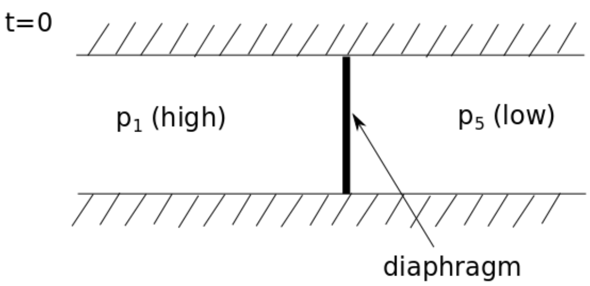
4. Sod’s Test Problems: The Shock Tube Problem

This set of problems was introduced in the paper by Gary Sod in 1978 called “A Survey of Several Finite Difference Methods for Systems of Non-linear Hyperbolic Conservation Laws”

4.1. Assumptions

- 1D
- Infinitely long tube
- Inviscid fluid

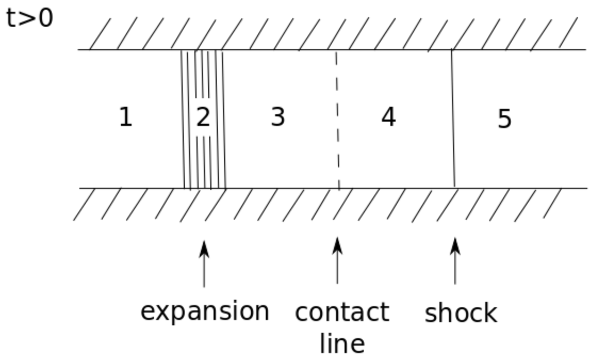
4.2. Initial Conditions



- At t=0 the diaphragm is instantaneously removed (this is done experimentally using a a thin sheet of metal and a small explosion bursts the diaphragm)

4.3. Regions of Flow

- The bursting of the diaphragm causes a 1D unsteady flow consisting of a steadily moving **shock** - A Riemann Problem.
- 1 discontinuity is present
- The solution is self-similar with 5 regions



- Region 1 & 5 - left and right sides of initial states
- Region 2 - expansion or rarefaction wave (x-dependent state)
- Regions 3 & 4 - steady states independent of x within the region (uniform)

Contact line between 3 and 4 separates fluids of different entropy (but they have the same pressure and velocity) i.e. it’s an invisible line - e.g. two fluids one side with water and the other with dye - contact line is moving.

$$p_3 = p_4$$
$$u_3 = u_4$$

4.4. Sod’s Test Number 1

Unknowns:

- Pressure
- Velocity
- Speed of sound
- Density
- Entropy
- Mach Number

Can also use Euler Equations in Primitive Form with:

- Pressure
- Velocity
- Density

Vector notation for the Euler Equations with Primitive Variables, p, u, ρ

4.4.1. Initial Conditions

$$\mathbf{V}(x,0) = \begin{cases} \mathbf{V}_L & x < 0 \\ \mathbf{V}_R & x \geq 0 \end{cases}$$

$$\mathbf{V}_L = \begin{bmatrix} \rho_L \\ u_L \\ p_L \end{bmatrix} = \begin{bmatrix} 1kg/m^3 \\ 0m/s \\ 100kN/m^2 \end{bmatrix}$$

$$\mathbf{V}_R = \begin{bmatrix} \rho_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} 0.125kg/m^3 \\ 0m/s \\ 10kN/m^2 \end{bmatrix}$$

- Everything is quiet until you break the diaphragm (u=0)
- The pressure ratio is 10

4.4.2. Discretisation

- N = 50 points in [-10m, 10m]
- Δx = 20m / 50 = 0.4m
- Initial CFL = 0.3
- Initial wave speed = 374.17m/s
- Timestep Δt = 0.4(0.4/374.17) = 4.276 $\times 10^{-4}$

$\Delta t / \Delta x$ = 1.069 $\times 10^{-3}$

Solution at t = 0.01s (in about 23 timesteps)

Now the problem is described, the numerical schemes can be applied.

4.5. Sod’s Test Number 2

Unknowns are same as Test Number 1

4.5.1. Initial Conditions

$$\mathbf{V}_L = \begin{bmatrix} \rho_L \\ u_L \\ p_L \end{bmatrix} = \begin{bmatrix} 1kg/m^3 \\ 0m/s \\ 100kN/m^2 \end{bmatrix}$$

$$\mathbf{V}_R = \begin{bmatrix} \rho_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} 0.01kg/m^3 \\ 0m/s \\ 1kN/m^2 \end{bmatrix}$$

Pressure ratio is 100 - this test is harder

4.5.2. Discretisation

N = 50 points in [-10m, 15m]
Δx = 25m / 50 = 0.5m
Initial CFL = 0.3
Initial wave speed = 374.17m/s
Timestep Δt = 0.3(0.5/374.17) = 4.01 $\times 10^{-4}$
$\Delta t / \Delta x$ = 8.02 $\times 10^{-4}$

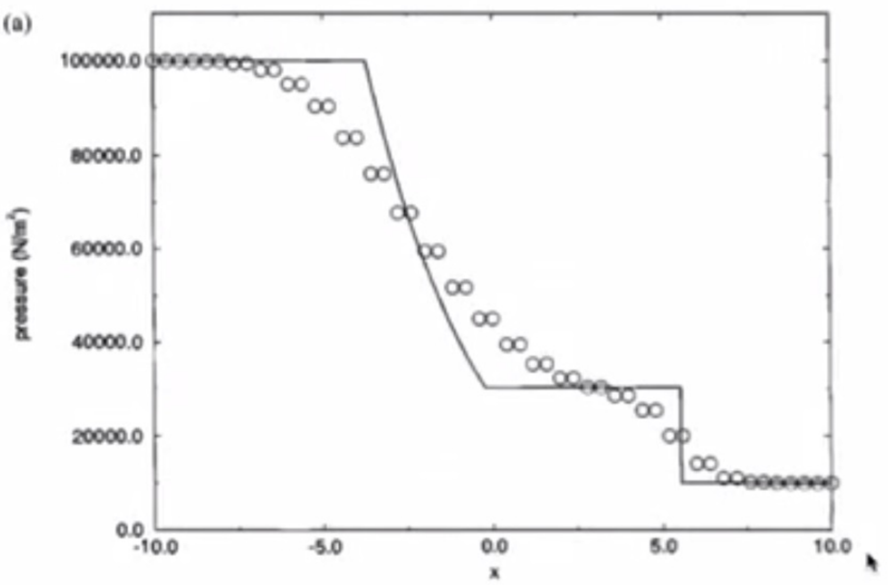
Solution at t = 0.01s (in about 25 timesteps)

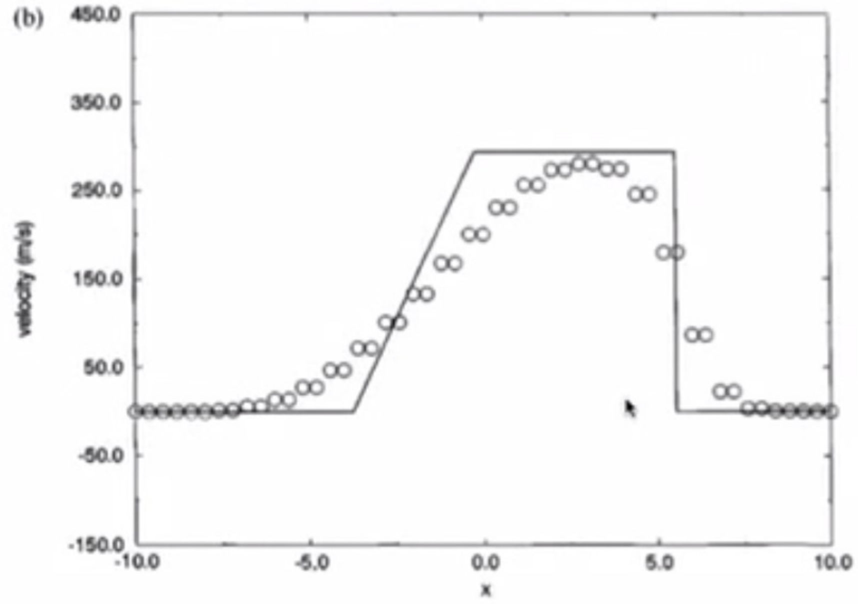
Now the problem is described, the numerical schemes can be applied.

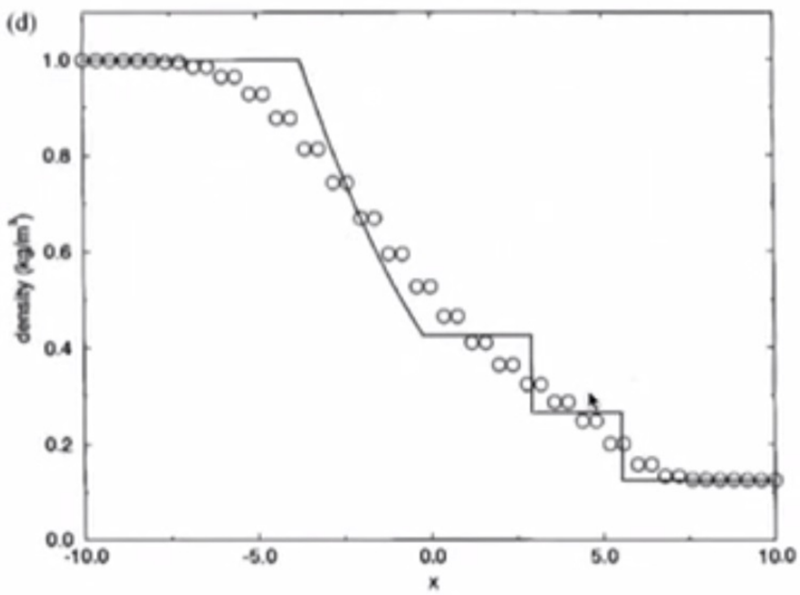
4.6. Test 1

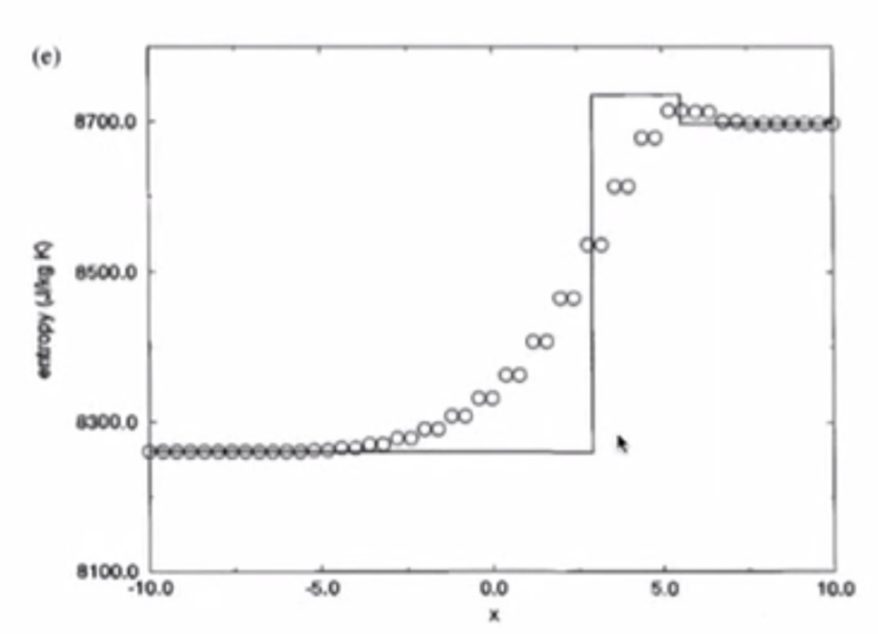
4.6.1. Lax-Friedrichs

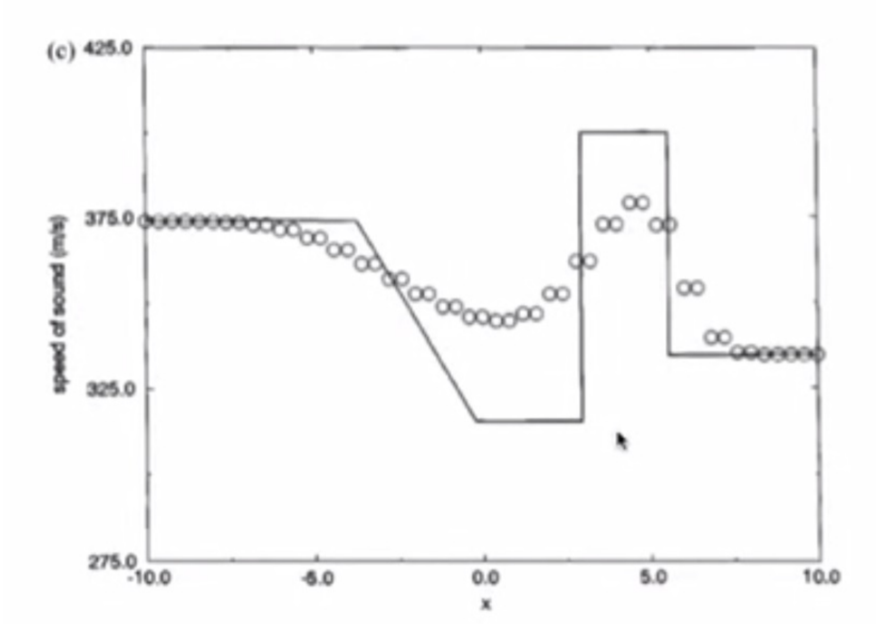
Pressure has a jump due to shockwave
Solution has numerical dissipation
Odd-even decoupling is present (staircase pattern)
Burgers Equation simulated all the important features of the Euler Equations

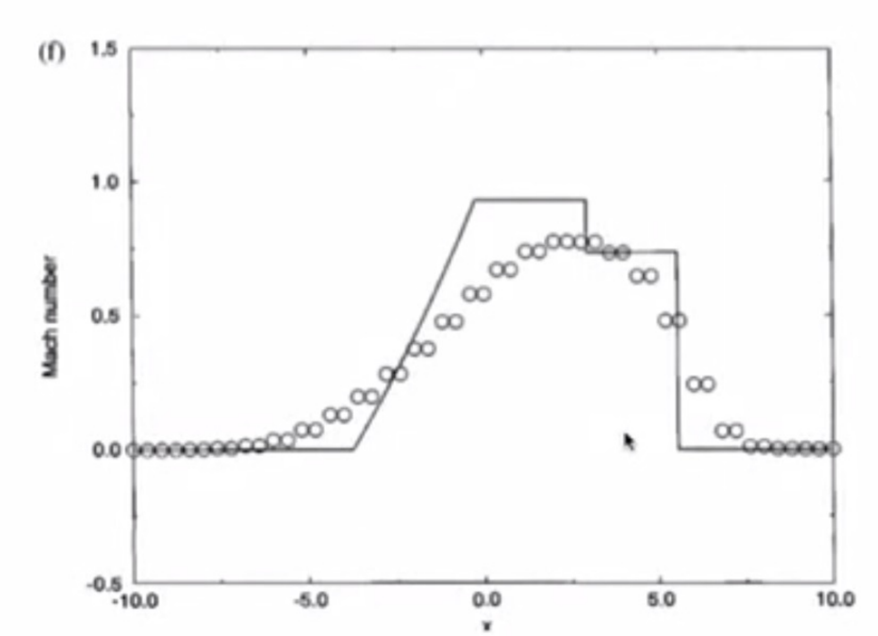

(a)
100000.0
80000.0
60000.0
40000.0
20000.0
0.0
pressure (N/m²)
-10.0
-5.0
0.0
5.0
10.0
x


(b)
450.0
350.0
250.0
150.0
50.0
-50.0
-150.0
velocity (m/s)
-10.0
-5.0
0.0
5.0
10.0
x


(d)
1.0
0.8
0.6
0.4
0.2
0.0
density (kg/m³)
-10.0
-5.0
0.0
5.0
10.0
x

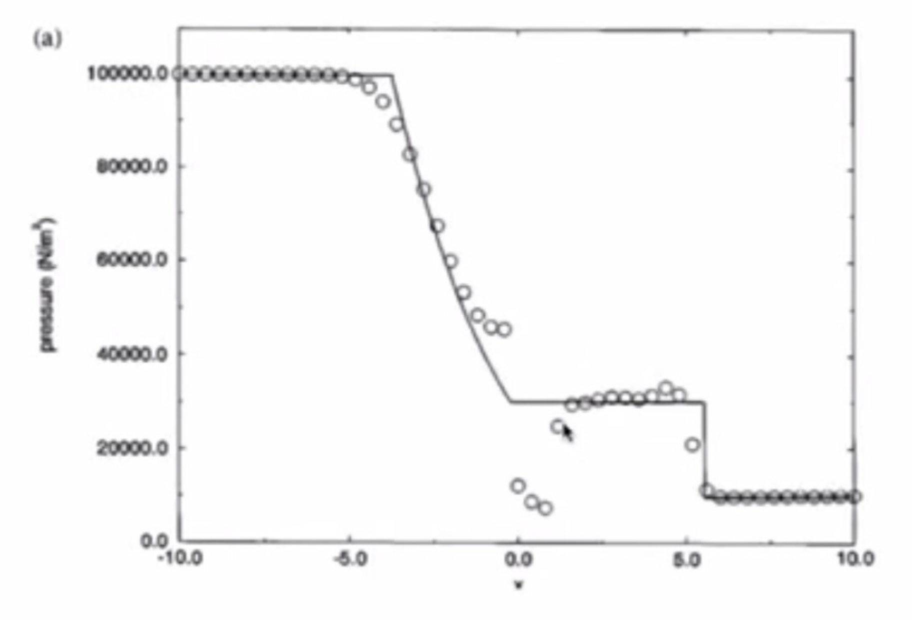


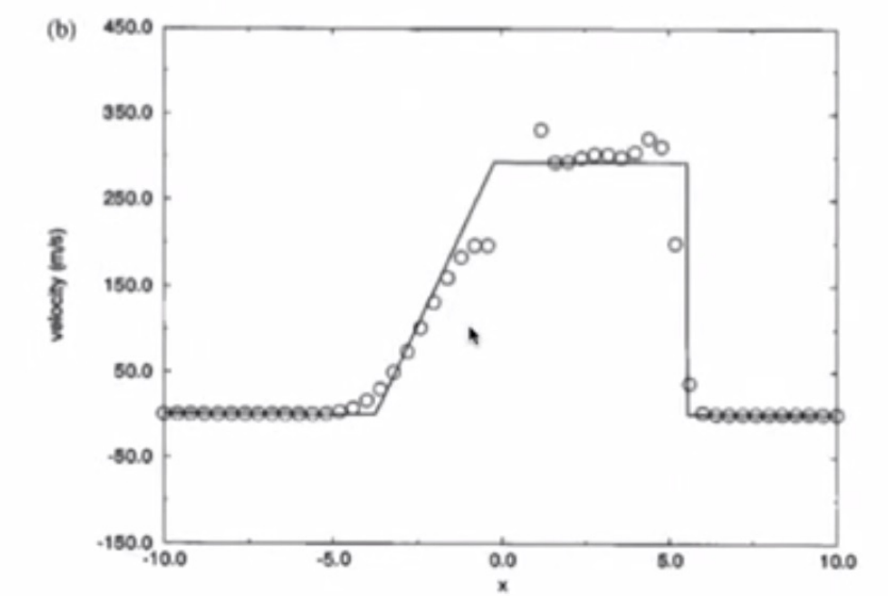


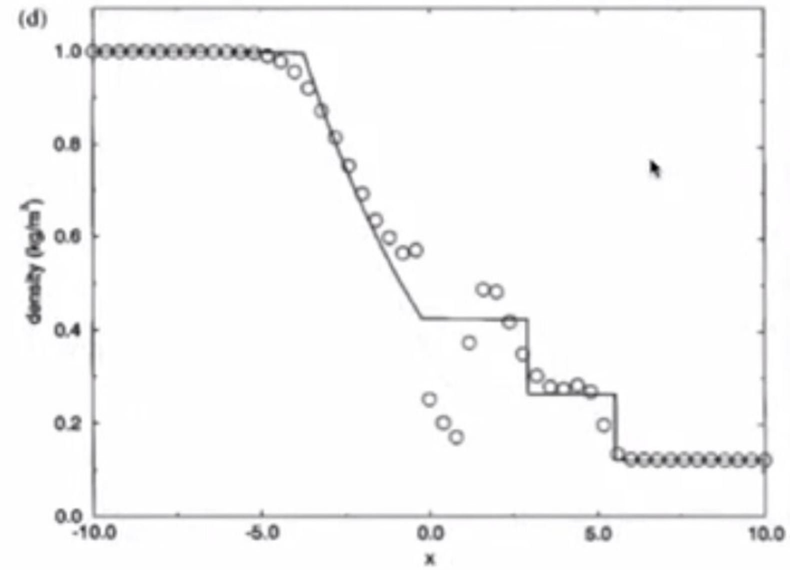


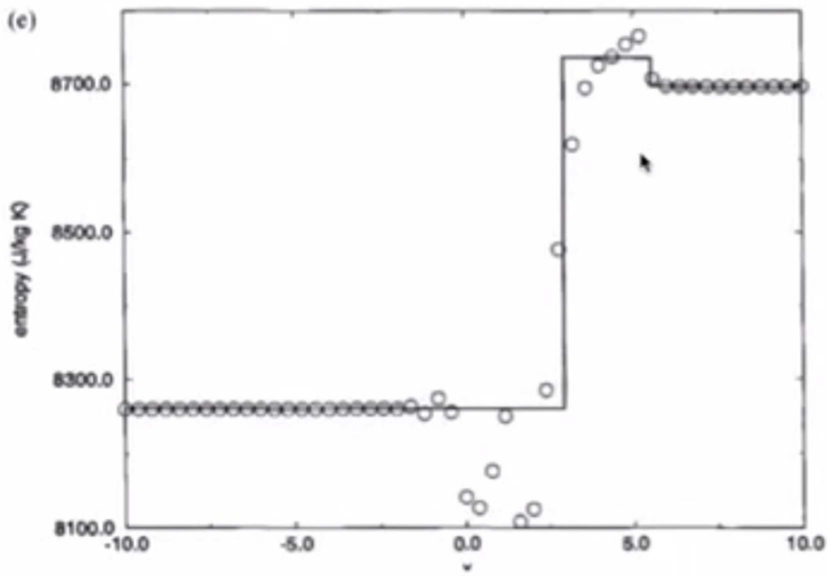
4.6.2. MacCormack

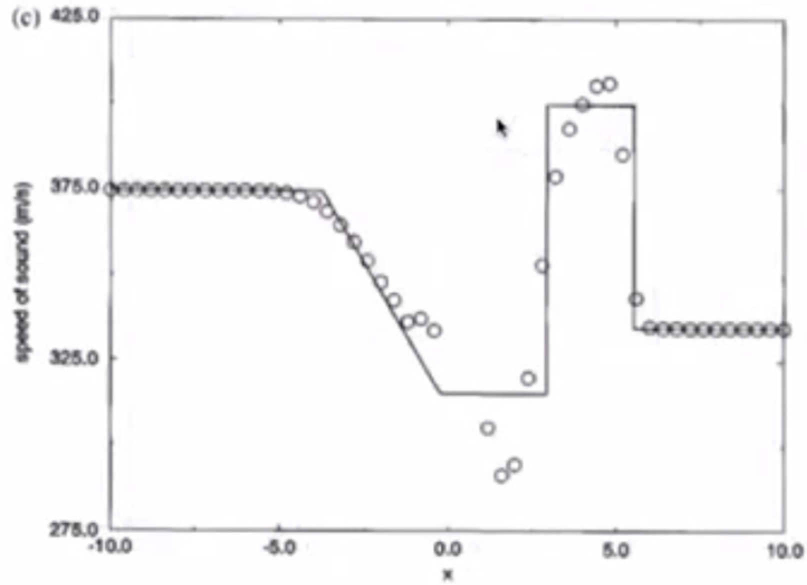
Similar to inviscid Burgers
Overshoot in pressure, speed of sound, density, entropy is bad
Lax-Friedrichs is better than MacCormack

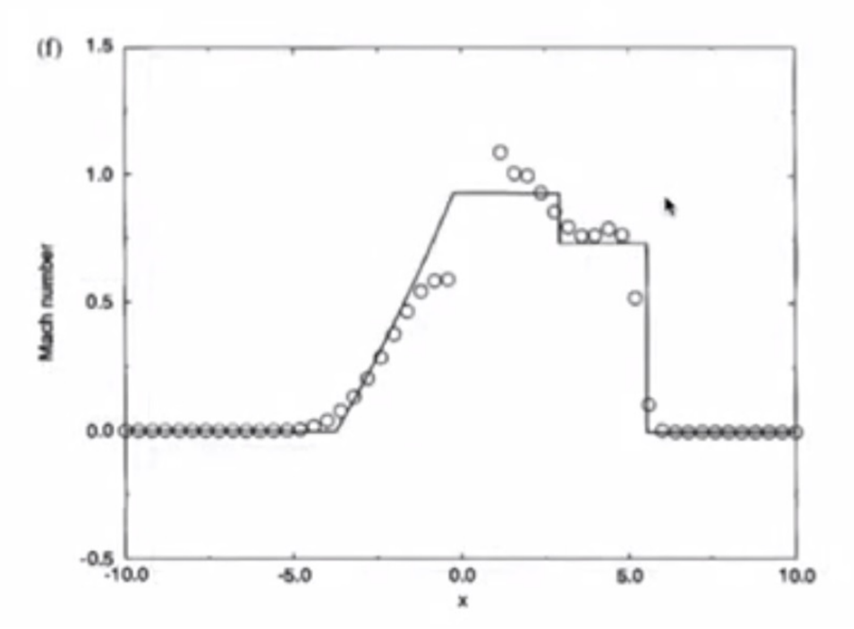





(d)
density (kg/m³)
1.0
0.8
0.6
0.4
0.2
0.0
-10.0
-5.0
0.0
5.0
10.0
x

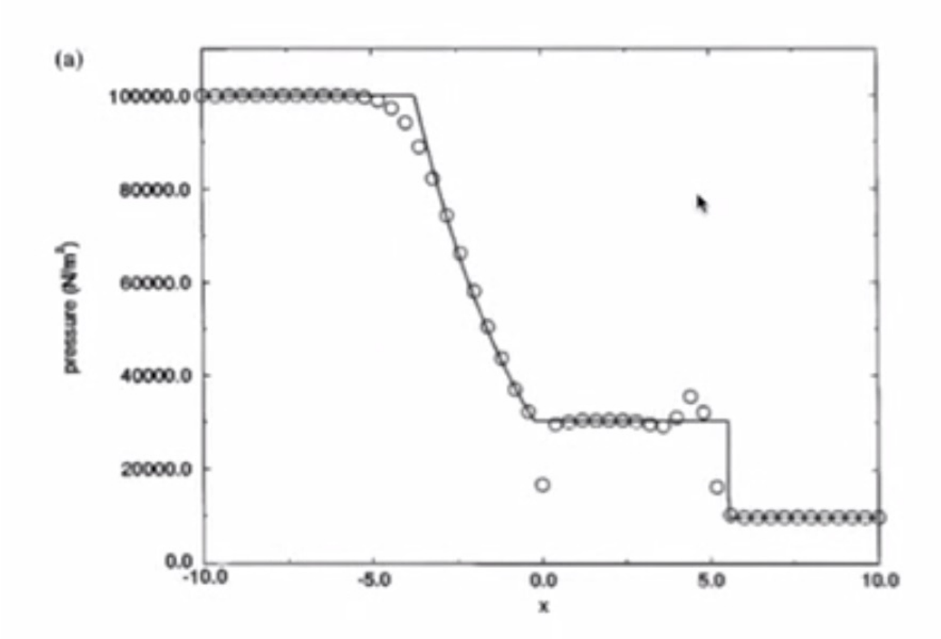

(e)
entropy (J/kg K)
8700.0
8500.0
8300.0
8100.0
-10.0
-5.0
0.0
5.0
10.0

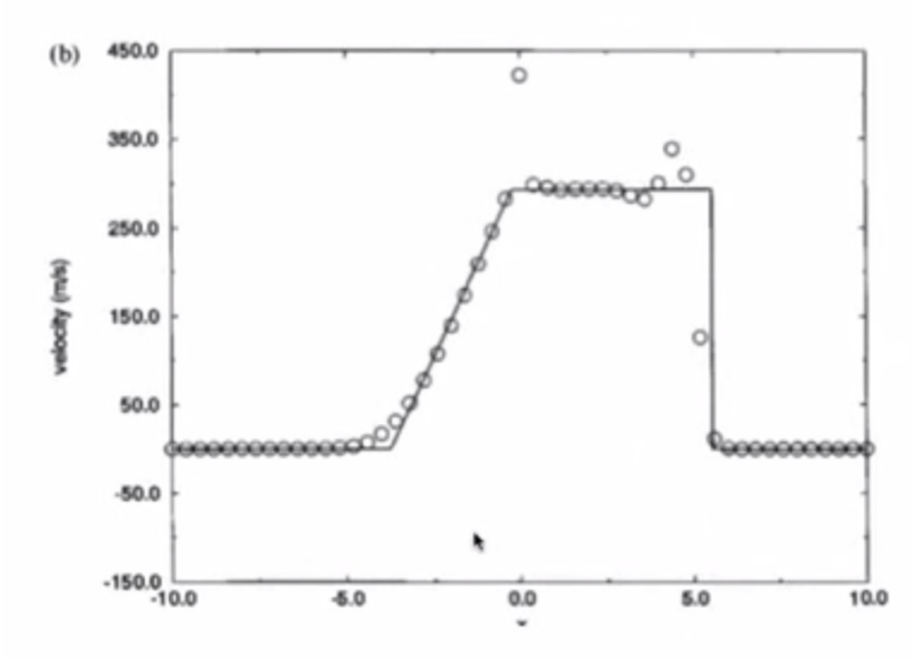

(c)
speed of sound (m/s)
425.0
375.0
325.0
275.0
-10.0
-5.0
0.0
5.0
10.0
x

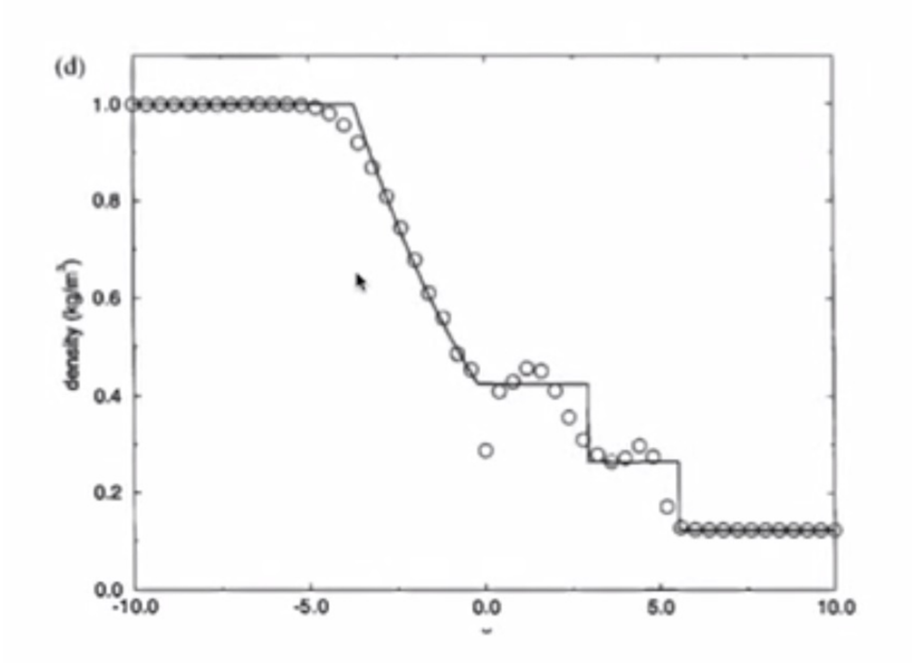


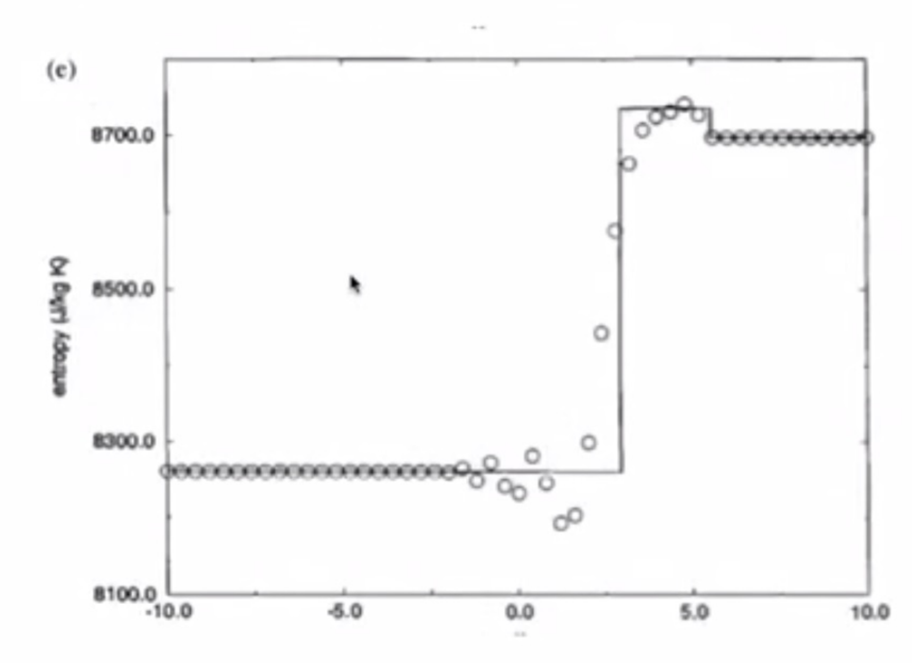
4.6.3. Richtmyer

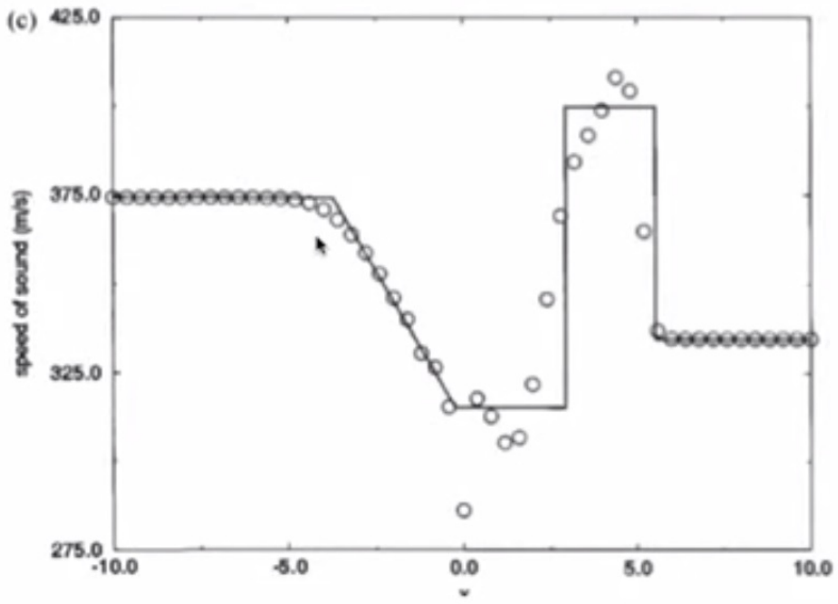
Less overshooting than MacCormack
Undershoot in pressure is bad
Overshoot in velocity is bad

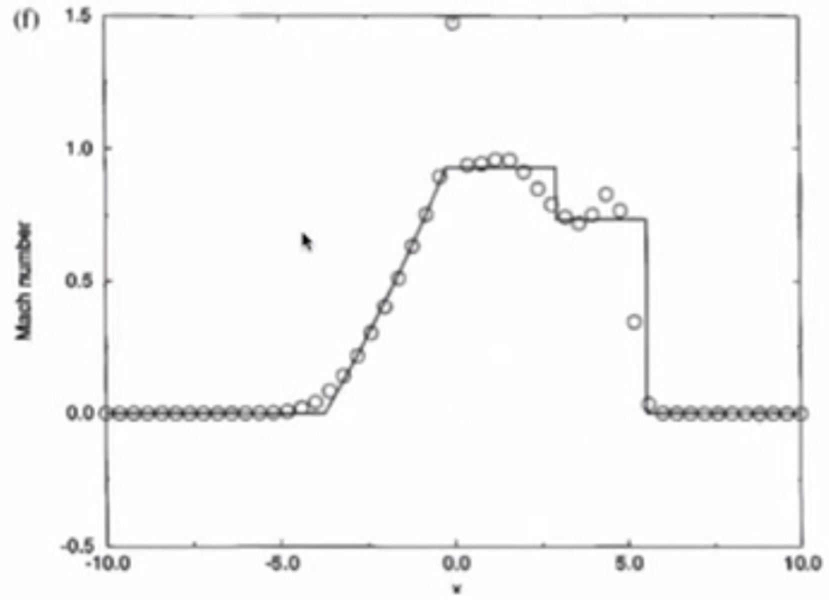



(b)
velocity (m/s)
450.0
350.0
250.0
150.0
50.0
-50.0
-150.0
-10.0
-5.0
0.0
5.0
10.0


(d)
density (kg/m³)
1.0
0.8
0.6
0.4
0.2
0.0
-10.0
-5.0
0.0
5.0
10.0


(e)
entropy (J/kg K)
8700.0
8500.0
8300.0
8100.0
-10.0
-5.0
0.0
5.0
10.0

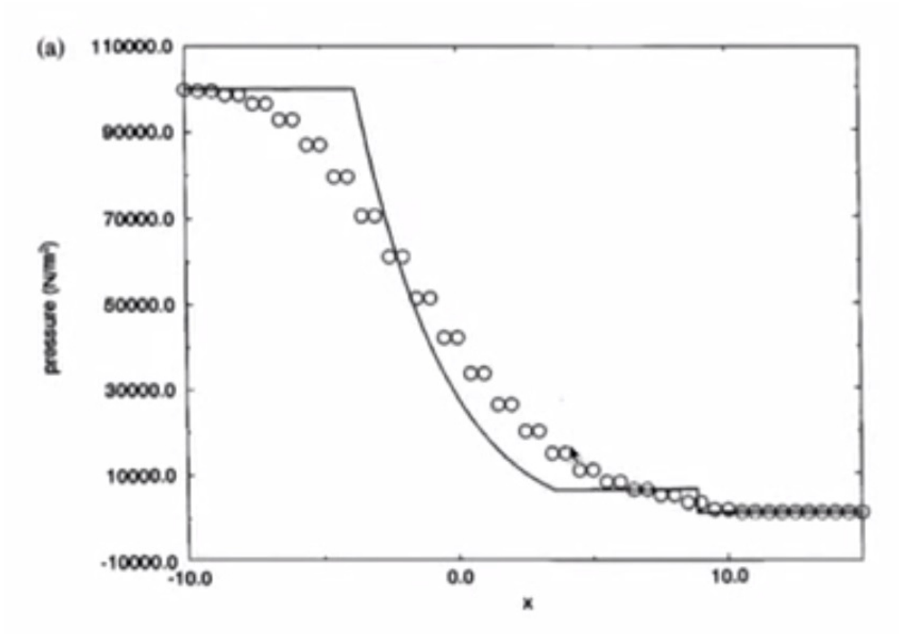


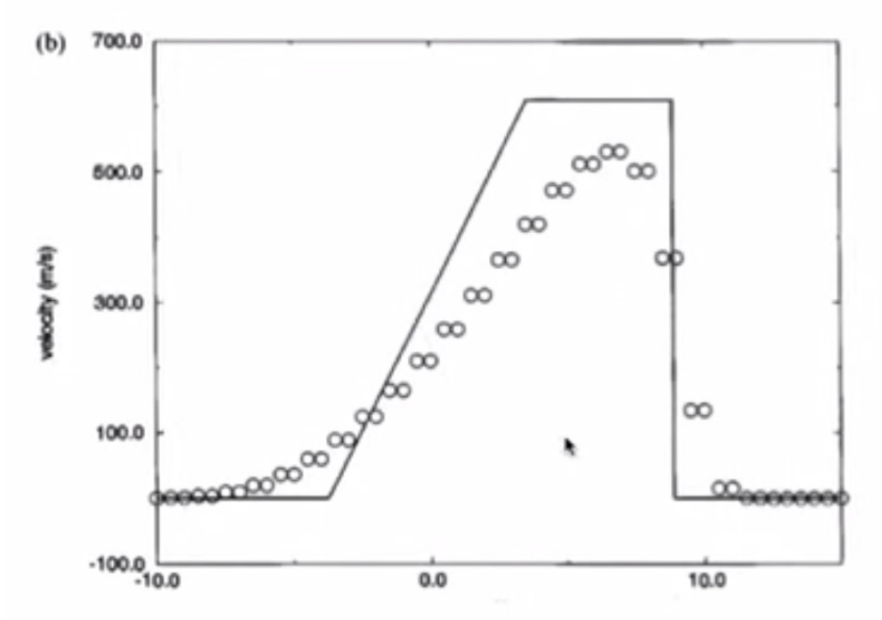


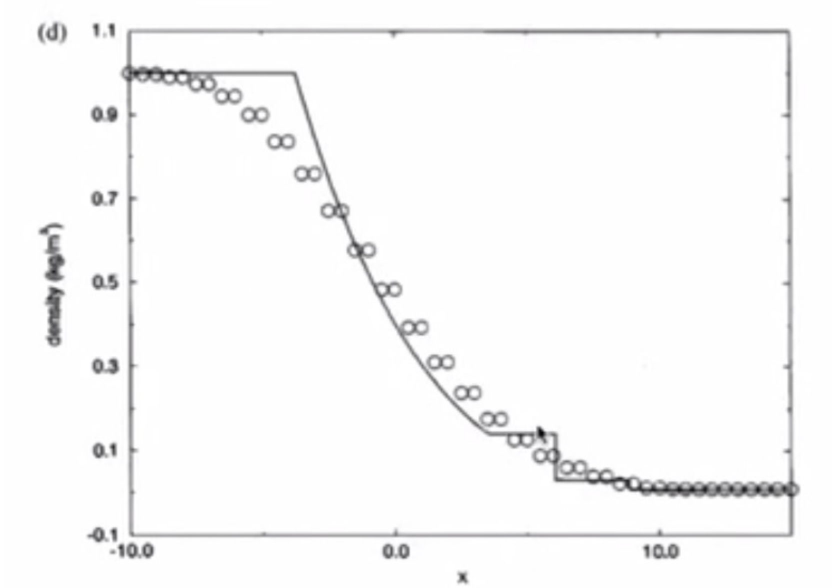
4.7. Test 2

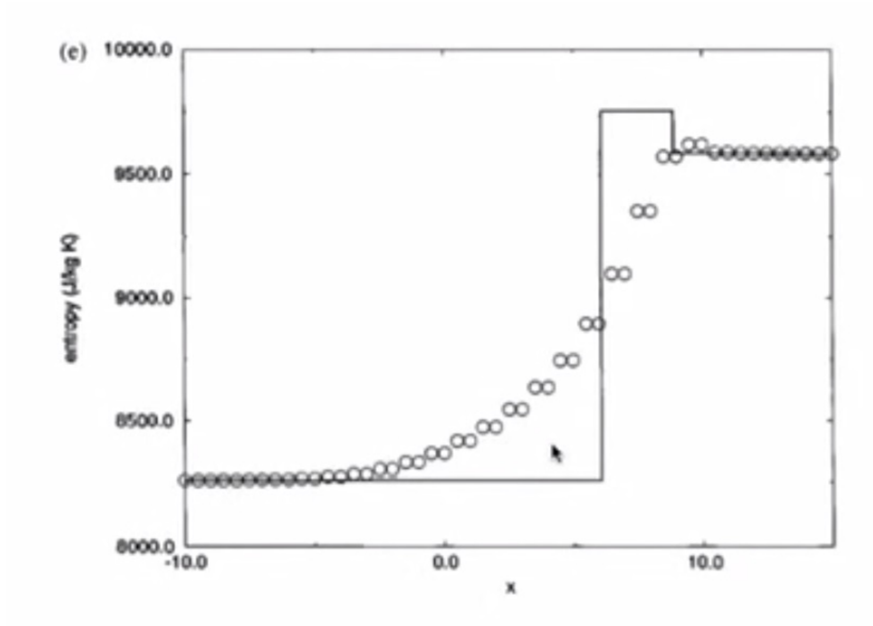
4.7.1. Lax-Friedrichs

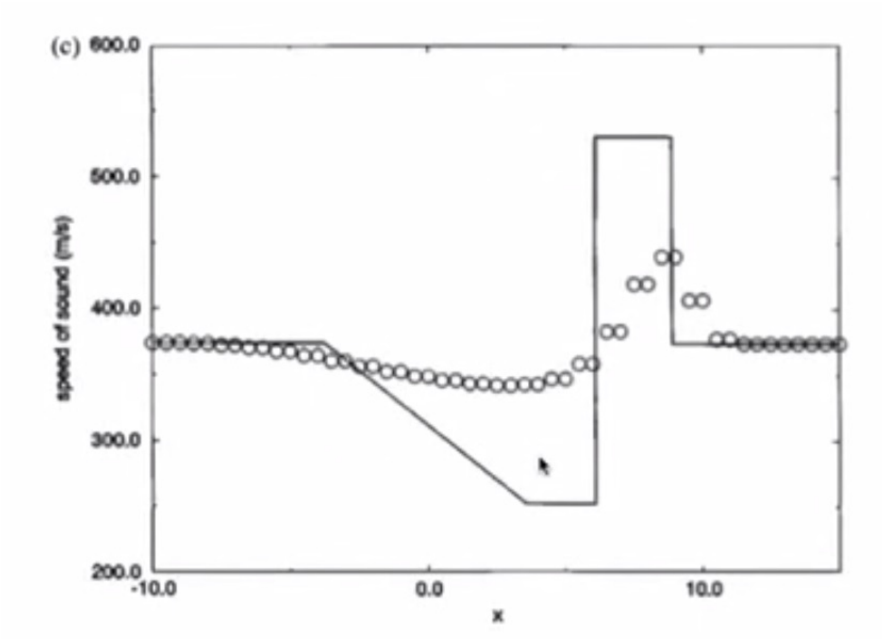
Diffusion
Odd-even decoupling
Speed of sound very diffused

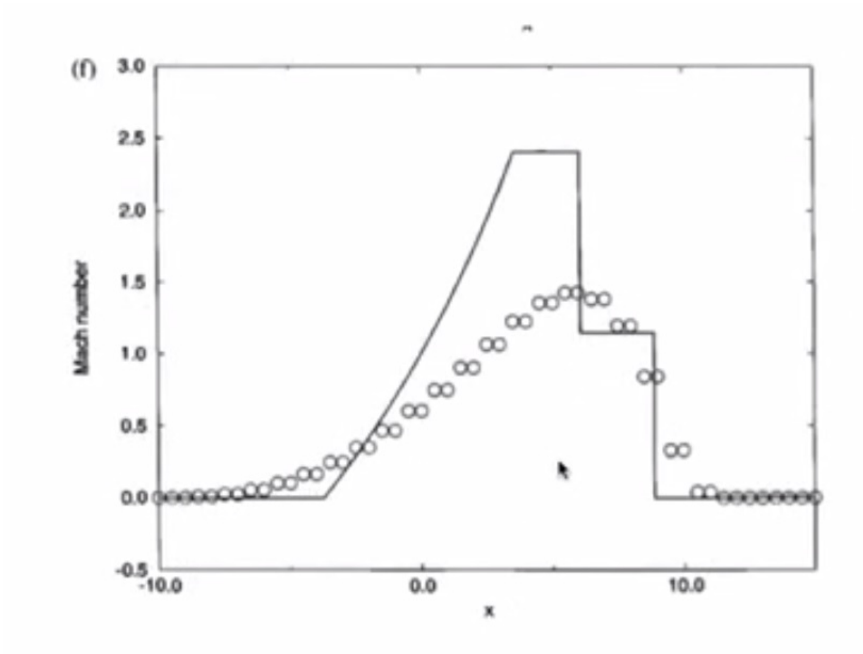

(a)
110000.0
90000.0
70000.0
50000.0
30000.0
10000.0
-10000.0
pressure (N/m²)
-10.0
0.0
10.0
x


(b)
700.0
500.0
300.0
100.0
-100.0
velocity (m/s)
-10.0
0.0
10.0


(d)
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
density (kg/m³)
-10.0
0.0
10.0
x

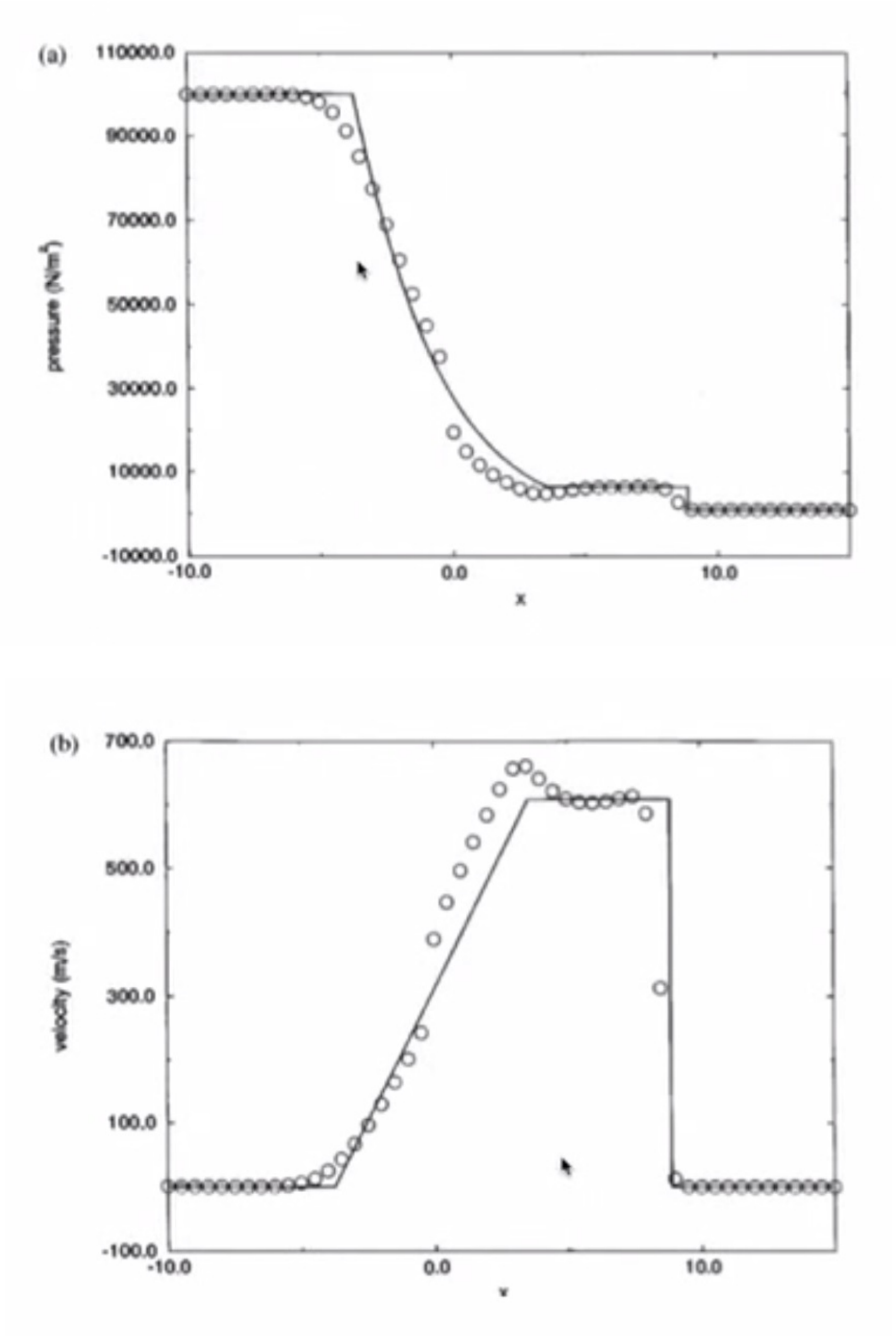

(e)
entropy (J/kg K)
x
10000.0
9500.0
9000.0
8500.0
8000.0
-10.0
0.0
10.0

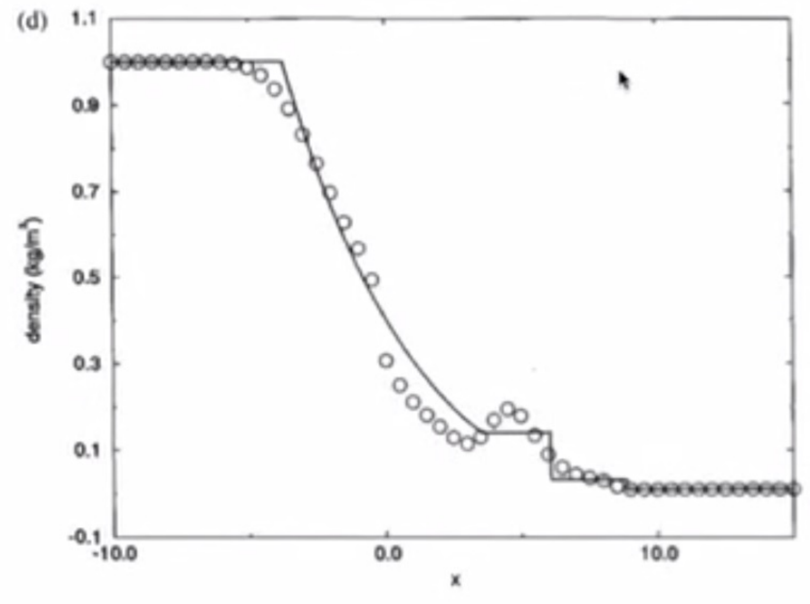

(c)
speed of sound (m/s)
x
600.0
500.0
400.0
300.0
200.0
-10.0
0.0
10.0

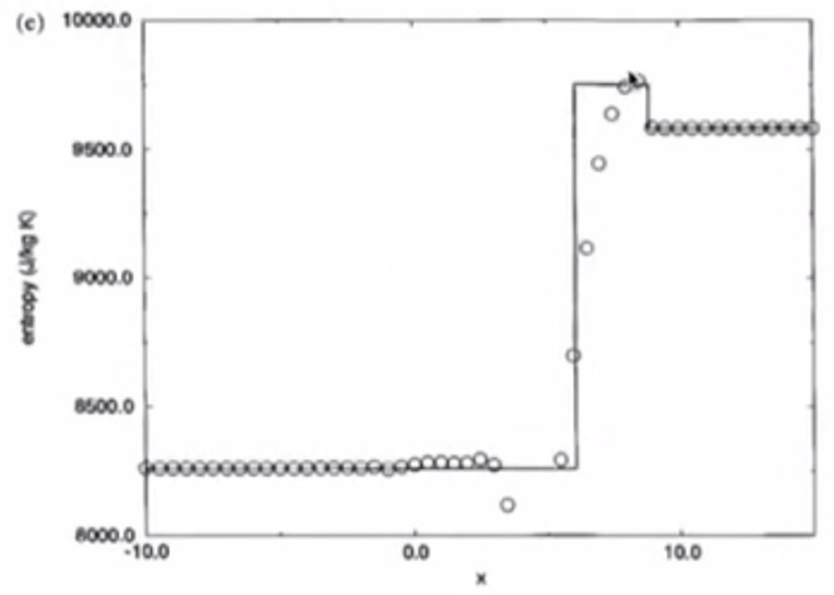

(f)
Mach number
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-10.0
0.0
10.0

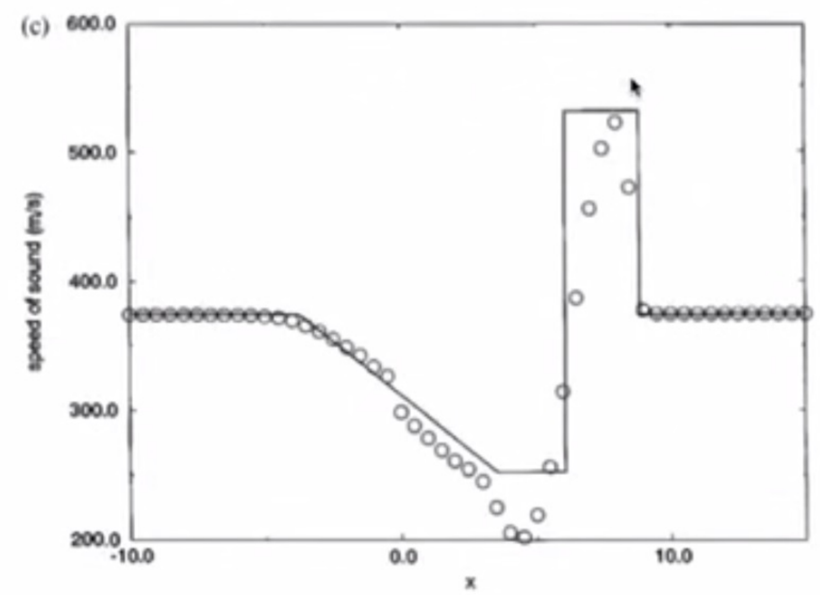
4.7.2. MacCormack with Artificial Viscosity

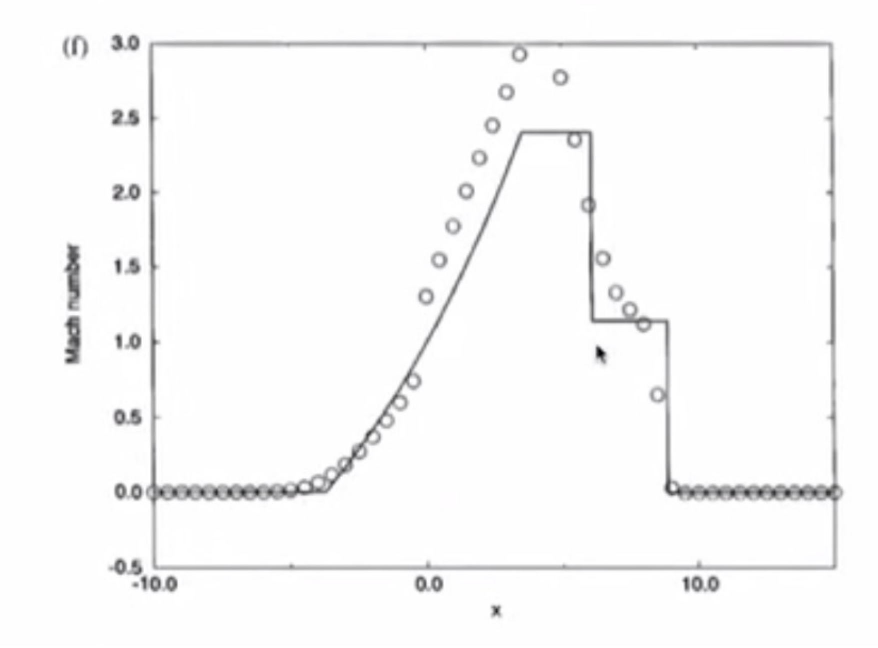
Smaller amplitude of oscillations even in Test 2
Small number of points - is a hard test for numerical scheme (coarse mesh)
Overshoot in velocity




(d)
density (kg/m³)
x

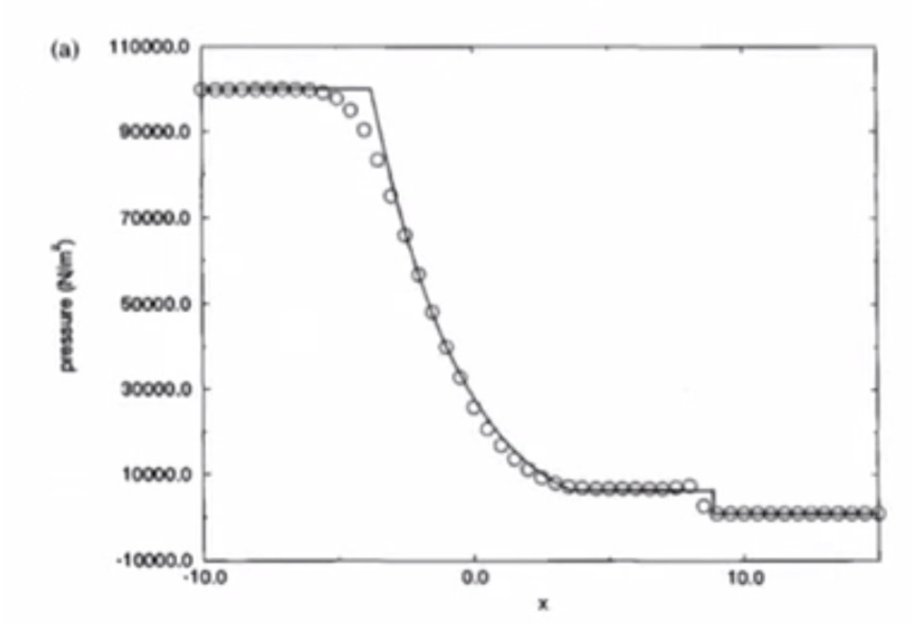

(e)
entropy (J/kg K)
x

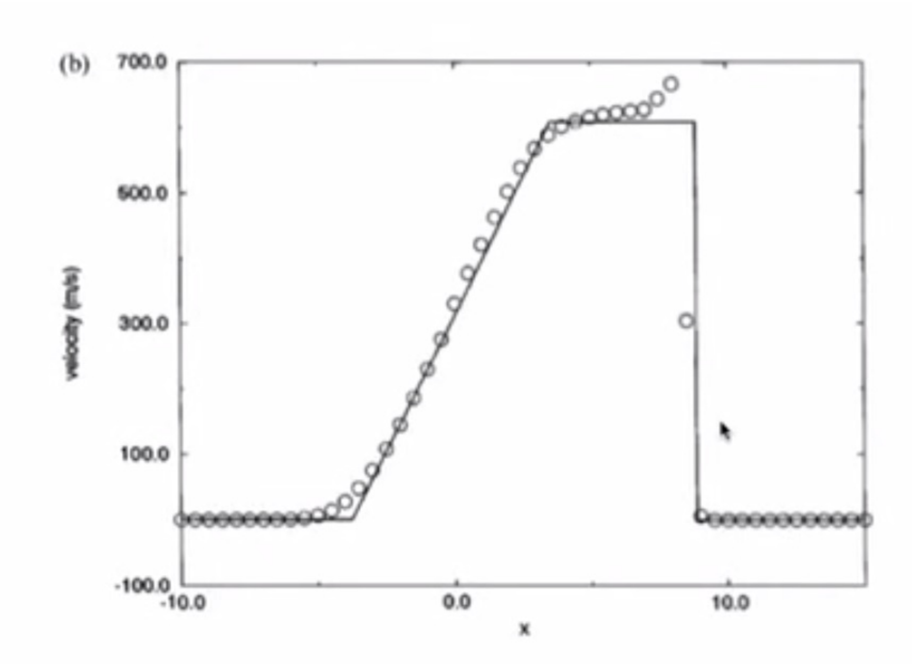

(c)
speed of sound (m/s)
x

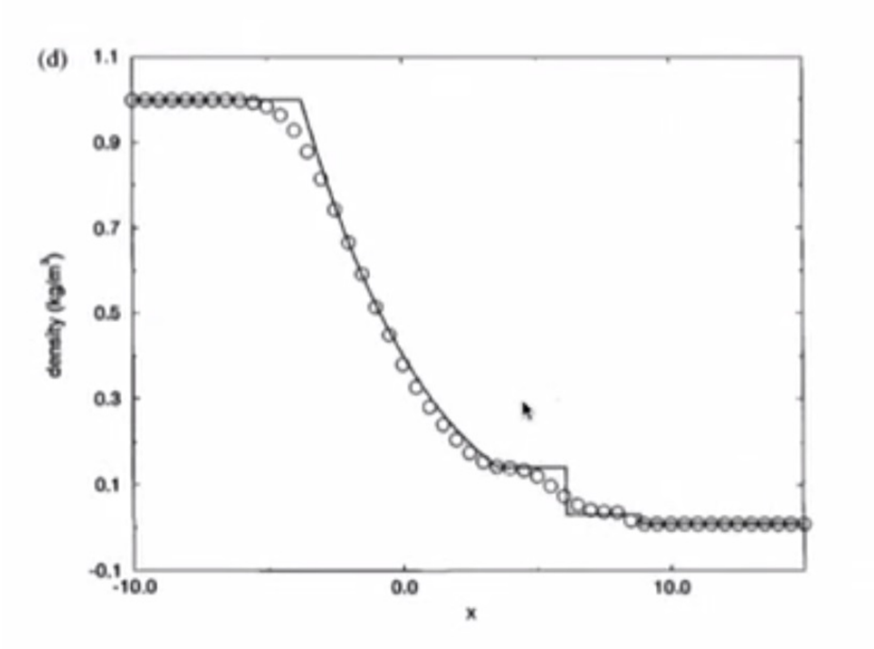


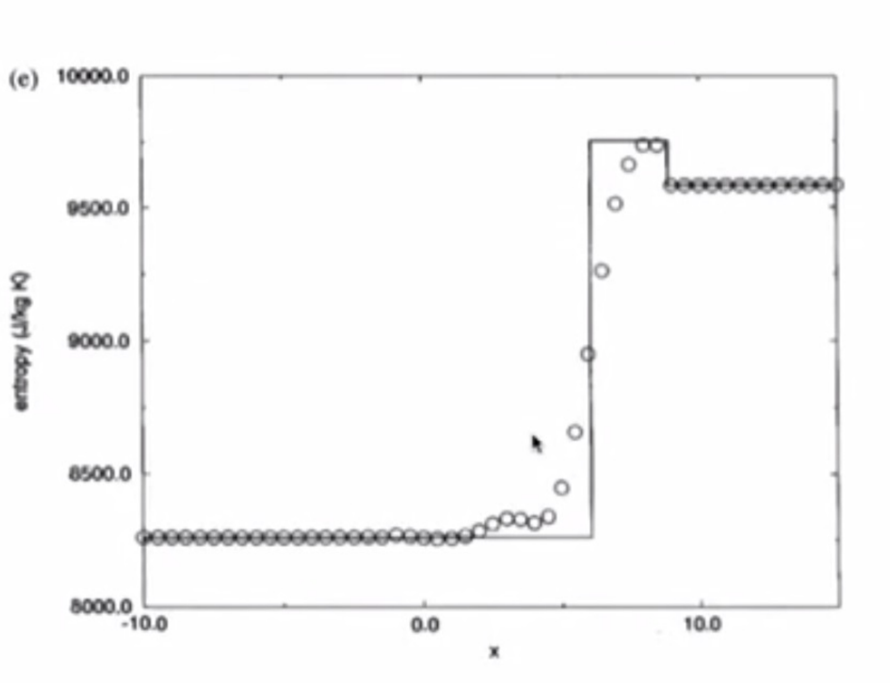
4.7.3. Richtmyer with Artificial Viscosity

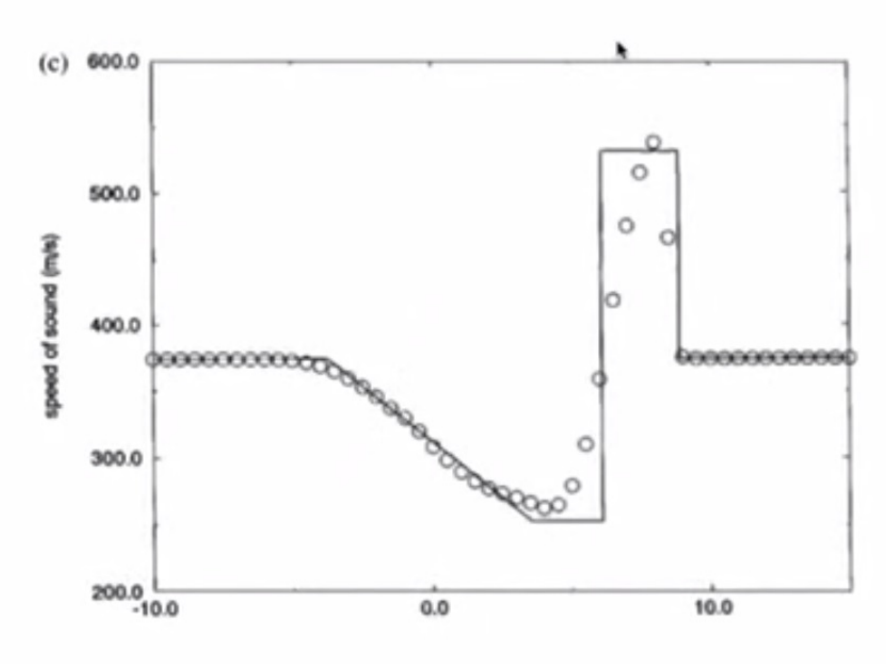
Nice result - better than MacCormack
Smaller about of overshoot
No oscillations in density - negative density might result in mass not being conserved

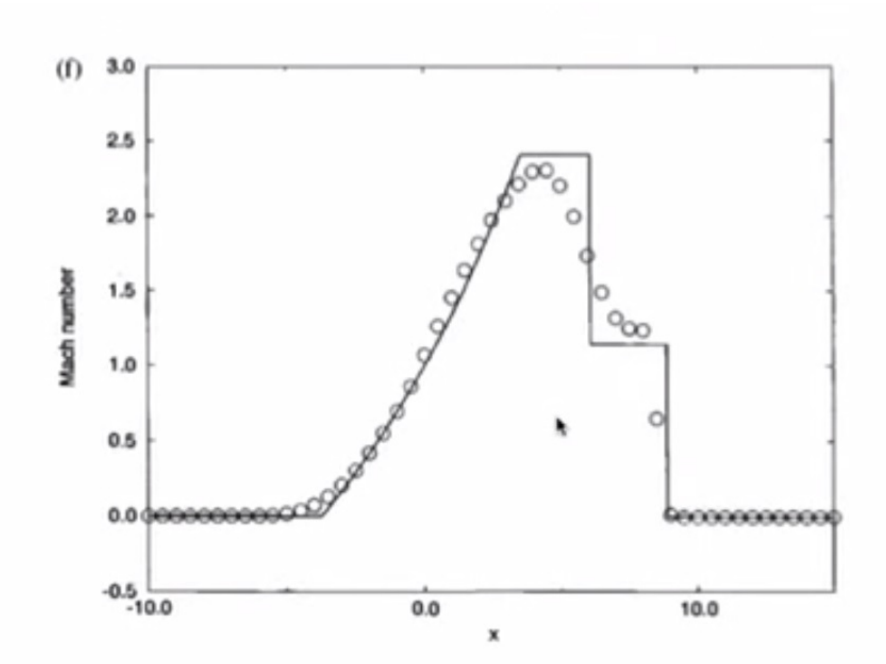



(b)
700.0
500.0
300.0
100.0
-100.0
velocity (m/s)
-10.0
0.0
10.0
x


(d)
1.1
0.9
0.7
0.5
0.3
0.1
-0.1
density (kg/m³)
-10.0
0.0
10.0
x


(e)
10000.0
9500.0
9000.0
8500.0
8000.0
entropy (J/kg K)
-10.0
0.0
10.0
x





4.8. Conclusion

Conclusions from Burgers Equation apply to Euler Equations
This is the usefulness of the model equations