

# A LEAST SQUARES SPECTRAL ELEMENT METHOD FOR INCOMPRESSIBLE FLOW SIMULATIONS

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## INTRODUCTION

Many aerospace products operate in a dynamic flow environment. The structural integrity, performance and development costs of these products are affected by the unsteady flow fields they encounter. In a rocket propulsion system, dynamic loads are attributed as the cause of many life limiting and failure mechanisms. In light of the importance in characterizing the dynamic flow environment, Rocketdyne has initiated a multiyear effort to develop a general purpose computational fluid dynamic analysis system for dynamic load prediction. This system will provide high-fidelity predictive capability through the development of a high order numerical algorithm and utilization of distributed parallel computing. High order methods (spectral accuracy) provide the capability to model complex geometries with high curvature as well as rapidly varying flow fields; parallel computing provides the necessary memory capacity and speed required by large scale computations. The objective of this paper is to highlight the current numerical algorithm and demonstrate its efficiency and accuracy through numerical examples.

## DESCRIPTION OF NUMERICAL METHOD

The least squares spectral element method (LSSEM) is an extension of the finite element method proposed by Jiang et. al [1] who cast the Navier-Stokes equations as a first order system  $\mathcal{L}\vec{u} = \vec{f}$  in a domain bounded by  $\Omega \subset \Re^{n_d}$  which is subjected to the condition  $\mathcal{B}\vec{u} = \vec{g}$  along a piecewise smooth boundary  $\Gamma$ .  $\mathcal{L}$  is a first-order partial differential operator,  $\mathcal{L}\vec{u} = \sum_{i=1}^{n_d} \mathcal{A}_i \frac{\partial \vec{u}}{\partial x_i} + \mathcal{A}_0 \vec{u}$ , where  $n_d = 2$  or  $3$ , depending on the spatial dimensions,  $x_i$ 's are the Cartesian coordinates,  $\vec{u}$  has a length  $n$ , where  $n$  is the number of dependent variables,  $\vec{f}$  is the forcing function and both  $B$  and  $\vec{g}$  describe the appropriate boundary conditions.  $\mathcal{A}$ 's are  $m \times n$  matrices which describe the characteristics of the system of equations being solved. With this approach only  $C^0$  continuity is required along element interfaces. For incompressible viscous flows, the working variables are velocity, pressure and vorticity and, for brevity, one can write the equations in the two-dimensional Cartesian coordinate system

as

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \\ \frac{1}{\delta\tau} + u^0 \frac{\partial}{\partial x} + v^0 \frac{\partial}{\partial y} + \frac{\partial u^0}{\partial x} & \frac{\partial u^0}{\partial y} & \frac{\partial}{\partial x} & \frac{1}{Re} \frac{\partial}{\partial y} \\ \frac{\partial v^0}{\partial x} & \frac{1}{\delta\tau} + u^0 \frac{\partial}{\partial x} + v^0 \frac{\partial}{\partial y} + \frac{\partial v^0}{\partial y} & \frac{\partial}{\partial y} & -\frac{1}{Re} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & 1 \end{bmatrix} \times \begin{Bmatrix} u^{s+1} \\ v^{s+1} \\ p^{s+1} \\ \omega^{s+1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{u^0}{\delta\tau} - \alpha_1 \frac{u^0}{\delta t} - \alpha_2 \frac{u^s}{\delta t} - \alpha_3 \frac{u^{s-2}}{\delta t} + u^0 \frac{\partial u^0}{\partial x} + v^0 \frac{\partial u^0}{\partial y} \\ \frac{v^0}{\delta\tau} - \alpha_1 \frac{v^0}{\delta t} - \alpha_2 \frac{v^s}{\delta t} - \alpha_3 \frac{v^{s-2}}{\delta t} + u^0 \frac{\partial v^0}{\partial x} + v^0 \frac{\partial v^0}{\partial y} \\ 0 \end{Bmatrix}$$

where  $\alpha_1 = 0.5$ ,  $\alpha_2 = -2.0$  and  $\alpha_3 = 1.5$  for second order temporal accuracy. Superscripts represent different time levels. In addition,  $\delta t$  is the physical time step whereas  $\delta\tau$  is the pseudo-time step. One could adjust  $\delta\tau$  to introduce certain level of diffusion in the streamwise direction to stabilize sharp gradient that might occur in an under-resolving grid. This idea has been demonstrated by Chan [2] but is not exercised here. We employ the isoparametric mapping to transform the above equations from the Cartesian coordinate system  $(x, y)$  to a generalized coordinate system  $(\xi, \eta)$  where the spatial discretization is performed implicitly with the convective terms linearized by the Newton-Raphson procedure. Sub-iterations are required at each time step for the purpose of minimizing the linearization error. For high spatial accuracy, we employ the Legendre polynomials type basis function developed by Rønquist and Patera [3]. The resulting system of algebraic equations are solved by the conjugate gradient method with Jacobi preconditioner. The major computational kernel is the matrix-vector multiplication which can be performed very efficiently with today's pipelined microprocessor design. Gauss quadrature rule is employed for the numerical integration on each element. For instance, one can evaluate the integral  $\int \frac{\partial u}{\partial x} d\Omega_e$  as

$$\begin{aligned} \int_{-1}^1 \int_{-1}^1 \frac{2}{B_e} \frac{\partial u}{\partial \xi} \frac{H_e}{2} d\xi \frac{H_e}{2} d\eta &= \sum_l \sum_m \sum_i \sum_j \frac{\partial \Psi(\xi_l)}{\partial \xi} \Phi_j(\eta_m) W(\xi_l) W(\eta_m) \frac{H_e}{2} u_{ij} \\ &= \sum_l \sum_m W(\xi_l) W(\eta_m) \frac{H_e}{2} \sum_i \frac{\partial \Psi(\xi_l)}{\partial \xi} u_{im} \end{aligned}$$

where  $W$ 's are quadrature weights and  $\Psi$  and  $\Phi$  are shape functions. The dependent variable,  $u$ , is represented as the tensor product of a set of one-dimensional shape functions as  $u(\xi_l, \eta_m) = \sum_i \Psi_i(\xi_l) \sum_j \Phi_j(\eta_m) u_{ij}$  where  $\Psi_i(\xi_l) = \delta_{il}$  and  $\Phi_j(\eta_m) = \delta_{jm}$ . For illustration purpose, a simple rectilinear transformation has been employed, where  $H_e$  and  $B_e$  is the height and width of an element, respectively. The Gauss-Lobatto points are used as collocation points (where the dependent variables are evaluated and stored) and integration points. Some of the advantages offered by LSSEM are: (1) the use of one set of basis functions for all dependent variables result in a simpler code structure, (2) the discretization leads to a set of positive definite and symmetric algebraic equations which can be solved efficiently, (3) the tight coupling among all the governing equations can accelerate convergence, (4) well-defined boundary conditions that are consistent with flow physics and mathematical constraints, (5)

the absence of special treatment along wall boundary, such as the utilization of ghost points, for maintaining high order accuracy.

## RESULTS AND DISCUSSION

To evaluate the effectiveness of the LSSEM, we first apply it to the flow problem behind an array of cylinders with the analytical solution derived by Kovasznay [4] and given as

$$u = 1 - e^{\lambda x} \cos(2\pi y); \quad v = \frac{\lambda e^{\lambda x} \sin(2\pi y)}{2\pi}; \quad p = \frac{1 - e^{2\lambda x}}{2}$$

where,  $\lambda = \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}$ , with  $Re = 40$ . The rectangular flow domain is bounded by the Cartesian coordinates  $[-0.5, 1.0] \times [-0.5, 0.5]$ . Computations have been performed on the IBM SP2 computer that is equipped with 160 microprocessors of type RS6000-590. Both  $\delta t$  and  $\delta\tau$  are set to  $10^{30}$ . Convergence is achieved after the residual of governing equations has dropped below  $10^{-10}$ . We perform the first set of computations by varying the number of elements and having the order of polynomials fixed at two and tabulate the results in terms of the  $\epsilon_u$ ,  $\epsilon_v$  and  $\epsilon_p$  which is the r.m.s. value of the error for  $u$ ,  $v$  and  $p$ , respectively. Timings are measured in the wallclock time for single node. Using a second order finite difference method and a  $61 \times 41$  grid, Chan [5] has computed the same flow case and reported an error of  $7.5 \times 10^{-2}$  and  $5 \times 10^{-2}$  for  $\epsilon_u$  and  $\epsilon_v$ , respectively. The required CPU time on a CRAY-XMP is about 50 seconds ( $2 \times 10^{-4}$  sec/point/iteration, with 100 iterations required). The accuracy of this result is comparable to the current case where  $N_x \times N_y = 15 \times 10$  and the required time is 32 seconds, that is a saving of 40 percent. Therefore, even when low order polynomials are used, LSSEM can still be very competitive with an existing second order finite difference method.

$N_x \times N_y$	$\epsilon_u$	$\epsilon_v$	$\epsilon_p$	No. of Steps	Timing (sec.)	Mflops
$15 \times 10$	$5.49 \times 10^{-2}$	$8.34 \times 10^{-3}$	0.25	18	31.6	52.4
$30 \times 20$	$1.07 \times 10^{-2}$	$1.77 \times 10^{-3}$	$7.29 \times 10^{-2}$	19	258	59.7
$60 \times 40$	$1.56 \times 10^{-3}$	$2.69 \times 10^{-4}$	$1.66 \times 10^{-2}$	19	1916	60.5

The second set of computations are performed with 8 elements (4 in the streamwise and 2 in the vertical directions) and order of polynomials ( $N$ ) between 4 and 14. The tolerance for the case with  $N = 14$  is set to  $10^{-13}$ . These results bring out the advantage of using a high order method. For instance, by using  $N = 14$  and 353 seconds, we are able to reduce the numerical errors to nearly the machine accuracy of a 64-bit computer. In addition, the computational speed also improves by 50 percent, from 60 to 96 Mflops, making high order LSSEM an efficient flow solver for current generation of computers.

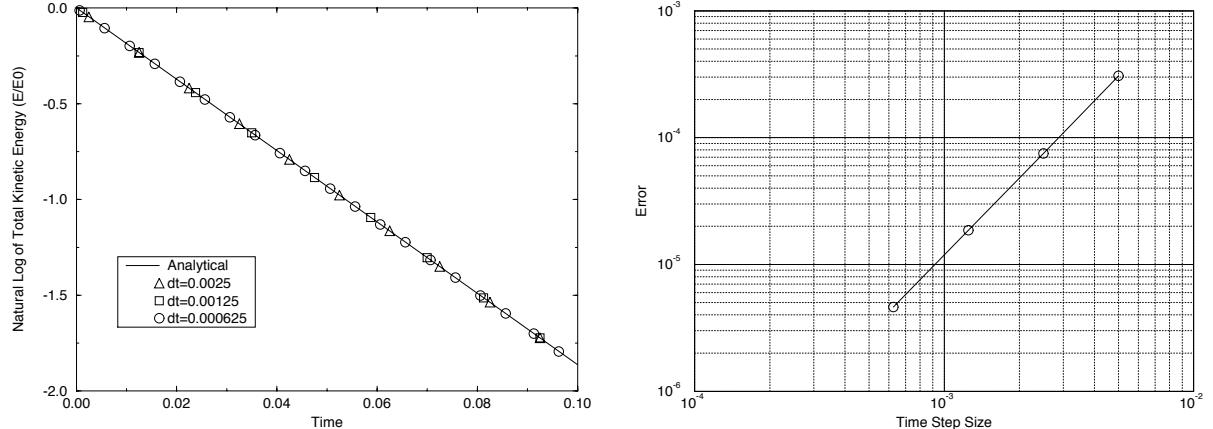


Figure 1: Accuracy in predicting the evolution of small disturbances in a channel. Left: predicted decay rate using three different time step sizes. Right: temporal accuracy of the LSSEM

$N$	$\epsilon_u$	$\epsilon_v$	$\epsilon_p$	No. of Steps	Timing (sec.)	Mflops
4	$6.44 \times 10^{-2}$	$1.31 \times 10^{-2}$	0.211	19	8.7	25.7
9	$1.56 \times 10^{-6}$	$3.58 \times 10^{-7}$	$3.76 \times 10^{-6}$	19	30.3	59.7
14	$9.22 \times 10^{-13}$	$4.72 \times 10^{-13}$	$1.47 \times 10^{-11}$	26	353	96

To characterize the temporal and spatial accuracy of LSSEM, we use it to analyze the unsteady flow inside a two-dimensional planar channel. The unsteadiness is generated as a response to external perturbations. Two different problems are solved. The first problem is associated with the Stokes flow which has an analytical solution [6] for detailed comparison. For the second problem, we solve the Navier-Stokes equations and compare the result with that given by the linear stability theory. To perform the Stokes flow problem, we use a computational grid that consists of two equally spaced elements in the streamwise direction, which spans from 0 to  $2\pi$ , and four elements in the wall-to-wall direction, which has a dimension of one. Within each element, we use 6<sup>th</sup> order Legendre polynomials in both directions to approximate all the dependent variables. Periodic boundary condition is imposed in the streamwise direction and no-slip condition is imposed along both walls. In addition, the pressure is set to zero at one point inside the channel. The initial condition is set to the analytical solution of the associated eigenvalue problem. Figure 1 shows the evolution of the total kinetic energy,  $E(t)$ , which is defined as  $E(t) = \frac{1}{2} \int (u^2 + v^2) dx dy$  for time step sizes of 0.0025, 0.00125 and 0.000625. The total kinetic energy changes as  $\frac{E(t)}{E(t_0)} = e^{2\sigma t}$ , where  $E(t_0)$  is the initial energy and  $\sigma$  is the decay rate. These results demonstrate that the current grid resolution is adequate in resolving all the flow features. The predicted decay rate for a time step size of 0.000625 is 9.313316 which, when compared to the analytical solution, corresponds to only 0.0045 percent in numerical error. Figure 1 also shows that the current numerical method achieves a second order temporal accuracy without the need

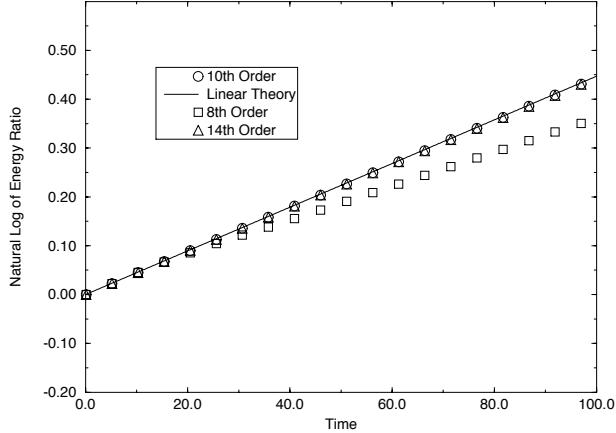


Figure 2: Predicted Growth Rate Using Three Different Orders of Polynomials

of a special treatment near the wall. This feature provides a distinct advantage over fractional step methods which require special boundary conditions for the velocity along solid boundaries [6] to maintain second order temporal accuracy. A similar test problem is set up for the Navier-Stokes system. The only exception is that the eigenfunction cannot be determined analytically and is computed as a solution to the Orr-Sommerfeld equation by Streett [7]. The Reynolds number is 7500. The initial condition consists of a parabolic profile superimposed with the eigenfunction that has a maximum amplitude of  $10^{-4}$ . Three elements are used in the vertical direction and one element is used in the streamwise direction. The elements that are adjacent to the walls cover only 30 percent of the channel width. Within each element, 8<sup>th</sup>, 10<sup>th</sup> and 14<sup>th</sup> order Legendre polynomials have been used. The total perturbed kinetic energy is monitored at each time step. With the above initial condition, the Orr-Sommerfeld solution predicts a growth rate of 0.00223497 and a phase speed of 0.24989154. Figure 2 shows the evolution of the kinetic energy predicted by LSSEM using three different orders of polynomials. A time step size of 0.1 is used and a convergence criterion of  $10^{-10}$  is used for each time step. The drastic improvement in accuracy as the order of polynomials increases is attributed to the exponential convergence behavior of the LSSEM. Furthermore, these results also demonstrate the stability of our method for long time integration; the duration covered by these computations represents the time it takes for the disturbance to propagate through the computational domain nearly four times. The computation using 14<sup>th</sup> order polynomials requires about one hour and predicts a growth rate with an error of 0.76 percent. Canuto et. al [8] suggest that this type of problem cannot be efficiently solved by second order accurate finite difference methods. They conclude that nearly 10,000 points are needed in the vertical direction to achieve the same level of accuracy as our results that require merely thirty-two points. Beaudan and Moin [9] use Rai's fifth order upwinding method to compute the same problem. With a careful choice of grid clustering in the wall-to-wall direction, they conclude that 129 points are needed in the vertical direction and 32 points are needed in the horizontal direction for them to minimize the error to one percent. In comparison, LSSEM provides a saving of 11 times in the number

of grid points, and the error of prediction is nearly one order of magnitude lower.

In summary, we have developed a stable and accurate numerical platform for the solution of the incompressible Navier-Stokes equations. Preliminary two-dimensional results indicate that the LSSEM is more efficient than the second order finite difference method in providing highly accurate numerical solution. The performance of this method is excellent on distributed memory parallel computer equipped with pipelined microprocessor. Testings on large scale three-dimensional problems are in progress.

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