Systems Thinking - Mini Project
Team Name: Systems Sleeping

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2-Link Manipulator Dynamics: *Consider the following system dynamics of a 2-link manipulator:*

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{1}$$

where the inertia matrix $\boldsymbol{M}$ and joint vector $\boldsymbol{q}$ are defined as:

$$\boldsymbol{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{2}$$

The individual matrix elements are:

$$M_{11} = (m_1 + m_2)l_1^2 + m_2 l_2^2(l_2 + 2l_1 \cos q_2) \tag{3}$$

$$M_{12} = M_{21} = m_2 l_2(l_2 + l_1 \cos q_2) \tag{4}$$

$$M_{22} = m_2 l_2^2 \tag{5}$$

$$\boldsymbol{C} = \begin{bmatrix} -m_2 l_1 l_2 \sin q_2 \cdot \dot{q}_2 & -m_2 l_1 l_2 \sin q_2(\dot{q}_1 + \dot{q}_2) \\ 0 & m_2 l_1 l_2 \sin q_2 \cdot \dot{q}_1 \end{bmatrix} \tag{6}$$

$$\boldsymbol{G} = \begin{bmatrix} m_1 l_1 g \cos q_1 + m_2 g(l_2 \cos(q_1 + q_2) + l_1 \cos q_1) \\ m_2 g l_2 \cos(q_1 + q_2) \end{bmatrix} \tag{7}$$

System Parameters:

$$m_1 = 5\,\mathrm{kg}, \qquad m_2 = 3\,\mathrm{kg} \tag{8}$$

$$l_1 = 0.25\,\mathrm{m}, \qquad l_2 = 0.15\,\mathrm{m} \tag{9}$$

$$g = 9.81\,\mathrm{m/s^2} \tag{10}$$

Initial Conditions:

$$\boldsymbol{q}(0) = \begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.15 \end{bmatrix} \text{ rad} \tag{11}$$

Control Objective: *Drive the manipulator to the desired configuration:*

$$\boldsymbol{q}_{desired} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ rad} \tag{12}$$

1 Model and notation

We use the standard manipulator model

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau},$$

where

- $\boldsymbol{q}(t) \in \mathbb{R}^n$ is the vector of joint positions,
- $\dot{\boldsymbol{q}}(t), \ddot{\boldsymbol{q}}(t)$ are joint velocities and accelerations,
- $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the inertia matrix,
- $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is the centrifugal matrix
- $\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^n$ is the gravity torque vector,
- $\boldsymbol{\tau}(t) \in \mathbb{R}^n$ is the control torque applied.

We assume $\boldsymbol{M}(\boldsymbol{q})$ is invertible.

2 General Idea

We have a desired trajectory $\boldsymbol{q}_d(t)$ (with known $\dot{\boldsymbol{q}}_d(t), \ddot{\boldsymbol{q}}_d(t)$).We define the error as

$$\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$$

1. We pick a simple, stable, linear differential equation that we want the error $\boldsymbol{e}$ to satisfy
2. We rewrite the equation to find $\ddot{\boldsymbol{q}}$ that satisfies it
3. We substitue the obtained $\ddot{\boldsymbol{q}}$ expression in the given $\boldsymbol{\tau}$ equation

This yields a torque law that makes the closed-loop error behave like the target linear dynamics.

3 PD Controller

PD control improves the transient response by combining proportional and derivative actions. The proportional term reacts to the present error, while the derivative term predicts the system's future behavior by considering the rate of change of error. This combination improves stability and helps reduce overshoot. However, PD control by itself cannot eliminate steady-state error when constant disturbances are present.
Given,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) \\ \implies \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} &= \boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q}) \\ \implies \ddot{\boldsymbol{q}} &= \boldsymbol{M}^{-1}(\boldsymbol{q})\,[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})] \end{aligned}$$

We have a desired trajectory $\boldsymbol{q}_d(t)$ (with known $\dot{\boldsymbol{q}}_d(t), \ddot{\boldsymbol{q}}_d(t)$).We define the error as

$$\boldsymbol{e}_i = \boldsymbol{q}_{id} - \boldsymbol{q}_i$$

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$\dot{\boldsymbol{e}}_i = \dot{\boldsymbol{q}}_{id} - \dot{\boldsymbol{q}}_i$

We choose the second-order target (damped spring):

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_d \dot{\boldsymbol{e}} + \boldsymbol{K}_p \boldsymbol{e} = \boldsymbol{0}, \tag{13}$$

where $\boldsymbol{K}_p, \boldsymbol{K}_d$ are constant positive-definite matrices (typically diagonal).
Given,

$$\boldsymbol{q}_{desired} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ rad}$$

$\dot{\boldsymbol{q}}_{id} = 0$

$$\begin{aligned} \implies e_i &= -q_i \\ \dot{e}_i &= -\dot{q}_i \end{aligned}$$

for derivative and proportional control,
WKT $\tau_i = K_{ip} e_i + K_{id} \dot{e}_i$
$\implies \tau_i = -K_{ip} q_i - K_{id} \dot{q}_i$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = - \begin{bmatrix} K_{1p} q_1 + K_{1d} \dot{q}_1 \\ K_{2p} q_2 + K_{2d} \dot{q}_2 \end{bmatrix}$$

$$K_p = \begin{bmatrix} K_{1p} & 0 \\ 0 & K_{2p} \end{bmatrix}, \quad K_d = \begin{bmatrix} K_{1d} & 0 \\ 0 & K_{2d} \end{bmatrix}$$

$$\boldsymbol{\tau} = -K_p \boldsymbol{q} - K_d \dot{\boldsymbol{q}} \tag{14}$$

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

From equation (14),

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) &= \boldsymbol{\tau} \\ \boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) &= -K_p \boldsymbol{q} - K_d \dot{\boldsymbol{q}} \\ \boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + K_p \boldsymbol{q} + K_d \dot{\boldsymbol{q}} &= 0 \end{aligned}$$

State space variables are as follows,

$$\begin{aligned} x_1 &= q_1 \\ x_2 &= \dot{q}_1 \\ x_3 &= q_2 \\ x_4 &= \dot{q}_2 \end{aligned}$$

$$\ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{q}) \left[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q}) \right]$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$$

We know that

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = -K_p\boldsymbol{q} - K_d\dot{\boldsymbol{q}}$$
$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + K_p\boldsymbol{q} + K_d\dot{\boldsymbol{q}} = 0$$

For steady state $\dot{q} = 0$ & $\ddot{q} = 0$

$$\implies \boldsymbol{G}(\boldsymbol{q}_{ss}) + K_p\boldsymbol{q}_{ss} = 0$$
$$\implies q_{ss} = -K_p^{-1}\boldsymbol{G}(\boldsymbol{q}_{ss})$$

If there is any constant external force or torque (like gravity), the controller must generate a steady torque to balance it.
Since the PD controller can only generate torque through the proportional term $-K_pq$, it needs a nonzero error q to produce that torque. Thus, the manipulator reaches equilibrium with a small but nonzero steady-state position error.

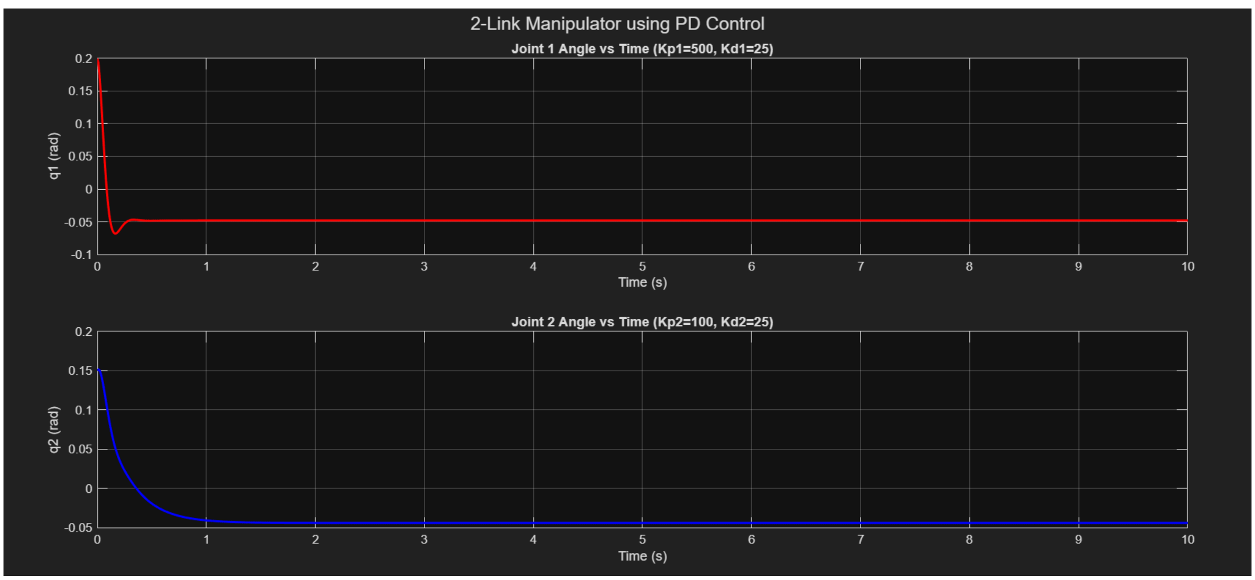

Figure 1: K1p = 500, K2p = 100, K1d = K2d = 25 (Joint Angle)

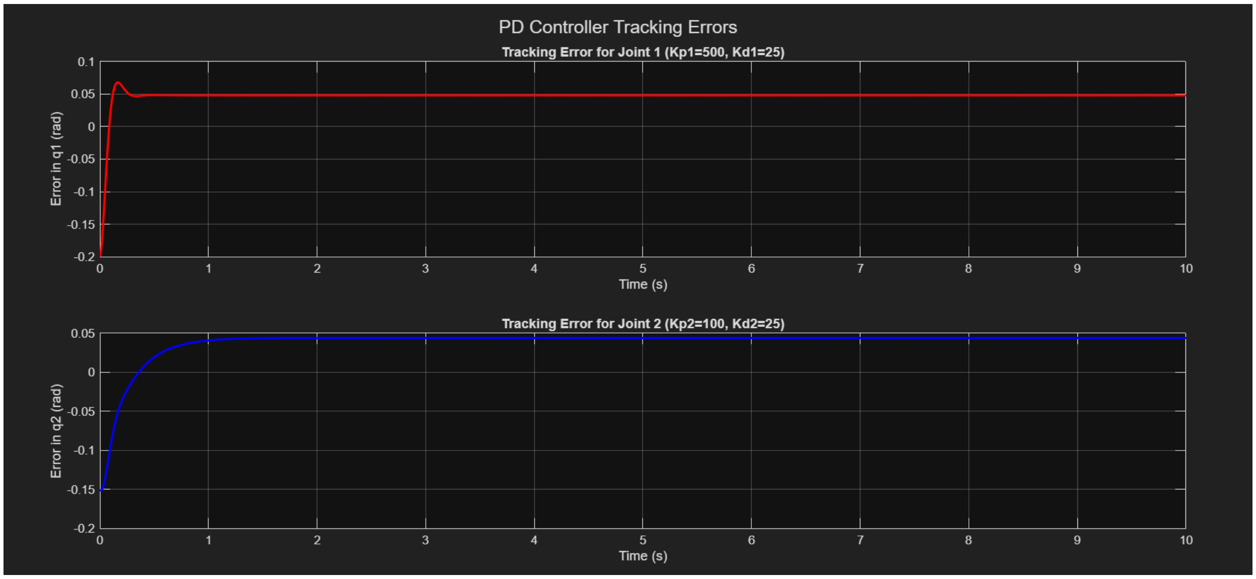

Figure 2: K1p = 500, K2p = 100, K1d = K2d = 25(Tracking Error)

- **Joint 1 (q_1):** Responds moderately fast but shows small oscillations before settling.
- **Joint 2 (q_2):** Shows a slower response because of smaller Kp, we also see minimal overshoots.
- **Interpretation:** With low Kp values, the control action is gentle. Kd provides enough damping to prevent oscillations, but the response remains relatively slow and is underdamped.

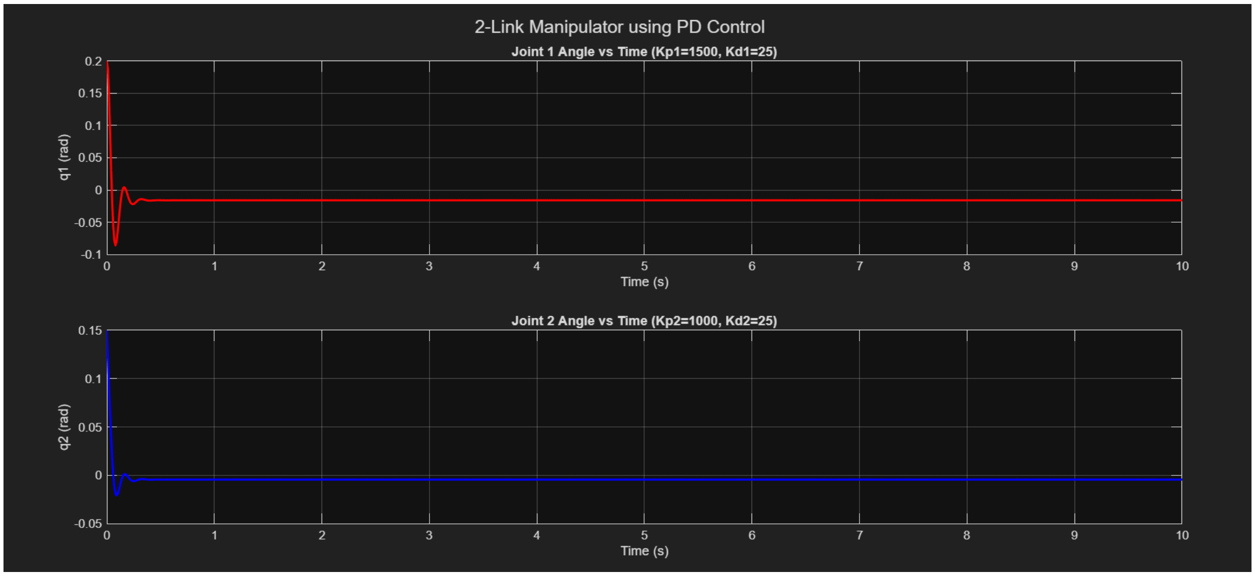

Figure 3: K1p = 1500, K2p = 1000, K1d = K2d = 25(Joint Angle)

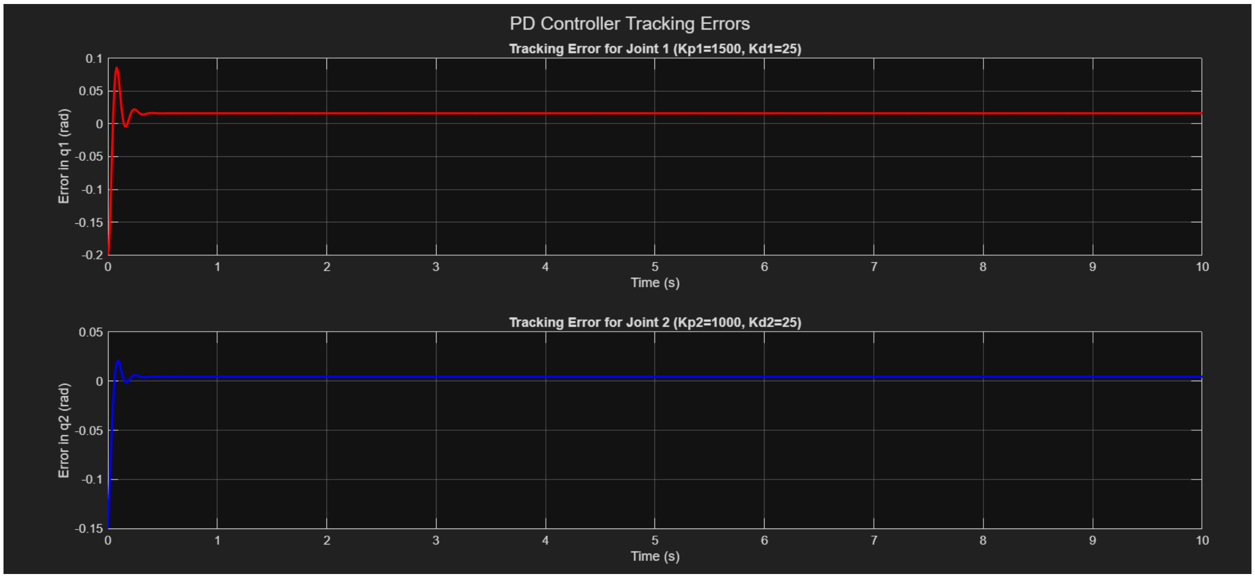

Figure 4: K1p = 1500, K2p = 1000, K1d = K2d = 25(Tracking Error)

- **Joint 1 (q_1):** Shows oscillations but gives fast response
- **Joint 2 (q_2):** More aggressive when compared to Figure1, and it also shows overshoots before settling.
- **Interpretation:** Increasing Kp results in faster response, but if Kd is much smaller than Kp then we still overshooting and oscillations since the damping is not sufficient.

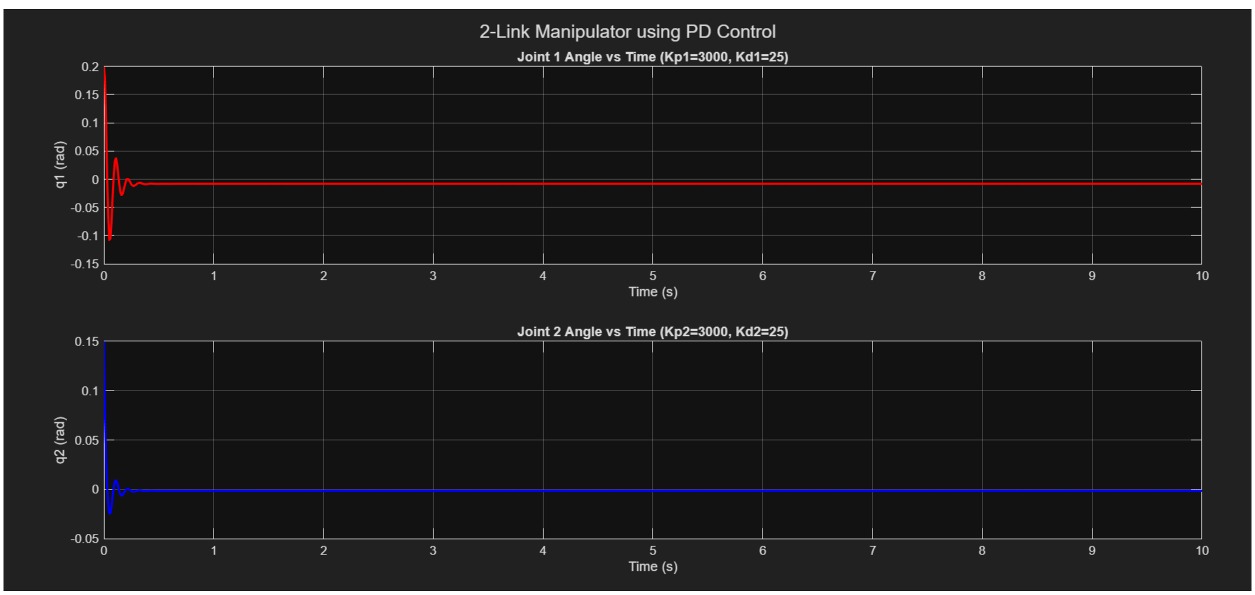

Figure 5: K1p = 3000, K2p = 3000, K1d = K2d = 25(Joint Angle)

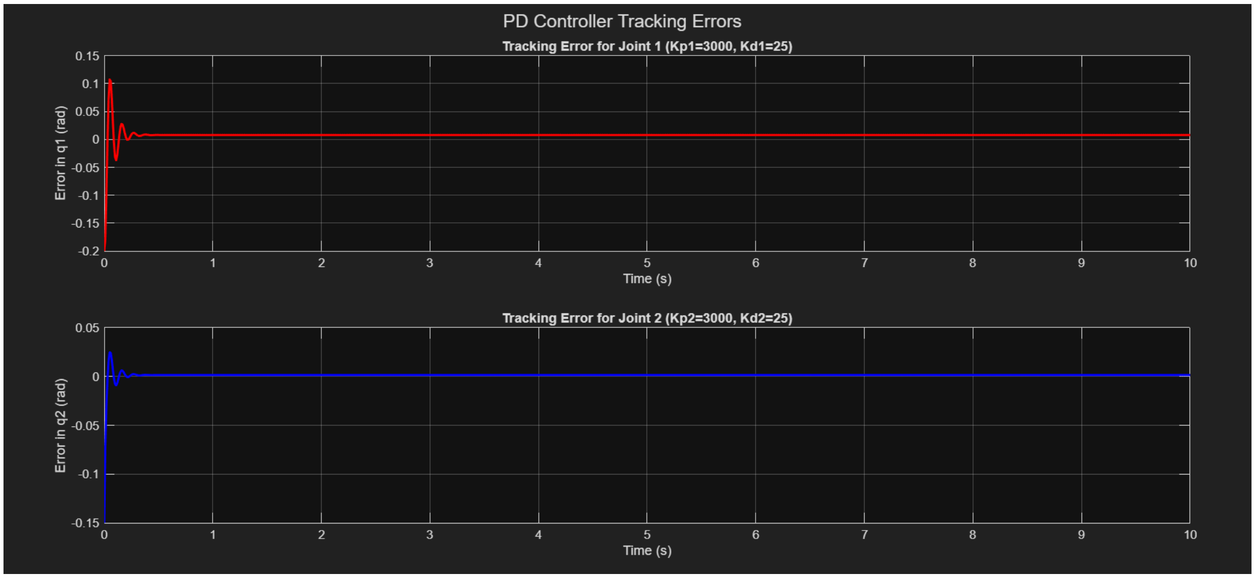

Figure 6: K1p = 3000, K2p = 3000, K1d = K2d = 25(Tracking Error)

- **Joint 1 (q_1) and Joint 2 (q_2):** Very aggressive when compared to the above two. Visible oscillations and overshooting is present.
- **Interpretation:** High Kp values make the system oscillate rapidly, the Kd values chosen aren't high enough, causing underdamping at high Kp.

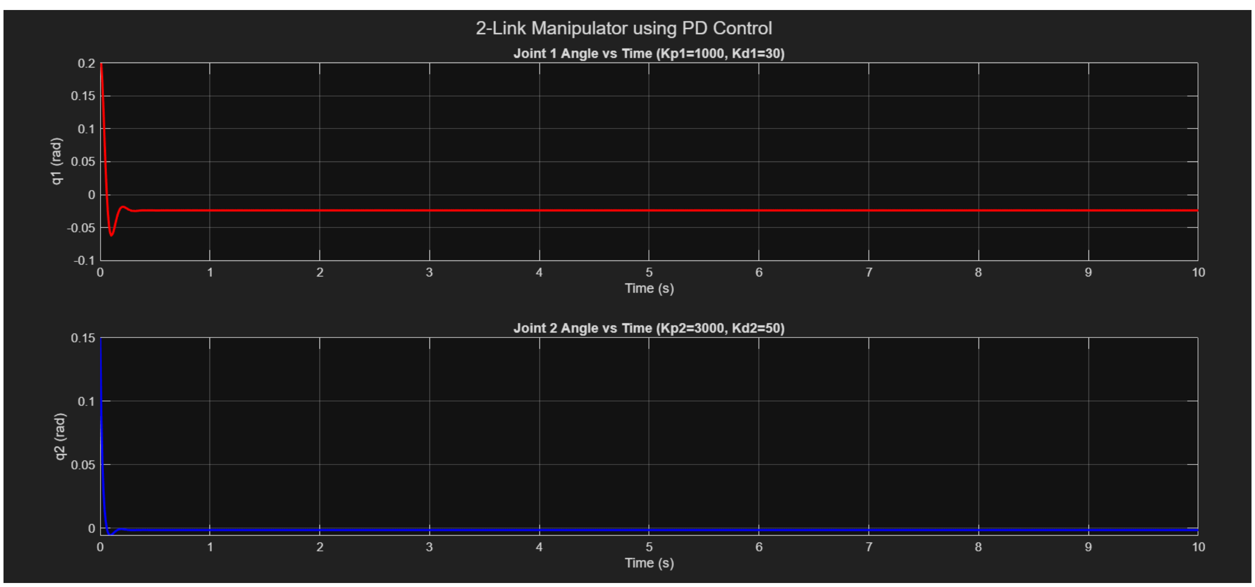

Figure 7: K1p = 1000,K2p = 3000, K1d = 30, K2d = 50(Joint Angle)

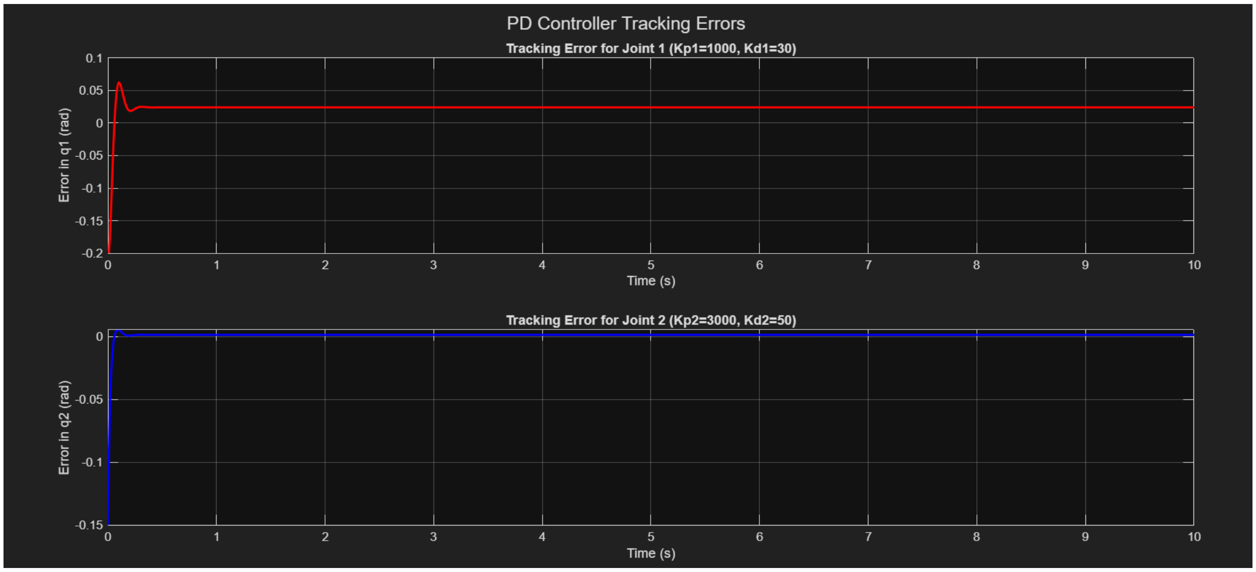

Figure 8: K1p = 1000,K2p = 3000, K1d = 30, K2d = 50(Tracking Error)

- **Joint 1 (q_1):** We get a damped response, with minimal oscillations, but slow speed.
- **Joint 2 (q_2):** Very smooth response that has no oscillations, but is slow.
- **Interpretation:** High Kd values increase damping and reduce overshoots. This system has a good balance between speed and overshooting.

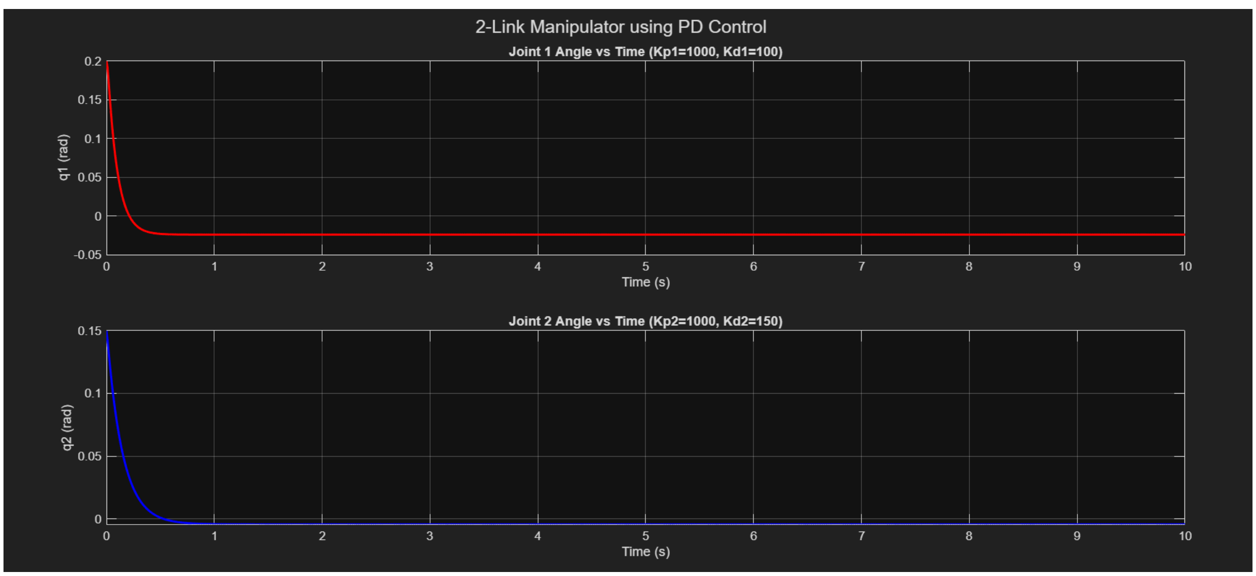

Figure 9: K1p = K2p = 1000, K1d = 100, K2d = 150(Joint Angle)

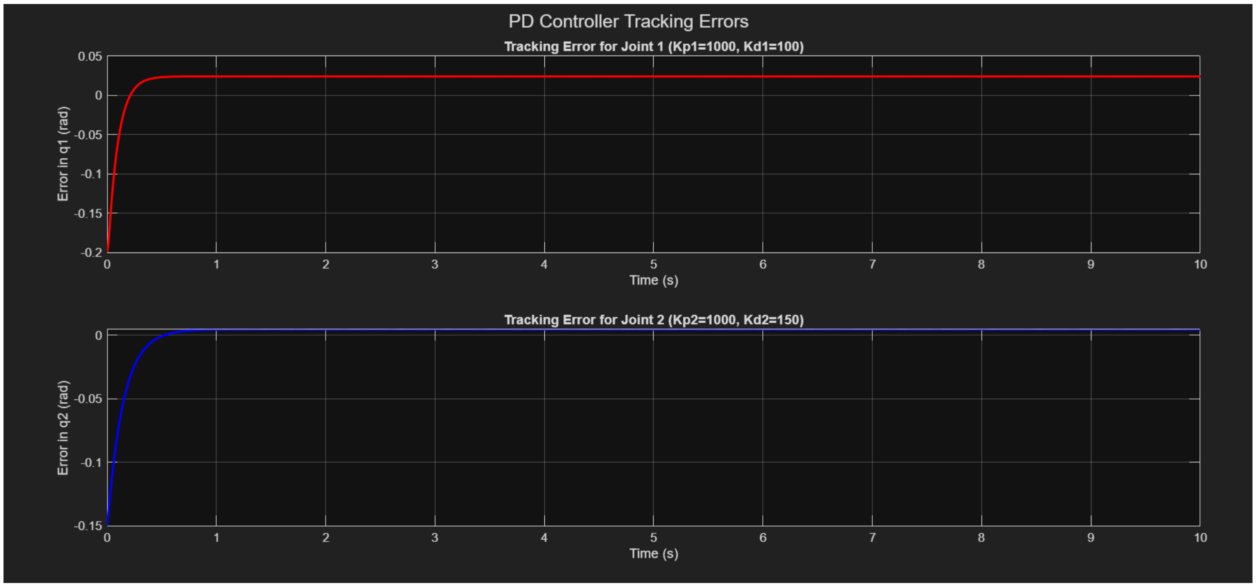

Figure 10: K1p = K2p = 1000, K1d = 100, K2d = 150(Tracking Error)

- **Joint 1 (q_1):** High damping is seen, but slow response.
- **Joint 2 (q_2):** Stable and smooth, but slow response.
- **Interpretation:** High Kd values cause more damping, The system becomes stable and oscillation-free, but at the cost of slower response.

4 PI Controller

PD control can leave a *steady-state error* when there are constant disturbances (gravity mismatch, friction, or modelling errors). An integrator accumulates persistent error and drives the steady-state error to zero.

Given,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) \\ \implies \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} &= \boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q}) \\ \implies \ddot{\boldsymbol{q}} &= \boldsymbol{M}^{-1}(\boldsymbol{q})\,[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})] \end{aligned}$$

We have a desired trajectory $\boldsymbol{q}_d(t)$ (with known $\dot{\boldsymbol{q}}_d(t), \ddot{\boldsymbol{q}}_d(t)$).We define the error as

$$\boldsymbol{e}_i = \boldsymbol{q}_{id} - \boldsymbol{q}_i$$

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$\dot{\boldsymbol{e}}_i = \dot{\boldsymbol{q}}_{id} - \dot{\boldsymbol{q}}_i$
$\ddot{\boldsymbol{e}}_i = \ddot{\boldsymbol{q}}_{id} - \ddot{\boldsymbol{q}}_i$

Given,

$$\boldsymbol{q}_{desired} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ rad}$$

$$\implies e_i = -q_i$$

$$\boldsymbol{\xi}(t) = \int_0^t \boldsymbol{e}(s)\,ds, \qquad \dot{\boldsymbol{\xi}} = \boldsymbol{e}. \tag{15}$$

We choose the target dynamics that include integral action:

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_p\boldsymbol{e} + \boldsymbol{K}_i\boldsymbol{\xi} = \boldsymbol{0}, \qquad \dot{\boldsymbol{\xi}} = \boldsymbol{e}. \tag{16}$$

for integral and proportional control,
WKT $\tau_i = K_{ip}e_i + K_{iI}\int e_i\,dt$
$\implies \tau_i = -K_{ip}q_i - K_{iI}\int q_i\,dt$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = -\begin{bmatrix} K_{1p}q_1 + K_{1I}\int q_1\,dt \\ K_{2p}q_2 + K_{2I}\int q_2\,dt \end{bmatrix}$$

$$K_p = \begin{bmatrix} K_{1p} & 0 \\ 0 & K_{2p} \end{bmatrix}, \quad K_I = \begin{bmatrix} K_{1I} & 0 \\ 0 & K_{2I} \end{bmatrix}$$

$$\boldsymbol{\tau} = -K_p\boldsymbol{q} - K_I\int \boldsymbol{q}\,dt \tag{17}$$

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad \int \boldsymbol{q}\,dt = \begin{bmatrix} \int q_1\,dt \\ \int q_2\,dt \end{bmatrix}$$

From equation (17),

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau}$$
$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = -K_p\boldsymbol{q} - K_I\int \boldsymbol{q}\,dt$$
$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + K_p\boldsymbol{q} + K_I\int \boldsymbol{q}\,dt = 0$$

State space variables are as follows,

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$
$$x_4 = \dot{q}_2$$
$$x_5 = \int e_1(q_1)\,dt$$
$$x_6 = \int e_2(q_2)\,dt$$

$$\ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{q})\left[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})\right]$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \\ \int e_1(q_1)\,dt \\ \int e_2(q_2)\,dt \end{bmatrix}$$

$$
\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \\ e_1(q_1) \\ e_2(q_2) \end{bmatrix}
$$

PI Controller by varying the K_p and keeping K_I constant

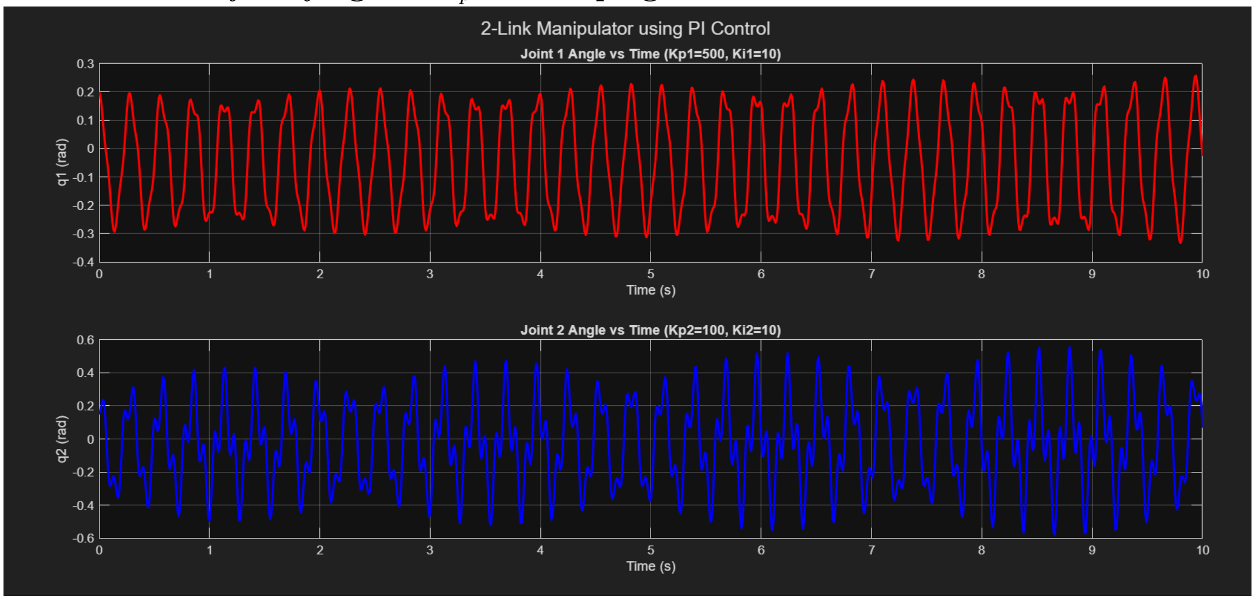

Figure 11: K1p = 500, K2p = 100, K1I = K2I = 10(Joint Angle)

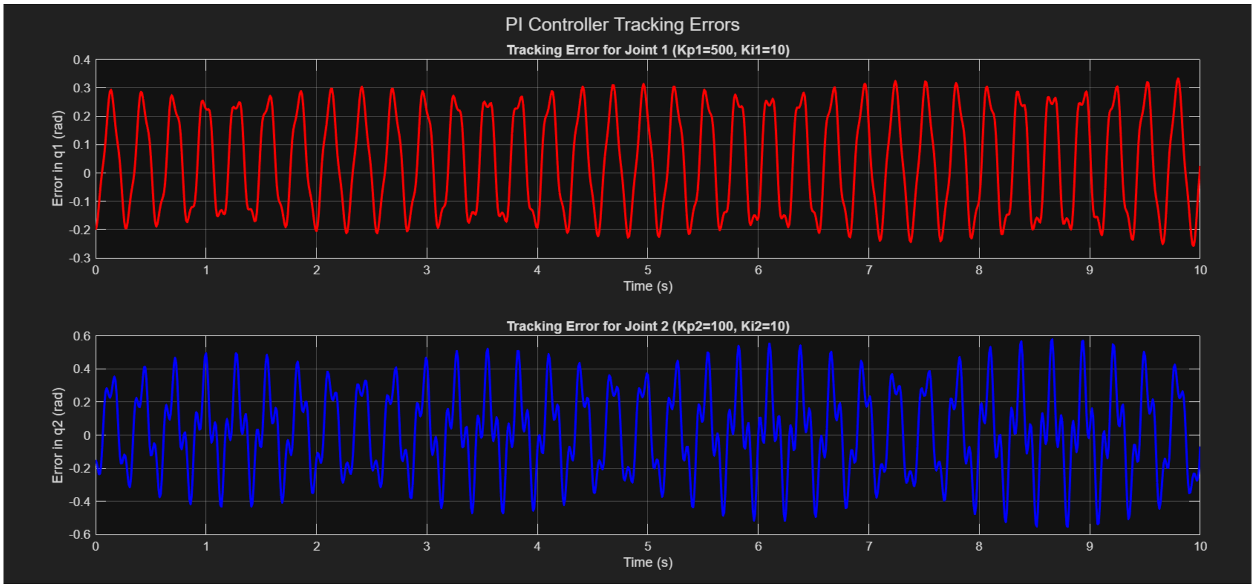

Figure 12: K1p = 500, K2p = 100, K1I = K2I = 10(Tracking Error)

- **Joint 1 (q_1):** Shows a moderately responsive trajectory with small oscillations and gradual convergence toward the desired angle.
- **Joint 2 (q_2):** Shows minimal oscillations and slow movement toward the target.
- **Interpretation:** The low K_{p2} value results in weak corrective torque for joint 2, leading to delayed response and poor tracking. Joint 1 performs better due to its relatively higher gain.

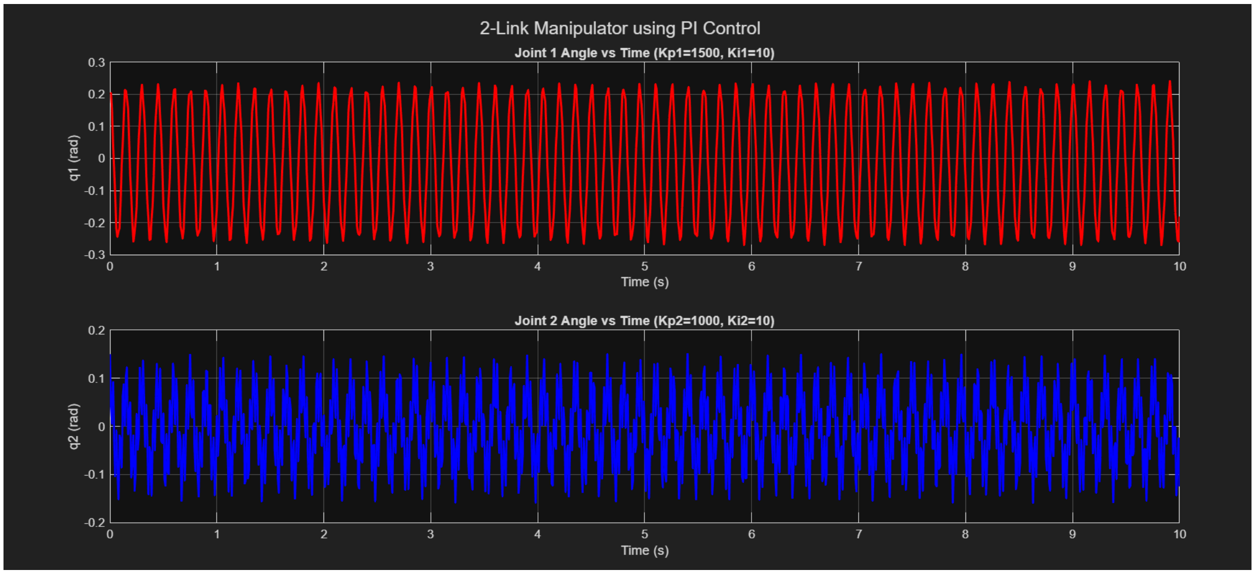

Figure 13: K1p = 1500, K2p = 1000, K1I = K2I = 10(Joint Angle)

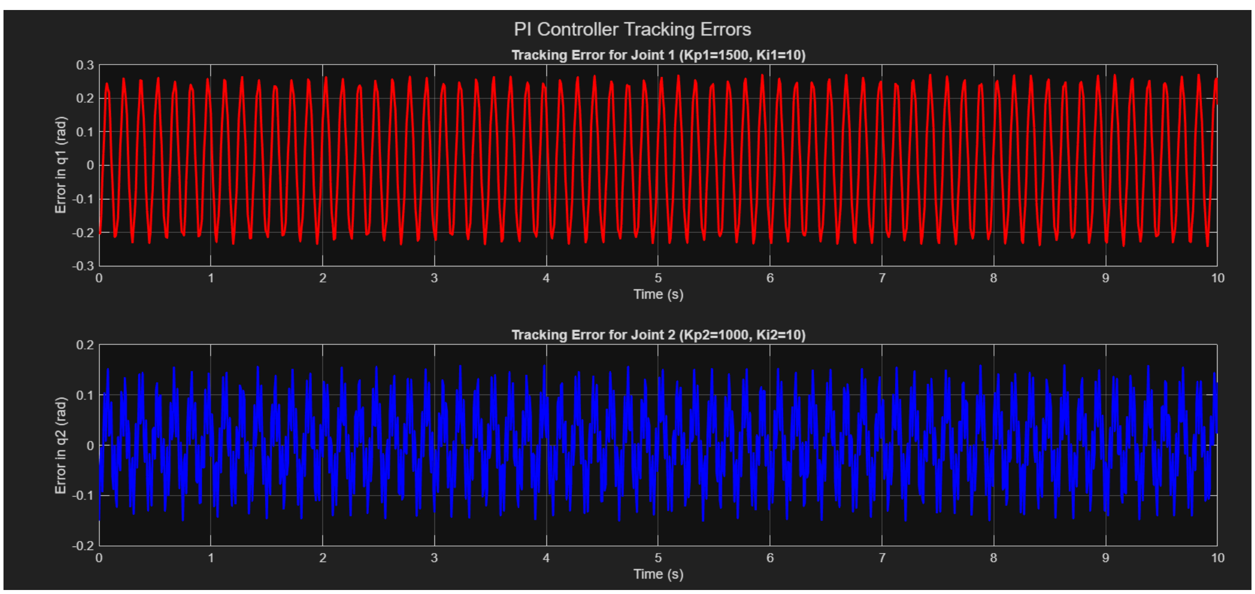

Figure 14: K1p = 1500, K2p = 1000, K1I = K2I = 10(Tracking Error)

- **Joint 1 (q_1):** responds aggressively, with high-frequency oscillations and faster convergence.

- **Joint 2 (q_2):** displays complex oscillatory behavior with increased amplitude and frequency, indicating potential instability.
- **Interpretation:** High K_p values lead to strong corrective actions. While this improves speed, it also risks overshoot and instability especially in joint 2, which is more sensitive to dynamic coupling.

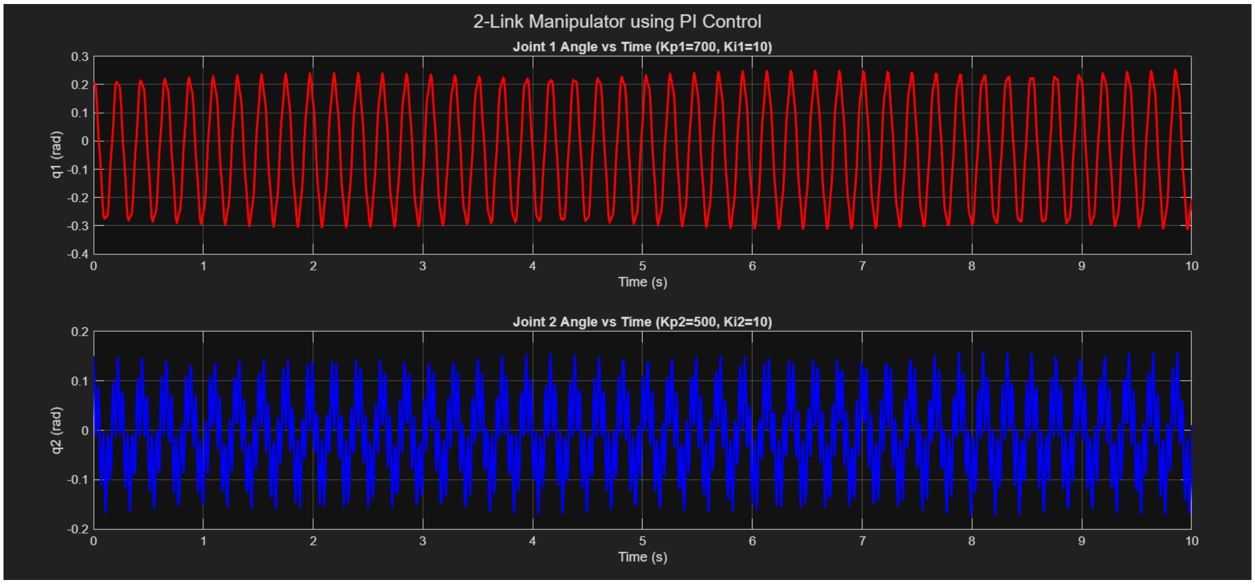

Figure 15: K1p = 700, K2p = 500, K1I = K2I = 10(Joint Angle)

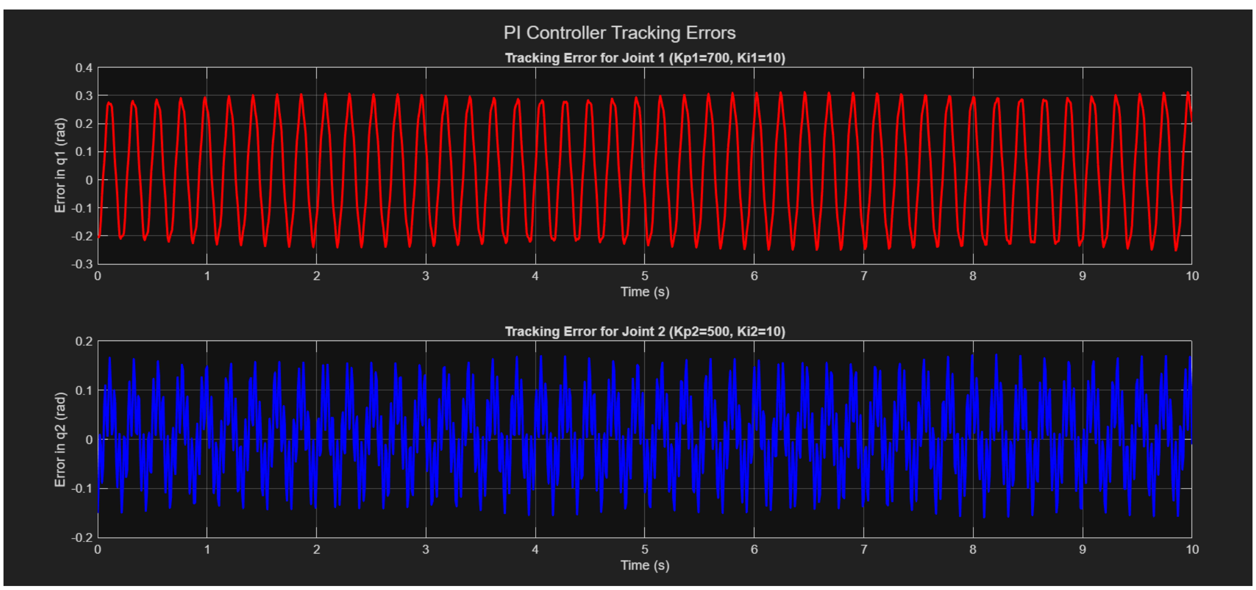

Figure 16: K1p = 700, K2p = 500, K1I = K2I = 10(Tracking Error)

- **Joint 1 (q_1):** demonstrates improved responsiveness with slightly faster convergence and controlled oscillations.

- **Joint 2 (q_2):** becomes more active, showing increased oscillatory behavior but still within stable bounds.

- **Interpretation:** Increasing K_{p2} enhances the responsiveness of joint 2, but also introduces more dynamic fluctuations. The system remains stable, and both joints show better tracking performance.

Conclusion: When we increase K_p while K_I is constant. Increasing K_p improves the speed of the response and reduces steady-state error. If we further increase K_p: We observe overshoot and oscillations and instability, especially if K_I is not tuned to balance the integral action. Moderate K_p values offer a good trade-off between responsiveness and stability.

PI Controller by varying the K_I and keeping K_p constant

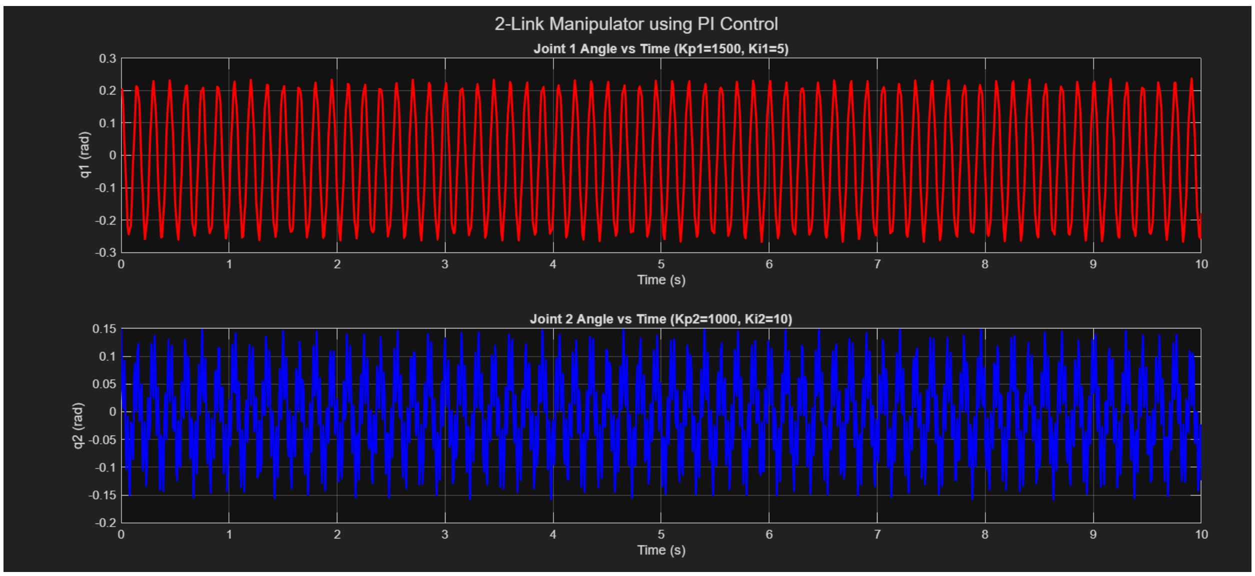

Figure 17: K1p = 1500, K2p = 1000, K1I = 5, K2I = 10(Joint Angle)

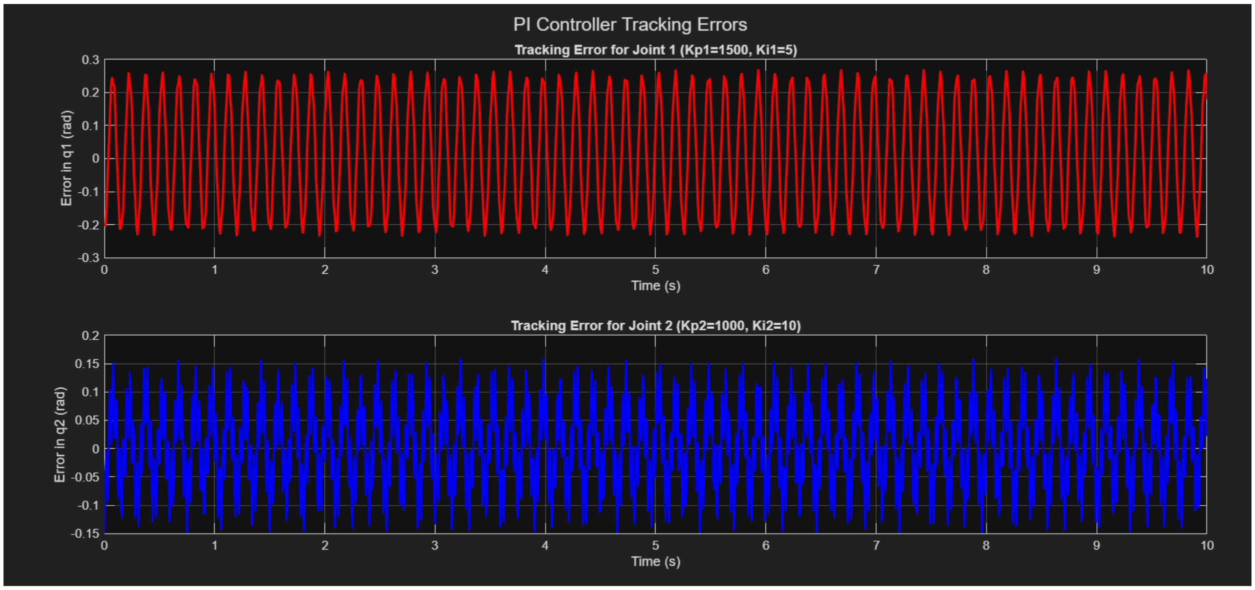

Figure 18: K1p = 1500, K2p = 1000, K1I = 5, K2I = 10(Tracking Error)

- **Joint 1 (q_1):** shows fast response due to high Kp, but may not settle exactly at zero.
- **Joint 2 (q_2):** responds moderately, with small oscillations and slow convergence.
- **Interpretation:** Low Ki values mean the integral term contributes weakly. The system relies mostly on Kp, leading to fast but imprecise settling. Steady-state error may persist.

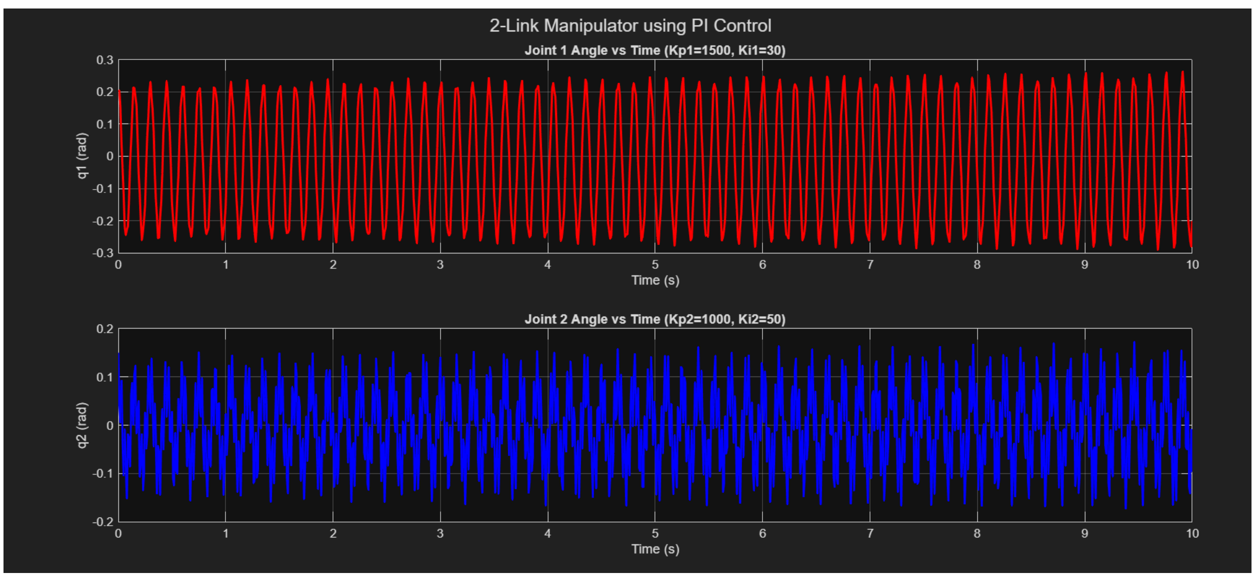

Figure 19: K1p = 1500, K2p = 1000, K1I = 30, K2I = 50(Joint Angle)

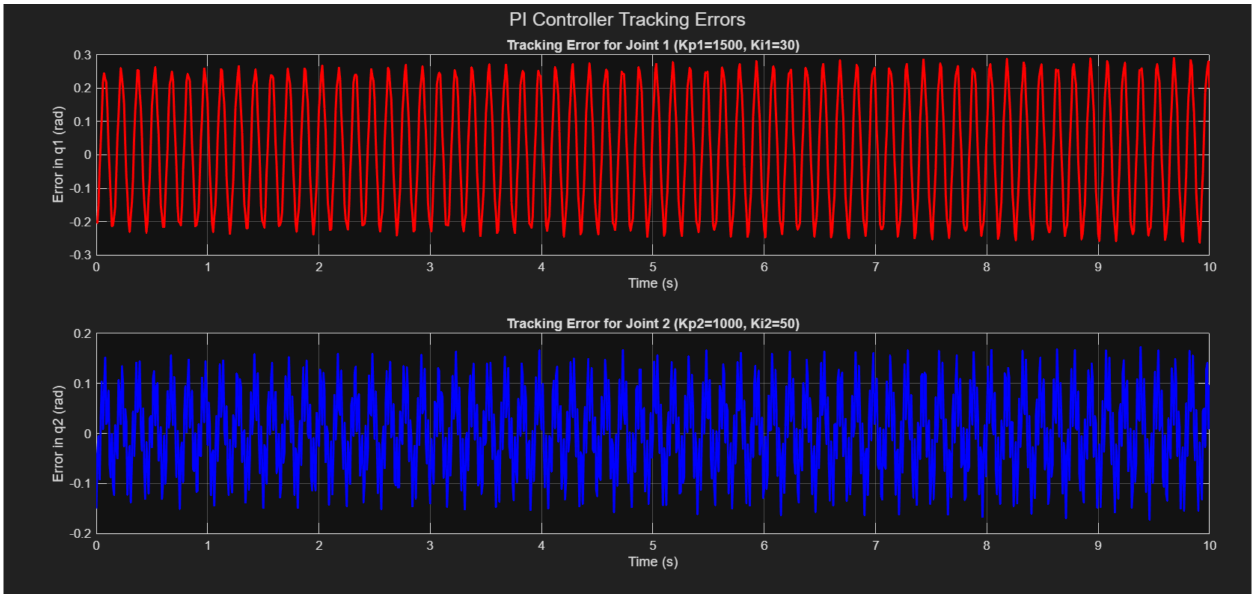

Figure 20: K1p = 1500, K2p = 1000, K1I = 30, K2I = 50(Tracking Error)

- **Joint 1 (q_1):** shows improved convergence with mild oscillations
- **Joint 2 (q_2):** becomes more active, with sharper oscillations and faster correction.
- **Interpretation:**Moderate Ki values enhance the controller's ability to eliminate residual error. The system becomes more reactive, but oscillations increase due to stronger integral action.

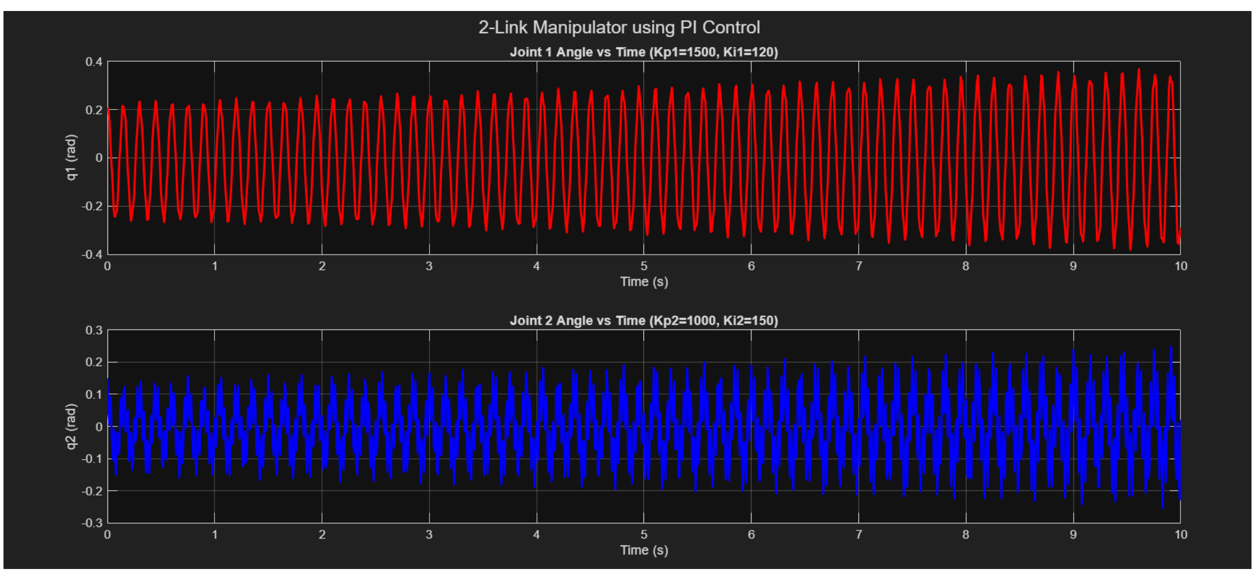

Figure 21: K1p = 1500, K2p = 1000, K1I = 120, K2I = 150(Joint Angle)

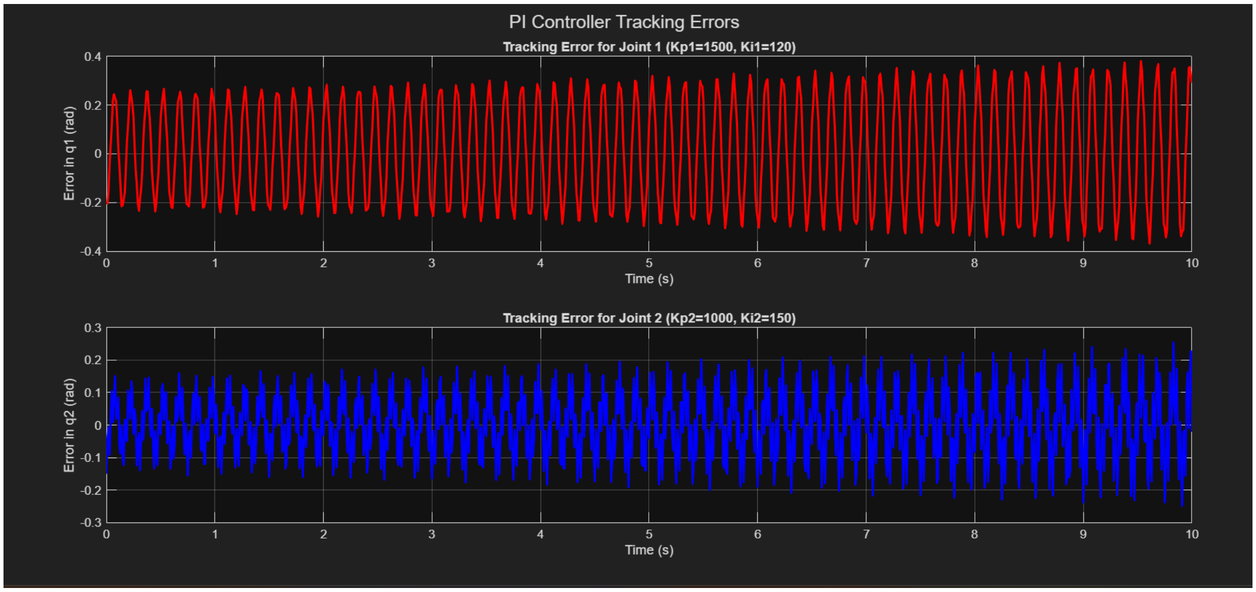

Figure 22: K1p = 1500, K2p = 1000, K1I = 120, K2I = 150(Tracking Error)

- **Joint 1 (q_1):** displays aggressive oscillations and possible instability.
- **Joint 2 (q_2):** shows complex, high-frequency oscillations with growing amplitude.
- **Interpretation:**: High Ki causes rapid accumulation of error, leading to excessive torque and overshoot. The system becomes unstable due to integral windup and dynamic coupling.

Conclusion: when we increase Ki while Kp is constant:Increasing Ki improves the system's ability to eliminate steady-state error over time.If we further increase Ki: We observe Overshoot ,Oscillations and Instability, especially if Kp is not tuned to counteract aggressive integral action. Moderate Ki values offer a good trade-off between accuracy and stability.

5 PID Controller

PID control combines proportional, integral, and derivative actions to improve both transient and steady-state performance. The proportional term provides immediate response to errors, the integral term eliminates steady-state errors caused by constant disturbances, and the derivative term predicts system behavior and improves stability by damping oscillations.

Given,

$$\begin{aligned}
\boldsymbol{\tau} &= \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) \\
\implies \boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} &= \boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q}) \\
\implies \ddot{\boldsymbol{q}} &= \boldsymbol{M}^{-1}(\boldsymbol{q})\left[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q})\right]
\end{aligned}$$

We have a desired trajectory $\boldsymbol{q}_d(t)$ (with known $\dot{\boldsymbol{q}}_d(t), \ddot{\boldsymbol{q}}_d(t)$).We define the error as

$$\boldsymbol{e}_i = \boldsymbol{q}_{id} - \boldsymbol{q}_i$$

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$\dot{\boldsymbol{e}}_i = \dot{\boldsymbol{q}}_{id} - \dot{\boldsymbol{q}}_i$
$\ddot{\boldsymbol{e}}_i = \ddot{\boldsymbol{q}}_{id} - \ddot{\boldsymbol{q}}_i$

$$\xi = \int e(q)\, dt, \qquad \dot{\xi} = e$$

We choose the PID target:

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_d \dot{\boldsymbol{e}} + \boldsymbol{K}_p \boldsymbol{e} + \boldsymbol{K}_i \boldsymbol{\xi} = \mathbf{0}, \qquad \dot{\boldsymbol{\xi}} = \boldsymbol{e}. \tag{18}$$

This is a third-order linear ODE per joint (when diagonal gains are used) with damping controlled by $\boldsymbol{K}_d$ and steady-state elimination by $\boldsymbol{K}_i$.

Given,

$$\boldsymbol{q}_{desired} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ rad}$$

$$\begin{aligned} \implies e_i &= -q_i \\ \dot{e}_i &= -\dot{q}_i \end{aligned}$$

for integral and proportional control,
WKT $\tau_i = K_{ip} e_i + K_{iI} \int e_i \, dt + K_{id} \dot{e}_i$
$\implies \tau_i = -K_{ip} q_i - K_{iI} \int q_i \, dt - K_{id} \dot{q}_i$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = - \begin{bmatrix} K_{1p} q_1 + K_{1I} \int q_1 \, dt + +K_{1d} \dot{q}_1 \\ K_{2p} q_2 + K_{2I} \int q_2 \, dt + +K_{2d} \dot{q}_2 \end{bmatrix}$$

$$K_p = \begin{bmatrix} K_{1p} & 0 \\ 0 & K_{2p} \end{bmatrix}, \quad K_I = \begin{bmatrix} K_{1I} & 0 \\ 0 & K_{2I} \end{bmatrix}, \quad K_d = \begin{bmatrix} K_{1d} & 0 \\ 0 & K_{2d} \end{bmatrix}$$

$$\boldsymbol{\tau} = -K_p \boldsymbol{q} - K_I \int \boldsymbol{q} \, dt - K_d \dot{q} \tag{19}$$

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad \int \boldsymbol{q} \, dt = \begin{bmatrix} \int q_1 \, dt \\ \int q_2 \, dt \end{bmatrix}$$

From equation (19),

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) &= \boldsymbol{\tau} \\ \boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) &= -K_p \boldsymbol{q} - K_I \int \boldsymbol{q} \, dt - K_d \dot{\boldsymbol{q}} \end{aligned}$$

$$\boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) + K_p \boldsymbol{q} + K_I \int \boldsymbol{q} \, dt + K_d \dot{\boldsymbol{q}} = 0$$

State space variables are as follows,

$$
\begin{aligned}
x_1 &= q_1 \\
x_2 &= \dot{q}_1 \\
x_3 &= q_2 \\
x_4 &= \dot{q}_2 \\
x_5 &= \int e_1(q_1)\, dt \\
x_6 &= \int e_2(q_2)\, dt
\end{aligned}
$$

$$\ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{q}) \left[\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{G}(\boldsymbol{q}) \right]$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \\ \int e_1(q_1)\, dt \\ \int e_2(q_2)\, dt \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \\ e_1(q_1) \\ e_2(q_2) \end{bmatrix}$$

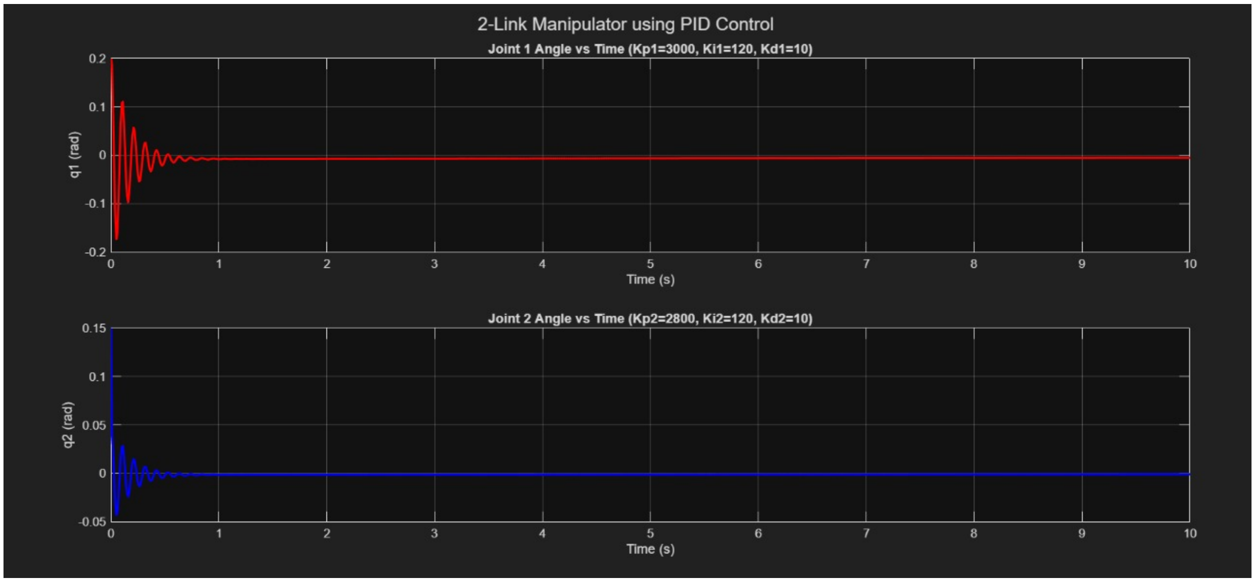

Figure 23: K1p = 3000, K1I = 120, K1d = 10, K2p = 2800, k2I = 120, k2d = 10(Joint angle)

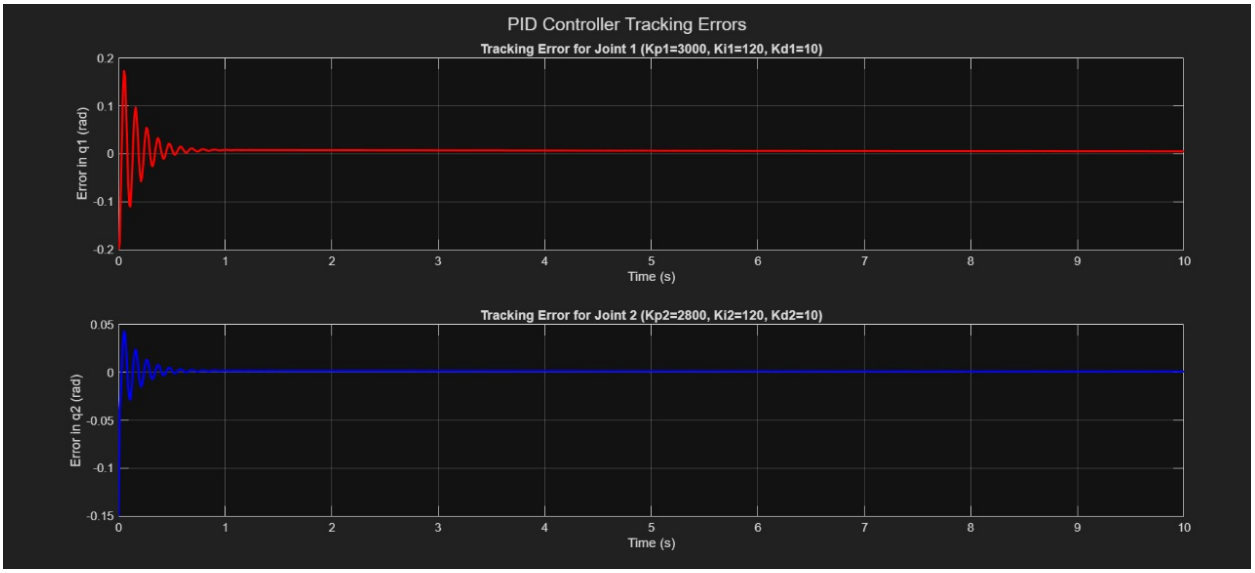

Figure 24: K1p = 3000, K1I = 120, K1d = 10, K2p = 2800, k2I = 120, k2d = 10(Tracking error)

- **Joint 1 (q_1):** shows a fast rise with limited overshoot and quick damping.
- **Joint 2 (q_2):**displays slower but smoother convergence toward the reference.
- **Interpretation:**:The large proportional gains dominate the transient dynamics, giving sharp responses, while derivative action suppresses oscillations. Integral gain ensures both joints settle near the target with minimal error.

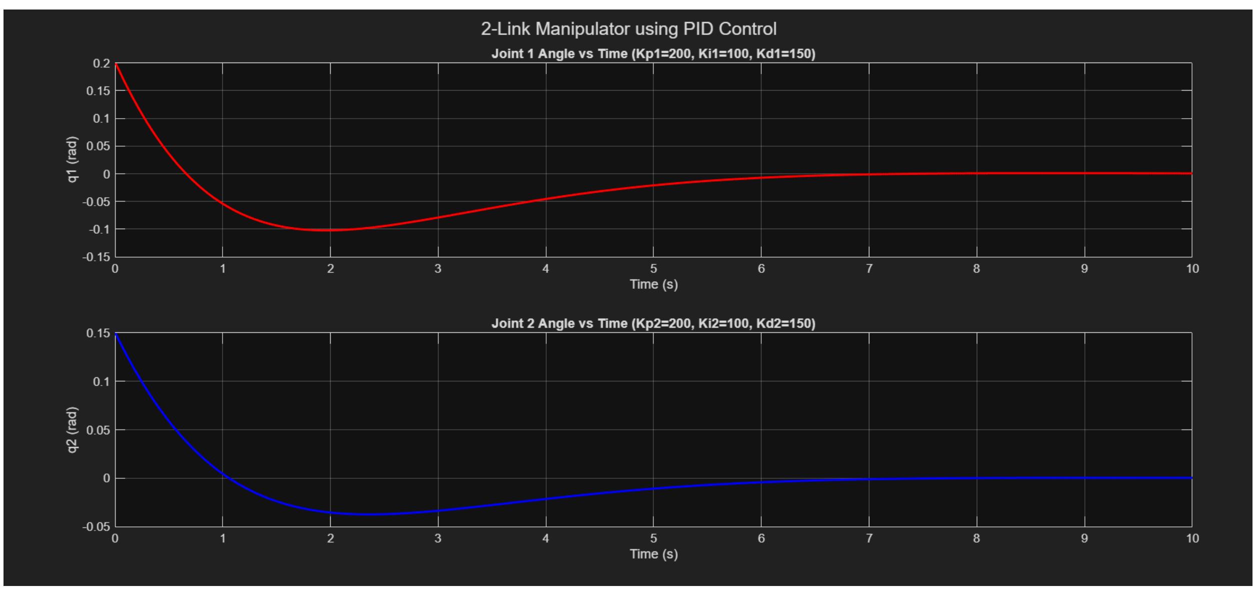

Figure 25: K1p = 200, K1I = 100, K1d = 150, K2p = 200, k2I = 100, k2d = 150(Joint angle)

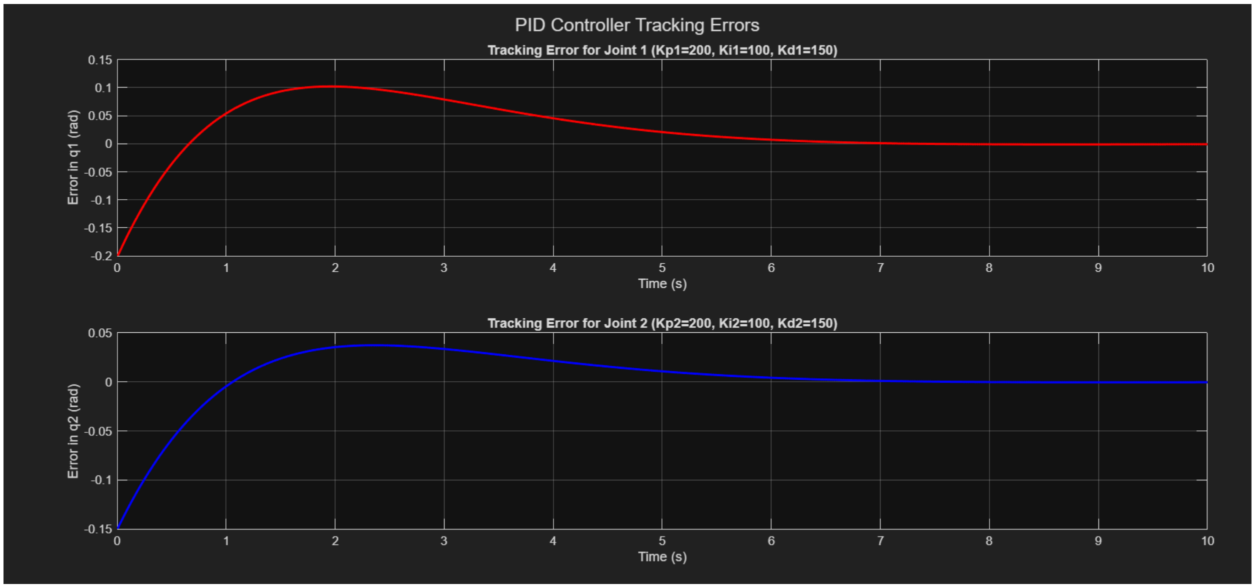

Figure 26: K1p = 200, K1I = 100, K1d = 150, K2p = 200, k2I = 100, k2d = 150(Tracking error)

- **Joint 1 (q_1):** Shows a smooth and well-damped response with minimal overshoot. The trajectory converges steadily toward the desired angle, indicating stable control with balanced proportional, integral, and derivative actions.
- **Joint 2 (q_2):**Exhibits a slightly slower response compared to Joint 1 but maintains stability with no significant oscillations. The angle settles gradually with a small steady-state offset.
- **Interpretation:**Kp ensures decent responsiveness. Kd ensures in damp oscillations and stabilizes both joints. KI eliminates steady-state error. Overall, it shows stable convergence toward the target and shows good error correction.

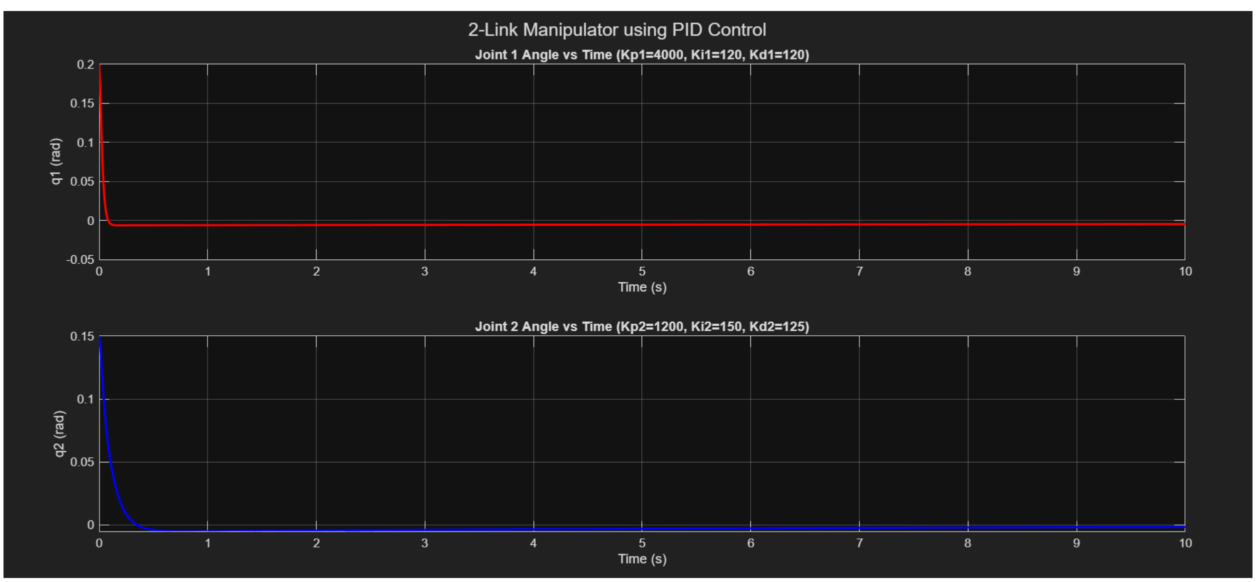

Figure 27: K1p = 4000, K1I = 120, K1d = 120, K2p = 1200, k2I = 150, k2d = 125(Joint angle)

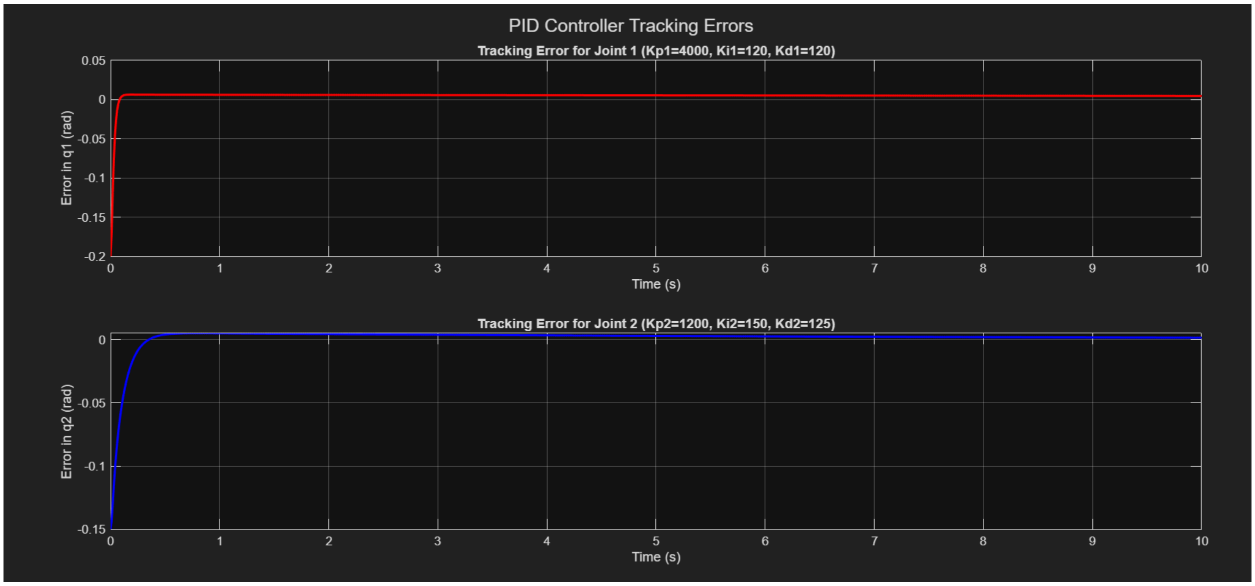

Figure 28: K1p = 4000, K1I = 120, K1d = 120, K2p = 1200, k2I = 150, k2d = 125(Tracking error)

- **Joint 1 (q_1):** Exhibits smooth, non-oscillatory convergence with rapid settling near zero error.
- **Joint 2 (q_2):** Also converges quickly with minimal overshoot and stable steady-state response.
- **Interpretation:**These gains provide a balance of effective damping and stability. KI eliminates steady-state error. Overall, the system shows good tracking performance and stability.

Justification for Selected PID Gains

Joint 1 (q_1): $K_{p1} = 4000,\ K_{i1} = 120,\ K_{d1} = 120$

- **High K_{p1} (4000):**
  - Provides strong proportional correction, ensuring fast convergence to the desired angle.
  - Helps reduce large initial errors quickly.
- **Moderate K_{i1} (120):**
  - Accumulates error over time to eliminate steady-state offset.
  - Tuned to avoid integral windup while ensuring final accuracy.
- **High K_{d1} (120):**
  - Offers strong damping to suppress overshoot and oscillations caused by aggressive K_p.
  - Ensures smooth settling and system stability.

Result: Joint 1 reaches the desired position rapidly with minimal overshoot and zero steady-state error.

Joint 2 (q_2): $K_{p2} = 1200,\ K_{i2} = 150,\ K_{d2} = 125$

- **Moderate K_{p2} (1200):**
  - Ensures responsive behavior without excessive aggressiveness.
  - Balances speed and control for smoother convergence.
- **Strong K_{i2} (150):**
  - Drives the system to eliminate steady-state error effectively.
  - Tuned to dominate low-frequency error without destabilizing the system.
- **High K_{d2} (125):**
  - Provides robust damping to counteract the effects of high K_i.
  - Prevents oscillations and ensures a critically damped response.

Result: Joint 2 maintains excellent stability and settles precisely at the desired position.

Simulink Implementation

Initially, the input joint angles $r_1(t)$ and $r_2(t)$ are set to 0 rad. The outputs obtained from the plant are the actual joint angles q_1 and q_2 at each instant of time. These angles are fed back through unity feedback to compute the instantaneous error signals as

$$e_1 = r_1 - q_1, \qquad e_2 = r_2 - q_2.$$

The corresponding errors are then sent into the PID controllers, which are tuned appropriately based on the desired performance criteria. The outputs of the controllers are the desired angular accelerations $\ddot{q}$, which are multiplied by the inertia matrix $M(q)$ to obtain the joint torques τ:

$$\tau = M(q)\ddot{q}.$$

These torques act as the actual inputs to the plant. The plant block models the system dynamics based on the equation

$$\ddot{q} = M^{-1}(q)\tau - M^{-1}(q)\big[c(q,\dot{q})\dot{q} + G(q)\big],$$

where $M(q)$ is the inertia matrix, $c(q,\dot{q})$ represents centrifugal effects, and $G(q)$ denotes the gravitational torque.

The plant integrates $\ddot{q}$ to obtain the joint velocities $\dot{q}$ and the joint angles q. Through continuous feedback of the joint positions, the control loop updates the torque commands at every instant. As a result, the PID controller gradually reduces the tracking error, bringing the joint angles to their steady-state values.

The overall closed-loop system thus estimates the joint angle errors at each instant and generates appropriate torque commands through the PID (or PD/PI) controllers to achieve the desired motion.

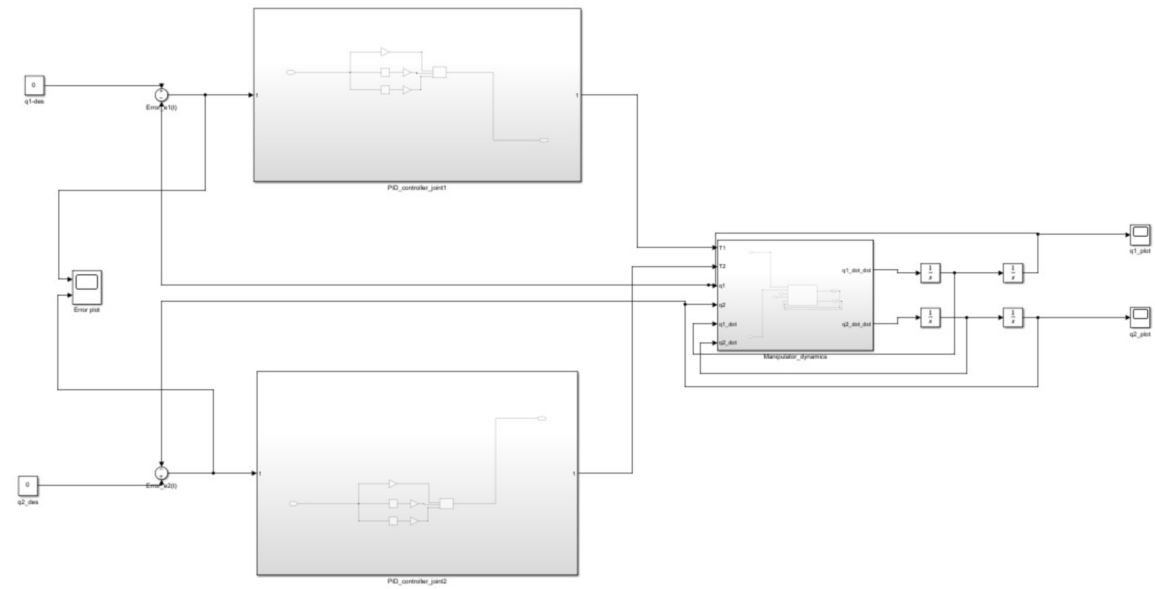

Figure 29: PID Controller Simulink

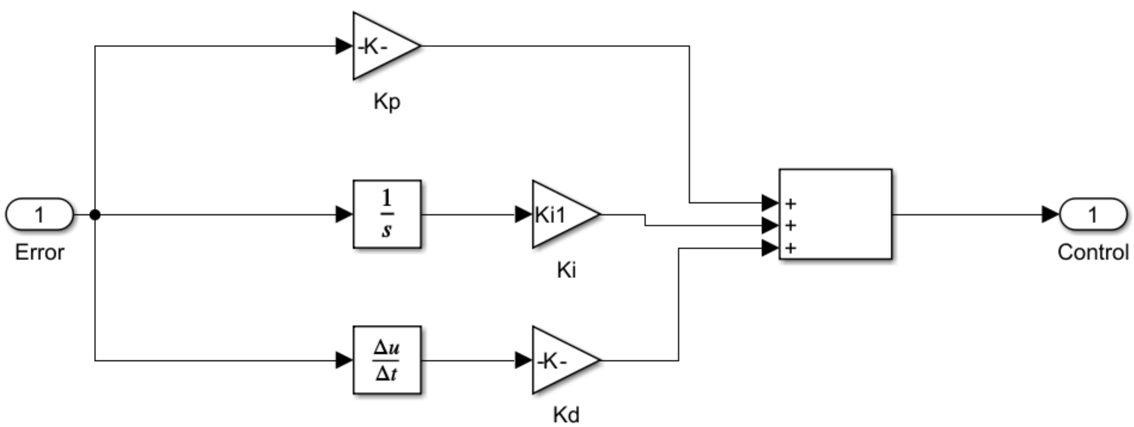

Figure 30: Joint 1 PID Controller

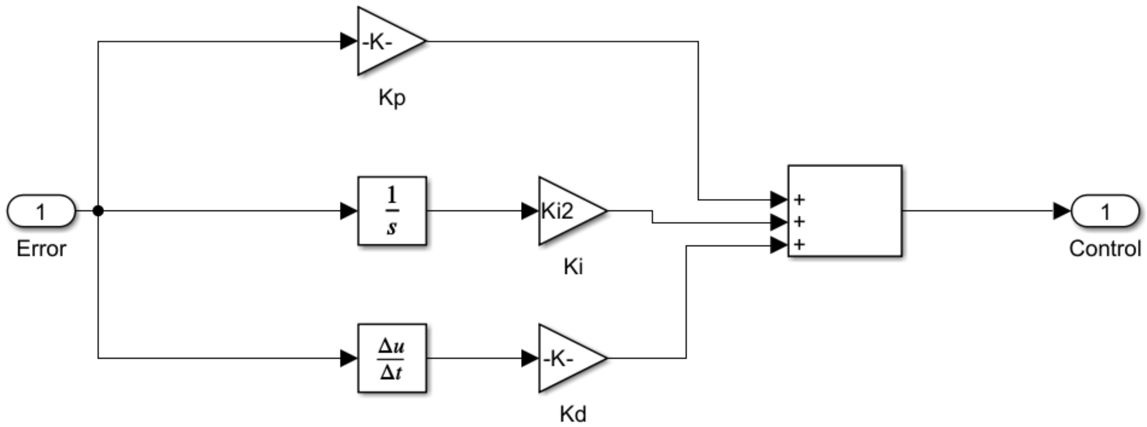

Figure 31: Joint 2 PID Controller

PID Plots in Simulink

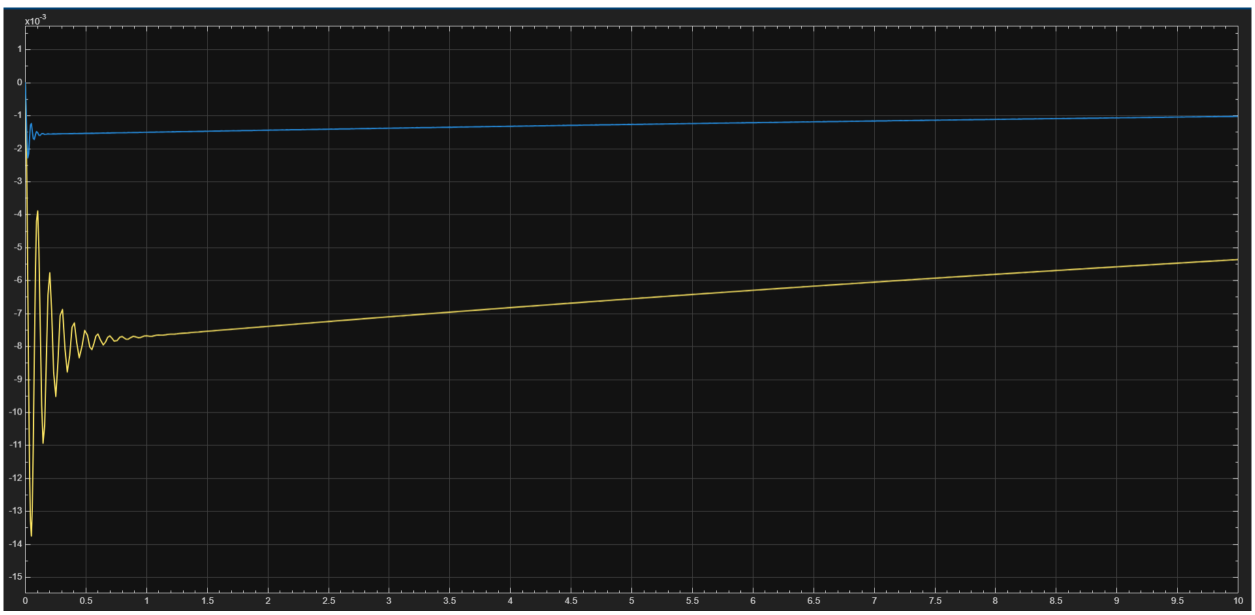

Figure 32: K1p = 3000, K1I = 120, K1d = 10, K2p = 2800, k2I = 120, k2d = 10

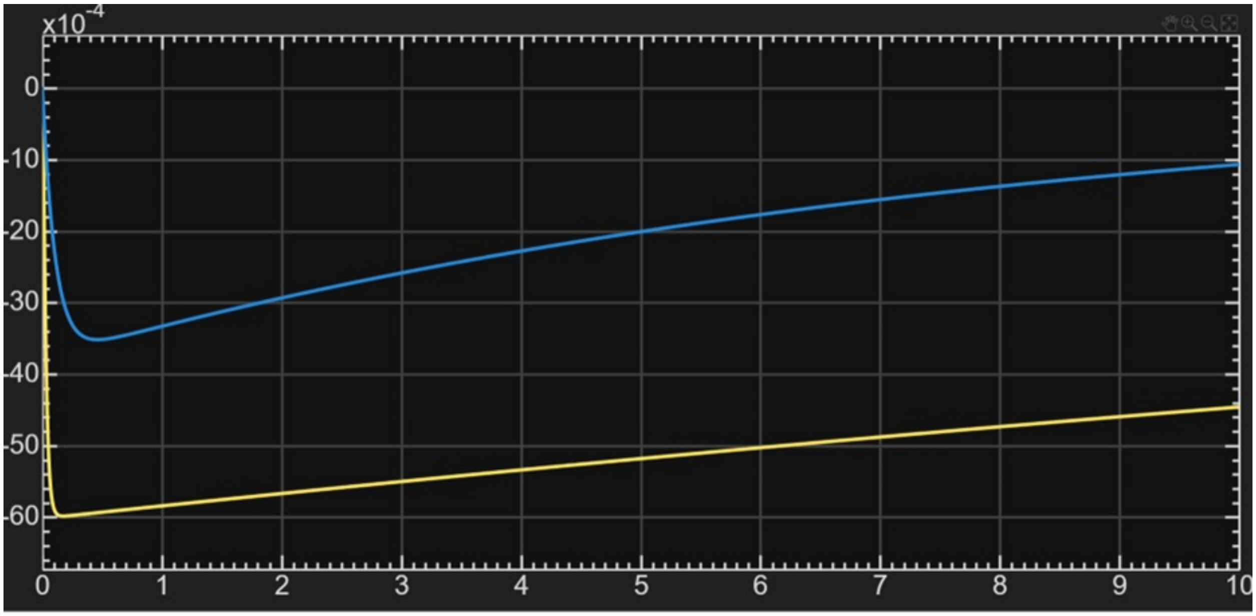

Figure 33: K1p = 4000, K1I = 120, K1d = 120, K2p = 1200, k2I = 150, k2d = 125

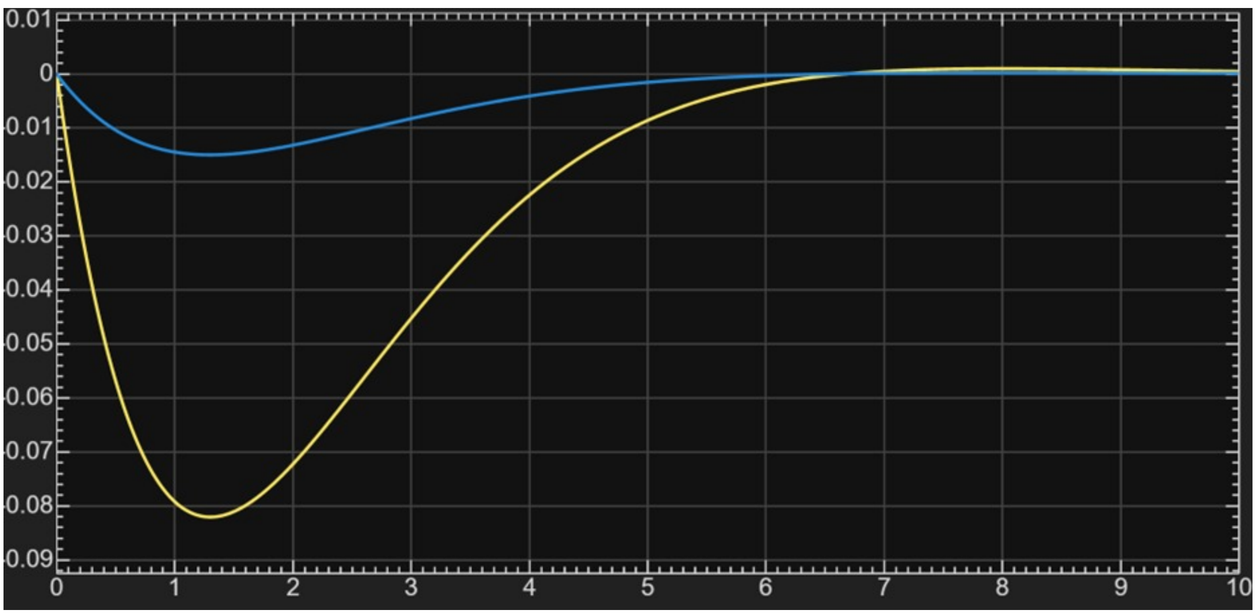

Figure 34: K1p = 200, K1I = 100, K1d = 150, K2p = 200, k2I = 100, k2d = 150

PD Plots in Simulink

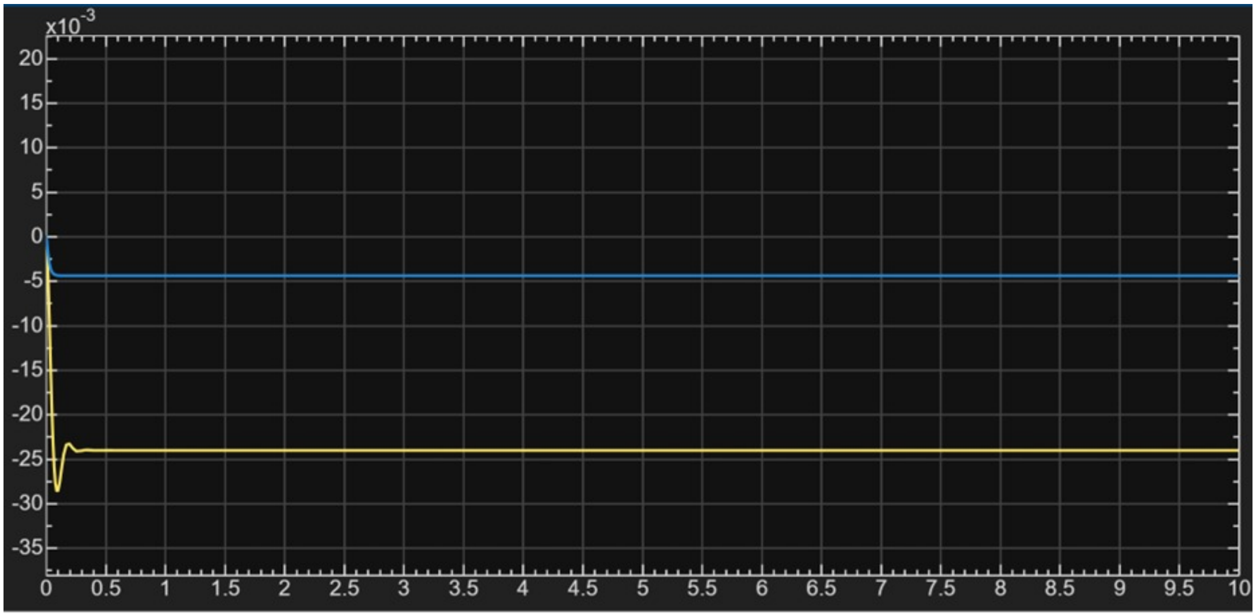

Figure 35: K1p = 1500, K1I = 0, K1d = 25, K2p = 1000, k2I = 0, k2d = 25

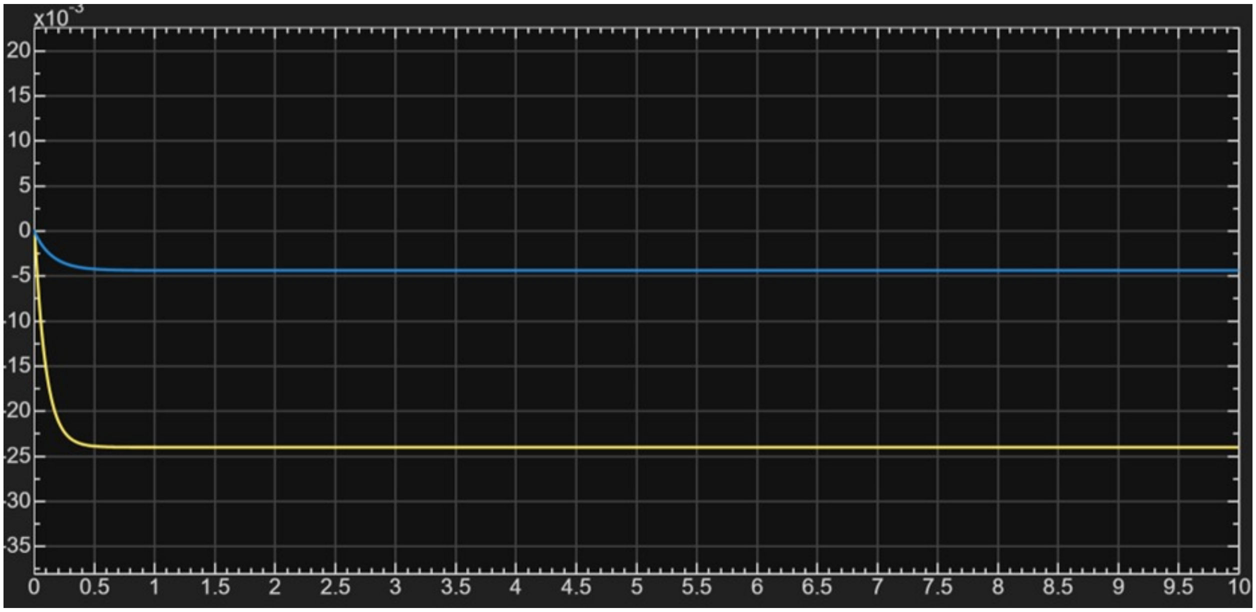

Figure 36: K1p = 1000, K1I = 0, K1d = 100, K2p = 1000, k2I = 0, k2d = 150

PI Plots in Simulink

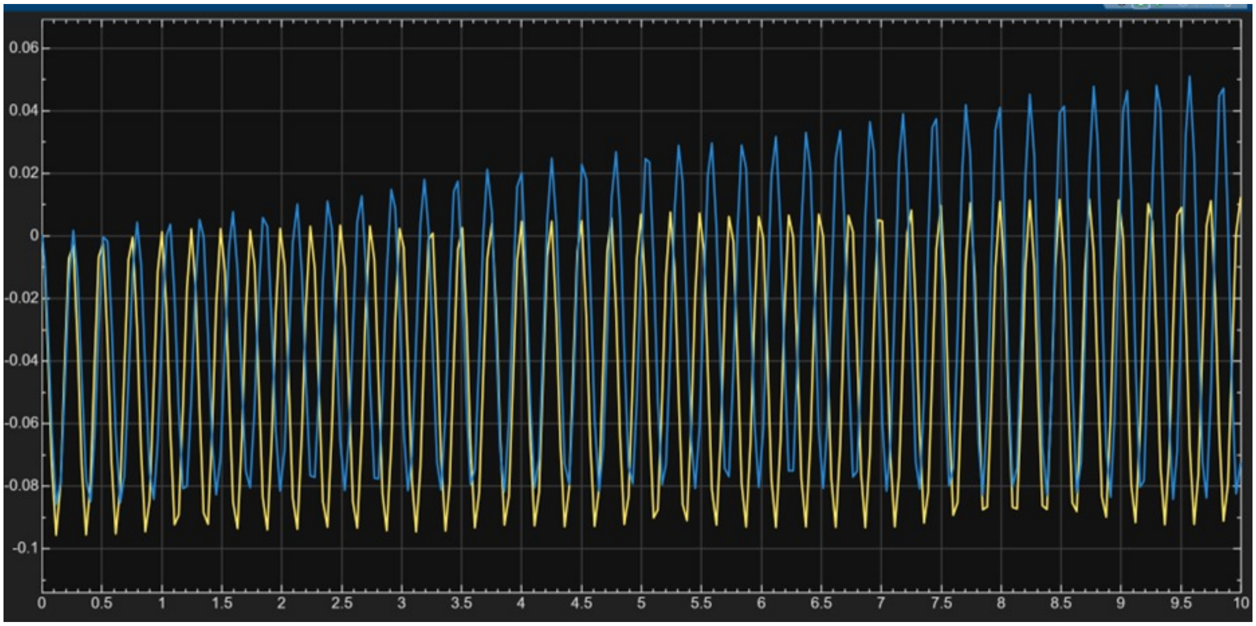

Figure 37: K1p = 500, K1I = 10, K1d = 0, K2p = 100, k2I = 10, k2d = 0

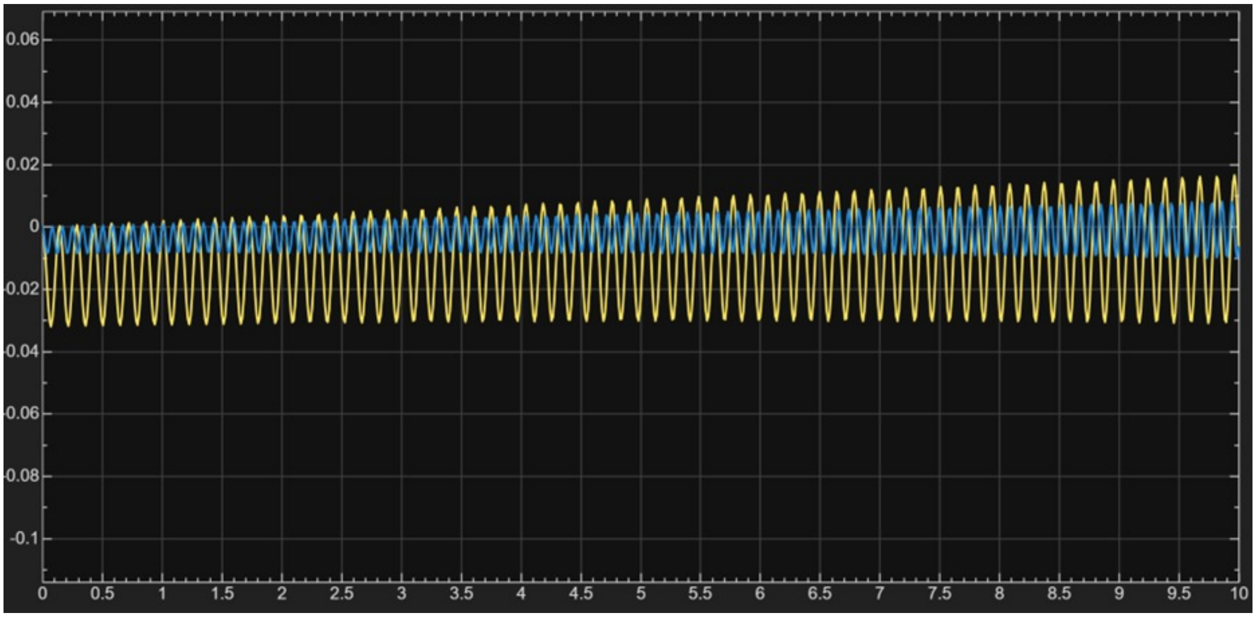

Figure 38: K1p = 1000, K1I = 120, K1d = 0, K2p = 1000, k2I = 150, k2d = 0

Gain Type	High Gain Behavior	Low Gain Behavior
Proportional Gain (KP)	• Quickly minimizes steady-state error by proportionally amplifying the control signal. • Enhances accuracy and responsiveness. • May introduce oscillations or instability. • Can cause overshoot and erratic system behavior due to excessive correction.	• Produces a more stable and smoother response. • Less prone to oscillations and instability. • Slower error correction and longer settling time. • Reduced accuracy in reaching the target value.
Derivative Gain (KD)	• Reduces oscillations and improves damping characteristics. • Provides smooth transitions and enhances stability when tuned properly. • May lead to an overly damped or sluggish response.	• Enables fast initial reaction to changes. • Ineffective at damping oscillations, possibly causing instability.
Integral Gain (KI)	• Strong corrective action for persistent or long-term errors. • Improves accuracy by eliminating steady-state errors. • May cause instability or unpredictable behavior in noisy systems.	• Provides smooth, gradual adjustments. • Lowers the risk of oscillations or instability. • May result in remaining steady-state errors.

Table 1: Combined Effect of Proportional, Derivative, and Integral Gains on System Behavior

Contributions

This project was carried out through the collective effort of all team members(**Veekshita, Chandralekhya, Chandini, Sukeerthi and Padmanjali**), with each individual actively participating in every component.

Throughout the project, Sukeerthi worked on PD controller,Chandralekhya worked on PI controller,Veekshita worked on PID controller, Chandini and Padmanjali together worked on the Simulink part for all of them. Each member wrote the report for their corresponding coding part and All of us went through each other's work so that we could get the full understanding and make our report better.