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Homework:
For question 4: determine

$$D_n h(y) = c_1 u(x_1) + c_2 u(x_2) + ... + c_n u(x_n)$$

What we will want to do is take the vanderbaun matrix and multiply it by our constants to get a vector of all 0s and a 1
Example:

$$u''(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = err$$

$$|E| = |u''(x) - \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h)|$$

$$= |u''(x) - \frac{1}{h^2}\{(u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u''''(\zeta_1))$$

$$-2u(u(x) - hu(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3(x) + \frac{h^4}{24}u''''(\zeta_2))\}$$

$$= |u'' - \frac{1}{h^2}\{h^2u''(x) + \frac{1}{24}h^4u''''(\zeta_3)\}|$$

Heat Equation: Heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial k}k(x)\frac{\partial u}{\partial x} + t(x,t)$$

If we assume k is continuous

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + t(x,t)$$

steady state = 1 $\Rightarrow \frac{\partial u}{\partial t} = 0 \Rightarrow Ku'' + t(x,t) = 0$

$$\begin{cases} u'' = f(x) \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$$

Exact solution:

$$\int u''(x)dx = \int f(x)dx = g(x) + c$$

$$u' = g(x) + c_1$$

$$u = \int g(x) + c_1x + c_2$$

We could decide to get an approximation at discrete points in the domain. Lets our domain be $[0,1]$.

So we will use equally spaced (for now) points in $[0,1]$, say $m+2$ points. Then

$$h = \frac{1}{m+1} \Rightarrow \{u_0, u_1, ..., u_m, u_{m+1}\} \quad \text{(size = } m+2\text{)}$$

$$x_j = j * h$$

$$u_0 = \alpha$$

$$u_{m+1} = \beta$$

our u_0, u_{m+1} variables will be exact values.
$u'' = f(x)$ at $x_0, x_1, ...x_n$

$$D^2 u_j = \frac{u_{j-1} 2u_j + u_{j+1}}{h^2}$$

so

$$\frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}) \approx f(x_j)$$

for $j = 0, 1, 2, ..., m+1$

$$\begin{array}{cc} j = 0 & u_0 = \alpha \\ j = 1 & \frac{1}{h^2}(u_0 - 2u_1 + u_2) = f(x_1) \\ j = 2 & \frac{1}{h^2}(u_1 - 2u_2 + u_3) = f(x_1) \\ ... & ... \\ j = m & \frac{1}{h^2}(u_{m-1} - 2u_m + u_{m+1}) = f(x_m) \end{array}$$

$$\begin{bmatrix} 1 & 0 & ... & 0 & 0 \\ -\frac{1}{h^2} & -\frac{2}{h^2} & ... & 0 & \\ 0 & -\frac{1}{h^2} & -\frac{2}{h^2} & ... & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_n \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ f_2 \\ f_n \\ \beta \end{bmatrix}$$

Then we get

$$\begin{bmatrix} \frac{\alpha}{n^2} \\ 0 \\ ... \\ 0 \\ \frac{\beta}{n^2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & ... & 0 & 0 \\ 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & & \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ ... \\ f_m \end{bmatrix}$$

To summarize we get a matrix A that we multiply by a vector U to get the vector of functions F.
