y=0}^{b} dx$$

$$= \frac{\sigma}{a} \int_{x=0}^{a} \left(x^2 b + \frac{xb^2}{2} \right) dx$$

$$= \frac{\sigma}{a} \left[\frac{bx^3}{3} + \frac{b^2 x^2}{4} \right]_{x=0}^{a} = \frac{\sigma}{a}\left(\frac{ba^3}{3} + \frac{b^2 a^2}{4} \right) = \frac{\sigma a^2 b}{a}\left(\frac{a}{3} + \frac{b}{4} \right)$$

Hence, $\iint_A x\rho\, dx\, dy = \dfrac{\sigma ab}{12}(4a+3b)$ ... (3)

Using (2) and (3) in (1) we have,

$$\bar{x} = \frac{\frac{\sigma ab}{12}(4a+3b)}{\frac{\sigma b}{2}(a+b)} = \frac{a(4a+3b)}{6(a+b)}$$

Similarly we can obtain, $\bar{y} = \dfrac{b(3a+4b)}{6(a+b)}$

Thus the required position of centre of gravity is

$$\boxed{\frac{a(4a+3b)}{6(a+b)},\ \frac{b(3a+4b)}{6(a+b)}}$$

3.4 Beta and Gamma functions

In this topic we define two special functions namely *Beta function* and *Gamma function* by means of an integral and study the associated properties. These help us to evaluate certain definite integrals which are either difficult or impossible to evaluate by various known methods of integration.

3.41 Definitions

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ (m,n>0) \qquad \ldots (1)$$

is called the **Beta function.**

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx,\ (n>0) \qquad \ldots (2)$$

is called the **Gamma function.**

These definitions can be put in the following alternative forms.

In (1), put $x = \sin^2\theta$. $dx = 2\sin\theta\cos\theta\, d\theta$

If, $x = 0, \sin^2\theta = 0 \Rightarrow \theta = 0$

$x = 1, \sin^2\theta = 1 \Rightarrow \theta = \pi/2$

$$\therefore \quad \beta(m,n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$ie., \quad \beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \qquad \ldots(3)$$

In (2), put $x = t^2$. $\therefore \quad dx = 2t\,dt$, t also varies from 0 to ∞.

$$\therefore \quad \Gamma(n) = \int_{t=0}^{\infty} e^{-t^2}(t^2)^{n-1}\, 2t\, dt$$

$$ie., \quad \Gamma(n) = 2\int_0^{\infty} e^{-t^2} t^{2n-1}\, dt \qquad \ldots(4)$$

(3) and (4) are also regarded as definitions of Beta function and Gamma function respectively.

3.42 Properties of Beta and Gamma functions

1. $\beta(m,n) = \beta(n,m)$

Proof : We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $x = 1-y$ or $1-x = y$ $\therefore$ $dx = -dy$

When $x = 0$, $y = 1$ and when $x = 1$, $y = 0$.

$$\therefore \quad \beta(m,n) = \int_{y=1}^{0} (1-y)^{m-1} y^{n-1}(-dy)$$

$$ie., \quad \beta(m,n) = \int_{y=0}^{1} y^{n-1}(1-y)^{m-1}\, dy = \beta(n,m)$$

Thus, $\boxed{\beta(m,n) = \beta(n,m)}$

2. (i) $\Gamma(n+1) = n\,\Gamma(n)$ (ii) $\Gamma(n+1) = n!$ for a positive integer n.

Proof (i) By definition $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$\therefore \quad \Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx \text{ or } \int_0^\infty x^n e^{-x}\, dx$$

Integrating by parts we get,

$$\Gamma(n+1) = \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1}\, dx$$

Note : $x^n/e^x \to 0$ as $x \to \infty$ by L' Hospital's rule.

$$\Gamma(n+1) = (0-0) + n\int_0^\infty e^{-x} x^{n-1}\, dx = n\Gamma(n)$$

$$\therefore \quad \boxed{\Gamma(n+1) = n\Gamma(n)}$$

(ii) Continuing from the above, we have similarly

$$\Gamma(n) = (n-1)\Gamma(n-1), \Gamma(n-1) = (n-2)\Gamma(n-2)\ldots$$

$$\Gamma(3) = 2\Gamma(2), \Gamma(2) = 1\Gamma(1).$$

Now we have,

$$\Gamma(n+1) = n\{(n-1)\cdot\Gamma(n-1)\} = n(n-1)\{(n-2)\Gamma(n-2)\} \text{ etc.}$$

Hence, $\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot\Gamma(1) = n!\Gamma(1)$

But $\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = -\left[e^{-x}\right]_0^\infty = -(0-1) = 1$

Thus, $\boxed{\Gamma(n+1) = n!}$ for a positive integer n.

Note :

1. *This result can be remembered in the form* $\Gamma(n) = (n-1)\Gamma(n-1)$ *where* $n \neq 1$ *and* $\Gamma(n) = (n-1)!$ *where* n *is a positive integer.*
2. $\Gamma(n)$ *is not defined for* $n = 0$ *and also for a negative integer* n.
3. $\Gamma(n+1) = n\Gamma(n)$ *or* $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ *and this expression is used for finding* $\Gamma(n)$ *when* n *is a negative real number.*

3.43 Relationship between Beta and Gamma functions

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

[June 2016, 17, 18, Dec 17]

Proof : We have by the definition of Beta and Gamma functions

$$\beta(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \qquad \ldots (1)$$

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx \qquad \ldots (2)$$

$$\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\, dy \qquad \ldots (3)$$

$$\Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \qquad \ldots (4)$$

Now, $\Gamma(m)\cdot\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\, dx\, dy \qquad \ldots (5)$

Let us evaluate RHS by changing into polars.
Putting $x = r\cos\theta$, $y = r\sin\theta$, we have, $x^2 + y^2 = r^2$
Also $dx\,dy = r\,dr\,d\theta$. r varies from 0 to ∞, θ varies from 0 to $\pi/2$.
(Analogous to Problem-35)
We now have (5) in the form,

$$\Gamma(m)\cdot\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1} r\,dr\,d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m+2n-1}\sin^{2m-1}\theta\cos^{2n-1}\theta\, dr\, d\theta$$

$$= \left[2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr\right]\left[2\int_{\theta=0}^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta\right]$$

$\therefore\quad \Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n)$ by using (1) and (4).

Thus,
$$\boxed{\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}$$

Corollary : To show that $\Gamma(1/2) = \sqrt{\pi}$

Putting, $m = n = 1/2$ in this result we get,

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2) \cdot \Gamma(1/2)}{\Gamma(1)} \quad \text{But } \Gamma(1) = 1.$$

$$\therefore \quad \beta(1/2, 1/2) = \{\Gamma(1/2)\}^2 \qquad \ldots (6)$$

Now consider, $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta = 2\int_0^{\pi/2} 1 \, d\theta = 2[\theta]_0^{\pi/2} = \pi$$

Now we have from (6) $\pi = \{\Gamma(1/2)\}^2$

Thus, $\boxed{\Gamma(1/2) = \sqrt{\pi}}$

Note : *We can independently prove that* $\Gamma(1/2) = \sqrt{\pi}$. *The proof is as follows.*

Question : *Prove that* $\Gamma(1/2) = \sqrt{\pi}$ *using the definition of* $\Gamma(n)$. **[June 2017,18]**

We have by the definition, $\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1} \, dx$

$$\therefore \quad \Gamma(1/2) = 2\int_0^{\infty} e^{-x^2} \, dx = 2\int_0^{\infty} e^{-y^2} \, dy$$

Hence, $\{\Gamma(1/2)\}^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} \, dx \, dy = 4 \cdot \frac{\pi}{4} = \pi$

(*We need to retrace the steps of Problem- 35*)

Thus,

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

3.44 **Duplication formula**

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma(m+1/2)$$

or $\quad \beta(m, 1/2) = 2^{2m-1}\,\beta(m, m)$

Proof : We have,

$$\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Putting $n = 1/2$ in (1) we get,

$$\frac{\Gamma(m)\cdot\Gamma(1/2)}{\Gamma(m+1/2)} = \beta(m,1/2) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{0}\theta\,d\theta$$

ie., $$\frac{\Gamma(m)\cdot\sqrt{\pi}}{\Gamma(m+1/2)} = \beta(m,1/2) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta$$

Also by putting $n = m$ in (1) we get,

$$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$$

$$= 2.\int_0^{\pi/2} \left\{\frac{\sin 2\theta}{2}\right\}^{2m-1} d\theta$$

ie., $$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,m) = 2\cdot\frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta$$

In the integral, put $2\theta = \phi \quad \therefore d\theta = d\phi/2$. ϕ varies from 0 to π

$$\therefore \quad \frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,m) = 2\cdot\frac{1}{2^{2m-1}}\int_{\phi=0}^{\pi} \sin^{2m-1}\phi\cdot\frac{d\phi}{2}$$

ie., $$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,m) = \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi$$

Using the fact that, $\int_0^{\pi} \sin^k \theta \, d\theta = 2\int_0^{\pi/2} \sin^k \theta \, d\theta$, we have,

$$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,m) = \frac{1}{2^{2m-1}} 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

or

$$2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2^{2m-1}\beta(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi \qquad \ldots(3)$$

With reference to (2) and (3) we can say that

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

since the variable is arbitrary in a definite integral.

Hence we must have,

$$\frac{\Gamma(m)\sqrt{\pi}}{\Gamma(m+1/2)} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} \text{ or } \beta(m,1/2) = 2^{2m-1}\beta(m,m)$$

$$\therefore \quad \boxed{\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+1/2)} \qquad \ldots(4)$$

$$\text{or} \quad \boxed{\beta(m,1/2) = 2^{2m-1}\beta(m,m)} \qquad \ldots(5)$$

(4) is the duplication formula in terms of gamma functions and (5) is the duplication formula in terms of beta functions.

Corollary : Putting $m = 1/4$ in (4) we get,

$$\sqrt{\pi}\,\Gamma(1/2) = 2^{-1/2}\,\Gamma(1/4)\Gamma(3/4)$$

ie., $$\sqrt{\pi}\cdot\sqrt{\pi} = 1/\sqrt{2}\cdot\Gamma(1/4)\Gamma(3/4)$$

Thus, $$\boxed{\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}}$$

Results to remember

(i) $\Gamma(n)=(n-1)\Gamma(n-1), \Gamma(n)=(n-1)!$ if n is a positive integer.

(ii) $\beta(m,n)=\dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

(iii) $\Gamma(1)=1, \Gamma(1/2)=\sqrt{\pi}, \Gamma(1/4)\Gamma(3/4)=\pi\sqrt{2}$

WORKED PROBLEMS

[47] *Show that* $\beta(m+1,n)+\beta(m,n+1)=\beta(m,n)$

☞ Using the relationship between beta and gamma functions in LHS we have

$$\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}+\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$\text{LHS}=\frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}+\frac{\Gamma(m)n\Gamma(n)}{(m+n)\Gamma(m+n)}, \text{ since } \Gamma(p+1)=p\Gamma(p)$$

$$\text{LHS}=\frac{\Gamma(m)\cdot\Gamma(n)}{(m+n)\Gamma(m+n)}(m+n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\beta(m,n)=\text{RHS}$$

Note : *We can also obtain the result from the basic definition of Beta function.*

[48] *Evaluate* (i) $\dfrac{\Gamma(3)\Gamma(2\cdot 5)}{\Gamma(5\cdot 5)}$ (ii) $\dfrac{6\Gamma(8/3)}{5\Gamma(2/3)}$

(iii) $\Gamma(-7/2)$ (iv) $\beta(7/2,-1/2)$

☞ (i) $\dfrac{\Gamma(3)\Gamma(2\cdot 5)}{\Gamma(5\cdot 5)}=\dfrac{(2!)(1\cdot 5)(0\cdot 5)\Gamma(0\cdot 5)}{(4\cdot 5)(3\cdot 5)(2.5)(1\cdot 5)(0\cdot 5)\Gamma(0\cdot 5)}$

$$\frac{\Gamma(3)\Gamma(2\cdot 5)}{\Gamma(5\cdot 5)}=\frac{2}{(4\cdot 5)(3\cdot 5)(2\cdot 5)}=\frac{2}{9/2\cdot 7/2\cdot 5/2}=\boxed{\frac{16}{315}}$$

(ii) $\dfrac{6\Gamma(8/3)}{5\Gamma(2/3)}=\dfrac{6\cdot 5/3\cdot 2/3\cdot\Gamma(2/3)}{5\Gamma(2/3)}=\boxed{\dfrac{4}{3}}$

(iii) To find $\Gamma(-7/2)$ we consider the relation in the form

$$\Gamma(n)=\frac{\Gamma(n+1)}{n}$$

$$\text{LHS} = \int_{t=0}^{\infty} (e^{-t})^{m+1} t^n \, dt = \int_{t=0}^{\infty} e^{-(m+1)t} t^n \, dt \qquad \ldots(1)$$

Now, let $(m+1)t = u \therefore dt = du/(m+1)$

u also varies from 0 to ∞ and hence (1) becomes

$$\text{LHS} = \int_{u=0}^{\infty} e^{-u} \left(\frac{u}{m+1}\right)^n \frac{du}{(m+1)} = \frac{1}{(m+1)^{n+1}} \int_0^{\infty} e^{-u} u^n \, du$$

$$\text{LHS} = \frac{1}{(m+1)^{n+1}} \cdot \Gamma(n+1) = \frac{n!}{(m+1)^{n+1}} = \text{RHS}$$

Thus, LHS = RHS

Evaluation of definite integrals by converting into gamma functions

Step-1 : We need to take a suitable substitution keeping in mind the definition of gamma function in two of the standard forms :

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} \, dx \; ; \; \Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1} \, dx$$

Step-2 : We need to correlate the value correspondig to $(n-1)$ or $(2n-1)$ as the case may be and find n.

[51] *Show that* $\int_0^{\infty} \sqrt{y}\, e^{-y^2} dy \times \int_0^{\infty} \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$

☞ Let $I_1 = \int_0^{\infty} \sqrt{y}\, e^{-y^2} dy = \int_0^{\infty} e^{-y^2} y^{1/2} dy \qquad \ldots(1)$

$$I_2 = \int_0^{\infty} \frac{e^{-y^2}}{\sqrt{y}} dy = \int_0^{\infty} e^{-y^2} y^{-1/2} dy \qquad \ldots(2)$$

We have, $\frac{1}{2}\Gamma(n) = \int_0^{\infty} e^{-x^2} x^{2n-1} dx \qquad \ldots(3)$

Comparing (1) and (3) we have $2n-1 = 1/2 \Rightarrow n = 3/4$

$\therefore \quad I_1 = 1/2 \cdot \Gamma(3/4)$

Comparing (2) and (3) we have $2n-1=-1/2 \Rightarrow n=1/4$

$\therefore \quad I_2 = 1/2 \cdot \Gamma(1/4)$

Hence the required, $I_1 \times I_2 = \dfrac{1}{4}\Gamma(3/4)\Gamma(1/4)$

But, $\Gamma(3/4)\cdot\Gamma(1/4) = \pi\sqrt{2}$

Thus,
$$\boxed{I_1 \times I_2 = \frac{\pi\sqrt{2}}{4} = \frac{\pi}{2\sqrt{2}}}$$

Evaluate the following integrals

[52] $\displaystyle\int_0^\infty x^{3/2}\, e^{-x}\, dx$ [53] $\displaystyle\int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$

Solutions

[52] Let $I = \displaystyle\int_0^\infty x^{3/2}\, e^{-x}\, dx$

We have, $\Gamma(n) = \displaystyle\int_0^\infty e^{-x}\, x^{n-1}\, dx$

Taking $(n-1)=3/2$ we get $n=5/2$

Hence, $I = \Gamma(5/2) = 3/2 \cdot 1/2 \cdot \Gamma(1/2) = 3\sqrt{\pi}/4$

Thus,
$$\boxed{I = 3\sqrt{\pi}/4}$$

[53] Let $I = \displaystyle\int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$

Put $\sqrt{x} = t$ or $x = t^2$ $\therefore$ $dx = 2t\, dt$ and t varies from 0 to ∞.

Hence, $I = \displaystyle\int_{t=0}^\infty (t^2)^{1/4}\, e^{-t}\, 2t\, dt = 2\int_0^\infty e^{-t}\, t^{3/2}\, dt = 2\int_0^\infty e^{-t}\, t^{(5/2)-1}\, dt$

That is, $I = 2\Gamma(5/2) = 2\cdot 3/2 \cdot 1/2\, \Gamma(1/2) = 3\sqrt{\pi}/2$

Thus,
$$\boxed{I = 3\sqrt{\pi}/2}$$

Problems on converting an integral into beta function and evaluation by transforming into gamma function

We have, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$

Putting, $2m-1 = p$ and $2n-1 = q$ we get,

$$m = (p+1)/2,\ n = (q+1)/2$$

$$\therefore \quad \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta$$

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad \cdots (1)$$

Integrals involving trigonometric functions are converted into the above form and in turn is expressed in terms of beta function. Further some of the integrals involving algebraic functions are converted into the trigonometric form by suitable substitution.

Two important forms along with the substitution is as follows.

(i) $(a - x^n)$: Substitution : $x^n = a\sin^2\theta$

(ii) $(a + x^n)$: Substitution : $x^n = a\tan^2\theta$

The integral expressed in terms of beta function is converted into gamma function by using the relationship between them for the purpose of evaluation.

[54] *Show that* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$ **[June 2015, 16, 17, 18, Dec 1**

☞ Let, $I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta$

and $I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$

We have, $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Hence $I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$I_2 = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$\therefore \quad I_1 \times I_2 = \frac{1}{4}\beta(1/4, 1/2)\cdot\beta(3/4, 1/2)$

$$= \frac{1}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}\cdot\frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(5/4)}$$

$$= \frac{1}{4}\Gamma(1/4)\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{1/4\cdot\Gamma(1/4)} = \pi$$

Thus,

$$\boxed{I_1 \times I_2 = \pi}$$

[55] *Evaluate* $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$ *by expressing in terms of gamma functions.*

☞ Let, $I = \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \int_0^{\pi/2}\frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\,d\theta = \int_0^{\pi/2}\sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$

Hence, $I = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$

ie., $I = \frac{1}{2}\beta(1/4, 3/4) = \frac{1}{2}\frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1)} = \frac{1}{2}\cdot\frac{\pi\sqrt{2}}{1} = \frac{\pi}{\sqrt{2}}$

Thus,

$$\boxed{I = \pi/\sqrt{2}}$$

Note : $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$ *is also equal to* $\pi/\sqrt{2}$. [June 2018]

Express the following integrals in terms of beta function and hence evaluate them.

[56] $\int_0^2 (4-x^2)^{3/2}\,dx$

[57] $\int_0^\infty \frac{dx}{1+x^4}$

[60] *Show that* $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}$

☞ Put $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$,

$\therefore \quad dx = 2/n \cdot \sin^{(2/n)-1}\theta\cos\theta\, d\theta$, θ varies from 0 to $\pi/2$.

$$\text{LHS} = I = \int_{\theta=0}^{\pi/2} \frac{2/n \cdot \sin^{(2/n)-1}\theta\cos\theta}{\cos\theta}\, d\theta$$

$$I = \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta\cos^0\theta\, d\theta$$

$$I = \frac{2}{n}\cdot\frac{1}{2}\beta\left(\frac{2/n-1+1}{2}, \frac{0+1}{2}\right) = \frac{1}{n}\beta\left(\frac{1}{n}, \frac{1}{2}\right) = \frac{1}{n}\frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(1/n+1/2)}$$

Thus, $$\boxed{I = \frac{\sqrt{\pi}}{n}\frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}}$$

ASSIGNMENT

Evaluate the following :

1. $\int_0^\infty x^6 e^{-3x}\, dx$

2. $\int_0^\infty x^{-7/4} e^{-\sqrt{x}}\, dx$

3. $\int_0^{\pi/2} \sin^{1/2} x \cos^{3/2} x\, dx$

4. $\int_0^\infty \frac{x}{1+x^6}\, dx$

ANSWERS

1. $80/243$
2. $8\sqrt{\pi}/3$
3. $\pi/4\sqrt{2}$
4. $\pi/3\sqrt{3}$