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Iterative methods for solutions of linear systems of equations

Lets say we have an equation of the form $Ax = b$, we can decompose the matrix A into the following form:

$$L = \begin{bmatrix} 0 & 0 & ... & 0 \\ a_{21} & 0 & ... & 0 \\ a_{31} & a_{32} & ... & 0 \\ ... & ... & ... & ... \\ a_{n-1} & ... & a_{n,n-1} & 0 \end{bmatrix} D = \begin{bmatrix} a_{11} & 0 & ... & 0 \\ 0 & a_{21} & ... & 0 \\ ... & ... & ... & ... \\ ... & ... & ... & a_{nm} \end{bmatrix}$$

...and an upper triangular matrix; that I am not going to write but you get the idea.

Theorem: If A is a diagonally dominant matrix then jacobi iteration converges to the solution of $Ax = b$.

$x_1 = D^{-1}(b - (L + U)x_0)$
$x_{k+1} = D^{-1}(b - (L + U)x_k)$
$x_{k+m} = D^{-1}(b - (L + U)x_k) = D^{-1}(b - (L + D + U)x_k + Dx_k)$
$\rightarrow= D^{-1}(b - Ax_k + Dx_k) = D^{-1}r_k + x_k = x_k + D^{-1}r_k$

Note: $b - Ax_k$ is he residual vector. Recall the residual is defined as:
$Ax = b \rightarrow r = b - Ax$

We can use this to create the conditional: if $||r_k||_2 \leq \texttt{tol} \rightarrow$ STOP.

Jacobi Iteration

Inputs: A, x_0

Loop:
$r = b - Ax$ for all $k = 0, 1, 2, ..$
$x_{k+1} = x_k + D^{-1}r_k$
error $= ||b - Ax_k||$
$x_k = x_{k+1}$

Since D is a diagonal matrix, D^{-1} will just be $\frac{1}{D}$, which gives us our x_{k+1} modified matrix.

Lets talk more about some tricks with the residual:
$r_{k+1} = b - Ax_{k+1} = b - A(x_k + D^{-1}r_k)$
$\rightarrow= (b - Ax_k) - AD^{-1}r_k$

Guass-Seidel

$A = (L + D + U)$
$Ax = b$
$(D + U)x = b - Lx$
$(D + U)x_{k+1} = (b - Lx) \rightarrow (D + U)x_{k+1} = (b - Lx_k)$
$x = (D + U)^{-1}(b - Lx)$
$x^{(k+1)} = (D + U)^{-1}(b - Lx^k)$

We can use our Back-substitution routine to find $(D + U)^{-1}$