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revisiting the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K\frac{\partial u}{\partial t} + 4(\gamma t) \\ u(x,0) = g(x) \\ u(a,t) = \alpha(t) \\ u(b,t) = \beta(t) \end{cases}$$

$$\begin{cases} u'' = f \quad x \in (0,1) \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \Rightarrow u'' = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f(x_j) \Rightarrow U_j \approx u(\alpha i) \Rightarrow A^h U^h = F^h$$

Where A is the fucking tridiagonal matrix, U is the approximation at a given value, and F is the result of $U^h * A^h$.

$$\therefore U^h = (A^h)^{-1} F^h$$

This gives approximations at

$$M_h = \{x_1, x_2, ..., x_m\}, \quad h = \text{mesh size}$$

Newmann condition

$$\begin{cases} u'' = f \\ u'(0) = \sigma \Rightarrow (U_0) = \sigma \\ u(1) = \beta \end{cases}$$

We still set up a mesh

$$M_k = \{x_0, x_1, x_2, ..., x_k\}, \quad h = 1/m + 1$$

For $j = 0$:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = \frac{U_{-1} - 2U_0 + U_1}{h^2}$$

The idea is to modify the difference quotient for $j = 0$.

For $j = 1$:

$$\frac{U_0 - 2U_1 + U_2}{h^2} = f(x_j)$$

modifying differnce for $j = 0$
Method 1:

$$u'(0) = \beta \approx \frac{U_1 - U_0}{h}$$

$$A^h U^h = 1/h^2 \begin{bmatrix} -h & h & ... & ... & ... & \\ 1 & -2 & 1 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & .... & 0 \\ ... & ... & ... & ... & ... & \\ 0 & 0 & ... & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U-1 \\ ... \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) \\ ... \\ f(x_{m-k}) \\ \beta \end{bmatrix}$$

From we can analyze

$$|u'(0) - \frac{u(h) - u(0)}{h}| \leq Ch'$$

This forces all local truncation errors to be first order accurate, even though the central differences are more accurate.

Method 2: introduce fictitious nodes
This requires extrapolating outside of our domain, which we do not generally want to do.

$$u'(0) = \sigma \approx \frac{U_1 - U_{-1}}{2h} \quad \text{LTE at } x = 0 \rightarrow \text{2nd order accurate}$$

$$\frac{U_1 - U_{-1}}{2h} = \sigma \Rightarrow U_1 - 2h\sigma = U_{-1}$$

From here, we solve for the fictitious node U_{-1}.

Method 3: Interpolate within
Try to take $0, h, 2h$ and interpolate a derivative within desired range.

$$\frac{1}{h}\left(\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2\right) = \sigma$$

Using this method, the structure of the matrix remains the same,

$$1/h^2 \begin{bmatrix} \frac{3}{2}h & -2h & \frac{1}{2}h & ... & 0 & 0 \\ 1 & -2 & 1 & ... & 0 & 0 \\ 0 & 1 & -2 & 1 & ... & 0 \\ ... & & & & & \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ ... \\ U_m \\ U_{m+1} \end{bmatrix}$$

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Reordering of nodes

$$1/h^2 \begin{bmatrix} \frac{3}{2}h & -2h & \frac{1}{2}h & \dots & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \dots \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \dots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ f(x_3) \\ \dots \\ f(x_{m-1}) \\ f(x_m) - \frac{\beta}{h^2} \end{bmatrix}$$

We can reorder things in terms of odd terms first and then even terms after.

$$\begin{aligned}
&j = 1 \\
&\quad = \frac{1}{h^2}(-2U_1 + U_2) = f(x_1) - \frac{\alpha}{h^2} \\
&j = 3 \\
&\quad = \frac{1}{h^2}(U_2 - 2U_3 + U_4) = f(x_3) \\
&j = 5 \\
&\quad = \frac{1}{h^2}(U_4 - 2U_5 + U_6) = f(x_3) \\
&\dots \\
&j = m-1 \\
&\quad = \frac{1}{h^2}(U_{m-2} - 2U_{m-1} + U_m) = f(x_{m-1}) - \frac{\alpha}{h^2} \\
&\quad \dots \\
&j = 2 \\
&\quad = \frac{1}{h^2}(U_1 - 2U_2 + U_3) = f(x_2) \\
&\quad \dots \\
&j = m \\
&\quad = \frac{1}{h^2}(U_m - 1 - 2U_m) = f(x_m) - \frac{\beta}{h^2}
\end{aligned}$$

Which results in a matrix following the shape

$$1/h^2 = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_3 \\ \dots \\ U_{m-1} \\ U_2 \\ \dots \\ U_M \end{bmatrix} = \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_3) \\ \dots \\ f(x_{m-1}) \\ f(x_2) \\ \dots \\ f(x_m) \end{bmatrix}$$

This divides the matrix into 4 equal regions.