Chandler Justice - A02313187

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<u>Local Truncation Error</u>

τ_j - the error obtained by substituting the exact solution into the finite difference equation

for j=1,2,...,m for a given "discretization" we want

$$AU = f \Rightarrow AU - F = 0$$

and for the LTE

$$A\hat{U} = F + \tau_j = A\hat{U} - F = \tau_j$$

subtract:

$$(AU - F) - (A\hat{U} - F) = -\tau_j \Rightarrow A(U - \hat{U}) = -\tau_j$$

$$AE = -\tau$$

$$E = -A^{-1}\tau \Rightarrow ||E|| = ||A^{-1}\tau||$$

$$||A^{-1}\tau|| \leq ||A^{-1}|| * ||\tau||$$

aside:
<u>Frobini's Norm</u>

$$||A||_F = (\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$$

$$||Ax|| \leq ||A|| * ||x|| \leq C * ||x||$$

$$\frac{||Ax||}{||x||} \leq C$$

where
$||A|| = max_{x \in \mathbb{R};||x|| \neq 0} \frac{||Ax||}{||x||} = max_{x \in \mathbb{R};||x||=1}||Ax||$

<u>P-norms</u>

$$||x|| = (\sum_{j=1}^{m}|x_j|^p)^{\frac{1}{p}}$$

<u>Equation of Norm</u>

$$||E|| \leq ||A^{-1}|| * ||\tau|| \leq C||\tau||$$

<u>Stability:</u> $||A^{-1}|| \leq C$

properties we desire

1. consistency: $||\tau^h|| \rightarrow 0$ as $h \rightarrow 0$
2. Stability: $||E^h|| \leq ||(A^h)^{-1}|| * ||\tau||$

If we have consistency + stability, that implies the method is convergent.

Example:

$$\frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 & ... & 0 \\ 0 & 1 & -2 & 1 & ... \\ ... & ... & ... & ... & ... \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & ... & 0 \\ 0 & 1 & -2 & 1 & ... \\ ... & ... & ... & ... & ... \end{bmatrix} U = h^2 F$$

$$h \rightarrow 0 \Rightarrow m \rightarrow \infty$$

Aside:

$$\begin{cases} u'' = f \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$$

Assume

$$u(\alpha) = \sum_{j=1}^{m} a_j \phi_j(x)$$

Where $\phi_j(x)$ is a basis function for a function space

<u>Condition number matrix</u>

let $A \in R^{m \times m}$, then $\kappa(A) = ||A|| * ||A^{-1}||$

- if $\kappa(A) \approx 1 \Rightarrow$ good condition
- if $\kappa(A) >> 1 \Rightarrow$ the matrix is poorly conditioned

Compute the eigenvalues of A (as defined earlier; a tridiagonal matrix with $[1, -2, 1]$)

$$Av = \lambda v$$

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We want a finite difference method to be

1. consistent: $|\tau_j \leq Ch^p, \quad p > 0, j = 1, 2, ..., m|$
2. stability: $||(A^h)^{-1}|| \leq C$

<u>stability</u> (for 2-norm for matrices)

notes: $||A||_2 = \sqrt{p(A^t A)} \Rightarrow$ we need the eigenvalues and eigenvector of A.
note: $A \in \mathbb{R}^{m \times m}$ is symmetric

$$\begin{aligned} &\Rightarrow ||A||_2 = p(a) \\ &\Rightarrow \text{if } \lambda \text{ is an eigenvalue then } \lambda \in \mathbb{R} \\ &\Rightarrow ||A^{-1}||_2 = p(A^{-1}) = \overset{\max}{1 \leq p \leq m} |(\lambda_1)^{-1}| = (\overset{\min}{1 \leq p \leq m} |\lambda_0|)^{-1} \end{aligned}$$

The eigenvalues for A and the compoents of the eigenvector of A are

$$\begin{aligned}
\lambda_p &= \frac{2}{h^2}(\cos(p\pi h) - 1) \quad p = 1, 2, ..., m \\
u_i^p &= \sin(p\pi jh) \quad p = 1, 2, ..., m, j = 1, 2, ..., m \\
(Au^p)_j &= \frac{1}{h^2}(u_{j-1}^p - 2u_j^p + u + j + 1^p) \\
&= \frac{1}{h^2}(\sin(p\pi(j-1)h) - 2\sin(p\pi jh) + \sin(p\pi(j+1)h)) \\
&= \frac{1}{h^2}(\sin(p\pi jh)\cos(p\pi h) - 2\sin(p\pi jh) + \sin(p\pi jh)\cos(p\pi jh)) \\
&= \frac{1}{h^2}\sin(p\pi jh)(2\cos(p\pi h) - 2) \\
&= (\frac{2}{h^2}(\cos(p\pi h) - 1))\sin(p\pi jh) \\
&\Rightarrow Av_p = \lambda_p v_p \\
&||A^{-1}|| \leq c
\end{aligned}$$

Lets look at λ_1

$$\begin{aligned}
\lambda_1 &= \frac{2}{h^2}(\cos(\pi h) - 1) \\
&= \frac{2}{h^2}((1 - \frac{(\pi h)^2}{2!} + \frac{(\pi h)^4}{4!} + ...) - 1) \\
&= -\frac{2}{h^2}(\frac{\pi^2 h^2}{2} + O(h^4)) \\
&= -\pi^2 + O(h^2) \\
|\lambda_1| &\approx |-\pi^2| \\
&= \pi^2 \leq C \\
||E^h|| &\leq ||(A^h)^{-1}|| * ||\tau^4|| \\
&\leq \pi^2 \leq O(h^2) \Rightarrow \text{implies convergence}
\end{aligned}$$

Sooooooo

$$||\tau^h||_2 = \frac{1}{12}h^2||u''''||_2 \leq Ch^2$$

And we know

$$\begin{aligned}
u'' &= f \\
u''' &= f' \\
u'''' &= f''
\end{aligned}$$

which we can use to express

$$\Rightarrow ||\tau^h||_2 = \frac{1}{12}h^2||f''||_2$$

Lets get our own Eigenvalues/Eigenvectors. We have

$$Av = \lambda v$$

Where A is the *beloved* tridiagonal matrix (with entries $[c, a, b]$ along their respective diagonals) and v is our approximate solutions.

$$\begin{aligned} Av &= \lambda v \Rightarrow (A - \lambda I)v = 0 \\ &= (a-\lambda)v_1 + bv_2 \\ A_1 &= cv_1 + (a-\lambda)v_2 + bv_3 \\ A_{...} &= ... \\ A_m &= cv_{m-1} + (a-\lambda)v_m \end{aligned}$$

given

$$\begin{cases} v_0 = 0 \\ v_{m+1} = 0 \end{cases} \quad \Rightarrow cv_{j-1} + (a-\lambda)cj + b_{j+1} \text{ for all } j = 1, 2, ..., m$$

<u>Second order approximations</u>

$$\begin{cases} \alpha y'' + \beta y'' + \gamma y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases} \qquad \Rightarrow y = e^{rx}$$

$$\alpha r^2 e^{rx} + \beta r e^{rx} + \gamma e^{rx} = 0 \Rightarrow \gamma = \frac{-\beta + / - \sqrt{\beta^2 - 4x\alpha}}{2\alpha}$$

$$\alpha v_{j+1} + \beta v_j + \gamma v_{j-1} = 0$$

$$\begin{cases} v_j = z^j \\ v_{j-1} = z^{j-1} \\ c_{j+1} = z^{j+1} \end{cases}$$
