CHAPTER 8

Vector Calculus and Its Applications

1. Differentiation of vectors. 2. Curves in space. 3. Velocity and acceleration, Tangential and normal acceleration, Relative velocity and acceleration. 4. Scalar and vector point functions—Vector operator del. 5. Del applied to scalar point functions—Gradient. 6. Del applied to vector point functions—Divergence and Curl. 7. Physical interpretations of div **F** and curl **F**. 8. Del applied twice to point functions. 9. Del applied to products of point functions. 10. Integration of vectors. 11. Line integral—Circulation—Work. 12. Surface integral—Flux. 13. Green's theorem in the plane. 14. Stoke's theorem. 15. Volume integral. 16. Divergence theorem. 17. Green's theorem. 18. Irrotational and Solenoidal fields. 19. Orthogonal curvilinear coordinates, Del applied to functions in orthogonal curvilinear coordinates. 20. Cylindrical coordinates. 21. Spherical polar coordinates. 22. Objective Type of Questions.

8.1 (1) DIFFERENTIATION OF VECTORS

If a vector $\mathbf{R}$ varies continuously as a scalar variable t changes, then $\mathbf{R}$ is said to be a function of t and is written as $\mathbf{R} = \mathbf{F}(t)$.

Just as in scalar calculus, *we define* **derivative** *of a vector function* $\mathbf{R} = \mathbf{F}(t)$ *as*

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\mathbf{F}(t + \delta t) - \mathbf{F}(t)}{\delta t} \text{ and write it as } \frac{d\mathbf{R}}{dt} \text{ or } \frac{d\mathbf{F}}{dt} \text{ or } \mathbf{F}'(t).$$

(2) **General rules of differentiation** are similar to those of ordinary calculus *provided the order of factors in vector products is maintained.* Thus, if ϕ, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ are scalar and vector functions of a scalar variable t, we have

(*i*) $\dfrac{d}{dt}(\mathbf{F} + \mathbf{G} - \mathbf{H}) = \dfrac{d\mathbf{F}}{dt} + \dfrac{d\mathbf{G}}{dt} - \dfrac{d\mathbf{H}}{dt}$

(*ii*) $\dfrac{d}{dt}(\mathbf{F}\phi) = \mathbf{F}\dfrac{d\phi}{dt} + \dfrac{d\mathbf{F}}{dt}\phi$

(*iii*) $\dfrac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \cdot \mathbf{G}$

(*iv*) $\dfrac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \times \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \times \mathbf{G}$

(*v*) $\dfrac{d}{dt}[\mathbf{FGH}] = \left[\dfrac{d\mathbf{F}}{dt}\mathbf{GH}\right] + \left[\mathbf{F}\dfrac{d\mathbf{G}}{dt}\mathbf{H}\right] + \left[\mathbf{FG}\dfrac{d\mathbf{H}}{dt}\right]$

(*vi*) $\dfrac{d}{dt}[(\mathbf{F} \times \mathbf{G}) \times \mathbf{H}] = \left(\dfrac{d\mathbf{F}}{dt} \times \mathbf{G}\right) \times \mathbf{H} + \left(\mathbf{F} \times \dfrac{d\mathbf{G}}{dt}\right) \times \mathbf{H} + (\mathbf{F} \times \mathbf{G}) \times \dfrac{d\mathbf{H}}{dt}$

As an illustration, let us prove (*iv*), while the others can be proved similarly :

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \underset{\delta t \to 0}{\text{Lt}} \frac{(\mathbf{F} + \delta\mathbf{F}) \times (\mathbf{G} + \delta\mathbf{G}) - \mathbf{F} \times \mathbf{G}}{\delta t} = \underset{\delta t \to 0}{\text{Lt}} \frac{\mathbf{F} \times \delta\mathbf{G} + \delta\mathbf{F} \times \mathbf{G} + \delta\mathbf{F} \times \delta\mathbf{G}}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \left[\mathbf{F} \times \frac{\delta\mathbf{G}}{\delta t} + \frac{\delta\mathbf{F}}{\delta t} \times \mathbf{G} + \frac{\delta\mathbf{F}}{\delta t} \times \delta\mathbf{G}\right] = \mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times G \qquad [\because \ \delta\mathbf{G} \to 0 \text{ as } \delta t \to 0]$$

Obs. 1. *If* $\mathbf{F}(t)$ *has a constant magnitude, then* $\mathbf{F} : \dfrac{d\mathbf{F}}{dt} = 0$

For $\mathbf{F}(t) \cdot \mathbf{F}(t) = [\mathbf{F}(t)]^2 =$ constant

$\therefore \qquad \mathbf{F} \cdot \dfrac{d\mathbf{F}}{dt} = 0, \quad$ *i.e.*, $\quad \dfrac{d\mathbf{F}}{dt} \perp \mathbf{F}.$

Obs. 2. *If* $\mathbf{F}(t)$ *has constant (fixed) direction, then* $\mathbf{F} \times \dfrac{d\mathbf{F}}{dt} = \mathbf{0}$

Let $\mathbf{G}(t)$ be a unit vector in the direction of $\mathbf{F}(t)$ so that

$$\mathbf{F}(t) = f(t)\,\mathbf{G}(t) \quad \text{where} \quad f(t) = |\,\mathbf{F}(t)\,|.$$

$$\therefore \qquad \frac{d\mathbf{F}}{dt} = f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G} \quad \text{and} \quad \mathbf{F} \times \frac{d\mathbf{F}}{dt} = f\,\mathbf{G} \times \left[f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G}\right]$$

$$= f^2\,\mathbf{G} \times \frac{d\mathbf{G}}{dt} = \mathbf{0}. \qquad \text{[since } \mathbf{G} \text{ is constant, } d\mathbf{G}/dt = \mathbf{0}.]$$

Example 8.1. *If* $\mathbf{A} = 5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, $\mathbf{B} = \sin t\mathbf{I} - \cos t\mathbf{J}$, *find (i)* $\dfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B})$; *(ii)* $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B})$.

Solution. (*i*) $\dfrac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt} \cdot \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \cdot [\cos t\mathbf{I} - (-\sin t)\,\mathbf{J}] + (10t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \cdot (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= (5t^2 \cos t + t \sin t) + (10t \sin t - \cos t) = 5t^2 \cos t + 11t \sin t - \cos t.$$

(*ii*) $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \dfrac{d\mathbf{B}}{dt} + \dfrac{d\mathbf{A}}{dt} \times \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (\cos t\mathbf{I} + \sin t\mathbf{J}) + (10\,t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \times (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= [5t^2 \sin t\mathbf{K} + r \cos t\,(-\mathbf{K}) - t^3 \cos t\mathbf{J} - t^3 \sin t\,(-\mathbf{I})]$$

$$+ [-10t \cos t\mathbf{K} + \sin t(-\mathbf{K}) - 3t^2 \sin t\mathbf{J} + 3t^2 \cos t\,(-\mathbf{I})]$$

$$= (t^3 \sin t - 3t^2 \cos t)\,\mathbf{I} - t^2(t \cos t + 3 \sin t)\,\mathbf{J} + [(5t^2 - 1) \sin t - 11t \cos t]\,\mathbf{K}.$$

8.2 CURVES IN SPACE

(1) **Tangent.** Let $\mathbf{R}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$ be the position vector of a point P. Then as the scalar parameter t takes different values, the point P traces out a *curve in space* (Fig. 8.1). If the neighbouring point Q corresponds to $t + \delta t$, then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along the chord PQ. As $\delta t \to 0$, Lt $\delta\mathbf{R}/\delta t$ becomes the tangent (vector) to the curve at P whenever it exists and is not zero.

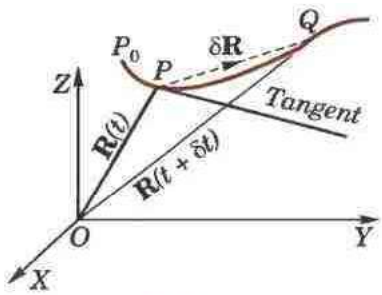

Fig. 8.1

Thus the vector $\mathbf{R}' = d\mathbf{R}/dt$ is a tangent to the space curve $\mathbf{R} = \mathbf{F}(t)$.

Let P_0 be a fixed point of this curve corresponding to $t = t_0$. If s be the length of the arc P_0P, then

$$\frac{\delta s}{\delta t} = \frac{\delta s}{|\,\delta\mathbf{R}\,|} \cdot \frac{|\,\delta\mathbf{R}\,|}{\delta t} = \frac{\text{arc } PQ}{\text{chord } PQ}\left|\frac{\delta\mathbf{R}}{\delta t}\right|$$

As $Q \to P$ along the curve QR *i.e.*, $\delta t \to 0$, then arc PQ/chord $PQ \to 1$ and

$$\frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| \quad \text{or} \quad |\,\mathbf{R}'(t)\,|.$$

If $\mathbf{R}'(t)$ is continuous, then arc P_0P is given by

$$s = \int_{t_0}^{t} |\,\mathbf{R}'\,|\,dt = \int_{t_0}^{t} \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt$$

If we take s the parameter in place of t then the magnitude of the tangent vector, *i.e.*, $|\,d\mathbf{R}/ds\,| = 1$. Thus denoting the unit tangent vector by $\mathbf{T}$, we have

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \qquad \text{...(1)}$$

(2) **Principal normal.** Since $\mathbf{T}$ is a unit vector, we have

$$d\mathbf{T}/ds \cdot \mathbf{T} = 0$$

i.e., $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$. Or else $d\mathbf{T}/ds = 0$, in which case $\mathbf{T}$ is a constant vector w.r.t. the arc length s and so has a fixed direction, *i.e.*, the curve is a straight line.

If we denote a unit normal vector to the curve at P by $\mathbf{N}$ then $d\mathbf{T}/ds$ is in the direction of $\mathbf{N}$ which is known as the *principal normal* to the space curve at P. The plane of $\mathbf{T}$ and $\mathbf{N}$ is called the *osculating plane* of the curve at P (Fig. 8.2).

(3) Binormal. A third unit vector $\mathbf{B}$ defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, is called the *binormal at P*. Since $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is also a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and hence normal to the *osculating plane at P*.

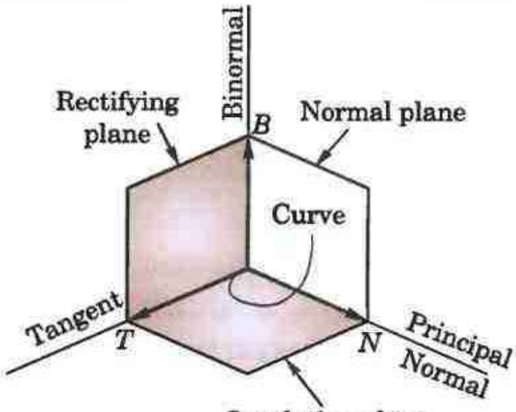

Fig. 8.2

Thus at each point P of a space curve there are three mutually perpendicular unit vectors, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ which form a moving trihedral such that

$$\mathbf{T} = \mathbf{N} \times \mathbf{B}, \mathbf{N} = \mathbf{B} \times \mathbf{T}, \mathbf{B} = \mathbf{T} \times \mathbf{N} \quad \text{...(2)}$$

This moving trihedral determines the following three fundamental planes at each point of the curve :

(*i*) The osculating plane containing $\mathbf{T}$ and $\mathbf{N}$

(*ii*) The normal plane containing $\mathbf{N}$ and $\mathbf{B}$

(*iii*) The rectifying plane containing $\mathbf{B}$ and $\mathbf{T}$.

(4) Curvature. The arc rate of turning of the tangent (*i.e.*, the magnitude of $d\mathbf{T}/ds$) is called the *curvature* of the curve and is denoted by k.

Since $d\mathbf{T}/ds$ is in the direction of the principal normal $\mathbf{N}$, therefore,

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N} \quad \text{...(3)}$$

(5) Torsion. Since $\mathbf{B}$ is a unit vector, we have $\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0$

Also $\mathbf{B} \cdot \mathbf{T} = 0$, therefore $\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot \dfrac{d\mathbf{T}}{ds} = 0.$

or $$\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot (k\mathbf{N}) = 0, \qquad i.e., \quad \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = 0 \qquad [\because \mathbf{B} \cdot \mathbf{N} = 0]$$

Hence $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$ and is, therefore, parallel to $\mathbf{N}$.

The arc rate of turning of the binormal (*i.e.*, the magnitude of $d\mathbf{B}/ds$) is called *torsion* of the curve and is denoted by τ. We may, therefore, write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \quad \text{...(4)}$$

(The negative sign indicates that for $\tau > 0$, $d\mathbf{B}/ds$ has direction of $-\mathbf{N}$).

Finally to find $d\mathbf{N}/ds$, we differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

$$\therefore \quad \frac{d\mathbf{N}}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} = -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times k\mathbf{N}$$

Using the relation (2), it reduces to $\dfrac{d\mathbf{N}}{ds} = \tau\mathbf{B} - k\mathbf{T}$ ...(5)

The equations (3), (4) and (5) constitute the well-known *Frenet formulae** for space curves.

Obs. 1. $\rho = 1/k$ and $\sigma = 1/\tau$ are respectively called the radii of curvature and torsion.

2. For a plane curve $\tau = 0$.

Example 8.2. *Find the angle between the tangents to the curve* $\mathbf{R} = t^2\mathbf{I} + 2t\mathbf{J} - t^3\mathbf{K}$ *at the point* $t = \pm 1$. *(V.T.U., 2010)*

Solution. The tangent at any point 't' is given by

$$\frac{d\mathbf{R}}{dt} = 2t\mathbf{I} + 2\mathbf{J} - 3t^2\mathbf{K}$$

$\therefore$ the tangents $\mathbf{T}_1$, $\mathbf{T}_2$ at $t = 1$ and $t = -1$ are respectively given by

$$\mathbf{T}_1 = 2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}\ ;\ \mathbf{T}_2 = -2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K},$$

* Named after a French mathematician *Jean-Frederic Frenet* (1816–1900).

Then the required $\angle\theta$ is given by $T_1 T_2 \cos\theta = \mathbf{T}_1 . \mathbf{T}_2 = 2(-2) + 2 . 2 + (-3)(-3)$

i.e., $\sqrt{17}\sqrt{17}\cos\theta = 9 \quad \therefore \quad \theta = \cos^{-1}(9/17)$.

Example 8.3. *Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$.*

(This curve is drawn on a circular cylinder cutting its generators at a constant angle and is known as a *circle helix*).

Solution. The vector equation of the curve is $\mathbf{R} = a\cos t\mathbf{I} + a\sin t\mathbf{J} + bt\mathbf{K}$

$\therefore$ $$d\mathbf{R}/dt = -a\sin t\mathbf{I} + a\cos t\mathbf{J} + b\mathbf{K}$$

Its arc length from P_0 $(t = 0)$ to any point $P(t)$ (Fig. 8.3) is given by

$$s = \int_0^t |d\mathbf{R}/dt|\,dt = \sqrt{(a^2+b^2)}\,t$$

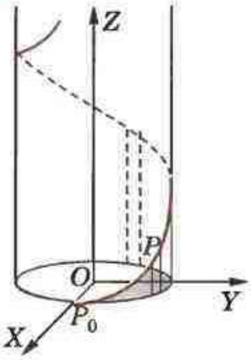

Fig. 8.3

$\therefore$ $$\frac{ds}{dt} = \sqrt{(a^2+b^2)}$$

Then $$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt}\Big/\frac{ds}{dt} = \frac{-a\sin t\mathbf{I} + a\cos t\mathbf{J} + b\mathbf{K}}{\sqrt{(a^2+b^2)}}$$

and $$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt} = \frac{-a(\cos t\mathbf{I} + \sin t\mathbf{J})}{a^2+b^2}$$

Thus $$k = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{a}{a^2+b^2} \quad ...(i) \qquad \text{and} \quad \mathbf{N} = -(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Also $$\mathbf{B} = \mathbf{T}\times\mathbf{N} = (b\sin t\mathbf{I} - b\cos t\mathbf{J} + a\mathbf{K})/\sqrt{(a^2+b^2)}$$

$\therefore$ $$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt}\Big/\frac{ds}{dt} = b(\cos t\mathbf{I} + \sin t\mathbf{J})/(a^2+b^2) = -\tau\mathbf{N} = \tau(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Hence $$\tau = \frac{b}{a^2+b^2}. \qquad ...(ii)$$

PROBLEMS 8.1

1. Show that, if $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$, where $\mathbf{A}, \mathbf{B}, \omega$ are constants, then (*i*) $\dfrac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{R}$ (*Bhopal, 2007 S*)

   (*ii*) $\mathbf{R}\times\dfrac{d\mathbf{R}}{dt} = -\omega\mathbf{A}\times\mathbf{B}$.
2. Given $\mathbf{R} = t^m\mathbf{A} + t^n\mathbf{B}$, where $\mathbf{A}, \mathbf{B}$ are constant vectors, show that, if $\mathbf{R}$ and $d^2\mathbf{R}/dt^2$ are parallel vectors, then $m + n = 1$, unless $m = n$.
3. If $\mathbf{P} = 5t^2\mathbf{I} + t^3\mathbf{J} - t\mathbf{K}$ and $\mathbf{Q} = 2\mathbf{I}\sin t - \mathbf{J}\cos t + 5t\mathbf{K}$, find (*i*) $\dfrac{d}{dt}(\mathbf{P}.\mathbf{Q})$; (*ii*) $\dfrac{d}{dt}(\mathbf{P}\times\mathbf{Q})$.
4. If $\dfrac{d\mathbf{U}}{dt} = \mathbf{W}\times\mathbf{U}$ and $\dfrac{d\mathbf{V}}{dt} = \mathbf{W}\times\mathbf{V}$, prove that $\dfrac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{W}\times(\mathbf{U}\times\mathbf{V})$. (*Mumbai, 2009*)
5. If $\mathbf{A} = x^2yz\mathbf{I} - 2xz^3\mathbf{J} + xz^2\mathbf{K}$ and $\mathbf{B} = 2z\mathbf{I} + y\mathbf{J} - x^2\mathbf{K}$, find $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})$ at $(1, 0, -2)$.
6. If $\mathbf{R} = (a\cos t)\,\mathbf{I} + (a\sin t)\,\mathbf{J} + (at\tan\alpha)\,\mathbf{K}$, find the value of

   (*i*) $\left|\dfrac{d\mathbf{R}}{dt}\times\dfrac{d^2\mathbf{R}}{dt^2}\right|$ (*ii*) $\left|\dfrac{d\mathbf{R}}{dt}, \dfrac{d^2\mathbf{R}}{dt^2}, \dfrac{d^3\mathbf{R}}{dt^3}\right|$ (*Rohtak, 2005*)

   Also find the unit tangent vector at any point t of the curve.
7. Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable. Also determine the unit tangent vector at the point $t = 2$.
8. Find the equation of the tangent line to the curve $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\tan\alpha$ at $\theta = \pi/4$.
9. Find the curvature of the (*i*) ellipse $\mathbf{R}(t) = a\cos t\mathbf{I} + b\sin t\mathbf{J}$; (*ii*) parabola $\mathbf{R}(t) = 2t\mathbf{I} + t^2\mathbf{J}$ at the point $t = 1$.

10. Find the education of the osculating plane and binormal to the curve
(i) $x = 2 \cosh (t/2), y = 2 \sinh (t/2), z = 2t$ at $t = 0$; (ii) $x = e^t \cos t, y = e^t \sin t, z = e^t$ at $t = 0$.
11. A circular helix is given by the equation $\mathbf{R}(t) = (2 \cos t)\,\mathbf{I} + (2 \sin t)\,\mathbf{J} + \mathbf{K}$. Find the curvature and torsion of the curve at any point and show that they are constant.
12. Show that for the curve $\mathbf{R} = a(3t - t^3)\,\mathbf{I} + 3at^2\,\mathbf{J} + a(3t + t^2)\,\mathbf{K}$, the curvature equals torsion.

8.3 (1) VELOCITY AND ACCELERATION

Let the position of a particle P at time t on a path (curve) C be $\mathbf{R}(t)$. At time $t + \delta t$, let the particle be at Q (Fig. 8.1), then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along PQ. As $Q \to P$ along C, the line PQ becomes the tangent at P to C.

$$\therefore \qquad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta \mathbf{R}}{\delta t} = \frac{d\mathbf{R}}{dt} = \mathbf{V}$$

is the tangent vector of C at P which is the *velocity* (vector) $\mathbf{V}$ of the motion and its magnitude is the *speed* $v = ds/dt$, where s is the arc length of P from a fixed point P_0 $(s = 0)$ on C.

The derivative of the velocity vector $\mathbf{V}(t)$ is called the *acceleration* (vector) $\mathbf{A}(t)$, which is given by

$$\mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}.$$

(2) Tangential and normal accelerations. It is important to note that the magnitude of acceleration is not always the rate of change of $|\mathbf{V}|$ because $\mathbf{A}(t)$ is not always tangential to the path C. Infact

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds} \cdot \frac{ds}{dt}. \text{ where } d\mathbf{R}/ds \text{ is a unit tangent vector of } C.$$

$$\therefore \qquad \mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left[\frac{ds}{dt} \cdot \frac{d\mathbf{R}}{ds}\right] = \frac{d^2 s}{dt^2} \cdot \frac{d\mathbf{R}}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2\mathbf{R}}{ds^2}$$

Now since $d\mathbf{R}/dt \cdot d^2\mathbf{R}/dt^2 = 0$, $d^2\mathbf{R}/dt^2$ is perpendicular to $d\mathbf{R}/dt$. Hence the acceleration $\mathbf{A}(t)$ is comprised of (*i*) the tangential component $d^2s/dt^2 \cdot d\mathbf{R}/ds$, called the *tangential acceleration, and*

(*ii*) the normal component $(ds/dt)^2 \cdot d^2\mathbf{R}/ds^2$, called the *normal acceleration.*

Obs. The acceleration is the time rate change of $|\mathbf{V}| = ds/dt$, if the normal acceleration is zero, for then

$$|A| = \left|\frac{d^2 s}{dt^2}\right| \cdot \left|\frac{d\mathbf{R}}{ds}\right| = \left|\frac{d^2 s}{dt^2}\right|.$$

(3) Relative velocity and acceleration. Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ at time t, (Fig. 8.4), so that $\mathbf{R} = \overrightarrow{P_1 P_2} = \mathbf{R}_2 - \mathbf{R}_1$

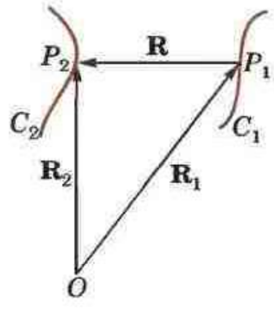

Fig. 8.4

Differentiating w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{R}_2}{dt} - \dfrac{d\mathbf{R}_1}{dt}$ ...(*iii*)

This defines the *relative velocity* (vector) of P_2 w.r.t. P_1 and states that the *velocity (vector) of P_2 relative to P_1 = velocity (vector) of P_2 − velocity (vector) of P_1.*

Again differentiating (*iii*), we have $\dfrac{d^2\mathbf{R}}{dt^2} = \dfrac{d^2\mathbf{R}_2}{dt^2} - \dfrac{d^2\mathbf{R}_1}{dt^2}$ ...(*iv*)

i.e., acceleration (vector) of P_2 relative to P_1 = acceleration (vector) of P_2 − acceleration (vector) of P_1.

Example 8.4. *A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 3$ where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$.*

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = \dfrac{d}{dt}\,[(t^3 + 1)\,\mathbf{I} + t^2\mathbf{J} + (2t + 3)\mathbf{K}]$

$= 3t^2\,\mathbf{I} + 2t\mathbf{J} + 2\mathbf{K} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ at $t = 1$

and *acceleration* $= \dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + 0\mathbf{K} = 6\mathbf{I} + 2\mathbf{J}$ at $t = 1$.

Now unit vector in the direction of $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$ is $\dfrac{\mathbf{I} + \mathbf{J} + 3\mathbf{K}}{\sqrt{(1^2 + 1^2 + 3^2)}} = \dfrac{1}{\sqrt{11}}(\mathbf{I} + \mathbf{J} + 3\mathbf{K})$.

$\therefore$ component of velocity at $t = 1$ in the direction $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$

$$= \frac{(3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}) \cdot (\mathbf{I} + \mathbf{J} + 3\mathbf{K})}{\sqrt{(11)}} = \frac{3 + 2 + 6}{\sqrt{(11)}} = \sqrt{(11)}$$

and component of acceleration at $t = 1$ in the direction

$$\mathbf{I} + \mathbf{J} + 3\mathbf{K} = (6\mathbf{I} + 2\mathbf{J}) \cdot (\mathbf{I} + \mathbf{J} + 3\mathbf{K})/\sqrt{(11)} = \frac{6 + 2}{\sqrt{(11)}} = \frac{8}{\sqrt{(11)}}.$$

Example 8.5. *A particle moves along the curve* $\mathbf{R} = (t^3 - 4t)\,\mathbf{I} + (t^2 + 4t)\mathbf{J} + (8t^2 - 3t^3)\mathbf{K}$ *where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time t = 2.* (*V.T.U., 2003 S*)

Solution. Velocity $\dfrac{d\mathbf{R}}{dt} = (3t^2 - 4)\mathbf{I} + (2t + 4)\mathbf{J} + (16t - 9t^2)\mathbf{K}$

and acceleration $\dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + (16 - 18t)\mathbf{K}$

$\therefore$ at $t = 2$, velocity $\mathbf{V} = 8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}$ and acceleration $\mathbf{A} = 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}$.

Since the velocity is along the tangent to the curve, therefore, the component of $\mathbf{A}$ along the tangent

$$= \mathbf{A} \cdot \frac{\mathbf{V}}{|\mathbf{V}|} = (12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}) \cdot \frac{8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}}{\sqrt{(64 + 64 + 16)}}$$

$$= \frac{12 \times 8 + 2 \times 8 + (-20) \times (-4)}{12} = 16.$$

Now the component of $\mathbf{A}$ along the normal

$= |\,\mathbf{A}$ – Resolved part of $\mathbf{A}$ along the tangent $|$

$$= \left| 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K} - 16\,\frac{8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}}{12} \right| = \frac{1}{3}\,|\,4\mathbf{I} - 26\mathbf{J} - 44\mathbf{K}\,| = 2\sqrt{73}.$$

Example 8.6. *The position vector of a particle at time t is* $\mathbf{R} = \cos(t - 1)\,\mathbf{I} + \sinh(t - 1)\,\mathbf{J} + \alpha t^3\mathbf{K}$. *Find the condition imposed on α by requiring that at time t = 1, the acceleration is normal to the position vector.*

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = -\sin(t - 1)\,\mathbf{I} + \cosh(t - 1)\,\mathbf{J} + 3\alpha t^2\mathbf{K}$

Acceleration $= \dfrac{d^2\mathbf{R}}{dt^2} = -\cos(t - 1)\,\mathbf{I} + \sinh(t - 1)\,\mathbf{J} + 6\alpha t\mathbf{K} = -\mathbf{I} + 6\alpha\mathbf{K}$ at $t = 1$.

Also $\mathbf{R} = \mathbf{I} + \alpha\mathbf{K}$ at $t = 1$.

If $\mathbf{R}$ and acceleration at $t = 1$ are normal, then their scalar product is zero.

$\therefore$ $(-\mathbf{I} + 6\alpha\mathbf{K}) \cdot (\mathbf{I} + \alpha\mathbf{K}) = 0$ or $-1 + 6\alpha^2 = 0$

or $\alpha^2 = 1/6$ or $\alpha = 1/\sqrt{6}$.

Example 8.7. *Find the radial and transverse acceleration of a particle moving in a plane curve.* (*Kurukshetra, 2006 ; Rajasthan, 2006*)

Solution. At any time t, let the position vector of the moving particle $P(r, \theta)$ be $\mathbf{R}$ (Fig. 8.5) so that

$$\mathbf{R} = r\hat{\mathbf{R}} = r(\cos\theta\mathbf{I} + \sin\theta\mathbf{J})$$

$\therefore$ its velocity $\mathbf{V} = \dfrac{d\mathbf{R}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\hat{\mathbf{R}}}{dt}$ ...(*i*)

As $\hat{\mathbf{R}} = \cos\theta\mathbf{I} + \sin\theta\mathbf{J}$

and $\dfrac{d\hat{\mathbf{R}}}{dt} = (-\sin\theta\mathbf{I} + \cos\theta\mathbf{J})\,\dfrac{d\theta}{dt}$

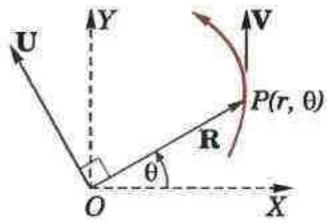

Fig. 8.5

$\therefore$ $\frac{d\hat{\mathbf{R}}}{dt} \perp \hat{\mathbf{R}}$ and $\left|\frac{d\hat{\mathbf{R}}}{dt}\right| = \frac{d\theta}{dt}$, *i.e.*, if **U** is a unit vector $\perp$ **R**, then

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{d\theta}{dt}\mathbf{U}$$

$\therefore$ (*i*) becomes, $\mathbf{V} = \frac{dr}{dt}\hat{\mathbf{R}} + r\frac{d\theta}{dt}\mathbf{U}$ ...(*ii*)

Thus the radial and transverse components of the velocity are dr/dt and $r\, d\theta/dt$.

Also
$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{R}} + \frac{dr}{dt}\frac{d\hat{\mathbf{R}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{U} + r\frac{d^2\theta}{dt^2}\mathbf{U} + r\frac{d\theta}{dt}\frac{d\mathbf{U}}{dt}$$
$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{R}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{U} \qquad \left[\because\ \mathbf{U} = -\sin\theta\mathbf{I} + \cos\theta\mathbf{J} \text{ gives } \frac{d\mathbf{U}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{R}}\right]$$

Thus the radial and transverse components of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}.$$

Example 8.8. *A person going eastwards with a velocity of 4 km per hour, finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.*

Solution. Let the actual velocity of the wind be $x\mathbf{I} + y\mathbf{J}$, where **I**, **J** represent velocities of 1 km per hour towards the east and north respectively. As the person is going eastwards with a velocity of 4 km per hour, his actual velocity is $4\mathbf{I}$.

Then the velocity of the wind relative to the man is $(x\mathbf{I} + y\mathbf{J}) - 4\mathbf{I}$, which is parallel to $-\mathbf{J}$, as it appears to blow from the north. Hence $x = 4$. ...(*i*)

When the velocity of the person becomes $8\mathbf{I}$, the velocity of the wind relative to man is $(x\mathbf{I} + y\mathbf{J}) - 8\mathbf{I}$. But this is parallel to $-(\mathbf{I} + \mathbf{J})$.

$\therefore$ $(x - 8)/y = 1$, which by (*i*) gives $y = -4$.

Hence the actual velocity of the wind is $4(\mathbf{I} - \mathbf{J})$, *i.e.*, $4\sqrt{2}$ km. per hour towards the south-east.

PROBLEMS 8.2

1. A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time variable. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$. (*P.T.U., 2003 ; V.T.U., 2003 S*)
2. The position vector of a particle at time t is $\mathbf{R} = \cos(t-1)\,\mathbf{I} + \sinh(t-1)\,\mathbf{J} + at^3\mathbf{K}$. Find the condition imposed on a by requiring that at time $t = 1$, the acceleration is normal to the position vector.
3. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$. (*V.T.U., 2008*)
4. A particle moves so that its position vector is given by $\mathbf{R} = \mathbf{I}\cos\omega t + \mathbf{J}\sin\omega t$. Show that the velocity **V** of the particle is perpendicular to **R** and $\mathbf{R} \times \mathbf{V}$ is a constant vector.
5. A particle (position vector **R**) is moving in a circle with constant angular velocity ω. Show by vector methods, that the acceleration is equal to $-\omega^2\mathbf{R}$.
6. (*a*) Find the tangential and normal accelerations of a point moving in a plane curve. (*Rajasthan, 2005*)
   (*b*) The position vector of a moving particle at a time t is $\mathbf{R} = 3\cos t\mathbf{I} + 3\sin t\mathbf{J} + 4t\mathbf{K}$. Find the tangent and normal components of its acceleration at $t = 1$. (*Marathwada, 2008*)
7. The velocity of a boat relative to water is represented by $3\mathbf{I} + 4\mathbf{J}$ and that of wate relative to earth is $\mathbf{I} - 3\mathbf{J}$. What is the velocity of the boat relative to the earth if **I** and **J** represent one km an hour east and north respectively.
8. A vessel A is sailing with a velocity of 11 knots per hour in the direction S.E. and a second vessel B is sailing with a velocity of 13 knots per hour in a direction 30°E of N. Find the velocity of A relative to B.
9. A person travelling towards the north-east with a velocity of 6 km per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} 2$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ km per hour towards the east.

8.4 SCALAR AND VECTOR POINT FUNCTIONS

(1) *If to each point P*(**R**) *of a region E in space there corresponds a definite scalar denoted by f* (**R**), then *f* (**R**) *is called a* **scalar point function** *in E. The region E so defined is called a* **scalar field.**

The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

(2) *If to each point P*(**R**) *of a region E in space there corresponds a definite vector denoted by* **F**(**R**), *then it is called the* **vector point function** *in E. The region E so defined is called a* **vector field.**

The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Differentiation of vector point functions follows the same rules as those of ordinary calculus. Thus if $\mathbf{F}(x, y, z)$ be a vector point function, then

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}$$ (See (*iii*) p. 203]

and $$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{F} \quad ...(i)$$

(3) Vector operator del. The operator on the right side of the equation (*i*) is in the form of a scalar product of $\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ and $\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

If ∇ *(read as del) be defined by the equation* $\nabla = \mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ ...(*ii*)

then (*i*) may be written as $d\mathbf{F} = (\nabla \cdot d\mathbf{R})\,\mathbf{F}$ for when $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

8.5 DEL APPLIED TO SCALAR POINT FUNCTIONS—GRADIENT

(1) Def. *The vector function* ∇f *is defined as the gradient of the scalar point function f and is written as grad f.*

Thus $$\text{grad } f = \nabla f = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}$$

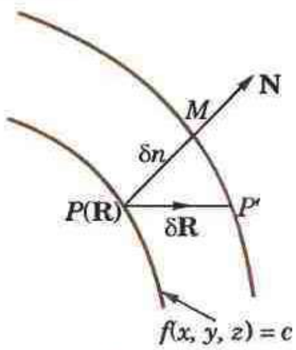

Fig. 8.6

(2) Geometrical interpretation. Consider the scalar point function $f(\mathbf{R})$, where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

If a surface $f(x, y, z) = c$ be drawn through any point $P(\mathbf{R})$ such that at each point on it, the function has the same value as at P, then such a surface is called a *level surface* of the function f through P, *e.g.*, equipotential or isothermal surface (Fig. 8.6).

Let $P'(\mathbf{R} + \delta\mathbf{R})$ be a point on a neighbouring level surface $f + \delta f$. Then

$$\nabla f \cdot \delta\mathbf{R} = \left[\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right] \cdot (\mathbf{I}\delta x + \mathbf{J}\delta y + \mathbf{K}\delta z)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = \delta f.$$

Now if P' lies on the same level surface as P, then $\delta f = 0$, *i.e.*, $\nabla f \cdot \delta\mathbf{R} = 0$. This means that ∇f is perpendicular to every $\delta\mathbf{R}$ lying on this surface. Thus ∇f *is normal to the surface* $f(x, y, z) = c$.

$\therefore$ $$\nabla f = |\nabla f|\,\mathbf{N}$$

where $\mathbf{N}$ is a unit vector normal to this surface. If the perpendicular distance PM between the surfaces through P and P' be δn, then the rate of change of f normal to the surface through P

$$= \frac{\partial f}{\partial n} = \underset{\delta n \to 0}{\text{Lt}} \frac{\delta f}{\delta n} = \underset{\delta n \to 0}{\text{Lt}} \nabla f \cdot \frac{\delta\mathbf{R}}{\delta n}$$

$$= |\nabla f| \underset{\delta n \to 0}{\text{Lt}} \frac{\mathbf{N}\cdot\delta\mathbf{R}}{\delta n} = |\nabla f|. \qquad [\because \ \mathbf{N}\cdot\delta\mathbf{R} = |\delta\mathbf{R}|\cos\theta = \delta n]$$

Hence the magnitude of $\nabla f = \partial f/\partial n$.

Thus *grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along this normal.*

(3) Directional derivative. If δr denotes the length PP' and $\mathbf{N}'$ is a unit vector in the direction PP', then the limiting value of $\delta f/\delta r$ as $\delta r \to 0$ (*i.e.*, $\partial f/\partial r$) is known as the *directional derivative of f at P along the direction PP'*.

Since $\qquad \delta r = \delta n/\cos\alpha = \delta n/\mathbf{N}\,.\,\mathbf{N}'$

$$\therefore \qquad \frac{\partial f}{\partial r} = \underset{\delta r \to 0}{\text{Lt}} \left[\mathbf{N}\,.\,\mathbf{N}'\,\frac{\delta f}{\delta n}\right] = \mathbf{N}'\,.\,\frac{\partial f}{\partial n}\,\mathbf{N} = \mathbf{N}'\,.\,\nabla f$$

Thus the directional derivation of f in the direction of $\mathbf{N}'$ is the resolved part of ∇f in the direction $\mathbf{N}'$.

Since $\qquad \nabla f\,.\,\mathbf{N}' = |\,\nabla f\,|\,\cos\alpha \le |\,\nabla f\,|$

It follows that ∇f *gives the maximum rate of change of f.*

Example 8.9. *Prove that* $\nabla r^n = nr^{n-2}\,\mathbf{R}$, *where* $\mathbf{R} + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

(*Bhopal, 2007 ; Anna, 2003 S ; V.T.U., 2000*)

Solution. We have $\quad f(x, y, z) = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\therefore \qquad \frac{\partial f}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}.\,2x = nxr^{n-2}.\ \text{Similarly,}\ \frac{\partial f}{\partial y} = nyr^{n-2}\ \text{and}\ \frac{\partial f}{\partial z} = nzr^{n-2}$$

$$\text{Thus} \qquad \nabla r^n = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z} = nr^{n-2}(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) = nr^{n-2}\,\mathbf{R}.$$

Otherwise : The level surfaces for f = constant, *i.e.*, r^n = constant are concentric spheres with centre O and hence unit normal $\mathbf{N}$ to the level surface through P is along the radius $\mathbf{R}$

i.e., $\qquad \mathbf{N} = \hat{\mathbf{R}}$.

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial n}\,.\,\mathbf{N} = \frac{df}{dr}\hat{\mathbf{R}} = nr^{n-1}\,\hat{\mathbf{R}} \qquad [\because\ f = r^n]$$

$$= nr^{n-1}(\mathbf{R}/r) = nr^{n-2}\mathbf{R}.$$

Example 8.10. *If* $\nabla u = 2r^4\,\mathbf{R}$, *find u.* (*Mumbai, 2008*)

Solution. We have $\nabla u = 2(x^2 + y^2 + z^2)^2\,\mathbf{R} \qquad [\because\ r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= 2(x^2 + y^2 + z^2)^2\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) \qquad ...(i)$$

$$\text{But} \qquad \nabla u = \frac{\partial u}{\partial x}\mathbf{I} + \frac{\partial u}{\partial y}\mathbf{J} + \frac{\partial u}{\partial z}\mathbf{K} \qquad ...(ii)$$

Comparing (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} = 2x(x^2 + y^2 + z^2)^2,\ \frac{\partial u}{\partial y} = 2y(x^2 + y^2 + z^2)^2\,;\ \frac{\partial u}{\partial z} = 2z(x^2 + y^2 + z^2)$$

$$\text{Also} \qquad du(x, y, z) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 2(x^2 + y^2 + z^2)^2\,(xdx + ydy + zdz)$$

$$= 2t^2\,.\,\frac{dt}{2},\ \text{taking}\ x^2 + y^2 + z^2 = t \quad \text{and} \quad 2(xdx + ydy + zdz) = dt$$

$$\text{Integrating both sides,}\ u = \int t^2\,dt + c = \frac{1}{3}t^3 + c = \frac{1}{3}(x^2 + y^2 + z^2)^3 + c$$

$$\text{Hence} \qquad u = \frac{1}{3}r^{3/2} + c.$$

Example 8.11. *If* $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = yz + zx + xy$, *prove that grad u, grad v and grad w are coplanar.* (*U.T.U., 2010 ; U.P.T.U., 2002*)

$$\textbf{Solution.} \qquad \text{grad}\ u = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x + y + z) = \mathbf{I} + \mathbf{J} + \mathbf{K}$$

$$\text{grad}\ v = 2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K},\ \text{grad}\ w = (y + z)\,\mathbf{I} + (z + x)\,\mathbf{J} + (z + x)\,\mathbf{K}$$

We know that three vectors are coplanar if their scalar triple product is zero.

Here [grad u, grad v, grad w]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & y+z+x & z+x+y \\ y+z & z+x & x+y \end{vmatrix} \quad \text{[Operate } R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v and grad w are coplanar.

Example 8.12. *Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.* *(Mumbai, 2008)*

Solution. A vector normal to the given surface is $\nabla (xy^3z^2)$

$$= \mathbf{I}\frac{\partial}{\partial x}(xy^3z^2) + \mathbf{J}\frac{\partial}{\partial y}(xy^3z^2) + \mathbf{K}\frac{\partial}{\partial z}(xy^3z^2) = \mathbf{I}(y^3z^2) + \mathbf{J}(3xy^2z^2) + \mathbf{K}(2xy^3z)$$

$$= -4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K} \text{ at the point } (-1, -1, 2).$$

Hence the desired unit normal to the surface

$$= \frac{-4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K}}{\sqrt{[(-4)^2 + (-12)^2 + 4^2]}} = -\frac{1}{\sqrt{11}}(\mathbf{I} + 3\mathbf{J} - \mathbf{K}).$$

Example 8.13. *Find the directional derivative of $f(x, y, z) = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of vector* $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$. *(Bhopal, 2008 ; Kurukshetra, 2006 ; Rohtak, 2003)*

Solution. Here $\nabla f = \mathbf{I}(y^3) + \mathbf{J}(3xy^2 + z^3) + \mathbf{K}(3yz^2) = \mathbf{I} - 3\mathbf{J} - 3\mathbf{K}$ at the point $(2, -1, 1)$.

$\therefore$ directional derivative of f in the direction $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$

$$= (\mathbf{I} - 3\mathbf{J} - 3\mathbf{K}) . \frac{\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}}{\sqrt{(1^2 + 2^2 + 2^2)}} = (1 . 1 - 3 . 2 - 3 . 2)/3 = -3\frac{2}{3}.$$

Example 8.14. *Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4). Also calculate the magnitude of the maximum directional derivative.*

Solution. We have $\nabla f = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\mathbf{I} - 2y\mathbf{J} + 4z\mathbf{K}$

$$= 2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K} \text{ at } P(1, 2, 3)$$

Also $\vec{PQ} = \vec{OQ} - \vec{OP} = (5\mathbf{I} + 0\mathbf{J} + 4\mathbf{K}) - (\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}) = 4\mathbf{I} - 2\mathbf{J} + \mathbf{K} = \mathbf{A}$ (say)

$\therefore$ unit vector of $\mathbf{A} = \hat{A} = \dfrac{\mathbf{A}}{a} = \dfrac{4\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{\sqrt{(16 + 4 + 1)}} = \dfrac{4\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{21}}$

Thus the directional derivative of f in the direction of $\vec{PQ}$

$$\nabla f . \hat{A} = (2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}) . (4\mathbf{I} - 2\mathbf{J} + \mathbf{K})/\sqrt{21}$$

$$= (8 + 8 + 12)/\sqrt{21} = 28/\sqrt{21}$$

The directional derivative of its maximum in the direction of the normal to the surface *i.e.*, in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\nabla f| = |2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}| = (4 + 16 + 144) = \sqrt{164}.$$

Example 8.15. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1, 1, 1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.* (Bhopal, 2008 ; U.P.T.U., 2004)

Solution. We have
$$\nabla\phi = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$
$$= (10xy + 2.5z^2)\,\mathbf{I} + (5x^2 - 10yz)\,\mathbf{J} + (-5y^2 + 5zx)\,\mathbf{K}$$
$$= 12.5\mathbf{I} - 5\mathbf{J} \text{ at } P(1, 1, 1)$$

Also direction of the given line is $\hat{A} = \dfrac{2\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{3}$

Hence the required directional derivative
$$= \nabla\phi \,.\, \hat{A} = (12.5\mathbf{I} - 5\mathbf{J})\,.\,(2\mathbf{I} - 2\mathbf{J} + \mathbf{K})/3 = (25 + 10)/3 = 11\frac{2}{3}.$$

Example 8.16. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (V.T.U., 2010 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Let $f_1 = x^2 + y^2 + z^2 - 9 = 0$ and $f_2 = x^2 + y^2 - z - 3 = 0$

Then $N_1 = \nabla f_1$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$

and $N_2 = \nabla f_2$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} - \mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} - \mathbf{K}$

Since the angle θ between the two surfaces at a point is the angle between their normals at that point and $\mathbf{N}_1$, $\mathbf{N}_2$ are the normals at $(2, -1, 2)$ to the given surfaces, therefore
$$\cos\theta = \frac{\mathbf{N}_1 \,.\, \mathbf{N}_2}{n_1 n_2} = \frac{(4I - 2J + 4K)\,.\,(4I - 2J - K)}{\sqrt{(16 + 4 + 16)}\,\sqrt{(16 + 4 + 1)}}$$
$$= \frac{4(4) + (-2)(-2) + 4(-1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$.

Example 8.17. *Find the values of a and b such that the surface $ax^2 - byz = (a + 2)\,x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.* (Madras, 2004)

Solution. Let $f_1 = ax^2 - byz - (a + 2)\,x = 0$ ...(i)

and $f_2 = 4x^2y + z^3 - 4 = 0$ ...(ii)

Then $\nabla f_1 = (2ax - a - 2)\mathbf{I} - 4z\mathbf{J} - by\mathbf{K} = (a - 2)\,\mathbf{I} - 2b\mathbf{J} + b\mathbf{K}$ at $(1, -1, 2)$.

$\nabla f_2 = 8xy\mathbf{I} + 4x^2\mathbf{J} + 3z^2\mathbf{K} = -8\mathbf{I} + 4\mathbf{J} + 12\mathbf{K}$ at $(1, -1, 2)$.

The surfaces (i) and (ii) will cut orthogonally if $\nabla f_1 \,.\, \nabla f_2 = 0$, *i.e.*, $-8\,(a - 2) - 8b + 12b = 0$

or $-2a + b + 4 = 0$ ...(iii)

Also since the point $(1, -1, 2)$ lies on (i) and (ii),

$\therefore$ $a + 2b - (a + 2) = 0$ or $b = 1$

From (iii), $-2a + 5 = 0$ or $a = 5/2$.

Hence $a = 5/2$ and $b = 1$.

PROBLEMS 8.3

1. (a) Find $\nabla\phi$, if $\phi = \log\,(x^2 + y^2 + z^2)$. (b) Show that grad $(1/r) = -\mathbf{R}/r^3$.
2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. (P.T.U., 1999)
3. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\mathbf{I} - \mathbf{J} - 2\mathbf{K}$. (V.T.U., 2007 ; Rohtak 2006 S ; J.N.T.U., 2006 ; U.P.T.U., 2006)
4. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$? (S.V.T.U., 2009)

5. Find the values of constants a, b, c so that the directional derivative of $p = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the z-axis. *(Rajasthan, 2006)*
6. Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at the point $A(1, -2, 1)$ in the direction AB where B is $(2, 6, -1)$. Also find the maximum directional derivative of ϕ at $(1, -2, 1)$. *(Mumbai, 2009)*
7. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$, find the values of a, b and c. *(U.P.T.U., 2002)*
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum. *(Rohtak, 2003)*
9. What is the greatest rate of increase of $u = xyz^2$ at the point $(1, 0, 3)$? *(Bhopal, 2008)*
10. The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?
11. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.
12. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$, $x^2y = 2 - z$ at the point $(1, 1, 1)$. *(Hissar, 2005 S ; J.N.T.U., 2003)*
13. Find the values of a and b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$. *(Nagpur, 2009)*
14. If f and $\mathbf{G}$ are point functions, prove that the components of the latter normal and tangential to the surface $f = 0$ are
$$\frac{(\mathbf{G} \cdot \nabla f)\, \nabla f}{(\nabla f)^2} \text{ and } \frac{\nabla f \times (\mathbf{G} \times \nabla f)}{(\nabla f)^2}$$ *[Cf. Ex. 3.24]*

8.6 DEL APPLIED TO VECTOR POINT FUNCTIONS

(1) Divergence. *The divergence of a continuously differentiable vector point function* $\mathbf{F}$ *is denoted by div* $\mathbf{F}$ *and is defined by the equation*

$$\textbf{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$

then $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \cdot (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

(2) Curl. *The curl of a continuously differentiable vector point function* $\mathbf{F}$ *is defined by the equation*

$$\textbf{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$ then $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \times (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \mathbf{I}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \mathbf{J}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \mathbf{K}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Example 8.18. *If* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, *show that*
(i) $\nabla \cdot \mathbf{R} = 3$ *(ii)* $\nabla \times \mathbf{R} = 0$. *(V.T.U. 2008 ; P.T.U., 2006 ; U.P.T.U., 2006)*

Solution. *(i)* $\nabla \cdot \mathbf{R} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$

(ii) $$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{I}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \mathbf{J}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \mathbf{K}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$
$$= \mathbf{I}(0 - 0) - \mathbf{J}(0 - 0) + \mathbf{K}(0 - 0) = \mathbf{0}.$$

[Remember : *div* $\mathbf{R} = \mathbf{3}$; *curl* $\mathbf{R} = \mathbf{0}$**]**

Example 8.19. *Find div* **F** *and curl* **F**, *where* $\mathbf{F} = grad\,(x^3 + y^3 + z^3 - 3xyz)$.

(*V.T.U., 2008 ; Kurukshetra, 2006 ; Burdwan, 2003*)

Solution. If $u = x^3 + y^3 + z^3 - 3xyz$, then

$$\mathbf{F} = \nabla u = \mathbf{I}\frac{\partial u}{\partial x} + \mathbf{J}\frac{\partial u}{\partial y} + \mathbf{K}\frac{\partial u}{\partial z} = \mathbf{I}(3x^2 - 3yz) + \mathbf{J}(3y^2 - 3zx) + \mathbf{K}(3z^2 - 3xy)$$

$$\therefore \quad \text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6(x + y + z)$$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= \mathbf{I}(-3x + 3x) - \mathbf{J}(-3y + 3y) + \mathbf{K}(-3z + 3z) = \mathbf{0}.$$

8.7 (1) PHYSICAL INTERPRETATION OF DIVERGENCE

Consider the motion of the fluid having velocity $\mathbf{V} = v_x\mathbf{I} + v_y\mathbf{J} + v_z\mathbf{K}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes in the mass of fluid, with one of its corners at P (Fig. 8.7).

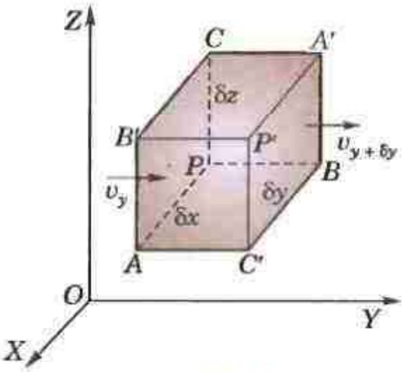

Fig. 8.7

$\therefore$ the amount of fluid entering the face PB' in unit time $= v_y\delta z\delta x$ and the amount of fluid leaving the face $P'B$ in unit time

$$= v_{y+\delta y}\delta z\delta x = \left(v_y + \frac{\partial v_y}{\partial y}\delta y\right)\delta z\delta x \quad \text{nearly}$$

$\therefore$ the net decrease of the amount of fluid due to flow across these two faces $= \dfrac{\partial v_y}{\partial y}\delta x\delta y\delta z$.

Finding similarly the contributions of other two pairs of faces, we have the total decrease of amount of fluid inside the parallelopiped per unit time $= \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}\right)\delta x\delta y\delta z$.

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div}\,\mathbf{V}.$$

Hence *div* **V** *gives the rate at which fluid is originating at a point per unit volume.*

Similarly, if V represents an electric flux, div **V** is the amount of flux which diverges per unit volume in unit time. If **V** represents heat flux, div **V** is the rate at which heat is issuing from a point per unit volume. In general, the divergence of a vector point function representing any physical quantity gives at each point, the rate per unit volume at which the physical quantity is issuing from that point. This explains the justification for the name *divergence of a vector point function.*

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\mathbf{V} = 0$, which is known in Hydrodynamics as the **equation of continuity** for incompressible fluids.

Def. *If the flux entering any element of space is the same as that leaving it, i.e., div* $\mathbf{V} = 0$ *everywhere then such a point function is called a* **solenoidal vector function.**

(2) Physical interpretation of curl. Consider the motion of a rigid body rotating about a fixed axis through O. If Ω be its angular velocity, then the velocity **V** of any particle $P(\mathbf{R})$ of the body is given by $\mathbf{V} = \Omega \times \mathbf{R}$. [See p. 91]

If $\quad \Omega = \omega_1\mathbf{I} + \omega_2\mathbf{J} + \omega_3\mathbf{K}$ and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$

then $$\mathbf{V} = \Omega \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \mathbf{I}(\omega_2 z - \omega_3 y) + \mathbf{J}(\omega_3 x - \omega_1 z) + \mathbf{C}(\omega_1 y - \omega_2 x)$$

$$\therefore \qquad \text{curl } \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y, & \omega_3 x - \omega_1 z, & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \mathbf{I}(\omega_1 + \omega_1) + \mathbf{J}(\omega_2 + \omega_2) + \mathbf{K}(\omega_3 + \omega_3) \qquad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants.}]$$

$$= 2(\omega_1 \mathbf{I} + \omega_2 \mathbf{J} + \omega_3 \mathbf{K}) = 2\Omega. \quad \text{Hence} \quad \Omega = \frac{1}{2} \text{ curl } \mathbf{V}$$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector which justifies the name *rotation* used for curl.

In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Def. *Any motion in which the curl of the velocity vector is zero is said to be* **irrotational**, *otherwise* **rotational.**

Example 8.20. *Prove that div* $(r^n \mathbf{R}) = (n + 3)\, r^n$. *Hence show that* $\mathbf{R}/r^3$ *is solenoidal.*
(V.T.U., 2006 ; U.P.T.U., 2006 ; P.T.U., 2005)

Solution. We have $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $r = \sqrt{(x^2 + y^2 + z^2)}$

$$\therefore \qquad div\ (r^n \mathbf{R}) = \nabla \cdot (x^2 + y^2 + z^2)^{n/2} (x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$= \frac{\partial}{\partial x}[x(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2 + y^2 + z^2)^{n/2}]$$

$$= \Sigma \left\{ 1 . (x^2 + y^2 + z^2)^{n/2} + x . \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} . 2x \right\}$$

$$= \Sigma r^n + n\, \Sigma x^2 (x^2 + y^2 + z^2)^{\frac{n}{2}-1} = 3r^n + nr^2 . r^{n-2}$$

Thus $div\ (r^n \mathbf{R}) = (n + 3)\, r^n$

When $n = -3$, $div\ (\mathbf{R}/r^3) = 0$ *i.e.*, $\mathbf{R}/r^3$ is solenoidal.

Example 8.21. *Show that* $r^\alpha \mathbf{R}$ *is any irrotational vector for any value of* α *but is solenoidal if* $\alpha + 3 = 0$ *where* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ *and r is the magnitude of* $\mathbf{R}$. *(V.T.U., 2006 ; Kottayam, 2005)*

Solution. Let $\mathbf{A} = r^\alpha \mathbf{R} = (x^2 + y^2 + z^2)^{\alpha/2} (x\mathbf{I} + y\mathbf{I} + z\mathbf{K}) = \Sigma x\, (x^2 + y^2 + z^2)^{\alpha/2}\, \mathbf{I}$

$$\therefore \qquad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma\, \mathbf{I} \left\{ \frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1}(2y) - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1} . 2z \right\} = 0$$

Hence $\mathbf{A}$ is irrotational for any value of α.

But $\qquad div\ \mathbf{A} = \nabla . (r^\alpha \mathbf{R}) = (\alpha + 3)\, r^\alpha$

which is zero for $\alpha + 3 = 0$, *i.e.*, $\mathbf{A}$ is solenoidal if $\alpha + 3 = 0$.

8.8 DEL APPLIED TWICE TO POINT FUNCTIONS

∇f and $\nabla \times \mathbf{F}$ being vector point functions, we can form their divergence and curl whereas $\nabla . \mathbf{F}$ being a scalar point function, we can have its gradients only. Thus we have the following five formulae :

(1) $div\ grad\ f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

(2) $curl\ grad\ f = \nabla \times \nabla f = \mathbf{0}$

(3) $div\ curl\ \mathbf{F} = \nabla . \nabla \times \mathbf{F} = 0$

(4) *curl curl* $\mathbf{F}$ = *grad div* $\mathbf{F} - \nabla^2\mathbf{F}$, *i.e.*, $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla . \mathbf{F}) - \nabla^2\mathbf{F}$

(5) *grad div* $\mathbf{F}$ = *curl curl* $\mathbf{F} + \nabla^2\mathbf{F}$, *i.e.*, $\nabla(\nabla . \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2\mathbf{F}$.

Proofs. **(1)** $\nabla^2 f = \nabla . \nabla f = \nabla . \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2 f = 0$ is called the *Laplace's equation.*

(2) $$\nabla \times \nabla f = \nabla \times \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Sigma\mathbf{I}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = \mathbf{0}$$ *(V.T.U., 2007)*

(3) $$\nabla \cdot \nabla \times \mathbf{F} = \left(\Sigma\mathbf{I}\frac{\partial}{\partial x}\right) \cdot \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma\mathbf{I} \cdot \left(\mathbf{I} \times \frac{\partial^2\mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2\mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2\mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left(\mathbf{I} \times \mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x^2} + \mathbf{I} \times \mathbf{J} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial y} + \mathbf{I} \times \mathbf{K} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial z}\right) = \Sigma\left(\mathbf{K} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial y} - \mathbf{J} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial z}\right) = 0.$$

(4) $$\nabla \times (\nabla \times \mathbf{F}) = \left(\Sigma\mathbf{I}\frac{\partial}{\partial x}\right) \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma\mathbf{I} \times \left(\mathbf{I} \times \frac{\partial^2\mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2\mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2\mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left[\left\{\left(\mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x^2}\right)\mathbf{I} - (\mathbf{I} \cdot \mathbf{I})\frac{\partial^2\mathbf{F}}{\partial x^2}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial y}\right)\mathbf{J} - (\mathbf{I} \cdot \mathbf{J})\frac{\partial^2\mathbf{F}}{\partial x \partial y}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial z}\right)\mathbf{K} - (\mathbf{I} \cdot \mathbf{K})\frac{\partial^2\mathbf{F}}{\partial x \partial z}\right\}\right]$$

$$= \Sigma\left[\left(\mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x^2}\right)\mathbf{I} + \left(\mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial y}\right)\mathbf{J} + \left(\mathbf{I} \cdot \frac{\partial^2\mathbf{F}}{\partial x \partial z}\right)\mathbf{K}\right] - \Sigma\frac{\partial^2\mathbf{F}}{\partial x^2}$$

$$= \Sigma\mathbf{I}\frac{\partial}{\partial x}\left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}\right) - \Sigma\frac{\partial^2\mathbf{F}}{\partial x^2} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$ *(Madras, 2006)*

(5) is just another way of writing (4) above.

Obs. *Interpretation of* ∇ *as a vector according to rules of vector products leads to correct results so far so the repeated application of* ∇ *is concerned.*

e.g., **1.** $\nabla . \nabla f = \nabla^2 f$ $(\because \nabla . \nabla = \nabla^2)$

2. $\nabla \times \nabla f = \mathbf{0}$ $(\because \nabla \times \nabla = \mathbf{0})$

3. $\nabla . \nabla \times \mathbf{F} = 0$ $(\because [\nabla\nabla\mathbf{F}] = 0)$

4. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla . \mathbf{F}) - \nabla^2\mathbf{F}$ by expanding it as a vector triple product.

8.9 DEL APPLIED TO PRODUCTS OF POINT FUNCTIONS

To prove that

(1) grad $(fg) = f$ (grad g) + g(grad f) *i.e.* $\nabla(fg) = f\nabla g + g\nabla f$.

(2) *div* $(f\mathbf{G})$ = (grad f) . $\mathbf{G}$ + f (div $\mathbf{G}$) *i.e.* $\nabla(f\mathbf{G}) = \nabla f . \mathbf{G} + f\nabla . \mathbf{G}$

(3) *curl* $(f\mathbf{G})$ = (*grad* f) $\times \mathbf{G}$ + f (*curl* $\mathbf{G}$) *i.e.* $\nabla \times (f\mathbf{G}) = \nabla f \times \mathbf{G} + f\nabla \times \mathbf{G}$

(4) *grad* $(\mathbf{F}.\mathbf{G}) = (\mathbf{F}.\nabla)\mathbf{G} + (\mathbf{G}.\nabla)\mathbf{F} + \mathbf{F} \times$ *curl* $\mathbf{G} + \mathbf{G} \times$ *curl* $\mathbf{F}$

i.e., $\nabla(\mathbf{F}.\mathbf{G}) = (\mathbf{F}.\nabla)\mathbf{G} + (\mathbf{G}.\nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

(5) *div* $(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (curl\ \mathbf{F}) - \mathbf{F} \cdot (curl\ \mathbf{G})$ *i.e.,* $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(6) *curl* $(\mathbf{F} \times \mathbf{G}) = \mathbf{F}\ (div\ \mathbf{G}) - \mathbf{G}\ (div\ \mathbf{F}) + (\mathbf{G} \cdot \nabla)\ \mathbf{F} - (\mathbf{F} \cdot \nabla)\ \mathbf{G}$

i.e., $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}\ (\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\ \mathbf{F} - (\mathbf{F} \cdot \nabla)\ \mathbf{G}$

Proofs **(1)** $\nabla\ (fg) = \Sigma \mathbf{I} \cdot \dfrac{\partial}{\partial x}\ (fg) = \Sigma \mathbf{I} \left(f \dfrac{\partial g}{\partial x} + g\ \dfrac{\partial f}{\partial x} \right)$

$$= f\, \Sigma \mathbf{I}\ \frac{\partial g}{\partial x} + g\ \Sigma \mathbf{I}\ \frac{\partial f}{\partial x} = f\, \nabla g + g \nabla f$$

(2) $$\nabla \cdot (f\, \mathbf{G}) = \Sigma \mathbf{I} \cdot \frac{\partial}{\partial x} (f\, \mathbf{G}) = \Sigma \mathbf{I} \cdot \left(\frac{\partial f}{\partial x}\, \mathbf{G} + f\ \frac{\partial \mathbf{G}}{\partial x} \right)$$

$$= \left(\Sigma \mathbf{I}\ \frac{\partial f}{\partial x} \right) \cdot \mathbf{G} + f \left(\Sigma \mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) = \nabla f \cdot \mathbf{G} + f\, \nabla \cdot \mathbf{G}$$ *(V.T.U., 2011)*

(3) $$\nabla \times (f\, \mathbf{G}) = \Sigma \mathbf{I} \times \frac{\partial}{\partial x} (f\, \mathbf{G}) = \Sigma \mathbf{I} \times \left(f\, \frac{\partial \mathbf{G}}{\partial x} + \frac{\partial f}{\partial x}\ \mathbf{G} \right)$$

$$= f\, \Sigma \mathbf{I} \times f\, \frac{\partial \mathbf{G}}{\partial x} + \Sigma \mathbf{I}\, \frac{\partial f}{\partial x} \times \mathbf{G} = f\ \nabla \times \mathbf{G} + \nabla f \times \mathbf{G}$$ *(V.T.U. 2008)*

(4) $$\nabla(\mathbf{F} \cdot \mathbf{G}) = \Sigma \mathbf{I}\ \frac{\partial}{\partial x} (\mathbf{F} \cdot \mathbf{G}) = \Sigma \mathbf{I} \left(\frac{\partial \mathbf{F}}{\partial x} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) = \Sigma \mathbf{I} \frac{\partial \mathbf{F}}{\partial x} \cdot \mathbf{G} + \Sigma \mathbf{I} \left(\mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \qquad ...(i)$$

Now $\mathbf{G} \times \left(\mathbf{I} \times \dfrac{\partial \mathbf{F}}{\partial x} \right) = \left(\mathbf{G} \cdot \dfrac{\partial \mathbf{F}}{\partial x} \right)\ \mathbf{I} - (\mathbf{G} \cdot \mathbf{I})\ \dfrac{\partial \mathbf{F}}{\partial x}$

or $$\left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x} \right)\ \mathbf{I} = \mathbf{G} \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} \right) + (\mathbf{G} \cdot \mathbf{I})\ \frac{\partial \mathbf{F}}{\partial x}$$

$\therefore$ $$\Sigma \left(\mathbf{G} \cdot \frac{\partial \mathbf{F}}{\partial x} \right)\ \mathbf{I} = \mathbf{G} \times \Sigma \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \Sigma\ (\mathbf{G} \cdot \mathbf{I})\ \frac{\partial \mathbf{F}}{\partial x} = \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} \qquad ...(ii)$$

Interchanging **F** and **G**, $\Sigma \left(\mathbf{F} \cdot \dfrac{\partial \mathbf{G}}{\partial x} \right)\ \mathbf{I} = \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} \cdot \nabla)\ \mathbf{G}$ ...*(iii)*

Substituting in *(i)* from *(ii)* and *(iii)*, we get

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$

(5) $$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \cdot \frac{\partial}{\partial x} (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + F \times \frac{\partial \mathbf{G}}{\partial x} \right) = \Sigma \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} - \Sigma \mathbf{I} \cdot \left(\frac{\partial \mathbf{G}}{\partial x} \times \mathbf{F} \right)$$

$$= \Sigma \left(I \times \frac{\partial \mathbf{F}}{\partial x} \right) \cdot \mathbf{G} - \Sigma \left(I \times \frac{\partial \mathbf{G}}{\partial x} \right) \cdot \mathbf{F}$$ $[\because\ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A \times B) \cdot C]$

$$= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

(6) $$\nabla \times (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \times \frac{\partial}{\partial x}\ (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \times \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial x} \right)$$

$$= \Sigma \left[(\mathbf{I} \cdot \mathbf{G}) \frac{\partial \mathbf{F}}{\partial x} - \left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) \mathbf{G} \right] + \Sigma \left[\left(\mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \mathbf{F} - (\mathbf{I} \cdot \mathbf{F}) \frac{\partial \mathbf{G}}{\partial x} \right]$$

$$= \Sigma\ (\mathbf{G} \cdot \mathbf{I}) \frac{\partial \mathbf{F}}{\partial x} - \mathbf{G} \Sigma \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{F} \Sigma \mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} - \Sigma(\mathbf{F} \cdot \mathbf{I}) \frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F} \left(\Sigma \mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) - \mathbf{G} \Sigma \left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) + \Sigma\ (\mathbf{G} \cdot \mathbf{I})\ \frac{\partial \mathbf{F}}{\partial x} - \Sigma\ (\mathbf{F} \cdot \mathbf{I})\ \frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F}\ (\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\ \mathbf{F} - (\mathbf{F} \cdot \nabla)\ \mathbf{G}$$

Rule to reproduce the above formulae easily :

(i) Treating each of the factors as constants separately, expresss the results of ∇*-operation as a sum of the two terms.*

(ii) Transform each of the two terms, noting that ∇ *always appears before a function and keeping in mind whether the result of operation is a scalar or a vector.* To carry out the simplification, we may sometimes, employ the properties of triple products.

(iii) Restore the change of treating the functions as constants.

Let us illustrate the application of this rule to (2), (4) and (6) above :

(2) $\nabla \cdot (f\mathbf{G}) = \nabla \cdot (f_c \mathbf{G} + f\mathbf{G}_c) = f_c \nabla \cdot \mathbf{G} + \mathbf{G}_c \cdot \nabla f = f \nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla f$

(4) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \nabla(\mathbf{F}_c \cdot \mathbf{G}) + \nabla(\mathbf{F} \cdot \mathbf{G}_c)$

$$= [\mathbf{F}_c \times (\nabla \times \mathbf{G}) + (\mathbf{F}_c \cdot \nabla)\mathbf{G}] + [\mathbf{G}_c \times (\nabla \times \mathbf{F}) + (\mathbf{G}_c \cdot \nabla)\mathbf{F}]$$

$$= \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} \cdot \nabla)\,\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F}$$

(6) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \nabla \times (\mathbf{F}_c \times \mathbf{G}) + \nabla \times (\mathbf{F} \times \mathbf{G}_c) = [\nabla \cdot \mathbf{G}\mathbf{F}_c - (\mathbf{F}_c \cdot \nabla)\mathbf{G}] + (\mathbf{G}_c \cdot \nabla)\,\mathbf{F} - \nabla \cdot \mathbf{F}\mathbf{G}_c]$

$$= \mathbf{F}(\nabla \cdot \mathbf{G}) - (\mathbf{F} \cdot \nabla)\,\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}).$$

Example 8.22. *Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$* *(S.V.T.U., 2006 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. $\nabla^2 r^n = \nabla \cdot (\nabla r^n)$

$$= \nabla \cdot \left(nr^{n-1}\frac{\mathbf{R}}{r}\right) = n\nabla \cdot (r^{n-2}\mathbf{R}) = n[(\nabla r^{n-2}) \cdot \mathbf{R} + r^{n-2}(\nabla \cdot \mathbf{R})] \quad \text{[By § 8.9 (2)]}$$

$$= n\left[(n-2)r^{n-3}\frac{\mathbf{R}}{r} \cdot \mathbf{R} + r^{n-2}(3)\right] \quad \text{[Using Ex. 8.18 (i)]}$$

$$= n[(n-2)r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)\,r^{n-2} \quad [\because \ \mathbf{R} \cdot \mathbf{R} = r^2]$$

Otherwise : $\nabla^2(r^n) = \dfrac{\partial^2(r^n)}{\partial x^2} + \dfrac{\partial^2(r^n)}{\partial y^2} + \dfrac{\partial^2(r^n)}{\partial z^2}$ [By § 8.8 (1)] ...(*i*)]

Now $\dfrac{\partial(r^n)}{\partial x} = nr^{n-1}\dfrac{\partial r}{\partial x} = nr^{n-1}\dfrac{x}{r} = nr^{n-2}x$ $[\because \ r^2 = x^2 + y^2 + z^2]$

$$\therefore \quad \frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}x\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right] \quad \text{...(ii)}$$

Similarly, $\dfrac{\partial^2(r^n)}{\partial y^2} = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$ ...(*iii*)

$$\frac{\partial^2(r^n)}{\partial z^2} = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right] \quad \text{...(iv)}$$

Adding (*ii*), (*iii*) and (*iv*), (*i*) gives

$$\nabla^2(r^n) = n\,[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)]$$

$$= n\,[3r^{n-2} + (n-2)r^{n-4}r^2] = n(n+1)r^{n-2}.$$

In particular $\nabla^2(1/r) = 0$. *(U.P.T.U., 2003 ; P.T.U., 2003)*

Example 8.23. *If $u\mathbf{F} = \nabla v$, where u, v are scalar fields and $\mathbf{F}$ is a vector field, show that $\mathbf{F}$. curl $\mathbf{F} = 0$.*

Solution. Since $\mathbf{F} = \dfrac{1}{u}\nabla v \quad \therefore \quad \text{curl}\,\mathbf{F} = \nabla \times \left(\dfrac{1}{u}\nabla v\right)$

or $$\text{curl}\,\mathbf{F} = \nabla\frac{1}{u} \times \nabla v + \frac{1}{u}\nabla \times (\nabla v) \quad \text{[By § 8.9 (3)]}$$

$$= \nabla\frac{1}{u} \times \nabla v \quad [\because \ \nabla \times \nabla v = 0$$

Hence $\mathbf{F} \cdot \text{curl}\,\mathbf{F} = \dfrac{1}{u}\nabla v \cdot \left(\nabla\dfrac{1}{u} \times \nabla v\right) = 0$, for it is a scalar triple product in which two factors are equal.

Example 8.24. *If r and $\mathbf{R}$ have their usual meanings and $\mathbf{A}$ is a constant vector, prove that* $\nabla \times \left(\dfrac{\mathbf{A} \times \mathbf{R}}{r^n}\right) = \dfrac{2-n}{r^n}\mathbf{A} + \dfrac{n(\mathbf{A} \cdot \mathbf{R})}{r^{n+2}}\mathbf{R}.$ *(Mumbai, 2009 ; Kurukshetra, 2006 ; J.N.T.U., 2005)*

Solution. $\nabla \times [r^{-n}(\mathbf{A} \times \mathbf{R})] = r^{-n}[\nabla \times (\mathbf{A} \times \mathbf{R})] + \nabla r^{-n} \times (\mathbf{A} \times \mathbf{R})]$ [By § 8.9 (3)]

$$= r^{-n}[(\nabla \cdot \mathbf{R})\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{R}] + (-nr^{-(n+1)}\,\mathbf{R}/r) \times (\mathbf{A} \times \mathbf{R})$$

$$= r^{-n}(3\mathbf{A} - \mathbf{A}) - nr^{-(n+2)}\,\mathbf{R} \times (\mathbf{A} \times \mathbf{R}) \qquad [\because \ \nabla \,.\, \mathbf{R} = 3, (\mathbf{A} \,.\, \nabla)\,\mathbf{R} = \mathbf{A}]$$

$$= 2\mathbf{A}r^{-n} - nr^{-(n+2)}\,[(\mathbf{R} \,.\, \mathbf{R})\,\mathbf{A} - (\mathbf{A} \,.\, \mathbf{R})\,\mathbf{R}]$$

$$= \frac{2\mathbf{A}}{r^n} - \frac{n}{r^{n+2}}\,[\,r^2\mathbf{A} - (\mathbf{A} \,.\, \mathbf{R})\,\mathbf{R}] = \frac{2-n}{r^n}\,\mathbf{A} + \frac{n(\mathbf{A} \,.\, \mathbf{R})}{r^{n+2}}\,\mathbf{R}.$$

Example 8.25. *If r is the distance of a point (x, y, z) from the origin, prove that curl* $\left(\mathbf{K} \times grad\,\frac{1}{r}\right)$ *+ grad* $\left(\mathbf{K} \,.\, grad\,\frac{1}{r}\right)$ *= 0, where* **K** *is the unit vector in the direction OZ.* (U.P.T.U., 2001)

Solution. $grad\,\dfrac{1}{r} = \left(\mathbf{I}\dfrac{\partial}{\partial x} + \mathbf{J}\dfrac{\partial}{\partial y} + \mathbf{K}\dfrac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$ $\qquad [\because \ r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}\,(2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$$

$$= -(x^2 + y^2 + z^2)^{-3/2}\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$curl\left(\mathbf{K} \times grad\frac{1}{r}\right) = \nabla \times [-(x^2 + y^2 + z^2)^{-3/2}\,(x\mathbf{J} - y\mathbf{I})]$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y/(x^2 + y^2 + z^2)^{3/2} & -x/(x^2 + y^2 + z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \mathbf{I}\,\frac{\partial}{\partial z}\left\{\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right\} + \mathbf{J}\,\frac{\partial}{\partial z}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$- \mathbf{K}\left\{\frac{\partial}{\partial x}\left[\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right]\right\}$$

$$= \frac{-3xz\mathbf{I} - 3yz\mathbf{J} + (x^2 + y^2 - 2z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}} \qquad ...(i)$$

$$\text{grad}\left(\mathbf{K} \,.\, \text{grad}\,\frac{1}{r}\right) = \nabla\left\{-\mathbf{K} \,.\, \frac{(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)\left\{\frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$

$$= \frac{3xz\mathbf{I}}{(x^2 + y^2 + z^2)^{5/2}} + \frac{3yz\mathbf{J}}{(x^2 + y^2 + z^2)^{5/2}} + \frac{(3z^2 - x^2 - y^2 - z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{3xz\mathbf{I} + 3yz\mathbf{J} - (x^2 + y^2 - 2z^2)\mathbf{K}}{(x^2 + y^2 + z^2)^{5/2}} \qquad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$\text{curl}\left(\mathbf{K} \times \text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\mathbf{K} \,.\, \text{grad}\,\frac{1}{r}\right) = \mathbf{0}.$$

Example 8.26. *In electromagnetic theory, we have* $\nabla \,.\, \mathbf{D} = \rho$, $\nabla \,.\, \mathbf{H} = 0$, $\nabla \times D = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}$, $\nabla \times \mathbf{H} = \dfrac{1}{c}\left(\rho\mathbf{V} + \dfrac{\partial \mathbf{D}}{\partial t}\right)$. *Prove that*

(*i*) $\nabla^2\mathbf{D} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{D}}{\partial t^2} = \nabla\rho + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}(\rho\mathbf{V})$ $\qquad$ (*ii*) $\nabla^2\mathbf{H} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2} = -\dfrac{1}{c}\,\nabla \times \rho\mathbf{V}$

Solution. (*i*) We have $\frac{1}{c^2}\left\{\frac{\partial^2 \mathbf{D}}{\partial t^2}+\frac{\partial}{\partial t}(\rho\mathbf{V})\right\}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}+\rho\mathbf{V}\right)$

$$=\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{H})=\frac{1}{c}\nabla\times\frac{\partial \mathbf{H}}{\partial t} \qquad \left[\because \ \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial \mathbf{D}}{\partial t}\right)\right]$$

$$=-\nabla\times(\nabla\times\mathbf{D}) \qquad \left[\because \ \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right]$$

$$=-[\nabla(\nabla\cdot\mathbf{D})-\nabla^2\mathbf{D}] \qquad \text{[Using § 8.8 (4)]}$$

$$=-\nabla\rho+\nabla^2\mathbf{D}$$

Hence $\nabla^2\mathbf{D}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho+\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V})$

(*ii*) $$\text{L.H.S.}=\nabla^2\text{H}-\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}=\nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right)$$

$$=\nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{D}) \qquad \left[\because \ \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right.$$

$$=\nabla^2\mathbf{H}+\frac{1}{c}\left(\nabla\times\frac{\partial \mathbf{D}}{\partial t}\right) \qquad \left[\because \ \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho V+\frac{\partial \mathbf{D}}{\partial t}\right)\right.$$

$$=\nabla^2\mathbf{H}+\nabla\times\left(\nabla\times\mathbf{H}-\frac{1}{C}\rho\mathbf{V}\right)=\nabla^2\mathbf{H}+\nabla\times(\nabla\times\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V})$$

$$=\nabla^2\mathbf{H}+\nabla(\nabla\cdot\mathbf{H})-\nabla^2\mathbf{H}-\frac{1}{c}\nabla\times(\rho\mathbf{V}), \qquad \text{[Using § 8.9 (4)]}$$

$$=\nabla(\nabla\cdot\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V}) \qquad [\because \ \nabla\cdot\mathbf{H}=0]$$

$$=-\frac{1}{c}\nabla\times\rho V=\text{R.H.S.}$$

PROBLEMS 8.4

1. Evaluate div **F** and curl **F** at the point (1, 2, 3) given (*i*) $\mathbf{F}=x^2yz\mathbf{I}+xy^2z\mathbf{J}+xyz^2\mathbf{K}$. (*B.P.T.U., 2005*)
   (*ii*) $\mathbf{F}=3x^2\mathbf{I}+5xy^2\mathbf{J}+5xyz^3\mathbf{K}$. (*S.V.T.U., 2009*)
   (*iii*) $\mathbf{F}=\text{grad}\,[x^3y+y^3z+z^3x-x^2y^2z^2]$ (*V.T.U., 2007*)
2. If $\mathbf{V}=(x\mathbf{I}+y\mathbf{J}+z\mathbf{K})/\sqrt{(x^2+y^2+z^2)}$, show that $\nabla.\mathbf{V}=2/\sqrt{(x^2+y^2+z^2)}$ and $\nabla\times\mathbf{V}=\mathbf{0}$. (*Osmania, 2002*)
3. If $\mathbf{F}=(x+y+1)\,\mathbf{I}+\mathbf{J}-(x+y)\mathbf{K}$, show that **F**. curl **F** = 0. (*V.T.U., 2000 S*)
4. Find the value of a if the vector $(ax^2y+yz)\,\mathbf{I}+(xy^2-xz^2)\,\mathbf{J}+(2xyz-2x^2y^2)\,\mathbf{K}$ has zero divergence. Find the curl of the above vector which has zero divergence.
5. Show that each of following vectors are solenoidal :
   (*i*) $(-x^2+yz)\,\mathbf{I}+(4y-z^2x)\,\mathbf{J}+(2xz-4z)\,\mathbf{K}$ (*Delhi, 2002*)
   (*ii*) $3y^4z^2\mathbf{I}+4x^3z^2\mathbf{J}+3x^2y^2\mathbf{K}$ (*iii*) $\nabla\phi\times\nabla\psi$.
6. If **A** and **B** are irrotational, prove that $\mathbf{A}\times\mathbf{B}$ is solenoidal. (*Madras, 2003 ; V.T.U., 2001*)
7. If $u=x^2+y^2+z^2$ and $\mathbf{V}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$, show that div $(u\mathbf{V})=5u$.
8. If $\mathbf{R}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$ and $r\neq 0$, show that (*i*) $\nabla/(1/r^2)=-2\mathbf{R}/r^4$; $\nabla\,.\,(\mathbf{R}/r^2)=1/r^2$
   (*ii*) div $(r^n\,\mathbf{R})=(n+3)r^n$; curl $(r^n\,\mathbf{R})=\mathbf{0}$ (*P.T.U., 2006 ; Kottayam, 2005*)
   (*iii*) *grad* $\left(\text{div}\,\frac{\mathbf{R}}{r}\right)=-\frac{2\mathbf{R}}{r^3}$. (*V.T.U., 2010 S*)
9. If $\mathbf{V}_1$ and $\mathbf{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that
   (*i*) div $(\mathbf{V}_1\times\mathbf{V}_2)=0$, (*ii*) grad $(\mathbf{V}_1\,.\,\mathbf{V}_2)=\mathbf{V}_1+\mathbf{V}_2$,
   (*iii*) curl $(\mathbf{V}_1\times\mathbf{V}_2)=2(\mathbf{V}_1-\mathbf{V}_2)$

10. Show that (*i*) $\nabla \cdot \left[\frac{f(r)}{r}\mathbf{R}\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$ (*Mumbai, 2008*)

(*ii*) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$ (*U.T.U., 2010 ; Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

(*iii*) $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

11. If **A** is a constant vector and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, prove that

(*i*) grad $(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$ (*Delhi, 2002*) (*ii*) div $(\mathbf{A} \times \mathbf{R}) = 0$ (*Burdwan, 2003*)

(*iii*) curl $(\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$ (*V.T.U., 2010 S*) (*iv*) curl $[(\mathbf{A} \cdot \mathbf{R})\mathbf{R}] = \mathbf{A} \times \mathbf{R}$. (*Kurukshetra, 2009 S*)

12. Prove that (*i*) $\nabla A^2 = 2(\mathbf{A} \cdot \nabla)\mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A})$, where **A** is a constant vector.

(*ii*) $\nabla \times (\mathbf{R} \times \mathbf{U}) = \mathbf{R}(\nabla \cdot \mathbf{U}) - 2\mathbf{U} - (\mathbf{R} \cdot \nabla)\mathbf{U}$.

13. Calculate (*i*) curl (grad f), given $f(x, y, z) = x^2 + y^2 - z$. (*B.P.T.U., 2006*)

(*ii*) curl (curl A) given $A = x^2y\mathbf{I} + y^2z\mathbf{J} + z^2y\mathbf{K}$ (*V.T.U., 2003*)

14. (*a*) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$. (*S.V.T.U., 2009 ; J.N.T.U. 2003*)

(*b*) Show that div (grad r^n) $= n(n+1)r^{n-2}$ where $r^2 = x^2 + y^2 + z^2$. (*Bhopal, 2008 ; U.P.T.U., 2003*)

15. For a solenoidal vector **F**, show that curl curl curl curl $\mathbf{F} = \nabla^4\mathbf{F}$.

16. If $u = x^2yz$, $v = xy - 3z^2$, find (*i*) $\nabla(\nabla u \cdot \nabla v)$; (*ii*) $\nabla \cdot (\nabla u \times \nabla v)$.

17. Find the directional derivative of $\nabla \cdot (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. (*Raipur, 2005*)

18. Prove that $\mathbf{A} \cdot \nabla\left(\mathbf{B} \cdot \nabla\frac{1}{r}\right) = \frac{3(\mathbf{A} \cdot \mathbf{R})(\mathbf{B} \cdot \mathbf{R})}{r^5} - \frac{\mathbf{A} \cdot \mathbf{B}}{r^3}$ where **A** and **B** are constant vectors.

8.10 INTEGRATION OF VECTORS

If two vector functions $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be such that

$$\frac{d\mathbf{G}(t)}{dt} = \mathbf{F}(t),$$

then $\mathbf{G}(t)$ is called an integral of $\mathbf{F}(t)$ with respect to the scalar variable t and we write

$$\int \mathbf{F}(t)\,dt = \mathbf{G}(t).$$

If **C** be an arbitrary constant vector, we have

$$\mathbf{F}(t) = \frac{d\mathbf{G}(t)}{dt} = \frac{d}{dt}[\mathbf{G}(t) + \mathbf{C}] \quad \text{then} \quad \int \mathbf{F}(t)\,dt = \mathbf{G}(t) + \mathbf{C}$$

This is called the *indefinite integral of* $\mathbf{F}(t)$ and its *definite integral is*

$$\int_a^b \mathbf{F}(t)\,dt = [\mathbf{G}(t) + \mathbf{C}]_a^b = \mathbf{G}(b) - \mathbf{G}(a).$$

Example 8.27. *Given* $\mathbf{R}(t) = 3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, *evaluate* $\int_0^1 (\mathbf{R} \times d^2\mathbf{R}/dt^2)\,dt$.

Solution.
$$\frac{d}{dt}\left(\mathbf{R} \times \frac{d\mathbf{R}}{dt}\right) = \frac{d\mathbf{R}}{dt} \times \frac{d\mathbf{R}}{dt} + \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2} = \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}$$

$$\therefore \quad \int \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right) dt = \mathbf{R} \times \frac{d\mathbf{R}}{dt}$$

$$= (3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (6t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 3t^2 & t & -t^3 \\ 6t & 1 & -3t^2 \end{vmatrix} = -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K}$$

Thus
$$\int_0^1 \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right) dt = \left| -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K} \right|_0^1$$

$$= -2\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$$

PROBLEMS 8.5

1. Given $\mathbf{F}(t) = (5t^2 - 3t)\mathbf{I} + 6t^3\mathbf{J} - 7t\,\mathbf{K}$, evaluate $\int_{t=2}^{t=4} \mathbf{F}(t)\,dt$.
2. If $\dfrac{d^2\mathbf{P}}{dt^2} = 6t\mathbf{I} - 12t^2\mathbf{J} + 4\cos t\mathbf{K}$, find $\mathbf{P}$, Given that $\dfrac{d\mathbf{P}}{dt} = -1 - 3\mathbf{K}$ and $\mathbf{P} = 2\mathbf{I} + \mathbf{J}$ when $t = 0$.
3. The acceleration of a particle at any time $t \geq 0$ is given by $12\cos 2t\mathbf{I} - 8\sin 2t\mathbf{J} + 16t\mathbf{K}$, the velocity and acceleration are initially zero. Find the velocity and displacement at any time.
4. If $\mathbf{R}(t) = \begin{cases} 2\mathbf{I} - \mathbf{J} + 2\mathbf{K} & \text{when } t = 1 \\ 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K} & \text{when } t = 2, \end{cases}$

   show that $\int_1^2 \left(\mathbf{R}\,.\,\dfrac{d\mathbf{R}}{dt}\right) dt = 10.$

8.11 (1) LINE INTEGRAL

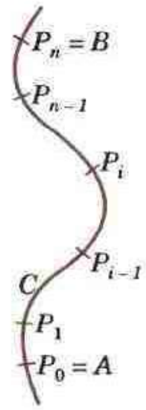

Fig. 8.8

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ which is defined at each point of curve C in space. Divide C into n parts at the points $A = P_0, P_1, ..., P_{i-1}, P_i ..., P_n = B$ (Fig. 8.8). Let their position vectors be $\mathbf{R}_0, \mathbf{R}_1, ..., \mathbf{R}_{i-1}, \mathbf{R}_i, ..., \mathbf{R}_n$. Let $\mathbf{U}_i$, be the position vector of any point on the arc $P_{i-1}P_i$.

Now consider the sum $S = \sum_{i=0}^{n} \mathbf{F}(\mathbf{U}_i)\,.\,\delta\mathbf{R}_i$, *where* $d\mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.

The limit of this sum as $n \to \infty$ *in such a way that* $|\,\delta\mathbf{R}_i\,| \to 0$, *provided it exists, is called the* **tangential line integral** of $\mathbf{F}(\mathbf{R})$ *along* C and is symbolically written as

$$\int_C \mathbf{F}(\mathbf{R})\,.\,d\mathbf{R} \quad \text{or} \quad \int_C \mathbf{F}\,.\,\frac{d\mathbf{R}}{dt}\,dt.$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\quad \mathbf{F}(\mathbf{R}) = \mathbf{I}f(x, y, z) + \mathbf{J}\phi(x, y, z) + \mathbf{K}\psi(x, y, z)$

and $\quad d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$

then $\quad \int_C \mathbf{F}(\mathbf{R})\,.\,d\mathbf{R} = \int_C (f\,dx + \phi dy + \psi dz).$

Two other types of line integrals are $\int_C \mathbf{F} \times d\mathbf{R}$ and $\int_C f d\mathbf{R}$ which are both vectors.

(2) **Circulation.** *If* $\mathbf{F}$ *represents the velocity of a fluid particle then the line integral* $\int_C \mathbf{F}\,.\,d\mathbf{R}$ *is called the* **circulation of F** *around the curve. When the circulation of* $\mathbf{F}$ *around every closed curve in a region E vanishes,* **F** *is said to be* **irrotational** *in E.*

(3) **Work.** If $\mathbf{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta\mathbf{R} = \mathbf{F}\,.\,\delta\mathbf{R}$.

$\therefore$ *the total work done by* $\mathbf{F}$ *during the displacement from A to B is given by the line integral* $\int_A^B \mathbf{F}.\,d\mathbf{R}$.

Example 8.28. *If* $\mathbf{F} = 3xy\mathbf{I} - y^2\mathbf{J}$, *evaluate* $\int \mathbf{F}\,.\,d\mathbf{R}$, *where C is the curve in the xy-plane* $y = 2x^2$ *from (0, 0) to (1, 2).* (V.T.U., 2010)

Solution. Since the particle moves in the xy-plane ($z = 0$), we take $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$. Then $\int_C \mathbf{F}\,.\,d\mathbf{R}$, where C is the parabola $y = 2x^2$

$$= \int_C (3xy\mathbf{I} - y^2\mathbf{J})\,.\,(dx\mathbf{I} + dy\mathbf{J}) = \int_C (3xydx - y^2dy) \qquad ...(i)$$

Substituting $y = 2x^2$, where x goes from 0 to 1, (i) becomes

$$= \int_{x=0}^{1} [3x(2x^2)\,dx - (2x^2)^2\,d(2x^2)] = \int_0^1 (6x^3 - 16x^5)\,dx = -7/6.$$

Otherwise, let $x = t$ in $y = 2x^2$. Then the parametric equation of C are $x = t$, $y = 2t^2$. The points (0, 0) and (1, 2) correspond to $t = 0$ and $t = 1$ respectively. Then (i) becomes

$$= \int_{t=0}^{1} [3t(2t^2)\,dt - (2t^2)^2 d(2t^2)] = \int_0^1 (6t^3 - 16t^5)\,dt = -7/6.$$

Example 8.29. *A vector field is given by* $\mathbf{F} = \sin y\mathbf{I} + x(1 + \cos y)\,\mathbf{J}$. *Evaluate the line integral over a circular path given by* $x^2 + y^2 = a^2$, $z = 0$. *(Rohtak, 2006 S ; P.T.U., 2003)*

Solution. As the particle moves in xy-plane ($z = 0$), let $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$ so that $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J}$. Also the circular path is $x = a\cos t$, $y = a \sin t$, $z = 0$ where t varies from 0 to 2π.

$$\therefore \quad \oint_c \mathbf{F}\,.\,d\mathbf{R} = \oint_c [\sin y\mathbf{I} + x(1 + \cos y)\mathbf{J}]\,.\,(dx\mathbf{I} + dy\mathbf{J})$$

$$= \oint_c [\sin y\,dx + x(1 + \cos y)\,dy] = \oint_c [(\sin y\,dx + x\cos y\,dy) + xdy]$$

$$= \oint [d(x \sin y) + x\,dy] = \int_0^{2\pi} [d\{a\cos t \sin (a \sin t)\} + a^2 \cos^2 t\,dt]$$

$$= \left| a\cos t \sin (a \sin t)\right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt = \frac{a^2}{2}\left| t + \frac{\sin 2t}{2}\right|_0^{2\pi} = \pi a^2.$$

Example 8.30. *Find the work done in moving a particle in the force field* $\mathbf{F} = 3x^2\mathbf{I} + (2xz - y)\,\mathbf{J} + z\mathbf{K}$, *along*
(a) the straight line from (0, 0, 0) to (2, 1, 3). *(S.V.T.U., 2007 ; J.N.T.U., 2002)*
(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. *(Delhi, 2002)*

Solution. $$\int_C \mathbf{F}\,.\,d\mathbf{R} = \int_C [3x^2\mathbf{I} + (2xz - y)\mathbf{J} + z\mathbf{K}]\,.\,(dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K})$$

$$= \int_C [3x^2dx + (2xz - y)dy + zdz] \qquad \ldots(i)$$

(a) The equations of the straight line from (0, 0, 0) to (2, 1, 3) are $x/2 = y/1 = z/3 = t$ (say)

$\therefore$ $x = 2t$, $y = t$, $z = 3t$ are its parametric equations. The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$, respectively

$$\therefore \qquad \text{work done} = \int_C \mathbf{F}\,.\,d\mathbf{R} = \int_0^1 [3(2t)^2\,2dt + \{(4t)(3t) - t\}dt + (3t)\,3dt]$$

$$= \int_0^1 (36t^2 + 8t)\,dt = 16.$$

(b) Let $x = t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of C are $x = t$, $y = t^2/4$, $z = 3t^3/8$ and t varies from 0 to 2.

$$\therefore \qquad \text{work done} = \int_C \mathbf{F}\,.\,d\mathbf{R} = \int_0^2 \left[3t^2\,dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\} d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}\,d\left(\frac{3t^2}{8}\right)\right]$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right) dt = \left| t^3 - \frac{t^4}{32} + \frac{17}{128}t^6 \right|_0^2 = 16.$$

PROBLEMS 8.6

1. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$. *(Delhi, 2002)*

2. If $\mathbf{F} = (5xy - 6x^2)\mathbf{I} + (2y - 4x)\mathbf{J}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8). *(J.N.T.U., 2006)*
3. Compute the line integral $\int_C (y^2 dx - x^2 dy)$ about the triangle whose vertices are (1, 0), (0, 1) and (– 1, 0).
4. If $A = (3x^2 + 6y)\mathbf{I} - 14yz\mathbf{J} + 20xz^2\mathbf{K}$, evaluate $\int \mathbf{A} \cdot d\mathbf{R}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t, y = t^2, z = t^3$. *(V.T.U., 2001)*
5. Evaluate $\int_C (xy + z^2)\, ds$ where C is the arc of the helix $x = \cos t, y = \sin t, z = t$ which joins the points (1, 0, 0) and $(-1, 0, \pi)$.
6. Find the total work done by the force $\mathbf{F} = 3xy\mathbf{I} - y\mathbf{J} + 2zx\mathbf{K}$ in moving a particle around the circle $x^2 + y^2 = 4$. *(V.T.U., 2010)*
7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\mathbf{I} - 5z\mathbf{J} + 10x\mathbf{K}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$. *(Bhopal, 2008)*
8. Using the line integral, compute the work done by the force $\mathbf{F} = (2y + 3)\mathbf{I} + xz\mathbf{J} + (yz - x)\mathbf{K}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$. *(Madras, 2000)*
9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = [2z, x, -y]$ and C is $\mathbf{R} = [\cos t, \sin t, 2t]$ from (1, 0, 0) to $(1, 0, 4\pi)$. *(B.P.T.U., 2006)*
10. If $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + x\mathbf{K}$, evaluate $\int_C \mathbf{F} \times d\mathbf{R}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \pi/2$.

8.12 (1) SURFACES

As seen in § 5.8, a surface S may be represented by $F(x, y, z) = 0$.

The *parametric representation of S* is of the form $\mathbf{R}(u, v) = x(u, v)\,\mathbf{I} + y(u, v)\,\mathbf{J} + z(u, v)\mathbf{K}$ and the continuous functions $u = \phi(t)$ and $v = \psi(t)$ of a real parameter t represent a curve C on this surface S.

For example, the parametric representation of the circular cylinder $x^2 + y^2 = a^2, -1 \le z \le 1$, (radius a and height 2), is

$$\mathbf{R}(u, v) = a\cos u\mathbf{I} + a\sin u\mathbf{J} + v\mathbf{K}$$

where the parameters u and v vary in the rectangle $0 \le u \le 2\pi$ and $-1 \le v \le 1$. Also $u = t, v = bt$ represent a *circular helix* (Fig. 8.3) on this circular cylinder. The equation of the circular helix is $\mathbf{R} = a\cos t\mathbf{I} + a\sin t\mathbf{J} + bt\mathbf{K}$.

Differentiating $\mathbf{R} = \mathbf{R}(u, v)$, w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{du}{dt} + \dfrac{\partial \mathbf{R}}{\partial v} \cdot \dfrac{dv}{dt}$

The vectors $\partial\mathbf{R}/\partial u$ and $\partial\mathbf{R}/\partial v$ are tangential to S at P and determine the tangent plane of S at P. $\mathbf{N} = \partial\mathbf{R}/\partial u \times \partial\mathbf{R}/\partial v\ (\ne 0)$ gives a normal vector $\mathbf{N}$ of S at P.

Def. *If S has a unique normal at each of its points whose direction depends continuously on the points of S, then the surface S is called a* **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions, then it is called a **piecewise smooth surface.**

For instance, the surface of a sphere is *smooth* while the surface of a cube is *piecewise smooth.*

Def. *A surface S is said to be* **orientable or two sided** *if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface.* If the positive direction of the normal is reversed as we move around a curve on S passing through P, then the surface is **non-orientable** (*i.e.*, **one-sided**).

An example of a non-orientable surface is the *Mobius strip**. If we take a long rectangular strip of paper and giving a half-twist join the shorter sides so that the two points A and the two points B in Fig. 8.9 coincide, then the surface generated is non-orientable. Such a surface is a model of a Mobius strip.

Fig. 8.9

(2) Surface integral. Consider a continuous function $\mathbf{F}(\mathbf{R})$ and a surface S. Divide S into a finite number of sub-surfaces. Let the surface element surrounding any point $P(\mathbf{R})$ be $\delta\mathbf{S}$ which can be regarded as a vector ; its magnitude being the area and its direction that of the outward normal to the element.

*Named after a German mathematician *August Ferdinand Möbius (1790–1868)* who was a student of Gauss and professor of astronomy at Leipzig. His important contributions are in projective geometry, theory of surfaces and mechanics.

Consider the sum $\Sigma\mathbf{F}(\mathbf{R})\,.\,\delta\mathbf{S}$, *where the summation extends over all the sub-surfaces. The limit of this sum as the number of sub-surfaces tends to infinity and the area of each sub-surface tends to zero, is called the* **normal surface integral** *of* $\mathbf{F}(\mathbf{R})$ *over S and is denoted by*

$$\int_S \mathbf{F}\,.\,d\mathbf{S} \quad \text{or} \quad \int_S \mathbf{F}\,.\,\mathbf{N}ds \text{ where } \mathbf{N} \text{ is a unit outward normal at } P \text{ to } S.$$

Other types of surface integrals are $\int_S \mathbf{F}\times d\mathbf{S}$ and $\int_S f\,d\mathbf{S}$ which are both vectors.

Notation : Only one integrals sign is used when there is one differential (say $d\mathbf{R}$ or $d\mathbf{S}$) and two (or three) signs when there are two (or three) differentials.

(3) **Flux across a surface.** *If* **F** *represent the velocity of a fluid particle then the total outward flux of* **F** *across a closed surface S is the surface integral* $\int_S \mathbf{F}\,.\,d\mathbf{S}$.

When the flux of **F** *across every closed surface S in a region E vanishes,* **F** *is said to be a* **solenoidal vector point function** *in E.*

It may be noted that **F** could equally well be taken as any other physical quantity *e.g.*, gravitational force, electric force and magnetic force.

Example 8.31. *Evaluate* $\int_S \mathbf{F}\,.\,\mathbf{N}ds$ *where* $\mathbf{F} = 2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}$ *and S is the closed surface of the region in the first octant bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0, x = 2, y = 0$ *and* $z = 0$.

Solution. The given closed surface S is piecewise smooth and is comprised of S_1 – the rectangular face $OAEB$ in xy-plane ; S_2–the rectangular face $OADC$ in xz-plane ; S_3–the circular quadrant ABC in yz-plane, S_4–the circular quadrant AED and S_5–the curved surface $BCDE$ of the cylinder in the first octant (Fig. 8.10).

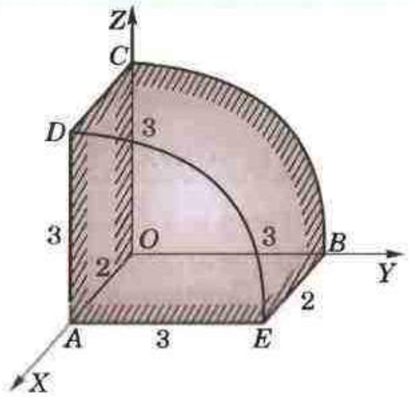

Fig. 8.10

$$\therefore \quad \int_S \mathbf{F}\,.\,\mathbf{N}\,ds = \int_{S_1} \mathbf{F}\,.\,\mathbf{N}\,ds + \int_{S_2} \mathbf{F}\,.\,\mathbf{N}\,ds + \int_{S_3} \mathbf{F}\,.\,\mathbf{N}\,ds + \int_{S_4} \mathbf{F}\,.\,\mathbf{N}\,ds + \int_{S_5} \mathbf{F}\,.\,\mathbf{N}\,ds \quad ...(i)$$

Now $\int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} (2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K})\,.\,(-\mathbf{K})\,ds$

$$= -4\int_{S_1} xz^2\,ds = 0 \qquad [\because \; z = 0 \text{ in the } xy\text{–plane}]$$

Similarly, $\int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds = 0$ and $\int_{S_3} \mathbf{F}\,.\,\mathbf{N}ds = 0$

$$\int_{S_4} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_4} (2x^2y\mathbf{I} - y^2\,\mathbf{J} + 4xz^2\,\mathbf{K})\,.\,\mathbf{I}\,ds$$

$$= \int_{S_4} 2x^2y\,ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\,dydz = 4\int_0^3 (9 - z^2)\,dz = 72$$

To find **N** in S_5, we note that $\nabla(y^2 + z^2) = 2y\mathbf{J} + 2z\mathbf{K}$

$$\therefore \qquad \mathbf{N} = \frac{2y\mathbf{J} + 2z\mathbf{K}}{\sqrt{(4y^2 + 4z^2)}} = \frac{y\mathbf{J} + z\mathbf{K}}{3} \qquad [\because \; y^2 + z^2 = 9]$$

and $\quad |\,\mathbf{N}\,.\,\mathbf{K}\,| = z/3 \quad$ so that $ds = dxdy/(z/3)$

Thus $$\int_{S_5} \mathbf{F}\,.\,\mathbf{N}\,ds = \int_0^2 \int_0^3 \frac{(-y^3 + 4xz^3)}{3}\,.\,dydx/(z/3) = \int_0^2 \int_0^3 \left(\frac{-y^3}{z} + 4xz^2\right)dydx$$

$$\begin{bmatrix} \text{Put } y = 3\sin\theta, z = 3\cos\theta \\ \therefore \; dy = 3\cos\theta\,d\theta \end{bmatrix}$$

$$= \int_0^2 \int_0^{\pi/2} \left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x(9\cos^2\theta\right]3\cos\theta\,d\theta dx = \int_0^2 \left[-27\times\frac{2}{3} + 108x\times\frac{2}{3}\right]dx = 108$$

Hence (*i*) gives $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds = 0 + 0 + 0 + 72 + 108 = 180.$

PROBLEMS 8.7

1. If velocity vector is $\mathbf{F} = y\mathbf{I} + 2\mathbf{J} + xz\mathbf{K}$ m/sec., show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m³/sec.
2. Evaluate $\int_S \mathbf{F}\,.\,d\mathbf{S}$ where $\mathbf{F} = x\mathbf{I} + (z^2 - zx)\,\mathbf{J} - xy\mathbf{K}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).
3. Evaluate $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds$ where $\mathbf{F} = 6z\mathbf{I} - 4\mathbf{J} + y\mathbf{K}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.
4. If $\mathbf{F} = 2y\mathbf{I} - 3\mathbf{J} + x^2\,\mathbf{K}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds = 132$.

8.13 GREEN'S THEOREM IN THE PLANE*

If $\phi(x, y)$, $\psi(x, y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy \qquad \text{...(1)}$$

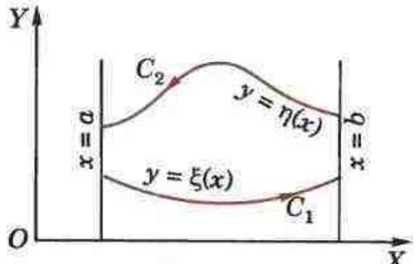

Fig. 8.11

Consider the region E bounded by a single closed curve C which is cut by any line parallel to the axes at the most in two points.

Let E be bounded by $x = a$, $y = \xi\,(x)$, $x = b$ and $y = \eta(x)$, where $\eta \ge \xi$, so that C is divided into curves C_1 and C_2 (Fig. 8.11).

$$\therefore \qquad \iint_E \frac{\partial \phi}{\partial y}\,dxdy = \int_a^b dx\left[\int_\xi^\eta \frac{\partial \phi}{\partial y}\,dy\right] = \int_a^b dx \mid \phi \mid_\xi^\eta$$

$$= \int_a^b [\phi(x, \eta) - \phi(x, \xi)]\,dx = -\int_{C_2} \phi(x, y)\,dx - \int_{C_1} \phi(x, y)\,dx$$

$$= -\int_C \phi(x, y)\,dx \qquad \text{...(2)}$$

Similarly, it can be shown that

$$\iint_E \frac{\partial \psi}{\partial x}\,dxdy = \int_C \psi(x, y)\,dy \qquad \text{...(3)}$$

On subtracting (2) from (3), we get (1).

This result can be extended to regions which may be divided into a finite number of sub-regions such that the boundary of each is cut at the most in two points by any line parallel to either axis. Applying (1) to each of these sub-regions and adding the results, the surface integrals combine into an integral over the whole region ; the line integrals over the common boundaries cancel (for each is covered twice but in opposite directions), whereas the remaining line integrals combine into the line integral over the external curve C.

Obs. This theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem. (See Cor. p. 342)

Example 8.32. *Verify Green's theorem for* $\int_C [(xy + y^2)\,dx + x^2 dy]$, *where C is bounded by* $y = x$ *and* $y = x^2$.

(V.T.U., 2011 ; S.V.T.U., 2009 ; Rohtak, 2003)

Solution. Here $\qquad \phi = xy + y^2$ and $\psi = x^2$

$$\therefore \qquad \int_C (\phi\,dx + \psi\,dy) = \int_{C_1} + \int_{C_2}$$

*Named after the English mathematician *George Green* (1793–1841) who taught at Cambridge and is known for his work on potential theory in connection with waves, vibrations, elasticity, electricity and magnetism.

Along C_1, $y = x^2$ and x varies from 0 to 1 (Fig. 8.12)

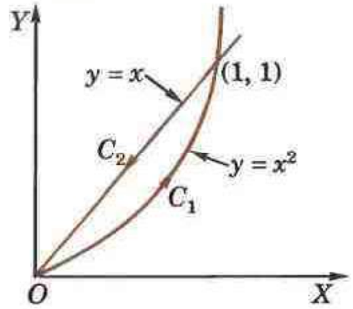

Fig. 8.12

$$\therefore \quad \int_{C_1} = \int_0^1 [\{x(x)^2 + (x^2)^2\}] \, dx + x^2 d(x^2)]$$

$$= \int_0^1 (3x^3 + x^4) \, dx = \frac{19}{20}$$

Along C_2, $y = x$ and x varies from 1 to 0.

$$\therefore \quad \int_{C_2} = \int_1^0 [\{x(x) + (x)^2\} \, dx + x^2 \, d(x)] = \int_1^0 3x^2 \, dx = -1.$$

Thus $\displaystyle\int_C (\phi \, dx + \psi \, dy) = \frac{19}{20} - 1 = -\frac{1}{20}$ ...(*i*)

Also $\displaystyle\iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy = \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dxdy$

$$= \int_0^1 \int_{x^2}^x (2x - x - 2y) \, dydx = \int_0^1 \left[xy - y^2\right]_{x^2}^x dx = \int_0^1 (x^4 - x^3) \, dx = -\frac{1}{20} \qquad ...(ii)$$

Hence, Green theorem is verified from the equality of (*i*) and (*ii*).

Example 8.33. *If C is a simple closed curve in the xy-plane not enclosing the origin, show that*

$$\int_C \mathbf{F} \, . \, d\mathbf{R} = 0, \quad \text{where} \quad \mathbf{F} = \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}$$ *(P.T.U., 2005)*

Solution. $\displaystyle\int_C \mathbf{F} \, . \, d\mathbf{R} = \int_C \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}(dx\mathbf{I} + dy\mathbf{J})$ $[\because \ \mathbf{R} = x\mathbf{I} + y\mathbf{J}]$

$$= \int_C \frac{ydx - xdy}{x^2 + y^2} = \int_C (\phi dx + \psi dy) \text{ where } \phi = \frac{y}{x^2 + y^2}, \psi = \frac{-x}{x^2 y^2}$$

$$= \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$ [By Green's theorem]

$$= \iint_S \left[\frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] dxdy$$

$$= \iint \left[\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dxdy = 0.$$

Example 8.34. *Using Green's theorem, evaluate* $\displaystyle\int_C [(y - \sin x) \, dx + \cos x \, dy]$ *where C is the plane triangle enclosed by the lines* $y = 0$, $x = \pi/2$ *and* $y = \dfrac{2}{\pi}x$. *(J.N.T.U., 2005 ; Anna, 2003)*

Solution. Here $\phi = y - \sin x$ and $\psi = \cos x$.

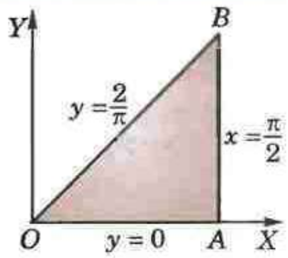

Fig. 8.13

By Green's theorem $\displaystyle\int_C [(y - \sin x) \, dx + \cos x \, dy]$

$$= \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx \, dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=2x/\pi} (-\sin x - 1) \, dy \, dx = -\int_0^{\pi/2} (\sin x + 1) \left| \, y \, \right|_0^{2x/\pi} dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x(\sin x + 1) \, dx = -\frac{2}{\pi}\left\{\left| \, x(-\cos x + x) \, \right|_0^{\pi/2} - \int_0^{\pi/2} 1 \, . \, (-\cos x + x) \, dx\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left| -\sin x + \frac{x^2}{2} \right|_0^{\pi/2}\right\} = -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right) = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Example 8.35. *Apply Green's theorem to evaluate* $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle* $x^2 + y^2 = a^2$. (*U.P.T.U., 2005*)

Solution. By Green's theorem

$$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$$

$$= \iint_A \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dxdy$$

$$= 2\iint_A (x + y)\, dxdy, \text{ where } A \text{ is the region of Fig. 8.14}$$

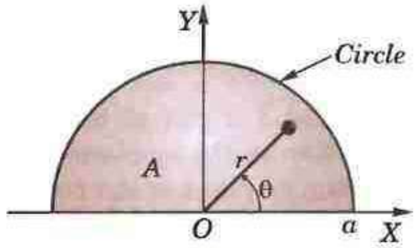

Fig. 8.14

$$= 2\int_0^a \int_0^\pi r\,(\cos\theta + \sin\theta)\cdot r d\theta\, dr$$

[Changing to polar coordinates (r, θ), r varies from 0 to a and θ varies from 0 to π]

$$= 2\int_0^a r^2\, dr \int_0^\pi (\cos\theta + \sin\theta)\, d\theta = 2\cdot\frac{a^3}{3}\cdot(1+1) = \frac{4a^3}{3}.$$

PROBLEMS 8.8

1. Verify Green's theorem for $\int_C [(3x - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region bounded by $x = 0, y = 0$ and $x + y = 1$. (*Nagpur, 2008 ; Kerala, 2005 ; Anna, 2003 S*)
2. Verify Green's theorem for $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$ where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$. (*Nagpur, 2009 ; P.T.U., 2006*)
3. Verify Green's theorem for $\int (x^2 ydx + x^2\, dy)$ where C is the boundary described counter clockwise of triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$. (*U.T.U., 2010*)
4. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int_C (xdy - ydx)$.

   Hence find the area of an ellipse whose semi-major and semi-minor axes are of lengths a and b. (*Kerala, 2005 ; V.T.U., 2000 S*)
5. Find the area of a circle of radius a using Green's theorem. (*Madras, 2003*)
6. Evaluate $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $x = \pm 1, y = \pm 1$. (*S.V.T.U., 2008 ; Marathwada, 2008*)
7. Evaluate $\int_C [(x^2 - 2xy)dx + (x^2y + 3)dy]$, around the boundary of the region defined by $y^2 = 8x$ and $x = 2$.
8. Evaluate by Green's theorem $\int_C \mathbf{F}\cdot d\mathbf{R}$ where $\mathbf{F} = -xy(x\mathbf{I} - y\mathbf{J})$ and C is $r = a(1 + \cos\theta)$. (*Mumbai, 2006*)

8.14 STOKE'S THEOREM* (*Relation between line and surface integrals*)

If S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$ *be any continuously differentiable vector point function, then*

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_S curl\, \mathbf{F}\cdot \mathbf{N} ds$$

where $\mathbf{N} = \cos\alpha\mathbf{I} + \cos\beta\mathbf{J} + \cos\gamma\mathbf{K}$ *is a unit external normal at any point of S.*

* Named after an Irish mathematician *Sir George Gabriel Stokes* (1819–1903) who became professor in Cambridge. His important contributions are to infinite series, geodesy and theory of viscous fluids.

Writing $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K}$, it may be reduced to the form

$$\int_C (f_1 dx + f_2 dy + f_3 dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] ds \quad ...(1)$$

Let us first prove that

$$\oint_C f_1 dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds \quad ...(2)$$

Let $z = g(x, y)$ be the equation of the surface S whose projection on the xy-plane is the region E. Then the projection of C on the xy-plane is the curve C' enclosing region E.

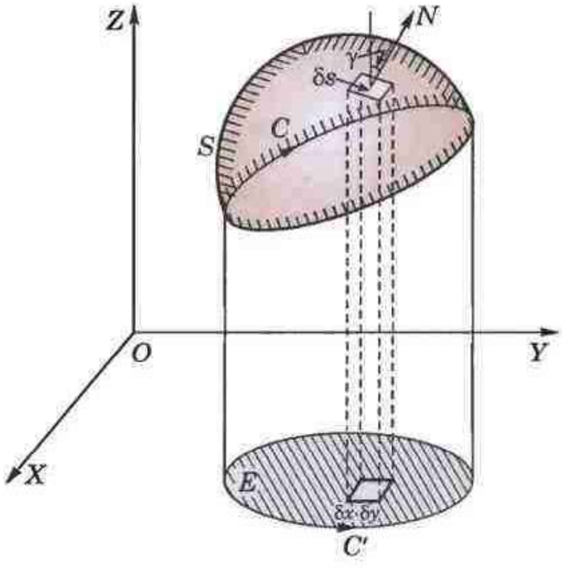

Fig. 8.15

$$\therefore \quad \int_C f_1(x, y, z) dx = \int_C f_1\{x, y, g(x, y)\} dx$$

$$= -\iint_E \frac{\partial}{\partial y} f_1(x, y, g)\, dxdy, \text{ by Green's theorem}$$

$$= -\iint_E \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right) dxdy \quad ...(3)$$

The direction cosines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{-\partial g/\partial x} = \frac{\cos\beta}{-\partial g/\partial y} = \frac{\cos\gamma}{1} \quad \text{(See p. 219)} \quad ...(4)$$

Moreover

$dxdy$ = projection of ds on the xy-plane
$= ds \cos\gamma$, *i.e.*, $ds = dxdy/\cos\gamma$.

$\therefore$ right side of (2)

$$= \iint_E \left(\frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial f_1}{\partial y}\right) dxdy = -\iint \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\cdot\frac{\partial g}{\partial y}\right) dxdy \qquad \left[\frac{\cos\beta}{\cos\gamma} = -\frac{\partial g}{\partial y} \text{ by (4)}\right]$$

= Left side of (2), by (3).

Thus we have proved (2). Similarly, we can prove the other corresponding relations for f_2 and f_3. Adding these three results, we get (1).

Cor. Green's theorem in a plane as a special case of Stokes theorem. Let $\mathbf{F} = \phi\mathbf{I} + \psi\mathbf{J}$ be a vector function which is continuously differentiable in a region S of the xy-plane bounded by a closed curve C. Then

$$\int_C \mathbf{F}\,.\,dR = \int_C (\phi\mathbf{I} + \psi\mathbf{J})\cdot(dx\mathbf{I} + dy\mathbf{J}) = \int_C (\phi\, dx + \psi\, dy)$$

and

$$\text{curl}\,\mathbf{F}\cdot\mathbf{N} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \partial/\partial x & \partial/\partial y & 0 \\ \phi & \psi & 0 \end{vmatrix} \cdot \mathbf{K} = \frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}$$

Hence *Stoke's theorem* takes the form $\int_C (\phi\, dx + \psi\, dy) = \int_C \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy$ which is *Green's theorem* in a plane.

Example 8.36. *Verify Stoke's theorem for* $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}$ *taken around the rectangle bounded by the lines* $x = \pm a$, $y = 0$, $y = b$. *(Bhopal, 2008 S ; V.T.U., 2007 ; J.N.T.U., 2003 ; U.P.T.U., 2003)*

Solution. Let $ABCD$ be the given rectangle as shown in Fig. 8.16.

$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = \int_{AB} \mathbf{F}\cdot d\mathbf{R} + \int_{BC} \mathbf{F}\cdot d\mathbf{R} + \int_{CD} \mathbf{F}\cdot d\mathbf{R} + \int_{DA} \mathbf{F}\cdot d\mathbf{R}$$

and

$$\mathbf{F}\cdot d\mathbf{R} = [(x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}]\cdot(\mathbf{I}dx + \mathbf{J}dy) = (x^2 + y^2)dx - 2xydy$$

Along AB, $x = a$ (*i.e.*, $dx = 0$) and y varies from 0 to b.

$$\therefore \quad \int_{AB} \mathbf{F}\cdot d\mathbf{R} = -2a\int_0^b y\, dy = -2a\cdot\frac{b^2}{2} = -ab^2.$$

Similarly,
$$\int_{BC} \mathbf{F}\cdot d\mathbf{R} = \int_{a}^{-a}(x^2+b^2)dx = -\frac{2a^3}{3} - 2ab^2 \cdot$$
$$\int_{CD} \mathbf{F}\cdot d\mathbf{R} = 2a\int_{b}^{0} y\,dy = -ab^2$$
and
$$\int_{DA} \mathbf{F}\cdot d\mathbf{R} = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3}\cdot$$

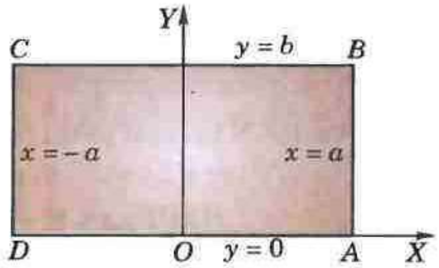

Fig. 8.16

Thus
$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = -4ab^2 \qquad ...(i)$$
Also since $\quad \text{curl } \mathbf{F} = -4\mathbf{K}y$

$$\therefore \quad \int_S \text{curl } \mathbf{F}\cdot\mathbf{N}ds = \int_0^b\int_{-a}^{a} -4\mathbf{K}y\cdot\mathbf{K}\,dxdy = -4\int_0^b\int_{-a}^{a} y\,dxdy$$
$$= -4\int_0^b |x|_{-a}^{a}\, y\,dy = -8a\left|\frac{y^2}{2}\right|_0^b = -4ab^2 \qquad ...(ii)$$

Hence Stoke's theorem is verified from the equality of (*i*) and (*ii*).

Example 8.37. *Verify Stoke's theorem for the vector field $F = (2x-y)\mathbf{I} - yz^2\mathbf{J} - y^2z\,\mathbf{K}$ over the upper half surface of $x^2+y^2+z^2=1$, bounded by its projection on the xy-plane.*

(Bhopal, 2008 ; Madras, 2006 ; S.V.T.U., 2006)

Solution. The projection of the upper half of given sphere on the xy-plane ($z = 0$) is the circle $c[x^2+y^2=1]$ (Fig. 8.17).

$$\oint_c \mathbf{F}.\,d\mathbf{R} = \oint_c [(2x-y)dx - yz^2\,dy - y^2\,zdz] = \oint_c (2x-y)dx \qquad [z = 0 \text{ in the } xy\text{-plane}]$$
$$= \int_{\theta=0}^{2\pi}(2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) \qquad [\text{Putting } x=\cos\theta,\ y=\sin\theta]$$
$$= \int_0^{2\pi}(-\sin 2\theta + \sin^2\theta)\,d\theta = \times 0 + 4\int_0^{\pi/2}\sin^2\theta\,d\theta = \pi. \qquad ...(i)$$

Now
$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$
$$= (-2yz+2yz)\,\mathbf{I} + 0\,\mathbf{J} + \mathbf{K} = \mathbf{K}$$

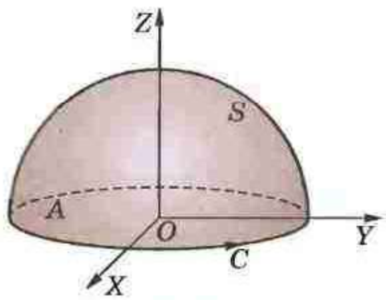

Fig. 8.17

$$\therefore \quad \int \text{curl } \mathbf{F}.\,Nds = \int_S K.\mathbf{N}\,ds = \int_A \mathbf{K}.\mathbf{N}\frac{dxdy}{|\mathbf{N}.\mathbf{K}|}$$
where A is the projection of S on xy-plane and $ds = dxdy/\mathbf{N}.\mathbf{K}$
$$= \int_A dx\,dy = \text{area of circle } C = \pi \qquad ...(ii)$$

Hence, the Stokes theorem is verified from the equality of (*i*) and (*ii*).

Example 8.38. *Uses Stoke's theorem evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

(Nagpur, 2009 ; Kurukshetra, 2009 S ; Kerala, 2005)

Solution. Here $\quad \mathbf{F} = (x+y)\,\mathbf{I} + (2x-z)\,\mathbf{J} + (y+z)\,\mathbf{K}$

$$\therefore \quad \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\mathbf{I} + \mathbf{K}$$

Also equation of the plane through A, B, C (Fig. 8.18) is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1 \text{ or } 3x+2y+z=6$$

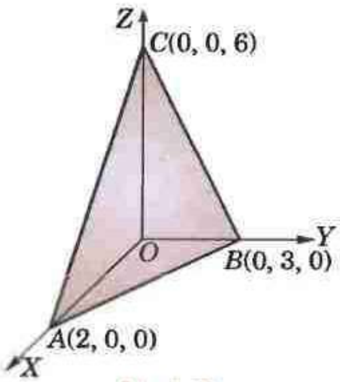

Fig. 8.18

Vector **N** normal to this plane is

$$\nabla(3x+2y+z-6) = 3\mathbf{I}+2\mathbf{J}+\mathbf{K}$$

$$\therefore \quad \hat{\mathbf{N}} = \frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{(9+4+1)}} = \frac{1}{\sqrt{14}}(3\mathbf{I}+2\mathbf{J}+\mathbf{K})$$

Hence $\int_C [(x+y)dx+(2x-z)dy+(y+z)dz] = \int_C \mathbf{F}\,.\,d\mathbf{R}$

$$= \int_S \text{curl}\,\mathbf{F}\,.\,\hat{\mathbf{N}}\,ds \qquad \text{where } S \text{ is the triangle } ABC$$

$$= \int_S (2\mathbf{I}+\mathbf{K})\,.\left(\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{14}}\right)ds = \frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$= \frac{7}{\sqrt{14}}\,(\text{Area of } \Delta ABC) = \frac{7}{\sqrt{14}}\,.\,3\sqrt{14} = 21.$$

Example 8.39. *If* $\mathbf{F} = 3y\mathbf{I} - xz\mathbf{J} + yz^2\mathbf{K}$ *and S is the surface of the paraboloid* $2z = x^2+y^2$ *bounded by* $z = 2$, *evaluate* $\iint_S (\nabla\times\mathbf{F})\,.\,d\mathbf{S}$ *using Stoke's theorem.*

Solution. By Stokes theorem, $I = \iint_S (\nabla\times\mathbf{F})\,.\,d\mathbf{S} = \oint_C \mathbf{F}\,.\,d\mathbf{R}$

where S is the surface $2z = x^2+y^2$ bounded by $z = 2$.

$$\therefore \quad I = \oint_C \mathbf{F}\,.\,d\mathbf{R} = \oint_C (3yI - xzJ + yz^2K)\,.\,(dxI+dyJ+dzK)$$

$$= \oint_C (3y\,dx - xz\,dy + yz^2 dz)$$

$\because \; S \equiv x^2+y^2 = 4,\; z = 2$

$\therefore$ Put $x = 2\cos\theta,\; y = 2\sin\theta$

$C \equiv x^2+y^2 = 4,\; \theta = 0$ to 2π.

$$= \int_0^{2\pi} [6\sin\theta(-2\cos\theta\,d\theta) - 4\cos\theta\,(2\cos\theta\,d\theta) + 8\sin\theta\,(0)]$$

$$= -4\int_0^{2\pi} (12\sin^2\theta + 8\cos^2\theta)\,d\theta$$

$$= -4\left(12\,.\,\frac{1}{2}\frac{\pi}{2} + 8\,.\,\frac{1}{2}\,.\,\frac{\pi}{2}\right) = -20\pi.$$

Example 8.40. *Apply Stoke's theorem to evaluate* $\int_C (ydx + zdy + xdz)$ *where C is the curve of intersection of* $x^2+y^2+z^2 = a^2$ *and* $x + z = a$. *(Bhopal, 2008)*

Solution. The curve C is evidently a circle lying in the plane $x + z = a$, and having $A(a, 0, 0)$, $B(0, 0, a)$ as the extremities of the diameter (Fig. 8.19).

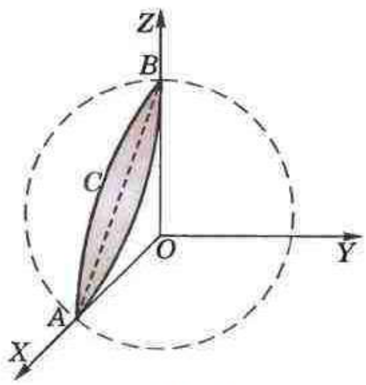

Fig. 8.19

$$\therefore \quad \int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\,.\,d\mathbf{R}$$

$$= \int_S \text{curl}\,(y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\,.\,\mathbf{N}ds$$

where S is the circle on AB as diameter and $\mathbf{N} = \frac{1}{\sqrt{2}}\mathbf{I} + \frac{1}{\sqrt{2}}\mathbf{K}$

$$= \int_S -(\mathbf{I}+\mathbf{J}+\mathbf{K})\,.\left(\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}\right)ds$$

$$= -\frac{2}{\sqrt{2}}\int_S ds = -\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}.$$

Example 8.41. *If S be any closed surface, prove that* $\int_S$ *curl* $\mathbf{F} . d\mathbf{S} = 0$.

Solution. Cut open the surface S by any plane and let S_1, S_2 denote its upper and lower portions. Let C be the common curve bounding both these portions.

$$\therefore \qquad \int_S \text{curl } \mathbf{F} \,.\, d\mathbf{S} = \int_{S_1} \text{curl } \mathbf{F} \,.\, d\mathbf{S} + \int_{S_2} \text{curl } \mathbf{F} \,.\, d\mathbf{S} = \int_C \mathbf{F} \,.\, d\mathbf{R} - \int_C \mathbf{F} \,.\, d\mathbf{R} = 0,$$

on applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of the first.

PROBLEMS 8.9

1. Verify Stoke's theorem for the vector field (*i*) $\mathbf{F} = (x^2 - y^2)\mathbf{I} + 2xy\mathbf{J}$ over the box bounded by the planes $x = 0, x = a$; $y = 0, y = b; z = 0, z = c$; if the face $z = 0$ is cut. (*B.P.T.U., 2006 ; Delhi, 2002*)
(*ii*) $\mathbf{F} = (z^2, 5x, 0)$ and $S : 0 \le x \le 1, 0 \le y \le 1, z = 1$.
2. Verify Stoke's theorem for a vector field defined by $\mathbf{F} = -y^3\mathbf{I} + x^3\mathbf{J}$, in the region $x^2 + y^2 \le 1, z = 0$.
3. Evaluate $\int_c \mathbf{F} \,.\, d\mathbf{R}$ where $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\,\mathbf{J}$ and c is the rectangle in the xy-plane bounded by $y = 0, x = a, y = b$, $x = 0$. (*Mumbai, 2007*)
4. Verity Stoke's theorem for $\mathbf{F} = (y - z + 2)\mathbf{I} + (yz + 4)\mathbf{J} - xz\mathbf{K}$ where S is the surface of the cube $x = 0, y = 0, z = 0$, $x = 2, y = 2, z = 2$ above the xy-plane. (*Andhra, 2000*)
5. Evaluate $\int_C \mathbf{F} \,.\, d\mathbf{R}$ where $\mathbf{F} = y\mathbf{I} + xz^3\mathbf{J} - zy^3\mathbf{K}$, C is the circle $x^2 + y^2 = 4, z = 1.5$.
6. Evaluate by Stoke's theorem $\oint_C (yz\,dz + zx\,dy + zx\,dz)$ where C is the curve $x^2 + y^2 = 1, z = y^2$. (*J.N.T.U., 2005*)
7. If S be the surface of the sphere $x^2 + y^2 + z^2 = 1$, prove that $\int_S \text{curl } \mathbf{F} \,.\, d\mathbf{S} = 0$. (*J.N.T.U., 1999*)
8. Prove that $\int_C A \times \mathbf{R} \,.\, d\mathbf{R} = 2\mathbf{A} \,.\, \int d\mathbf{S}$, $\mathbf{A}$ being any constant vector, and deduce that $\oint_C \mathbf{R} \times d\mathbf{R}$ is twice the vector area of the surface enclosed by C.
9. If ϕ is a scalar point function, use Stoke's theorem to prove that (*i*) curl (grad ϕ) = 0. (*ii*) div curl $\mathbf{F} = 0$. (*Kerala, 2005*)
10. Evaluate $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi$. $0 \le y \le 1, z = 3$. (*Rohtak, 2005*)
11. Use Stoke's theorem to evaluate $(\nabla \times \mathbf{F}) \,.\, \mathbf{N}\,ds$, where $\mathbf{F} = y\mathbf{I} + (x - 2xz)\,\mathbf{J} - xy\mathbf{K}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. (*Kottayam, 2005*)
12. Evaluate $\int \nabla \times \mathbf{V} \,.\, d\mathbf{S}$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \ge 0$ where $\mathbf{V} = y\mathbf{I} + z\mathbf{J} + x\mathbf{K}$.

8.15 VOLUME INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ and surface S enclosing the region E. Divide E into finite number of sub-regions $E_1, E_2, \ldots, E_n$. Let δv_i be the volume of the sub-region E_i enclosing any point whose position vector is $\mathbf{R}_i$.

Consider the sum $$\mathbf{V} = \sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\,\delta v_i$$

The limit of this sum as $n \to \infty$ in such a way that $\delta v_i \to 0$, is called the volume integral of $\mathbf{F}(\mathbf{R})$ *over E and is symbolically written as* $\int_E \mathbf{F}\,dv$.

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$ so that $\delta v = \delta x \delta y \delta z$, then

$$\int_E \mathbf{F}\,dv = \mathbf{I} \iiint_E f dxdydz + \mathbf{J} \iiint_E \phi dxdydz + \mathbf{K} \iiint_E \psi dxdydz.$$

8.16 GAUSS DIVERGENCE THEOREM* (*Relation between surface and volume integrals*)

If **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_E div\, \mathbf{F}\, dv$$

where **N** *is the unit external normal vector.*

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$

then it is required to prove that

$$\iint_S (f dydz + \phi dzdx + \psi dxdy)$$

$$= \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz \qquad ...(1)$$

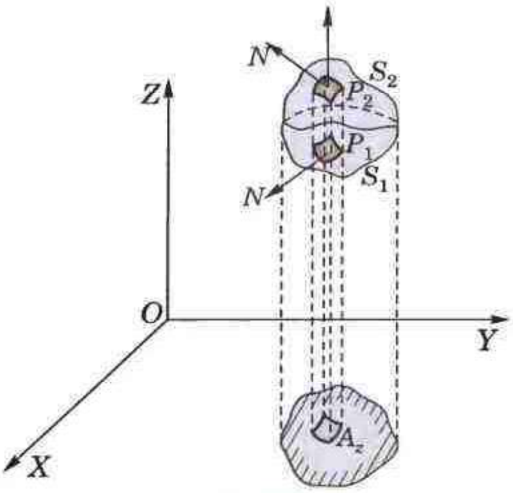

Fig. 8.20

Firstly consider such a surface S that a line parallel to z-axis cuts it in two points; say $P_1(x, y, z_1)$ *and* $P_2(x, y, z_2)$ $(z_1 \le z_2)$ (Fig. 8.20).

If S projects into the area A_z on the xy-plane, then

$$\iiint_E \frac{\partial \psi}{\partial z} dxdydz = \iint_{A_z} dxdy \int_{z_1}^{z_2} \frac{\partial \psi}{\partial z} dz$$

$$= \iint_{A_z} [\Psi(x, y, z_2) - \Psi(x, y, z_1)]\, dxdy = \iint_{A_z} \Psi(x, y, z_2)\, dxdy - \iint_{A_z} \Psi(x, y, z_1)\, dxdy \qquad ...(2)$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to the points P_1 and P_2 respectively and **N** be the unit external normal vector at any point of S. As the external normal at any point of S_2 makes an acute angle with the positive direction of z-axis and that at any point of S_1 an obtuse angle, therefore

$$\iint_{A_z} \Psi(x, y, z_2)\, dxdy = \int_{S_2} \psi \mathbf{N} \cdot \mathbf{K} ds \qquad ...(3)$$

$$\iint_{A_z} \Psi(x, y, z_1)\, dxdy = -\int_{S_1} \psi \mathbf{N} \cdot \mathbf{K} ds \qquad ...(4)$$

Using (3) and (4), (2) now becomes

$$\iiint_E \frac{\partial \psi}{\partial z} dxdydz = \int_{S_2} \psi \mathbf{N} \cdot \mathbf{K} ds + \int_{S_1} \psi \mathbf{N} \cdot \mathbf{K} ds = \int_S \psi \mathbf{N} \cdot \mathbf{K} ds \qquad ...(5)$$

Similarly, we have

$$\iiint_E \frac{\partial f}{\partial x} dxdydz = \int_S f \mathbf{N} \cdot \mathbf{I} ds \qquad ...(6)$$

$$\iiint_E \frac{\partial \phi}{\partial y} dxdydz = \int_S \phi \mathbf{N} \cdot \mathbf{J} ds \qquad ...(7)$$

Addition of (5), (6) and (7) gives

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz = \int_S (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) \cdot \mathbf{N} ds \text{ which is same as (1).}$$

Secondly, consider a general region E. Assume that it can be split up into a finite number of sub-regions each of which is met by a line parallel to any axis in only two points. Applying (1) to each of these sub-regions and adding the results, the volume integrals will combine to give the volume integral over the whole region E. Also the surface integrals over the common boundaries of two sub-regions cancel because each occurs twice and having corresponding normals in opposite directions whereas the remaining surface integrals combine to give the surface integral over the entire surface S.

Finally consider a region E bounded by two closed surfaces S_1, S_2 (S_1 *being within* S_2). Noting that outward normal at points of S_1 is directed inwards (*i.e.*, away from S_2) and introducing an additional surface cutting S_1, S_2 so that all parts of E are bounded by a single closed surface, the truth of the theorem follows as before. Thus theorem also holds for regions enclosed by several surfaces.

Hence the theorem is completely established.

*See footnote p. 37.

Example 8.42. *Verify Divergence theorem for* $\mathbf{F} = (x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$ *taken over the rectangular parallelepiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. (*Rohtak, 2006 S ; Madras, 2000 S*)

Solution. As
$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$
$$= 2(x + y + z)$$

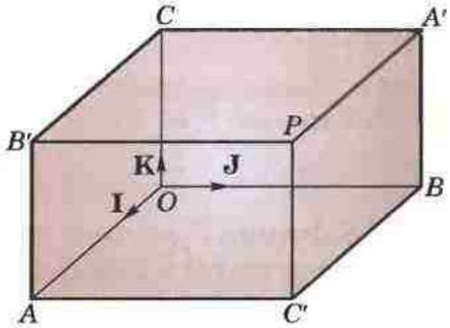

Fig. 8.21

$\therefore$
$$\int_R \text{div } \mathbf{F}\, dv = 2\int_0^c \int_0^b \int_0^a (x + y + z)\, dxdydz$$
$$= 2\int_0^c dz \int_0^b dy\left(\frac{a^2}{2} + ya + za\right)$$
$$= 2\int_0^c dz\left(\frac{a^2}{2}b + \frac{ab^2}{2} + abz\right)$$
$$= 2\left(\frac{a^2b}{2}c + \frac{ab^2}{2}c + ab\frac{c^2}{2}\right)$$
$$= abc\,(a + b + c) \qquad \text{...}(i)$$

Also
$$\int_S \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds + \int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds + \ldots + \int_{S_6} \mathbf{F}\,.\,\mathbf{N}ds$$
where S_1 in the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$ and S_6 the face $BAP'C'$ (Fig. 8.21).

Now
$$\int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} \mathbf{F}\,.\,(-\mathbf{K})\, ds = -\int_0^b \int_0^a (0 - xy)\, dxdy = \frac{a^2b^2}{4}$$
$$\int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_2} \mathbf{F}\,.\,\mathbf{K}ds = \int_0^b \int_0^a (c^2 - xy)\, dxdy = abc^2 - \frac{a^2b^2}{4}$$

Similarly,
$$\int_{S_3} \mathbf{F}\,.\,\mathbf{N}ds = \frac{b^2c^2}{4},\ \int_{S_4} \mathbf{F}\,.\,\mathbf{N}ds = a^2bc - \frac{b^2c^2}{4},$$
$$\int_{S_5} \mathbf{F}\,.\,\mathbf{N}ds = \frac{c^2a^2}{4} \text{ and } \int_{S_6} \mathbf{F}\,.\,\mathbf{N}ds = ab^2c - \frac{c^2a^2}{4}$$

Thus
$$\int_S \mathbf{F}\,.\,\mathbf{N}ds = abc(a + b + c) \qquad \text{...}(ii)$$

Hence the theorem is verified from the equality of (*i*) and (*ii*).

Example 8.43. *Evaluate* $\int_S \mathbf{F}\,.\,d\mathbf{s}$ *where* $\mathbf{F} = 4x\mathbf{I} - 2y^2\mathbf{J} + z^2\mathbf{K}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$. (*S.V.T.U., 2007 S ; Mumbai, 2006 ; J.N.T.U., 2006*)

Solution. By divergence theorem,
$$\int_S \mathbf{F}\,.\,d\mathbf{s} = \int_V div\, \mathbf{F}\, dv$$
$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$
$$= \iiint_V ((4 - 4y + 2z)\, dxdydz$$
$$= \int_{-2}^{2} \int_{\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_0^3 (4 - 4y + 2z)\, dzdydx$$

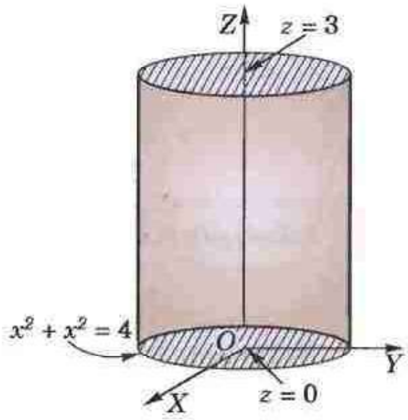

Fig. 8.22

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left|4z - 4yz + z^2\right|_0^3 dydx$$
$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12 - 12y + 9)\, dydx$$

$$= \int_{-2}^{2} \left| 21y - 6y^2 \right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{(4-x^2)}\, dx = 84 \int_{0}^{2} \sqrt{(4-x^2)}\, dx = 84 \left| \frac{x\sqrt{(4-x^2)}}{2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right|_0^2 = 84\pi.$$

Example 8.44. *Evaluate* $\int_S (yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}) \, .\, d\mathbf{S}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* *(U.P.T.U., 2004 S)*

Solution. The surface of the region V: $OABC$ is piecewise smooth (Fig. 8.23) and is comprised of four surfaces (*i*) S_1 – circular quadrant OBC in the yz-plane,

(*ii*) S_2 – circular quadrant OCA in the zx-plane,

(*iii*) S_3 – circular quadrant OAB in the xy-plane,

and (*iv*) S–surface ABC of the sphere in the first octant.

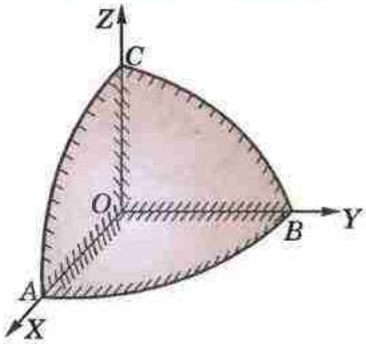

Fig. 8.23

Also $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$

By Divergence theorem,

$$\int_V div\, \mathbf{F}\, dv = \int_{S_1} \mathbf{F} \,.\, d\mathbf{S} + \int_{S_2} \mathbf{F} \,.\, d\mathbf{S} + \int_{S_3} \mathbf{F} \,.\, d\mathbf{S} + \int_{S} \mathbf{F} \,.\, d\mathbf{S} \quad ...(1)$$

Now $$div\, \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$$

For the surface S_1, $x = 0$

$$\therefore \quad \int_{S_1} \mathbf{F} \,.\, d\mathbf{S} = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} (yz\mathbf{I}) \,.\, (-\,dydz\mathbf{I}) = -\int_0^a \int_0^{\sqrt{(a^2-y^2)}} yzdydz = = -\frac{a^4}{8}$$

Thus (1) becomes $0 = -\dfrac{3a^4}{8} + \int_S \mathbf{F} \,.\, d\mathbf{S}$ whence $\int_S \mathbf{F} \,.\, d\mathbf{S} = 3a^4/8.$

Example 8.45. *Apply divergence theorem to evaluate* $\int (lx^2 + my^2 + nz^2)\, ds$ *taken over the sphere* $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$*; l, m, n being the direction cosines of the external normal to the sphere.*

Solution. The parametric equations of the sphere are $x = a + \rho \sin\theta \cos\phi$, $y = b + \rho \sin\theta \sin\phi$, $z = c + \rho\cos\theta$ and to cover the whole sphere, r varies from 0 to ρ, θ varies from 0 to π and ϕ from 0 to 2π.

$$\therefore \quad \int_S (lx^2 + my^2 + nz^2)\, ds = \int_S (x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}) \,.\, \mathbf{N}ds$$

$$= \int_V \text{div}\,(x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K})dv = 2\int_V (x + y + z)dv$$

$$= 2\int_0^{2\pi} \int_0^{\pi} \int_0^{\rho} [(a + b + c) + \rho(\sin\theta\cos\phi + \sin\theta\ \sin\phi + \cos\theta)] \times \rho^2 \sin\theta\, dr d\theta d\phi$$

$$= 2(a + b + c)\ \frac{\rho^3}{3} \left| -\cos\theta \right|_0^{\pi} .\ 2\pi = \frac{8\pi}{3}(a + b + c)\,\rho^3.$$

Example 8.46. *Evaluate* $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$*, where S is the surface of the ellipsoid* $ax^2 + by^2 + cz^2 = 1$.

Solution. Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$, $\nabla\phi = 2ax\mathbf{I} + 2by\mathbf{J} + 2cz\mathbf{K}$

$\therefore$ Unit vector normal to the ellipsoid $= \hat{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}$

Since $\mathbf{F} \,.\, \hat{\mathbf{N}} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$, $\quad \therefore\ \mathbf{F} \,.\, (ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}) = 1$

Obviously $\mathbf{F} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ $[\because ax^2 + by^2 + cz^2 = 1$

$\therefore$ By Divergence theorem,

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V div\, \mathbf{F} dv = \int_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dv = 3\int_V dv = 3V$$

$$= 3 \cdot \frac{4\pi}{3}\frac{1}{\sqrt{(abc)}} = \frac{4\pi}{\sqrt{(abc)}}. \qquad \left[\because \text{ Vol. of ellipsoid} = \frac{4\pi}{3}\frac{1}{\sqrt{(abc)}}\right]$$

Example 8.47. *If the position vector of any point (x, y, z) within a closed surface S, be* $\mathbf{R}$ *measured from an origin O, then show that*

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = \begin{cases} 0, \text{ if } O \text{ lies outside } S \\ 4\pi, \text{ if } O \text{ lies inside } S \end{cases}$$

Solution. (*a*) *When O is outside S.* Here $\mathbf{F} = \mathbf{R}/r^3$ is continuously differentiable throughout the volume V enclosed by S. Hence by Divergence theorem, we have

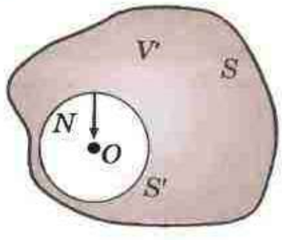

Fig. 8.24

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = \iiint_V \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV = 0 \qquad \left[\because \text{div}\left(\frac{\mathbf{R}}{r^3}\right) = 0\right.$$

(*b*) *When O is inside S.* Hence $F = \mathbf{R}/r^3$ has a point of discontinuity at O and as such Divergence theorem cannot be applied to the region V enclosed by S. To remove this point of discontinuity, we enclose O by a small sphere S' of radius ρ.

Now $\mathbf{F}$ is continuously differentiable throughout the region V' enclosed between S and S'. Therefore applying Divergence theorem to region V', we get

$$\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds + \iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' = \iiint_{V'} \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV' = 0 \qquad \left[\because \text{ div}\left(\frac{\mathbf{R}}{r^3}\right) = 0\right]$$

$$\therefore \qquad \iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = -\iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' \qquad ...(i)$$

Now the outward normal $\mathbf{N}$ on the sphere S' is directed towards the centre O. Therefore $\mathbf{N} = -\mathbf{R}/\rho$ on S' (Fig. 8.24).

$$\therefore \qquad -\iint_{S'} \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds' = -\iint_{S'} \frac{\mathbf{R}}{\rho^3} \cdot \left(-\frac{\mathbf{R}}{\rho}\right) ds' \qquad [\because \text{ on } S', r = \rho]$$

$$= \iint_{S'} \frac{r^2}{\rho^4} ds' = \iint_{S'} \frac{\rho^2}{\rho^4} ds' = \frac{1}{\rho^2}\iint_{S'} ds' = \frac{1}{\rho^2} \cdot 4\pi\rho^2 = 4\pi$$

Hence from (*i*), $\iint_S \frac{\mathbf{R}}{r^3} \cdot \mathbf{N}\, ds = 4\pi.$

8.17 GREEN'S THEOREM*

If ϕ and ψ are scalar point functions possessing continuous derivatives of first and second orders, then

$$\int_E (\phi\nabla^2\psi - \psi\nabla^2\phi) dv = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\psi}{\partial n}\right) ds \qquad ...(1)$$

where $\partial/\partial n$ denotes differentiation in the direction of the external normal to the bounding surface S enclosing the region E.

Applying Divergence theorem : $\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_S \nabla \cdot \mathbf{F} dv$ to the function $\phi\nabla\psi$, we get

$$\int_S \phi\nabla\psi \cdot \mathbf{N} ds = \int_E \nabla \cdot (\phi\nabla\psi) dv = \int_E (\nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi) dv \qquad \text{[By (2) page 329]}$$

$$= \int_E \nabla\phi \cdot \nabla\psi dv + \int_E \phi\nabla^2\psi dv \qquad ...(2)$$

*See footnote p. 339.

Interchanging ϕ and ψ, (*ii*) gives

$$\int_S \psi\nabla\phi \,.\, \mathbf{N}ds = \int_E \nabla\psi \,.\, \nabla\phi dv + \int_E \psi\nabla^2\phi dv \qquad ...(3)$$

Subtracting (3) from (2), we have $\int_S (\phi\nabla\psi - \psi\nabla\phi) \,.\, \mathbf{N}ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$

But $\nabla\psi \,.\, \mathbf{N} = \partial\psi/\partial n$ the directional derivative of ψ along the external normal at any point of S. Hence

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$$ which is the required result (1).

Obs. Harmonic function. *A scalar point function ϕ satisfying the Laplace's equation $\nabla^2\phi = 0$ at every point of a region E, is called a harmonic function in E.*

If ϕ and ϕ be both harmonic functions in E, (1) gives

$$\int_S \phi\frac{\partial\psi}{\partial n}ds = \int_S \psi\frac{\partial\phi}{\partial n}ds$$ which is known as **Green's reciprocal theorem.**

PROBLEMS 8.10

1. Verify divergence theorem for $\mathbf{F}$ taken over the cube bounded by $x = 0, x = 1$; $y = 0, y = 1$; $z = 0, z = 1$ where
   (*i*) $\mathbf{F} = 4xz\mathbf{I} - y^2\mathbf{J} + yz\mathbf{K}$ (*Madras, 2006*) (*ii*) $x^2\mathbf{I} + z\mathbf{J} + yz\mathbf{K}$ (*Bhopal, 2008*)
2. Verify Gauss divergence theorem for the function $\mathbf{F} = y\mathbf{I} + x\mathbf{J} + z^2\mathbf{K}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$ and $z = 2$.
3. Using divergence theorem, prove that
   (*i*) $\int_S \mathbf{R} \,.\, d\mathbf{S} = 3V$ (*ii*) $\int_S \nabla r^2 \,.\, d\mathbf{S} = 6V$ (*U.P.T.U., 2003*)
   where S is any closed surface enclosing a volume V and $r^2 = x^2 + y^2 + z^2$.
4. Using divergence theorem, evaluate $\int_S \mathbf{R} \,.\, \mathbf{N}ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.
5. If S is any closed surface enclosing a volume V and $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, prove that
   $$\int_S \mathbf{F} \,.\, \mathbf{N}ds = (a + b + c)V$$ (*Madras, 2003*)
6. For any closed surface S, prove that $\int [x(y - z)\mathbf{I} + y(z - x)\mathbf{J} + z(x - y)\mathbf{K}] \,.\, ds = 0$.
7. Use divergence theorem to evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$, where
   (*i*) $\mathbf{F} = x^3\mathbf{I} + y^3\mathbf{J} + z^3\mathbf{K}$, and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. (*V.T.U., 2008 ; P.T.U., 2005*)
   (*ii*) $\mathbf{F} = [e^x, e^y, e^z]$ and S is the surface of the cube $|x| \le 1, |y| \le 1, |z| \le 1$. (*B.P.T.U., 2005*)
8. Evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of a sphere of radius a. (*Kurukshetra, 2008 S*)
9. Evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$ where $F = y^2z^2\mathbf{I} + z^2x^2\mathbf{J} + x^2y^2\mathbf{K}$ and S is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$ above XOY plane.
10. By transforming to triple integral, evaluate $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$. (*Burdwan, 2003*)
11. Evaluate $\int_S (\nabla \times \mathbf{F}) \,.\, d\mathbf{S}$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane, and $\mathbf{F} = (x^2 + y - 4)\mathbf{I} + 3xy\mathbf{J} + (2xz + z^2)\mathbf{K}$.
12. If $\mathbf{F} = (2x^2 - 3z)\mathbf{I} - 2xy\mathbf{J} - 4x\mathbf{K}$, then evaluate $\iiint_V \nabla \,.\, \mathbf{F}\, dv$, where V is bounded by $x = y = z = 0$ and $2x + 2y + z = 4$. (*Bhopal, 2008*)
13. If $\mathbf{F} = \text{grad}\, \phi$ and $\nabla^2\phi = -4\pi\rho$, prove that $\int_S \mathbf{F} \,.\, \mathbf{N}ds = -4\pi\rho \int_V dV$ where the symbol have their usual meanings.

8.18 (1) IRROTATIONAL FIELDS

An irrotational field **F** *is characterised by any one of the following conditions* :

(*i*) $\Delta \times \mathbf{F} = \mathbf{0}$. (*ii*) *Circulation* $\int \mathbf{F} \cdot d\mathbf{R}$ *along every closed surface is zero.*

(*iii*) $\mathbf{F} = \Delta\phi$, *if the domain is simply connected.**

If $\nabla \times \mathbf{F} = \mathbf{0}$, then by Stoke's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \mathbf{0},$$ *i.e.*, the circulation along every closed surface is zero.

Again since $\nabla \times \nabla\phi = 0$

$\therefore$ in an irrotational field for which $\Delta \times \mathbf{F} = 0$, the vector **F** can always be expressed as the gradient of a scalar function ϕ provided the domain is simply connected. Thus

$$\mathbf{F} = \nabla\phi.$$

Such a scalar function ϕ is called the *potential*. In a rotational field, **F** cannot be expressed as the gradient of a scalar potential.

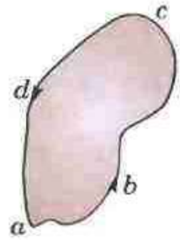

Fig. 8.25

Obs. 1. *In an irrotational field, the line integral* **F** *between two points is independent of the path of integration and is equal to the potential difference between these points.*

If *aced* be any closed contour in an irrotational field F (Fig. 8.25), then

$$\int_{abcd} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R} + \int_{cda} \mathbf{F} \cdot d\mathbf{R} = 0$$

or
$$\int_{abc} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R}$$

i.e. the value of the line integral is independent of the path joining the end points.

Further, substituting $\mathbf{F} = \nabla\phi$, we have

$$\int_a^c \mathbf{F} \cdot d\mathbf{R} = \int_a^c \nabla\phi \cdot d\mathbf{R} = \int_a^c \left(\mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z} \right) \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz)$$

$$= \int_a^c \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right) = \int_a^c d\phi = \phi_c - \phi_a .$$

Obs. 2. If **F** is a vector force acting on a particle, then $\oint_C \mathbf{F} \cdot d\mathbf{R}$ represents the work done in moving the particle around a closed path. [See p. 328]

When $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$, *the field is said to be* **conservative,** *i.e.*, no work is done in displacement from a point *a* to another point in the field and back to *a* and the mechanical energy is conserved.

Thus *every irrotational field is conservative.*

Obs. 3. *The well-known equations of the Poisson and Laplace hold good for every irrotational field.*

Suppose $\nabla \cdot \mathbf{F} = f(x, y, z)$. Then $\nabla \cdot \nabla\phi = f(x, y, z)$ *i.e.*, $\nabla^2\phi = f(x, y, z)$ ...(*i*)

which is known as *Poisson's equation*. Its solutions for electrostatic fields enable us to determine the potential ϕ as a function of the charge distribution $f(x, y, z)$.

If $f(x, y, z) = 0$ then (*i*) reduces to $\nabla^2\phi = 0$ which is the *Laplace's equation*. The solutions of this equation are of great importance in modern engineering and physics, some of which we'll study in § 18.11 and 18.12.

(2) **Solenoidal fields.** *A solenoidal field* **F** *is characterised by any one of the following conditions :*

(*i*) $\nabla \cdot \mathbf{F} = 0$. (*ii*) *flux* $\int \mathbf{F} \cdot \mathbf{N}\, ds$ *across every closed surface is zero.* (*iii*) $\mathbf{F} = \nabla \times \mathbf{V}$.

If $\nabla \cdot \mathbf{F} = 0$ then by the Divergence theorem,

$$\int_S \mathbf{F} \cdot \mathbf{N}ds = \int_V \nabla \cdot \mathbf{F}dv = 0,$$ *i.e.*, the flux across every closed surface is zero.

Again since $\nabla \cdot \nabla \times \mathbf{V} = 0$.

$\therefore$ in a solenoidal field for which $\nabla \cdot \mathbf{F} = 0$, the vector **F** can always be expressed as the curl of a vector function **V**; thus $\mathbf{F} = \nabla \times \mathbf{V}$.

*A domain *D* is said to be *simply connected* if every closed curve in *D* can be shrunk to any point within *D*.

Example 8.48. *A vector field is given by* $\mathbf{F} = (x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J}$. *Show that the field is irrotational and find its scalar potential. Hence evaluate the line integral from (1, 2) to (2, 1).*

Solution. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + x & -(2xy + y) & 0 \end{vmatrix} = \mathbf{0}$

∴ this field is *irrotational* and the vector $\mathbf{F}$ can be expressed as the gradient of a scalar potential,

i.e., $(x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J} = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{I} + \frac{\partial \phi}{\partial y}\mathbf{J}$

whence $\partial\phi/\partial x = x^2 - y^2 + x$ ...(*i*)

$\partial\phi/\partial y = -(2xy + y)$ ...(*ii*)

Integrating (*i*) w.r.t. x, keeping y constant, we get $\phi = \frac{x^3}{3} - y^2 x + \frac{x^2}{2} + f(y)$ ...(*iii*)

Similarly integrating (*ii*) w.r.t. y, keeping x constant, we obtain $\phi = -xy^2 - \frac{y^2}{2} + g(x)$ ...(*iv*)

Equating (*iii*) and (*iv*), we get $\frac{x^3}{3} - y^2 x + \frac{x^2}{2} + f(y) = -xy^2 - \frac{y^2}{2} + g(x)$

∴ $f(y) = -\frac{y^2}{2}$ and $g(x) = \frac{x^3}{3} + \frac{x^2}{2}$

Hence $\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2}$

Since the field is irrotational,

∴ $\int \mathbf{F} \,.\, d\mathbf{R}$ from (1, 2) to (2, 1) $= \phi_{1,2} - \phi_{2,1} = \left(\frac{1}{3} - 1 \times 4 + \frac{1}{2} - \frac{4}{2}\right) - \left(\frac{8}{3} - 2 \times 1 + \frac{4}{2} - \frac{1}{2}\right) = -7\frac{1}{3}$.

Example 8.49. *A fluid motion is given by* $\mathbf{V} = (y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K}$.
(*a*) *Is this motion irrotational? If so, find the velocity potential.* (*U.P.T.U., 2004*)
(*b*) *Is the motion possible for an incompressible fluid?*

Solution. We have $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix} = \mathbf{I}(1 - 1) - \mathbf{J}(1 - 1) + \mathbf{K}(1 - 1) = \mathbf{0}.$

∴ this motion is irrotational and if ϕ is the velocity potential then $\mathbf{V} = \nabla\phi$. [§ 20.6]

i.e., $(y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K} = \frac{\partial\phi}{\partial x}\mathbf{I} + \frac{\partial\phi}{\partial y}\mathbf{J} + \frac{\partial\phi}{\partial z}\mathbf{K}$

∴ $\frac{\partial\phi}{\partial x} = y + z, \frac{\partial\phi}{\partial y} = z + x, \frac{\partial\phi}{\partial z} = x + y$

Integrating these, we get $\phi = (y + z)x + f_1(y, z)$ ...(*i*)

$\phi = (z + x)y + f_2(z, x)$ ...(*ii*)

and $\phi = (x + y)z + f_3(x, y)$ ...(*iii*)

Equality of (*i*), (*ii*) and (*iii*), requires that

$$f_1(y, z) = yz, f_2(z, x) = zx, f_3(x, y) = xy.$$

Hence $\phi = yz + zx + xy$.

(*b*) The fluid motion is possible if $\mathbf{V}$ satisfies the equation of continuity which for an incompressible fluid is $\nabla \,.\, \mathbf{V} = 0$. [See § 8.7 (1)

Here $$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y} = (z+x) + \frac{\partial}{\partial z}(x+y) = 0.$$

Hence, the fluid motion is possible.

Example 8.50. *Find whether $\int_C [2xyz^2 dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz]$ is independent of the path joining $(0, \pi/2, 1)$ and $(1, 0, 1)$. If so, evaluate this line integral.*

Solution. The line integral of **F** is independent of path of integration if $\nabla \times \mathbf{F} = \mathbf{0}$.

$$= \int_C [2xyz^2\,\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\mathbf{K}] \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz) = \int_C \mathbf{F} \cdot d\mathbf{R}$$

and $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \mathbf{I}\,[2x^2z + \cos yz - yz\sin yz - (2x^2z + \cos yz - yz\sin yz)] - \mathbf{J}[4xyz - 4xyz] + \mathbf{K}[2xz^2 - 2xz^2] = \mathbf{0}$$

$\therefore$ the given integral is independent of the path C.

Now let $\mathbf{F} = \nabla\phi$

i.e., $$(2xyz^2)\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\,\mathbf{K} = \mathbf{I}\frac{\partial \phi}{\partial x} + \mathbf{J}\frac{\partial \phi}{\partial y} + \mathbf{K}\frac{\partial \phi}{\partial z}$$

$$\therefore \quad 2xyz^2 = \frac{\partial \phi}{\partial x},\; x^2z^2 + z\cos yz = \frac{\partial \phi}{\partial y},\; 2x^2yz + y\cos yz = \frac{\partial \phi}{\partial z}$$

Integrating first w.r.t. x partially, we get

$$\phi = x^2y^2z^2 + \Psi_1(y, z) \quad \text{...}(i)$$

Integrating second w.r.t. y partially, we get

$$\phi = x^2yz^2 + \sin yz + \Psi_2(z, x) \quad \text{...}(ii)$$

Integrating third w.r.t. z partially, we get

$$\phi = x^2yz^2 + \sin yz + \Psi_3(x, y) \quad \text{...}(iii)$$

Comparing (*i*), (*ii*), (*iii*), we have

$\Psi_1(y, z)$ = terms in ϕ independent of $x = \sin yz$

$\Psi_2(z, x)$ = terms in ϕ independent of $y = 0$

$\Psi_3(z, x)$ = terms in ϕ independent of $z = 0$

Thus $\phi = x^2yz^2 + \sin yz$

Hence the value of the given integral $= |\phi|_{(0,\pi/2,1)}^{(1,0,1)}$

$$= (0 + 0) - (0 + \sin \pi/2) = -1.$$

Example 8.51. *Determine whether $\mathbf{F} = (y^2\cos x + z^3)\mathbf{I} + (2y\sin x - 4)\mathbf{J} + (3xz^2 + 2)\mathbf{K}$ is a conservative vector field? If so find the scalar potential ϕ. Also compute the work done in moving the particle from $(0, 1, -1)$ to $(\pi/2, -1, 2)$.* (*Mumbai, 2006*)

Solution. F is a conservative vector field when curl **F** = **0**. Here

$$Curl = F = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \mathbf{I}(0 - 0) - \mathbf{J}(3z^2 - 3z^2) + \mathbf{K}(2y\cos x - 2y\cos x) = \mathbf{0}$$

$\therefore$ **F** is a conservative field.

Now let $\mathbf{F} = \nabla\phi$

i.e.,
$$(y^2 \cos x + z^3)\,\mathbf{I} + (2y \sin x - 4)\,\mathbf{J} + (3xz^2 + 2)\,\mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad y^2 \cos x + z^3 = \frac{\partial\phi}{\partial x},\ 2y \sin x - 4 = \frac{\partial\phi}{\partial y},\ 3xz^2 + 2 = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get

$$\phi = y^2 \sin x + xz^3 + \psi_1(y, z) \qquad ...(i)$$

Integrating second w.r.t. y partially, we get

$$\phi = y^2 \sin x - 4y + \Psi_2(z, x) \qquad ...(ii)$$

Integrating third w.r.t. z partially, we obtain

$$\phi = xz^3 + 2z + \Psi_3(x, y) \qquad ...(iii)$$

Comparing (i), (ii), (iii), we get

$$\Psi_1(y, z) = \text{terms in } \phi \text{ independent of } x = -4y + 2z$$
$$\Psi_2(z, x) = \text{terms in } \phi \text{ independent of } y = xz^3 + 2z$$
$$\Psi_3(z, x) = \text{terms in } \phi \text{ independent of } z = y^2 \sin x - 4y$$

Thus $\qquad \phi = xz^3 + y^2 \sin x - 4y + 2z$

In a conservative field, the work done $= \phi_B - \phi_A$

$$= \phi\left(\frac{\pi}{2}, -1, 2\right) - \phi(0, 1, -1)$$
$$= (4\pi + 1 + 4 + 4) - (-4 - 2) = 4\pi + 15.$$

PROBLEMS 8.11

1. If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.
2. Show that the vector field defined by $\mathbf{F} = (x^2 + xy^2)\mathbf{I} + (y^2 + x^2y)\mathbf{J}$ is conservative and find the scalar potential. Hence evaluate $\int \mathbf{F} \cdot d\mathbf{R}$ from (0, 1) to (1, 2).
3. Find the work done by the variable force $\mathbf{F} = 2y\mathbf{I} + xy\mathbf{J}$ on a particle when it is displaced from the origin to the point $\mathbf{R} = 4\mathbf{I} + 2\mathbf{J}$ along the parabola $y^2 = x$.
4. Show that the vector field given by $\mathbf{A} = 3x^2y\mathbf{I} + (x^3 - 2yz^2)\mathbf{J} + (3z^2 - 2y^2z)\mathbf{K}$ is irrotational but not solenoidal. Also find $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{A}$.
5. Show that the following vectors are irrotational and find the scalar potential in each case :
   (*i*) $(x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$
   (*ii*) $2xy\mathbf{I} + (x^2 + 2yz)\mathbf{J} + (y^2 + 1)\mathbf{K}$ *(V.T.U., 2007)*
   (*iii*) $(6xy + z^3)\mathbf{I} + (3x^2 - z)\mathbf{J} + (3xz^2 - y)\mathbf{K}$ *(Raipur, 2005 ; V.T.U., 2003 S)*
   (*iv*) $(2xy^2 + yz)\mathbf{I} + (2x^2y + xz + 2yz^2)\mathbf{J} + (2y^2z + xy)\mathbf{K}$. *(V.T.U., 2010)*
6. Fluid motion is given by $\mathbf{V} = ax\mathbf{I} + ay\mathbf{J} - 2az\mathbf{K}$.
   (*i*) Is it possible to find out the velocity potential ? If so, find it.
   (*ii*) Is the motion possible for an incompressible fluid ?
7. Show that the vector field defined by $\mathbf{F} = (y \sin z - \sin x)\mathbf{I} + (x \sin z + 2yz)\mathbf{J} + (xy \cos z + y^2)\mathbf{K}$ is irrotational and find its velocity potential. *(Kottayam, 2005)*
8. Show that $\mathbf{F} = (2xy + z^3)\mathbf{I} + x^2\mathbf{J} + 3xz^2\mathbf{K}$ is a conservative vector field and find a function ϕ such that $\mathbf{F} = \nabla\phi$. Also find the work done in moving an object in this field from (1, – 2, 1) to (3, 1, 4). *(Nagpur, 2009)*
9. If $\mathbf{F} = (x + y + az)\mathbf{I} + (bx + 2y - z)\mathbf{J} + (x + cy + 2z)\mathbf{K}$, find a, b, c such that curl $\mathbf{F} = \mathbf{0}$, then find ϕ such that $\mathbf{F} = \nabla\phi$. *(V.T.U., 2000)*
10. Find the constant a so that **V** is a conservative vector field, where
$$\mathbf{V} = (axy - z^3)\mathbf{I} + (a - 2)x^2\mathbf{J} + (1 - a)xz^2\mathbf{K}.$$
    Calculate its scalar potential and work done in moving a particle from (1, 2, – 3) to (1, – 4, 2) in the field. *(Mumbai, 2006 ; Rajasthan, 2006)*

8.19 (1) ORTHOGONAL CURVILINEAR COORDINATES

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of u, v, w so that

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \quad ...(1)$$

Suppose that (1) can be solved for u, v, w in terms of x, y, z, so that

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z) \quad ...(2)$$

We assume that the functions in (1) and (2) are single-valued and have continuous partial derivatives so that the correspondence between (x, y, z) and (u, v, w) is unique. Then (u, v, w) are called *curvilinear coordinates* of (x, y, z).

Each of u, v, w has a level surface through an arbitrary point. The surfaces $u = u_0, v = v_0, w = w_0$ are called *coordinate surfaces* through $P(u_0, v_0, w_0)$. Each pair of these coordinate surfaces intersect in curves called the *coordinate curves*. The curve of intersection of $u = u_0$ and $v = v_0$ will be called the w-curve, for only w changes along this curve. Similarly we define u and v-curves.

In vector notation, (1) can be written as $\mathbf{R} = x(u, v, w)\mathbf{I} + y(u, v, w)\mathbf{J} + z(u, v, w)\mathbf{K}$

$$\therefore \qquad d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u} du + \frac{\partial \mathbf{R}}{\partial v} dv + \frac{\partial \mathbf{R}}{\partial w} dw \quad ...(3)$$

Then $\partial\mathbf{R}/\partial u$ is a tangent vector to the u-curve at P. If $\mathbf{T}_u$ is a unit vector at P in this direction, then $\partial\mathbf{R}/\partial u = h_1\mathbf{T}_v$ where $h_1 = |\partial\mathbf{R}/\partial u|$.

Similarly if $\mathbf{T}_v$ and $\mathbf{T}_w$ be unit tangent vectors to v- and w-curves at P, then

$$\frac{\partial \mathbf{R}}{\partial v} = h_2\mathbf{T}_v \text{ and } \frac{\partial \mathbf{R}}{\partial w} = h_3\mathbf{T}_w$$

where $h_2 = |\partial\mathbf{R}/\partial v|$ and $h_3 = |\partial\mathbf{R}/\partial w|$. [$h_1, h_2, h_3$ are called scalar factors.]

Then (3) can be written as

$$d\mathbf{R} = h_1 du\ \mathbf{T}_u + h_2 dv\ \mathbf{T}_v + h_3\ dw\ \mathbf{T}_w \quad ...(4)$$

Since ∇u is normal to the surface $u = u_0$ at P, therefore, a unit vector in this direction is given by $\mathbf{N}_u = \dfrac{\nabla u}{|\nabla u|}$.

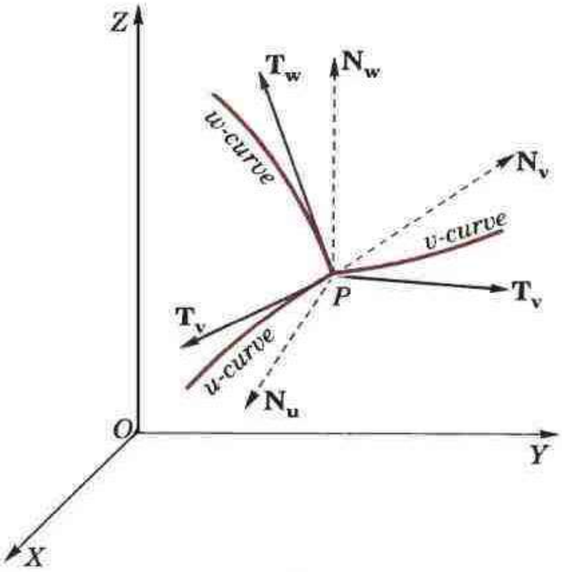

Fig. 8.26

Similarly, the unit vectors $\mathbf{N}_v = \dfrac{\nabla v}{|\nabla v|}$ and $\mathbf{N}_w = \dfrac{\nabla w}{|\nabla w|}$ are normal to the surfaces $v = v_0$ and $w = w_0$ at P respectively. Thus at each point P of a curvilinear coordinate system there exist two triads of unit vectors : $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ tangents to u, v, w-curves and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ normals to the co-ordinates surfaces (Fig. 8.26).

In particular, when the coordinate surfaces intersect a right angles, the three coordinate curves are also mutually orthogonal and u, v, w are called the *orthogonal curvilinear coordinates*. In this case $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ are mutually perpendicular unit vector triads and hence become identical. Henceforth, we shall refer to orthogonal curvilinear coordinates only.

Multiplying (3) scalarly by ∇u, we get

$$\nabla u \,.\, d\mathbf{R} = du = \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u}\right) du + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v}\right) dv + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w}\right) dw$$

whence $$\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u} = 1, \nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v} = 0, \nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w} = 0$$

Similarly, $$\nabla v \,.\, \frac{\partial \mathbf{R}}{\partial u} = 0, \nabla v \,.\, \frac{\partial \mathbf{R}}{\partial v} = 1, \nabla v \,.\, \frac{\partial \mathbf{R}}{\partial w} = 0$$

and $$\nabla w \,.\, \frac{\partial \mathbf{R}}{\partial u} = 0, \nabla w \,.\, \frac{\partial \mathbf{R}}{\partial v} = 0, \nabla w \,.\, \frac{\partial \mathbf{R}}{\partial w} = 1.$$

These relations show that the sets $\partial\mathbf{R}/\partial u$, $\partial\mathbf{R}/\partial v$, $\partial\mathbf{R}/\partial w$ and $\nabla u, \nabla v, \nabla w$ constitute reciprocal system of vectors.

$$\therefore \qquad \nabla u = \frac{\dfrac{\partial \mathbf{R}}{\partial v} \times \dfrac{\partial \mathbf{R}}{\partial w}}{\left[\dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{\partial \mathbf{R}}{\partial v} \times \dfrac{\partial \mathbf{R}}{\partial w}\right]} = \frac{(h_2 \mathbf{T_v}) \times (h_3 \mathbf{T_w})}{[(h_1 \mathbf{T_u}) \,.\, (h_2 \mathbf{T_v}) \times (h_3 T_\mathbf{w})]}$$

$$= \frac{h_2 h_3 \; \mathbf{T}_v \times \mathbf{T}_w}{h_1 h_2 h_3 \, [\mathbf{T}_u \mathbf{T}_v \mathbf{T}_w]} = \frac{\mathbf{T}_u}{h_1} \qquad [\because \quad \mathbf{T}_u \mathbf{T}_v \mathbf{T}_w = 1]$$

or $\qquad \mathbf{T}_v = h_1 \, \nabla u$

Similarly $\qquad \mathbf{T}_v = h_2 \nabla v$ and $\mathbf{T}_w = h_3 \, \nabla w \qquad ...(5)$

Also $\qquad = \mathbf{T}_v \times \mathbf{T}_w = h_2 h_3 \; \nabla v \times \nabla w$

Similarly $\qquad \mathbf{T}_v = h_3 h_1 \; \nabla w \times \nabla u$ and $\mathbf{T}_w = h_1 h_2 \; \nabla u \times \nabla v \qquad ...(6)$

Arc, area and volume elements

(*i*) *Arc element*. The element of arc length ds is determined from (4).

$$\therefore \qquad ds^2 = d\mathbf{R} \,.\, d\mathbf{R} = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2 \qquad ...(7)$$

The arc length ds_1 along u-curve at P is $h_1 du$ for v and w are constants. Therefore the vector arc element along the u-curve is $d\mathbf{u} = h_1 du \mathbf{T}_u$. Similarly vector arc elements along v and w curves at P are $d\mathbf{v} = h_2 dv \; \mathbf{T}_v$ and $dw = h_3 dw \; \mathbf{T}_w$. The arc element ds therefore corresponds to the length of the diagonal of the rectangular parallelopiped of Fig. 8.27.

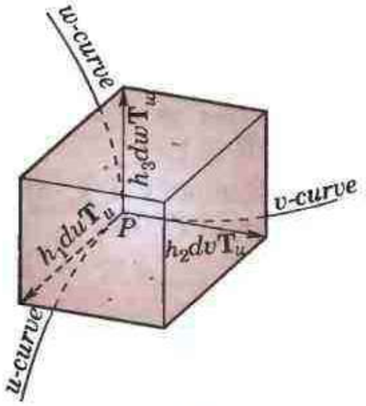

Fig. 8.27

(*ii*) *Area elements*. The area of the parallelogram formed by $d\mathbf{u}$ and $d\mathbf{v}$ is called the area element on the uv surface which is perpendicular to w-curve and we denote it by dS_w. Hence, $dS_w = |du \times dv| = h_1 h_2 du dv$. Similarly, $dS_u = h_2 h_3 \, dv dw$, $dS_v = h_3 h_1 dw du$.

(*iii*) *Volume element* is the volume of the parallelopiped formed by $d\mathbf{u}$, $d\mathbf{v}$, $d\mathbf{w}$.

$$\therefore \qquad dV = [h_1 du \mathbf{T}_u) \,.\, (h_2 dv \mathbf{T}_v) \times (h_3 dw \mathbf{T}_w)]$$

$$= h_1 h_2 h_3 \; du dv dw \qquad ...(8) \; [\because \; [\mathbf{T}_u \mathbf{T}_v \mathbf{T}_w] = 1$$

This can also be written as

$$dV = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} \times \frac{\partial \mathbf{R}}{\partial w} \; du dv dw = \frac{\partial(x, y, z)}{\partial(u, v, w)} \, du dv dw \qquad ...(9)$$

where $\partial(x, y, z)/\partial(u, v, w)$ is called the *Jacobian of the transformation* from (x, y, z) to (u, v, w) coordinates.

(2) Del applied to Functions in Orthogonal Curvilinear coordinates

To prove that

(1) $\nabla f = \dfrac{\mathbf{T}_u}{h_1} \dfrac{\partial f}{\partial u} + \dfrac{\mathbf{T}_v}{h_2} \dfrac{\partial f}{\partial v} + \dfrac{\mathbf{T}_w}{h_3} \dfrac{\partial f}{\partial w}$

(2) $\nabla \,.\, \mathbf{F} = \dfrac{1}{h_1 h_2 h_3} \left[\dfrac{\partial}{\partial u} (h_2 h_3 f_1) + \dfrac{\partial}{\partial v} (h_3 h_1 f_2) + \dfrac{\partial}{\partial w} (h_1 h_2 f_3) \right]$

(3) $\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2 h_3} & \dfrac{\mathbf{T}_v}{h_3 h_1} & \dfrac{\mathbf{T}_w}{h_1 h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$ where $\mathbf{F} = f_1 \mathbf{T}_u + f_2 \mathbf{T}_v + f_3 \mathbf{T}_w$.

(1) Let $f(u, v, w)$ be any scalar point function in terms of u, v, w, the orthogonal curvilinear coordinates. Taking u, v, w as functions of x, y, z, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} \qquad ...(i)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial y} \qquad ...(ii)$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial z} \qquad ...(iii)$$

Multiplying (*i*) by **I**, (*ii*) by **J**, (*iii*) by **K** and adding, we have

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w \qquad ...(iv)$$

$$= \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w} \qquad \text{[By (5) p. 356]}$$

which is the required result.

(2) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = \Sigma f_1 h_2 h_3\, \nabla v \times \nabla w \qquad \text{[By (6) p. 356]}$$

$$\therefore \quad \nabla\,.\,\mathbf{F} = \Sigma\nabla\,.\,\{(f_1h_2h_3)(\nabla v \times \nabla w)\}$$

$$= \Sigma[(f_1h_2h_3)\,\nabla\,.\,(\nabla v \times \nabla w) + (\nabla v \times \nabla w)\,\nabla(f_1h_2h_3)] \qquad ...(v)$$

Now $\nabla\,.\,(\nabla v \times \nabla w) = \nabla w\,.\,\nabla \times (\nabla v) - \nabla v\,.\,\nabla \times (\nabla w) = 0$ [By (5) p. 330]

and

$$\nabla(f_1h_2h_3) = \frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w \qquad \text{[By (iv) above]}$$

$\therefore$ (*v*) now becomes

$$\nabla\,.\,\mathbf{F} = \sum(\nabla v \times \nabla w)\,.\left\{\frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w\right\}$$

$$= [\nabla u, \nabla v, \nabla w]\sum\frac{\partial(f_1h_2h_3)}{\partial u} = \frac{1}{h_1h_2h_3}\sum\frac{\partial(f_1h_2h_3)}{\partial u}$$ which is the required result.

Cor. Laplacian. $\nabla^2 f = \nabla\,.\,(\nabla f)$

$$= \nabla\,.\left(\frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}\right) = \frac{1}{h_1h_2h_3}\sum\frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2h_3\right)$$

(3) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = f_1h_1\nabla u + f_2h_2\nabla v + f_3h_3\,\nabla w \qquad \text{[By (5) p. 356]}$$

$$\nabla \times \mathbf{F} = \Sigma\nabla \times (f_1h_1\nabla u) \qquad \text{[Using (3) p. 329]}$$

$$= \Sigma[\nabla(f_1h_1) \times \nabla u + f_1h_1\,.\,\nabla \times \nabla u] = \sum\left[\frac{\partial(f_1h_1)}{\partial u}\nabla u + \frac{\partial(f_1h_1)}{\partial v}\nabla v + \frac{\partial(f_1h_1)}{\partial w}\nabla w\right]\times\nabla u$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\nabla v \times \nabla u + \frac{\partial(f_1h_1)}{\partial w}\nabla w \times \nabla u\right]$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\left(-\frac{\mathbf{T}_u \times \mathbf{T}_v}{h_1h_2}\right) + \frac{\partial(f_1h_1)}{\partial w}\left(\frac{\mathbf{T}_w \times \mathbf{T}_u}{h_3h_1}\right)\right]$$

$$= -\frac{\partial(f_1h_1)}{\partial v}\frac{\mathbf{T}_w}{h_1h_2} + \frac{\partial(f_1h_1)}{\partial w}\frac{\mathbf{T}_v}{h_3h_1} - \frac{\partial(f_2h_2)}{\partial w}\frac{\mathbf{T}_u}{h_2h_3} + \frac{\partial(f_2h_2)}{\partial u}\frac{\mathbf{T}_w}{h_1h_2} - \frac{\partial(f_3h_3)}{\partial u}\frac{\mathbf{T}_v}{h_3h_1} + \frac{\partial(f_3h_3)}{\partial v}\frac{\mathbf{T}_u}{h_2h_3}$$

$$= \frac{\mathbf{T}_u}{h_2h_3}\left[\frac{\partial(f_3h_3)}{\partial v} - \frac{\partial(f_2h_2)}{\partial w}\right]$$ + two similar terms, whence follows the required result.

TWO SPECIAL CURVILINEAR SYSTEMS

8.20 (1) CYLINDRICAL COORDINATES

Any point $P(x, y, z)$ whose projection on the *xy*-plane is $Q(x, y)$ has the *cylindrical coordinates* (ρ, ϕ, z), where $\rho = OQ$, $\phi = \angle XOQ$ and $z = QP$.

The level surfaces $\rho = \rho_0$, $\phi = \phi_0$, $z = z_0$ are respectively cylinders about the *Z*-axis; planes through the *Z*-axis and planes perpendicular to the *Z*-axis.

The coordinate curves for ρ are rays perpendicular to the *Z*-axis; for ϕ, horizontal circles with centres on the *Z*-axis; for *z*, lines parallel to the *Z*-axis.

From Fig. 8.28, we have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

(i) *Arc element.*

$\therefore \qquad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$

so that the scale factors are $h_1 = 1, h_2 = \rho, h_3 = 1.$

(ii) *Area elements* $dS_\rho = \rho d\phi dz$, $dS_\phi = dz d\rho$, $dS_z = \rho d\rho d\phi$ where dS_ρ is the area element $\perp$ to ρ-direction, etc.

(iii) *Volume element* $dV = \rho d\rho d\phi dz$.

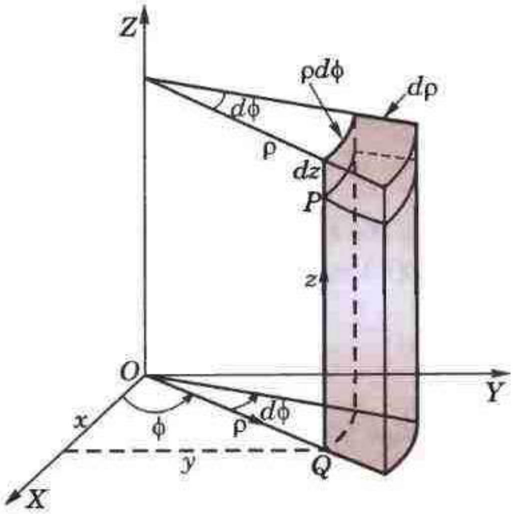

Fig. 8.28

(2) Cylindrical co-ordinate system is orthogonal

At any point P, we have $x = \rho \cos \phi, y = \rho \sin \phi, z = z,$

so that $\qquad \mathbf{R} = \rho \cos \phi \mathbf{I} + \rho \sin \phi \mathbf{J} + z\mathbf{K}$

If $\mathbf{T}_\rho, \mathbf{T}_\phi, \mathbf{T}_z$ be the unit vectors at P in the directions of the tangents to the ρ, ϕ, z-curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial \mathbf{R}/\partial \rho}{|\partial \mathbf{R}/\partial \rho|} = \frac{\cos \phi \mathbf{I} + \sin \phi \mathbf{J}}{\sqrt{(\cos^2 \phi + \sin^2 \phi)}} = \cos \phi \mathbf{I} + \sin \phi \mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial \mathbf{R}/\partial \phi}{|\partial \mathbf{R}/\partial \phi|} = \frac{-\rho \sin \phi \mathbf{I} + \rho \cos \phi \mathbf{J}}{\sqrt{[(-\rho \sin \phi)^2 + (\rho \cos \phi)^2]}} = -\sin \phi \mathbf{I} + \cos \phi \mathbf{J}$$

and $$\mathbf{T}_z = \frac{\partial \mathbf{R}/\partial z}{|\partial \mathbf{R}/\partial z|} = \mathbf{K}$$

Now $\mathbf{T}_\rho \cdot \mathbf{T}_\phi = (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) \cdot (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) = -\cos \phi \sin \phi + \sin \phi \cos \phi = 0,$

$\mathbf{T}_\phi \cdot \mathbf{T}_z = (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) \cdot \mathbf{K} = 0$, and $\mathbf{T}_z \cdot \mathbf{T}_\rho = \mathbf{K} \cdot (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) = 0.$

Hence the cylindrical coordinate system is orthogonal.

Also $\quad \mathbf{T}_\rho \times \mathbf{T}_\phi = (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) \times (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) = (\cos^2 \phi + \sin^2 \phi)\, \mathbf{I} \times \mathbf{J} = \mathbf{K} = T_z$

$\mathbf{T}_\phi \times \mathbf{T}_z = (-\sin \phi \mathbf{I} + \cos \phi \mathbf{J}) \times \mathbf{K} = \sin \phi \mathbf{J} + \cos \phi \mathbf{I} = \mathbf{T}_\rho$

$\mathbf{T}_z \times \mathbf{T}_\rho = \mathbf{K} \times (\cos \phi \mathbf{I} + \sin \phi \mathbf{J}) = \cos \phi \mathbf{J} - \sin \phi \mathbf{I} = \mathbf{T}_\phi$

These conditions satisfied by T_ρ, T_ϕ, and T_z, show that the *cylindrical coordinates system is a right handed orthogonal coordinate system.* (V.T.U., 2008)

(3) Del applied to functions in Cylindrical coordinates

We have $u = \rho, v = \phi, w = z$ and $h_1 = 1, h_2 = \rho, h_3 = 1.$

Let T_ρ, T_ϕ, T_z be the unit vectors in the directions of the tangents to the ρ, ϕ, z curves.

(i) *Expression for grad f.*

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}w}{h_3}\frac{\partial f}{\partial w}$$

$\therefore$ $$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{T}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{T}_\phi + \frac{\partial f}{\partial z}\mathbf{T}_z$$

(ii) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial f}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$$

$\therefore$ $$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho f_1) + \frac{\partial f_2}{\partial \phi} + \frac{\partial}{\partial z}(\rho f_3)\right\}$$

(iii) *Expression for curl* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{T}_\rho/\rho & \mathbf{T}_\phi & \mathbf{T}_z/\rho \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

$$= \mathbf{T}_\rho\left(\frac{1}{\rho}\frac{\partial f_3}{\partial \phi} - \frac{\partial f_2}{\partial z}\right) + \mathbf{T}_\phi\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial \rho}\right) + \mathbf{T}_z\left(\frac{\partial f_2}{\partial \rho} - \frac{1}{\rho}\frac{\partial f_1}{\partial \phi}\right)$$

(iv) *Expression for* $\nabla^2 f$

Since $$\nabla^2 f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2h_3\right)+\frac{\partial}{\partial v}\left(\frac{1}{h_2}\frac{\partial f}{\partial v}h_3h_1\right)+\frac{\partial}{\partial w}\left(\frac{1}{h_3}\frac{\partial f}{\partial w}h_1h_2\right)\right\}$$

$\therefore$ $$\nabla^2 f = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right)+\frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right\} = \frac{\partial^2 f}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2}+\frac{\partial^2 f}{\partial z^2}.$$

Example 8.52. *Express the vector* $z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ *in cylindrical coordinates.* (V.T.U., 2010)

Solution. We have $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $z = z$.

so that $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \rho\cos\phi\mathbf{I} + \rho\sin\phi\mathbf{J} + z\mathbf{K}$$

If $\mathbf{T}_\rho$, $\mathbf{T}_\phi$, $\mathbf{T}_z$ be the unit vectors along the tangents to ρ, ϕ and z curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial\mathbf{R}/\partial\rho}{|\partial\mathbf{R}/\partial\rho|} = \frac{\cos\phi\mathbf{I}+\sin\phi\mathbf{J}}{\sqrt{(\cos^2\phi+\sin^2\phi)}} = \cos\phi\mathbf{I}+\sin\phi\mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-\rho\sin\phi\mathbf{I}+\rho\cos\phi\mathbf{J}}{\sqrt{[(-\rho\sin\phi)^2+(\rho\cos\phi)^2]}} = -\sin\phi\mathbf{I}+\cos\phi\mathbf{J}$$

$$\mathbf{T}_z = \frac{\partial\mathbf{R}/\partial z}{|\partial\mathbf{R}/\partial z|} = \mathbf{K}$$

Let the expression for $\mathbf{F} = z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ in cylindrical coordinates be

$$\mathbf{F} = f_1\mathbf{T}_\rho + f_2\mathbf{T}_\phi + f_3\mathbf{T}_z \qquad ...(i)$$

Then $$f_1 = \mathbf{F}\cdot\mathbf{T}_\rho = z\cos\phi - 2x\sin\phi$$
$$f_2 = \mathbf{F}\cdot\mathbf{T}_\phi = -z\sin\phi - 2x\cos\phi$$
$$f_3 = \mathbf{F}\cdot\mathbf{T}_z = y$$

Substituting the values of f_1, f_2, f_3 in (i), we get

$$\mathbf{F} = (z\cos\phi - 2x\sin\phi)\,\mathbf{T}_\rho - (z\sin\phi + 2x\cos\phi)\,\mathbf{T}_\phi + y\mathbf{T}_z$$
$$= (z\cos\phi - \rho\sin 2\phi)\,\mathbf{T}_\rho - (z\sin\phi + 2\rho\cos^2\phi)\,\mathbf{T}_\phi + \rho\sin\phi\,\mathbf{T}_z$$

Example 8.53. *Show that* $\nabla(\log\rho)$ *and* $\nabla\phi$, $\rho \neq 0$, $\phi \neq 0$ *are solenoidal vectors.*

Solution. (i) $f = \log\rho$ is a function of ρ only. We have to prove that $\nabla\cdot(\nabla f)$, *i.e.*, $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2}{\partial\rho^2}(\log\rho)+\frac{1}{\rho}\frac{\partial(\log\rho)}{\partial\rho}+0+0 = -\frac{1}{\rho^2}+\frac{1}{\rho^2} = 0$$

Hence $\nabla(\log\rho)$ is a solenoidal vector.

(ii) $f = \nabla\phi$ is a function of ϕ only. We have to show that $\nabla\cdot(\nabla f)$, *i.e.*, $\nabla^2 f = 0$.

$$\nabla^2 f = \frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2} = 0+0+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2}+0 = 0.$$

Hence the result.

8.21 (1) SPHERICAL POLAR COORDINATES

Let $P(x, y, z)$ be any point whose projection on the XY-plane is $Q(x, y)$. Then the spherical polar co-ordinates of P are (r, θ, ϕ) such that $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.

The level surfaces $r = r_0$, $\theta = \theta_0$, $\phi = \phi_0$ are respectively spheres about O, cones about the Z-axis with vertex at O and planes through the Z-axis.

The co-ordinate curves for r are rays from the origin; for θ, vertical circles with centre at O (called *meridians*); for ϕ, horizontal circles with centres on the Z-axis

From Fig. 8.29, we have

$$x = OQ\cos\phi = OP\cos(90° - \theta)\cos\phi = r\sin\theta\cos\phi,$$
$$y = OQ\sin\phi = r\sin\theta\sin\phi;\ z = r\cos\theta.$$

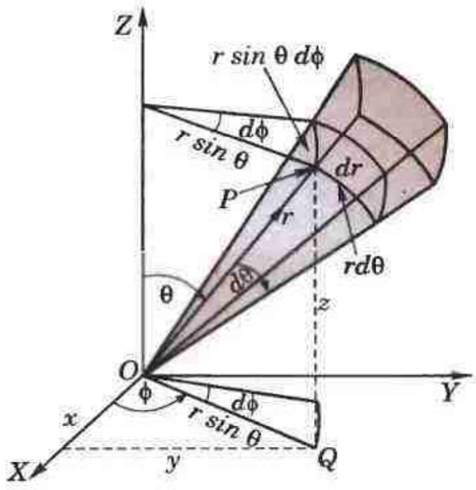

Fig. 8.29

(*i*) *Arc element*

$\therefore \quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2 (d\theta)^2 + (r \sin\theta)^2 (d\phi)^2$

so that the scale factors are

$$h_1 = 1, h_2 = r, h_3 = r \sin\theta.$$

(*ii*) *Area elements*

$$dS_r = r^2 \sin\theta\, d\theta d\phi,\ dS_\theta = r \sin\theta d\phi dr,\ dS_\phi = rdrd\theta$$

where dS_r is the area element perpendicular to the r-direction, etc.

(*iii*) *Volume element* $dV = r^2 \sin\theta\, drd\theta d\phi$.

(2) Spherical polar coordinate system is orthogonal

At any point P, we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

so that $\quad \mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}$

If $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors at P in the directions of the tangents to the r, θ, ϕ-curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{r\sqrt{(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

and $$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{r\sin\theta} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

Now $\quad \mathbf{T}_r \cdot \mathbf{T}_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta = 0$

$\mathbf{T}_\theta \cdot \mathbf{T}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$

$\mathbf{T}_\phi \cdot \mathbf{T}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0$

Also $\quad \mathbf{T}_r \times \mathbf{T}_\theta = \sin\theta\cos\phi\cos\theta\sin\phi\mathbf{K} + \sin^2\theta\cos\phi\mathbf{J} - \sin\theta\sin\phi\cos\theta\cos\phi\mathbf{K}$

$\qquad - \sin^2\theta\sin\phi\mathbf{I} + \cos^2\theta\cos\phi\mathbf{J} - \cos^2\theta\sin\phi\mathbf{I}$

$\qquad = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J} = \mathbf{T}_\phi$

$\mathbf{T}_\theta \times \mathbf{T}_\phi = \cos\theta\cos^2\phi\mathbf{K} + \sin^2\phi\cos\theta\mathbf{K} + \sin\theta\sin\phi\mathbf{J} + \sin\theta\cos\phi\mathbf{I} = \mathbf{T}_r$

and $\quad \mathbf{T}_\phi \times \mathbf{T}_r = -\sin\theta\sin^2\phi\mathbf{K} + \sin\phi\cos\theta\mathbf{J} - \sin\theta\cos^2\phi\mathbf{K} + \cos\phi\cos\theta\mathbf{I} = \mathbf{T}_\theta$

The above conditions satisfied by $\mathbf{T}_r, \mathbf{T}_\theta$, and $\mathbf{T}_\phi$ show that *the spherical polar coordinate system is a right handed orthogonal coordinate system.* (*V.T.U., 2008*)

(3) Del applied to functions in spherical polar coordinates

We have $u = r$, $v = \theta$, $w = \phi$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

Let $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors in the directions of the tangents to the r, θ, ϕ-curves.

(*i*) *Expression for grad f*

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}$$

$\therefore$ $$\nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi$$

(*ii*) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$$

$\therefore$ $$\nabla\cdot\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta f_1) + \frac{\partial}{\partial\theta}(r\sin\theta f_2) + \frac{\partial}{\partial\phi}(rf_3)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(f_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial\phi}$$

(iii) Expression for curl $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

$\therefore$ $$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_r}{r^2 \sin\theta} & \dfrac{\mathbf{T}_\theta}{r\sin\theta} & \dfrac{\mathbf{T}\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

$$= \frac{\mathbf{T}_r}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}(r\sin\theta f_3) - \frac{\partial}{\partial\phi}(rf_2)\right\} - \frac{\mathbf{T}_\theta}{r\sin\theta}\left\{\frac{\partial}{\partial r}(r\sin\theta f_3) - \frac{\partial f_1}{\partial\phi}\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

$$= \frac{\mathbf{T}_r}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(f_3\sin\theta) - \frac{\partial f_2}{\partial\phi}\right\} + \frac{\mathbf{T}_\theta}{r}\left\{\frac{1}{\sin\theta}\frac{\partial f_1}{\partial\phi} - \frac{\partial}{\partial r}(rf_3)\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

(iv) Expression for $\nabla^2 f$.

Since $$\nabla^2 f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right\}$$

$\therefore$ $$\nabla^2 f = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}.$$

Example 8.54. *Express the vector field* $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ *in spherical polar coordinate system.*

Solution. We have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

so that $$\mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}.$$

If $\mathbf{T}_r$, $\mathbf{T}_\theta$, $\mathbf{T}_\phi$ be the unit vectors along the tangents to r, θ, ϕ, curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{[(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta]}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{\sqrt{[(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2]}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{\sqrt{[(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}.$$

Let the expression for $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ in spherical polar coordinates be

$$\mathbf{F} = f_1\mathbf{T}_r + f_2\mathbf{T}_\theta + f_3\mathbf{T}_\phi \qquad ...(i)$$

Then $f_1 = \mathbf{F}\,.\,\mathbf{T}_r = (2r\sin\theta\sin\phi\mathbf{I} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K})\,.\,(\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K})$

$= 2r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$

$$f_2 = \mathbf{F} \cdot \mathbf{T}_\theta = (2r \sin\theta \sin\phi \mathbf{I} - r\cos\theta \mathbf{J} + 3r \sin\theta \cos\phi \mathbf{K}) \cdot (\cos\theta\cos\phi \mathbf{I} + \cos\theta \sin\phi \mathbf{J} - \sin\theta \mathbf{K})$$
$$= 2r \sin\theta\cos\theta \sin\phi\cos\phi - r\cos^2\theta \sin\phi - 3r\sin^2\theta\cos\phi.$$

and
$$f_3 = \mathbf{F} \cdot \mathbf{T}_\phi = (2r \sin\theta \sin\phi \mathbf{K} - r\cos\theta \mathbf{J} + 3r\sin\theta\cos\phi \mathbf{K}) \cdot (-\sin\phi \mathbf{I} + \cos\phi \mathbf{J})$$
$$= -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

Substituting the values of f_1, f_2, f_3 in (*i*), we get the desired expression.

Example 8.55. *Prove that* $\nabla(\cos\theta) \times \nabla\phi = \nabla(1/r)$. $r \neq 0$.

Solution. In spherical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{T}_\phi$$

$$\therefore \quad \nabla(\cos\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta)\mathbf{T}_\theta = -\frac{1}{r}\sin\theta\,\mathbf{T}_\theta \qquad ...(i)$$

$$\nabla\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi)\mathbf{T}_\phi = \frac{1}{r\sin\theta}\mathbf{T}_\phi \qquad ...(ii)$$

and
$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}(r^{-1})\mathbf{T}_r = -\frac{1}{r^2}\mathbf{T}_r$$

Now from (*i*) and (*ii*), we get

$$\nabla(\cos\theta) \times \nabla\phi = -\frac{1}{r^2}\mathbf{T}_\theta \times \mathbf{T}_\phi = -\frac{1}{r^2}\mathbf{T}_r = \nabla\left(\frac{1}{r}\right).$$

Example 8.56. *If* $F = r^2\cos\theta\,\mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi$, *find the value of* $\mathbf{F} \times$ *curl* $\mathbf{F}$.

Solution. In spherical coordinates,

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{T}_r / r^2\sin\theta & \mathbf{T}_\theta / r\sin\theta & \mathbf{T}_\phi / r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

Here $f_1 = r^2\cos\theta$, $f_2 = -1/r$, $f_3 = 1/r\sin\theta$.

$$\therefore \quad \text{curl } \mathbf{F} = \frac{2}{r^2\sin\theta}\begin{vmatrix} \mathbf{T}_r & r\mathbf{T}_\theta & r\sin\theta\,\mathbf{T}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r^2\cos\theta & -1 & 1 \end{vmatrix} = r\sin\theta\mathbf{T}_\phi$$

$$\therefore \quad \mathbf{F} \times \text{curl } \mathbf{F} = \left(r^2\cos\theta\mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi\right) \times (r\sin\theta\mathbf{T}_\phi) = -(r^3\sin\theta\cos\theta\mathbf{T}_\theta + \sin\theta\mathbf{T}_r).$$

PROBLEMS 8.12

1. Express the following vectors in cylindrical coordinates
   (*i*) $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ (*ii*) $2x\mathbf{I} - 3y^2\mathbf{J} + zx\mathbf{K}$ (*V.T.U.*, *2009*)
2. Express the following vectors in spherical polar coordinates
   (*i*) $x\mathbf{I} + 2y\mathbf{J} + yz\mathbf{K}$ (*ii*) $xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K}$
3. Evaluate $\nabla\phi = xyz$ in cylindrical coordinates.
4. Show that $\nabla(r/\sin\theta) \times \nabla\theta = \nabla\phi$.
5. Prove that $\mathbf{V} = \frac{\cos\theta}{r^3}(\mathbf{T}_r/\sin\theta - \mathbf{T}_\theta/\cos\theta + r^4\mathbf{T}_\phi)$ is solenoidal.
6. Show that (*i*) $\nabla^2(\log r) = 1/r^2$ (*ii*) $\nabla \times [(\cos\theta)(\nabla\phi)] = \nabla(1/r)$.

7. Prove that $\mathbf{V} = \rho z \sin 2\phi \left[\mathbf{T}_\rho + \cot 2\phi \mathbf{T}_\phi + \frac{\rho}{2z}\mathbf{T}_z\right]$ is irrotational.
8. If u, v, w are orthogonal curvilinear coordinates with h_1, h_2, h_3 as scale factors, prove that
$$\left[\frac{\partial \mathbf{R}}{\partial u}, \frac{\partial \mathbf{R}}{\partial v}, \frac{\partial \mathbf{R}}{\partial w}\right] = \frac{1}{[\nabla u, \nabla v, \nabla w]} = h_1 h_2 h_3.$$

8.22 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 8.13

Fill up the blanks or choose the correct answer from the following problems :

1. A unit tangent vector to the surface $x = t, y = t^2, z = t^3$ at $t = 1$ is ....
2. The equation of the normal to the surface $2x^2 + y^2 + 2z = 3$ at $(2, 1, -3)$ is ....
3. If $u = u(x, y)$ and $v = v(x, y)$, then the area-element $dudv$ is related to the area-element $dxdy$ by the relation ....
4. If $A = 2x^2\mathbf{I} - 3yz\mathbf{J} + xz^2\,\mathbf{K}$, then $\nabla.\,\mathbf{A}$ = ...
5. div curl $\mathbf{F}$ = ...
6. Area bounded by a simple closed curve C is ....
7. If S is a closed surface enclosing a volume V and if $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then
$$\int_S \mathbf{R}\,.\,\mathbf{N}\,ds = ...$$
8. div $\mathbf{R}$ = ...; curl $\mathbf{R}$ = ...
9. If $\mathbf{A}$ is such that $\nabla \times \mathbf{A} = 0$, then $\mathbf{A}$ is called ....
10. If $\nabla\,.\,\mathbf{F} = 3$, then $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds$ where S is a surface of a unit sphere, is ....
11. If $\nabla\,.\,\mathbf{F} = 0$, then $\mathbf{F}$ is called.......
12. The directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction PQ where $P = (1, 2, -1)$ and $Q = (-1, 2, 3)$ is ....
13. If $u = x^2yz$, $v = xy - 3z^2$, then $\nabla\,.\,(\nabla u \times \nabla v)$ =.......
14. curl $(xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K})$ = ...
15. If $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$, then $\nabla\,.\,\mathbf{F}$ = ...
16. If $\mathbf{F}$ is a conservative force field then curl $\mathbf{F}$ is ....
17. If $\phi = 3x^2y - y^3z^2$, grad ϕ at the point $(1, -2, -1)$ is ....
18. curl $(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$ = ...
19. Workdone by a particle along the square formed by the lines $y = \pm 1$ and $x = \pm 1$ under the force
$\mathbf{F} = (x^2 + xy)\mathbf{I} + (x^2 + y^2)\mathbf{J}$ is.......
20. Curl (grad ϕ) = ...
21. If $\mathbf{A}$ is a constant vector, then *div* $(\mathbf{A} \times \mathbf{R})$ = ...
22. If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla \log r$ = ... ; $\nabla(r^n)$ = ...
23. A level surface is defined as ....
24. Unit normal vector to the surface $z = 2xy$ at the point $(2, 1, 4)$ is ....
25. If the directional derivative of $f = ax + by + cz$ at $(1, 1, 1)$ has maximum magnitude 4 in direction parallel to x-axis, then the values of a, b, c are ....
26. Maximum value of the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ is ....
27. If $r^2 = x^2 + y^2 + z^2$, then $\nabla\,.\,(\mathbf{R}/r)$ = ...
28. Directional derivative of $f = xyz$ at the point $(1, -1, -2)$ in the direction of the vector $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ is ....
29. If $\mathbf{V} = x^2\mathbf{I} + xye^x\mathbf{J} + \sin z\mathbf{K}$, then $\nabla\,.\,(\nabla \times F)$ = ...
30. If $f = \tan^{-1}(y/x)$ then div (grad f) is equal to
(*a*) 1 (*b*) -1 (*c*) 0 (*d*) 2.
31. The value of curl (grad f), where $f = 2x^2 - 3y^2 + 4z^2$ is
(*a*) $4x - 6y + 8z$, (*b*) $4x\mathbf{I} - 6y\mathbf{J} + 8z\mathbf{K}$ (*c*) 0 (*d*) 3.

32. The value of $\int \text{grad}\,(x + y - z)\, d\mathbf{R}$ from (0, 1, –1) to (1, 2, 0) is
(*a*) 0 (*b*) 3 (*c*) – 1 (*d*) not obtainable.

33. If $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, then $\int_S \mathbf{F} \cdot d\mathbf{S}$, S being the surface of a unit sphere, is
(*a*) $(4/3)\, \pi\, (a + b + c)^2$ (*b*) 0 (*c*) $4\pi/3\, (a + b + c)$ (*d*) none of these.

34. A necessary and sufficient condition that the line integral $\int_L \mathbf{F} \cdot d\mathbf{R}$ for every closed C vanishes, is
(*a*) curl $\mathbf{F} = 0$ (*b*) div $\mathbf{F} = 0$ (*c*) curl $\mathbf{F} \neq 0$ (*d*) div $\mathbf{F} \neq 0$.

35. The value of $\iint_S (yzdydz + zxdz\, dx + xy\, dxdy)$, where S is the surface of unit sphere $x^2 + y^2 + z^2 = 1$ is
(*a*) 0 (*b*) 4π (*c*) $4\pi/3$ (*d*) 10π.

36. If $u = x^2 + y^2 + z^2$ and $\mathbf{V} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla\,(u\mathbf{V})$ = ...

37. For any scalar function ψ, $\nabla \times \nabla\, \psi$ = ...

38. $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of the path joining any two points if and only if it is ....

39. The value of the line integral $\int_C (y^2 dx + x^2 dy)$ where C is the boundary of the square $-1 \leq x \leq 1, -1 \leq y \leq 1$ is
(*a*) 0 (*b*) $2(x + y)$ (*c*) 4 (*d*) 4/3. (*V.T.U., 2010*)

40. If $\mathbf{V}$ is the instantaneous velocity vector of the moving fluid at a point P, then div $\mathbf{V}$ represents ....

41. The spherical coordinate system is
(*a*) Orthogonal (*b*) Coplanar (*c*) Non-coplanar (*d*) Not orthogonal. (*V.T.U., 2010*)

42. Physical interpretation of $\nabla\, \phi$ is that ....

43. The magnitude of the vector drawn perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point (1, – 1, 2) is
(*a*) 2/3 (*b*) 3/2 (*c*) 3 (*d*) 6.

44. The value of λ so that the vector $(x + 3y)\mathbf{I} + (y - 2z)\mathbf{J} + (x + \lambda z)\mathbf{K}$ is a solenoidal vector, is
(*a*) – 2 (*b*) 3 (*c*) 1 (*d*) none of these.

45. The work done by the force $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$, in moving a particle from the point (1, 1, 1) to the point (3, 3, 2) along the path c is
(*a*) 17 (*b*) 10 (*c*) 0 (*d*) cannot be found.

46. Value of $\int_c (y^2 dx + x^2 dy)$ where c is the boundary of the square $-1 \leq x \leq 1, -1 \leq y \leq 1$, is
(*a*) 4 (*b*) 0 (*c*) $2(x + y)$ (*d*) 4/3.

47. The directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at (1, 1) is zero along a ray making an angle with the positive direction of x-axis :
(*a*) 45° (*b*) 60° (*c*) 135° (*d*) none of these.

48. The vector $\mathbf{V} = e^x \sin y\mathbf{I} + e^x \cos y\mathbf{J}$, is
(*a*) solenoidal (*b*) irrotational (*c*) rotational.

49. If $u = 1/r$ where $r^2 = x^2 + y^2$, then $\nabla^2 u = 0$. (True or False)

50. $\mathbf{F} = (x + 3y)\,\mathbf{I} + (z - 3y)\mathbf{J} + (x + 2z)\mathbf{K}$ is a solenoidal vector function. (True or False)

51. $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$ is irrotational. (True or False)