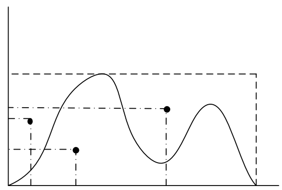
Numerical Integration Using Monte Carlo Method



Given a single valued function f(x) as shown in the figure above, the goal is to determine the integral

$$I = \int_{x_1=0}^{x_2=x_{\max}} f(x)\,dx$$

The above integral is the area under the curve represented by a solid line in the above figure.

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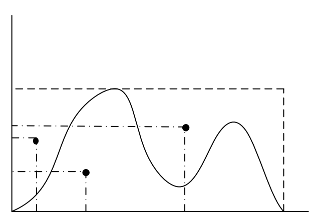

In order to use the Monte method, we need two parameters:

(I) Range of integration. In the above case it runs from $x_1 = 0$ to $x_2 = x_{max}$. Therefore the full range of integration:

$$x_2 - x_1 = x_{\max} - 0 = x_{\max}$$

(II) Maximum value of the function f(x) in the range of integration: f_{max}. Values larger than the exact f_{max} are acceptable.

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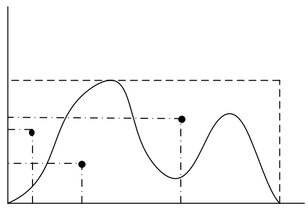
$$I = \int_{x_1=0}^{x_2=x_{\max}} f(x)dx$$



The parameters f_{max} and x_{max} define the sides of a rectangle as shown above. The area of the rectangle is given by:

$$Area\_A = f_{\max} * x_{\max}$$

The integral I of the function f(x) is part of the rectangle defined by f_{max} and x_{max}.

Using Monte Carlo to perform the integration amounts to generating a random sequence of points (xr , fr) and checking to see if the points are under the curve defined by f(x) or not.

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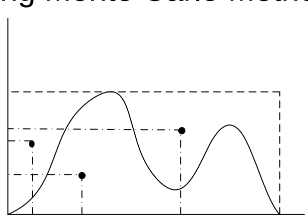
$$I = \int_{x_1=0}^{x_2=x_{\max}} f(x)dx$$



1. generate a pair of random numbers r_1 and r_2 . Note that: $0 \le r_1 \le 1$ and $0 \le r_2 \le 1$
2. Calculate $x_r = r_1 * x_{max}$ and $f_r = r_2 * f_{max}$.
3. Check if the point is under the curve. Check if $f_r \le f(x_r)$
4. If the condition in step (3) is true, then accept the point and update the counter for points under curve (N_accept).
   Note that out of the three points in the above figure only point (3) falls below the curve. For points (1) and (2) $f_r \rangle f(x_r)$
5. Repeat steps (1) through (4) large number of times (N_trials).
   Typical values of N_trials range from 10,000 to 1,000,000.
6. Compute the integral I (=Area under the curve): $I = \dfrac{N\_accept}{N\_trials} * (f_{\max} * x_{\max})$

Numerical Integration Using Monte Carlo Method

Example:
Evaluate the following integral using Monte Carlo method.

$$I = \int_0^{\pi} \cos^2 \theta d\theta$$

This can evaluated analytically and results in $I = I_{Actual} = \frac{\pi}{2} = 1.57$

Solution: We first determine the range of integration and the maximum value $f_{\max}$

1. Range of integration: $\pi - 0 = \pi$ $\quad x_{\max} = \pi$
2. Maximum value of the function $f(\theta) = \cos^2 \theta$ $\quad f_{\max} = 1$

(You may also use f_{max} greater than 1)

The number of trails was varied from 1,000 to 1,000,000.
The error in the integration was also calculated.

Numerical Integration Using Monte Carlo Method

$$I = \int_0^{\pi} \cos^2 \theta d\theta$$

$$I = I_{Actual} = \frac{\pi}{2} = 1.57$$

$$\Delta\% = \frac{\left| I_{Aactual} - I_{Monte} \right|}{I_{Actual}} x100$$

N_trials	I_{Monte}	Error (%)
1,000	1.529955	2.60
10,000	1.568911	0.12
100,000	1.566586	0.27
1,000,000	1.571855	0.07

Numerical Integration Using Monte Carlo Method

Random Number generation program:
(1) SERRN: sets the sequence of random numbers using a real number Q
(2) Rannum: generates random numbers r uniformly distributed in 0<r<1.

```
c----- set random number sequence using Q as seed.
c-----
      SUBROUTINE SETRN(Q)
      IMPLICIT REAL*8(A-H,O-Z)
           common /RANDY/   QA1,QA2,QB1,QB2,QBASE
C*  INITIALIZE WITH A CALL TO SETRN(0.D0-1.D0)
      QA1=2057713.0D0
      QA2=16676923.0D0
      QBASE=2.**24
      QC=DINT(QBASE*(QBASE*Q))
      QB1=DINT(QC/QBASE)
      QB2=QC-QB1*QBASE
      QB1=DMOD(QB1,QBASE)
      QB2=DINT(QB2/2.D0) * 2.D0 + 1.D0
      RETURN
      END
c
```


Numerical Integration Using Monte Carlo Method

```
C*******************************************************
C-- Random number generator
c--
      FUNCTION RANNUM(I)
      IMPLICIT REAL*8(A-H,O-Z)
           common /RANDY/   QA1,QA2,QB1,QB2,QBASE
C***  FROM CLAMPS AT NRCC - FROM KALOS
 10   QD2=QA2*QB2
      QE2=DINT(QD2/QBASE)
      QC2=QD2-QBASE*QE2
      QB1=DMOD(QE2+DMOD(QA1*QB2,QBASE)+DMOD(QA2*QB1,QBASE),QBASE)
      QB2=QC2
      RANNUM=QB1/QBASE
      IF(RANNUM.EQ.0.0D0)GO TO 10
      RETURN
      END
```

Problems:
Determine the following integrals using Monte Carlo method
(write a Matlab code or C):

1) $\int_0^{\pi} \sin^2(\pi \cos(3\theta)) \cos^2 \theta d\theta$

2) $\int_0^{10} \frac{x^3}{x^4+16} dx$

(plot and find an approximate maximum and add 0.1 to it). I_{actual}=1.60984. Estimate the error (%).

3) $\int_0^{\pi} \sin^4(3x) dx$