Definition 1 (Orthogonal Diagonalizable) *Let A be a $n \times n$ matrix. A is orthogonal diagonalizable if there is an orthogonal matrix S(i.e. $S^T S = I_n$) such thet $S^{-1}AS$ is diagonal.*

Theorem 2 (Spectral Theorem) *Let A be a $n \times n$ matrix. A is orthogonal diagonalizable if and only if A is symmetric(i.e. $A^T = A$).*

Theorem 3 *If A is a symmetric matrix. If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A with distinct eigenvales λ_1 and λ_2, respectively, then $\vec{v}_1 \cdot \vec{v}_2 = 0$.*

Summary 4 (Orthogonal Diagonalization) *Let A be a $n \times n$ matrix.*

1. *Solve the characteristic equation $\det(A - \lambda I_n) = 0$ for λ with multiplicity to find out eigenvalues.*
2. *For each eigenvalue λ, find a basis of the eigenspace $E_\lambda = \ker(A - \lambda I_n)$.*
3. *Use Gram-Schmidt process to get an orthonormal basis from a basis of (2) for each eigenspace $E_\lambda = \ker(A - \lambda I_n)$.*
4. *Collect all bases of all eigenspaces to get an eigenbasis $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.*
5. *Write $S = [\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n]$ which is orthogonal.*
6. *Write $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$, where λ_i is the eigenvalue corresponding to $\vec{v}_i$ for $i = 1, 2, \cdots, n$.*
7. *Then we have the diagonalization $S^{-1}AS = D$.*

Example 5 *Orthogonally diagonalize $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$.*

[**Solution**] Let us follow the above steps.

1. The characteristic equation is

$$\begin{aligned} 0 &= \det(A - \lambda I_n) = \det \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 4-\lambda & -4 \\ 2 & -4 & 4-\lambda \end{pmatrix} \\ &= (1-\lambda)(4-\lambda)^2 + 16 + 16 - 4(4-\lambda) - 16(1-\lambda) - 4(4-\lambda) \\ &= \lambda^2(\lambda - 9). \end{aligned}$$

We have three solutions for λ, which are $\lambda_1 = 9$, $\lambda_2 = 0$ and $\lambda_3 = 0$.

2. To find a basis for each eigenspace.

- For $\lambda_1 = 9$, we have to find a basis for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in

$$\begin{bmatrix} 1-9 & -2 & 2 \\ -2 & 4-9 & -4 \\ 2 & -4 & 4-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

that is,

$$\begin{bmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

So we have

$$\begin{bmatrix} -4 & -1 & 1 \\ 2 & 5 & 4 \\ 0 & -9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} -4 & 0 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which tell us $x_3 = 2x_1$ and $x_2 = -x_3 = -2x_1$. Therefore, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \\ 2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} x_1$. Hence, $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ is the basis of eigenspace E_{λ_1}.

- For $\lambda_2 = 0$, we have to find a basis for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in

$$\begin{bmatrix} 1-0 & -2 & 2 \\ -2 & 4-0 & -4 \\ 2 & -4 & 4-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

that is,

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

So we have

$$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which tell us $x_1 = 2x_2 - 2x_3$. Therefore, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} x_3$. Hence, $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is the basis of eigenspace E_{λ_2}.

3. To find an orthonormal basis for each eigenspace.

   - For E_{λ_1}, we have a basis $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. Hence, we have an orthonormal basis $\frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ of eigenspace E_{λ_1}.

   - For $\lambda_2 = 0$, we have a basis $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$. By using Gram-Schmidt process, we have an orthonormal basis $\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{3\sqrt{5}} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} \right\}$.
     Hence, $\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{3\sqrt{5}} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} \right\}$ is the basis of eigenspace E_{λ_2}.

4. Write our eigenbasis as $\vec{v}_1 = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$ and $\vec{v}_3 = \begin{pmatrix} \frac{-2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{pmatrix}$.

5. Write $S = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$ which is orthogonal.

6. Write $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

7. So the orthogonal diagonalization is $S^{-1}AS = D$.