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Example: We assume that f has a derivative at $x = a$.
$f'(a) = \lim_{x\to\infty} \frac{f(x)-f(a)}{x-a} \approx \frac{f(x)-f(a)}{x-a}, x \neq a$

We can also say $x = a + h$ and we get $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \approx \frac{f(a+h)-f(a)}{h}$

Note: in the textbook, they may refer to a as x

We can then construct the equation $D_+f(x) = \frac{f(x+h)-f(x)}{h}, h \geq h_0 \geq 0$

Example: Maltheas Population Model

$$\begin{cases} \frac{dA}{dt} = kA \\ A(0) = A_0 \end{cases}$$

$$\frac{dA}{dt} \approx \frac{A(t+h) - A(t)}{h}$$

$$A'(t) \approx D_+A(t)$$

$$\begin{cases} \frac{A(t+h)-A(t)}{h} \approx kA(t^*) \\ A(0) = A_0 \end{cases}$$

$$h = \Delta t$$

$$\frac{A(t+\Delta t) - A(t)}{h} \approx kA(t^*)$$

$$A(t+\Delta t) \approx A(t) + khA(t^*)$$

$$t = 0 \to A(\Delta t) \approx A(0) + k\Delta t A(t^*)$$

$$A(\Delta t) \approx A_0 + khA_0$$

Example: another model

$$\begin{cases} \frac{dA}{dt} = kA \\ A(0) = a_0 \end{cases}$$

We can then use a bunch of techniques for the approximation of the limit that we discuessed last semester. recall the *forward, backwards, and central* difference quotients.

Errors in Our Approximations

$$|f'(x) - D_+f(x)|$$

$$= |f'(x) - \frac{f(x+h) - f(x)}{h}|$$

$$f(x+h) = f(x) + f'(x)(x+h-x) + \frac{1}{2}f''(x)(x+h-x)^2 + ...$$

$$f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + ...$$

$$= |f'(x) - \frac{1}{h}(hf'(x) + \frac{1}{2}f''(\zeta)h^2) \to | - \frac{1}{2}hf''(\zeta)|$$

This allows us to establish the relationship

$$|f'(x) - D_+f(x)| \leq Ch$$

Homework: compute the error in

$$|f'(x) - D_-f(x)|$$

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Example: $D_+f(x) = \frac{f(x+h)-f(x)}{h} \approx f'(x)$

$$\text{err} = |f'(x) - \frac{f(x+h) - f(x)}{h}|$$

$$= |f'(x) - \frac{1}{h}(f(x) + f'(x)h + \frac{1}{2}f''(\zeta)h^2) - f(x)|$$

$$| - \frac{1}{2}hf''(\zeta)| \leq ch'$$

The idea above is called a **local truncatoin error analysis**.
Implication: To use this difference quotient and have

$$\text{err} \leq Ch$$

Theorem: f must be twice continous differentiable on some interval containing x

Example:

$$D_0f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$\text{err} = |f'(x) - \frac{1}{2h}(f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{6}h^3f'''(\zeta_1)) - (f(x) + hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f'''(\zeta_2)|$$

$$= | - \frac{1}{12}h^2(f'(\zeta_1) + f'(\zeta_2)|$$

$$= \frac{1}{12}h^2|f'''(\theta)|$$

$$\leq Ch^2$$

where

$$C = C(f'''(\zeta_1) + f'''(\zeta_2))$$

Lets recap. We want to find

$$f'(x) = D_+f(x) = \frac{1}{h}f(x+h) - \frac{1}{h}f(x)$$

Example: suppose $f'(x) \approx a_{-1}f(x-h) + a_0f(x) + a_1f(x+h)$
This means

$$f(xh) = f(x) + f'(x)(x-h-x) + \frac{1}{2}f''(x)(x-h-x)^2 + \frac{1}{6}f'''(\zeta_1)(x-h-x)^3$$

$$f(x) = f(x)$$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(\zeta_2)h^3$$

$$f'(x) = a_0(f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(\zeta_2)h^3) + a_0f(x) + a_1(f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(\zeta_2)h^3)$$

$$= f(x)(a_{-1} + a_0 + a_1) + hf'(x)(-a_{-1} + a_1) + \frac{1}{2}h^2 f''(x)(a_{-1} + a_0) + \frac{1}{6}(f'''(\zeta_1) + f'''(\zeta_2))h^3$$

now, let

$$a_{-1} + a_0 + a_1 = 0$$
$$h(-a_{-1} + a_1) = 1$$
$$h^2(a_{-1} + a_1) = 0 \to a_{-1} + a_1 = 0$$

We can now use this system of equations to find h

$$a_{-1} = -a_1$$
$$h(a_1 + a_1) = 2ha_1 = 1 \to a_1 = \frac{1}{2h}$$
$$a_{-1} = -\frac{1}{2h}$$
$$a_0 = 0$$

We can now put this all together

$$f'(x) = -\frac{1}{2h} f(x-h) + \frac{1}{2h} f(x+h)$$

And we get back the **central difference theorem** (amazin').

Example: $\begin{cases} \frac{dA}{dt} = A(t) \\ A(0) = A_0 \end{cases}$

$$\frac{dA}{dt} \approx \frac{A(t+h) - A(t-h)}{2h}$$

Example: Define a function at a set of distributed points $x_1, x_2, ...x_n$ with the idea of approximating $u'(x)$ using

$$u'(x) = c_1 u(x_1) + c_2 u(x_2) + ... + c_n(x_n)$$

$$|u'(x) - \sum_{k=1}^{n} c_k u(x_k)| \le Ch^p$$

$$u(x_i) = u(x) + u'(x)(x_i - x) + \frac{1}{2}u''(x)(x_i - x)^2 + ...$$
$$= \sum_{k=0}^{n} \frac{u^{(k)}(x)}{h!}(x_i - x)^2$$

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Topic: Errors

Example: $d\hat{u}/dt = f(\hat{u}(t)); \hat{u}(0) = n \in \mathbb{R}$
Find all $u(t)$ that approximates $\hat{u}(t)$, try F.D.

$$d\hat{u}/dt = \frac{\hat{u}(t+h) - \hat{u}(t)}{h}$$
$$\begin{cases} \frac{\hat{u}(t+h) - \hat{u}(t)}{h} \approx f(\hat{u}(t)) \\ \hat{u}(0) = n \end{cases}$$
$$\hat{u}(t+h) \approx \hat{u}(t) + hf(\hat{u}(t)$$

Define an approximation, u(t), by

$$\frac{u(t+h)-u(t)}{h}f(u(t))$$
$$u(t+h)=u(t)+hf(u(t))$$
$$u(0)=\mu \Rightarrow u(u+h)=u(0)+hf(u(0))$$

The error is defined as

$$E=\hat{u}(t)-u(t)$$

where $\hat{u}$ represents the approximation of u A measure of the error

$$|E|=|\hat{u}(t)-u(t)|=abs.(E)$$

Example: $\hat{z}=2.2, z=2.20345$

$$E=2.2-2.20345=0.00345=|E|$$

Now, suppose z and $\hat{z}$ are in terms of meters, and we decide to change from meters to nanometres

$$\hat{z}=2.2*10^9, z=2.20345*10^9, |E|=0.00345*10^9=3.45*10^6$$

Now, lets suppose we wanted to transform from meters into kilometers

$$\hat{z}=2.2*10^{-3}, z=2.20345*10^{-3}, |E|=0.0345*10^{-3}$$

This example is *poorly scaled.*

Definition: relative error

$$E_{|e|}=\frac{|\hat{z}-z|}{|\hat{z}|}$$

This gives us a *dimensionless error.*

The problems we will work in will be assumed to be *properly scaled.* This allows us to use the *absolute error* ($E_{abs}=|E|=|\hat{z}-z|$)

Definition: Big-O notation

If $f(h)$ and $g(h)$ are functions of h, then

$$f(h)=O(g(h))$$

as $h \to 0$ if

$$|\frac{f(h)}{g(h)}| \leq c; h \to 0$$

or

$$|f(h)| \leq C|g(h)|$$

Example: $|D_+f(x)-f'(x)|$

$$=|f'(x)-\frac{f(x+h)-f(x)}{h}|$$
$$\leq |1/2hf''(\zeta)|$$
$$\leq Ch$$
$$\Rightarrow |f'(x)-D_+f(x) \leq Ch$$
$$\frac{|f'(x)-D_+f(x)|}{h} \leq c$$
$$|f'(x)-D_+f(x)|=O(h)$$

Definition: Little-o notation
For o(h)

$$f(h) = o(g(h)); \quad h \to 0$$

if

$$|\frac{f(h)}{g(h)}| \to 0$$

for h sufficiently small.

Example: $2h^3 = O(h^2)$

$$2h^3/h2 = 2h < c; \quad h < 1/c$$

Example: $2h^3 = o(h^2)$

$$2h^3/h^2 = 2h \to 0; \quad h \to 0$$

Example: $\sin(h) = O(h); \quad h \to 0$

$$\sin(h) = h - h^3/3! + h^5/5!...; \quad h \in h$$

$$\sin(h) < 1h = Ch$$

Example: $\sin(h) = h + O(h) \Rightarrow \sin(h) - h = O(h)$

$$(\sin(h) - h)/h = O(n^3)$$

Example: $\sqrt{h} = O(1) \Rightarrow \sqrt{h} \to 0 < Ch$

$$\sqrt{h} = O(1) \to \sqrt{h} \text{ to} 0 < Ch$$

$$\sqrt{h} = o(1) \text{ to} \sqrt{h} \to 0$$

$$\sqrt{h} \neq O(h) \to \sqrt{h}/h \to 1/\sqrt{h}$$
