Lecture 22

Game Theory :
Simplex Method

22.1 Simplex Method

Let us consider the 3 x 3 matrix

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array} \begin{array}{c} \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

As per the assumptions, A always attempts to choose the set of strategies with the non-zero probabilities say p_1, p_2, p_3 where $p_1 + p_2 + p_3 = 1$ that maximizes his minimum expected gain.

Similarly B would choose the set of strategies with the non-zero probabilities say q_1, q_2, q_3 where $q_1 + q_2 + q_3 = 1$ that minimizes his maximum expected loss.

Step 1
Find the minimax and maximin value from the given matrix

Step 2
The objective of A is to maximize the value, which is equivalent to minimizing the value 1/V. The LPP is written as

Min $1/V = p_1/V + p_2/V + p_3/V$
and constraints ≥ 1

It is written as

Min $1/V = x_1 + x_2 + x_3$
and constraints ≥ 1

Similarly for B, we get the LPP as the dual of the above LPP

Max $1/V = Y_1 + Y_2 + Y_3$
and constraints ≤ 1
Where $Y_1 = q_1/V$, $Y_2 = q_2/V$, $Y_3 = q_3/V$

Step 3
Solve the LPP by using simplex table and obtain the best strategy for the players

Example 1
Solve by Simplex method

$$
\begin{array}{cc}
 & \text{B} \\
\text{A} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}
\end{array}
$$

Solution

$$
\begin{array}{ccc}
 & \text{B} & \\
\text{A} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} & \begin{array}{l} -2 \\ -1 \\ 2 \ \text{Maximin} \end{array} \\
 & \begin{array}{ccc} 3 & 4 & 6 \\ \text{Minimax} & & \end{array} &
\end{array}
$$

We can infer that $2 \leq V \leq 3$. Hence it can be concluded that the value of the game lies between 2 and 3 and the $V > 0$.

LPP

Max $1/V = Y_1 + Y_2 + Y_3$

Subject to

$3Y_1 - 2Y_2 + 4Y_3 \leq 1$

$-1Y_1 + 4Y_2 + 2Y_3 \leq 1$

$2Y_1 + 2Y_2 + 6Y_3 \leq 1$

$Y_1, Y_2, Y_3 \geq 0$

SLPP

Max $1/V = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$3Y_1 - 2Y_2 + 4Y_3 + s_1 = 1$

$-1Y_1 + 4Y_2 + 2Y_3 + s_2 = 1$

$2Y_1 + 2Y_2 + 6Y_3 + s_3 = 1$

$Y_1, Y_2, Y_3, s_1, s_2, s_3 \geq 0$

Basic Variables	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min Ratio Y_B / Y_K
		$C_j \rightarrow$	1	1	1	0	0	0	
S_1	0	1	$\boxed{3}$	-2	4	1	0	0	1/3→
S_2	0	1	-1	4	2	0	1	0	-
S_3	0	1	2	2	6	0	0	1	1/2
	1/V = 0		↑ -1	-1	-1	0	0	0	
Y_1	1	1/3	1	-2/3	4/3	1/3	0	0	-
S_2	0	4/3	0	10/3	10/3	1/3	1	0	2/5
S_3	0	1/3	0	$\boxed{10/3}$	10/3	-2/3	0	1	1/10→
	1/V =1/3		0	↑ -5/3	1/3	1/3	0	0	
Y_1	1	2/5	1	0	2	1/5	0	1/5	
S_2	0	1	0	0	0	1	1	-1	
Y_2	1	1/10	0	1	1	-1/5	0	3/10	
	1/V = 1/2		0	0	2	0	0	1/2	

$1/V = 1/2$
$V = 2$

$y_1 = 2/5 * 2 = 4/5$
$y_2 = 1/10 * 2 = 1/5$
$y_3 = 0 * 2 = 0$

$x_1 = 0*2 = 0$
$x_2 = 0*2 = 0$
$x_3 = 1/2*2 = 1$

$S_A = (0, 0, 1)$
$S_B = (4/5, 1/5, 0)$
Value = 2

Example 2

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{cc} & \begin{array}{ccc} & B & \end{array} \\ A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \begin{array}{c} -1 \\ -1 \\ -1 \end{array} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \end{array}$$

Maximin = -1
Minimax = 1

We can infer that $-1 \leq V \leq 1$

Since maximin value is -1, it is possible that value of the game may be negative or zero, thus the constant 'C' is added to all the elements of matrix which is at least equal to the negative of maximin.

Let C = 1, add this value to all the elements of the matrix. The resultant matrix is

$$\begin{array}{cc} & \begin{array}{ccc} & B & \end{array} \\ A & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix} \end{array}$$

LPP
Max $1/V = Y_1 + Y_2 + Y_3$
Subject to

$2Y_1 + 0Y_2 + 0Y_3 \leq 1$
$0Y_1 + 0Y_2 + 4Y_3 \leq 1$
$0Y_1 + 3Y_2 + 0Y_3 \leq 1$
$Y_1, Y_2, Y_3 \geq 0$

SLPP
Max $1/V = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$
Subject to

$2Y_1 + 0Y_2 + 0Y_3 + s_1 = 1$
$0Y_1 + 0Y_2 + 4Y_3 + s_2 = 1$
$0Y_1 + 3Y_2 + 0Y_3 + s_3 = 1$
$Y_1, Y_2, Y_3, s_1, s_2, s_3 \geq 0$

Basic Variables	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min Ratio Y_B / Y_K
		$C_j \rightarrow$	1	1	1	0	0	0	
S_1	0	1	$\boxed{2}$	0	0	1	0	0	$1/2 \rightarrow$
S_2	0	1	0	0	4	0	1	0	-
S_3	0	1	0	3	0	0	0	1	-
	1/V =0		↑ -1	-1	-1	0	0	0	
Y_1	1	1/2	1	0	0	1/2	0	0	-
S_2	0	1	0	0	4	0	1	0	-
S_3	0	1	0	$\boxed{3}$	0	0	0	1	$1/3 \rightarrow$
	1/V =1/2		0	↑ -1	-1	1/2	0	0	
Y_1	1	1/2	1	0	0	1/2	0	0	-
S_2	0	1	0	0	$\boxed{4}$	0	1	0	$1/4 \rightarrow$
Y_2	1	1/3	0	1	0	0	0	1/3	-
	1/V = 5/6		0	0	↑ -1	1/2	0	1/3	
Y_1	1	1/2	1	0	0	1/2	0	0	
Y_3	1	1/4	0	0	1	0	1/4	0	
Y_2	1	1/3	0	1	0	0	0	1/3	
	1/V =13/12		0	0	0	1/2	1/4	1/3	

1/V =13/12
V = 12/13

$y_1 = 1/2 * 12/13 = 6/13$
$y_2 = 1/3 * 12/13 = 4/13$
$y_3 = 1/4 * 12/13 = 3/13$

$x_1 = 1/2*12/13 = 6/13$
$x_2 = 1/4 * 12/13 = 3/13$
$x_3 = 1/3 * 12/13 = 4/13$

$S_A = (6/13, 3/13, 4/13)$
$S_B = (6/13, 4/13, 3/13)$

Value = 12/13 – C =12/13 -1 = -1/13